

Next-Next-Gen Notes

Object-Oriented Maths

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Format: $characteristic((subjects), (dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$

Note: All weaker objects automatically induces notions inherited from stronger objects.

TODO assign free variables as parameters

TODO define \parallel abs cross-product and other missing refs

TODO distinguish new condition vs implied proposition - separate propositions into new line thms

TODO silent link expressions! - e.g. *backslashsilentPLPL_X*

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s, (RegEx)) \iff well-formedString(s, ()) \quad (1)$$

$$proposition((p, t), ()) \iff \left(statement(p, ()) \wedge \begin{aligned} &(t = eval(p)) \wedge \\ &(t = true \vee t = false) \end{aligned} \right) \quad (2)$$

$$operator\left(o, \left((p)_{n \in \mathbb{N}}\right)\right) \iff proposition\left(o\left((p)_{n \in \mathbb{N}}\right), ()\right) \quad (3)$$

$$operator(\neg, (p_1)) \iff \left(proposition((p_1, true), ()) \implies ((\neg p_1, false), ()) \right) \wedge \left(proposition((p_1, false), ()) \implies ((\neg p_1, true), ()) \right) \\ \# \text{ an operator takes in propositions and returns a proposition} \quad (4)$$

$$operator(\neg) \iff \mathbf{NOT} ; operator(\vee) \iff \mathbf{OR} ; operator(\wedge) \iff \mathbf{AND} ; operator(\vee) \iff \mathbf{XOR} \\ operator(\implies) \iff \mathbf{IF} ; operator(\iff) \iff \mathbf{OIF} ; operator(\iff) \iff \mathbf{IFF} \quad (5)$$

$$proposition((false \implies true), true, ()) \wedge proposition((false \implies false), true, ()) \\ \# \text{ truths based on a false premise is not false; ex falso quodlibet principle} \quad (6)$$

$$(\text{THM}) : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \quad (7)$$

$$predicate(P, (V)) \iff \forall_{v \in V} \left(proposition\left(P(v), t, ()\right) \right) \quad (8)$$

$$0thOrderLogic(P, ()) \iff proposition((P, t), ()) \\ \# \text{ individual proposition} \quad (9)$$

$$1stOrderLogic(P, (V)) \iff \left(\forall_{v \in V} \left(0thOrderLogic(v, ()) \right) \right) \wedge \left(\forall_{v \in V} \left(proposition \left((P(v), t), () \right) \right) \right)$$

propositions defined over a set of the lower order logical statements (10)

$$quantifier(q, (p, V)) \iff \left(predicate(p, (V)) \right) \wedge \left(proposition \left((q(p), t), () \right) \right)$$

a quantifier takes in a predicate and returns a proposition (11)

$$quantifier(\forall, (p, V)) \iff proposition \left(\left(\bigwedge_{v \in V} (p(v)), t \right), () \right)$$

universal quantifier (12)

$$quantifier(\exists, (p, V)) \iff proposition \left(\left(\bigvee_{v \in V} (p(v)), t \right), () \right)$$

existential quantifier (13)

$$quantifier(\exists!, (p, V)) \iff \exists_{x \in V} \left(P(x) \wedge \neg \left(\exists_{y \in V \setminus \{x\}} (P(y)) \right) \right)$$

uniqueness quantifier (14)

$$(THM) : \forall_x p(x) \iff \neg \exists_x \neg p(x)$$

De Morgan's law (15)

$$(THM) : \forall_x \exists_y p(x, y) = \forall_x \neg \forall_y \neg p(x, y) \neq \exists_y \forall_x p(x, y) = \neg \forall_y \neg (\forall_x p(x, y)) = \neg \forall_y \exists_x \neg p(x, y)$$

different quantifiers are not interchangeable (16)

$$===== \text{ N O T } = \text{ U P D A T E D } =====$$

(17)

$$\text{proof} = \text{truths derived from a finite number of axioms and deductions}$$

