

Transformation of Primordial Cosmological Perturbations under the Invertible Higher-Derivative Disformal Transformation

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ABSTRACT: Primordial cosmological perturbations are quantum fluctuations that served as seeds in the early universe for what we nowadays observe as galaxies and clusters thereof. Cosmic inflation and the accompanying dynamics of the expanding Universe, stretched and allowed them to evolve, leading to their current state. Being the "beginning" of us all, their importance cannot be overemphasised. Focusing on their mathematical properties, in this study, we explore their possible variation under different forms of disformal transformation. This goes from reviews for the simplest special disformal transformal, passing through the original Bekenstein disformal transformation, and then leading to investigation of the effect of the Invertible Higher-Derivative Disformal Transformation (TMM disformal transformation). We examine the variations of scalar and tensor perturbations within the framework of the Horndeski theory. While footprints of the change of these perturbations may be apparent at the leading order, we find that the stretching of the early universe may remove these marks of variance in the superhorizon limit.

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1 Read Me First

Some Reminders

This document is a **work in progress**. Citations, figure sources, and the figures themselves are still being finalized (I plan to create original versions of all figures in this document, at hand, some are already original). At present, several parts still lack proper attribution. All

necessary citations and credits will be included in the final version. This version is uploaded to a GitHub repository for continuous backup and version control.

Furthermore, I (the author, marked as (*)) am still developing my understanding of the topic at hand. The concepts, derivations, and exposition presented here have not yet been reviewed or verified by my supervisor (the co-author, marked as (1)). Errors are expected at this stage.

This section will be removed once the document has been fully reviewed and approved by my supervisor. Might I also thank my supervisor for giving me a topic that I find *truly interesting*. Thank you for giving me the great opportunity of learning about the Universe, its history, evolution, and a peek into its unknowns.

The Purpose of this Document

Although this document is currently formatted in the style of JHEP, it is not *yet* meant to be a submission to JHEP or any journal thereof. This document, serves as two things. Firstly, it is where the second author fleshes out his thoughts regarding the topic, and thus the calculations and discussions may be lengthy and verbose. The second author wishes to grasp and understand the topics at hand. Secondly, it serves as a template and source material for a future formal document that may be submitted to JHEP or any other journal; it might even be a springboard for the author's undergraduate thesis.

2 Overview and thoughts

The Big Bang singularity was an infinitely hot and dense object. For some inconclusive reason, the singularity expanded, or more accurately inflated. If we are to follow the initial mathematical model offered by the Big Bang Theory (BBT), it would fail to predict that universe that we see today – homogenous, isotropic, and euclidean. This is where **Inflationary Cosmology** (IC) steps in. One of the problems of the initial BBT is the rate of inflation is too slow. For the sake of imagination, let us consider a deformable fabric. Suppose you place a really heavy sphere along the surface of the fabric then slowly stretch the fabric. The heaviness of this sphere would over power the stretching speed, still leaving a dented surface. However, what if we increase the stretching so fast around the order of e^{60} ? You'd find that the sphere does not have a time to "leave" a signifanct dent. Inflationary Cosmology introduces a particle called an *inflaton*, that drove the inflation rate of the universe exponentially, leaving us with a homogenous, isotropic, and euclidean space-time.

Let us take a step-back again when the univese was rather young. The primordial universe was a soup of plasma and we often refer to this state as a collection of quantum fluctuations – random variations in energy density. The Universe was simply too hot to accomodate the particles that we know today as presented in the Standard Model of Particle Physics. These quantum fluctuations are the "seedlings" of the present universe. It would rather be an interesting question to ask: "How did these quantum fluctuations influenced the inflaton field, and vice versa?"

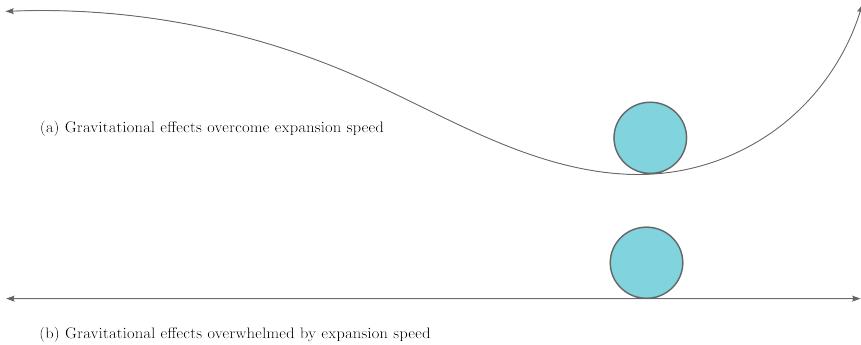


Figure 1: An analogy for the flatness problem. I am not even sure if this is a valid analogy.

