Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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MANAGEMENT SCIENCE
Vol. 59, No. 9, September 2013, pp. 2079–2092
ISSN 0025-1909 (print) | ISSN 1526-5501 (online)
http://dx.doi.org/10.1287/mnsc.1120.1660

Inventory Management for an Assembly System Subject to Supply Disruptions

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We consider an assembly system with a single end product and a general assembly structure, where one or more of the component suppliers or (sub)assembly production processes is subject to random supply disruptions. We present a method for reducing the system to an equivalent system with some subsystems replaced by a series structure. This reduction simplifies the computation of optimal ordering policies and can also allow for comparison of disruption impacts across systems with different supply chain structures. We identify conditions under which a state-dependent echelon base-stock policy is optimal. Based on this result, we propose a heuristic policy for solving the assembly system with disruptions and test its performance in numerical trials. Using additional numerical trials, we explore a variety of strategic questions. For example, contrary to what is typically observed in systems without disruptions, we find that choosing a supplier with a longer lead time can sometimes yield lower system costs. We also find that backup supply is more valuable for a supplier with a shorter lead time than one with a longer lead time. In addition, because of component complementarities, we find that choosing suppliers whose disruptions are perfectly correlated yields lower system costs than choosing suppliers whose disruptions are independent, in contrast to the strategy that is typically preferred when choosing backup suppliers for a single product.

Key words: inventory; production; multi-item; multiechelon; multistage; stochastic; approximations; heuristics; dynamic programming
History: Received December 31, 2008; accepted January 23, 2012, by Gérard P. Cachon, operations management. Published online in *Articles in Advance* March 4, 2013.

1. Introduction

As firms increasingly source from suppliers all over the world, supply chains are becoming more extended and complex. Although bringing new opportunities, these global supply chains may be more vulnerable to forces that can cause supply disruptions. Such disruptions can have major negative impacts for the firms involved. For example, the Japanese earthquake and tsunami in March of 2011 significantly reduced production of automobile components in that country, leading to interruptions in auto manufacturing throughout the world (see, e.g., Ramsey and Moffett 2011). As a result, it is increasingly important for firms to have sound strategies for dealing with potential supply disruptions.

Disruptions of components or subassemblies for manufacturing operations can present particular challenges. In many cases, the items whose supply is shut down may represent a small subset of the many components used in final assembly of the product. The unavailability of those items not only delays the ability to produce and sell the finished good, but can also cause investment in inventories of other complementary components and subassemblies to be wasted as those items sit unused.

In this paper we present a model of an assembly system with a single end product, where one or more of the items (components, subassemblies, etc.) is subject to random supply disruptions. We demonstrate how the presence of disruptions interferes with a property known as long-run balance. We show how and when that property is partially preserved by the optimal policy when disruptions are present. We use these results to demonstrate how to reduce an assembly system with disruptions to an equivalent assembly system where some subsystems have been replaced by subsystems with a series structure. This reduction facilitates easier computation of the optimal ordering policy and also allows for comparing the impact of disruptions across systems with different structures. For some special cases, complete reduction to a series system is possible. For general series systems with disruptions, we establish conditions under which the optimal ordering policy is a state-dependent echelon base-stock policy, and we describe how to compute the optimal policy parameters.

For general assembly systems with disruptions, computing the optimal policy is computationally intractable for realistically sized problems. For such systems we propose a heuristic approach. We test the performance of this heuristic on a set of small problems (where the optimal ordering policy and associated system costs can be computed) and find that it yields near-optimal policies.

Through additional numerical trials, we find that a longer supplier lead time can sometimes reduce system costs. We also find that potential supply disruptions at a supplier with a short lead time is more costly than at a supplier with a longer lead time. This indicates that supplier development efforts (or identification of backup suppliers) may be more valuable at short lead time suppliers. Because of the complementarity of component inventories, we observe that choosing two different component suppliers whose disruptions are perfectly correlated—for example, by sourcing both components from the same firm or geographic region—yields lower system costs than choosing suppliers whose disruptions are statistically independent. This highlights a fundamental difference between the sourcing of multiple components in an assembly system and the choice of multiple (backup) suppliers in the sourcing of a single product.

2. Literature Review

Several recent papers provide general discussions of supply chain risk management issues, or reviews of the literature in this area. Kleindorfer and Saad (2005) present a conceptual framework for risk assessment and risk mitigation for supply chains facing disruptions, whereas Chopra and Sodhi (2004) discuss managerial issues related to a broad range of supply chain risks. Tang (2006) provides a review of the academic literature dealing with various types of supply chain risks. Snyder et al. (2010) review research that uses quantitative models to examine supply chain disruptions.

The majority of the analytical research on supply disruptions has focused on single-product, single-location systems. The earliest work in this area is that by Meyer et al. (1979). Many subsequent papers also study system performance and identify optimal or near-optimal policies in settings with deterministic demand and stochastic disruptions. Examples include Parlar and Berkin (1991), Parlar and Perry (1995, 1996), Moinzadeh and Aggarwal (1997), Snyder (2006), and Gürlür and Parlar (1997). Papers that allow for stochastic demand include Gupta (1996), Parlar (1997), and Arreola-Risa and DeCroix (1998).

Several papers expand the single-product setting to include backup suppliers—for example, Li et al. (2004), Tomlin (2006), Tomlin and Snyder (2007), and Schmitt and Snyder (2012). The last of these builds on the work of Song and Zipkin (1996), who present a general model of supply processes that can represent supply disruptions, stochastic lead times, or other state-dependent uncertaines.

Several researchers have studied assembly or other multiechelon systems that exhibit some form of uncertainty in the supply process. Some of this work has focused on systems with stochastic lead times rather than supply disruptions—for example, Song et al. (1999), Gallien and Wein (2001), Song and Yao (2002), Benjaafar and ElHafsi (2006), Nadar et al. (2010), Benjaafar et al. (2011), and Ceryan et al. (2012). Although supply disruptions are a type of stochastic lead time, a key difference is that awareness of the disruption allows a firm to adjust its ordering policy, which is typically not the case with stochastic lead times. Gerchak et al. (1994) analyze assembly systems where supplier deliveries, and possibly the assembly process, are subject to random yield losses. Gurnani et al. (1996) study one- and two-period versions of a two-component assembly system where the supplier delivery process is uncertain. Gurnani et al. (2000) consider a two-component assembly system where supplier deliveries are subject to random yield losses, and determine the ordering and production quantities that minimize an approximate cost function. Kim et al. (2011) consider yield uncertainty in a three-echelon supply chain consisting of a retailer, a distribution center, and multiple suppliers producing identical products. Masih-Tehrani et al. (2011) examine the impact of correlated disruptions in a single-period, multiple-supplier, single-retailer setting and explore how that impact depends on whether the suppliers provide identical products or different components that are assembled into a single end product.

The literature dealing with total disruption of the supply process in a multiperiod, multiechelon setting is somewhat limited. Bollapragada et al. (2004) study an assembly system where component supplier capacities are random, and a fixed base-stock policy is used. Snyder and Shen (2006) use simulation studies to compare the impacts of supply and demand uncertainty under various supply chain structures, including series and distribution systems. Both of these papers assume a fixed ordering policy—i.e., the policy is not dependent on the current state of the suppliers. In contrast, this paper allows for different ordering policies depending on whether supply sources are available. Hopp and Yin (2006) and Gao et al. (2011) consider state-dependent ordering/production policies, but in settings that are simpler than ours in key dimensions, and thus some of the central issues addressed in this paper do not arise. Hopp and Yin (2006) assume deterministic demand and focus on determining the best quantity and placement of additional inventory and/or capacity buffers in the system. Gao et al. (2011) consider an assemble-to-order system with a single tier of component suppliers. Disruptions may occur, and inventories are held, only at the component level, and all times between events (e.g., demands, disruptions, repairs) are exponential. The authors show that state-dependent base-stock policies are optimal, but then propose base-stock heuristics that do not depend on the state of the suppliers.

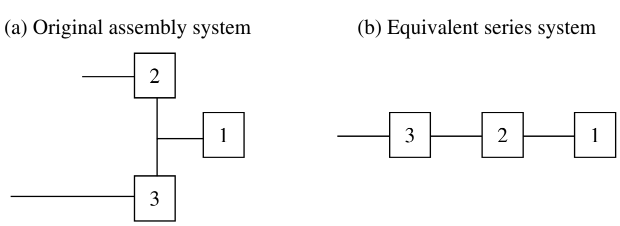
Finally, this paper builds on the literature on optimal policies in periodic review assembly systems with stochastic demand but no disruptions. The most relevant papers in this area are those by Schmidt and Nahmias (1985) and Rosling (1989). The former considers a two-component system with a finite horizon, whereas the latter studies a more general assembly system over an infinite horizon. Rosling (1989) shows that under an optimal policy, inventories in the system satisfy a condition called long-run balance so that the system can be reduced to an equivalent series system. As a result, an optimal policy can be computed using the series-system method of Federgruen and Zipkin (1984) and Chen and Zheng (1994). See Zipkin (2000) for a more detailed discussion of these results.

