1 The TMM Transformation

Let us decompose the perturbed FLRW metric (with SVT decomposition) as

$$ds^2 = -(1+2\varphi)\,dt^2 + 2a(\alpha_{,i}+\beta_i)\,dt\,dx^i + a^2\Big[(1-2\psi)\delta_{ij}+\gamma_{ij}+2E_{,ij}+2F_{(i,j)}\Big]\,dx^i dx^j, \quad (1.1)$$

where $a = a(t)$ is the scale factor, $(\varphi, \alpha, \psi, E)$ are scalar perturbations, β_i, F_i are vector perturbations, and γ_{ij} are tensor perturbations. Note that throughout this transformation a partial derivative is denoted by a comma subscript ",". Meanwhile, a covariant derivative is denoted by a semi-colon subscript ";".

Under a disformal transformation, the line element becomes

$$d\hat{s}^2 = \hat{g}_{\mu\nu}\,dx^\mu dx^\nu. \quad (1.2)$$

The invertible disformal transformation with higher derivatives presented by Takahashi, Motohashi, and Minamitsuji (which we call the TMM transformation for convenience) is

$$\hat{g}_{\mu\nu} = A\,g_{\mu\nu} + B\,\phi_{;\mu}\phi_{;\nu} + C\big(\phi_{;\mu}X_{;\nu} + X_{;\mu}\phi_{;\nu}\big) + D\,X_{;\mu}X_{;\nu}, \quad (1.3)$$

where A, B, C, D are functionals of ϕ, X, Y, Z, and X, Y, Z are defined as

$$X \equiv -\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}, \qquad Y \equiv g^{\mu\nu}\phi_{;\mu}X_{;\nu}, \qquad Z \equiv g^{\mu\nu}X_{;\mu}X_{;\nu}. \quad (1.4)$$

2 Expanding $d\hat{s}^2$ term-by-term

Under (1.3), the transformed line element can be expanded as

$$\begin{aligned}
d\hat{s}^2 &= \hat{g}_{\mu\nu}\,dx^\mu dx^\nu \\
&= \big(Ag_{00} + B\phi_{;0}^2 + C(2\phi_{;0}X_{;0}) + DX_{;0}^2\big)dt^2 \\
&= 2\big(Ag_{0i} + B\phi_{;0}\phi_{;i} + C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i}) + DX_{;0}X_{;i}\big)dt\,dx^i \\
&= \big(Ag_{ij} + B\phi_{;i}\phi_{;j} + C(\phi_{;i}X_{;j} + X_{;i}\phi_{;j}) + DX_{;i}X_{;j}\big)dx^i dx^j
\end{aligned} \quad (2.1)$$

3 Background-perturbation split

We perturb the scalar field as

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}) \quad (3.1)$$

Since this is a scalar field the covariant derivative reduces to partial derivatives, it follows that

$$\phi_{;0} = \partial_0\phi = \dot{\bar{\phi}} + \dot{\delta\phi}, \qquad \phi_{;i} = \partial_i\phi = \delta\phi_{,i}, \quad (3.2)$$

where we used $\bar{\phi}_{,i} = 0$ by background homogeneity.

Likewise for X we have

$$X(x, \mathbf{x}) = \bar{X} + \delta X \quad (3.3)$$

Consequently,

$$X_{;0} = \dot{\bar{X}} + \delta\dot{X}, \qquad X_{;i} = \delta X_{,i} \tag{3.4}$$

It should also make sense to split the coefficients since they are functionals of (ϕ, X, Y, Z). So we have

$$A = \bar{A} + \delta A, \qquad B = \bar{B} + \delta B, \qquad C = \bar{C} + \delta C, \qquad D = \bar{D} + \delta D, \tag{3.5}$$

3.1 Linear Expression for X

$$\begin{aligned}
X &= -\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} \\
&= -\frac{1}{2}g^{00}\dot{\phi}^2 \\
&= -\frac{1}{2}\big(-(1-2\varphi)\big)(\dot{\bar{\phi}} + \delta\dot{\phi})^2 \\
&= \left(\frac{1}{2} - \varphi\right)\left(\dot{\bar{\phi}}^2 + 2\dot{\bar{\phi}}\,\delta\dot{\phi} + (\delta\dot{\phi})^2\right) \\
&= \frac{1}{2}\dot{\bar{\phi}}^2 + \dot{\bar{\phi}}\,\delta\dot{\phi} - \varphi\,\dot{\bar{\phi}}^2,
\end{aligned} \tag{3.6}$$

where $g^{00} = -(1-2\varphi)$ follows from inverting then expanding g_{00} at linear order. We have also dropped higher-order perturbations (e.g. $(\delta\dot{\phi})^2$).