Let us say these quantum fluctuations perturb the inflation field, which we denote as *phi*, as follows:

$$\phi = \bar{\phi} + \delta\phi \quad (2.1)$$

Where, $\bar{\phi}$ is the unperturbed portion of the field, while $\delta\phi$, is our object of interest. Now the thing is, the inflaton is still a theoretical framework, and it is rather of astronomical difficulty to probe this field experimentally. We then resort to our bestfriend that is mathematics. Now, how do we determine the precise mathematical properties of these perturbations? There is no consensus, *yet* under what mathematical framework should we probe these perturbations; however, some frameworks are better than other. Take for example the scalar-tensor theories, which may be represented in different "frames" that are mathematically related by metric transformations such as:

- **Conformal transformation.** A transformation that uniformly rescales the metric.
- **Disformal transformation.** Describes a more complex and anisotropic stretching of the metric.

2

Some issues arise. Firstly, in our current General Relativity textbooks, we only know about what we call as Conformal Transformation. Handwaving our way around, it is simply a symmetric transformation in space, and it is quite the gymnastics to think about this formalism where there is constant changes in space due to perturbations. We owe the great Bekenstein with this one, as he developed a mathematical tool which we now call as the Disformal Transformation. This transformation of the metric under takes the form:

$$A(\phi)g_{\mu\nu} + B(\phi)\phi_{;\mu}\phi_{;\nu} \quad (2.2)$$

Notice that we go straight to the metric! The metric measures the changes in the geometry in our spacetime, and it allows us to solve the LHS Einstein Field Equations (EFE) given by

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (2.3)$$

Knowing how spacetime behaves also give us some clue regarding the nature of the matter that influenced the geometry. We'd find to encounter another pressing issue: $\delta\phi$ is not gauge invariant. In other words, it is not physically measurable. We'd first want to package it mathematically into its gauge invariant variant (get-it, get-it?); we then slot it in our FLRW metric (a metric that aligns with the universe we see today). Once we slot that in the metric, we then apply our chosen Disformal Transformation, then solve the Einstein Field Equations.

2.1 Overture

1. We describe the **homogeneous and isotropic** universe that we see today, encapsulated by the FLRW metric. This is our "stage" where the physics takes place.
2. We use the Inflationary Model, which hypothesizes the existence of a scalar field (the inflaton, denoted ϕ) that drives the initial expansion and leads to the universe we observe.
3. We note that during the early universe, **quantum fluctuations** perturbed both the background metric and the inflaton field.
4. To better analyze these perturbations, we use the scalar-vector-tensor decomposition. We take only the scalar and tensor components, as they are the relevant ones for this study.
 - We must keep track of the **scalar perturbation, encapsulated in ζ** . This single, gauge-invariant variable packages together the scalar fluctuations from both the quantum fluctuations and the inflaton field.
 - We must also keep an eye on the **tensor perturbation, denoted as γ_{ij}** , which represents primordial gravitational waves.
5. We slot these perturbations into our background metric, which gives us the full, physically perturbed metric, $g_{\mu\nu}$.
6. We take this perturbed metric $g_{\mu\nu}$ and then apply the **TMM disformal transformation**.
7. This gives us a new metric, denoted as $\hat{g}_{\mu\nu}$.
8. Essentially, we want to look at the perturbations from two different perspectives, or "frames." First, we look at them in the original frame ($g_{\mu\nu}$) and then compare this to what the TMM frame sees ($\hat{g}_{\mu\nu}$). The central question is: **are these perturbations invariant** when viewed from these different perspectives?
9. We take this a step further and ask what happens at the superhorizon limit.

3 The Expanding Universe

It is definitely established that the universe is expanding and flat (in large scales of around $100 Mpc$). How do we describe this seemingly flat, homogenous, and isotropic universe?

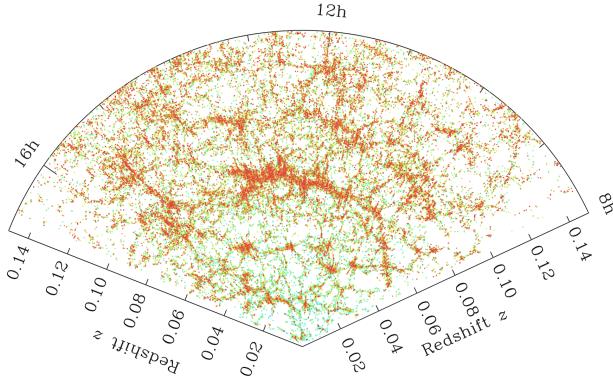


Figure 2: aion

4 Mathematical Preliminaries

4.1 The Riemann Curvature Tensor and its Contractions

A **manifold** is a topological space that is locally Euclidian. On the other hand a **Riemannian manifold** is a type of manifold where measurement of distance and angle is possible.

