Next-Next-Gen Notes

Object-Oriented Maths

JP Guzman

November 21, 2017

Format: $characteristic((subjects),(dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$
Note: All weaker objects automatically induces notions inherited from stronger objects.
TODO assign free variables as parameters
TODO define || abs cross-product and other missing refs
TODO distinguish new condition vs implied proposition
TODO link thms?

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement\big(s,(RegEx)\big) \iff well\text{-}formedString\big(s,()\big) \tag{1}$$

$$\begin{aligned} proposition\big((p,t),()\big) \iff \Big(statement\big(p,()\big)\Big)\wedge \\ \big(t = eval(p)\big)\wedge \\ \big(t = true \veebar t = false\big) \end{aligned} \tag{2}$$

$$operator\Big(o,\Big((p)_{n\in\mathbb{N}}\Big)\Big) \iff proposition\Big(o\Big((p)_{n\in\mathbb{N}}\Big),()\Big) \tag{3}$$

$$\begin{aligned} operator\big(\neg,(p_1)\big) \iff \Big(proposition\big((p_1,true),()\big) \implies \big((\neg p_1,false),()\big)\Big)\wedge \\ \Big(proposition\big((p_1,false),()\big) \implies \big((\neg p_1,true),()\big)\Big) \\ \text{\# an operator takes in propositions and returns a proposition} \end{aligned} \tag{4}$$

$$\begin{aligned} operator(\neg) \iff \textbf{NOT} \text{ ; } operator(\vee) \iff \textbf{OR} \text{ ; } operator(\wedge) \iff \textbf{AND} \text{ ; } operator(\veebar) \iff \textbf{XOR} \\ operator(\implies) \iff \textbf{IF} \text{ ; } operator(\impliedby) \iff \textbf{OIF} \text{ ; } operator(\iff) \iff \textbf{IFF} \end{aligned} \tag{5}$$

$$\begin{aligned} proposition\big((false \implies true),true,()\big) \wedge proposition\big((false \implies false),true,()\big) \\ \text{\# truths based on a false premise is not false; ex falso quodlibet principle} \end{aligned} \tag{6}$$

$$\text{(THM)} : (a \implies b \implies c) \iff \big(a \implies (b \implies c)\big) \iff \big((a \wedge b) \implies c\big) \tag{7}$$

$$predicate\big(P,(V)\big) \iff \forall_{v\in V}\Big(proposition\Big(\big(P(v),t\big),()\Big)\Big) \tag{8}$$

$$\begin{aligned} 0thOrderLogic\big(P,()\big) \iff proposition\big((P,t),()\big) \\ \text{\# individual proposition} \end{aligned} \tag{9}$$

$$1stOrderLogic\big(P,(V)\big) \iff \Big(\forall_{v\in V}\Big(0thOrderLogic\big(v,()\big)\Big)\Big)\wedge$$
$$\Bigg(\forall_{v\in V}\Big(proposition\Big(\big(P(v),t\big),()\Big)\Big)\Bigg)$$
$$\text{\# propositions defined over a set of the lower order logical statements} \tag{10}$$

$$quantifier\big(q,(p,V)\big) \iff \Big(predicate\big(p,(V)\big)\Big)\wedge$$
$$\Big(proposition\Big(\big(q(p),t\big),()\Big)\Big)$$
$$\text{\# a quantifier takes in a predicate and returns a proposition} \tag{11}$$

$$quantifier\big(\forall,(p,V)\big) \iff proposition\Big(\Big(\wedge_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\text{\# universal quantifier} \tag{12}$$

$$quantifier\big(\exists,(p,V)\big) \iff proposition\Big(\Big(\vee_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\text{\# existential quantifier} \tag{13}$$

$$quantifier\big(\exists!,(p,V)\big) \iff \exists_{x\in V}\Big(P(x)\wedge\neg\Big(\exists_{y\in V\setminus\{x\}}\big(P(y)\big)\Big)\Big)$$
$$\text{\# uniqueness quantifier} \tag{14}$$

$$\text{(THM)} : \forall_x p(x) \iff \neg\exists_x\neg p(x)$$
$$\text{\# De Morgan's law} \tag{15}$$

$$\text{(THM)} : \forall_x\exists_y p(x,y)=\forall_x\neg\forall_y\neg p(x,y)\neq\exists_y\forall_x p(x,y)=\neg\forall_y\neg\big(\forall_x p(x,y)\big)=\neg\forall_y\exists_x\neg p(x,y)$$
$$\text{\# different quantifiers are not interchangeable} \tag{16}$$

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{17}$$

$$\textbf{proof}=\textbf{truths derived from a finite number of axioms and deductions} \tag{18}$$

$$\textbf{elementary arithmetics}=\textbf{system with substitutions, and some notion of}$$
$$\textbf{addition, multiplication, and prime nuumbers for encoding metamathematics} \tag{19}$$

$$\textbf{Gödel theorem} \implies \textbf{axiomatic systems equivalent in power to elementary mathematics}$$
$$\textbf{either has unprovable statements or has contradictions} \tag{20}$$

$$sequenceSet\big((A)_{\mathbb{N}},(A)\big) \iff (Amapinputn)\big((A)_{\mathbb{N}}=\{A(1),A(2),A(3),\ldots\}\big) \tag{21}$$

$$\textbf{TODO: define union, intersection, complement, etc.} \tag{22}$$

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{23}$$

1.1 Axiomatic Set Theory

$$\text{============} \textbf{ N O T = U P D A T E D } \text{============} \tag{24}$$

$$\textbf{ZFC set theory} = \textbf{usual form of axiomatic set theory} \tag{25}$$

$$A \subseteq B = \forall_x x \in A \Longrightarrow x \in B \tag{26}$$

$$(A = B) = A \subseteq B \land B \subseteq A \tag{27}$$

$$\in \textbf{basis} \Longrightarrow \{x, y\} = \{y, x\} \land \{x\} = \{x, x\} \tag{28}$$

$$\in \textbf{and sets works following the 9 ZFC axioms:} \tag{29}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \text{ \# E: } \in \text{ is only a proposition on sets} \tag{30}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \text{ \# E: existence of empty set} \tag{31}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \Longleftrightarrow u = x \lor u = y \text{ \# C: pair set construction} \tag{32}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \land y \in x \Longrightarrow y \in u) \text{ \# C: union set construction} \tag{33}$$

$$x = \{\{a\}, \{b\}\} \text{ \# from the pair set axiom} \tag{34}$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \tag{35}$$

$$\forall_x \exists!_y R(x, y) \text{ \# functional relation } R \tag{36}$$

$$\begin{gathered} \exists_i \forall_x \exists!_y R(x, y) \Longrightarrow y \in i \text{ \# C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \Longrightarrow \{y \in m \,|\, P(y)\} \text{ \# Restricted Comprehension} \not\Longrightarrow \{y \,|\, P(y)\} \text{ \# Universal Comprehension} \end{gathered} \tag{37}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \Longrightarrow P(x)\big) \text{ \# ignores out of scope} \neq \forall_x \big(x \in m \land P(x)\big) \text{ \# restricts entirety} \tag{38}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \Longrightarrow n \subseteq \mathcal{P}(m)\big) \text{ \# C: existence of power set} \tag{39}$$

$$\exists_I \Big(\emptyset \in I \land \forall_{x \in I} \big(\{x\} \in I\big)\Big) \text{ \# I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \Longrightarrow \mathbb{N} \textbf{ is a set} \tag{40}$$

$$\forall_x \Big(\big(\emptyset \notin x \land x \cap x' = \emptyset\big) \Longrightarrow \exists_y (\textbf{set of each e} \in x)\Big) \text{ \# C: axiom of choice} \tag{41}$$

$$\forall_x x \neq \emptyset \Longrightarrow x \notin x \text{ \# F: axiom of foundation covers further paradoxes} \tag{42}$$

$$\text{============} \textbf{ N O T = U P D A T E D } \text{============} \tag{43}$$

1.2 Classification of sets

$$space((set, structure), ()) \iff structure(set)$$
$$\text{\# a space a set equipped with some structure}$$
$$\text{\# various spaces can be studied through structure preserving maps between those spaces} \tag{44}$$

$$map(\phi, (A, B)) \iff \Big(\forall_{a \in A} \exists!_{b \in B}\big(\phi(a, b)\big)\Big) \vee \Big(\forall_{a \in A} \exists!_{b \in B}\big(b = \phi(a)\big)\Big)$$
$$\text{\# maps elements of a set to elements of another set} \tag{45}$$

$$domain(A, (\phi, A, B)) \iff \Big(map(\phi, (A, B))\Big) \tag{46}$$

$$codomain(B, (\phi, A, B)) \iff \Big(map(\phi, (A, B))\Big) \tag{47}$$

$$image(B, (A, q, M, N)) \iff \Big(map(q, (M, N)) \wedge A \subseteq M\Big) \wedge \Big(B = \{n \in N \mid \exists_{a \in A}\big(q(a) = n\big)\}\Big) \tag{48}$$

$$preimage(A, (B, q, M, N)) \iff \Big(map(q, (M, N)) \wedge B \subseteq N\Big) \wedge \Big(A = \{m \in M \mid \exists_{b \in B}\big(b = q(m)\big)\}\Big) \tag{49}$$

$$injection(q, (M, N)) \iff \Big(map(q, (M, N))\Big) \wedge \forall_{u, v \in M}\big(q(u) = q(v) \implies u = v\big)$$
$$\text{\# every } m \text{ has at most 1 image} \tag{50}$$

$$surjection(q, (M, N)) \iff \Big(map(q, (M, N))\Big) \wedge \forall_{n \in N} \exists_{m \in M}\big(n = q(m)\big)$$
$$\text{\# every } n \text{ has at least 1 preimage} \tag{51}$$

$$bijection(q, (M, N)) \iff \Big(injection(q, (M, N))\Big) \wedge \Big(surjection(q, (M, N))\Big)$$
$$\text{\# every unique } m \text{ corresponds to a unique } n \tag{52}$$

$$isomorphicSets((A, B), ()) \iff \exists_{\phi}\Big(bijection(\phi, (A, B))\Big) \tag{53}$$

$$infiniteSet(S, ()) \iff \exists_{T \subset S}\Big(isomorphicSets((T, S), ())\Big) \tag{54}$$

$$finiteSet(S, ()) \iff \Big(\neg infiniteSet(S, ())\Big) \vee \big(|S| \in \mathbb{N}\big) \tag{55}$$

$$countablyInfinite(S, ()) \iff \Big(infiniteSet(S, ())\Big) \wedge \Big(isomorphicSets((S, \mathbb{N}), ())\Big) \tag{56}$$

$$uncountablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big) \wedge \Big(\neg isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \tag{57}$$

$$\begin{aligned} inverseMap\Big(q^{-1},(q,M,N)\Big) \iff \Big(bijection\big(q,(M,N)\big)\Big)\wedge \\ \Big(map\Big(q^{-1},(N,M)\Big)\Big)\wedge \\ \Big(\forall_{n\in N}\exists!_{m\in M}\Big(q(m)=n \implies q^{-1}(n)=m\Big)\Big) \end{aligned} \tag{58}$$

$$\begin{aligned} mapComposition\big(\phi\circ\psi,(\phi,\psi,A,B,C)\big) \iff map\big(\psi,(A,B)\big)\wedge map\big(\phi,(B,C)\big)\wedge \\ \forall_{a\in A}\Big(\phi\circ\psi(a)=\phi\big(\psi(a)\big)\Big) \end{aligned} \tag{59}$$

$$\begin{aligned} equivalenceRelation\Big(\sim(\$1,\$2),(M)\Big) \iff (\forall_{m\in M}(m\sim m))\wedge \\ (\forall_{m,n\in M}(m\sim n \implies n\sim m))\wedge \\ (\forall_{m,n,p\in M}(m\sim n\wedge n\sim p \implies m\sim p)) \\ \#\text{ behaves as equivalences should} \end{aligned} \tag{60}$$

$$\begin{aligned} equivalenceClass\big([m]_\sim,(m,M,\sim)\big) \iff [m]_\sim=\{n\in M\,|\,n\sim m\} \\ \#\text{ set of elements satisfying the equivalence relation with } m \end{aligned} \tag{61}$$

$$\begin{aligned} \text{(THM)} : a\in[m]_\sim \implies [a]_\sim=[m]_\sim \;;\; [m]_\sim=[n]_\sim \veebar [m]_\sim\cap[n]_\sim=\emptyset \\ \#\text{ equivalence class properties} \end{aligned} \tag{62}$$

$$\begin{aligned} quotientSet\big(M/\sim,(M,\sim)\big) \iff M/\sim=\{equivalenceClass\big([m]_\sim,(m,M,\sim)\big)\in\mathcal{P}(M)\,|\,m\in M\} \\ \#\text{ set of all equivalence classes} \end{aligned} \tag{63}$$

$$\begin{aligned} \text{(THM)} : \bigg((M,\sim,+) \implies \Big(quotientSet\big(M/\sim,(M,\sim)\big),+_\sim\Big)\bigg) \iff \forall_{[r],[s]\in M/\sim}\forall_{a\in[r]}\forall_{b\in[s]}\big([r]+_\sim[s]=[a+b]\big) \\ \#\text{ a quotient set can inherit the operations on the original set if it is well-defined} \end{aligned} \tag{64}$$

$$\begin{aligned} \text{(THM)} : \textbf{axiom of choice} \implies \forall_{[m]_\sim\in M/\sim}\exists_r\big(r\in[m]_\sim\big) \\ \#\text{ well-defined maps may be defined in terms of chosen representative elements } r \end{aligned} \tag{65}$$

1.3 Construction of number sets

$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{66}$$

$$\textbf{axiom of infinity} \implies \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\ldots\}\cong\mathbb{N} \tag{67}$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \tag{68}$$

$$\textbf{addition}=\textbf{successor map:}\ \mathbb{N}\to\mathbb{N}=S(n)=\{n\}\ \#\text{ adds a layer of brackets} \tag{69}$$

$$\textbf{subtraction}=\textbf{predecessor map:}\ \mathbb{N}^\star\to\mathbb{N}=P(n)=m\,|\,m\in n\ \#\text{ removes a layer of brackets} \tag{70}$$

$$S^0=id\ ;\ n\in\mathbb{N}^\star \implies S^n=S\circ S^{P(n)} \tag{71}$$

$$\textbf{addition}=+:\mathbb{N}\times\mathbb{N}\to\mathbb{N}=+(m,n)=m+n=S^n(m) \tag{72}$$

$$S^x=id=S^0 \implies x=\textbf{additive identity}=0 \tag{73}$$

$$S^n(x)=0 \implies x=\textbf{additive inverse}\notin\mathbb{N}\ \#\ \text{git gud smh -\_-} \tag{74}$$

$$\begin{gathered}\mathbb{Z}=\mathbb{N}\times\mathbb{N}/\sim\textbf{, s.t.:}\\ (m,n)\sim(p,q)\iff m+q=p+n\ \#\ \text{span } \mathbb{Z} \text{ using differences then group equal differences}\end{gathered} \tag{75}$$

$$\mathbb{N}\hookrightarrow\mathbb{Z}:\forall_{n\in\mathbb{N}}n\to[(n,0)]\ \#\ \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{76}$$

$$+_{\mathbb{Z}}=[(m+_{\mathbb{N}}p,n+_{\mathbb{N}}q)]\ \#\ \text{well-defined and consistent} \tag{77}$$

$$\textbf{multiplication}\ldots M^x=id\implies x=\textbf{multiplicative identity}=1\ldots\textbf{multiplicative inverse}\notin\mathbb{N} \tag{78}$$

$$\mathbb{Q}=(\mathbb{Z}\times\mathbb{Z}^\star)/\sim\textbf{, s.t.: } (x,y)\sim(u,v)\iff x\cdot v=u\cdot y \tag{79}$$

$$\mathbb{Z}\hookrightarrow\mathbb{Q}\forall_{q\in\mathbb{Q}}q\to[(q,1)]\ ;\ \ldots\{x\,|\,x^2=2\}\notin\mathbb{Q} \tag{80}$$

$$\mathbb{R}=\textbf{almost homomorphisms on } \mathbb{Z}/\sim\ \#\ \text{http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{81}$$