3. Model

Before formally introducing a detailed model, we use the simple example in Figure 1(a) to illustrate key issues that arise in our assembly system. Item 1 is the end product, which is assembled from component items 2 and 3, which are in turn procured from outside suppliers. In the absence of disruptions, Rosling (1989) shows that the optimal ordering policy preserves balance among the component pipelines—which in this simple system means that a unit of the shorter lead time component (item 2) is ordered only if a matching unit of the longer lead time component (item 3) is scheduled to arrive within a time window equal to the lead time for item 2. This makes intuitive sense, because a unit of item 2 requires a matching unit of item 3 for final assembly. Rosling (1989) then shows that this relationship between inventory positions implies that the system can be reduced to the series system in Figure 1(b).

This same logic continues to hold if item 1 and/or item 3 are subject to random disruptions. However, suppose instead that item 2's supplier is subject to disruptions. To hedge against future loss of supply, it may be optimal to order extra units of item 2 when the supplier is available, violating the balance property.

Figure 1 Simple Three-Item Assembly System

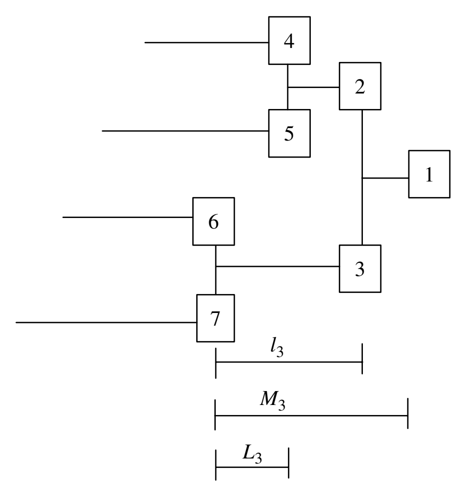


To formalize and expand on the above ideas, we now present the general model; the notation follows that of Rosling (1989). Consider an assembly system with N items indexed $i = 1, \ldots, N$, where again $i = 1$ is the end product. Each item i has a unique immediate successor $s(i)$ (where $s(1) = 0$) and may have a number of immediate predecessors denoted $P(i)$ (where $P(0) = \{1\}$). Assembling a unit of item i that has predecessors requires one unit (without loss of generality) each of all items in $P(i)$. Items without predecessors are ordered from outside suppliers. There is a lead time $l_i \geq 1$ for delivery or assembly of item i. Let $A(i)$ be the set of all successors of item i (i.e., all nodes downstream from i, not just immediate successors), and let $B(i)$ be the set of all predecessors (not just immediate) of item i. Figure 2 illustrates an example of such a system.

Time is discrete, and in each period t the system experiences stochastic demand D_t for the end product (item 1). Demand is stationary over time, demand in each period is bounded above by $\bar{D} < \infty$, and k-period demand is denoted by $D^{(k)}$. Let M_i be the total lead time for item i and all its successors, where $M_0 = 0$ and $M_i = l_i + \sum_{j \in A(i)} l_j$ for $i = 1, \ldots, N$, i.e., the minimum time between ordering a unit of item i and having that unit become part of a finished product. If necessary, reindex items so that $M_i \geq M_{i-1}$ for all i, and $j < i$ for $j \in A(i)$. Also, let $L_i = M_i - M_{i-1}$ be the difference in total lead times between two sequentially indexed items. (In a series system, this would be equal to l_i.) The lead times l_i, M_i, and L_i are illustrated for item $i = 3$ in Figure 2.

A subset $J \subseteq \{1, \ldots, N\}$ of the items is subject to stochastic disruptions. For a component (item with no predecessor), a disruption means the supplier cannot ship that item in the current period. Let K be the set of all components. Disruption of other items means it is not possible to initiate shipment/assembly

Figure 2 Example of General Assembly System



of predecessor items to create the item. At the beginning of each period, the supply source for each $i \in J$ is observed to be either available or unavailable. To simplify exposition, assume for now that disruptions/recoveries are independent across items. (All results continue to hold if simultaneous disruptions/recoveries are allowed. We examine such cases in §6.) If item i's supply is available in a given period, it becomes unavailable in the next period with probability β_i; if it is unavailable, it becomes available with probability γ_i.

Even in pure series systems without disruptions, processing capacities greatly complicate the analysis (see, e.g., Parker and Kapuscinski 2004). For this reason, and to simplify the discussion, we assume that all stages have unlimited processing capacities. (All results except Proposition 4 continue to hold with finite capacities as long as nodes $i \in J$ have enough capacity for "normal" periods—i.e., capacity of at least $\bar{D}$—and nodes $i \notin J$ have enough capacity that they do not constrain orders.)

There is a cost c_i for each unit of item i acquired, due upon arrival. Shortages of the end item are backordered at a unit cost of p' per period, and each unit of item i in inventory or being assembled into its successor incurs a (physical) installation holding cost H_i' per period. Echelon holding costs are defined as $h_i' = H_i' - \sum_{k\in P(i)} H_k'$. Future costs are discounted using a discount factor $0 < \alpha \le 1$, where $\alpha = 1$ is interpreted as the average cost per period case. We work with echelon inventories (on-hand, inventory position), which are defined in the usual way.

It will be useful to refer to the following quantities for item i in period t.

X_{it}: echelon inventory position before ordering/assembly decisions;

Y_{it}: echelon inventory position after ordering/assembly decisions;

X_{it}^l: echelon inventory on hand before ordering/assembly decisions, but after orders arrive;

X_{it}^L: (partial) echelon inventory position, counting only those on-order items that were ordered L_i periods ago or earlier.

Note that

$$X_{it} = Y_{i,t-1} - D_{t-1}, \tag{1}$$

and similarly,

$$X_{it}^l = Y_{i,t-l_i} - \sum_{a=t-l_i}^{t-1} D_a \quad \text{and}$$
$$X_{it}^L = Y_{i,t-L_i} - \sum_{a=t-L_i}^{t-1} D_a. \tag{2}$$

Orders for item i must satisfy $Y_{it} \ge X_{it}$ (disposal or negative orders are not allowed) and also $Y_{it} \le \bar{X}_{it}$, where

$$\bar{X}_{it} = \begin{cases} X_{it} & \text{if } i \in J \text{ and supply is unavailable,} \\ \infty & \text{if } i \in J \cap K \text{ and supply is available,} \\ \min_{k\in P(i)}\{X_{kt}^l\} & \text{if } i \in J,\, i \notin K \text{ and supply is available,} \\ \infty & \text{if } i \notin J,\ i \in K, \\ \min_{k\in P(i)}\{X_{kt}^l\} & \text{if } i \notin J,\ i \notin K. \end{cases}$$

We seek a policy (represented by Y_{it}) that minimizes the expected discounted cost of operating the system over an infinite horizon.

Using the standard approach (e.g., see Rosling 1989) of taking expected values, redefining cost parameters $h_i = h_i' + (1-\alpha)c_i$, $H_i = h_i + \sum_{k\in P(i)} H_k$, and $p = p' - (1-\alpha)\sum_{i=1}^N c_i$, and dropping constant terms that do not affect the decisions, the assembly problem can be stated as follows.

Assembly Problem:

$$\min_Y E\left\{\sum_{t=1}^\infty \alpha^{t-1}\left(\sum_{i=1}^N \alpha^{l_i} h_i Y_{it} + \alpha^{l_1}(p+H_1)[D^{(l_1+1)} - Y_{1t}]^+\right)\right\} \tag{3}$$

subject to

$$X_{it} \le Y_{it} \le \bar{X}_{it} \quad \text{for all } i \text{ and } t. \tag{4}$$

In the case of $\alpha = 1$, (3) is multiplied by the discount rate $\alpha/(1-\alpha)$, and then $\alpha \to 1$.

Assume the cost parameters satisfy $h_i > 0$ for all i and $\sum_{i=1}^N h_i \alpha^{-M_{s(i)}} < p + H_1$. The latter assures that it is optimal to fill existing backorders, whereas the former reflects higher physical and financial holding costs associated with items that have progressed further through the system.

Define $X_{it}^{M-\mu}$ to be the echelon inventory position at time t of item i ordered $M_i - \mu$ periods ago or earlier. This quantity is an upper bound, based on the current echelon inventory and units on order of item i, on the amount of end product that could be made available within μ time periods. Letting $s = M_i - \mu$, we have

$$X_{it}^{M-\mu} = Y_{i,t-s} - \sum_{a=t-s}^{t-1} D_a, \quad s = 0, 1, \ldots, M_i.$$

The following properties of $X_{it}^{M-\mu}$, established by Rosling (1989) for systems without disruptions, continue to hold here. The logic is the same as in Rosling (1989), so we omit the proof.

Lemma 1. (a) $X_{it}^{M-\mu} \ge X_{it}^{M-\mu+1}$, (b) $X_{it} = X_{it}^{M-(M_i-1)}$, (c) $X_{it}^{M-M_{s(i)}} = X_{it}^l$, *and* (d) $X_{it}^L = X_{it}^{M-M_{i-1}}$.

