Primordial Cosmological Perturbations under Invertible Higher-Derivative Disformal Transformation and other Disformal Transformations

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ABSTRACT: Primordial comsmological perturbations are quantum fluctuations that served as seeds in the early universe for what we nowadays observe as galaxies and clusters thereof. Cosmic inflation and the accompanying dynamics of the expanding Universe, stretched and allowed them to evolve, leading to their current state. Being the "beginning" of us all, their importance cannot be overemphasised. Focusing on their mathematical properties, in this study, we explore their possible variation under different forms of disformal transformation. This goes from reviews for the simplest special disformal transformal, passing through the original Bekenstein disformal transformation, and then leading to investigation of the effect of the Invertible Higher-Derivative Disformal Transformation (TMM disformal transformation). We examine the variations of scalar and tensor perturbations within the framework of the Horndeski theory. While footprints of the change of these perturbations may be apparent at the leading order, we find that the stretching of the early universe may remove these marks of variance in the superhorizon limit.

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[2]what in the world

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1 The purpose of this document

Although this document is currently formatted in the style of JHEP, it is not yet meant to be a submission to JHEP or any journal thereof. This document, serves as two things. Firstly, it is where the second author fleshes out his thoughts regarding the topic, and thus the calculations and discussions may be lengthy and verbose. The second author wishes to grasp and understand the topics at hand. Secondly, it serves as a template and source material for a future formal document that may be submitted to JHEP or any other journal; it might even be a springboard for the author's undergraduate thesis.

2 Overview and thoughts

3 Mathematical Preliminaries

3.1 The Riemann Curvature Tensor and its Contractions

A **manifold** is a topological space that is locally Euclidian. On the other hand a **Riemannian manifold** is a type of manifold where mesaurement of distance and angle is possible. This is made possible by the metric.

An aside. Gauss figured out that any point on a 2D surface can be summarized by a single number which we denote as K and is called as the **Gaussian curvature**. This is the product of the two principal curvatures at a particular point. Simply put, the principal curvatures are rough approximations of the 2D surface.

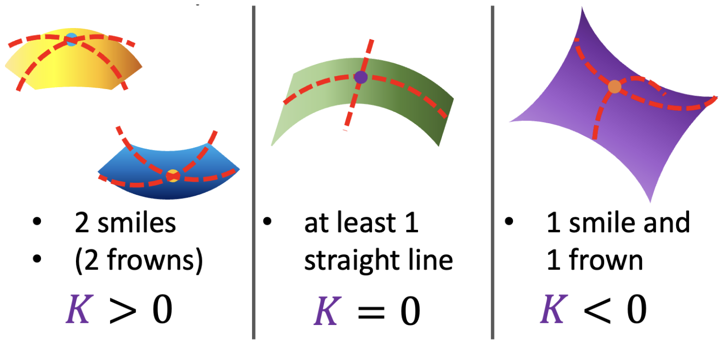

Figure 1: According to Gaussian Curvature, if you have a surface that can be approximated by two "frowns" or "smiles", then the intersection thereof has positive curvature ($K > 0$). If one is a "frown" and the other a "smile", the intersection has negative curvature ($K < 0$). If one direction is flat, the curvature is zero ($K = 0$).

The issue with the Gaussian Curvate is that it only works for 2D surfaces. This motivates us to develop something that can work for higher dimensions. This is where we use the **Riemann Curvature Tensor** (RCT) given by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{3.1}$$

Although we say that the Christofell Symbols are non-tensorial in nature, the sum thereof ammounting to the Riemann Curvature Tensor is indeed a tensor. We can think of the Christofell Symbols as mathematical objects that describe how the coordinate basis vectors change from point to point on a given manifold. Note that the covariant derivative (which is related to the original definition of the RCT) is related to the Christofell Symbols as follows:

$$\nabla_{\mu}V^{\rho} = \partial_{\mu}V^{\rho} + \Gamma^{\rho}_{\mu\sigma}V^{\sigma}. \tag{3.2}$$

4 History and Properties of the Universe

The Universe as we see it today is **homogenous** and **isotropic**. Homogeneity talks about

4.1 What the Cosmic Microwave Background Tells us

For the first $\sim 400,000$ years of the Universe, it was too hot to allow atoms to form, at this stage electrons have yet to couple with protons. Due to quantum mechanical effects, free electrons scatter the photons of the early univers, making it opaque, thereby "trapping" the photons. However, as the Universe expanded and cooled, it eventually reached a temperature where electrons could finally couple with protons to form atoms. At this point, photons could finally travel freely through space. This photons, given the extreme nature of the early universe were highly energetic, and so we must be careful in observing them today.

The continuous expansion of the universe stretched the photons, thereby increasing their wavelength. If we traceback this stretching from the time when photons started to propagate freely up to the present time, we find these photons fall in the microwave region. This is what we now call as the Cosmic Microwave Background (CMB). It is a strong evidence that, indeed, the universe started from a hot and dense state, and has been expanding ever since.

5 The Expanding and Flat Universe

The FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{5.1}$$

As discussed in 3.1 a Gaussian Curvature of $K = 0$ denotes a flat space. Would it make sense to probe perturbations in a flat universe? Observations show (take for example the CMB), indicate that our universe is close to being spatially flat. We can actually study the perturbations by looking into the deviations from a flat, homogenous, and isotropic "background". Hence in this exposition, we will consider the case where we have a flat universe, i.e., $K = 0$ in eq. (5.1).

