

1 The TMM Transformation

Let us decompose the perturbed FLRW metric (with SVT decomposition) as

$$ds^2 = -(1+2\varphi) dt^2 + 2a(\alpha_{,i} + \beta_i) dt dx^i + a^2 \left[(1-2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)} \right] dx^i dx^j, \quad (1.1)$$

where $a = a(t)$ is the scale factor, $(\varphi, \alpha, \psi, E)$ are scalar perturbations, β_i, F_i are vector perturbations, and γ_{ij} are tensor perturbations. Note that throughout this transformation a partial derivative is denoted by a comma subscript ", ". Meanwhile, a covariant derivative is denoted by a semi-colon subscript ";".

Under a disformal transformation, the line element becomes

$$d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu. \quad (1.2)$$

The invertible disformal transformation with higher derivatives presented by Takahashi, Motohashi, and Minamitsuji (which we call the TMM transformation for convenience) is

$$\hat{g}_{\mu\nu} = A g_{\mu\nu} + B \phi_{;\mu} \phi_{;\nu} + C (\phi_{;\mu} X_{;\nu} + X_{;\mu} \phi_{;\nu}) + D X_{;\mu} X_{;\nu}, \quad (1.3)$$

where A, B, C, D are functionals of ϕ, X, Y, Z , and X, Y, Z are defined as

$$X \equiv -\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu}, \quad Y \equiv g^{\mu\nu} \phi_{;\mu} X_{;\nu}, \quad Z \equiv g^{\mu\nu} X_{;\mu} X_{;\nu}. \quad (1.4)$$

2 Expanding $d\hat{s}^2$ term-by-term

Under (1.3), the transformed line element can be expanded as

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{\mu\nu} dx^\mu dx^\nu \\ &= (A g_{00} + B \phi_{;0}^2 + C(2\phi_{;0} X_{;0}) + D X_{;0}^2) dt^2 \\ &= 2(A g_{0i} + B \phi_{;0} \phi_{;i} + C(\phi_{;0} X_{;i} + X_{;0} \phi_{;i}) + D X_{;0} X_{;i}) dt dx^i \\ &= (A g_{ij} + B \phi_{;i} \phi_{;j} + C(\phi_{;i} X_{;j} + X_{;i} \phi_{;j}) + D X_{;i} X_{;j}) dx^i dx^j \end{aligned} \quad (2.1)$$

3 Background-perturbation split

We perturb the scalar field as

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}) \quad (3.1)$$

Since this is a scalar field the covariant derivative reduces to partial derivatives, it follows that

$$\phi_{;0} = \partial_0 \phi = \dot{\bar{\phi}} + \delta\dot{\phi}, \quad \phi_{;i} = \partial_i \phi = \delta\phi_{,i}, \quad (3.2)$$

Likewise for X we have

$$X(x, \mathbf{x}) = \bar{X} + \delta X \quad (3.3)$$

Consequently,

$$X_{;0} = \dot{\bar{X}} + \delta\dot{X}, \quad X_{;i} = \delta X_{;i} \quad (3.4)$$

It should also make sense to split the coefficients since they are functionals of (ϕ, X, Y, Z) . So we have

$$A = \bar{A} + \delta A, \quad B = \bar{B} + \delta B, \quad C = \bar{C} + \delta C, \quad D = \bar{D} + \delta D, \quad (3.5)$$

3.1 Linear Expression for X

$$\begin{aligned} X &= -\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} \\ &= -\frac{1}{2}g^{00}\dot{\phi}^2 \\ &= -\frac{1}{2}(-(1-2\varphi))(\dot{\phi} + \delta\dot{\phi})^2 \\ &= \left(\frac{1}{2} - \varphi\right)(\dot{\phi}^2 + 2\dot{\phi}\delta\dot{\phi} + (\delta\dot{\phi})^2) \\ &= \frac{1}{2}\dot{\phi}^2 + \dot{\phi}\delta\dot{\phi} - \varphi\dot{\phi}^2, \end{aligned} \quad (3.6)$$

where g^{00} comes from the perturbed metric, and we have dropped higher-order perturbations (e.g. $(\delta\dot{\phi})^2$).