(18)

$$\text{elementary arithmetics} = \text{system with substitutions, and some notion of addition, multiplication, and prime numbers for encoding metamathematics}$$

(19)

$$\text{Gödel theorem} \implies \text{axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions}$$

(20)

$$sequenceSet((A)_{\mathbb{N}}, (A)) \iff (Amapinputn)((A)_{\mathbb{N}} = \{A(1), A(2), A(3), \dots\})$$

(21)

$$\text{TODO: define union, intersection, complement, etc.}$$

(22)

$$===== \text{ N O T } = \text{ U P D A T E D } =====$$

(23)

1.1 Axiomatic Set Theory

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (24)$$

$$\text{ZFC set theory} = \text{usual form of axiomatic set theory} \quad (25)$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \quad (26)$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \quad (27)$$

$$\in \text{ basis} \implies \{x, y\} = \{y, x\} \wedge \{x\} = \{x, x\} \quad (28)$$

$$\in \text{ and sets works following the 9 ZFC axioms:} \quad (29)$$

$$\forall_x \forall_y (x \in y \vee \neg(x \in y)) \# \text{ E: } \in \text{ is only a proposition on sets} \quad (30)$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \# \text{ E: existence of empty set} \quad (31)$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \# \text{ C: pair set construction} \quad (32)$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \# \text{ C: union set construction} \quad (33)$$

$$x = \{\{a\}, \{b\}\} \# \text{ from the pair set axiom} \quad (34)$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \quad (35)$$

$$\forall_x \exists!_y R(x, y) \# \text{ functional relation } R \quad (36)$$

$$\begin{aligned} \exists_i \forall_x \exists!_y R(x, y) \implies y \in i \# \text{ C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \mid P(y)\} \# \text{ Restricted Comprehension } \not\Rightarrow \{y \mid P(y)\} \# \text{ Universal Comprehension} \end{aligned} \quad (37)$$

$$\forall_{x \in m} P(x) = \forall_x (x \in m \implies P(x)) \# \text{ ignores out of scope } \neq \forall_x (x \in m \wedge P(x)) \# \text{ restricts entirety} \quad (38)$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} (n \subseteq m \implies n \subseteq \mathcal{P}(m)) \# \text{ C: existence of power set} \quad (39)$$

$$\exists_I (\emptyset \in I \wedge \forall_{x \in I} (\{x\} \in I)) \# \text{ I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}; I \cong \mathbb{N} \implies \mathbb{N} \text{ is a set} \quad (40)$$

$$\forall_x ((\emptyset \notin x \wedge x \cap x' = \emptyset) \implies \exists_y (\text{set of each } \mathbf{e} \in x)) \# \text{ C: axiom of choice} \quad (41)$$

$$\forall_x x \neq \emptyset \implies x \notin x \# \text{ F: axiom of foundation covers further paradoxes} \quad (42)$$

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (43)$$

1.2 Classification of sets

$$\begin{aligned} \text{space}((\text{set}, \text{structure}), ()) &\iff \text{structure}(\text{set}) \\ \# \text{ a space a set equipped with some structure} \\ \# \text{ various spaces can be studied through structure preserving maps between those spaces} \end{aligned} \quad (44)$$

$$\begin{aligned} \text{map}(\phi, (A, B)) &\iff \left(\forall_{a \in A} \exists!_{b \in B} (\phi(a, b)) \right) \vee \\ &\quad \left(\forall_{a \in A} \exists!_{b \in B} (b = \phi(a)) \right) \\ \# \text{ maps elements of a set to elements of another set} \end{aligned} \quad (45)$$

$$\text{domain}(A, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (46)$$

$$\text{codomain}(B, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (47)$$

$$\begin{aligned} \text{image}(B, (A, q, M, N)) &\iff \left(\text{map}(q, (M, N)) \wedge A \subseteq M \right) \wedge \\ &\quad \left(B = \{n \in N \mid \exists_{a \in A} (q(a) = n)\} \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \text{preimage}(A, (B, q, M, N)) &\iff \left(\text{map}(q, (M, N)) \wedge B \subseteq N \right) \wedge \\ &\quad \left(A = \{m \in M \mid \exists_{b \in B} (b = q(m))\} \right) \end{aligned} \quad (49)$$