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{82}$$

1.4 Topology

$$\begin{aligned}\textit{topology}\big(\mathcal{O},(M)\big)\iff\big(\mathcal{O}\subseteq\mathcal{P}(M)\big)\land\\ \big(\emptyset,M\in\mathcal{O}\big)\land\\ \Big(\big(F\in\mathcal{O}\land|F|<|\mathbb{N}|\big)\implies\cap F\in\mathcal{O}\Big)\land\\ (C\subseteq\mathcal{O}\implies\cup C\in\mathcal{O})\\ \#\ \text{topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.}\\ \#\ \text{arbitrary unions of open sets always result in an open set}\\ \#\ \text{open sets do not contain their boundaries and infinite intersections of open sets may approach and}\\ \#\ \text{induce boundaries resulting in a closed set}\end{aligned} \tag{83}$$

$$\textit{topologicalSpace}\big((M,\mathcal{O}),()\big)\iff\textit{topology}\big(\mathcal{O},(M)\big) \tag{84}$$

$$\begin{aligned}\textit{open}\big(S,(M,\mathcal{O})\big)\iff\Big(\textit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\land\\ (S\subseteq M)\land(S\in\mathcal{O})\\ \#\ \text{an open set do not contains its own boundaries}\end{aligned} \tag{85}$$

$$closed(S,(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$(S \subseteq M) \wedge (S \in \mathcal{P}(M) \setminus \mathcal{O})$$
$$\text{\# a closed set contains the boundaries an open set} \quad (86)$$

$$clopen(S,(M,\mathcal{O})) \iff \Big(closed(S,(M,\mathcal{O}))\Big) \wedge \Big(open(S,(M,\mathcal{O}))\Big) \quad (87)$$

$$neighborhood(U,(a,\mathcal{O})) \iff (a \in U \in \mathcal{O})$$
$$\text{\# another name for open set containing } a \quad (88)$$

$$M = \{a,b,c,d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\} \implies$$
$$\Big(open(X,(M,\mathcal{O})) \iff X = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\}\Big) \wedge$$
$$\Big(closed(Y,(M,\mathcal{O})) \iff Y = \{\emptyset, \{a,b,d\}, \{c,d\}, \{a,b\}, \{d\}, M\}\Big) \wedge$$
$$\Big(clopen(Z,(M,\mathcal{O})) \iff Z = \{\emptyset, \{a,b\}, \{c,d\}, M\}\Big) \quad (89)$$

$$chaoticTopology(M) = \{0, M\} \; ; \; discreteTopology = \mathcal{P}(M) \quad (90)$$

1.5 Induced topology

$$metric\Big(d(\$1,\$2),(M)\Big) \iff \Bigg(map\Big(d, \Big(M \times M, \mathbb{R}_0^+\Big)\Big)\Bigg)$$
$$\Big(\forall_{x,y \in M} \left(d(x,y) = d(y,x)\right)\Big) \wedge$$
$$\Big(\forall_{x,y \in M} \left(d(x,y) = 0 \iff x = y\right)\Big) \wedge$$
$$\Big(\forall_{x,y,z} \Big(\left(d(x,z) \leq d(x,y) + d(y,z)\right)\Big)\Big)$$
$$\text{\# behaves as distances should} \quad (91)$$

$$metricSpace((M,d),()) \iff metric(d,(M)) \quad (92)$$

$$openBall(B,(r,p,M,d)) \iff \Big(metricSpace((M,d),())\Big) \wedge (r \in \mathbb{R}^+, p \in M) \wedge (B = \{q \in M \mid d(p,q) < r\}) \quad (93)$$

$$metricTopology(\mathcal{O},(M,d)) \iff \Big(metricSpace((M,d),())\Big) \wedge$$
$$\Big(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+} \Big(openBall(B,(r,p,M,d)) \wedge B \subseteq U\Big)\}\Big)$$
$$\text{\# every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (94)$$

$$metricTopologicalSpace((M,\mathcal{O},d),()) \iff metricTopology(\mathcal{O},(M,d)) \quad (95)$$

$$limitPoint(p,(S,M,d)) \iff (S \subseteq M) \wedge \forall_{r \in \mathbb{R}^+} \Big(openBall(B,(r,p,M,d)) \cap S \neq \emptyset\Big)$$
$$\text{\# every open ball centered at } p \text{ contains some intersection with } S \quad (96)$$

$$interiorPoint(p,(S,M,d)) \iff (S \subseteq M) \wedge \Big(\exists_{r \in \mathbb{R}^+} \Big(openBall(B,(r,p,M,d)) \subseteq S\Big)\Big)$$

$$\# \text{ there is an open ball centered at } p \text{ that is fully enclosed in } S \tag{97}$$

$$closure\big(\bar{S},(S,M,d)\big) \iff \bar{S}=S\cup\{limitPoint\big(p,(S,M,d)\big)\,|\,p\in M\} \tag{98}$$

$$dense\big(S,(M,d)\big) \iff (S\subseteq M)\wedge\Bigg(\forall_{p\in M}\Big(p\in closure\big(\bar{S},(S,M,d)\big)\Big)\Bigg)$$
$$\# \text{ every of point in } M \text{ is a point or a limit point of } S \tag{99}$$

$$eucD\big(d,(n)\big) \iff \big(\forall_{i\in\mathbb{N}\wedge i\le n}(x_i\in\mathbb{R})\big)\wedge\left(d=\sqrt[2]{\sum_{i=1}^{n}{x_i}^2}\right) \tag{100}$$

$$metricTopology\Bigg(euclideanTopology,\Big(\mathbb{R}^n,eucD\big(d,(n)\big)\Big)\Bigg)$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============$$
$$\textbf{L1: } \forall_{p\in U=\emptyset}(\dots)\implies\forall_p\Big(\big(p\in\emptyset\big)\implies\dots\Big)\implies\forall_p\big((\textbf{False})\implies\dots\big)\implies\emptyset\in\mathcal{O}_{euclidean}$$
$$\textbf{L2: } \forall_{p\in\mathbb{R}^n}B(r,p,\mathbb{R}^n,d)\subseteq\mathbb{R}^n\implies M\in\mathcal{O}_{euclidean}$$
$$\textbf{L4: } C\subseteq\mathcal{O}_{euclidean}\implies\forall_{U\in C}\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\big(B_r(p)\subseteq U\subseteq\cup C\big)\implies\cup C\in\mathcal{O}_{euclidean}$$
$$\textbf{L3: } U,V\in\mathcal{O}_{euclidean}\implies p\in U\cap V\implies p\in U\wedge p\in V\implies$$
$$\exists_{r\in\mathbb{R}^+}B(r,p,\mathbb{R}^n,d)\wedge\exists_{s\in\mathbb{R}^+}B(s,p,\mathbb{R}^n,d)\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\subseteq U\wedge B\big(min(r,s),q,\mathbb{R}^n,d\big)\subseteq V\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\in U\cap V\implies U\cap V\in\mathcal{O}_{euclidean}$$
$$\# \text{ natural topology for } \mathbb{R}^d$$
$$\# \text{ could fail on infinite sets since } min \text{ could approach } 0$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{101}$$

$$subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\iff topology\big(\mathcal{O},(M)\big)\wedge(N\subseteq M)\wedge\big(\mathcal{O}|_N=\{U\cap N\,|\,U\in\mathcal{O}\}\big)$$
$$\# \text{ crops open sets outside } N \tag{102}$$

$$\text{(THM)}: subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\wedge topology\big(\mathcal{O}|_N,(N)\big)\impliedby$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============$$
$$\textbf{L1: } \emptyset\in\mathcal{O}\implies U=\emptyset\implies\emptyset\cap N=\emptyset\implies\emptyset\in\mathcal{O}|_N$$
$$\textbf{L2: } M\in\mathcal{O}\implies U=M\implies M\cap N=N\implies N\in\mathcal{O}|_N$$
$$\textbf{L3: } S,T\in\mathcal{O}|_N\implies\exists_{U\in\mathcal{O}}(S=U\cap N)\wedge\exists_{V\in\mathcal{O}}(T=V\cap N)\implies S\cap T=(U\cap N)\cap(V\cap N)$$
$$=(U\cap V)\cap N\wedge U\cap V\in\mathcal{O}\implies S\cap T\in\mathcal{O}|_N$$
$$\textbf{L4: } TODO:EXERCISE$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{103}$$

$$productTopology\Big(\mathcal{O}_{A\times B},\big((A,\mathcal{O}_A),(B,\mathcal{O}_B)\big)\Big)\iff\Big(topology\big(\mathcal{O}_A,(A)\big)\Big)\wedge\Big(topology\big(\mathcal{O}_B,(B)\big)\Big)\wedge$$
$$\big(\mathcal{O}_{A\times B}=\{(a,b)\in A\times B\,|\,\exists_S(a\in S\in\mathcal{O}_A)\exists_T(b\in T\in\mathcal{O}_B)\}\big)$$
$$\# \text{ open in cross iff open in each} \tag{104}$$

1.6 Convergence

$$\mathit{sequence}\big(q,(M)\big) \iff \mathit{map}\big(q,(\mathbb{N},M)\big) \tag{105}$$

$$\mathit{sequenceConvergesTo}\big((q,a),(M,\mathcal{O})\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\Big(\mathit{sequence}\big(q,(M)\big)\Big)\wedge(a\in M)\wedge\Big(\forall_{U\in\mathcal{O}|a\in U}\exists_{N\in\mathbb{N}}\forall_{n>N}\big(q(n)\in U\big)\Big)$$
each neighborhood of a has a tail-end sequence that does not map to outside points (106)

(THM) : **convergence generalizes to: the sequence** $q:\mathbb{N}\to\mathbb{R}^d$ **converges against** $a\in\mathbb{R}^d$ **in** $\mathcal{O}_S$ **if:**
$\forall_{r>0}\exists_{N\in\mathbb{N}}\forall_{n>N}\big(||q(n)-a||<\epsilon\big)$ # distance based convergence (107)

1.7 Continuity

$$\mathit{continuous}\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}_M),()\big)\Big)\wedge$$
$$\Big(\mathit{topologicalSpace}\big((N,\mathcal{O}_N),()\big)\Big)\wedge\Big(\forall_{V\in\mathcal{O}_N}\Big(\mathit{preimage}\big(A,(V,\phi,M,N)\big)\in\mathcal{O}_M\Big)\Big)$$
preimage of open sets are open (108)

$$\mathit{homeomorphism}\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big) \iff \Big(\mathit{inverseMap}\Big(\phi^{-1},(\phi,M,N)\Big)\Big)$$
$$\Big(\mathit{continuous}\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big)\Big)\wedge\Big(\mathit{continuous}\Big(\phi^{-1},(N,\mathcal{O}_N,M,\mathcal{O}_M)\Big)\Big)$$
structure preserving maps in topology, ability to share topological properties (109)

$$\mathit{isomorphicTopologicalSpace}\Big(\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big),()\Big) \iff$$
$$\exists_\phi\Big(\mathit{homeomorphism}\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big)\Big) \tag{110}$$

1.8 Separation

$$\mathit{T0Separate}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U\in\mathcal{O}}\Big(\big(x\in U\wedge y\notin U\big)\vee\big(y\in U\wedge x\notin U\big)\Big)\Big)$$
each pair of points has a neighborhood s.t. one is inside and the other is outside (111)

$$\mathit{T1Separate}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U,V\in\mathcal{O}\wedge U\neq V}\Big(\big(x\in U\wedge y\notin U\big)\wedge\big(y\in V\wedge x\notin V\big)\Big)\Big)$$
every point has a neighborhood that does not contain another point (112)

$$\mathit{T2Separate}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U,V\in\mathcal{O}\wedge U\neq V}\big(U\cap V=\emptyset\big)\Big)$$
every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space (113)

$$\text{(THM)} : \mathit{T2Separate} \Longrightarrow \mathit{T1Separate} \Longrightarrow \mathit{T0Separate} \quad (114)$$

1.9 Compactness

$$\mathit{openCover}(C,(M,\mathcal{O})) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M)$$
collection of open sets whose elements cover the entire space (115)

$$\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(\widetilde{C} \subseteq C\Big) \wedge \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big) \wedge \Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big) \wedge \Big(\mathit{finiteSet}\Big(\widetilde{C},()\Big)\Big)$$
finite subset of a cover that is also a cover (116)

$$\mathit{compact}((M,\mathcal{O}),()) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big) \wedge \left(\forall_{C \subseteq \mathcal{O}}\left(\mathit{openCover}(C,(M,\mathcal{O})) \Longrightarrow \exists_{\widetilde{C} \subseteq C}\Big(\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big)\Big)\right)\right)$$
every covering of the space is represented by a finite number of nhbhds (117)

$$\mathit{compactSubset}(N,(M,\mathcal{O})) \iff \Big(\mathit{compact}((M,\mathcal{O}),())\Big) \wedge \Big(\mathit{subsetTopology}(\mathcal{O}|_N,(M,\mathcal{O},N))\Big) \wedge \Big(\mathit{compact}\Big((N,\mathcal{O}|_N),()\Big)\Big) \quad (118)$$

$$\mathit{bounded}(N,(M,d)) \iff \Big(\mathit{metricSpace}((M,d),())\Big) \wedge (N \subseteq M) \wedge \Big(\exists_{r \in \mathbb{R}^+} \forall_{p,q \in n}\big(d(p,q) < r\big)\Big) \quad (119)$$

$$\text{(THM) Heine-Borel thm.: } \mathit{metricTopologicalSpace}((M,\mathcal{O}_d,d),()) \Longrightarrow \forall_{S \subseteq M}\Big(\Big(\mathit{closed}(S,(M,\mathcal{O}_d)) \wedge \mathit{bounded}(S,(M,\mathcal{O}_d))\Big) \iff \mathit{compactSubset}(S,(M,\mathcal{O}_d))\Big)$$
when metric topologies are involved, compactness is equivalent to being closed and bounded (120)

1.10 Paracompactness

$$\mathit{openRefinement}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big) \wedge \Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big) \wedge \Big(\forall_{\widetilde{U} \in \widetilde{C}} \exists_{U \in C}\Big(\widetilde{U} \subseteq U\Big)\Big)$$
a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space (121)

$$\text{(THM)} : \mathit{finiteSubcover} \Longrightarrow \mathit{openRefinement} \quad (122)$$

$$\mathit{locallyFinite}(C,(M,\mathcal{O})) \iff \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big) \wedge \forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U}\Big(\mathit{finiteSet}(\{U_c \in C | U \cap U_c \neq \emptyset\},())\Big)$$
each point has a neighborhood that intersects with only finitely many sets in the cover (123)

$$paracompact((M,\mathcal{O}),()) \iff \forall_C \left(openCover(C,(M,\mathcal{O})) \implies \exists_{\widetilde{C}} \left(locallyFinite\left(openRefinement\left(\widetilde{C},(C,M,\mathcal{O})\right),(M,\mathcal{O})\right)\right)\right)$$
every open cover has a locally finite open refinement (124)

$$\text{(THM)} : metricTopologicalSpace \implies paracompact \quad (125)$$

============ **N O T = U P D A T E D** ============ (126)

$$partitionOfUnitySubjCover(\mathcal{F},(C,M,\mathcal{O})) \iff \left(locallyFinite(C,(M,\mathcal{O}))\right) \wedge (f \in \mathcal{F}) \wedge \left(continuous\left(f, \left(M,\mathcal{O},[0,1], subsetTopology\left(\mathcal{O}|_{[0,1]},([0,1],\mathbb{R},euclideanTopology)\right)\right)\right)\right) \wedge \left(\exists_{U_f \in C} \forall_{p \in M} \left(f(p) \neq 0 \implies p \in U_f\right)\right) \wedge \left(\forall_{p \in M} \exists_{U \in \mathcal{O}|p \in U} \left((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\}\right)\right) \wedge \left(locallyFinite(C,M,\mathcal{O}) \implies finiteSet((f_U)_n,())\right) \wedge \left(\forall_{p \in M} \exists_{U \in \mathcal{O}|p \in U} \left(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1\right)\right)$$
useful for defining integrals between overlapping neighborhoods (127)

$$T2Separate((M,\mathcal{O}),()) \implies \left(paracompact((M,\mathcal{O}),())\right) \iff \forall_C \left(openCover(C,(M,\mathcal{O})) \implies partitionOfUnitySOTCover(\mathcal{F},(C,M,\mathcal{O}))\right) \quad (128)$$

============ **N O T = U P D A T E D** ============ (129)