4. Policy Properties

In general, solving the assembly problem requires tracking all inventories on hand or in transit, making the problem computationally intractable for all but very small systems. To mitigate this, we show that in many cases at least parts of the system can be reduced to series subsystems, as Rosling (1989) shows for the entire system in the absence of disruptions. Rosling's (1989) results depend on a condition he calls long-run balance. We modify that condition by defining it for each item individually, as well as for the system as a whole.

Definition (Long-Run Balance). Item i exhibits long-run balance in period t if

$$X_{it}^{M-\mu} \le X_{kt}^{M-\mu} \quad \text{for } k > i \text{ and } \mu = 0, \ldots, M_i - 1. \quad (5)$$

If the condition holds for all $i = 1, 2, \ldots, N-1$, we say that the system is in long-run balance.

Although the optimal policy may not preserve systemwide long-run balance, it does preserve *item-specific* long-run balance for items not affected by disruptions—i.e., neither the item itself nor any of its successors are subject to disruptions. For items that are affected by disruptions, a similar version of balance is preserved, but only relative to other items that are affected by disruptions at the same downstream node. These results are formalized in the following proposition. All proofs are provided in the appendix.

Proposition 1. *For each item i such that* $\{\{i\} \cup A(i)\} \cap J = \varnothing$, *the optimal ordering policy in each period satisfies* $Y_{it} \le \max\{X_{kt}^{M-M_i}, X_{it}\}$ *for all* $k > i$. *For each item i such that* $\{\{i\} \cup A(i)\} \cap J \ne \varnothing$, *the optimal ordering policy in each period satisfies* $Y_{it} \le \max\{X_{jt}^{M-M_i}, X_{it}\}$ *for any* $j > i$ *such that* $\max\{k\colon k \in \{\{i\} \cup A(i)\} \cap J\} = \max\{k\colon k \in \{\{j\} \cup A(j)\} \cap J\}$.

Note that if item i is not affected by disruptions, but does not exhibit long-run balance in period t—i.e., if $X_{it} > X_{kt}^{M-M_i}$ for some $k > i$—then $Y_{it} = X_{it}$ is optimal. After a number of periods of this, the system will reach long-run balance with respect to item i.

This partial preservation of long-run balance allows a partial reduction of the assembly system to an equivalent system with some subsystems reduced to a series structure. To formally present this reduction, define $b = \min\{i\colon i \in J\}$. Also, for $i \ge b$, define the modified lead time $\hat{L}_i = M_i - M_{b-1}$. The following algorithm describes the reduction process.

Algorithm (Partial Series Reduction)

1. For $i = 1, 2, \ldots, b-1$, the items take on a series structure with the lead time for item i given by $L_i = M_i - M_{i-1}$. (If $b = 1$, go to step 4 with $G = P(1)$.)

2. All nodes $i \in G = \{j \ge b\colon s(j) \le b-1\}$ become part of a single echelon now with $s(i) = b-1$ for all $i \in G$. The lead time for each item $i \in G$ is given by $\hat{L}_i$.

3. Modify the holding costs so that the new coefficients become $h_i = h_i \alpha^{l_i - L_i}$ for $i \le b-1$, $h_i = h_i \alpha^{l_i - \hat{L}_i}$ for $i \in G$, and h_i remains unchanged for all other i. The coefficient $(p + H_1)$ remains unchanged.

4. For each $i \in G$, consider the subsystem consisting of $\{i\} \cup B(i)$. Temporarily assign a new index $\hat{k}$ to each item in the subsystem as if that subsystem were the entire system—for example, $\hat{i} = 1$, etc. Repeat steps 1–3 for the subsystem.

5. Continue moving out the branches of the assembly tree, repeating step 4 at each echelon.

6. For any pair of items $i < j$ with $s(i) = s(j)$, if $\{i \cup B(i)\} \cap J = \{j \cup B(j)\} \cap J = \varnothing$, consider the subsystem consisting of $F \equiv \{i \cup B(i)\} \cup \{j \cup B(j)\}$. Temporarily re-index items in F similar to step 4, starting with $\hat{i} = 1$. Replace the subsystem with a series structure with lead time for item $\hat{k}$ given by $L_{\hat{k}} = M_{\hat{k}} - M_{\hat{k}-1}$ and new cost coefficients $h_{\hat{k}} = h_{\hat{k}} \alpha^{l_{\hat{k}} - L_{\hat{k}}}$.

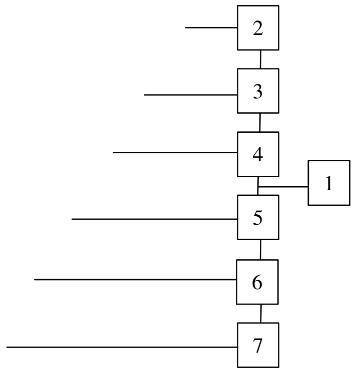
The following proposition establishes the correctness of this procedure.

Proposition 2. *In an assembly system with disruptions, if all items exhibit long-run balance at the beginning of the planning horizon, then the optimal policy for the assembly system is equivalent to the optimal policy of the system obtained through the partial series reduction algorithm.*

Partial reduction to a series system reduces computational effort by reducing the state space and facilitating the use of more streamlined series solution techniques. For example, consider the system in Figure 2, and suppose $J = \{6\}$. By Proposition 2, the subsystem involving items 1–5 can be reduced to a series subsystem. The bottom-up approach of Chen and Zheng (1994) and Zipkin (2000) can be used to construct an implicit cost function that allows compression of that subsystem into a single stage (stage 5), effectively reducing the overall system to one with just three items—items 5, 6, and 7—similar to the one depicted in Figure 1(a).

Proposition 2 also may allow comparison of disruption impacts on systems with different structures. For example, compare the system in Figure 2 with the one in Figure 3 (where both systems have the same total lead times M_i). Using Proposition 2, one can show that for any $J \subseteq \{1, 4, 5, 6, 7\}$, the two systems can be reduced to the same modified assembly system, so the optimal policies would be the same. This is no longer the case if items 2 or 3 are disrupted—for example, if $J = \{3\}$, Figure 2 reduces to Figure 4(a), whereas Figure 3 reduces to Figure 4(b).

Figure 3 Assembly System with Flat Structure



In some special cases, it is possible to fully reduce the assembly system to a series system with disruptions, as stated in the following proposition.

Proposition 3. *If $J = \{i\}$ for some i such that for all $k > i$ we have $k \in B(i)$, and if all items exhibit long-run balance at the beginning of the planning horizon, then the optimal policy in the assembly system is equivalent to that of a series system with disruptions at echelon i.*

Two special cases of the proposition are $i = 1$ and $i = N$, but some other cases satisfy the conditions as well—for example, the system in Figure 5 with $J = \{3\}$.

Although Proposition 3 applies only to systems where a single item experiences disruptions, the following result shows that it is possible to characterize the form of the optimal policy for a series system with general disruption set J.

Proposition 4. *For a series system with disruptions, a state-dependent echelon base-stock policy is optimal, where the optimal base-stock level at echelon i depends on the state of the supply sources at echelons $k < i$.*

Figure 4 Partial Series Reduction Algorithm—Example with Different Outcomes

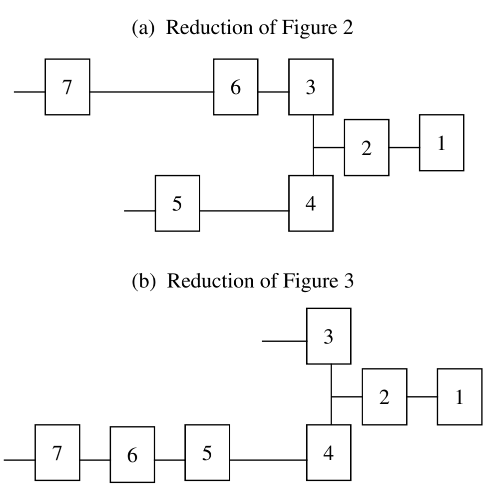
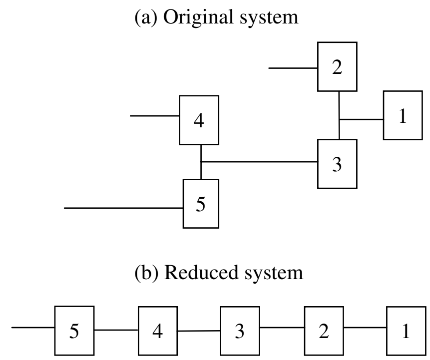


Figure 5 Illustration of Proposition 3



The optimal policy for a series system with disruptions can be computed using a variant of the bottom-up approach of Chen and Zheng (1994) and Zipkin (2000, Section 8.3.3), where the computations at stage N are modified to incorporate state information and the potential for disruptions. (For details, see the description of the heuristic policy in the next section. The heuristic policy is optimal for a series system with disruptions.)