$$\begin{aligned} \text{injection}(q, (M, N)) &\iff \left(\text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{u, v \in M} (q(u) = q(v) \implies u = v) \\ \# \text{ every } m \text{ has at most 1 image} \end{aligned} \quad (50)$$

$$\begin{aligned} \text{surjection}(q, (M, N)) &\iff \left(\text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{n \in N} \exists_{m \in M} (n = q(m)) \\ \# \text{ every } n \text{ has at least 1 preimage} \end{aligned} \quad (51)$$

$$\begin{aligned} \text{bijection}(q, (M, N)) &\iff \left(\text{injection}(q, (M, N)) \right) \wedge \\ &\quad \left(\text{surjection}(q, (M, N)) \right) \\ \# \text{ every unique } m \text{ corresponds to a unique } n \end{aligned} \quad (52)$$

$$\text{isomorphicSets}((A, B), ()) \iff \exists_{\phi} (\text{bijection}(\phi, (A, B))) \quad (53)$$

$$\text{infiniteSet}(S, ()) \iff \exists_{T \subseteq S} (\text{isomorphicSets}((T, S), ())) \quad (54)$$

$$\text{finiteSet}(S, ()) \iff \left(\neg \text{infiniteSet}(S, ()) \right) \vee (|S| \in \mathbb{N}) \quad (55)$$

$$\text{countablyInfinite}(S, ()) \iff \left(\text{infiniteSet}(S, ()) \right) \wedge \left(\text{isomorphicSets}((S, \mathbb{N}), ())) \quad (56)$$

$$\text{uncountablyInfinite}(S, ()) \iff (\text{infiniteSet}(S, ())) \wedge (\neg \text{isomorphicSets}((S, \mathbb{N}), ())) \quad (57)$$

$$\begin{aligned} \text{inverseMap}(q^{-1}, (q, M, N)) &\iff (\text{bijection}(q, (M, N))) \wedge \\ &\quad (\text{map}(q^{-1}, (N, M))) \wedge \\ &\quad \left(\forall_{n \in N} \exists!_{m \in M} (q(m) = n \implies q^{-1}(n) = m) \right) \end{aligned} \quad (58)$$

$$\begin{aligned} \text{mapComposition}(\phi \circ \psi, (\phi, \psi, A, B, C)) &\iff \text{map}(\psi, (A, B)) \wedge \text{map}(\phi, (B, C)) \wedge \\ &\quad \forall_{a \in A} (\phi \circ \psi(a) = \phi(\psi(a))) \end{aligned} \quad (59)$$

$$\begin{aligned} \text{equivalenceRelation}(\sim (\$1, \$2), (M)) &\iff (\forall_{m \in M} (m \sim m)) \wedge \\ &\quad (\forall_{m, n \in M} (m \sim n \implies n \sim m)) \wedge \\ &\quad (\forall_{m, n, p \in M} (m \sim n \wedge n \sim p \implies m \sim p)) \\ &\quad \# \text{ behaves as equivalences should} \end{aligned} \quad (60)$$

$$\begin{aligned} \text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) &\iff [m]_{\sim} = \{n \in M \mid n \sim m\} \\ &\quad \# \text{ set of elements satisfying the equivalence relation with } m \end{aligned} \quad (61)$$

$$\begin{aligned} (\text{THM}) : a \in [m]_{\sim} &\implies [a]_{\sim} = [m]_{\sim} ; [m]_{\sim} = [n]_{\sim} \vee [m]_{\sim} \cap [n]_{\sim} = \emptyset \\ &\quad \# \text{ equivalence class properties} \end{aligned} \quad (62)$$