1.11 Connectedness and path-connectedness

$$connected((M,\mathcal{O}),()) \iff \left(topologicalSpace((M,\mathcal{O}),())\right) \wedge \left(\neg \exists_{A,B \in \mathcal{O} \setminus \emptyset} (A \cap B \neq \emptyset \wedge A \cup B = M)\right)$$
if there is some covering of the space that does not intersect (130)

$$\text{(THM)} : \neg connected\left(\left(\mathbb{R} \setminus \{0\}, subsetTopology\left(\mathcal{O}_{euclidean}|_{\mathbb{R}\setminus\{0\}},(\mathbb{R},euclideanTopology,\mathbb{R}\setminus\{0\})\right)\right),()\right) \Longleftarrow \left(A = (-\infty,0) \in \mathcal{O}_{euclidean}|_{\mathbb{R}\setminus\{0\}}\right) \wedge \left(B = (0,\infty) \in \mathcal{O}_{euclidean}|_{\mathbb{R}\setminus\{0\}}\right) \wedge (A \cap B = \emptyset) \wedge (A \cup B = \mathbb{R} \setminus \{0\}) \quad (131)$$

$$\text{(THM)} : connected((M,\mathcal{O}),()) \iff \forall_{S \in \mathcal{O}} \left(clopen\left(S,(M,\mathcal{O}) \implies (S = \emptyset \vee S = M)\right)\right) \quad (132)$$

$$pathConnected((M,\mathcal{O}),()) \iff \left(subsetTopology\left(\mathcal{O}_{euclidean}|_{[0,1]},(\mathbb{R},euclideanTopology,[0,1])\right)\right) \wedge$$

$$\left(\forall_{p,q\in M}\exists_{\gamma}\left(continuous\left(\gamma,\left([0,1],\mathcal{O}_{euclidean}|_{[0,1]},M,\mathcal{O}\right)\right)\wedge\gamma(0)=p\wedge\gamma(1)=q\right)\right) \quad (133)$$

$$\text{(THM)}: pathConnected \implies connected \quad (134)$$

1.12 Homotopic curve and the fundamental group

$$============ \textbf{N O T = U P D A T E D} ============ \quad (135)$$

$$\begin{aligned}
homotopic(\sim,(\gamma,\delta,M,\mathcal{O})) \iff (map(\gamma,([0,1],M))\wedge map(\delta,([0,1],M)))\wedge\\
(\gamma(0)=\delta(0)\wedge\gamma(1)=\delta(1))\wedge\\
(\exists_H\forall_{\lambda\in[0,1]}(continuous(H,(([0,1]\times[0,1],\mathcal{O}_{euclidean^2}|_{[0,1]\times[0,1]}),(M,\mathcal{O}))\wedge H(0,\lambda)=\gamma(\lambda)\wedge H(1,\lambda)=\delta(\lambda))))\\
\#\ H \text{ is a continuous deformation of one curve into another}
\end{aligned} \quad (136)$$

$$homotopic(\sim)\implies equivalenceRelation(\sim) \quad (137)$$

$$loopSpace(\mathcal{L}_p,(p,M,\mathcal{O}))\iff\mathcal{L}_p=\{map(\gamma,([0,1],M))|continuous(\gamma)\wedge\gamma(0)=\gamma(1)\} \quad (138)$$

$$\begin{aligned}
concatination(\star,(p,\gamma,\delta))\iff(\gamma,\delta\in loopSpace(\mathcal{L}_p))\wedge\\
(\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda)=\begin{cases}\gamma(2\lambda) & 0\le\lambda<0.5\\ \delta(2\lambda-1) & 0.5\le\lambda\le1\end{cases}))
\end{aligned} \quad (139)$$

$$\begin{aligned}
group((G,\bullet),())\iff(map(\bullet,(G\times G,G))\wedge\\
(\forall_{a,b\in G}(a\bullet b\in G))\\
(\forall_{a,b,c\in G}((a\bullet b)\bullet C=a\bullet(b\bullet c)))\\
(\exists_{\boldsymbol{e}}\forall_{a\in G}(\boldsymbol{e}\bullet a=a=a\bullet\boldsymbol{e}))\wedge\\
(\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1}=\boldsymbol{e}=a^{-1}\bullet a))\\
\#\ \text{characterizes symmetry of a set structure}
\end{aligned} \quad (140)$$

$$isomorphic(\cong,(X,\odot),(Y,\ominus)))\iff\exists_f\forall_{a,b\in X}(bijection(f,(X,Y))\wedge f(a\odot b)=f(a)\ominus f(b)) \quad (141)$$

$$\begin{aligned}
fundamentalGroup((\pi_{1,p},\bullet),(p,M,\mathcal{O}))\iff(\pi_{1,p}=\mathcal{L}_p/\sim)\wedge\\
(map(\bullet,(\pi_{1,p}\times\pi_{1,p},\pi_{1,p})))\wedge\\
(\forall_{A,B\in\pi_{1,p}}([A]\bullet[B]=[A\star B]))\wedge\\
(group((\pi_{1,p},\bullet),()))\\
\#\ \text{an equivalence class of all loops induced from the homotopic equivalence relation}
\end{aligned} \quad (142)$$

$$fundamentalGroup_1\ncong fundamentalGroup_2\implies topologicalSpace_1\ncong topologicalSpace_2 \quad (143)$$

there exists no known list of topological properties that can imply homeomorphisms (144)

CONTINUE @ Lecture 6: manifolds (145)

$$============ \textbf{N O T = U P D A T E D} ============ \quad (146)$$

1.13 Measure theory

$$\textit{sigmaAlgebra}(\sigma,(M)) \iff (M \neq \emptyset) \wedge (\sigma \subseteq \mathcal{P}(M)) \wedge (M \in \sigma) \wedge \left(\forall_{A\in\sigma}(M \setminus A \in \sigma)\right) \wedge \left(\left(A \subseteq \sigma \wedge \neg \textit{uncountablyInfinite}(A,())\right) \implies \cup A \in \sigma\right)$$

behaves as measurable sets should; provides the sufficient structure for defining a measure μ (147)

$$\textit{measurableSpace}((M,\sigma),()) \iff \textit{sigmaAlgebra}(\sigma,(M)) \tag{148}$$

$$\textit{measurableSet}(A,(M,\sigma)) \iff \left(\textit{measurableSpace}((M,\sigma),())\right) \wedge (A \in \sigma) \tag{149}$$

$$\textit{measure}(\mu,(M,\sigma)) \iff \left(\textit{measurableSpace}((M,\sigma),())\right) \wedge \left(\textit{map}\left(\mu,\left(\sigma,\left(\overline{\mathbb{R}}\right)_0^+\right)\right)\right) \wedge \left(\mu(\emptyset)=0\right) \wedge \left(\left((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}\forall_{j\in\mathbb{N}\setminus\{i\}}\left(A_i \cap A_j = \emptyset\right)\right) \implies \mu\left(\cup_{i\in\mathbb{N}}(A_i)\right) = \sum_{i\in\mathbb{N}}(\mu(A_i))\right)$$

enforces meaningful concepts of measures such as precise additivity (150)

$$(\mathrm{THM}) : \textit{measure}(\mu,(M,\sigma)) \implies \left(\forall_{A,B\in\sigma}\left(A \subseteq B \implies \mu(A) \leq \mu(B)\right)\right) \wedge \left((A)_{\mathbb{N}} \subseteq \sigma \implies \mu\left(\cup_{i\in\mathbb{N}}(A_i)\right) \leq \sum_{i\in\mathbb{N}}(\mu(A_i))\right) \wedge \left(\left((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}(B_i \subseteq B_{i+1}) \wedge B = \cup(B)_{\mathbb{N}}\right) \implies \lim_{n\to\infty}\left(\mu(B_n)\right) = \mu(B)\right) \wedge \left(\left((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}(C_{i+1} \subseteq C_i) \wedge C = \cap(C)_{\mathbb{N}}\right) \implies \lim_{n\to\infty}\left(\mu(C_n)\right) = \mu(C)\right)$$

immediate implications of the measurable set $A \in \sigma$ axioms and the measure μ axioms (151)

$$\textit{measureSpace}((M,\sigma,\mu),()) \iff \textit{measure}(\mu,(M,\sigma)) \tag{152}$$

$$\textit{finiteMeasure}(\mu,(M,\sigma)) \iff \left(\textit{measure}(\mu,(M,\sigma))\right) \wedge \left(\exists_{(A)_{\mathbb{N}}\subseteq\sigma}\left(\cup\left((A)_{\mathbb{N}}\right) = M \wedge \forall_{n\in\mathbb{N}}\left(\mu(A_n) < \infty\right)\right)\right) \tag{153}$$

$$\textit{generatedSigmaAlgebra}(\sigma(\zeta),(\zeta,M)) \iff \left(G = \{\sigma \subseteq \mathcal{P}(M) \,|\, \textit{sigmaAlgebra}(\sigma,(M))\}\right) \wedge \left(\sigma(\zeta) = \cap G\right)$$

smallest σ-algebra containing the generating set ζ (154)

$$(\mathrm{THM}) : \exists_{\zeta\subseteq M}\left(\textit{generatedSigmaAlgebra}(\sigma(\zeta),(\zeta,M)) = \textit{sigmaAlgebra}(\sigma,(M))\right) \tag{155}$$

$$\textit{borelSigmaAlgebra}(\sigma(\mathcal{O}),(M,\mathcal{O})) \iff \left(\textit{topologicalSpace}((M,\mathcal{O}),())\right) \wedge \left(\textit{generatedSigmaAlgebra}(\sigma(\mathcal{O}),(\mathcal{O},M))\right)$$

σ-algebra induced by a topology (156)

$$euclideanSigma(\sigma_s, ()) \iff \left(borelSigmaAlgebra\left(\sigma_s, \left(\mathbb{R}^d, euclideanTopology\right)\right)\right) \tag{157}$$

$$lebesgueMeasure(\lambda, ()) \iff \left(measure\left(\lambda, \left(\mathbb{R}^d, euclideanSigma\right)\right)\right)\wedge \left(\lambda\left(\times_{i=1}^{d}([a_i, b_i))\right) = \sum_{i=1}^{d}\left(\sqrt[2]{(a_i - b_i)^2}\right)\right)$$
$$\# \text{ natural measure for } \mathbb{R}^d \tag{158}$$

$$measurableMap(f, (M, \sigma_M, N, \sigma_N)) \iff \left(measurableSpace((M, \sigma_M), ())\right)\wedge \left(measurableSpace((N, \sigma_N), ())\right)\wedge\left(\forall_{B\in\sigma_N}\left(preimage(A, (B, f, M, N)) \in \sigma_M\right)\right)$$
$$\# \text{ preimage of measurable sets are measurable} \tag{159}$$

$$pushForwardMeasure(f\star\lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \iff \left(measureSpace((M, \sigma_M, \mu_M), ())\right)\wedge \left(measurableSpace((N, \sigma_N), ())\right)\wedge\left(measurableMap(f, (M, \sigma_M, N, \sigma_N))\right)\wedge \left(\forall_{B\in N}\left(f\star\lambda_M(B) = \mu_M\left(preimage(A, (B, f, M, N))\right)\right)\right)\wedge\left(measure(f\star\lambda_M, (N, \sigma_N))\right)$$
$$\# \text{ natural construction of a measure based primarily on measurable map} \tag{160}$$

$$nullSet(A, (M, \sigma, \mu)) \iff \left(measureSpace((M, \sigma, \mu), ())\right)\wedge(A\in\sigma)\wedge(\mu(A) = 0) \tag{161}$$

$$almostEverywhere(p, (M, \sigma, \mu)) \iff \left(measureSpace((M, \sigma, \mu), ())\right)\wedge\left(predicate(p, (M))\right)\wedge \left(\exists_{A\in\sigma}\left(nullSet(A, (M, \sigma, \mu)) \implies \forall_{n\in M\setminus A}(p(n))\right)\right)$$
$$\# \text{ the predicate holds true for all points except the points in the null set}$$
$$\# \text{ in terms of measure, almost nothing is not equivalent to nothing} \tag{162}$$

1.14 Lebesque integration

$$simpleTopology(\mathcal{O}_{simple}, ()) \iff \mathcal{O}_{simple} = subsetTopology\left(\mathcal{O}|_{\mathbb{R}_0^+}, \left(\mathbb{R}, euclideanTopology, \mathbb{R}_0^+\right)\right) \tag{163}$$

$$simpleSigma(\sigma_{simple}, ()) \iff borelSigmaAlgebra\left(\sigma_{simple}, \left(\mathbb{R}_0^+, simpleTopology\right)\right) \tag{164}$$

$$simpleFunction(s, (M, \sigma)) \iff \left(measurableMap\left(s, \left(M, \sigma, \mathbb{R}_0^+, simpleSigma\right)\right)\right)\wedge$$

$$\left(finiteSet\left(image\left(B,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right)$$

if the map takes on finitely many values on $\mathbb{R}_0^+$

(165)

$$characteristicFunction(X_A,(A,M)) \iff (A \subseteq M) \wedge \left(map(X_A,(M,\mathbb{R}))\right) \wedge \left(\forall_{m \in M}\left(X_A(m) = \begin{cases} 1 & m \in A \\ 0 & m \notin A \end{cases}\right)\right)$$

(166)

$$(\text{THM}): simpleFunction(s,(M,\sigma_M)) \implies \left(finiteSet\left(image\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right) \wedge \left(characteristicFunction(X_A,(A,M))\right) \wedge \left(\forall_{m \in M}\left(s(m) = \sum_{z \in Z}\left(z \cdot X_{preimage\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)}(m)\right)\right)\right)$$

(167)

$$exEuclideanSigma(\overline{\sigma_s},()) \iff \overline{\sigma_s} = \{A \subseteq \overline{\mathbb{R}} \mid A \cap R \in euclideanSigma\}$$

ignores $\pm\infty$ to preserve the points in the domain of the measurable map

(168)

$$nonNegIntegrable(f,(M,\sigma)) \iff \left(measurableMap\left(f,\left(M,\sigma,\overline{\mathbb{R}},exEuclideanSigma\right)\right)\right) \wedge \left(\forall_{m \in M}(f(m) \geq 0)\right)$$

(169)

$$nonNegIntegral\left(\int_M (f d\mu),(f,M,\sigma,\mu)\right) \iff \left(measureSpace((M,\sigma,\mu),())\right) \wedge \left(measureSpace\left(\left(\overline{\mathbb{R}},exEuclideanSigma,lebesgueMeasure\right),()\right)\right) \wedge \left(nonNegIntegrable(f,(M,\sigma))\right) \wedge (\int_M (f d\mu) = sup(\{\sum_{z \in Z}\left(z \cdot \mu\left(preimage\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)\right)\right) \mid \forall_{m \in M}(s(m) \leq f(m)) \wedge simpleFunction(s,(M,\sigma)) \wedge finiteSet\left(image\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\}))$$

lebesgue measure on z reduces to z

(170)

$$explicitIntegral \iff \int (f(x)\mu(dx)) = \int (f d\mu)$$

alternative notation for lebesgue integrals

(171)

$$(\text{THM}): nonNegIntegral\left(\int (f d\mu),(f,M,\sigma,\mu)\right) \wedge nonNegIntegral\left(\int (g d\mu),(g,M,\sigma,\mu)\right) \implies$$

$$\text{(THM) Markov inequality: } \left(\forall_{z\in\mathbb{R}_0^+}\left(\int(fd\mu)\geq z\cdot\mu\left(preimage\left(A,\left([z,\infty),f,M,\overline{\mathbb{R}}\right)\right)\right)\right)\right)\wedge$$
$$\left(almostEverywhere\left(f=g,(M,\sigma,\mu)\right)\implies\int(fd\mu)=\int(gd\mu)\right)$$
$$\left(\int(fd\mu)=0\implies almostEverywhere\left(f=0,(M,\sigma,\mu)\right)\right)\wedge$$
$$\left(\int(fd\mu)\leq\infty\implies almostEverywhere\left(f<\infty,(M,\sigma,\mu)\right)\right) \tag{172}$$

$$\text{(THM) Mono. conv.: } \left((f)_{\mathbb{N}}=\{f_n\,|\,\wedge measurableMap\left(f_n,\left(M,\sigma,\overline{R},exEuclideanSigma\right)\right)\wedge 0\leq f_{n-1}\leq f_n\}\right)\wedge$$
$$\left(map\left(f,\left(M,\overline{\mathbb{R}}\right)\right)\right)\wedge\left(\forall_{m\in M}\left(f(m)=sup\left(f_n(m)|f_n\in(f)_{\mathbb{N}}\right)\right)\right)\implies\left(\lim_{n\to\infty}\left(\int_M(f_nd\mu)\right)=\int_M(fd\mu)\right)$$
$$\text{\# lengths now depend on } M,\sigma \text{ and limits can be pulled in or out of an integral} \tag{173}$$

$$\text{(THM) : } nonNegIntegral\left(\int(fd\mu),(f,M,\sigma,\mu)\right)\wedge nonNegIntegral\left(\int(gd\mu),(g,M,\sigma,\mu)\right)\implies$$
$$\left(\forall_{\alpha\in\mathbb{R}_0^+}\left(\int((f+\alpha g)d\mu)=\int(fd\mu)+\alpha\int(gd\mu)\right)\right)$$
$$\text{\# integral acts linearly and commutes finite summations} \tag{174}$$

$$\text{(THM) : } \left((f)_{\mathbb{N}}=\{f_n\,|\,\wedge measurableMap\left(f_n,\left(M,\sigma,\overline{R},exEuclideanSigma\right)\right)\wedge 0\leq f_n\}\right)\implies$$
$$\left(\int\left(\left(\sum_{n=1}^{\infty}(f_n)\right)d\mu\right)=\sum_{n=1}^{\infty}\left(\int(f_nd\mu)\right)\right)$$
$$\text{\# } \sum_{n=1}^{\infty}f_n \text{ can be treated as } \lim_{n\to\infty}\sum_{i=1}^{n}f_n \text{ since } f_n\geq 0 \text{ and it commutes with integral from monotone conv.} \tag{175}$$