Gauss figured out that any point on a 2D surface can be summarized by a single number which we denote as K and is called as the **Gaussian curvature**. This is the product of the two principal curvatures at a particular point. Simply put, the principal curvatures are rough approximations of the 2D surface.

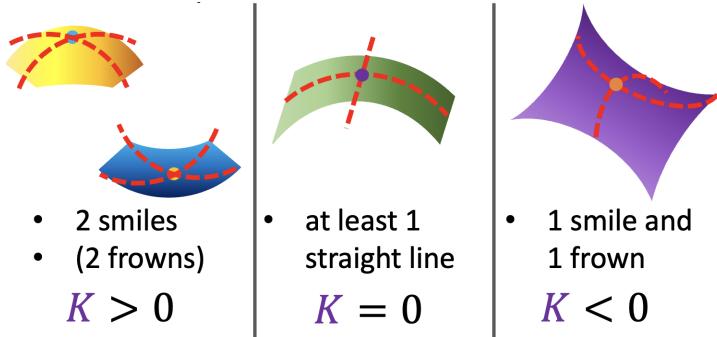


Figure 3: According to Gaussian Curvature, if you have a surface that can be approximated by two “frowns” or “smiles”, then the intersection thereof has positive curvature ($K > 0$). If one is a “frown” and the other a “smile”, the intersection has negative curvature ($K < 0$). If one direction is flat, the curvature is zero ($K = 0$).

The issue with the Gaussian Curvature is that it only works for 2D surfaces. This motivates us to develop something that can work for higher dimensions. This is where we use the **Riemann Curvature Tensor** (RCT) given by

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \quad (4.1)$$

Although we say that the Christoffel Symbols are non-tensorial in nature, the sum thereof amounting to the Riemann Curvature Tensor is indeed a tensor. We can think of the Christoffel Symbols as mathematical objects that describe how the coordinate basis vectors change from point to point on a given manifold. Note that the covariant derivative (which is related to the original definition of the RCT) is related to the Christoffel Symbols as follows:

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma_{\mu\sigma}^\rho V^\sigma. \quad (4.2)$$

equation

5 History and Properties of the Universe

The Universe as we see it today is **homogenous** and **isotropic**. Homogeneity talks about

5.1 What the Cosmic Microwave Background Tells us

For the first $\sim 400,000$ years of the Universe, it was too hot to allow atoms to form, at this stage electrons have yet to couple with protons. Due to quantum mechanical effects, free electrons scatter the photons of the early universe, making it opaque, thereby "trapping" the photons. However, as the Universe expanded and cooled, it eventually reached a temperature where electrons could finally couple with protons to form atoms. At this point, photons could finally travel freely through space. This photons, given the extreme nature of the early universe were highly energetic, and so we must be careful in observing them today.

The continuous expansion of the universe stretched the photons, thereby increasing their wavelength. If we traceback this stretching from the time when photons started to propagate freely up to the present time, we find these photons fall in the microwave region. This is what we now call as the Cosmic Microwave Background (CMB). It is a strong evidence that, indeed, the universe started from a hot and dense state, and has been expanding ever since.

[1–5]

6 How a Flat, Homogenous, and Isotropic Universe Evolves: The FLRW Metric

The FLRW metric is given by

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (6.1)$$

As discussed in 4.1 a Gaussian Curvature of $K = 0$ denotes a flat space. Would it make sense to probe perturbations in a flat universe? Observations show (take for example the CMB), indicate that our universe is close to being spatially flat. We can actually study the perturbations by looking into the deviations from a flat, homogenous, and isotropic

"background". Hence in this exposition, we will consider the case where we have a flat universe, i.e., $K = 0$ in (6.1).

$$5 + 4 + 3 + 2 \quad (6.2)$$

6.1 An Example in using the FLRW Metric

Consider a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric with c explicit,

$$ds^2 = -c^2 dt^2 + a(t)^2 (dr^2 + r^2 d\Omega^2).$$

A light signal emitted at cosmic time t_{em} from radial coordinate r_{em} and received at t_0 (so $r_{\text{obs}} = 0$) follows a radial null geodesic with $dr = -c dt/a(t)$. The comoving radial distance from observer to source is

$$\chi \equiv r_{\text{em}} = c \int_{t_{\text{em}}}^{t_0} \frac{dt}{a(t)}.$$

Assume the scale factor is a power law $a(t) = a_0 (t/t_0)^n$ with $n \neq 1$.