$$\begin{aligned} \text{quotientSet}(M/\sim, (M, \sim)) &\iff M/\sim = \{\text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) \in \mathcal{P}(M) \mid m \in M\} \\ &\quad \# \text{ set of all equivalence classes} \end{aligned} \quad (63)$$

$$\begin{aligned} (\text{THM}) : (M, \sim, +) &\implies (\text{quotientSet}(M/\sim, (M, \sim)), +_{\sim}) \iff \forall_{[r], [s] \in M/\sim} \forall_{a \in [r]} \forall_{b \in [s]} ([r] +_{\sim} [s] = [a + b]) \\ &\quad \# \text{ a quotient set can inherit the operations on the original set if it is well-defined} \end{aligned} \quad (64)$$

$$\begin{aligned} (\text{THM}) : \text{axiom of choice} &\implies \forall_{[m]_{\sim} \in M/\sim} \exists_r (r \in [m]_{\sim}) \\ &\quad \# \text{ well-defined maps may be defined in terms of chosen representative elements } r \end{aligned} \quad (65)$$

1.3 Construction of number sets

$$\text{===== N O T = U P D A T E D =====} \quad (66)$$

$$\text{axiom of infinity} \implies \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\} \cong \mathbb{N} \quad (67)$$

$$\mathbb{N}^* = \mathbb{N} \setminus \{0\} \quad (68)$$

$$\text{addition} = \text{successor map: } \mathbb{N} \rightarrow \mathbb{N} = S(n) = \{n\} \# \text{ adds a layer of brackets} \quad (69)$$

$$\text{subtraction} = \text{predecessor map: } \mathbb{N}^* \rightarrow \mathbb{N} = P(n) = m \mid m \in n \# \text{ removes a layer of brackets} \quad (70)$$

$$S^0 = id ; n \in \mathbb{N}^* \implies S^n = S \circ S^{P(n)} \quad (71)$$

$$\mathbf{addition} = + : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} = +(m, n) = m + n = S^n(m) \quad (72)$$

$$S^x = id = S^0 \implies x = \mathbf{additive\ identity} = 0 \quad (73)$$

$$S^n(x) = 0 \implies x = \mathbf{additive\ inverse} \notin \mathbb{N} \# \text{ git gud smh -_-} \quad (74)$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim, \text{ s.t.: } (m, n) \sim (p, q) \iff m + q = p + n \# \text{ span } \mathbb{Z} \text{ using differences then group equal differences} \quad (75)$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \rightarrow [(n, 0)] \# \mathbb{N} \text{ embedded in } \mathbb{Z} \quad (76)$$

$$+_Z = [(m +_N p, n +_N q)] \# \text{ well-defined and consistent} \quad (77)$$

$$\mathbf{multiplication} \dots M^x = id \implies x = \mathbf{multiplicative\ identity} = 1 \dots \mathbf{multiplicative\ inverse} \notin \mathbb{N} \quad (78)$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^*) / \sim, \text{ s.t.: } (x, y) \sim (u, v) \iff x \cdot v = u \cdot y \quad (79)$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \rightarrow [(q, 1)] ; \dots \{x \mid x^2 = 2\} \notin \mathbb{Q} \quad (80)$$

$$\mathbb{R} = \mathbf{almost\ homomorphisms\ on\ } \mathbb{Z} / \sim \# \text{ http://blog.sigfpe.com/2006/05/defining-reals.html} \quad (81)$$

$$\text{===== N O T = U P D A T E D =====} \quad (82)$$

1.4 Topology

$$\text{topology}(\mathcal{O}, (M)) \iff (\mathcal{O} \subseteq \mathcal{P}(M)) \wedge (\emptyset, M \in \mathcal{O}) \wedge$$

$$\left((F \in \mathcal{O} \wedge |F| < |\mathbb{N}|) \implies \cap F \in \mathcal{O} \right) \wedge (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O})$$

topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.