$$integrable\left(f,(M,\sigma)\right)\iff\left(measurableMap\left(f,\left(M,\sigma,\overline{\mathbb{R}},exEuclideanSigma\right)\right)\right)\wedge$$
$$\left(\forall_{m\in M}\left(f(m)=max\left(f(m),0\right)-max\left(0,-f(m)\right)\right)\right)\wedge$$
$$\left(measureSpace(M,\sigma,\mu)\implies\left(\int\left(max\left(f(m),0\right)d\mu\right)<\infty\wedge\int\left(max\left(0,-f(m)\right)d\mu\right)<\infty\right)\right)$$
$$\text{\# extra condition prevents the occurrence of the indeterminate } \infty-\infty \tag{176}$$

$$integral\left(\int(fd\mu),(f,M,\sigma,\mu)\right)\iff\left(nonNegIntegral\left(\int(f^+d\mu),\left(max(f,0),M,\sigma,\mu\right)\right)\right)\wedge$$
$$\left(nonNegIntegral\left(\int(f^-d\mu),\left(max(0,-f),M,\sigma,\mu\right)\right)\right)\wedge\left(integrable\left(f,(M,\sigma)\right)\right)\wedge$$

$$\left(\int(fd\mu)=\int(f^{+}d\mu)-\int(f^{-}d\mu)\right)$$

arbitrary integral in terms of nonnegative integrals (177)

$$\text{(THM)}:\left(map(f,(M,\mathbb{C}))\right)\implies\left(\int(fd\mu)=\int(Re(f)d\mu)-\int(Im(f)d\mu)\right) \tag{178}$$

$$\begin{aligned}\text{(THM)}:integral\left(\int(fd\mu),(f,M,\sigma,\mu)\right)\wedge integral\left(\int(gd\mu),(g,M,\sigma,\mu)\right)\implies\\ \left(almostEverywhere(f\leq g,(M,\sigma,\mu))\implies\int(fd\mu)\leq\int(gd\mu)\right)\wedge\\ \left(\forall_{m\in M}(f(m),g(m),\alpha\in\mathbb{R})\implies\int((f+\alpha g)d\mu)=\int(fd\mu)+\alpha\int(gd\mu)\right)\end{aligned} \tag{179}$$

$$\text{(THM) Dominant convergence: }\left((f)_{\mathbb{N}}=\{f_n\,|\wedge measurableMap\left(f_n,\left(M,\sigma,\overline{R},exEuclideanSigma\right)\right)\}\right)\wedge$$

$$\left(map(f,(M,\overline{\mathbb{R}}))\right)\wedge\left(almostEverywhere\left(f(m)=\lim_{n\to\infty}(f_n(m)),(M,\sigma,\mu)\right)\right)\wedge$$

$$\left(nonNegIntegral\left(\int(gd\mu),(g,M,\sigma,\mu)\right)\right)\wedge\left(\left|\int(gd\mu)\right|<\infty\right)\wedge\left(almostEverywhere(|f_n|\leq g,(M,\sigma,\mu))\right)$$

if all $f_n(m)$ are bounded by some integrable $|g(m)|\implies$

then all $f_n(m)$ including f satisfy bounded and integrable properties

$$\left(\forall_{\phi\in\{f\}\cup(f)_{\mathbb{N}}}\left(integrable(\phi,(M,\sigma))\right)\right)\wedge\left(\lim_{n\to\infty}\left(\int(|f_n-f|d\mu)=0\right)\right)\wedge\left(\lim_{n\to\infty}\left(\int(f_nd\mu)\right)=\int(fd\mu)\right) \tag{180}$$

1.15 Vector space and structures

$$\begin{aligned}vectorSpace((V,+,\cdot),())\iff\left(map(+,(V\times V,V))\right)\wedge\left(map(\cdot,(\mathbb{R}\times V,V))\right)\wedge\\ (\forall_{v,w\in v}(v+w=w+v))\wedge\\ \left(\forall_{v,w,x\in v}((v+w)+x=v+(w+x))\right)\wedge\\ (\exists_{\boldsymbol{0}\in V}\forall_{v\in V}(v+\boldsymbol{0}=v))\wedge\\ \left(\forall_{v\in V}\exists_{-v\in V}(v+(-v)=\boldsymbol{0})\right)\wedge\\ \left(\forall_{a,b\in\mathbb{R}}\forall_{v\in V}(a(b\cdot v)=(ab)\cdot v)\right)\wedge\\ (\exists_{1\in\mathbb{R}}\forall_{v\in V}(1\cdot v=v))\wedge\\ \left(\forall_{a,b\in\mathbb{R}}\forall_{v\in V}((a+b)\cdot v=a\cdot v+b\cdot v)\right)\wedge\\ \left(\forall_{a\in\mathbb{R}}\forall_{v,w\in V}(a\cdot(v+w)=a\cdot v+a\cdot w)\right)\end{aligned}$$

behaves similar as vectors should i.e., additive, scalable, linear distributive (181)

$$\begin{aligned}innerProduct(\langle\$1,\$2\rangle,(V,+,\cdot))\iff\left(vectorSpace((V,+,\cdot),())\right)\wedge\left(map(\langle\$1,\$2\rangle,(V\times V,\mathbb{R}))\right)\wedge\\ \left(\forall_{v,w\in V}(\langle v,w\rangle=\langle w,v\rangle)\right)\wedge\end{aligned}$$

$$\left(\forall_{v,w,x\in V}\forall_{a,b\in\mathbb{R}}\left(\langle av+bw,x\rangle=a\langle v,x\rangle+b\langle w,x\rangle\right)\right)\wedge$$
$$\left(\forall_{v\in V}\left(\langle v,v\rangle\right)\geq 0\right)\wedge\left(\forall_{v\in V}\left(\langle v,v\rangle\right)=0\iff v=\boldsymbol{0}\right)$$
$$\#\text{ the sesquilinear or 1.5 linear map inner product provides info. on distance and orthogonality} \tag{182}$$

$$innerProductSpace\Big(\left(V,+,\cdot,\langle\$1,\$2\rangle\right),()\Big)\iff innerProduct\left(\langle\$1,\$2\rangle,(V,+,\cdot)\right) \tag{183}$$

$$vectorNorm\left(||\$1||,(V,+,\cdot)\right)\iff\Big(vectorSpace\left((V,+,\cdot),()\right)\Big)\wedge\left(map\left(||\$1||,\left(V,\mathbb{R}_0^+\right)\right)\right)\wedge$$
$$\left(\forall_{v\in V}\left(||v||=0\iff v=\boldsymbol{0}\right)\right)\wedge$$
$$\left(\forall_{v\in V}\forall_{s\in\mathbb{R}}\left(||sv||=|s|||v||\right)\right)\wedge$$
$$\left(\forall_{v,w\in V}\left(||v+w||\leq||v||+||w||\right)\right)$$
$$\#\text{ magnitude of a point in a vector space} \tag{184}$$

$$normedVectorSpace\Big(\left(V,+,\cdot,||\$1||\right),()\Big)\iff\Big(vectorSpace\left((V,+,\cdot),()\right)\Big)\wedge\Big(vectorNorm\left(||\$1||,(V,+,\cdot)\right)\Big) \tag{185}$$

$$vectorMetric\Big(d(\$1,\$2),(V,+,\cdot)\Big)\iff\Big(vectorSpace\left((V,+,\cdot),()\right)\Big)\wedge$$
$$\left(metric\Big(d(\$1,\$2),(V)\Big)\vee\left(map\left(d,\left(V\times V,\mathbb{R}_0^+\right)\right)\right)\right)$$
$$\left(\forall_{x,y\in V}\left(d(x,y)=d(y,x)\right)\right)\wedge$$
$$\left(\forall_{x,y\in V}\left(d(x,y)=0\iff x=y\right)\right)\wedge$$
$$\left(\forall_{x,y,z\in V}\left(\left(d(x,z)\leq d(x,y)+d(y,z)\right)\right)\right)$$
$$\#\text{ behaves as distances should} \tag{186}$$

$$metricVectorSpace\left(\Big(V,+,\cdot,d(\$1,\$2)\Big),()\right)\iff\Big(vectorSpace\left((V,+,\cdot),()\right)\Big)\wedge$$
$$\left(vectorMetric\Big(d(\$1,\$2),(V,+,\cdot)\Big)\right) \tag{187}$$

$$innerProductNorm\Big(||\$1||,\left(V,+,\cdot,\langle\$1,\$2\rangle\right)\Big)\iff\left(innerProductSpace\Big(\left(V,+,\cdot,\langle\$1,\$2\rangle\right),()\Big)\right)\wedge$$
$$\left(\forall_{v\in V}\left(||v||=\sqrt[2]{\langle v,v\rangle}\right)\implies vectorNorm\left(||\$1||,(V,+,\cdot)\right)\right) \tag{188}$$

$$normInnerProduct\Big(\langle\$1,\$2\rangle,\left(V,+,\cdot,||\$1||\right)\Big)\iff\left(normedVectorSpace\Big(\left(V,+,\cdot,||\$1||\right),()\Big)\right)\wedge$$
$$\left(\forall_{u,v\in V}\left(2||u||^2+2||v||^2=||u+v||^2+||u-v||^2\right)\right)\wedge$$
$$\left(\forall_{v,w\in V}\left(\langle v,w\rangle=\frac{||v+w||^2-||v-w||^2}{4}\right)\implies innerProduct\left(\langle\$1,\$2\rangle,(V,+,\cdot)\right)\right) \tag{189}$$

$$normMetric\Big(d(\$1,\$2),\left(V,+,\cdot,||\$1||\right)\Big)\iff\left(normedVectorSpace\Big(\left(V,+,\cdot,||\$1||\right),()\Big)\right)\wedge$$

$$\left(\forall_{v,w\in V}\left(d(v,w)=||v-w||\right)\Longrightarrow vectorMetric\Big(d(\$1,\$2),(V,+,\cdot)\Big)\right) \tag{190}$$

$$metricNorm\Big(||\$1||,\Big(V,+,\cdot,d(\$1,\$2)\Big)\Big)\Longleftrightarrow\left(metricVectorSpace\Big(\Big(V,+,\cdot,d(\$1,\$2)\Big),()\Big)\right)\wedge$$
$$\left(\forall_{u,v,w\in V}\forall_{s\in\mathbb{R}}\Big(d\big(s(u+w),s(v+w)\big)=|s|d(u,v)\Big)\right)\wedge$$
$$\left(\forall_{v\in V}\left(||v||=d(v,\boldsymbol{0})\right)\Longrightarrow vectorNorm\big(||\$1||,(V,+,\cdot)\big)\right) \tag{191}$$

$$orthogonal\Big((v,w),\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Longleftrightarrow\left(innerProductSpace\Big(\big(V,+,\cdot,\langle\$1,\$2\rangle\big),()\Big)\right)\wedge$$
$$(v,w\in V)\wedge\big(\langle v,w\rangle=0\big)$$
$$\text{\# the inner product also provides info. on orthogonality} \tag{192}$$

$$normal\Big(v,\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Longleftrightarrow\left(innerProductSpace\Big(\big(V,+,\cdot,\langle\$1,\$2\rangle\big),()\Big)\right)\wedge(v\in V)\wedge\big(\langle v,v\rangle=1\big)$$
$$\text{\# the vector has unit length} \tag{193}$$

$$\text{(THM) Cauchy-Schwarz inequality: }\forall_{v,w\in V}\big(\langle v,w\rangle\leq||v||||w||\big) \tag{194}$$

$$basis\big((b)_n,(V,+,\cdot,)\big)\Longleftrightarrow\Big(vectorSpace\big((V,+,\cdot),()\big)\Big)\wedge\left(\forall_{v\in V}\exists_{(a)_n\in\mathbb{R}^n}\left(v=\sum_{i=1}^{n}(a_ib_i)\right)\right) \tag{195}$$

$$orthonormalBasis\Big((b)_n,\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Longleftrightarrow\left(innerProductSpace\Big(\big(V,+,\cdot,\langle\$1,\$2\rangle\big),()\Big)\right)\wedge$$
$$\Big(basis\big((b)_n,(V,+,\cdot)\big)\Big)\wedge\left(\forall_{v\in(b)_n}\left(normal\Big(v,\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\right)\right)\wedge$$
$$\left(\forall_{v\in(b)_n}\forall_{w\in(b)_n\setminus\{v\}}\left(orthogonal\Big((v,w),\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\right)\right) \tag{196}$$

1.16 Subvector space

$$subspace\big((U,\circ),(V,\circ)\big)\Longleftrightarrow\Big(space\big((V,\circ),()\big)\Big)\wedge(U\subseteq V)\wedge\Big(space\big((U,\circ),()\big)\Big) \tag{197}$$

$$subspaceSum\big(U+W,(U,W,V,+)\big)\Longleftrightarrow\Big(subspace\big((U,+),(V,+)\big)\Big)\wedge\Big(subspace\big((W,+),(V,+)\big)\Big)\wedge$$
$$\big(U+W=\{u+w\,|\,u\in U\wedge w\in W\}\big) \tag{198}$$

$$subspaceDirectSum\big(U\oplus W,(U,W,V,+)\big)\Longleftrightarrow(U\cap W=\emptyset)\wedge\Big(subspaceSum\big(U\oplus W,(U,W,V,+)\big)\Big) \tag{199}$$

$$orthogonalComplement\Big(W^{\perp},\big(W,V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Longleftrightarrow$$
$$\left(subspace\left(\big(W,+,\cdot,\langle\$1,\$2\rangle\big),\left(innerProductSpace\Big(\big(V,+,\cdot,\langle\$1,\$2\rangle\big),()\Big)\right)\right)\right)\wedge$$

$$\left(W^{\perp}=\left\{v\in V\,|\,w\in W\wedge \mathit{orthogonal}\Big((v,w),\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\right\}\right) \tag{200}$$

$$\mathit{orthogonalDecomposition}\Big(\big(W,W^{\perp}\big),\big(W,V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\iff$$
$$\Big(\mathit{orthogonalComplement}\Big(W^{\perp},\big(W,V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Big)\wedge\Big(\mathit{subspaceDirectSum}\Big(V,\big(W,W^{\perp},V,+\big)\Big)\Big) \tag{201}$$

(THM) if V is finite dimensional, then every vector has an orthogonal decomposition: (202)

1.17 Banach and Hilbert Space

$$\mathit{cauchy}\Big((s)_{\mathbb{N}},\big(V,d(\$1,\$2)\big)\Big)\iff\Big(\mathit{metricSpace}\Big(\big(V,d(\$1,\$2)\big),()\Big)\Big)\wedge\big((s)_{\mathbb{N}}\subseteq V\big)$$
$$\Big(\forall_{\epsilon>0}\exists_{N\in\mathbb{N}}\forall_{m,n\geq N}\big(d(s_m,s_n)<\epsilon\big)\Big)$$

distances between some tail-end point gets arbitrarily small (203)

$$\mathit{complete}\Big(\big(V,d(\$1,\$2)\big),()\Big)\iff\Big(\forall_{(s)_{\mathbb{N}}\subseteq V}\exists_{s\in V}\Big(\mathit{cauchy}\Big((s)_{\mathbb{N}},\big(V,d(\$1,\$2)\big)\Big)\implies\lim_{n\to\infty}\big(d(s,s_n)\big)=0\Big)\Big)$$

or converges within the induced topological space

in complete spaces, the weaker notion of cauchy is enforced to be equivalent to convergence (204)

$$\mathit{banachSpace}\Big(\big(V,+,\cdot,||\$1||\big),()\Big)\iff\Big(\mathit{normMetric}\Big(d(\$1,\$2),\big(V,||\$1||\big)\Big)\Big)\wedge\Big(\mathit{complete}\Big(V,d\big(\$1,\$2\big)\Big),()\Big)$$

a complete normed vector space (205)

$$\mathit{hilbertSpace}\Big(\big(V,+,\cdot,\langle\$1,\$2\rangle\big),()\Big)\iff\Big(\mathit{innerProductNorm}\Big(||\$1||,\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Big)\wedge$$
$$\Big(\mathit{normMetric}\Big(d(\$1,\$2),\big(V,||\$1||\big)\Big)\Big)\wedge\Big(\mathit{complete}\Big(V,d\big(\$1,\$2\big)\Big),()\Big)$$

a complete inner product space (206)