4 Linear Expansion of the Transformed Metric

We are now ready to expand (2.1), using the splits and their derivatives given by (3.2), (3.4) and (3.5), while discarding higher-order perturbations (in other words, $\mathcal{O}(\epsilon^2)$ terms)

4.1 The dt^2 terms

We wish to expand

$$Ag_{00} + B\phi_{;0}^2 + C(2\phi_{;0}X_{;0}) + DX_{;0}^2 \quad (4.1)$$

For Ag_{00}

From (1.1), we have $g_{00} = -(1+2\varphi)$

Hence,

$$\begin{aligned} Ag_{00} &= -(\bar{A} + \delta A)(1+2\varphi) \\ &= -(\bar{A} + 2\varphi\bar{A} + \delta A + 2\varphi\delta A) \\ &\approx -\bar{A} - \delta A - 2\bar{A}\varphi \end{aligned} \quad (4.2)$$

because $2\varphi\delta A$ is $\mathcal{O}(\epsilon^2)$.

Starting from here we will be introducing a new notation for cases where $\mathcal{O}(\epsilon^n)$, $n \geq 2$. For a particular term that inherits higher-ordered perturbations we will be cancelling them as follows: ~~term~~.

For $B\phi_{;0}^2$

Using (3.2) and (3.5) we have

$$\begin{aligned}
B\phi_{;0}^2 &= B\dot{\phi}^2 \\
&= (\bar{B} + \delta B)(\dot{\phi} + \delta\dot{\phi})^2 \\
&= (\bar{B} + \delta B)(\dot{\phi}^2 + 2\dot{\phi}\delta\dot{\phi} + \cancel{\delta\dot{\phi}^2}) \\
&\approx \bar{B}\dot{\phi}^2 + 2\bar{B}\dot{\phi}\delta\dot{\phi} + \delta B\dot{\phi}^2 + \cancel{2\dot{\phi}\delta\bar{B}\delta\dot{\phi}} \\
&\approx \bar{B}\dot{\phi}^2 + 2\bar{B}\dot{\phi}\delta\dot{\phi} + \delta B\dot{\phi}^2
\end{aligned} \tag{4.3}$$

For the cross-term $C\phi_{;0}X_{;0}$ we have

$$\begin{aligned}
C\phi_{;0}X_{;0} &= (\bar{C} + \delta C)(\dot{\phi} + \delta\dot{\phi})(\dot{X} + \delta\dot{X}) \\
&= (\bar{C} + \delta C)(\dot{\phi}\dot{X} + \dot{\phi}\delta\dot{X} + \delta\dot{\phi}\dot{X} + \cancel{\delta\dot{\phi}\delta\dot{X}}) \\
&\approx \bar{C}\dot{\phi}\dot{X} + \bar{C}\dot{\phi}\delta\dot{X} + \bar{C}\delta\dot{\phi}\dot{X} + \delta C\dot{\phi}\dot{X} + \cancel{\delta C\dot{\phi}\delta\dot{X}} + \cancel{\delta C\delta\dot{\phi}\dot{X}} \\
&\approx \bar{C}\dot{\phi}\dot{X} + \bar{C}\dot{\phi}\delta\dot{X} + \bar{C}\delta\dot{\phi}\dot{X} + \delta C\dot{\phi}\dot{X}
\end{aligned} \tag{4.4}$$

For $DX_{;0}^2$

Finally we have,

$$\begin{aligned}
DX_{;0}^2 &= (\bar{D} + \delta D)(\dot{X} + \delta\dot{X})^2 \\
&= (\bar{D} + \delta D)(\dot{X}^2 + 2\dot{X}\delta\dot{X} + \cancel{(\delta\dot{X})^2}) \\
&\approx \bar{D}\dot{X}^2 + 2\bar{D}\dot{X}\delta\dot{X} + \delta D\dot{X}^2 + \cancel{2\delta D\dot{X}\delta\dot{X}} \\
&\approx \bar{D}\dot{X}^2 + 2\bar{D}\dot{X}\delta\dot{X} + \delta D\dot{X}^2
\end{aligned} \tag{4.5}$$