1. Derive expressions for the comoving distance χ , the proper distance today $D_p(t_0)$, the particle horizon, the luminosity distance D_L , and the angular diameter distance D_A .
2. Then compute numerical values for these quantities for the matter-dominated case $n = \frac{2}{3}$ and redshift $z = 1$.
3. For the numerical example, use the normalization $a_0 = 1$, the relation $H_0 = n/t_0$, and a Hubble constant of $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

Solution:

Deriving χ for the travelling light

Note for a travelling light we have a null geodesic, hence $ds^2 = 0$. Furthermore, since we have a travelling light in the radial direction, there are no changes in the angular coordinates; in other words, $d\Omega^2 = 0$. Clearly then the FLRW metric reduces to

$$0 = -c^2 dt^2 + a(t)^2 dr^2 \quad (6.3)$$

If we move the first term to the right and take the square root of both sides (taking note of the \pm) then continuing to solve the differential equation, we have

$$\begin{aligned} cdt &= \pm a(t) dr \\ dr &= \pm \frac{c}{a(t)} dt \\ \chi &= \pm c \int_{t_0}^{t_{\text{em}}} \frac{1}{a(t)} dt \end{aligned} \quad (6.4)$$

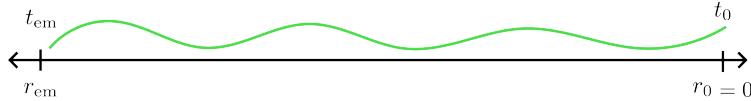


Figure 4: A light signal emitted at cosmic time t_{em} from radial coordinate r_{em} and received at t_0 .

If we refer to Fig. 4, since as far as we know, time moves in one direction then $dt > 0$. On the other hand, notice that since we set $r_0 = 0$ as the observer's position, when the light travel from r_{em} to the observer, the distance decreases – implying that $dr < 0$. It is now clear that in order to satisfy both conditions, we must take the negative solution.

$$\chi = -c \int_{t_0}^{t_{\text{em}}} \frac{1}{a(t)} dt \quad (6.5)$$

Does Eq. (6.5) make sense? In my first encounter of this problem I was quite confused how this measures the comoving distance. But once we realize that $a \in (0, 1)$ then we see that χ scales properly.

7 The Matter and Energy of a Homogenous and Isotropic Universe

The features of the universe on large scales force us to consider the matter and energy (energy-momentum tensor $T^{\mu\nu}$) to take the form of a perfect fluid. Clearly, the universe that we see in our daily lives is not homogenous and isotropic. Thankfully (or not), the universe is way larger than what we experience, and it follows homogeneity and isotropicity.

Why perfect fluids? A perfect fluid is a fluid with **no viscosity** and **no heat conduction**. In other words, the matter and energy must flow without internal friction and there must be no net flow of energy. The fluid in this state is described completely by its Energy Density (ρ) and Isotropic Pressure (P). This is a direct mathematical consequence of an isotropic universe. In the case where matter and energy had viscosity and/or heat flow, it would define a preferred spatial direction. And thus we are forced to define the covariant energy-momentum tensor as

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) U_\mu U_\nu + P g_{\mu\nu} \quad (7.1)$$

8 Cosmological Perturbations

We define the perturbations of the metric as follows

$$ds^2 = -(1 + 2\Phi)dt^2 + 2aB_idx^i dt + a^2[(\delta_{ij})(1 - 2\Psi) + 2C_{ij}]dx^i dx^j \quad (8.1)$$

Note that although the metric tensor given by $g_{\mu\nu}$ have 16 components, since it is symmetric we only have 10 independent components. Further, note that ds^2 (spacetime element) is invariant under coordinate transform; however, the metric tensor, in general is

variant under coordinate transform. From, 8.1, Φ, Ψ are scalar perturbations, while B_i and C_{ij} are vector and tensor perturbations, respectively.