(THM) : $\mathit{hilbertSpace}\implies\mathit{banachSpace}$ (207)

$$\mathit{separable}\big((V,d),()\big)\iff\Big(\exists_{S\subseteq V}\Big(\mathit{dense}\big(S,(V,d)\big)\wedge\mathit{countablyInfinite}\big(S,()\big)\Big)\Big)$$

needs only a countable subset to approximate any element in the entire space (208)

$$\text{(THM)}:\ \mathit{hilbertSpace}\Big(\Big(\big(V,+,\cdot,\langle\$1,\$2\rangle\big),()\Big),()\Big)\implies$$
$$\Big(\exists_{(b)_{\mathbb{N}}\subseteq V}\Big(\mathit{orthonormalBasis}\Big((b)_{\mathbb{N}},\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\wedge\mathit{countablyInfinite}\big((b)_{\mathbb{N}},()\big)\Big)\iff$$
$$\mathit{separable}\Big(\Big(V,\sqrt{\langle\$1-\$2,\$1-\$2\rangle}\Big),()\Big)\Big)$$

separability in hilbert spaces is equivalent to the existence of a countable orthonormal basis (209)

1.18 Matrices, Operators, and Functionals

$$\mathit{linearOperator}(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)) \iff \Big(\mathit{map}(L,(V,W))\Big)\wedge\Big(\mathit{vectorSpace}((V,+_V,\cdot_V),())\Big)\wedge \Big(\mathit{vectorSpace}((W,+_W,\cdot_W),())\Big)\wedge\Big(\forall_{v_1,v_2\in V}\forall_{s_1,s_2\in\mathbb{R}}(L(s_1\cdot_V v_1+_V s_2\cdot_V v_2)=s_1\cdot_W L(v_1)+_W s_2\cdot_W L(v_2))\Big) \quad (210)$$

$$\mathit{matrix}(L,(n,m)) \iff \Big(\mathit{linearOperator}(L,(\mathbb{R}^m,+_m,\cdot_m,\mathbb{R}^n,+_n,\cdot_n))\Big)$$

rows=dimensions, cols=vectors (211)

$$\mathit{eigenvector}(v,(L,V,+,\cdot)) \iff \Big(\mathit{linearOperator}(L,(V,+,\cdot,V,+,\cdot))\Big)\wedge\Big(\exists_{\lambda\in\mathbb{R}}(L(v)=\lambda v)\Big) \quad (212)$$

$$\mathit{eigenvalue}(\lambda,(v,L,V,+,\cdot)) \iff \Big(\mathit{eigenvector}(v,(L,V,+,\cdot))\Big) \quad (213)$$

$$\mathit{identityOperator}(I,(A)) \iff \Big(\mathit{matrix}(A,(n,n))\Big)\wedge(AI=IA=A) \quad (214)$$

$$\mathit{inverseOperator}\Big(A^{-1},(A)\Big) \iff \Big(A^{-1}A=AA^{-1}=I\Big)$$

gauss-jordan elimination: $E[A|I]=[I|E]=[I|A^{-1}]$ (215)

$$CONTHERETODOABSTRACTALGEB \quad (216)$$

$$\text{(THM)}:(AB)^{-1}(AB)=I=B^{-1}A^{-1}AB \quad (217)$$

$$\mathit{transposeOperator}\Big(A^T,(A)\Big) \iff \Big(\Big(A^T\Big)_{m,n}=(A)_{n,m}\Big)\vee\mathit{adjoint}\Big(A^T,(A)\Big) \quad (218)$$

$$\mathit{symmetricOperator}(A,()) \iff \Big(A=\mathit{transposeOperator}\Big(A^T,(A)\Big)\Big)\vee\Big(\mathit{selfAdjoint}(A,())\Big) \quad (219)$$

$$\text{(THM)}:(AB)^T=B^TA^T\wedge\Big(A^T\Big)^{-1}=\Big(A^{-1}\Big)^T \quad (220)$$

$$\mathit{triangularOperator}(A,()) \iff \Big(\mathit{matrix}(A,(n,n))\Big)\wedge\Big(\forall_{x<n}\forall_{0<i<x}(A_{i,i}=0)\Big) \quad (221)$$

$$\mathit{decomposeLU}(LU(A),(A)) \iff \Big(\mathit{matrix}(A,(n,n))\Big)\wedge\Big(\exists_E\Big(EA=\mathit{triangularOperator}(U,())\Big)\Big)\wedge \Big(LU(A)=E^{-1}U=A\Big)$$

lower triangle are all 0; useful for solving linear equations (222)

$$\mathit{Img}(Img(A),(A)) \iff \Big(\mathit{matrix}(A,(n,m))\Big)\wedge(Img(A)=\{Av\in\mathbb{R}^n\,|\,v\in\mathbb{R}^m\})$$

the column space; not always a subspace since A can map to a set not containing $\boldsymbol{0}$ (223)

$$\mathit{Ker}(Ker(A),(A)) \iff \Big(\mathit{matrix}(A,(n,m))\Big)\wedge(Ker(A)=\{v\in\mathbb{R}^m\,|\,Av=\boldsymbol{0}\in\mathbb{R}^n\})$$

the null or solution space; always a subspace due to linearity $Av+Aw=\boldsymbol{0}=A(v+w)$ (224)

$$\text{(THM) general linear solution: } (Ax_p = b) \wedge (x_n \in Ker(A)) \implies \Big(Ax_p + Ax_n = b + 0 = A(x_p + x_n) = b\Big) \quad (225)$$

$$independentOperator(A, ()) \iff \Big(matrix(A, (n, m))\Big) \wedge \Big(\neg \exists_{v \in \mathbb{R}^m \setminus \mathbf{0}_m} (Av = 0) \iff Ker(A) = \{\mathbf{0}_m\}\Big)$$
$$\text{\# also equivalent to invertible operator} \quad (226)$$

$$dimensonality(N, (A)) \iff \Big(matrix(A, (n, m))\Big) \wedge \bigg(N = inf\Big(\{|(b)_n| \mid basis((b)_n, (A))\}\Big)\bigg) \quad (227)$$

$$rank(r, (A)) \iff \Big(matrix(A, (n, m))\Big) \wedge \Big(dimensonality(r, (A))\Big) \quad (228)$$

$$\text{(THM)} : \Big(matrix(A, (n, m))\Big) \implies \Big(dimensionality(Ker(A)) = n - rank(r, (A))\Big)$$
$$\text{\# number of free variables} \quad (229)$$

$$transposeNorm(||x||, ()) \iff \Big(||x|| = \sqrt{x^T x}\Big) \quad (230)$$

$$\text{(THM)} : P = P^T = P^2 \quad (231)$$

$$orthogonalVectors((x, y), ()) \iff \Big(||x||^2 + ||y||^2 = ||x + y||^2\Big) \iff$$
$$\Big(x^T x + y^T y = (x + y)^T (x + y) = x^T x + y^T y + x^T y = y^T x\Big) \iff$$
$$\left(0 = \frac{x^T x + y^T y - (x^T x + y^T y)}{2} = \frac{x^T y + y^T x}{2} = x^T y\right) \iff \left(0 = \sum_i (x_i y_i) \vee \int (x(u) y(u) du)\right)$$
$$\text{\# vector and functional orthogonality} \quad (232)$$

$$orthogonalOperator\Big(Q, (V, +, \cdot, \langle \$1, \$2 \rangle)\Big) \iff \bigg(orthonormalBasis\Big(Q^T, \big(V, +, \cdot, \$1^T, \$2\big)\Big)\bigg) \vee \Big(Q^T Q = I\Big) \quad (233)$$

$$\text{(THM)} : orthogonalOperator\Big(Q, (V, +, \cdot, \langle \$1, \$2 \rangle)\Big) \implies \Big(Q^T Q Q^{-1} = I Q^{-1} = Q^T = Q^{-1}\Big) \quad (234)$$

$$orthogonalProjection(P_A b, (A, b)) \iff \Big(matrix(A, (n, m))\Big) \wedge \Big(matrix(b, (m, 1))\Big) \wedge$$
$$\bigg(\exists_{c \in \mathbb{R}^m} \Big(A^T (b - P_A b) = 0 = A^T (b - Ac)\Big) \iff$$
$$A^T b = A^T Ac \iff c = \Big(A^T A\Big)^{-1} A^T b \iff P_A b = Ac = \bigg(A\Big(A^T A\Big)^{-1} A^T\bigg) b\bigg)$$
$$\text{\# } A, A^T \text{ may not necessarily be invertible} \quad (235)$$

$$\text{(THM)} : independentOperator(A, ()) \implies independentOperator(A^T A, ()) \quad (236)$$

$$eigenvectors(X, (A, V, +, \cdot, ||\$1||)) \iff (normedVectorSpace((V, +, \cdot, ||\$1||), ())) \wedge$$
$$(X = \{v \in V \mid ||v|| = 1 \wedge eigenvector(v, (A, V, +, \cdot))\}) \quad (237)$$

$$det(det(A),(A,V,+,\cdot,||\$1||)) \iff (eigenvectors(X,(A,V,+,\cdot,||\$1||)))\wedge$$
$$(det(A)=\prod_{x\in X}(eigenvalue(\lambda,(x,A,V,+,\cdot))))$$
$$\# \text{ DEFINE; exterior algebra wedge product area??} \quad (238)$$

$$tr(tr(A),(A,V,+,\cdot,||\$1||)) \iff (eigenvectors(X,(A,V,+,\cdot,||\$1||)))\wedge$$
$$(tr(A)=\sum_{x\in X}(eigenvalue(\lambda,(x,A,V,+,\cdot))))$$
$$\# \text{ DEFINE} \quad (239)$$

$$\text{(THM)}: independentOperator(A,()) \iff det(A)\neq 0 \quad (240)$$

$$\text{(THM)}: A=A^T=A^2 \implies Tr(A)=dimensionality(N,(A)) \;\# \text{ counts dimensions} \quad (241)$$

$$(normalOperator(A,())) \iff A^TA=AA^T$$
$$\# \text{ DEFINE} \quad (242)$$

$$diagonalOperator(A,()) \iff (normalOperator(A,()))\wedge(triangularOperator(A,())) \quad (243)$$

$$characteristicEquation((A-\lambda I)x=0,(A)) \iff (Ax=\lambda x \implies Ax-\lambda x=(A-\lambda I)x=0)\wedge$$
$$(x\neq \boldsymbol{0} \implies eigenvalue(0,(x,A-\lambda I) \implies \prod_{\lambda_i\in\Lambda}=0=det(A-\lambda I)))$$
$$\# \text{ characterizes eigenvalues} \quad (244)$$

$$eigenDecomposition(S\Lambda S^{-1},(A,V,+,\cdot,||\$1||)) \iff (S\subseteq(eigenvectors(X,(A,V,+,\cdot,||\$1||))^T))\wedge$$
$$(diagonalOperator(\Lambda,())\{1\}^n=(\lambda)_n=\{\lambda\in\mathbb{R}\,|\,s\in S^T\wedge eigenvalue(\lambda,s,A,V)\})$$
$$(independentOperator(S,()))\wedge(\exists_{S^{-1}}(AS=S\Lambda \implies A=S\Lambda S^{-1})) \quad (245)$$

$$\text{(THM)}: eigenDecomposition(S\Lambda S^{-1},(A,V,+,\cdot,||\$1||)) \implies A^2=(A)(A)=S\Lambda S^{-1}S\Lambda S^{-1}=S\Lambda^2S^{-1} \quad (246)$$

$$\text{(THM)}: spectralDecomposition(Q\Lambda Q^T,(A,V,+,\cdot,||\$1||)) \iff (symmetricOperator(A,())) \implies$$
$$(\exists_Q(eigenDecomposition(Q\Lambda Q^{-1},(A,V,+,\cdot,\$1^T\$1))\wedge orthogonalOperator(Q,(V,+,\cdot,\$1^T\$2))\wedge(\lambda)_n\in\mathbb{R}^n))$$
$$\# \text{ if symmetric and eigenvalues are real, then there exists orthonormal eigenbasis} \quad (247)$$

$$hermitianAdjoint(A^H,(A)) \iff (A^H=\overline{A}^T) \Longleftarrow (\langle A,A\rangle=\overline{A}^TA\in\mathbb{R})$$
$$\# \text{ complex analog to adjoint} \quad (248)$$

$$hermitianOperator(A,()) \iff A=A^H$$
$$\# \text{ complex analog to symmetric operator} \quad (249)$$

$$unitaryOperator(Q^HQ,(Q)) \iff Q^HQ=I$$
$$\# \text{ complex analog to orthogonal operator} \quad (250)$$

$$positiveDefiniteOperator(A,(V,+,\cdot,||\$1||)) \iff (\forall_{x\in V\setminus\{\boldsymbol{0}\}}(x^TAx>0))\vee$$
$$(\forall_{x\in eigenvectors(X,(A,V,+,\$1^T\$1))}(eigenvalue(\lambda,(x,A,V,+,\cdot)) \implies \lambda>0))$$

$$\# \text{ acts like a positive scalar where any vector only scales and cannot reflect against its perpendicular axis} \tag{251}$$

$$(\text{THM}) : positiveDefiniteOperator(A^TA) \Longleftarrow \forall_{x\in V\setminus\{\boldsymbol{0}\}}(x^TA^TAx=(Ax)^T(Ax)=||Ax||>0) \tag{252}$$

$$\begin{aligned} semiPositiveDefiniteOperator(A,(V,+,\cdot,||\$1||)) \iff (\forall_{x\in V\setminus\{\boldsymbol{0}\}}(x^TAx\geq 0))\vee \\ (\forall_{x\in eigenvectors(X,(A,V,+,\$1^T\$1))}(eigenvalue(\lambda,(x,A,V,+,\cdot)) \implies \lambda\geq 0)) \\ \# \text{ acts like a nonnegative scalar} \end{aligned} \tag{253}$$

$$(\text{THM}) : symmetricOperator(A^TA) \Longleftarrow (A^TA=(A^TA)^T=A^TA^{TT}=A^TA) \tag{254}$$

$$similarOperators((A,B),()) \iff (matrix(A,(n,n)))\wedge(matrix(B,(n,n)))\wedge(\exists_M(B=M^{-1}AM)) \tag{255}$$

$$\begin{aligned} (\text{THM}) : (similarOperators((A,B),())\wedge Ax=\lambda x) \implies (\exists_M(M^{-1}Ax=\lambda M^{-1}x=M^{-1}AMM^{-1}x=BM^{-1}x)) \\ \# \text{ similar operators have the same eigenvalues but } M^{-1} \text{ shifted eigenvectors} \end{aligned} \tag{256}$$

$$\begin{aligned} singularValueDecomposition(Q\Sigma R^T,(A,V,+,\cdot,\langle\$1,\$2\rangle)) \iff (orthogonalOperator(R,(V,+,\cdot,\$1^T\$2)))\wedge \\ (orthogonalOperator(Q,(Img(A),+,\cdot,\$1^T\$2)))\wedge(semiPositiveDefiniteOperator(\Sigma,(V,+,\cdot,\$1^T\$1)))\wedge \\ (AR=Q\Sigma)\wedge(A=Q\Sigma R^{-1}=Q\Sigma R^T)\wedge(symmetricOperator(A^TA))\wedge(symmetricOperator(AA^T))\wedge \\ (A^TA=R\Sigma^TQ^TQ\Sigma R^T=R\Sigma^T\Sigma R^T)\wedge(spectralDecomposition(R(\Sigma^T\Sigma)R^T,(A^TA,V,+,\cdot,\$1^T\$1)))\wedge \\ (AA^T=Q\Sigma R^TR\Sigma^TQ^T=Q\Sigma\Sigma^TQ^T)\wedge(spectralDecomposition(Q(\Sigma\Sigma^T)Q^T,(AA^T,V,+,\cdot,\$1^T\$1)))\wedge \\ (diagonalOperator(\Sigma^T\Sigma) \implies normalOperator(\Sigma^T\Sigma)=\Sigma\Sigma^T=\Sigma_{\sigma^2})\wedge(\Sigma=\Sigma_{\sqrt[2]{\sigma^2}}=\Sigma_{|\sigma|}) \\ (\text{THM}) \text{ based on the spectral theorem:} \end{aligned} \tag{257}$$

$$\begin{aligned} leftInverseOperator(A_L^{-1},(A)) \iff (matrix(A,(n,m)))\wedge(rank(A)=n<m)\wedge \\ (A_L^{-1}A=I=((A^TA)^{-1}A^T)A) \end{aligned} \tag{258}$$

$$\begin{aligned} rightInverseOperator(A_R^{-1},(A)) \iff (matrix(A,(n,m)))\wedge(rank(A)=m<n)\wedge \\ (AA_R^{-1}=I=A(A^T(AA^T)^{-1})) \end{aligned} \tag{259}$$