Putting (4.2) to (4.5), therefore g_{00} in linear-order is

$$\begin{aligned}
\hat{g}_{00} &\approx -\bar{A} - \delta A - 2\bar{A}\varphi \\
&\quad + \bar{B}\dot{\phi}^2 + 2\bar{B}\dot{\phi}\delta\dot{\phi} + \delta B\dot{\phi}^2 \\
&\quad + \bar{C}\dot{\phi}\dot{X} + \bar{C}\dot{\phi}\delta\dot{X} + \bar{C}\delta\dot{\phi}\dot{X} + \delta C\dot{\phi}\dot{X} \\
&\quad + \bar{D}\dot{X}^2 + 2\bar{D}\dot{X}\delta\dot{X} + \delta D\dot{X}^2
\end{aligned}$$

(4.6)

5 For the $dt dx^i$ terms

We wish to expand $Ag_{0i} + B\phi_{;0}\phi_{;i} + C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i}) + DX_{;0}X_{;i}$. We notice that $\phi(t)_{;i}$ vanishes.

For Ag_{0i}

$$\begin{aligned}
Ag_{0i} &= a(\bar{A} + \delta A)(\alpha_{;i} + \beta_i) \\
&= a(\bar{A}\alpha_{;i} + \bar{A}\beta_i + \cancel{\delta A\alpha_{;i}} + \cancel{\delta A\beta_i}) \\
&\approx a\bar{A}(\alpha_{;i} + \beta_i)
\end{aligned} \tag{5.1}$$

For $B\phi_{;0}\phi_{;i}$

$$\begin{aligned}
B\phi_{;0}\phi_{;i} &= (\bar{B} + \delta B)(\dot{\bar{\phi}} + \delta\dot{\phi})(\delta\phi_{;i}) \\
&= (\bar{B} + \delta B)(\delta\phi_{;i}\dot{\bar{\phi}} + \cancel{\delta\phi_{;i}\delta\dot{\phi}}) \\
&\approx \bar{B}\delta\phi_{;i}\dot{\bar{\phi}} + \cancel{\delta B\delta\phi_{;i}\dot{\bar{\phi}}} \\
&\approx \bar{B}\dot{\bar{\phi}}\delta\phi_{;i}
\end{aligned} \tag{5.2}$$

For $C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i})$

$$\begin{aligned}
C(\phi_{;0}X_{;i} + X_{;0}\phi_{;i}) &= (\bar{C} + \delta C)\left[(\dot{\bar{\phi}} + \delta\dot{\phi})(\delta X_{;i}) + (\dot{\bar{X}} + \delta\dot{X})(\delta\phi_{;i})\right] \\
&= (\bar{C} + \delta C)\left[\delta X_{;i}\dot{\bar{\phi}} + \cancel{\delta X_{;i}\delta\dot{\phi}} + \delta\phi_{;i}\dot{\bar{X}} + \cancel{\delta\phi_{;i}\delta\dot{X}}\right] \\
&\approx \bar{C}\delta X_{;i}\dot{\bar{\phi}} + \bar{C}\delta\phi_{;i}\dot{\bar{X}} + \cancel{\delta C\delta X_{;i}\dot{\bar{\phi}}} + \cancel{\delta C\delta\phi_{;i}\dot{\bar{X}}} \\
&\approx \bar{C}(\delta X_{;i}\dot{\bar{\phi}} + \delta\phi_{;i}\dot{\bar{X}})
\end{aligned} \tag{5.3}$$

For $DX_{;0}X_{;i}$

$$\begin{aligned}
DX_{;0}X_{;i} &= (\bar{D} + \delta D)(\dot{\bar{X}} + \delta\dot{X})\delta X_{;i} \\
&= (\bar{D} + \delta D)(\delta X_{;i}\dot{\bar{X}} + \cancel{\delta X_{;i}\delta\dot{X}}) \\
&\approx \bar{D}\delta X_{;i}\dot{\bar{X}} + \cancel{\delta D\delta X_{;i}\dot{\bar{X}}} \\
&\approx \bar{D}\dot{\bar{X}}\delta X_{;i}
\end{aligned} \tag{5.4}$$