We wish to probe these scalar perturbations under the following transformation

$$\begin{aligned} t &\rightarrow t' + \alpha(t, \mathbf{x}) \\ x^i &\rightarrow x'^i + \beta^i(t, \mathbf{x}) \end{aligned} \quad (8.2)$$

Here, note that the apostrophe/prime symbol ($'$) denotes a transformed coordinate, while a comma (,) denotes a derivative. We wish to show that transforming the purely spatial component of the metric tensor gives us (up to first order perturbations)

$$\begin{aligned} \Psi' &= \Psi + \alpha H \\ C' &= C - \beta \end{aligned} \quad (8.3)$$

Here, $H := \frac{\dot{a}}{a}$ and is called the Hubble parameter. Now, recall that the metric tensor, (or any covariant tensor) follows the following transformation law

$$g_{\rho\sigma} = g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \quad (8.4)$$

For purely spatial we have

9 Disformal Transformation

9.1 Perturbing the FLRW Metric

Let us decompose the perturbed FLRW metric (with SVT decomposition) as

$$ds^2 = -(1+2\varphi) dt^2 + 2a(\alpha_{,i} + \beta_i) dt dx^i + a^2 \left[(1-2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)} \right] dx^i dx^j, \quad (9.1)$$

where $a = a(t)$ is the scale factor, $(\varphi, \alpha, \psi, E)$ are scalar perturbations, β_i, F_i are vector perturbations, and γ_{ij} are tensor perturbations.

Under a disformal transformation, the line element becomes

$$d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu. \quad (9.2)$$

The invertible disformal transformation with higher derivatives presented by Takahashi, Motohashi, and Minamitsuji (which we call the TMM transformation for convenience) is

$$\hat{g}_{\mu\nu} = A g_{\mu\nu} + B \phi_{;\mu} \phi_{;\nu} + C (\phi_{;\mu} X_{;\nu} + X_{;\mu} \phi_{;\nu}) + D X_{;\mu} X_{;\nu}, \quad (9.3)$$

where A, B, C, D are functionals of ϕ, X, Y, Z , and X, Y, Z are defined as

$$X \equiv -\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu}, \quad Y \equiv g^{\mu\nu} \phi_{;\mu} X_{;\nu}, \quad Z \equiv g^{\mu\nu} X_{;\mu} X_{;\nu}. \quad (9.4)$$

The TMM form can be simplified by a “completing-the-square” step. We have “squared” covariant-derivative terms of ϕ , cross terms in the middle, and “squared” covariant-derivative terms of X . Recall the elementary identity

$$ax^2 + 2bxy + cy^2 = a \left(x + \frac{b}{a} y \right)^2 + \left(c - \frac{b^2}{a} \right) y^2. \quad (9.5)$$

Identifying $a = B$, $b = C$, and $c = D$, the last three terms in (9.3) combine as

$$B\left(\phi_{;\mu} + \frac{C}{B}X_{;\mu}\right)\left(\phi_{;\nu} + \frac{C}{B}X_{;\nu}\right) + \left(D - \frac{C^2}{B}\right)X_{;\mu}X_{;\nu}. \quad (9.6)$$

Hence, (9.3) may be rewritten as

$$\hat{g}_{\mu\nu} = A g_{\mu\nu} + B\left(\phi_{;\mu} + \frac{C}{B}X_{;\mu}\right)\left(\phi_{;\nu} + \frac{C}{B}X_{;\nu}\right) + \left(D - \frac{C^2}{B}\right)X_{;\mu}X_{;\nu}. \quad (9.7)$$

If we define

$$\Phi_\mu \equiv \sqrt{B}\left(\phi_{;\mu} + \frac{C}{B}X_{;\mu}\right), \quad \Delta \equiv D - \frac{C^2}{B}, \quad (9.8)$$

then the transformation reduces cleanly to

$$\boxed{\hat{g}_{\mu\nu} = A g_{\mu\nu} + \Phi_\mu \Phi_\nu + \Delta X_{;\mu}X_{;\nu}.} \quad (9.9)$$

Under (9.9), the transformed line element can be expanded as

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{\mu\nu} dx^\mu dx^\nu \\ &= (A g_{00} + \Phi_0 \Phi_0 + \Delta X_{;0}X_{;0}) dt^2 \\ &\quad + 2(A g_{0i} + \Phi_0 \Phi_i + \Delta X_{;0}X_{;i}) dt dx^i \\ &\quad + (A g_{ij} + \Phi_i \Phi_j + \Delta X_{;i}X_{;j}) dx^i dx^j. \end{aligned} \quad (9.10)$$