1.19 Functional analysis

$$\begin{aligned} denseMap\Big(L,\big(D,H,+,\cdot,\langle\$1,\$2\rangle\big)\Big) \iff (D\subseteq H)\wedge\Big(linearOperator\big(L,(D,+,\cdot,H,+,\cdot)\big)\Big)\wedge \\ \Bigg(innerProductTopology\Big(\mathcal{O},\big(H,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Bigg)\wedge\Bigg(dense\Big(D,\big(H,\mathcal{O},d(\$1,\$2)\big)\Big)\Bigg) \end{aligned} \tag{260}$$

$$\begin{aligned} mapNorm\Big(||L||,\big(L,V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big) \iff \\ \Big(linearOperator\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big)\wedge \\ \Big(normedVectorSpace\big((V,+_V,\cdot_V,||\$1||_V),()\big)\Big)\wedge\Big(normedVectorSpace\big((W,+_W,\cdot_W,||\$1||_W),()\big)\Big)\wedge \\ \Bigg(||L||=sup\bigg(\Big\{\frac{||Lf||_W}{||f||_V}\,|\,f\in V\Big\}\bigg)=sup\Big(\big\{||Lf||_W\,|\,f\in V\wedge||f||=1\big\}\Big)\Bigg) \end{aligned} \tag{261}$$

$$\mathit{boundedMap}\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big) \iff \Big(\mathit{mapNorm}\Big(||L||,\big(L,V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)<\infty\Big) \quad (262)$$

$$\neg\mathit{boundedMap}\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big) \Longleftarrow (U\subset V)\wedge\Big(\infty=\mathit{mapNorm}\Big(||L||_U,\big(L,U,+_U,\cdot_U,||\$1||_U,W,+_W,\cdot_W,||\$1||_W\big)\Big)\leq||L||\Big) \quad (263)$$

$$\mathit{extensionMap}\Big(\widehat{L},\big(L,V,D,W\big)\Big) \iff (D\subseteq V)\wedge\Big(\mathit{linearOperator}\big(L,(D,+_D,\cdot_D,W,+_W,\cdot_W)\big)\Big)\wedge \Big(\mathit{linearOperator}\Big(\widehat{L},\big(V,+_V,\cdot_V,W,+_W,\cdot_W\big)\Big)\Big)\wedge\Big(\forall_{d\in D}\Big(\widehat{L}(d)=L(d)\Big)\Big) \quad (264)$$

$$\mathit{adjoint}\Big(L^T,\big(L,V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big) \iff \Big(\mathit{hilbertSpace}\Big(\big(V,+_V,\cdot_V,\langle\$1,\$2\rangle_V\big),()\Big)\Big)\wedge \Big(\mathit{hilbertSpace}\Big(\big(W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big),()\Big)\Big)\wedge\Big(\mathit{linearOperator}\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big)\wedge \Big(\forall_{v\in V}\forall_{w\in W}\Big(\Big(\langle Lv,w\rangle_W=\langle v,L^Tw\rangle_V\Big)\vee\Big((Lv)^Tw=v^TL^Tw\Big)\Big)\Big)$$

\# target operator that acts similar to the domain operator (265)

$$\mathit{selfAdjoint}\Big(L,\big(V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big) \iff L=\mathit{adjoint}\Big(L^T,\big(L,V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big)$$

\# also a generalization of symmetric matrices (266)

$$\mathit{compactMap}\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big) \iff \Big(\mathit{boundedMap}\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)\Big)\wedge \Big(\forall_{v\in V}\Big(\mathit{openBall}\Big(B,\Big(1.0,v,V,d_V(\$1,\$2)\Big)\Big)\implies \mathit{compactSubset}\Big(\mathit{closure}\Big(\overline{L(B)},\mathit{image}\big(L(B),(B,L,V,W)\big),W,d_W(\$1,\$2)\Big),(W,\mathcal{O}_W)\Big)\Big)\Big) \quad (267)$$

(THM) Spectral thm.:

$$\Big(\mathit{selfAdjoint}\Big(L,\big(V,+,\cdot,\langle\$1,\$2\rangle,V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Big)\wedge\Big(\mathit{compactMap}\big(L,(V,+,\cdot,V,+,\cdot)\big)\Big)\implies \Big(\exists_{(e)_{\mathbb{N}}\subseteq V}\Big(\mathit{orthonormalBasis}\Big((e)_{\mathbb{N}},\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\wedge\forall_{e_n\in(e)_{\mathbb{N}}}\Big(\mathit{eigenvector}\big(e_n,(L,V,+,\cdot)\big)\Big)\Big)\Big)\implies \Big(\exists_{(\lambda)_{\mathbb{N}}\subseteq\mathbb{R}^n}\forall_{e_n\in(e)_{\mathbb{N}}}\exists_{\lambda_n\in(\lambda)_{\mathbb{N}}}\Big(\mathit{eigenvalue}\big(\lambda_n,(e_n,L,V,+,\cdot)\big)\wedge\lim_{n\to\infty}(\lambda_n=0)\wedge L=\sum_{n=1}^{\infty}\Big(\lambda_n e_n {e_n}^T\Big)\Big)\Big)$$

\# DEFINE (268)

1.20 Function spaces

$$\mathit{curLp}\big(\mathcal{L}^p,(p,M,\sigma,\mu)\big) \iff (p\in\mathbb{R})\wedge(1\leq p<\infty)\wedge$$

$$\left(\mathcal{L}^p = \{map(f,(M,\mathbb{R})) \,|\, measurableMap(f,(M,\sigma,\mathbb{R},euclideanSigma)) \wedge \int (|f|^p d\mu) < \infty\}\right) \quad (269)$$

$$vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu)) \iff \left(curLp(\mathcal{L}^p,(p,M,\sigma,\mu))\right) \wedge \left(\forall_{f,g\in\mathcal{L}^p}\forall_{m\in M}((f+g)(m) = f(m)+g(m))\right) \wedge$$
$$\left(\forall_{f\in\mathcal{L}^p}\forall_{s\in\mathbb{R}}\forall_{m\in M}((s\cdot f)(m) = (s)f(m))\right) \wedge \left(vectorSpace((\mathcal{L}^p,+,\cdot),())\right) \quad (270)$$

$$integralNorm(\wr\wr\$1\wr\wr,(+,\cdot,p,M,\sigma,\mu)) \iff \left(vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu))\right) \wedge \left(map\left(\wr\wr\$1\wr\wr,\left(\mathcal{L}^p,\mathbb{R}_0^+\right)\right)\right) \wedge$$
$$\left(\forall_{f\in\mathcal{L}^p}\left(0 \le \wr\wr f\wr\wr = \left(\int (|f|^p d\mu)\right)^{1/p}\right)\right) \quad (271)$$

$$(\text{THM}): integralNorm(\wr\wr\$1\wr\wr,(+,\cdot,p,M,\sigma,\mu)) \implies$$
$$\left(\forall_{f\in\mathcal{L}^p}\left(\wr\wr f\wr\wr = 0 \implies almostEverywhere(f=\boldsymbol{0},(M,\sigma,\mu))\right)\right)$$
$$\#\ \text{not an expected property from a norm} \quad (272)$$

$$Lp\left(L^p,((+,\cdot,p,M,\sigma,\mu))\right) \iff \left(integralNorm(\wr\wr\$1\wr\wr,(+,\cdot,p,M,\sigma,\mu))\right) \wedge$$
$$\left(L^p = quotientSet\left(\mathcal{L}^p/\sim,\left(\mathcal{L}^p,\left(\wr\wr\$1+(-\$2)\wr\wr = 0\right)\right)\right)\right)$$
$$\#\ \text{functions in } L^p \text{ that have finite integrals above and below the x-axis} \quad (273)$$

$$(\text{THM}): banachSpace\left(\left(Lp(L^p,(+,\cdot,p,M,\sigma,\mu)),+,\cdot,\wr\wr\$1\wr\wr\right),()\right) \quad (274)$$

$$(\text{THM}): hilbertSpace\left(\left(Lp(L^p,(+,\cdot,2,M,\sigma,\mu)),+,\cdot,\frac{\wr\wr\$1+\$2\wr\wr^2 - \wr\wr\$1-\$2\wr\wr^2}{4}\right),()\right) \quad (275)$$

$$curL\left(\mathcal{L},(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right) \iff \left(banachSpace\left((W,+_W,\cdot_W,||\$1||_W),()\right)\right) \wedge$$
$$\left(normedVectorSpace\left((V,+_V,\cdot_V,||\$1||_V),()\right)\right) \wedge$$
$$\left(\mathcal{L} = \{f \,|\, boundedMap\left(f,(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right)\}\right) \quad (276)$$

$$(\text{THM}): banachSpace\left(\left(curL\left(\mathcal{L},(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right),+,\cdot,mapNorm\right),()\right) \quad (277)$$

$$(\text{THM}): ||L|| \ge \frac{||Lf||}{||f||}\ \#\ \text{from choosing an arbitrary element in the mapNorm sup} \quad (278)$$

$$(\text{THM}): \left(cauchy((f)_{\mathbb{N}},(\mathcal{L},+,\cdot,mapNorm)) \implies cauchy\left((f_n v)_{\mathbb{N}},(W,+_W,\cdot_W,||\$1||_W)\right)\right) \Longleftarrow$$
$$\left(\forall_{\epsilon'>0}\forall_{v\in V}(||f_n v - f_m v||_W = ||(f_n - f_m)v||_W \le ||f_n - f_m||\cdot||v||_V) < \epsilon\cdot||v||_V = \epsilon'\right)$$
$$\#\ \text{a cauchy sequence of operators maps to a cauchy sequence of targets} \quad (279)$$

$$\text{(THM) BLT thm.: }\left(\left(dense\big(D,(V,\mathcal{O},d_V)\big)\wedge boundedMap\Big(A,\big(D,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)\right)\Longrightarrow\right.$$
$$\left(\exists!_{\widehat{A}}\left(extensionMap\Big(\widehat{A},(A,V,D,W)\Big)\right)\wedge||\widehat{A}||=||A||\right)\Big)\Longleftarrow$$
$$\left(\forall_{v\in V}\exists_{(v)_{\mathbb{N}}\subseteq D}\left(\lim_{n\to\infty}(v_n=v)\right)\right)\wedge\left(\widehat{A}v=\lim_{n\to\infty}(Av_n)\right)\quad(280)$$

1.21 Probability Theory

$$randomExperiment\big(E,(\Omega)\big)\Longleftrightarrow\Omega=\{\omega|\textbf{experiment}=E\to\textbf{outcome}=\omega\}\quad(281)$$

$$probabilitySpace\big((\Omega,\mathcal{F},P),()\big)\Longleftrightarrow measureSpace\big((\Omega,\mathcal{F},P),()\big)\wedge\big(P(\Omega)=1\big)\quad(282)$$

$$event\big(F,(\Omega,\mathcal{F},P)\big)\Longleftrightarrow\Big(probabilitySpace\big((\Omega,\mathcal{F},P),()\big)\Big)\wedge(F\in\mathcal{F})$$

\# F can represent both singleton outcomes and outcome combinations and $\mathcal{F}$ can represent

\# a countable event that contains outcomes with even number of coin tosses before the first head

\# $\mathcal{P}(\mathbb{R})$ sets are not considered because definite uniform measures diverge everywhere

\# $\mathcal{P}(\mathbb{N})$ sets can be assigned a meaningful convergent measure e.g., $\forall_{k\in\mathbb{R}^+}\forall_{f\in F}P(\{f\})=k^{-f}$ (283)

$$\text{(THM) : }\Big(probabilitySpace\big((\Omega,\mathcal{F},P),()\big)\wedge F,A,B\in\mathcal{F}\Big)\Longrightarrow$$
$$\left(F^C\bigcup F=\Omega\wedge F^C\bigcap F=\emptyset\Longrightarrow P\left(F^C\right)+P(F)=1\Longrightarrow P\left(F^C\right)=1-P(F)\right)\wedge$$
$$\left(P\left(A\bigcup B\right)=P(A)+P(B)-P\left(A\bigcap B\right)=P(A)+P(B)-\left(1-P\left(A^C\bigcup B^C\right)\right)=\right.$$
$$P(A)+P(B)-1+P\left(A^C\right)+P\left(B^C\right)-P\left(A^C\bigcap B^C\right)=$$
$$\left.P(A)+P(B)-1+1-P(A)+1-P(B)-\left(1-P\left(A\bigcup B\right)\right)=P\left(A\bigcup B\right)\right)\wedge$$
$$\left(P\left(\bigcup_{i=1}^{n}(A_i)\right)=\sum_{k=1}^{n}\left((-1)^{k-1}\sum_{I\subset\mathbb{N}_1^n\wedge|I|=k}\left(P\left(\bigcap_{i\in I}(A_i)\right)\right)\right)\right)\quad(284)$$

$$\text{(THM) : }\Big(measureSpace\big((\Omega,\mathcal{F},P),()\big)\wedge(A)_{\mathbb{N}},(B)_{\mathbb{N}}\subseteq\mathcal{F}\wedge A,B\in\mathcal{F}\Big)\Longrightarrow$$
$$^{CL285}\left(B_n=A_n\setminus\bigcup_{i=1}^{n-1}(A_i)\right)\wedge^{DL285}_{CL285}\left(\forall_{i\in\mathbb{N}}\forall_{j\in\mathbb{N}\setminus\{i\}}\left(B_i\bigcap B_j=\emptyset\right)\right)\wedge^{EL285}_{CL285}\left(\bigcup_{i\in\mathbb{N}}(A_i)=\bigcup_{i\in\mathbb{N}}(B_i)\right)\wedge$$
$$^{1IL285}_{\substack{DL285\\measure}}\left(P\left(\bigcup_{i\in\mathbb{N}}(B_i)\right)=\sum_{i\in\mathbb{N}}(P(B_i))\right)\wedge^{2IL285}_{limit}\left(\sum_{i\in\mathbb{N}}(P(B_i))=\lim_{m\to\infty}\left(\sum_{i=1}^{m}(P(B_i))\right)\right)\wedge$$
$$^{3IL285}_{\substack{DL285\\measure}}\left(\lim_{m\to\infty}\left(\sum_{i=1}^{m}(P(B_i))\right)=\lim_{m\to\infty}\left(P\left(\bigcup_{i=1}^{m}(B_i)\right)\right)\right)\wedge$$

$$\substack{4IL285\\ EL285}\left(\lim_{m\to\infty}\left(P\left(\bigcup_{i=1}^{m}(B_i)\right)\right)=\lim_{m\to\infty}\left(P\left(\bigcup_{i=1}^{m}(A_i)\right)\right)\right)\wedge$$

$$\substack{MSCont\\ EL285\\ 1IL285\\ 2IL285\\ 3IL285\\ 4IL285}\left(P\left(\bigcup_{i\in\mathbb{N}}(A_i)\right)=\lim_{m\to\infty}\left(P\left(\bigcup_{i=1}^{m}(A_i)\right)\right)\right)\wedge$$

$$\substack{MSConvL\\ MSCont}\left(\forall_{j\in\mathbb{N}}(A_j\subseteq A_{j+1})\Longrightarrow P\left(\bigcup_{i\in\mathbb{N}}(A_i)\right)=\lim_{m\to\infty}\left(P(A_m)\right)\right)\wedge$$

$$\substack{MSConvU\\ MSConvL\\ DeMorgans}\left(\forall_{j\in\mathbb{N}}(A_{j+1}\subseteq A_j)\Longrightarrow P\left(\bigcap_{i\in\mathbb{N}}(A_i)\right)=\lim_{m\to\infty}\left(P(A_m)\right)\right)\wedge$$

$$\substack{MSSetOrder\\ measure}\big(A\subseteq B\Longrightarrow P(A)\leq P(B)\big)\wedge\substack{MSSetBound\\ measure}\left(\bigcup_{i\in\mathbb{N}}(A_i)\leq\sum_{i\in\mathbb{N}}(P(A_i))\right)\quad(285)$$

$$\mathit{generatedSigmaAlgebra}\big(\sigma(\mathcal{M}),(\mathcal{M},S)\big)\Longleftrightarrow\Big(\forall_{M\in\mathcal{M}}\Big(\mathit{sigmaAlgebra}\big(M,(S)\big)\Big)\Big)\wedge$$
$$\Big(\mathit{sigmaAlgebra}\big(\sigma(\mathcal{M}),(S)\big)=\bigcap(\mathcal{M})\Big)$$
$$\text{\# the smallest sigma algebra containing the generating sets}\quad(286)$$

$$\text{(THM)}:\big(\textbf{cantor set}\cong\mathcal{P}(\mathbb{N})\wedge(\mathbb{R},\mathit{eucledianSigma},\mathit{lebesgueMeasure})\big)\Longrightarrow P(\textbf{cantor set})=0\ \text{\# :O}\quad(287)$$

$$\mathit{conditionalProbability}\Big(P\big(A|B\big),(A,B,\Omega,\mathcal{F},P)\Big)\Longleftrightarrow\big(\mathit{probabilitySpace}(\Omega,\mathcal{F},P)\big)\wedge(A,B\in\mathcal{F})\wedge$$
$$\big(P(B)>0\big)\wedge\left(P\big(A|B\big)=\frac{P(A\cap B)}{P(B)}\vee P(B)P\big(A|B\big)=P(A\cap B)\right)$$
$$\text{\# calculates } P(A) \text{ for the subset spanned by } B$$
$$\text{\# conditioning on 0 probability sets leads to paradoxes}\quad(288)$$