Finally, putting together (5.1) to (5.4) \hat{g}_{0i} in linear order is given by

$$\boxed{\hat{g}_{0i} \approx a\bar{A}(\alpha_{;i} + \beta_i) + \bar{B}\dot{\bar{\phi}}\delta\phi_{;i} + \bar{C}(\delta X_{;i}\dot{\bar{\phi}} + \delta\phi_{;i}\dot{\bar{X}}) + \bar{D}\dot{\bar{X}}\delta X_{;i}} \tag{5.5}$$

5.1 For $dx^i dx^j$ terms

We wish to expand $Ag_{ij} + B\phi_{;i}\phi_{;j} + C(\phi_{;i}X_{;j} + X_{;i}\phi_{;j}) + DX_{;i}X_{;j}$. Now before we proceed, we have established from (3.2) and (3.4) that the perturbations with derivatives with respect to spatial components are already first-order. Hence this reduces to

$$Ag_{ij} + \cancel{B\phi_{;i}\phi_{;j}} + \cancel{C(\phi_{;i}X_{;j} + X_{;i}\phi_{;j})} + \cancel{DX_{;i}X_{;j}} \tag{5.6}$$

A huge shout-out to cosmological perturbations for making the math bearable. From (1.1) we have $g_{ij} = a^2[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}]$; then

$$\begin{aligned}
Ag_{ij} &= a^2(\bar{A} + \delta A)\left[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\right] \\
&= a^2\bar{A}\left[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\right] \\
&\quad + \delta A\left[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\right] \\
&\approx a^2\bar{A}\left[(1 - 2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\right] + a^2\delta A\delta_{ij} \\
&\approx a^2\bar{A}[(1 - 2\psi + \delta A/\bar{A})\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}]
\end{aligned} \tag{5.7}$$

Then it follows that

$$\boxed{\hat{g}_{ij} \approx a^2 \bar{A} [(1 - 2\psi + \delta A / \bar{A}) \delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}]} \quad (5.8)$$

6 Finally $d\hat{s}^2$

And so with

$$d\hat{s}^2 = \hat{g}_{00} dt^2 + 2\hat{g}_{0i} dt dx^i + \hat{g}_{ij} dx^i dx^j \quad (6.1)$$

and collecting our results given by (4.6), (5.5) and (5.8) we finally have

$$\begin{aligned} d\hat{s}^2 \simeq & \left[-\bar{A} - \delta A - 2\bar{A}\varphi + \bar{B}\dot{\bar{\phi}}^2 + 2\bar{B}\dot{\bar{\phi}}\delta\dot{\bar{\phi}} + \delta B\dot{\bar{\phi}}^2 \right. \\ & + \bar{C}\dot{\bar{\phi}}\dot{\bar{X}} + \bar{C}\dot{\bar{\phi}}\delta\dot{\bar{X}} + \bar{C}\delta\dot{\bar{\phi}}\dot{\bar{X}} + \delta C\dot{\bar{\phi}}\dot{\bar{X}} + \bar{D}\dot{\bar{X}}^2 + 2\bar{D}\dot{\bar{X}}\delta\dot{\bar{X}} + \delta D\dot{\bar{X}}^2 \Big] dt^2 \\ & + 2 \left[a\bar{A}(\alpha_{,i} + \beta_i) + \bar{B}\dot{\bar{\phi}}\delta\phi_{,i} + \bar{C}(\dot{\bar{\phi}}\delta X_{,i} + \dot{\bar{X}}\delta\phi_{,i}) + \bar{D}\dot{\bar{X}}\delta X_{,i} \right] dt dx^i \\ & + a^2\bar{A} \left[\left(1 - 2\psi + \frac{\delta A}{\bar{A}} \right) \delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)} \right] dx^i dx^j. \end{aligned} \quad (6.2)$$