Consequently, take the perturbed scalar field $\phi = \bar{\phi} + \delta\phi$. Then

$$\begin{aligned} X &= -\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} \\ &= -\frac{1}{2}g^{00}\dot{\phi}^2 \\ &= -\frac{1}{2}(-(1-2\varphi))(\dot{\bar{\phi}} + \delta\dot{\phi})^2 \\ &= \left(\frac{1}{2} - \varphi\right)(\dot{\bar{\phi}}^2 + 2\dot{\bar{\phi}}\delta\dot{\phi} + (\delta\dot{\phi})^2) \\ &= \frac{1}{2}\dot{\bar{\phi}}^2 + \dot{\bar{\phi}}\delta\dot{\phi} - \varphi\dot{\bar{\phi}}^2, \end{aligned} \quad (9.11)$$

where g^{00} comes from the perturbed metric, and we have dropped higher-order perturbations (e.g. $(\delta\dot{\phi})^2$).

It is natural to split A , Φ_0 , and Δ as

$$A = \bar{A} + \delta A, \quad \Phi_0 = \bar{\Phi}_0 + \delta\Phi_0, \quad \Delta = \bar{\Delta} + \delta\Delta, \quad (9.12)$$

Notice that from (9.8) we have

$$\begin{aligned} \Phi_0 &= \sqrt{B}\left(\phi_{;0} + \frac{C}{B}X_{;0}\right) \\ &= \sqrt{B}\left(\dot{\bar{\phi}} + \delta\dot{\phi} + \frac{C}{B}(\dot{\bar{X}} + \delta\dot{X})\right) \end{aligned} \quad (9.13)$$

Where we see to have zeroth order perturbations given back the background variables. On the other hand for the spatial component, notice

$$\begin{aligned}\Phi_i &= \sqrt{B} \left(\phi_{,i} + \frac{C}{B} X_{,i} \right) \\ &= \sqrt{B} \left(\frac{C}{B} (\bar{X}_{,i} + \delta X_{,i}) \right) \\ &= \sqrt{B} \left(\frac{C}{B} \delta X_{,i} \right)\end{aligned}\tag{9.14}$$

Where again, the background term vanishes under a spatial covariant derivative since it is only dependent upon time. On the other hand, we are left with $\delta X_{,i}$ which is a first order perturbation. Hence, we choose not to "split" the spatial components of Φ_μ

9.2 For the dt^2 terms

From the perturbed metric given by (9.1) we have $g_{00} = -(1 + 2\varphi)$ it follows that

$$\begin{aligned}Ag_{00} &= -(\bar{A} + \delta A)(1 + 2\varphi) \\ &= -(\bar{A} + 2\varphi \bar{A} + \delta A + 2\varphi \delta A) \\ &\approx -\bar{A} - \delta A - 2\bar{A}\varphi\end{aligned}\tag{9.15}$$

On the other hand

$$\Phi_0^2 = (\bar{\Phi}_0 + \delta \Phi_0)^2 = \bar{\Phi}_0^2 + 2\Phi_o\tag{9.16}$$

Finally,

$$\begin{aligned}\Delta X_{;0} X_{;0} &= (\bar{\Delta} + \delta \Delta)(\dot{\bar{X}} + \delta X)^2 \\ &= (\bar{\Delta} + \delta \Delta)(\dot{\bar{X}}^2 + 2\dot{\bar{X}} \delta X + \underbrace{(\delta X)^2}_{\text{2nd order}}) \\ &\approx \bar{\Delta} \dot{\bar{X}}^2 + \delta \Delta \dot{\bar{X}}^2 + 2\bar{\Delta} \dot{\bar{X}} \delta X + \underbrace{2\delta \Delta \dot{\bar{X}} \delta X}_{\text{2nd order}} \\ &\approx \bar{\Delta} \dot{\bar{X}}^2 + \delta \Delta \dot{\bar{X}}^2 + 2\bar{\Delta} \dot{\bar{X}} \delta X\end{aligned}\tag{9.17}$$

9.3 For $dtdx^i$ terms

Again, given by (9.1) we find that $g_{0i} = a(\alpha_{,i} + \beta_i)$. So for Ag_{0i} we have

$$\begin{aligned}Ag_{0i} &= a(\bar{A} + \delta A)(\alpha_{,i} + \beta_i) \\ &= a(\bar{A}\alpha_{,i} + \bar{A}\beta_i + \delta A\alpha_{,i} + \delta A\beta_i) \\ &\approx a\bar{A}(\alpha_{,i} + \beta_i)\end{aligned}\tag{9.18}$$