arbitrary unions of open sets always result in an open set

open sets do not contain their boundaries and infinite intersections of open sets may approach and

induce boundaries resulting in a closed set (83)

$$\text{topologicalSpace}((M, \mathcal{O}), ()) \iff \text{topology}(\mathcal{O}, (M)) \quad (84)$$

$$\text{open}(S, (M, \mathcal{O})) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge (S \subseteq M) \wedge (S \in \mathcal{O})$$

an open set do not contains its own boundaries (85)

$$\begin{aligned} \text{closed}(S, (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\quad (S \subseteq M) \wedge (S \in \mathcal{P}(M) \setminus \mathcal{O}) \\ \# \text{ a closed set contains the boundaries an open set} \end{aligned} \quad (86)$$

$$\text{clopen}(S, (M, \mathcal{O})) \iff \left(\text{closed}(S, (M, \mathcal{O})) \right) \wedge \left(\text{open}(S, (M, \mathcal{O})) \right) \quad (87)$$

$$\begin{aligned} \text{neighborhood}(U, (a, \mathcal{O})) &\iff (a \in U \in \mathcal{O}) \\ \# \text{ another name for open set containing } a \end{aligned} \quad (88)$$

$$\begin{aligned} M = \{a, b, c, d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} &\implies \\ \left(\text{open}(X, (M, \mathcal{O})) \iff X = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} \right) &\wedge \\ \left(\text{closed}(Y, (M, \mathcal{O})) \iff Y = \{\emptyset, \{a, b, d\}, \{c, d\}, \{a, b\}, \{d\}, M\} \right) &\wedge \\ \left(\text{clopen}(Z, (M, \mathcal{O})) \iff Z = \{\emptyset, \{a, b\}, \{c, d\}, M\} \right) \end{aligned} \quad (89)$$

$$\text{chaoticTopology}(M) = \{0, M\} ; \text{discreteTopology} = \mathcal{P}(M) \quad (90)$$

1.5 Induced topology

$$\begin{aligned} \text{metric}(d(\$1, \$2), (M)) &\iff \left(\text{map} \left(d, \left(M \times M, \mathbb{R}_0^+ \right) \right) \right) \\ &\quad \left(\forall_{x, y \in M} (d(x, y) = d(y, x)) \right) \wedge \\ &\quad \left(\forall_{x, y \in M} (d(x, y) = 0 \iff x = y) \right) \wedge \\ &\quad \left(\forall_{x, y, z} \left(d(x, z) \leq d(x, y) + d(y, z) \right) \right) \\ \# \text{ behaves as distances should} \end{aligned} \quad (91)$$

$$\text{metricSpace}((M, d), ()) \iff \text{metric}(d, (M)) \quad (92)$$

$$\text{openBall}(B, (r, p, M, d)) \iff \left(\text{metricSpace}((M, d), ()) \right) \wedge (r \in \mathbb{R}^+, p \in M) \wedge (B = \{q \in M \mid d(p, q) < r\}) \quad (93)$$

$$\begin{aligned} \text{metricTopology}(\mathcal{O}, (M, d)) &\iff \left(\text{metricSpace}((M, d), ()) \right) \wedge \\ &\quad \left(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (\text{openBall}(B, (r, p, M, d)) \wedge B \subseteq U)\} \right) \\ \# \text{ every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \end{aligned} \quad (94)$$

$$\text{metricTopologicalSpace}((M, \mathcal{O}, d), ()) \iff \text{metricTopology}(\mathcal{O}, (M, d)) \quad (95)$$

$$\begin{aligned} \text{limitPoint}(p, (S, M, d)) &\iff (S \subseteq M) \wedge \forall_{r \in \mathbb{R}^+} \left(\text{openBall}(B, (r, p, M, d)) \cap S \neq \emptyset \right) \\ \# \text{ every open ball centered at } p &\text{ contains some intersection with } S \end{aligned} \quad (96)$$

$$\text{interiorPoint}(p, (S, M, d)) \iff (S \subseteq M) \wedge \left(\exists_{r \in \mathbb{R}^+} \left(\text{openBall}(B, (r, p, M, d)) \subseteq S \right) \right)$$

