1 Disformal Transformation

1.1 Perturbing the FLRW Metric

Let us decompose the perturbed FLRW metric (with SVT decomposition) as

$$ds^2 = -(1+2\varphi)\,dt^2 + 2a(\alpha_{,i}+\beta_i)\,dt\,dx^i + a^2\Big[(1-2\psi)\delta_{ij} + \gamma_{ij} + 2E_{,ij} + 2F_{(i,j)}\Big]\,dx^i dx^j, \quad (1.1)$$

where $a = a(t)$ is the scale factor, $(\varphi, \alpha, \psi, E)$ are scalar perturbations, β_i, F_i are vector perturbations, and γ_{ij} are tensor perturbations.

Under a disformal transformation, the line element becomes

$$d\hat{s}^2 = \hat{g}_{\mu\nu}\,dx^\mu dx^\nu. \quad (1.2)$$

The invertible disformal transformation with higher derivatives presented by Takahashi, Motohashi, and Minamitsuji (which we call the TMM transformation for convenience) is

$$\hat{g}_{\mu\nu} = A\,g_{\mu\nu} + B\,\phi_{;\mu}\phi_{;\nu} + C\big(\phi_{;\mu}X_{;\nu} + X_{;\mu}\phi_{;\nu}\big) + D\,X_{;\mu}X_{;\nu}, \quad (1.3)$$

where A, B, C, D are functionals of ϕ, X, Y, Z, and X, Y, Z are defined as

$$X \equiv -\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}, \qquad Y \equiv g^{\mu\nu}\phi_{;\mu}X_{;\nu}, \qquad Z \equiv g^{\mu\nu}X_{;\mu}X_{;\nu}. \quad (1.4)$$

The TMM form can be simplified by a "completing-the-square" step. We have "squared" covariant-derivative terms of ϕ, cross terms in the middle, and "squared" covariant-derivative terms of X. Recall the elementary identity

$$ax^2 + 2bxy + cy^2 = a\Big(x + \frac{b}{a}y\Big)^2 + \Big(c - \frac{b^2}{a}\Big)y^2. \quad (1.5)$$

Identifying $a = B$, $b = C$, and $c = D$, the last three terms in (1.3) combine as

$$B\Big(\phi_{;\mu} + \frac{C}{B}X_{;\mu}\Big)\Big(\phi_{;\nu} + \frac{C}{B}X_{;\nu}\Big) + \Big(D - \frac{C^2}{B}\Big)X_{;\mu}X_{;\nu}. \quad (1.6)$$

Hence, (1.3) may be rewritten as

$$\hat{g}_{\mu\nu} = A\,g_{\mu\nu} + B\Big(\phi_{;\mu} + \frac{C}{B}X_{;\mu}\Big)\Big(\phi_{;\nu} + \frac{C}{B}X_{;\nu}\Big) + \Big(D - \frac{C^2}{B}\Big)X_{;\mu}X_{;\nu}. \quad (1.7)$$

If we define

$$\Phi_\mu \equiv \sqrt{B}\Big(\phi_{;\mu} + \frac{C}{B}X_{;\mu}\Big), \qquad \Delta \equiv D - \frac{C^2}{B}, \quad (1.8)$$

then the transformation reduces cleanly to

$$\boxed{\hat{g}_{\mu\nu} = A\,g_{\mu\nu} + \Phi_\mu\Phi_\nu + \Delta\,X_{;\mu}X_{;\nu}.} \quad (1.9)$$

Under (1.9), the transformed line element can be expanded as

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{\mu\nu}\,dx^\mu dx^\nu \\ &= \big(Ag_{00} + \Phi_0\Phi_0 + \Delta\,X_{;0}X_{;0}\big)\,dt^2 \\ &\quad + 2\big(Ag_{0i} + \Phi_0\Phi_i + \Delta\,X_{;0}X_{;i}\big)\,dt\,dx^i \\ &\quad + \big(Ag_{ij} + \Phi_i\Phi_j + \Delta\,X_{;i}X_{;j}\big)\,dx^i dx^j. \end{aligned} \quad (1.10)$$

Consequently, take the perturbed scalar field $\phi = \bar{\phi} + \delta\phi$. Then

$$\begin{aligned} X &= -\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} \\ &= -\frac{1}{2} g^{00} \dot{\phi}^2 \\ &= -\frac{1}{2} \big(-(1-2\varphi) \big) (\dot{\bar{\phi}} + \delta\dot{\phi})^2 \\ &= \Big(\frac{1}{2} - \varphi \Big) \Big(\dot{\bar{\phi}}^2 + 2\dot{\bar{\phi}}\,\delta\dot{\phi} + (\delta\dot{\phi})^2 \Big) \\ &= \frac{1}{2} \dot{\bar{\phi}}^2 + \dot{\bar{\phi}}\,\delta\dot{\phi} - \varphi\,\dot{\bar{\phi}}^2, \end{aligned} \tag{1.11}$$

where g^{00} comes from the perturbed metric, and we have dropped higher-order perturbations (e.g. $(\delta\dot{\phi})^2$).