Note that equivalence to a series system fails even in some settings where one might expect it to hold. For example, suppose the suppliers of items 2 and 3 in Figure 1(a) are always disrupted simultaneously. Despite the pure complementarity of those items, long-run balance may not be preserved. When a disruption occurs, units of item 3 in the (early part of the) supply pipeline may not be paired with units in item 2's pipeline. In anticipation of this, it may be optimal to inflate orders of item 2 (beyond item 3's pipeline levels) when the supplier is available, disrupting long-run balance.

5. Heuristic Policy

Except for the special cases in Proposition 3, or for very small systems, computation of the optimal policy is impractical. For this reason, we seek a heuristic policy that is easy to compute and that performs well for general systems. For simplicity we focus on the average cost case ($\alpha = 1$).

Recall that without disruptions it is optimal to follow the modified base-stock policy

$$Y_{it}^* = \begin{cases} \min\{\hat{S}_i, X_{i+1,t}^L\} & \text{if } X_{it} \le \hat{S}_i, \\ X_{it} & \text{if } X_{it} \ge \hat{S}_i, \end{cases} \tag{6}$$

where the $\hat{S}_i$, $i = 1, \ldots, N$, are optimal base-stock levels for the equivalent series system, and the upper bound $X_{i+1,t}^L$ preserves long-run balance. The heuristic uses this same structure, where the base-stock levels are now the optimal (state-dependent) base-stock levels for a series system with disruptions. The heuristic also uses the upper bounds $X_{i+1,t}^L$ to preserve long-run balance, even though this may not be optimal—numerical testing on some small problems suggests that there is little increased cost from doing so. This allows us to use a variant of the bottom-up approach of Chen and Zheng (1994) and Zipkin (2000) to compute the state-dependent base-stock levels.

Formal statement of this variant and the resulting heuristic requires derivation of multiperiod supply-state transition probabilities and steady-state distributions of inventory positions for items subject to disruptions. For the latter, consider a single item $i \in J$, and suppose for now that item i follows a pure echelon base-stock policy with target S_i. Define $W_{it} = S_i - Y_{it}$ under such a policy—i.e., the amount by which item i falls short of its order-up-to target due to disruption of supply. (Note that $W_{it} = 0$ whenever supply is available.) Letting W_i denote W_{it} in steady state, the distribution of W_i can be computed through a simple recursion. Defining $\pi_i^w = \Pr\{W_i = w\}$, $w = 0, 1, \ldots$, $\pi_i^{01} = \Pr\{W_i = 0$ and supply source of item i is available$\}$, and $\pi_i^{00} = \Pr\{W_i = 0$ and supply source of item i is unavailable$\}$, we have

$$\begin{aligned}
\pi_i^{01} &= \frac{\gamma_i}{\gamma_i + \beta_i},\\
\pi_i^{00} &= \left(\frac{\beta_i \Pr\{D_t = 0\}}{1-(1-\gamma_i)\Pr\{D_t = 0\}}\right)\pi_i^{01},\\
\pi_i^{0} &= \pi_i^{01} + \pi_i^{00},\\
\pi_i^{w} &= \left(\frac{1}{1-(1-\gamma_i)\Pr\{D_t = 0\}}\right)\\
&\quad\cdot\Biggl(\beta_i \Pr\{D_t = w\}\pi_i^{01} + (1-\gamma_i)\Pr\{D_t = w\}\pi_i^{00}\\
&\quad + (1-\gamma_i)\sum_{z=1}^{w-1}\Pr\{D_t = w - z\}\pi_i^{z}\Biggr).
\end{aligned} \tag{7}$$

For all $i \notin J$, define $W_i = 0$.

Temporarily, to simplify exposition, assume that only a single item is disrupted, i.e., $J = \{j\}$. (The approach extends in a natural way to more general disruption patterns.) Define the l-period transition probabilities for the state u of item j's supply source as

$$\rho(u'' \mid u'; l) = \Pr\{u(t+l) = u'' \mid u(t) = u'\}.$$

These can be computed recursively as follows:

$$\begin{aligned}
&\rho(1\mid 1;1) = (1-\beta_j), \quad \rho(0\mid 1;1) = \beta_j,\\
&\rho(1\mid 0;1) = \gamma_j, \quad \rho(0\mid 0;1) = (1-\gamma_j),
\end{aligned} \tag{8}$$

and, for $l > 1$ and $u = 0$ or 1,

$$\begin{aligned}
\rho(u\mid 1;l) &= (1-\beta_j)\rho(u\mid 1;l-1)\\
&\quad + \beta_j \rho(u\mid 0;l-1),\\
\rho(u\mid 0;l) &= \gamma_j \rho(u\mid 1;l-1)\\
&\quad + (1-\gamma_j)\rho(u\mid 0;l-1).
\end{aligned} \tag{9}$$

Using these quantities, the following recursion computes the state-dependent base-stock levels $S_i^*(u)$ used in the heuristic:

$$\begin{aligned}
&\underline{C}_0^*(x,u) = (p + H_1)[x]^-, \quad u = 0 \text{ or } 1;\\
&\hat{C}_i^*(x,1) = h_i x + \rho(1\mid 1;L_i)\underline{C}_{i-1}^*(x,1)\\
&\qquad + \rho(0\mid 1;L_i)\underline{C}_{i-1}^*(x,0);\\
&\hat{C}_i^*(x,0) = h_i x + \rho(1\mid 0;L_i)\underline{C}_{i-1}^*(x,1)\\
&\qquad + \rho(0\mid 0;L_i)\underline{C}_{i-1}^*(x,0);\\
&C_i^*(y,u) = E[\hat{C}_i^*(y - W_i - D^{(L_i)}, u)];\\
&S_i^*(u) = \arg\min\{C_i^*(y,u)\}, \quad i \le N;\\
&\underline{C}_i^*(x,u) = \begin{cases} 0 & \text{if } i = j,\ u = 0,\\ C_i^*(x,u) - C_i^*(S_i^* \vee x, u) & \\ \quad\text{otherwise.} \end{cases}
\end{aligned} \tag{10}$$

Note that, for $i \le j$, the cost functions in the recursion are independent of u, so the procedure generates a single base-stock level for those stages.

After computing the base-stock levels using (10) (or an analogous recursion with more general disruption patterns), the heuristic policy orders units of each item i so that

$$Y_{it} = \begin{cases} \min\{S_i^*(\mathbf{u}), X_{i+1,t}^L\} & \text{if supply source for } i \text{ is available and } X_{it} \le S_i^*(\mathbf{u}),\\ X_{it} & \text{otherwise.} \end{cases}$$

The $S_i^*(\mathbf{u})$ can be computed quickly—after computing the π and ρ probabilities, (10) is only a little more involved than the recursion for computing optimal echelon base-stock levels in a series system. (In the numerical study, the calculations were done in a spreadsheet in a matter of seconds.) In addition, because the policy is just a modified echelon base-stock policy with base-stock levels dependent on the state of supply sources, the policy has a relatively intuitive structure, and is nearly as easy to implement as the modified base-stock policy that is optimal for an assembly system without disruptions.

6. Numerical Study

In this section we perform a two-part numerical study. The first part explores performance of the heuristic relative to the optimal policy for small problems where the optimal policy can be computed. The second part explores several strategic questions related to supplier selection and assembly system design.

Table 1 **Parameters for Heuristic Performance Test**

Holding costs (h_1, h_2, h_3)	Average length of disruption	Average length of disruption / Average time between disruptions
(1, 1, 1)	1.33	0.02
(4, 1, 1)	2	0.05
(1, 4, 1)	2.67	0.10
(1, 1, 4)	4	

6.1. Heuristic Performance

We begin by testing the heuristic on variations of the three-item system in Figure 1(a) with lead times $(l_1, l_2, l_3) = (1, 1, 2)$, possible demand realizations $D_t = \{0, 1, 2, 3, 4\}$, and associated demand probabilities $\{0.1, 0.2, 0.4, 0.2, 0.1\}$. We consider three different disruption patterns—(i) $J = \{2\}$, (ii) $J = \{2, 3\}$ where supplier disruptions for the two items are perfectly correlated, and (iii) $J = \{2, 3\}$ where supplier disruptions for the two items are independent. (Recall that if $J = \{3\}$, Propositions 3 and 4 imply that the heuristic is optimal, so this case is not tested.) For each disruption pattern, we consider all 48 combinations of holding cost and disruption parameters in Table 1. (Unit backorder costs are fixed at $p = 10$ for all cases.)

For each of the resulting 144 test problems, we compute expected holding/backorder costs per period for the heuristic policy and the optimal policy (computed by dynamic programming). For both policies, we subtract expected in-transit holding costs, which are constant across policies. Performance is measured by the relative error:

$$\text{relative error} = 100\% \times \frac{\text{(exp. cost of heuristic policy)} - \text{(exp. cost of optimal policy)}}{\text{(exp. cost of optimal policy)}}.$$

The average, minimum and maximum errors are reported in Table 2.