$$\# \text{ there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (97)$$

$$\text{closure}(\bar{S}, (S, M, d)) \iff \bar{S} = S \cup \{\text{limitPoint}(p, (S, M, d)) \mid p \in M\} \quad (98)$$

$$\text{dense}(S, (M, d)) \iff (S \subseteq M) \wedge \left(\forall_{p \in M} \left(p \in \text{closure}(\bar{S}, (S, M, d)) \right) \right) \\ \# \text{ every of point in } M \text{ is a point or a limit point of } S \quad (99)$$

$$\text{eucD}(d, (n)) \iff (\forall_{i \in \mathbb{N} \wedge i \leq n} (x_i \in \mathbb{R})) \wedge \left(d = \sqrt[2]{\sum_{i=1}^n x_i^2} \right) \quad (100)$$

$$\text{metricTopology} \left(\text{euclideanTopology}, \left(\mathbb{R}^n, \text{eucD}(d, (n)) \right) \right) \\ \text{===== N O T = U P D A T E D =====} \\ \text{L1: } \forall_{p \in U = \emptyset} (\dots) \implies \forall_p \left((p \in \emptyset) \implies \dots \right) \implies \forall_p (\text{False} \implies \dots) \implies \emptyset \in \mathcal{O}_{\text{euclidean}} \\ \text{L2: } \forall_{p \in \mathbb{R}^n} B(r, p, \mathbb{R}^n, d) \subseteq \mathbb{R}^n \implies M \in \mathcal{O}_{\text{euclidean}} \\ \text{L4: } C \subseteq \mathcal{O}_{\text{euclidean}} \implies \forall_{U \in C} \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (B_r(p) \subseteq U \subseteq \cup C) \implies \cup C \in \mathcal{O}_{\text{euclidean}} \\ \text{L3: } U, V \in \mathcal{O}_{\text{euclidean}} \implies p \in U \cap V \implies p \in U \wedge p \in V \implies \\ \exists_{r \in \mathbb{R}^+} B(r, p, \mathbb{R}^n, d) \wedge \exists_{s \in \mathbb{R}^+} B(s, p, \mathbb{R}^n, d) \implies \\ B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \subseteq U \wedge B(\min(r, s), q, \mathbb{R}^n, d) \subseteq V \implies \\ B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \in U \cap V \implies U \cap V \in \mathcal{O}_{\text{euclidean}} \\ \# \text{ natural topology for } \mathbb{R}^d \\ \# \text{ could fail on infinite sets since } \min \text{ could approach } 0 \\ \text{===== N O T = U P D A T E D =====} \quad (101)$$

$$\text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \iff \text{topology}(\mathcal{O}, (M)) \wedge (N \subseteq M) \wedge (\mathcal{O}|_N = \{U \cap N \mid U \in \mathcal{O}\}) \\ \# \text{ crops open sets outside } N \quad (102)$$

$$(\text{THM}) : \text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \wedge \text{topology}(\mathcal{O}|_N, (N)) \iff \\ \text{===== N O T = U P D A T E D =====} \\ \text{L1: } \emptyset \in \mathcal{O} \implies U = \emptyset \implies \emptyset \cap N = \emptyset \implies \emptyset \in \mathcal{O}|_N \\ \text{L2: } M \in \mathcal{O} \implies U = M \implies M \cap N = N \implies N \in \mathcal{O}|_N \\ \text{L3: } S, T \in \mathcal{O}|_N \implies \exists_{U \in \mathcal{O}} (S = U \cap N) \wedge \exists_{V \in \mathcal{O}} (T = V \cap N) \implies S \cap T = (U \cap N) \cap (V \cap N) \\ = (U \cap V) \cap N \wedge U \cap V \in \mathcal{O} \implies S \cap T \in \mathcal{O}|_N \\ \text{L4: } \text{TODO: EXERCISE} \\ \text{===== N O T = U P D A T E D =====} \quad (103)$$

$$\text{productTopology} \left(\mathcal{O}_{A \times B}, ((A, \mathcal{O}_A), (B, \mathcal{O}_B)) \right) \iff \left(\text{topology}(\mathcal{O}_A, (A)) \right) \wedge \left(\text{topology}(\mathcal{O}_B, (B)) \right) \wedge \\ (\mathcal{O}_{A \times B} = \{(a, b) \in A \times B \mid \exists_S (a \in S \in \mathcal{O}_A) \exists_T (b \in T \in \mathcal{O}_B)\}) \\ \# \text{ open in cross iff open in each} \quad (104)$$