It is natural to split A, Φ_0, and Δ as

$$A = \bar{A} + \delta A, \qquad \Phi_0 = \bar{\Phi}_0 + \delta\Phi_0, \qquad \Delta = \bar{\Delta} + \delta\Delta, \tag{1.12}$$

Notice that from (1.8) we have

$$\begin{aligned} \Phi_0 &= \sqrt{B} \Big(\phi_{;0} + \frac{C}{B} X_{;0} \Big) \\ &= \sqrt{B} \Big(\dot{\bar{\phi}} + \delta\dot{\phi} + \frac{C}{B} (\dot{\bar{X}} + \delta\dot{X}) \Big) \end{aligned} \tag{1.13}$$

Where we see to have zeroth order perturbations given back the background variables. On the other hand for the spatial component, notice

$$\begin{aligned} \Phi_i &= \sqrt{B} \Big(\phi_{;i} + \frac{C}{B} X_{;i} \Big) \\ &= \sqrt{B} \Big(\frac{C}{B} (\bar{X}_{,i} + \delta X_{,i}) \Big) \\ &= \sqrt{B} \Big(\frac{C}{B} \delta X_{,i} \Big) \end{aligned} \tag{1.14}$$

Where again, the background term vanishes under a spatial covariant derivative since it is only dependent upon time. On the other hand, we are left with $\delta X_{,i}$ which is a first order perturbation. Hence, we choose not to "split" the spatial components of Φ_μ

1.2 For the dt^2 terms

From the perturbed metric given by (1.1) we have $g_{00} = -(1 = 2\varphi)$ it follows that

$$\begin{aligned} A g_{00} &= -(\bar{A} + \delta A)(1 + 2\varphi) \\ &= -(\bar{A} + 2\varphi\bar{A} + \delta A + 2\varphi\delta A) \\ &\approx -\bar{A} - \delta A - 2\bar{A}\varphi \end{aligned} \tag{1.15}$$

On the other hand

$$\Phi_0^2 = (\bar{\Phi}_0 + \delta\Phi_0)^2 = \bar{\Phi}_0^{\,2} + 2\Phi_o \tag{1.16}$$

Finally,

$$
\begin{aligned}
\Delta X_{;0}X_{;0} &= (\bar{\Delta} + \delta\Delta)(\dot{\bar{X}} + \delta X)^2 \\
&= (\bar{\Delta} + \delta\Delta)(\dot{\bar{X}}^2 + 2\bar{\dot{X}}\delta X + \overbrace{(\delta X)^2}^{\text{2nd order}}) \\
&\approx \bar{\Delta}\dot{\bar{X}}^2 + \delta\Delta\dot{\bar{X}}^2 + 2\bar{\Delta}\dot{\bar{X}}\delta X + \underbrace{2\delta\Delta\dot{\bar{X}}\delta X}_{\text{2nd order}} \\
&\approx \bar{\Delta}\dot{\bar{X}}^2 + \delta\Delta\dot{\bar{X}}^2 + 2\bar{\Delta}\dot{\bar{X}}\delta X
\end{aligned}
\tag{1.17}
$$

1.3 For $dtdx^i$ terms

Again, given by (1.1) we find that $g_{0i} = a(\alpha_{,i} + \beta_i)$. So for Ag_{0i} we have

$$
\begin{aligned}
Ag_{0i} &= a(\bar{A} + \delta A)(\alpha_{,i} + \beta_i) \\
&= a(\bar{A}\alpha_{,i} + \bar{A}\beta_i + \delta A\alpha_{,i} + \delta A\beta_i) \\
&\approx a\bar{A}(\alpha_{,i} + \beta_i)
\end{aligned}
\tag{1.18}
$$

For $\Phi_0\Phi_i$ note that we have already established that Φ_i is a already a first order perturbation. Then

$$
\begin{aligned}
\Phi_0\Phi_i &= (\bar{\Phi_0} + \delta\Phi_0)(\Phi_i) \\
&= \bar{\Phi_0}\Phi_i + \delta\Phi_0\Phi_i \\
&\approx \bar{\Phi_0}\Phi_i
\end{aligned}
\tag{1.19}
$$

On the other hand, for $\Delta X_{;0}X_{;i}$, note that $X_{;i} = \bar{X}_{;i} + \delta X_{;i} = \delta X_{;i}$ since background quantities are only time dependent. So,

$$
\begin{aligned}
\Delta X_{;0}X_{;i} &= (\bar{\Delta} + \delta\Delta)(\bar{X}_{;0} + \delta X_{;0})\delta X_{;i} \\
&= (\bar{\Delta}\dot{\bar{X}} + \bar{\Delta}\delta\dot{X} + \delta\Delta\dot{\bar{X}} + \delta\Delta\delta\dot{X})\delta X_{,i} \\
&\approx (\bar{\Delta}\dot{\bar{X}} + \bar{\Delta}\delta\dot{X} + \delta\Delta\dot{\bar{X}})\delta X_{,i} \\
&\approx \bar{\Delta}\dot{\bar{X}}\delta X_{,i}
\end{aligned}
\tag{1.20}
$$

1.4 For dx^idx^j terms