For $\Phi_0 \Phi_i$ note that we have already established that Φ_i is already a first order perturbation. Then

$$\begin{aligned}\Phi_0 \Phi_i &= (\bar{\Phi}_0 + \delta \Phi_0)(\Phi_i) \\ &= \bar{\Phi}_0 \Phi_i + \delta \Phi_0 \Phi_i \\ &\approx \bar{\Phi}_0 \Phi_i\end{aligned}\tag{9.19}$$

On the other hand, for $\Delta X_{;0}X_{;i}$, note that $X_{;i} = \bar{X}_{;i} + \delta X_{;i} = \delta X_{;i}$ since background quantities are only time dependent. So,

$$\begin{aligned}\Delta X_{;0}X_{;i} &= (\bar{\Delta} + \delta\Delta)(\bar{X}_{;0} + \delta X_{;0})\delta X_{;i} \\ &= (\bar{\Delta}\dot{\bar{X}} + \bar{\Delta}\delta\dot{X} + \delta\Delta\dot{\bar{X}} + \delta\Delta\delta\dot{X})\delta X_{;i} \\ &\approx (\bar{\Delta}\dot{\bar{X}} + \bar{\Delta}\delta\dot{X} + \delta\Delta\dot{\bar{X}})\delta X_{;i} \\ &\approx \bar{\Delta}\dot{\bar{X}}\delta X_{;i}\end{aligned}\tag{9.20}$$

9.4 For $dx^i dx^j$ terms

10 Disformal Transformation

Consider a spatially flat FLRW background given by

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j\tag{10.1}$$

We want to add all possible linear perturbations that behave as scalar or tensor under spatial rotations. This is what we call as SVT-decomposition in linear order. Doing so gives us

$$ds^2 = -(1 + 2\xi)dt^2 + 2\psi_{;i}dt dx^i + a^2(t) [(1 + 2\zeta)\delta_{ij} + \gamma_{ij} + 2E_{;ij}] dx^i dx^j\tag{10.2}$$

Working in unitary gauge and ADM variables, then

$$ds^2 = -(1 + 2\alpha)dt^2 + 2\partial_i\beta dt dx^i + a^2(t) [(1 + 2\mathcal{R}_c)\delta_{ij} + \gamma_{ij}] dx^i dx^j.\tag{10.3}$$

$$ds^2 = (-N^2 + h_{ij}N^i N^j)dt^2 + 2h_{ij}N^j dx^i dt + h_{ij}dx^i dx^j\tag{10.4}$$

The TMM states that

$$\hat{g}_{\mu\nu} = A g_{\mu\nu} + B\phi_{;\mu}\phi_{;\nu} + C(\phi_{;\mu}X_{;\nu} + X_{;\mu}\phi_{;\nu}) + DX_{;\mu}X_{;\nu}\tag{10.5}$$

Where A, B, C, D are dependent on ϕ, X, Y, Z . Following [2] we may define
The scalar field perturbation is defined to be

$$\zeta(t, \mathbf{x}) \equiv -\Psi(t, \mathbf{x}) - \frac{H}{\dot{\rho}(t)}\delta\rho(t, \mathbf{x}),\tag{10.6}$$

The lapse function

$$N \equiv \frac{1}{\sqrt{-g'^{\mu\nu}\nabla_a t \nabla_b t}} = \frac{1}{\sqrt{-g^{00}}}\tag{10.7}$$

Referring to Ref. [2], we may define the following terms as

$$\begin{aligned}X &:= -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ Y &:= \nabla^\mu\phi\nabla_\mu X \\ Z &:= \nabla^\mu X\nabla_\mu X\end{aligned}\tag{10.8}$$

Using the definition we have from 10.8, while being under the unitary gauge $\phi = \phi(t)$. Note that since the covariant derivative is defined to be

$$\begin{aligned}\nabla_\mu A^\nu &= \partial_\mu A^\nu + \Gamma_{\mu\lambda}^\nu A^\lambda, \\ \nabla_\mu B_\nu &= \partial_\mu B_\nu - \Gamma_{\mu\nu}^\lambda B_\lambda.\end{aligned}\tag{10.9}$$

Then it follows that for a scalar field, the covariant derivative reduces to that of a partial derivative.

Proof: Let φ be an arbitrary scalar field. Note that a scalar field is a rank 0 tensor. By the definition of the

We have to take note that for a scalar field, the covariant derivative reduces to a normal partial derivative. It follows that $\nabla_\mu \phi$ reduces to $\partial_t \phi$ and $\partial_j \phi$.