To test the effect of system size on heuristic performance, we also consider the two different five-item problems shown in Figure 6. For the two-echelon system, we assume $l_1 = 1$ and $l_i = i - 1$, $i = 2, \ldots, 5$, and consider disruption patterns with $J = \{3\}$ and $J = \{4\}$. For the three-echelon system, we assume $l_i = 1$, $i = 1, \ldots, 5$, and examine $J = \{3\}$. (Note that this latter system also allows us to test the performance of the heuristic in a setting with disruption of a noncomponent item.) For both systems we use echelon holding costs $(1, 1, 1, 1, 1)$, $(4, 1, 1, 1, 1)$, $(1, 4, 1, 1, 1)$, $(1, 1, 4, 1, 1)$, $(1, 1, 1, 4, 1)$, and $(1, 1, 1, 1, 4)$, and the disruption profiles from Table 1.

Table 2 **Relative Error for Heuristic—Three Items**

	$J = \{2\}$ (%)	$J = \{2, 3\}$ correlated (%)	$J = \{2, 3\}$ independent (%)
Average	0.07	0.08	0.09
Min	0	0	0
Max	1.17	1.17	1.08

Figure 6 **Test Problems with Five Items**

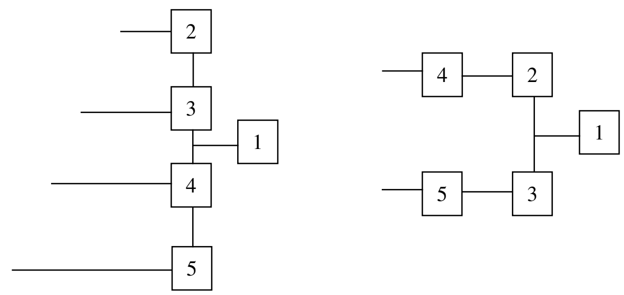


Table 3 **Relative Error for Heuristic—Five Items**

	Two-echelon, $J = \{3\}$ (%)	Two-echelon, $J = \{4\}$ (%)	Three-echelon, $J = \{3\}$ (%)
Average	0.18	0.07	0.14
Min	0	0	0
Max	2.43	2.86	2.02

Note that the average errors for the five-item system (Table 3) are similar to those in Table 2, suggesting very little deterioration in heuristic performance as the number of items grows from three to five.

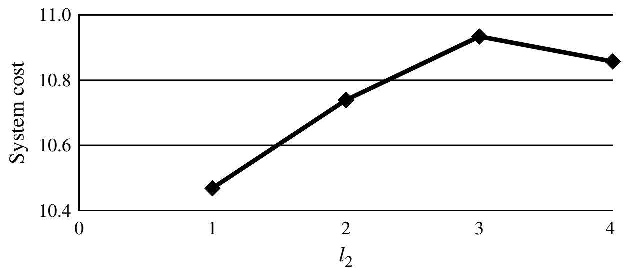
To assess the value added by the heuristic, we also consider two simpler policies—one with fixed base-stock levels and one where disruptions are ignored altogether. Tests of these policies on the system in Figure 1(a) (with $J = \{2\}$ and the parameters in Table 1, extended to include cases with average downtime as long as 16 periods) reveal that the heuristic can deliver substantial benefits, especially when disruption risks are significant. For example, for a downtime-to-uptime ratio of 0.1 and average disruption duration of 16 periods, both simpler policies yield relative errors greater than 17%, whereas the largest heuristic error for that case was only 0.60%.

6.2. Supplier Selection Issues

Although the focus so far has been on operational decisions, there are strategic choices that set the framework for those decisions. For example, an assembler must consider the impact of potential disruptions when selecting suppliers and structuring the supply chain. We explore these issues now.

6.2.1. Lead Times. Increased reliance on overseas suppliers leads to longer lead times and may also bring increased risk of disruptions (from weather, political instability, etc.). We know that longer lead times increase system costs in the absence of disruptions, and that increased disruption risk increases system costs for fixed lead times. What is less clear is the interaction between the two.

Figure 7 System Cost as Unreliable Supplier Lead Time Increases



We explore this by fixing a disruption pattern and seeing how system costs change with lead times. Using the system in Figure 1(a) with $J = \{2\}$, $(h_1, h_2, h_3, p) = (1, 1, 1, 10)$, an average disruption duration of two periods, a downtime-to-uptime ratio of 0.05, and $(l_1, l_3) = (1, 4)$, we calculate system costs for lead times $l_2 = 1$, 2, 3, or 4. These costs are shown in Figure 7.

One striking observation is that a longer lead time ($l_2 = 4$ versus $l_2 = 3$) can sometimes yield lower costs. This counterintuitive result is due to the challenge of balancing component inventories when a shorter lead time item is subject to disruptions. When $l_2 < l_3$, units of item 3 must be ordered before units of item 2. If a disruption of item 2's supplier occurs in the meantime, those units of item 3 will sit idle—increasing the lead time so that $l_2 = l_3$ avoids that possibility. This comes at a cost, however, because a longer lead time makes it more difficult to match supply of item 2 to end-product demand (as in all inventory models). The concavity of the curve in Figure 7 suggests that the component balancing benefit grows relative to the supply/demand mismatch cost as l_2 grows closer to l_3. (This pattern was consistent over a range of cases tested.) This makes inuitive sense—in many settings supply/demand mismatch costs tend to increase concavely in lead time, and it is plausible that the component balancing benefit is greatest when full balance is achieved (i.e., moving from $l_2 = 3$ to $l_2 = 4$ as opposed to moving from $l_2 = 1$ to $l_2 = 2$). In the cases tested, the net cost reduction from increasing l_2 was never large—no more than approximately 3%. But the existence of any benefit from a longer lead time stands in contrast to behavior seen in most inventory systems.

By varying other system parameters, we find that increasing l_2 is more attractive when (i) demand is less variable (matching supply to demand is easier), (ii) the system has a higher percentage of downtime (disruptions are a bigger factor, so the balancing benefit is greater), (iii) disruptions are rare but long, (iv) item 2 holding costs are low (item 2 supply/demand mismatch is less costly), and (v) item 3 holding costs are high (better component balance helps avoid excess inventory of item 3). Proposition 3 (and observations by Rosling 1989) implies that this three-item system is equivalent to a system with multiple identical copies of item 3—where the holding cost at item 3 is split among those suppliers. With that interpretation, observation (v) also implies that longer item 2 lead times are more valuable when the system contains more long lead time components.

6.2.2. Value of Backup Suppliers. Disruption risk can be mitigated by qualifying backup suppliers for some items. Because this may be too costly to do for all suppliers, which items should have priority? For example, if suppliers of two items differ only in their lead times, which one should have a backup? Backing up the shorter lead time supplier would bring component balancing benefits—with disruptions effectively eliminated, that supplier could respond effectively to the state of the other supplier. However, backing up the longer lead time supplier would eliminate a source of variability for that item, and variability usually has a more negative impact when lead times are longer. To explore which is better, we use the scenarios in Table 1 and compute optimal system costs for the cases $J = \{2\}$ and $J = \{3\}$. (These cases can be interpreted as starting with both suppliers subject to disruptions and then backing up the supplier for item 3 or item 2, respectively.) For all scenarios considered, we find that $J = \{2\}$ yields higher system costs—i.e., it is more valuable to back up the shorter lead time supplier. By examining system costs under different parameters, we find that the value of a backup supplier is also greater when the system has a higher percentage downtime, when disruptions are rare but long, and when item 3 holding costs are high (in the case of $J = \{2\}$).

6.2.3. Correlated Disruptions. Suppose a firm seeks suppliers for two components, both of which are subject to disruptions. The firm could source those components so that the disruptions are highly correlated (e.g., the items are produced in the same facility or must travel through the same shipping port), or so that the disruptions are largely independent (the items are made by different companies, in different geographic regions, etc.). Which would be preferred? Comparing the numerical results for $J = \{2, 3\}$ with perfectly correlated disruptions to the results with independent disruptions, we find that system costs are always higher under the scenario with independent disruptions (for all parameter combinations in Table 1). Because of the complementarity of the components, production can only continue if *both* suppliers are available—and joint uptime is maximized by choosing correlated disruptions. This stands in stark contrast to the situation where a firm seeks multiple (backup) suppliers for a *single* component. In that case, lower correlation among supplier disruptions is generally helpful, because production can continue if either supplier is available. (Masih-Tehrani et al. (2011)

subsequently obtained a similar result in a somewhat different setting.)

7. Conclusions

In this paper we explored the management of inventories in an assembly system where some items face random supply disruptions. We demonstrated how the presence of disruptions can interfere with long-run balance in the system, but showed that the property is partially preserved. These results allowed us to present an algorithm for reducing an assembly system with disruptions to an equivalent system with some subsystems replaced by a series structure. This reduction facilitates easier computation of the optimal policy and also allows for comparing the impact of disruptions across systems with different structures. For some special cases, complete reduction to a series system is possible. For a series system with a general disruption pattern, we showed that the optimal policy is a state-dependent echelon base-stock policy, and we described how to compute the policy parameters. Because of the difficulty of computing the optimal policy for more general assembly systems with disruptions, we used the structure of the optimal series system policy as the foundation for a heuristic policy, which performed well in numerical trials.