$$\text{(THM)}:\big(\mathit{probabilitySpace}(\Omega,\mathcal{F},P)\wedge P(B)>0\big)\Longrightarrow\forall_{F\in\mathcal{F}}\Big(P'(F)=P\big(F|B\big)\Big)\wedge\mathit{probabilitySpace}(\Omega,\mathcal{F},P')\quad(289)$$

$$\mathit{independentEvents}\big((A,B),(\Omega,\mathcal{F},P)\big)\Longleftrightarrow(A,B\in\mathcal{F})\wedge\big(P(A\cap B)=P(A)P(B)\big)$$
$$\text{\# depends on the } P\text{, not only on } A,B\quad(290)$$

$$\mathit{setPartition}\big((X)_{\mathbb{N}},(Y)\big)\Longleftrightarrow\left(\bigcup_{i\in\mathbb{N}}(X_i)=Y\right)\wedge\Big(\forall_{i\in\mathbb{N}}\forall_{j\in\mathbb{N}\setminus\{i\}}\big(X_i\cap X_j=\emptyset\big)\Big)\quad(291)$$

$$\text{(THM)}:\Big(\mathit{probabilitySpace}(\Omega,\mathcal{F},P)\wedge\{A\}\cup(B)_{\mathbb{N}}\subseteq\mathcal{F}\wedge\mathit{setPartition}\big((B)_{\mathbb{N}},(\Omega)\big)\Big)\Longrightarrow$$
$$\left(P(A)=\sum_{i\in\mathbb{N}}\Big(P\big(A|B_i\big)P(B_i)\Big)\right)\wedge$$
$$\left(\forall_{i\in\mathbb{N}}\left(P\big(A|B_i\big)P(B_i)=P(A)P\big(B_i|A\big)=\left(\sum_{j\in\mathbb{N}}\Big(P\big(B_i|A\big)\Big)\right)P\big(B_i|A\big)\right)\right)\wedge$$

$$\left(P\left(\bigcap_{i\in\mathbb{N}}(B_i)\right)=P(B_1)\prod_{i=2}^{\infty}\left(P\left(B_i|\bigcap_{j=1}^{i-1}(B_j)\right)\right)\right)$$

from the subspace definition of conditional probability and algebraic manipulations (292)

$$finIndEvents\Big((A)_{\mathbb{N}_k},(\Omega,\mathcal{F},P)\Big)\Longleftrightarrow\Big(probabilitySpace(\Omega,\mathcal{F},P)\Big)\wedge(k\in\mathbb{N})\wedge$$
$$(A_{\mathbb{N}_k}\subseteq\mathcal{F})\wedge\left(\forall_{I_0\in\mathcal{P}(\mathbb{N}_k)\setminus\emptyset}\left(P\left(\bigcap_{i\in I_0}(A_i)\right)=\prod_{i\in I_0}(P(A_i))\right)\right)$$

every combination of subsets must be independent (293)

$$infIndEvents\big((A)_I,(\Omega,\mathcal{F},P)\big)\Longleftrightarrow$$
$$\left(\forall_{I_F\subseteq I}\left(finiteSet(I_F)\Longrightarrow finIndEvents\Big((A)_{I_F},(\Omega,\mathcal{F},P)\Big)\right)\right) \quad (294)$$

$$subSigmaAlgebra\big(\mathcal{B},(\mathcal{F},\Omega)\big)\Longleftrightarrow\Big(sigmaAlgebra\big(\mathcal{F},(\Omega)\big)\Big)\wedge\Big(sigmaAlgebra\big(\mathcal{B},(\Omega)\big)\Big)\wedge(\mathcal{B}\subseteq\mathcal{A}) \quad (295)$$

$$independentSigmaAlgebras\big((\mathcal{A},\mathcal{B}),(\Omega,\mathcal{F},P)\big)\Longleftrightarrow\big(probabilitySpace(\Omega,\mathcal{F},P)\big)\wedge$$
$$\Big(subSigmaAlgebra\big(\mathcal{A},(\mathcal{F},\Omega)\big)\Big)\wedge\Big(subSigmaAlgebra\big(\mathcal{B},(\mathcal{F},\Omega)\big)\Big)\wedge$$
$$\Big(\forall_{A\in\mathcal{A}}\forall_{B\in\mathcal{B}}\Big(independentEvents\big((A,B),(\Omega,\mathcal{F},P)\big)\Big)\Big) \quad (296)$$

$$infIndSigmaAlgebras\big((\mathcal{A})_I,(\Omega,\mathcal{F},P)\big)\Longleftrightarrow\Big(\forall_{i\in I}\big(subSigmaAlgebra(\mathcal{A}_i),(\mathcal{F},\Omega)\big)\Big)\wedge$$
$$\big(\forall_{i\in I}(F_i\in\mathcal{A}_i)\big)\wedge\Big(infIndEvents\big((F)_I,(\Omega,\mathcal{F},P)\big)\Big) \quad (297)$$

$$infinitelyOften\big(\{A_n \text{ i-o}\},()\big)\Longleftrightarrow\left(B_n=\bigcup_{i=n}^{\infty}(A_i)\in\mathcal{F}\right)\wedge\left(\{A_n \text{ i-o}\}=\bigcap_{n\in\mathbb{N}}(B_n)=\bigcap_{n\in\mathbb{N}}\bigcup_{i=n}^{\infty}(A_i)\in\mathcal{F}\right)$$

the event that infinitely many A_n's will occur

B_n occur if some event within the nth-tail-end event $A_i|i\geq n$ occur, which follows from $\cup$

$\{A_n$ i-o$\}$ occur if every tail-end event B_n occur for all n, which follows from $\cap$

similarly, $\{A_n$ i-o$\}$ occur, for all values of n, the nth-tail-end event occur (298)

$$\text{(THM) BCL 1: }\left(^{Cond300}\left(\sum_{n\in\mathbb{N}}(P(A_n))<\infty\right)\Longrightarrow\Big(P\big(\{A_n \text{ i-o}\}\big)=0\Big)\right)\Longleftarrow$$
$$^{1IL300}_{\substack{infinitelyOften\\MSContU}}\left(P\left(\bigcap_{n\in\mathbb{N}}(B_n)\right)=\lim_{n\to\infty}(P(B_n))=\lim_{n\to\infty}\left(P\left(\bigcup_{i=n}^{\infty}(A_i)\right)\right)\right)\wedge$$
$$^{2IL300}_{MSSetBount}\left(\lim_{n\to\infty}\left(P\left(\bigcup_{i=n}^{\infty}(A_i)\right)\right)\leq\lim_{n\to\infty}\left(\sum_{i=n}^{\infty}(P(A)_i)\right)\right)\wedge$$

$$\overset{3IL300}{\underset{Cond300}{}}\left(\lim_{n\to\infty}\left(\sum_{i=n}^{\infty}(P(A)_i)\right)=0\right)\wedge^{Impl300}_{\substack{1IL300\\2IL300\\3IL300}}\left(0\leq P\big(\{A_n \text{ i-o}\}\big)\leq 0\right) \quad (299)$$

$$(\text{THM}):{}^{logp}\left(\forall_{x\in[0,1]}\big(\log(1-x)\leq -x\big)\right) \quad (300)$$

$$(\text{THM}):{}^{sump}\left(\left({}^{1Cond302}\big(\forall_{i\in\mathbb{N}}(p_i\in[0,1])\big)\wedge{}^{2Cond302}\left(\sum_{i\in\mathbb{N}}(p_i)=\infty\right)\right)\Longrightarrow\prod_{i\in\mathbb{N}}(1-p_i)=0\right)\Longleftarrow$$

$$^{1IL302}\left(\prod_{i\in\mathbb{N}}(1-p_i)=\exp\left(\log\left(\prod_{i\in\mathbb{N}}(1-p_i)\right)\right)=\exp\left(\log\left(\lim_{n\to\infty}\left(\prod_{i=1}^{n}(1-p_i)\right)\right)\right)\right)\wedge$$

$$^{2IL302}_{logp}\left(\exp\left(\log\left(\lim_{n\to\infty}\left(\prod_{i=1}^{n}(1-p_i)\right)\right)\right)=\exp\left(\lim_{n\to\infty}\left(\sum_{i=1}^{n}\big(\log(1-p_i)\big)\right)\right)\leq\exp\left(\lim_{n\to\infty}\left(\sum_{i=1}^{n}(-p_i)\right)\right)\right)\wedge$$

$$^{3IL302}_{2Cond302}\left(\exp\left(\lim_{n\to\infty}\left(\sum_{i=1}^{n}(-p_i)\right)\right)=\exp(-\infty)=0\right)\wedge^{Impl302}_{\substack{1Cond302\\1IL302\\2IL302\\3IL302}}\left(0\leq\prod_{i\in\mathbb{N}}(1-p_i)\leq 0\right) \quad (301)$$

$$(\text{THM})\text{ BCL 2: }\left(\left({}^{1Cond303}\left(\sum_{n\in\mathbb{N}}(P(A_n))=\infty\right)\wedge{}^{2Cond303}\big(\mathit{infIndEvents}((A)_{\mathbb{N}})\big)\right)\Longrightarrow P\big(\{A_n\text{ i-o}\}\big)=1\right)$$

$$\Longleftarrow{}^{1IL303}_{MSSetBound}\left(1-P\big(\{A_n\text{ i-o}\}\big)=P\left(\{A_n\text{ i-o}\}^{C}\right)=P\left(\bigcup_{n\in\mathbb{N}}\left({B_n}^{C}\right)\right)\leq\sum_{n\in\mathbb{N}}\left(P\left({B_n}^{C}\right)\right)\right)\wedge$$

$$^{2IL303}_{\substack{DeMorgans\\2Cond303}}\left(\sum_{n\in\mathbb{N}}\left(P\left({B_n}^{C}\right)\right)=\sum_{n\in\mathbb{N}}\left(P\left(\bigcap_{i=n}^{\infty}\left({A_i}^{C}\right)\right)\right)=\sum_{n=1}^{\infty}\left(\prod_{i=n}^{\infty}\left(P\left({A_i}^{C}\right)\right)\right)=\sum_{n=1}^{\infty}\left(\prod_{i=n}^{\infty}(1-P(A_i))\right)\right)\wedge$$

$$^{3IL303}_{\substack{1Cond303\\sump}}\left(\sum_{n=1}^{\infty}\left(\prod_{i=n}^{\infty}(1-P(A_i))\right)=\sum_{n=1}^{\infty}(0)=0\right)\wedge^{Impl303}_{\substack{1Il303\\2Il303\\3Il303}}\left(0\leq 1-P\big(\{A_n\text{ i-o}\}\big)\leq 0\Longleftrightarrow P\big(\{A_n\text{ i-o}\}\big)=1\right) \quad (302)$$

$$\mathit{randomVariable}(X,(\Omega,\mathcal{F},P))\Longleftrightarrow\big(\mathit{probabilitySpace}(\Omega,\mathcal{F},P)\big)\wedge(\mathit{map}(X,(\Omega,\mathbb{R})))\wedge$$
$$(\mathit{measurableMap}(X,(\Omega,\mathcal{F},\mathbb{R},\mathit{euclideanSigma}(\sigma_S,()))))$$

\# Random-Deterministic Variable-Function maps the measurable space to the real line and borel sets (303)

$$\mathit{PL}(P_X,(X,\Omega,\mathcal{F},P))\Longleftrightarrow(\mathit{randomVariable}(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(\forall_{B\in\sigma_S}(P_X(B)=P(\mathit{preimage}(A,(B,X,\Omega,\mathbb{R}))=(P\circ X^{-1})(B))=P(X\in B)))$$

\# probability of outcomes occuring in the Borel set (304)

$$\mathit{piSystem}(\mathcal{G},(\Omega))\Longleftrightarrow\mathcal{G}\subseteq\mathcal{P}(\Omega)\wedge\forall_{A,B\in\mathcal{G}}(A\cap B\in\mathcal{G}) \quad (305)$$

$$(\text{THM}):(\mathit{piSystem}(\mathcal{G},(\Omega))\wedge\mathcal{F}=\sigma(\mathcal{G})\wedge\mathit{probabilitySpace}(\Omega,\mathcal{F},P_1)\wedge\mathit{probabilitySpace}(\Omega,\mathcal{F},P_2))\Longrightarrow$$
$$(\forall_{G\in\mathcal{G}}(P_1(G)=P_2(G))\Longrightarrow\forall_{F\in\mathcal{F}}(P_1(F)=P_2(F))) \quad (306)$$

$$\text{(THM)}: \mathit{euclideanSigma}(\sigma_S) = \sigma(\{(-\infty, x] \mid x \in \mathbb{R}\}) \tag{307}$$

$$\begin{aligned}\mathit{CDF}(F_X, (X, \Omega, \mathcal{F}, P)) \iff (\mathit{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\forall_{x \in \mathbb{R}}(F_X(x) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) = P(X \leq x))) \\ \#\text{ this is from the generating borel sets } P(X \in (-\infty, x]))\end{aligned} \tag{308}$$

$$\text{(THM) DEFINE: } F_X \cong P_X \tag{309}$$

$$\begin{aligned}\text{(THM)}: \mathit{CDF}(F_X, (X, \Omega, \mathcal{F}, P)) \iff (\lim_{x \to -\infty}(F_X(x)) = 0) \wedge (\lim_{x \to \infty}(F_X(x)) = 1) \wedge \\ (\forall_{x, y \in \mathbb{R}}(x \leq y \implies F_X(x) \leq F_X(y))) \wedge (\forall_{x \in \mathbb{R}}(\lim_{\epsilon \to 0^+}(F(x + \epsilon) = F(x)))) \\ \#\text{ left-continuity will approach } P(X < x) \neq F_X \text{ and } P(\{x\}) = 0 \implies P(X \leq x) = F_X\end{aligned} \tag{310}$$

$$\begin{aligned}\mathit{PMF}(H_X, (X, \Omega, \mathcal{F}, P)) \iff (\mathit{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\forall_{x \in \mathbb{R}}(H_X(x) = P(\{\omega \in \Omega \mid X(\omega) = x\}) = P(X = x))) \\ \#\text{ type of probability law}\end{aligned} \tag{311}$$

$$\begin{aligned}\mathit{indicatorRandomVariable}(I_A, (\Omega, \mathcal{F}, P)) \iff (\mathit{randomVariable}(I_A, (\Omega, \mathcal{F}, P))) \wedge \\ (\forall_{A \in \mathcal{F}} \forall_{\omega \in \Omega}(I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}))\end{aligned} \tag{312}$$

$$\text{(THM)}: \textbf{measures on R = discrete, continous, and singular components} \tag{313}$$

$$\begin{aligned}\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\mathit{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\exists_{E \subseteq \mathbb{R}}(\mathit{countablyInfinite}(E) \wedge P_X(E) = 1)) \wedge (\cup((e)_{\mathbb{N}}) = E) \wedge (\forall_{i \in \mathbb{N}}(e_i \in E))\end{aligned} \tag{314}$$

$$\begin{aligned}\text{(THM)}: (\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \implies \\ (1 = P(E) = \sum_{i \in \mathbb{N}}(P_X(\{e_i\})) = \sum_{i \in \mathbb{N}}(P(X = e_i))) \wedge (\forall_{B \in \sigma_S}(P_X(B) = \sum_{x \in E \cap B}(P(X = x))))\end{aligned} \tag{315}$$

$$\begin{aligned}\mathit{bernoulliRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge (E = \{0, 1\}) \wedge \\ (p \in \mathbb{R}) \wedge (P_X = P(X = x) = \begin{cases} 1 & x = 1 \\ 0 & x = 0 \end{cases})\end{aligned} \tag{316}$$

$$\begin{aligned}\mathit{uniformRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (n = |\mathit{finiteSet}(E)|) \wedge (\forall_{i \in \mathbb{N} \wedge i \leq n}(P_X(\{e_i\}) = P(X = e_i) = \frac{1}{n}))\end{aligned} \tag{317}$$

$$\begin{aligned}\mathit{geometricRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\mathit{countablyInfinite}(E)) \wedge (p \in \mathbb{R}) \wedge (\forall_{i \in \mathbb{N}}(P_X(\{e_i\}) = P(X = e_i) = (1 - p)^{i-1} p))\end{aligned} \tag{318}$$

$$\begin{aligned}\mathit{binomialRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (n = |\mathit{finiteSet}(E)|) \wedge (p \in \mathbb{R}) \wedge (\forall_{i \in \mathbb{N}}(P_X(\{e_i\}) = P(X = e_i) = \binom{n}{i} p^i (1 - p)^{n-i}))\end{aligned} \tag{319}$$

$$\mathit{poissonRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\mathit{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge$$