4 Linear Expansion of the Transformed Metric

We are now ready to expand (2.1), using the splits and their derivatives given by (3.2), (3.4) and (3.5), while discarding higher-order perturbations (in other words, $\mathcal{O}(\epsilon^2)$ terms)

4.1 The dt^2 terms

We wish to expand

$$Ag_{00} + B\phi_{;0}^2 + C(2\phi_{;0}X_{;0}) + DX_{;0}^2 \tag{4.1}$$

For Ag_{00}

From (1.1), we have $g_{00} = -(1 + 2\varphi)$

Hence,

$$\begin{aligned}
Ag_{00} &= -(\bar{A} + \delta A)(1 + 2\varphi) \\
&= -(\bar{A} + 2\varphi\bar{A} + \delta A + 2\varphi\delta A) \\
&\approx -\bar{A} - \delta A - 2\bar{A}\varphi
\end{aligned} \tag{4.2}$$

because $2\varphi\delta A$ is $\mathcal{O}(\epsilon^2)$.

Starting from here we will be introducing a new notation for cases where $\mathcal{O}(\epsilon^n), n \geq 2$. For a particular term that inherits higher-ordered perturbations we will be cancelling them as follows: ~~term~~.

For $B\phi_{;0}^2$

Using (3.2) and (3.5) we have

$$
\begin{aligned}
B\phi_{;0}^2 &= (\bar{B} + \delta B)(\dot{\bar{\phi}} + \delta\dot{\phi})^2 \\
&= (\bar{B} + \delta B)(\dot{\bar{\phi}}^2 + 2\dot{\bar{\phi}}\delta\dot{\phi} + \delta\dot{\phi}^2) \\
&\approx \bar{B}\dot{\bar{\phi}}^2 + 2\bar{B}\dot{\bar{\phi}}\delta\dot{\phi} + \delta B\dot{\bar{\phi}}^2 + 2\dot{\phi}\delta\bar{B}\delta\dot{\phi} \\
&\approx \bar{B}\dot{\bar{\phi}}^2 + 2\bar{B}\dot{\bar{\phi}}\delta\dot{\phi} + \delta B\dot{\bar{\phi}}^2
\end{aligned}
\tag{4.3}
$$

For the cross-term $C\phi_{;0}X_{;0}$ **we have**

$$
\begin{aligned}
C\phi_{;0}X_{;0} &= (\bar{C} + \delta C)(\dot{\bar{\phi}} + \delta\dot{\phi})(\dot{\bar{X}} + \delta\dot{X}) \\
&= (\bar{C} + \delta C)(\dot{\bar{\phi}}\dot{\bar{X}} + \dot{\bar{\phi}}\delta\dot{X} + \delta\dot{\phi}\dot{\bar{X}} + \delta\dot{\phi}\delta\dot{X}) \\
&\approx \bar{C}\dot{\bar{\phi}}\dot{\bar{X}} + \bar{C}\dot{\bar{\phi}}\delta\dot{X} + \bar{C}\delta\dot{\phi}\dot{\bar{X}} + \delta C\dot{\bar{\phi}}\dot{\bar{X}} + \delta C\dot{\bar{\phi}}\delta\dot{X} + \delta C\delta\dot{\phi}\dot{\bar{X}} \\
&\approx \bar{C}\dot{\bar{\phi}}\dot{\bar{X}} + \bar{C}\dot{\bar{\phi}}\delta\dot{X} + \bar{C}\delta\dot{\phi}\dot{\bar{X}} + \delta C\dot{\bar{\phi}}\dot{\bar{X}}
\end{aligned}
\tag{4.4}
$$