5.1 An Example in using the FLRW Metric

Consider a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric with c explicit,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\big(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\big).$$

A light signal emitted at cosmic time $t_{\rm em}$ from radial coordinate $r_{\rm em}$ and received at t_0 (so $r_{\rm obs} = 0$) follows a radial null geodesic with $\mathrm{d}r = -c\,\mathrm{d}t/a(t)$. The comoving radial distance from observer to source is

$$\chi \equiv r_{\rm em} = c\int_{t_{\rm em}}^{t_0} \frac{\mathrm{d}t}{a(t)}.$$

Assume the scale factor is a power law $a(t) = a_0\big(t/t_0\big)^n$ with $n \neq 1$.

1. Derive expressions for the comoving distance χ, the proper distance today $D_p(t_0)$, the particle horizon, the luminosity distance D_L, and the angular diameter distance D_A.
2. Then compute numerical values for these quantities for the matter-dominated case $n = \frac{2}{3}$ and redshift $z = 1$.
3. For the numerical example, use the normalization $a_0 = 1$, the relation $H_0 = n/t_0$, and a Hubble constant of $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.

<u>**Solution:**</u>

Deriving χ for the travelling light

Note for a travelling light we have a null geodesic, hence $ds^2 = 0$. Furthermore, since we have a travelling light in the radial direction, there are no changes in the angular coordinates; in other words, $d\Omega^2 = 0$. Hence the FLRW metric reduces to

$$0 = -c^2 dt^2 + a(t)^2 dr^2 \tag{5.2}$$

If we move the first term to the right and take the square root of both sides (taking note of the $\pm$) then continuing to solve the differential equation, we have

$$cdt = \pm a(t) dr \tag{5.3}$$

$$dr = \pm \frac{c}{a(t)} dt \tag{5.4}$$

$$\chi = \pm c \int_{t_0}^{t_{em}} \frac{1}{a(t)} dt \tag{5.5}$$

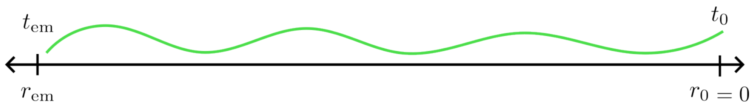

Figure 2: A light signal emitted at cosmic time t_{em} from radial coordinate r_{em} and received at t_0.

If we refer to Fig. 2, since as far as we know, time moves in one direction then $dt > 0$. On the other hand, notice that since we set $r_0 = 0$ as the obesrver's position, as the light travel from r_{em} to the observer, the distance decreases – impying that $dr < 0$. It is now clear that in order to satifsy both conditions, we take the negative solution.

$$\chi = -c \int_{t_0}^{t_{em}} \frac{1}{a(t)} dt \tag{5.6}$$

Does Eq. (5.6) make sense? In my first encounter of this problem I was quite confused how this measures the comoving distance. Isn't the comoving distance supposed to be the "fixed" value prior to expansion at $t = t_{em}$? B

6 Disformal Transformation

Table 1: Types of Disformal Transformations of the Metric

Transformation Name	Metric Transformation ($\hat{g}_{\mu\nu}$)	Short Description
Bekenstein	$\hat{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_{;\mu}\phi_{;\nu}$	The original proposal where both the conformal factor A and disformal factor B depend only on the scalar field ϕ.
Special	$\hat{g}_{\mu\nu} = g_{\mu\nu} + B(\phi, X)\phi_{;\mu}\phi_{;\nu}$	A purely disformal transformation with $A = 1$. The disformal factor B depends on ϕ and its kinetic term X.
Generalized with one arbitrary conformal	$\hat{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi, X)\phi_{;\mu}\phi_{;\nu}$	A generalization where A depends only on ϕ, but B depends on both ϕ and X.
Fully Generalized for First Derivatives	$\hat{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_{;\mu}\phi_{;\nu}$	Both the conformal factor A and the disformal factor B are allowed to depend on ϕ and its kinetic term X.
Fully General and Extended	$\hat{g}_{\mu\nu} = A(\dots)g_{\mu\nu}$ $+(C(\dots)\phi_{;\mu} + D(\dots)X_{;\mu})$ $\times(C(\dots)\phi_{;\nu} + D(\dots)X_{;\nu})$	The conformal factor A and functions C, D depend on ϕ and its invariants. The disformal part is a quadratic form involving gradients of ϕ and X.
TMM	$\hat{g}_{\mu\nu} = A(\dots)g_{\mu\nu} + B(\dots)\phi_{;\mu}\phi_{;\nu}$ $+C(\dots)(\phi_{;\mu}X_{;\nu} + X_{;\mu}\phi_{;\nu})$ $+D(\dots)X_{;\mu}X_{;\nu}$	The most general quadratic transformation in first derivatives. All coefficient functions (A, B, C, D) depend on ϕ and its invariants (X, Y, Z).

Note: $X = -\frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta}$ denotes the kinetic term. The arguments $(\dots)$ in the last two rows are shorthand for (ϕ, X, Y, Z), representing dependence on the scalar field and its invariants.

A Appendix

Please always give a title also for appendices.

Acknowledgments

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Note added. This is also a good position for notes added after the paper has been written.

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References

[1] Author, *Title*, *J. Abbrev.* **vol** (year) pg.

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