$$(countablyInfinite(E))\wedge(\lambda\in\mathbb{R}^+)\wedge(\forall_{i\in\mathbb{N}}(P_X(\{e_i\})=P(X=e_i)=\frac{e^{-\lambda}\lambda^i}{i!})) \quad (320)$$

$$absolutelyContinous((f,g),(M,\sigma)) \iff (measure(f,(M,\sigma)))\wedge(measure(g,(M,\sigma)))\wedge$$
$$(\forall_{A\in\sigma}(g(A)=0 \implies f(A)=0)) \quad (321)$$

$$\text{(THM) Radon-Nikodym: } (measurableSpace((M,\sigma),()))\wedge(finiteMeasure(\mu,(M,\sigma)))\wedge$$
$$(finiteMeasure(\nu,(M,\sigma)))\wedge(absolutelyContinous((\nu,\mu),(M,\sigma))) \implies$$
$$(\exists_{map(f,(M,\overline{\mathbb{R}}^+))}\forall_{A\in\sigma}(\nu(A)=\int_A(fd\mu)))$$
$$\text{\# connects } P_X=F_X=\int(f_x dx) \quad (322)$$

$$continuousRandomVariable(X,(\Omega,\mathcal{F},P)) \iff (randomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(absolutelyContinous((P_X,lebesgueMeasure),(\mathbb{R},euclideanSigma)))$$
$$\text{\# the probabilities lie on nonzero lebesgue measure sets} \quad (323)$$

$$contUniformRandomVariable(X,(\Omega,\mathcal{F},P)) \iff (continuousRandomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(a,b\in\mathbb{R})\wedge(a<b)\wedge(P_X=F_X(x)=\begin{cases}0 & x<a\\ \frac{x}{b-a} & a\leq x\leq b\\ 1 & x>b\end{cases}) \quad (324)$$

$$exponentialRandomVariable(X,(\Omega,\mathcal{F},P)) \iff (continuousRandomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(\lambda\in\mathbb{R}^+)\wedge(P_X=F_X(x)=\begin{cases}1-e^{-\lambda x} & x>0\\ 0 & x\leq 0\end{cases}) \quad (325)$$

$$memorylessRandomVariable(X,()) \iff (\forall_{\omega\in\Omega}(X(\omega)\geq 0))\wedge(\forall_{s,t\in\mathbb{R}_0^+}(P(X>s)=P(X>s+t\,|\,x>t))) \quad (326)$$

$$gaussianRandomVariable(X,(\Omega,\mathcal{F},P)) \iff (continuousRandomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(\mu\in\mathbb{R})\wedge(\sigma\in\mathbb{R}^+)\wedge(P_X=F_X(x)=\int(\frac{\exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sigma\sqrt{2\pi}}dx)) \quad (327)$$

(THM) : **DEFINE gaussian is stable and is an attractor** (328)

$$simplifiedCauchyRandomVariable(X,(\Omega,\mathcal{F},P)) \iff (continuousRandomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(P_X=F_X(x)=\int(\frac{1}{\pi(1+x^2)}dx)) \quad (329)$$

$$singularRandomVariable(X,(\Omega,\mathcal{F},P)) \iff (randomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(\forall_{x\in\mathbb{R}}(P_X(\{x\})=0))\wedge(\exists_{F\in euclideanSigma}(P_X(F)=1\wedge lebesgueMeasure(F)=0))$$
$$\text{\# an example is uniform measure on the Cantor set} \quad (330)$$

$$preimageSigma(\sigma(X),(X,\Omega,\mathcal{F},P)) \iff (randomVariable(X,(\Omega,\mathcal{F},P)))\wedge$$
$$(\sigma(X)=\{A\in\mathcal{F}\,|\,B\in euclideanSigma\wedge preimage(A,(B,X,\Omega,\mathbb{R}))\})\wedge(subSigmaAlgebra(\sigma(X),(\mathcal{F},\Omega)))$$
$$\text{\# checks all events occurred in Omega by checking } X(\omega)\in B\in euclideanSigma \text{ TODO} \quad (331)$$

============ **N O T = U P D A T E D** ============ (332)

$$S^n = (x,y)^n \subset Z \text{ \# sample set consists of } n \text{ input-output pairs} \tag{333}$$

$$S^n \implies map(f_{S^n}, (X,Y)) \text{ \# learned predictor function} \tag{334}$$

$$V \text{ \# loss function} \tag{335}$$

$$I_n[f] = \frac{1}{n}\sum_i (V(f(x_i), y_i)) \text{ \# empirical predictor error} \tag{336}$$

$$I[f] = \int_Z (V(f(x_i), y_i) d\mu(x_i, y_i)) \text{ \# expected predictor error} \tag{337}$$

$$f_\star \text{ \# optimal or lowest expected error hypothesis} \tag{338}$$

$$\lim_{n\to\infty}(I[f_n]) = I[f_\star] \text{ \# consistency: expected error of learned approaches best hypothesis} \tag{339}$$

$$\lim_{n\to\infty}(I_n[f_n]) = I[f_n] \text{ \# generalization: empirical error of learned hyptohesis approximates expected error} \tag{340}$$

$$\begin{gathered}|I_n[f_n] - I[f_n]| < \epsilon(n,\delta) \text{ with P } 1-\delta? \text{ \# generalization error: measure performance of learning algorithm} \\ \forall_{\epsilon>0}(\lim_{n\to\infty}(P(\{|I_n[f_n] - I[f_n]| \geq \epsilon\}) = 0)) \\ \#\end{gathered} \tag{341}$$

$$X \text{ \# random variable ; } \mu \text{ \# probability measure} \tag{342}$$

$$measureSpace(X, F, P) \tag{343}$$

$$\begin{gathered}IID(A,(X,P)) \iff (A \in F \subseteq X) \wedge P_{a_1,a_2,\ldots}(a_1 = t_1, a_2 = t_2, \ldots) = \prod_i (P_{a_1}(a_i = t_i)) \\ \text{\# outcomes are independent and equally likely}\end{gathered} \tag{344}$$

$$E[X] = \int_{Range} (xd(P(x))) \tag{345}$$

$$0 \tag{346}$$

1.22 Underview

$$\tag{347}$$

$$curve - fitting/explaining \neq prediction \tag{348}$$

$$ill - defined problem + solution space constraints \implies well - defined problem \tag{349}$$

$$x \text{ \# input ; } y \text{ \# output} \tag{350}$$

$$S_n = \{(x_1, y_1), \ldots, (x_n, y_n)\} \text{ \# training set} \tag{351}$$

$$f_S(x) \sim y \text{ \# solution} \tag{352}$$

$$each(x,y) \in p(x,y) \text{ \# training data } x,y \text{ is a sample from an unknown distribution } p \tag{353}$$

$$V(f(x),y) = d(f(x),y) \text{ \# loss function} \tag{354}$$

$$I[f] = \int_{X \times Y} V(f(x),y)p(x,y)dxdy \text{ \# expected error} \tag{355}$$

$$I_n[f] = \frac{1}{n} \sum_{i=1}^{n} V(f(x_i), y_i) \text{ \# empirical error} \tag{356}$$

$$probabilisticConvergence(X,()) \iff \forall_{\epsilon > 0} \lim_{n \to \infty} Pxn - x \leq \epsilon = 0 \tag{357}$$

$$I - Ingeneralizationerror \tag{358}$$

$$well - posed := exists, unique, stable; elseill - posed \tag{359}$$

2 Machine Learning

2.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X,Y) \text{ \# dataset} \tag{360}$$

learned parameters = parameters to be fixed by training with the dataset (361)

hyperparameters = parameters that depends on a dataset (362)

validation = partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition # useful for fixing hyperparameters (363)

cross-validation = average accuracy of validation for different choices of testing partition (364)

L1 = scales linearly ; L2 = scales quadratically (365)

d = **distance = quantifies the the similarity between data points** (366)

$$d_{L1}(A,B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance} \tag{367}$$

$$d_{L2}(A,B)=\sqrt{\sum_p (A_p-B_p)^2} \text{ \# Euclidean distance} \tag{368}$$

$$\textbf{kNN classifier}=\textbf{classifier based on } k \textbf{ nearest data points} \tag{369}$$

$$s=\textbf{class score}=\textbf{quantifies bias towards a particular class} \tag{370}$$

$$s_{linear}=f_{c\times 1}(x_{n\times 1},W_{c\times n},b_{c\times 1})=W_{c\times n}x_{n\times 1}+b_{c\times 1} \text{ \# linear score function} \tag{371}$$

$$l=\textbf{loss}=\textbf{quantifies the errors by the learned parameters} \tag{372}$$

$$l=\frac{1}{|c_i|}\sum_{c_i} l_i \text{ \# average loss for all classes} \tag{373}$$

$$l_{SVM_i}=\sum_{y_i\neq c_i}\max(0,s_{y_i}-s_{c_i}+1) \text{ \# SVM hinge class loss function:}$$
$$\text{\# ignores incorrect classes with lower scores including a non-zero margin} \tag{374}$$

$$l_{MLR_i}=-\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$
$$\text{\# lower scores correspond to lower exponentiated-normalized probabilities} \tag{375}$$

$$R=\textbf{regularization}=\textbf{optimizes the choice of learned parameters to minimize test error} \tag{376}$$

$$\lambda \text{ \# regularization strength hyperparameter} \tag{377}$$

$$R_{L1}(W)=\sum_{W_i}|W_i| \text{ \# L1 regularization} \tag{378}$$

$$R_{L2}(W)=\sum_{W_i}{W_i}^2 \text{ \# L2 regularization} \tag{379}$$

$$L'=L+\lambda R(W) \text{ \# weight regularization} \tag{380}$$

$$\nabla_W L=\overrightarrow{\frac{\partial}{\partial W_i}}L=\textbf{loss gradient w.r.t. weights} \tag{381}$$

$$\frac{\partial L_E}{\partial W_I}=\frac{\partial L_L}{\partial W_I}\frac{\partial L_E}{\partial L_L} \text{ \# loss gradient w.r.t. input weight in terms of external and local gradients} \tag{382}$$

$$s=\textbf{forward API} \; ; \; \frac{\partial L_L}{\partial W_I}=\textbf{backward API} \tag{383}$$

$$W_{t+1}=W_t-\nabla_{W_t}L \text{ \# weight update loss minimization} \tag{384}$$

$$\textbf{TODO:Research on Activation functions, Weight Initialization, Batch Normalization} \tag{385}$$

$$review5meanvardiscussion/hyperparameteroptimization/babysittinglearning \tag{386}$$

TODO loss L or l ??

3 Glossary

chaoticTopology
discreteTopology
topology
topologicalSpace
open
closed
clopen
neighborhood
chaoticTopology
discreteTopology
metric
metricSpace
openBall
metricTopology
metricTopologicalSpace
limitPoint
interiorPoint
closure
dense
eucD
euclideanTopology
subsetTopology
productTopology
metric
metricSpace
openBall
metricTopology
metricTopologicalSpace
limitPoint
interiorPoint
closure
dense
eucD
euclideanTopology
subsetTopology
productTopology
sequence
sequenceConvergesTo
sequence
sequenceConvergesTo
continuous
homeomorphism
isomorphicTopologicalSpace
continuous
homeomorphism
isomorphicTopologicalSpace
T0Separate
T1Separate
T2Separate
T0Separate
T1Separate
T2Separate
openCover
finiteSubcover
compact
compactSubset
bounded
openCover
finiteSubcover
compact
compactSubset
bounded
openRefinement
locallyFinite
paracompact
openRefinement
locallyFinite
paracompact
connected
pathConnected
connected
pathConnected
sigmaAlgebra
measurableSpace
measurableSet
measure
measureSpace
finiteMeasure
generatedSigmaAlgebra
borelSigmaAlgebra
euclideanSigma
lebesgueMeasure
measurableMap
pushForwardMeasure
nullSet
almostEverywhere
sigmaAlgebra
measurableSpace
measurableSet
measure
measureSpace
finiteMeasure
generatedSigmaAlgebra
borelSigmaAlgebra
euclideanSigma
lebesgueMeasure
measurableMap
pushForwardMeasure
nullSet
almostEverywhere
simpleTopology
simpleSigma
simpleFunction
characteristicFunction
exEuclideanSigma
nonNegIntegrable
nonNegIntegral
explicitIntegral
integrable
integral
simpleTopology
simpleSigma
simpleFunction
characteristicFunction
exEuclideanSigma
nonNegIntegrable
nonNegIntegral
explicitIntegral
integrable
integral
vectorSpace
innerProduct
innerProductSpace
vectorNorm
normedVectorSpace
vectorMetric
metricVectorSpace
innerProductNorm
normInnerProduct
normMetric
metricNorm
orthogonal
normal
basis
orthonormalBasis
vectorSpace
innerProduct
innerProductSpace
vectorNorm
normedVectorSpace
vectorMetric
metricVectorSpace
innerProductNorm
normInnerProduct
normMetric
metricNorm
orthogonal
normal
basis
orthonormalBasis
subspace
subspaceSum
subspaceDirectSum
orthogonalComplement
orthogonalDecomposition
subspace
subspaceSum
subspaceDirectSum
orthogonalComplement
orthogonalDecomposition
cauchy
complete
banachSpace
hilbertSpace
separable
cauchy
complete
banachSpace
hilbertSpace
separable
linearOperator
matrix
eigenvector
eigenvalue
identityOperator
inverseOperator
transposeOperator
symmetricOperator
triangularOperator
decomposeLU
Img
Ker
independentOperator
dimensonality
rank
transposeNorm
orthogonalVectors
orthogonalOperator
orthogonalProjection
eigenvectors
det
tr
diagonalOperator
characteristicEquation
eigenDecomposition
spectralDecomposition
hermitianAdjoint
hermitianOperator
unitaryOperator
positiveDefiniteOperator
semiPositiveDefiniteOperator
similarOperators
similarOperators
singularValueDecomposition
linearOperator
matrix
eigenvector
eigenvalue
identityOperator
inverseOperator
transposeOperator
symmetricOperator
triangularOperator
decomposeLU
Img
Ker
independentOperator
dimensonality
rank
transposeNorm

orthogonalVectors
orthogonalOperator
orthogonalProjection
eigenvectors
det
tr
diagonalOperator
characteristicEquation
eigenDecomposition
spectralDecomposition
hermitianAdjoint
hermitianOperator
unitaryOperator
positiveDefiniteOperator
semiPositiveDefiniteOperator
similarOperators
similarOperators
singularValueDecomposition
denseMap
mapNorm
boundedMap
extensionMap
adjoint
selfAdjoint
compactMap
denseMap
mapNorm
boundedMap
extensionMap
adjoint
selfAdjoint
compactMap
curLp
vecLp
integralNorm
Lp
curL
curLp
vecLp
integralNorm
Lp
curL
randomExperiment
probabilitySpace
measureSpace
event
CL285
DL285
EL285
1IL285
2IL285
3IL285
4IL285
MSCont
MSConvL
MSConvU
MSSetOrder
MSSetBound
generatedSigmaAlgebra
conditionalProbability
independentEvents
setPartition
finIndEvents
infIndEvents
subSigmaAlgebra
independentSigmaAlgebras
infIndSigmaAlgebras
infinitelyOften
Cond300
1IL300
2IL300
3IL300
Impl300
logp
sump
1Cond302
2Cond302
1IL302
2IL302
3IL302
Impl302
1Cond303
2Cond303
1IL303
2IL303
3IL303
Impl303
randomVariable
PL
piSystem
CDF
PMF
indicatorRandomVariable
discreteRandomVariable
bernoulliRandomVariable
uniformRandomVariable
geometricRandomVariable
binomialRandomVariable
poissonRandomVariable
absolutelyContinous
continuousRandomVariable
contUniformRandomVariable
exponentialRandomVariable
memorylessRandomVariable
gaussianRandomVariable
simplifiedCauchyRandomVariable
singularRandomVariable
preimageSigma
randomExperiment
probabilitySpace
measureSpace
event
CL285
DL285
EL285
1IL285
2IL285
3IL285
4IL285
MSCont
MSConvL
MSConvU
MSSetOrder
MSSetBound
generatedSigmaAlgebra
conditionalProbability
independentEvents
setPartition
finIndEvents
infIndEvents
subSigmaAlgebra
independentSigmaAlgebras
infIndSigmaAlgebras
infinitelyOften
Cond300
1IL300
2IL300
3IL300
Impl300
logp
sump
1Cond302
2Cond302
1IL302
2IL302
3IL302
Impl302
1Cond303
2Cond303
1IL303
2IL303
3IL303
Impl303
randomVariable
PL
piSystem
CDF
PMF
indicatorRandomVariable
discreteRandomVariable
bernoulliRandomVariable
uniformRandomVariable
geometricRandomVariable
binomialRandomVariable
poissonRandomVariable
absolutelyContinous
continuousRandomVariable
contUniformRandomVariable
exponentialRandomVariable
memorylessRandomVariable
gaussianRandomVariable
simplifiedCauchyRandomVariable
singularRandomVariable
preimageSigma