Based on additional numerical trials, we explored various strategic questions related to supplier selection and system design. In contrast to what is observed in systems without disruptions, we found that choosing a supplier with a longer lead time can sometimes yield lower system costs, and we explored how this phenomenon is affected by other system parameters. We also found that, given fixed lead times, an unreliable supplier with a short lead time is a better candidate for backup supply arrangements than a similar supplier with a longer lead time. In addition, we observed that choosing two different component suppliers whose disruptions are perfectly correlated—for example, by sourcing both components from the same firm or the same geographic region—yields lower system costs than choosing suppliers whose disruptions are statistically independent. This highlights a fundamental difference between the sourcing of multiple components in an assembly system and the choice of multiple (backup) suppliers in the sourcing of a single product.

Acknowledgments

The author thanks attendees of the 2007 INFORMS National Meeting and the 2008 MSOM Conference; seminar participants at the University of Wisconsin-Madison, Duke University, and the University of British Columbia; and the anonymous associate editor and referees for their constructive comments that helped improve this paper.

Appendix. Proofs of Results

The proof of Proposition 1 relies on an interim result—we state and prove this first.

Lemma 2. *If $k \in B(i)$, then for any $\mu = 0, 1, \ldots, M_i$, $X_{it}^{M-\mu} \le X_{kt}^{M-\mu}$. Also, $X_{it} \le X_{kt}^{M-M_i}$, so feasibility implies that $X_{it} \le Y_{it} \le X_{kt}^{M-M_i}$.*

Proof. First, suppose that $k \in P(i)$. Then

$$\begin{aligned} X_{it}^{M-\mu} &= Y_{i,\,t-(M_i-\mu)} - \sum_{a=t-(M_i-\mu)}^{t-1} D_a \\ &\le X_{k,\,t-(M_i-\mu)}^{M-M_i} - \sum_{a=t-(M_i-\mu)}^{t-1} D_a \\ &= Y_{k,\,t-(M_i-\mu)-(M_k-M_i)} - \sum_{a=t-(M_i-\mu)-(M_k-M_i)}^{t-(M_i-\mu)-1} D_a \\ &\quad - \sum_{a=t-(M_i-\mu)}^{t-1} D_a = X_{kt}^{M-\mu}. \end{aligned}$$

The definition of echelon inventory implies that $X_{it} \le X_{kt}^{l} = X_{kt}^{M-M_i}$, and feasibility implies that $X_{it} \le Y_{it} \le X_{kt}^{l} = X_{kt}^{M-M_i}$. This establishes the result for $k \in P(i)$. By choosing $j \in P(i)$ such that $j \in A(k)$, we can show that $X_{jt}^{M-\mu} \le X_{kt}^{M-\mu}$ for any $\mu = 0, 1, \ldots, M_i$ by repeatedly applying the argument above. This extends the result to $k \in B(i)$. □

Proof of Proposition 1. For the first part, suppose Y^* is an optimal policy such that $Y_{it}^* > X_{it}$ and $Y_{it}^* > X_{kt}^{M-M_i}$ for some $k > i$ and t. By Lemma 2, this implies $k \notin B(i)$. Define w such that $X_{wt}^{M-M_i} = \min_{k>i} X_{kt}^{M-M_i}$. Construct the alternate policy $\bar{Y}$ (with resulting states $\bar{X}$):

$$\bar{Y}_{it} = \max\{X_{wt}^{M-M_i}, X_{it}\} \quad \text{and}$$
$$\bar{Y}_{s(k),\,q} = \min\{Y_{s(k),\,q}^*, \bar{X}_{kq}^{l}\} \quad \text{for all } k \in \{i\} \cup A(i),$$
$$\text{where } q = t + M_i - M_{s(k)}.$$

The policy $\bar{Y}$ postpones some ordering of item i in period t, and also postpones orders of downstream items in future periods if necessary to preserve feasibility. Because item $i \notin J$ and $k \notin J$ for all $k \in A(i)$, $\bar{Y}$ will not postpone ordering into a period in which a supply source for item i or its successors is unavailable. As a result, $\bar{Y}$ is feasible and satisfies the required condition. Also, since $\bar{Y}_{it} < Y_{it}^*$, $\bar{Y}_{ks} \le Y_{ks}^*$, and $h_k > 0$ for all k and s, $\bar{Y}$ yields lower holding costs. The remainder of the proof follows the same arguments as in the proof of Lemma 1 of Rosling (1989) and is omitted.

For the second part, if $\max\{k\colon k \in \{\{i\} \cup A(i)\} \cap J\} = i$, then $j \in B(i)$. In that case, Lemma 2 implies that $Y_{it}^* \le X_{jt}^{M-M_i}$, so the result holds. If not, then $\max\{k\colon k \in \{\{i\} \cup A(i)\} \cap J\} < i$. In that case, again suppose that Y^* is an optimal policy such that $Y_{it}^* > X_{it}$ and $Y_{it}^* > X_{jt}^{M-M_i}$. Again construct an alternate policy $\bar{Y}$ (with resulting states $\bar{X}$):

$$\bar{Y}_{it} = \max\{X_{jt}^{M-M_i}, X_{it}\} \quad \text{and}$$
$$\bar{Y}_{s(k),\,q} = \min\{Y_{s(k),\,q}^*, \bar{X}_{kq}^{l}\} \quad \text{for all } k \in \{i\} \cup A(i),$$
$$\text{where } q = t + M_i - M_{s(k)}.$$

Because $\max\{k\colon k \in \{\{i\} \cup A(i)\} \cap J\} = \max\{k\colon k \in \{\{j\} \cup A(j)\} \cap J\}$, items i and j share the same set of successors facing disruptions. The policy $\bar{Y}$ postpones orders of item i only to match item j's pipeline, and so does not reduce availability of matched sets of items at any downstream nodes. So $\tilde{Y}$ is again feasible, satisfies the required condition, and yields lower holding costs. □

Proof of Proposition 2. First, consider steps 1–3. The objective function (3) can be rewritten as

$$\min_Y E\Bigg\{\sum_{t=1}^{\infty}\alpha^{t-1}\Bigg(\sum_{i=1}^{b-1}\alpha^{L_i}(h_i\alpha^{l_i-L_i})Y_{it}+\sum_{i\in G}\alpha^{\hat{L}_i}(h_i\alpha^{l_i-\hat{L}_i})Y_{it}+\sum_{\substack{i>b\\ i\notin G}}\alpha^{l_i}h_iY_{it}+\alpha^{l_1}(p+H_1)[D^{(l_1+1)}-Y_{1t}]^+\Bigg)\Bigg\},$$

which is the cost function of the system obtained through steps 1–3. Because the items $i\le b-1$ exhibit long-run balance at the beginning of the planning horizon, Proposition 1 says that for those i, $X_{it}\le Y_{it}\le \min_{k>i}X_{kt}^{M-M_i}$ for all t. In addition, Lemma 1(d) and long-run balance for items $i\le b-1$ imply that, for $i\le b-2$,

$$X_{i+1,t}^{L}=X_{i+1,t}^{M-M_i}=\min_{k>i}X_{kt}^{M-M_i}\le\min_{k\in P(i)}X_{kt}^{M-M_i}.$$

As a result, (4) can be replaced by $X_{it}\le Y_{it}\le X_{i+1,t}^{L}$ for $i\le b-2$. For $i=b-1$, Proposition 1 and long-run balance for that item imply that

$$X_{it}\le Y_{it}\le\min_{k>i}X_{kt}^{M-M_i}=\min_{k>i}X_{kt}^{M-M_{b-1}}=\min_{k>i}X_{kt}^{\hat{L}_k}.$$

For any $j>b$ such that $j\notin G$, by definition we have $s(j)\ge b$, and so $j\in P(i)$ for some $i\in G$. Lemma 2 then implies that for any such j, $X_{jt}^{M-M_{b-1}}\ge X_{it}^{M-M_{b-1}}$ for some $i\in G$. As a result, $\min_{k>i}X_{kt}^{M-M_{b-1}}=\min_{i\in G}X_{it}^{M-M_{b-1}}$, so (4) can be replaced by $X_{it}\le Y_{it}\le\min_{i\in G}X_{it}^{\hat{L}_i}$ for $i=b-1$. For $i\ge b$, (4) is unchanged. Repeating these arguments confirms the validity of steps 4 and 5.

