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MATH 111-X
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Problem Set 2

1 Let $\star$ be defined on $\mathbb{Z}$ by letting $a \star b = ab$. Verify whether $(\mathbb{Z}, \star)$ is a group by checking whether it satisfy the 3 group axioms.

Solution:
Observe that in our description, the operation $\star$ is already defined on $\mathbb{Z}$, hence we only need to show the 3 group axioms.

- Suppose $c \in \mathbb{Z}$. Thus we have $(a \star b) \star c = (a \cdot b) \cdot c = abc$; as well as $a \star (b \star c) = a \cdot (b \cdot c) = abc$. Hence $\star$ is associative.
- Now, consider $e \in \mathbb{Z}$ where $e = 1$. Hence for any integer we have: $a \star e = ae = a$, as well as $e \star a = ea = a$. Hence, there exists an identity element for the operation $\star$.
- Finally, let $a^{-1}, b^{-1} \in \mathbb{Z}$. By definition of an inverse we have, $a^{-1} \star a = a^{-1}a = e$, as well as $b^{-1} \star b = b^{-1}b = e$. Since for any $a, b \in \mathbb{Z}$, $a \star b = ab = ba = b \star a$, this operation is also commutative. And hence, the right hand inverse of a , b should also follow.

Since the binary operation $\star$ satisfies all three group axioms, therefore $\mathbb{Z}, \star$ is a group. □

2 Let $\star$ be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a \star b = a + b$. Verify whether $(2\mathbb{Z}, \star)$ is a group as in the item above.

3 Let G be a group and suppose that $(ab)^2 = a^2b^2$ for all a and for all b in G. Prove that G is an abelian group.

Proof. Since G is defined as a group we have: $a^2b^2 = aabb$. Multiplying the left by a^{-1} and the right by b^{-1} we get: $a^{-1}aabb = aabbb^{-1} \Rightarrow ab^2 = a^2b$ □

4 Let $\{H_i\}_{i \in I}$ be an arbitrary collection of subgroups of a group G for some index set I, show that $\bigcap_{i \in I} H_i$ is a subgroup of G. Is $\bigcup_{i \in I} H_i$ a subgroup of G? Justify your claims.

5 Let G be an abelian group. Show that the elements of finite order in G form a subgroup. This subgroup is called the torsion subgroup of G.

6 Let S be any subset of group G.

(a) Show that $H_s = \{x \in G | xs = sx \text{ for all } s \in S\}$ is a subgroup of G.

(b) Show that H_G is an abelian group, where H_G is called the center of G.

7 List all of the elements in each of the following subgroups and draw the lattice diagram* as needed.

(a) The subgroup of $\mathbb{Z}_{24}$ generated by 15

(b) All subgroups $\mathbb{Z}_{48}$ *draw lattice here*

(c) The subgroup generated by 5 in $U(18) = (\mathbb{Z}_{18})^\times$