For $DX_{;0}^2$

Finally we have,

$$
\begin{aligned}
DX_{;0}^2 &= (\bar{D} + \delta D)(\dot{\bar{X}} + \delta\dot{X})^2 \\
&= (\bar{D} + \delta D)(\dot{\bar{X}}^2 + 2\dot{\bar{X}}\delta\dot{X} + (\delta\dot{X})^2) \\
&\approx \bar{D}\dot{\bar{X}}^2 + 2\bar{D}\dot{\bar{X}}\delta\dot{X} + \delta D\dot{\bar{X}}^2 + 2\delta D\dot{\bar{X}}\delta\dot{X} \\
&\approx \bar{D}\dot{\bar{X}}^2 + 2\bar{D}\dot{\bar{X}}\delta\dot{X} + \delta D\dot{\bar{X}}^2
\end{aligned}
\tag{4.5}
$$

Putting (4.2) to (4.5), therefore $\hat{g_{00}}$ in linear-order is

$$
\boxed{
\begin{aligned}
\hat{g}_{00} \approx &-\bar{A} - \delta A - 2\bar{A}\varphi \\
&+ \bar{B}\dot{\bar{\phi}}^2 + 2\bar{B}\dot{\bar{\phi}}\delta\dot{\phi} + \delta B\dot{\bar{\phi}}^2 \\
&+ 2(\bar{C}\dot{\bar{\phi}}\dot{\bar{X}} + \bar{C}\dot{\bar{\phi}}\delta\dot{X} + \bar{C}\delta\dot{\phi}\dot{\bar{X}} + \delta C\dot{\bar{\phi}}\dot{\bar{X}}) \\
&+ \bar{D}\dot{\bar{X}}^2 + 2\bar{D}\dot{\bar{X}}\delta\dot{X} + \delta D\dot{\bar{X}}^2
\end{aligned}
}
\tag{4.6}
$$

5   For the $dtdx^i$ terms

We wish to expand $Ag_{0i} + B\phi_{;0}\phi_{;i} + C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i}) + DX_{;0}X_{;i}$.

For Ag_{0i}

$$
\begin{aligned}
Ag_{0i} &= a(\bar{A} + \delta A)(\alpha_{,i} + \beta_i) \\
&= a(\bar{A}\alpha_{,i} + \bar{A}\beta_i + \delta A\alpha_{,i} + \delta A\beta_i) \\
&\approx a\bar{A}(\alpha_{,i} + \beta_i)
\end{aligned}
\tag{5.1}
$$

For $B\phi_{;0}\phi_{;i}$

$$
\begin{aligned}
B\phi_{;0}\phi_{;i} &= (\bar{B} + \delta B)(\dot{\bar{\phi}} + \delta\dot{\phi})(\delta\phi_{,i}) \\
&= (\bar{B} + \delta B)(\delta\phi_{,i}\dot{\bar{\phi}} + \delta\phi_{,i}\delta\dot{\phi}) \\
&\approx \bar{B}\delta\phi_{,i}\dot{\bar{\phi}} + \delta B\delta\phi_{,i}\dot{\bar{\phi}} \\
&\approx \bar{B}\dot{\bar{\phi}}\delta\phi_{,i}
\end{aligned} \tag{5.2}
$$

For $C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i})$

$$
\begin{aligned}
C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i}) &= (\bar{C} + \delta C)\Big[(\dot{\bar{\phi}} + \delta\dot{\phi})(\delta X_{,i}) + (\dot{\bar{X}} + \delta\dot{X})(\delta\phi_{,i})\Big] \\
&= (\bar{C} + \delta C)\Big[\delta X_{,i}\dot{\bar{\phi}} + \delta X_{,i}\delta\dot{\phi} + \delta\phi_{,i}\dot{\bar{X}} + \delta\phi_{,i}\delta\dot{X}\Big] \\
&\approx \bar{C}\delta X_{,i}\dot{\bar{\phi}} + \bar{C}\delta\phi_{,i}\dot{\bar{X}} + \delta C\delta X_{,i}\dot{\bar{\phi}} + \delta C\delta\phi_{,i}\dot{\bar{X}} \\
&\approx \bar{C}(\delta X_{,i}\dot{\bar{\phi}} + \delta\phi_{,i}\dot{\bar{X}})
\end{aligned} \tag{5.3}
$$