For step 6, the objective function (3) can be rewritten as

$$\min_Y\ E\Bigg\{\sum_{t=1}^{\infty}\alpha^{t-1}\Bigg(\sum_{i\in F}\alpha^{L_{\hat{k}}}(h_{\hat{k}}\alpha^{l_{\hat{k}}-L_{\hat{k}}})Y_{it}+\sum_{i\notin F}\alpha^{l_i}h_iY_{it}+\alpha^{l_1}(p+H_1)[D^{(l_1+1)}-Y_{1t}]^+\Bigg)\Bigg\},$$

which is the cost function of the system obtained through step 6 of the algorithm. By Proposition 1, $X_{\hat{i},t}\le Y_{\hat{i},t}\le X_{\widehat{i+1},t}^{M-M_{\hat{i}}}$ for each $\hat{i}$, $\widehat{i+1}\in F$, so this relationship can replace (4). □

Proof of Proposition 3. By step 1 of the partial series reduction algorithm, the subsystem consisting of items $1,\ldots,i-1$ reduces to a series system. The set G defined in step 2 consists of only item i, so in fact items $1,\ldots,i$ have a series structure. Step 4 has no impact on the subsystem rooted at i, but continuing up the tree and applying step 4 to i's immediate predecessors $P(i)$ yields a collection of series subsystems, each rooted at an element of $P(i)$. Repeatedly applying step 6 to this collection of subsystems eventually reduces them to a single series subsystem rooted at $i+1$, thus completing the reduction of the entire system to a series system with disruptions of item i. □

Proof of Proposition 4. We prove the case of $N=2$—general N is similar. First, consider a T-period planning horizon. The following expanded notation will be useful.

$z_{jL}(t)$: shipment in transit to stage j, sent L periods ago (at time $t-L$), $L=1,\ldots,l_j-1$;

$\mathbf{z}_j(t)$: $(z_{jL}(t))_L$;

$z_j(t)$: shipment sent to stage j at time t ($z_j(t)\ge 0$, $z_1(t)\le X_{2t}^{l}$).

Also, assume that ordering costs c_j are paid at the time of the order, rather than when items arrive as assumed earlier. (This does not change the analysis; it just simplifies the expressions.)

Expected inventory and backorder costs incurred in period t are

$$h_2[X_{2t}^{l}-E(D_t)]+\hat{C}_1(t,X_{1t}^{l}),$$

where $\hat{C}_1(t,x)=h_1[x-E(D_t)]+[p+H_1]E[x-D_t]^-$.

First assume that $J=\{1\}$. The state of stage 1's supply source is denoted u, where $u=1$ ($u=0$) indicates that a new shipment from stage 2 is (is not) allowed in the current period. Let $\hat{V}(t,X_{2t}^{l},\mathbf{z}_2,X_{1t}^{l},\mathbf{z}_1,u)$ denote the optimal expected net present cost from time t onward, starting in the indicated initial state.

For convenience (and similar to Zipkin 2000), we artificially extend the horizon by l_1+l_2 periods, during which the system continues to receive demands and accrue costs. In addition, assume that $h_2=0$ for $t\ge T+l_2$, the last shipment to stage 2 is $z_2(T-1)$, and the last shipment from stage 2 to stage 1 is $z_1(T+l_2-1)$.

We assume terminal costs of $\hat{V}(T+l_1+l_2,X_{2t}^{l},\mathbf{z}_2,X_{1t}^{l},\mathbf{z}_1,u)=0$. Then, for $t<T+l_1+l_2$, if supply is available to stage 1, we have

$$\begin{aligned}&\hat{V}(t,X_{2t}^{l},\mathbf{z}_2,X_{1t}^{l},\mathbf{z}_1,1)\\&\quad=\min\big\{c_1z_1+c_2z_2+h_2[X_{2t}^{l}-E(D_t)]+\hat{C}_1(t,X_{1t}^{l})\\&\qquad+(1-\beta_1)\alpha E[\hat{V}(t+1,X_{2t}^{l}+z_{2,l_2-1}-D_t,\\&\qquad(z_2,z_{21},\ldots,z_{2,l_2-2}),X_{1t}^{l}+z_{1,l_1-1}-D_t,\\&\qquad(z_1,z_{11},\ldots,z_{1,l_1-2}),1)]\\&\qquad+\beta_1\alpha E[\hat{V}(t+1,X_{2t}^{l}+z_{2,l_2-1}-D_t,\\&\qquad(z_2,z_{21},\ldots,z_{2,l_2-2}),X_{1t}^{l}+z_{1,l_1-1}-D_t,\\&\qquad(z_1,z_{11},\ldots,z_{1,l_1-2}),0)]\colon z_j\ge 0,X_{1t}+z_1\le X_{2t}^{l}\big\},\end{aligned}\tag{11}$$

whereas if supply is unavailable at stage 1, we have

$$\begin{aligned}&\hat{V}(t,X_{2t}^{l},\mathbf{z}_2,X_{1t}^{l},\mathbf{z}_1,0)\\&=\min\big\{c_1z_1+c_2z_2+h_2[X_{2t}^{l}-E(D_t)]+\hat{C}_1(t,X_{1t}^{l})\\&\qquad+\gamma_1\alpha E[\hat{V}(t+1,X_{2t}^{l}+z_{2,l_2-1}-D_t,(z_2,z_{21},\ldots,z_{2,l_2-2}),\\&\qquad X_{1t}^{l}+z_{1,l_1-1}-D_t,(z_1,z_{11},\ldots,z_{1,l_1-2}),1)]\\&\qquad+(1-\gamma_1)\alpha E[\hat{V}(t+1,X_{2t}^{l}+z_{2,l_2-1}-D_t,\\&\qquad(z_2,z_{21},\ldots,z_{2,l_2-2}),X_{1t}^{l}+z_{1,l_1-1}-D_t,\\&\qquad(z_1,z_{11},\ldots,z_{1,l_1-2}),0)]\colon z_1=0,z_2\ge 0\big\}.\end{aligned}\tag{12}$$

Now define $C_1^{L}(t,y)=\alpha^{L}E[\hat{C}_1(t-L,y-D^{(L)})]$, $C_1(t,y)=C_1^{l_1}(t,y)$, and

$$\tilde{C}_1(t,X_{1t}^{l},\mathbf{z}_1)=C_1^{0}(t,X_{1t}^{l})+C_1^{1}(t,X_{1t}^{l}+z_{1,l_1-1})+\cdots+C_1^{l_1-1}(t,X_{1t}).$$

Since $z_1(t)=0$ for $t\geq T+l_2$, $z_2(t)=0$ for $t\geq T$, and $h_2(t)=0$ for $t\geq T+l_2$, the expected cost at time $t=T+l_2$ is

$$\hat{V}(T+l_2, X_{2t}^l, \mathbf{z}_2, X_{1t}^l, \mathbf{z}_1, u)=\tilde{C}_1(T+l_2, X_{1t}^l, \mathbf{z}_1).$$

We first establish that it is sufficient to work with echelon inventory position at stage 1. To that end, note that z_1 (at time t) only affects $\hat{C}_1(\cdot,\cdot)$ at time $t+l_1$, so we reassign those costs from time $t+l_1$ to time t. Also, define $\tilde{V}(T+l_2, X_{2t}^l, \mathbf{z}_2, X_{1t}, u)=0$, and for $t<T+l_2$,

$$\begin{aligned}
&\tilde{V}(t, X_{2t}^l, \mathbf{z}_2, X_{1t}, 1)\\
&\quad=\min\big\{c_1(Y_{1t}-X_{1t})+c_2z_2+h_2[X_{2t}^l-E(D_t)]+C_1(t, Y_{1t})\\
&\qquad+\alpha(1-\beta_1)E[\tilde{V}(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), Y_{1t}-D_t, 1)]\\
&\qquad+\alpha\beta_1E[\tilde{V}(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), Y_{1t}-D_t, 0)]:\\
&\qquad X_{1t}\leq Y_{1t}\leq X_{2t}^l, z_2\geq 0\big\},
\end{aligned}\tag{13}$$

and

$$\begin{aligned}
&\tilde{V}(t, X_{2t}^l, \mathbf{z}_2, X_{1t}, 0)\\
&\quad=\min\big\{c_1(Y_{1t}-X_{1t})+c_2z_2+h_2[X_{2t}^l-E(D_t)]+C_1(t, Y_{1t})\\
&\qquad+\alpha\gamma_1E[\tilde{V}(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), Y_{1t}-D_t, 1)]\\
&\qquad+\alpha(1-\gamma_2)E[\tilde{V}(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), Y_{1t}-D_t, 0)]:\\
&\qquad Y_{1t}=X_{1t}, z_2\geq 0\big\}.
\end{aligned}\tag{14}$$

Using standard arguments, one can show by induction that, for $t\leq T+l_2$,

$$\begin{aligned}
&\hat{V}(t, X_{2t}^l, \mathbf{z}_2, X_{1t}^l, \mathbf{z}_1, u)\\
&\quad=\tilde{C}_1(t, X_{1t}^l, \mathbf{z}_1)+\tilde{V}(t, X_{2t}^l, \mathbf{z}_2, X_{1t}, u).
\end{aligned}$$

Because the decision variables do not enter $\tilde{C}_1(t, X_{1t}^l, \mathbf{z}_1)$, the z_1 and z_2 that optimize (13) and (14) also optimize (11) and (12), respectively.