Taking note of Eq. 10.8, and using unitary gauge where $\phi = \phi(t)$, then we may write these as

By the definition of X from Eq. (10.8), in unitary gauge this reduces to

$$\begin{aligned}X &= -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ &= -\frac{1}{2}g_{00}\nabla_0\phi(t)\nabla_0\phi(t) \\ &= -\frac{1}{2}(-1)\partial_t\phi\partial_t\phi \\ &= \frac{1}{2}\dot{\phi}^2\end{aligned}\tag{10.10}$$

$$\begin{aligned}X &= \frac{\dot{\phi}^2}{2N^2} \\ Y &= -\frac{\dot{\phi}\dot{X}}{N^2} \\ Z &= -\frac{\dot{X}^2}{N^2}\end{aligned}\tag{10.11}$$

$$X_{;\mu} = \begin{cases} \dot{\bar{X}} - 2\dot{\bar{X}}\alpha - 2\bar{X}\dot{\alpha}, & \mu = 0, \\ -2\bar{X}\partial_i\alpha, & \mu = i. \end{cases}\tag{10.12}$$

or more concisely we may write this as

$$X_{;\mu} = \delta_\mu^0(\dot{\bar{X}} - 2\dot{\bar{X}}\alpha - 2\bar{X}\dot{\alpha}) - 2\bar{X}\delta_\mu^i\partial_i\alpha$$

Table 1: Types of Disformal Transformations of the Metric

Transformation Name	Metric Transformation ($\hat{g}_{\mu\nu}$)	Short Description
Bekenstein	$\hat{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_{;\mu}\phi_{;\nu}$	The original proposal where both the conformal factor A and disformal factor B depend only on the scalar field ϕ .
Special	$\hat{g}_{\mu\nu} = g_{\mu\nu} + B(\phi, X)\phi_{;\mu}\phi_{;\nu}$	A purely disformal transformation with $A = 1$. The disformal factor B depends on ϕ and its kinetic term X .
Generalized with one arbitrary conformal	$\hat{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi, X)\phi_{;\mu}\phi_{;\nu}$	A generalization where A depends only on ϕ , but B depends on both ϕ and X .
Fully Generalized for First Derivatives	$\hat{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_{;\mu}\phi_{;\nu}$	Both the conformal factor A and the disformal factor B are allowed to depend on ϕ and its kinetic term X .
Fully General and Extended	$\begin{aligned} \hat{g}_{\mu\nu} = & A(\dots)g_{\mu\nu} \\ & + (C(\dots)\phi_{;\mu} + D(\dots)X_{;\mu}) \\ & \times (C(\dots)\phi_{;\nu} + D(\dots)X_{;\nu}) \end{aligned}$	The conformal factor A and functions C, D depend on ϕ and its invariants. The disformal part is a quadratic form involving gradients of ϕ and X .
TMM	$\begin{aligned} \hat{g}_{\mu\nu} = & A(\dots)g_{\mu\nu} + B(\dots)\phi_{;\mu}\phi_{;\nu} \\ & + C(\dots)(\phi_{;\mu}X_{;\nu} + X_{;\mu}\phi_{;\nu}) \\ & + D(\dots)X_{;\mu}X_{;\nu} \end{aligned}$	The most general quadratic transformation in first derivatives. All coefficient functions (A, B, C, D) depend on ϕ and its invariants (X, Y, Z) .

Note: $X = -\frac{1}{2}g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta}$ denotes the kinetic term. The arguments (\dots) in the last two rows are shorthand for (ϕ, X, Y, Z) , representing dependence on the scalar field and its invariants.

11 Geodesics, Cosmological Redshift, Horizons, and Comoving Coordinates

How do the galaxies in the universe spread out overtime as the universe changes time according to the scale factor $a(t)$

11.1 Geodesics

The Geodesic equation as given by Eq. (11.1) below describes the path a free particle would take if there are no *forces acting* upon it. It is only affected by the curvature of spacetime.

$$\frac{d^2x^\sigma}{d\lambda^2} + \Gamma_{\mu\nu}^\sigma \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \quad (11.1)$$

Why bother with the geodesic? Since for the most part, the universe is an empty space of vacuum, if we find the geodesic describing the system, we can potentially track its evolution overtime. **Would it be possible to treat the *inflaton* as a free particle, then track its evolution?**

Notice the Connection Coefficients given in Eq. (11.1). Upon deriving the Connection Coefficients (CC) for the FLRW metric, we recover 13 non-vanishing and unique CC. To save us from going through this one-by-one, it might be smart to take an initial ansatz.

A Appendix

Please always give a title also for appendices.

Acknowledgments

Note added. I'm just cleaning up my workflow here for the references.

References

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