For $DX_{;0}X_{;i}$

$$
\begin{aligned}
DX_{;0}X_{;i} &= (\bar{D} + \delta D)(\dot{\bar{X}} + \delta\dot{X})\delta X_{,i} \\
&= (\bar{D} + \delta D)(\delta X_{,i}\dot{\bar{X}} + \delta X_{,i}\delta\dot{X}) \\
&\approx \bar{D}\delta X_{,i}\dot{\bar{X}} + \delta D\delta X_{,i}\dot{\bar{X}} \\
&\approx \bar{D}\dot{\bar{X}}\delta X_{,i}
\end{aligned} \tag{5.4}
$$

Finally, putting together (5.1) to (5.4) $\hat{g}_0 i$ in linear order is given by

$$
\boxed{\hat{g}_{0i} \approx a\bar{A}(\alpha_{,i} + \beta_i) + \bar{B}\dot{\bar{\phi}}\delta\phi_{,i} + \bar{C}(\delta X_{,i}\dot{\bar{\phi}} + \delta\phi_{,i}\dot{\bar{X}}) + \bar{D}\dot{\bar{X}}\delta X_{,i}} \tag{5.5}
$$

5.1 For $dx^i dx^j$ terms

We wish to expand $Ag_{ij} + B\phi_{;i}\phi_{;j} + C(\phi_{;i}X_{;j} + X_{;i}\phi_{;j}) + DX_{;i}X_{;j}$. Now before we proceed, we have established from (3.2) and (3.4) that the perturbations with derivatives with respect to spatial components are already first-order. Hence this reduces to

$$
Ag_{ij} + B\phi_{;i}\phi_{;j} + C(\phi_{;i}X_{;j} + X_{;i}\phi_{;j}) + DX_{;i}X_{;j} \tag{5.6}
$$

A huge shout-out to cosmological perturbations for making the math bearable. From (1.1) we have $g_{ij} = a^2\Big[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\Big]$; then

$$
\begin{aligned}
Ag_{ij} &= a^2(\bar{A} + \delta A)\Big[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\Big] \\
&= a^2\bar{A}\Big[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\Big] \\
&\quad + \delta A\Big[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\Big] \\
&\approx a^2\bar{A}\Big[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\Big] + a^2\delta A\delta_{ij} \\
&\approx a^2\bar{A}[(1 - 2\psi + \delta A/\bar{A})\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}]
\end{aligned} \tag{5.7}
$$

Then it follows that

$$\hat{g}_{ij} \approx a^2 \bar{A}[(1 - 2\psi + \delta A/\bar{A})\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}] \qquad (5.8)$$

6   Finally $d\hat{s}^2$

And so with

$$d\hat{s}^2 = \hat{g}_{00}\, dt^2 + 2\hat{g}_{0i}\, dt\, dx^i + \hat{g}_{ij}\, dx^i dx^j \tag{6.1}$$

and collecting our results given by (4.6), (5.5) and (5.8) we finally have

$$
\begin{aligned}
d\hat{s}^2 \simeq & \Big[-\bar{A} - \delta A - 2\bar{A}\varphi + \bar{B}\dot{\bar{\phi}}^2 + 2\bar{B}\dot{\bar{\phi}}\,\delta\dot{\phi} + \delta B\,\dot{\phi}^2 \\
& \quad + 2\Big(\bar{C}\dot{\bar{\phi}}\dot{\bar{X}} + \bar{C}\dot{\bar{\phi}}\,\delta\dot{X} + \bar{C}\delta\dot{\phi}\,\dot{\bar{X}} + \delta C\,\dot{\bar{\phi}}\dot{\bar{X}} \Big) \\
& \quad + \bar{D}\dot{\bar{X}}^2 + 2\bar{D}\dot{\bar{X}}\,\delta\dot{X} + \delta D\,\dot{\bar{X}}^2 \Big] dt^2 \\
& + 2\Big[a\bar{A}(\alpha_{,i} + \beta_i) + \bar{B}\dot{\bar{\phi}}\,\delta\phi_{,i} + \bar{C}\big(\dot{\bar{\phi}}\,\delta X_{,i} + \dot{\bar{X}}\,\delta\phi_{,i} \big) + \bar{D}\dot{\bar{X}}\,\delta X_{,i} \Big] dt\, dx^i \\
& + a^2 \bar{A}\Big[\Big(1 - 2\psi + \frac{\delta A}{\bar{A}} \Big)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)} \Big] dx^i dx^j.
\end{aligned}
\tag{6.2}
$$