Next we show that the $\tilde{V}$ functions are separable and establish the optimality of a base-stock policy at stage 1. To that end, define the functions V_1 and K_1 through a dynamic program, where

$$V_1(T+l_1, x, u)=0 \quad \text{for any } x, \text{ and for } u=0 \text{ or } 1$$

and for $t<T+l_1$,

$$\begin{aligned}
K_1(t, y, 1)&=c_1y+C_1(t, y)\\
&\quad+\alpha(1-\beta_1)E[V_1(t+1, y-D_t, 1)]\\
&\quad+\alpha\beta_1E[V_1(t+1, y-D_t, 0)],
\end{aligned}\tag{15}$$

$$\begin{aligned}
K_1(t, y, 0)&=c_1y+C_1(t, y)+\alpha\gamma_1E[V_1(t+1, y-D_t, 1)]\\
&\quad+\alpha(1-\gamma_1)E[V_1(t+1, y-D_t, 0)],
\end{aligned}\tag{16}$$

$$V_1(t, x, 1)=-c_1x+\min\{K_1(t, y, 1): y\geq x\},\tag{17}$$

$$V_1(t, x, 0)=-c_1x+\min\{K_1(t, y, 0): y\geq x\}.\tag{18}$$

Also define the functions $\hat{V}_2(T+l_2, X_{2t}^l, \mathbf{z}_2, u)=0$, and for $t\leq T+l_2$,

$$\begin{aligned}
&\hat{V}_2(t, X_{2t}^l, \mathbf{z}_2, 1)\\
&\quad=\min\big\{c_2z_2+\hat{C}_2(t, X_{2t}^l, 1)\\
&\qquad+\alpha(1-\beta_1)E[\hat{V}_2(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), 1)]\\
&\qquad+\alpha\beta_1E[\hat{V}_2(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), 0)]: z_2\geq 0\big\},
\end{aligned}$$

and

$$\begin{aligned}
&\hat{V}_2(t, X_{2t}^l, \mathbf{z}_2, 0)\\
&\quad=\min\big\{c_2z_2+\hat{C}_2(t, X_{2t}^l, 0)\\
&\qquad+\alpha\gamma_1E[\hat{V}_2(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), 1)]\\
&\qquad+\alpha(1-\gamma_1)E[\hat{V}_2(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad(z_2, z_{21}, \ldots, z_{2,l_2-2}), 0)]: z_2\geq 0\big\}.
\end{aligned}$$

We wish to show for all $t\leq T+l_2$ that

$$\tilde{V}(t, X_{2t}^l, \mathbf{z}_2, X_{1t}, u)=\hat{V}_2(t, X_{2t}^l, \mathbf{z}_2, u)+V_1(t, X_{1t}, u).$$

The argument is by induction. For $t=T+l_2$, $\tilde{V}(T+l_2, X_{2t}^l, \mathbf{z}_2, X_{1t}, u)=\hat{V}_2(T+l_2, X_{2t}^l, \mathbf{z}_2, u)=V_1(T+l_2, X_{1t}, u)=0$, so the result holds. Now suppose it holds for some $t-1\leq T+l_2$. For $u=1$,

$$\begin{aligned}
&\tilde{V}(t, X_{2t}^l, \mathbf{z}_2, X_{1t}, 1)\\
&\quad=\min\big\{c_1(Y_{1t}-X_{1t})+C_1(t, Y_{1t})\\
&\qquad+\alpha(1-\beta_1)E[V_1(t+1, Y_{1t}-D_t, 1)]\\
&\qquad+\alpha\beta_1E[V_1(t+1, Y_{1t}-D_t, 0)]\\
&\qquad+c_2z_2+h_2[X_{2t}^l-E(D_t)]\\
&\qquad+\alpha(1-\beta_1)E[\hat{V}_2(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad\quad(z_2, z_{21}, \ldots, z_{2,l_2-2}), 1)]\\
&\qquad+\alpha\beta_1E[\hat{V}_2(t+1, X_{2t}^l+z_{2,l_2-1}-D_t,\\
&\qquad\quad(z_2, z_{21}, \ldots, z_{2,l_2-2}), 0)]:\\
&\qquad X_{1t}\leq Y_{1t}\leq X_{2t}^l, z_2\geq 0\big\}.
\end{aligned}\tag{19}$$

We perform the optimization in two steps. First fix z_2 and optimize over Y_{1t}; that is, evaluate

$$-c_1X_{1t}+\min\{K_1(t, Y_{1t}, 1): X_{1t}\leq Y_{1t}\leq X_{2t}^l\},\tag{20}$$

where $K_1(t, Y_{1t}, 1)$ was defined in (15). Since $\hat{C}_1(t, x)$ is convex in x, it follows that $C_1(t, y)$ is convex in y, and standard induction arguments can be used to show that $K_1(t, y, u)$ is convex in y and $V_1(t, x, u)$ is convex in x for all t and $u=0$ or 1. As a result, there exists an $s_1^*(t)$ such that if $s_1^*(t)\leq X_{2t}^l$, the base-stock policy with base-stock level $s_1^*(t)$ is optimal for (20). In this case, (20) is equal to $V_1(t, X_{1t}, 1)$. If instead $s_1^*(t)>X_{2t}^l$, then the optimal solution is to set $Y_{1t}=X_{2t}^l$, and (20) is equal to $V_1(t, X_{1t}, 1)+[K_1(t, X_{2t}^l, 1)-K_1(t, s_1^*(t), 1)]$. Defining $\underline{K}_1(t, y, 1)=K_1(t, \min\{s_1^*(t), y\}, 1)-K_1(t, s_1^*(t), 1)$

and combining the two cases, we have that (20) is equal to $V_1(t, X_{1t}, 1) + \underline{K}_1(t, X^l_{2t}, 1)$. Substituting this into (19) and simplifying yields $\hat{V}(t, X^l_{2t}, \mathbf{z}_2, X_{1t}, 1) = V_1(t, X_{1t}, 1) + \hat{V}_2(t, X^l_{2t}, \mathbf{z}_2, 1)$. The argument for $u = 0$ is similar, but simplified by the fact that $Y_{1t} = X_{1t}$ (stage 1 cannot order); we omit the details for brevity.

Next define

$$C^l_2(t, y, u) = \alpha^l \rho(1 \mid u; l) E[\hat{C}_2(t+l, y - D[t, t+l), 1)] + \alpha^l \rho(0 \mid u; l) E[\hat{C}_2(t+l, y - D[t, t+l), 0)],$$

where $\rho(u'' \mid u'; l)$ is defined in (8) and (9). Set $C_2(t, y, u) = C^l_2(t, y, u)$ with $l = l_2$, and note that $C_2(t, y, u)$ is convex in y. Define the functions V_2 and K_2 through the dynamic program

$$V_2(T, x, u) = 0,$$

and for $t \le T$,

$$K_2(t, y, 1) = c_2 y + C_2(t, y, 1) + \alpha(1-\beta_1) E[V_2(t+1, y - D_t, 1)] + \alpha\beta_1 E[V_2(t+1, y - D_t, 0)],$$

$$K_2(t, y, 0) = c_2 y + C_2(t, y, 0) + \alpha\gamma_1 E[V_2(t+1, y - D_t, 1)] + \alpha(1-\gamma_1) E[V_2(t+1, y - D_t, 0)],$$

$$V_2(t, x, 1) = -c_2 x + \min\{K_2(t, y, 1)\colon y \ge x\}, \tag{21}$$

$$V_2(t, x, 0) = -c_2 x + \min\{K_2(t, y, 0)\colon y \ge x\}. \tag{22}$$

The function $V_2(T, x, u)$ is trivially convex in x, so $K_2(T-1, y, u)$ is convex in y. Now suppose that $V_2(t+1, x, u)$ is convex in x and $K_2(t, y, u)$ is convex in y. Because the minimands in (21) and (22) are both convex, the optimal solutions both have base-stock structures: there exist $s^{*1}_2(t)$ and $s^{*0}_2(t)$, respectively, such that the optimal actions given u are to set

$$y(x) = \begin{cases} s^{*u}_2(t) & \text{if } x \le s^{*u}_2(t), \\ x & \text{if } x > s^{*u}_2(t). \end{cases}$$

It then follows that

$$V_2(t, x, u) = \begin{cases} -c_2 x + H_2(t, s^{*u}_2(t), u) & \text{if } x \le s^{*u}_2(t), \\ -c_2 x + H_2(t, x, u) & \text{if } x > s^{*u}_2(t), \end{cases}$$

which is convex in x. As a result, $K_2(t+1, y, 1)$ and $K_2(t+1, y, 0)$ are both convex in y.

The case of $J = \{2\}$ (upstream disruption) is quite similar, so the proof is omitted. In that case, again the optimal ordering policy at stage 1 follows a base-stock policy with base-stock level $s^*_1(t)$, regardless of whether the supply source at stage 2 is available. At stage 2 it is also optimal to follow a base-stock policy with a single target base-stock level $s^*_2(t)$. Because stage 2 cannot order when supply is disrupted, there is no need for a second base-stock level in that event.

Similar arguments extend the result to $N > 2$ and more general disruption sets J, and standard arguments can be used to extend the result to an infinite horizon. □

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