1.6 Convergence

$$\text{sequence}(q, (M)) \iff \text{map}(q, (\mathbb{N}, M)) \quad (105)$$

$$\begin{aligned} \text{sequenceConvergesTo}((q, a), (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\text{sequence}(q, (M)) \right) \wedge (a \in M) \wedge \left(\forall U \in \mathcal{O} | a \in U \exists N \in \mathbb{N} \forall n > N (q(n) \in U) \right) \\ &\# \text{ each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \end{aligned} \quad (106)$$

(THM) : convergence generalizes to: the sequence $q: \mathbb{N} \rightarrow \mathbb{R}^d$ converges against $a \in \mathbb{R}^d$ in \mathcal{O}_S if:

$$\forall r > 0 \exists N \in \mathbb{N} \forall n > N (\|q(n) - a\| < r) \# \text{ distance based convergence} \quad (107)$$

1.7 Continuity

$$\begin{aligned} \text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}_M), ()) \right) \wedge \\ &\left(\text{topologicalSpace}((N, \mathcal{O}_N), ()) \right) \wedge \left(\forall V \in \mathcal{O}_N \left(\text{preimage}(A, (V, \phi, M, N)) \in \mathcal{O}_M \right) \right) \\ &\# \text{ preimage of open sets are open} \end{aligned} \quad (108)$$

$$\begin{aligned} \text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left(\text{inverseMap}(\phi^{-1}, (\phi, M, N)) \right) \\ &\left(\text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \wedge \left(\text{continuous}(\phi^{-1}, (N, \mathcal{O}_N, M, \mathcal{O}_M)) \right) \\ &\# \text{ structure preserving maps in topology, ability to share topological properties} \end{aligned} \quad (109)$$

$$\begin{aligned} \text{isomorphicTopologicalSpace}((M, \mathcal{O}_M), (N, \mathcal{O}_N), ()) &\iff \\ &\exists \phi \left(\text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \end{aligned} \quad (110)$$

1.8 Separation

$$\begin{aligned} T0Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U \in \mathcal{O} \left((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U) \right) \right) \\ &\# \text{ each pair of points has a neighborhood s.t. one is inside and the other is outside} \end{aligned} \quad (111)$$

$$\begin{aligned} T1Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V \left((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V) \right) \right) \\ &\# \text{ every point has a neighborhood that does not contain another point} \end{aligned} \quad (112)$$

$$\begin{aligned} T2Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V (U \cap V = \emptyset) \right) \\ &\# \text{ every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \end{aligned} \quad (113)$$

$$(THM) : T2Separate \implies T1Separate \implies T0Separate \quad (114)$$

1.9 Compactness

$$\begin{aligned} openCover(C, (M, \mathcal{O})) &\iff \left(topologicalSpace((M, \mathcal{O}), ()) \right) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M) \\ &\# \text{ collection of open sets whose elements cover the entire space} \end{aligned} \quad (115)$$

$$\begin{aligned} finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) &\iff (\tilde{C} \subseteq C) \wedge (openCover(C, (M, \mathcal{O}))) \wedge \\ &\left(openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge (finiteSet(\tilde{C}, ())) \\ &\# \text{ finite subset of a cover that is also a cover} \end{aligned} \quad (116)$$

$$\begin{aligned} compact((M, \mathcal{O}), ()) &\iff \left(topologicalSpace((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall C \subseteq \mathcal{O} \left(openCover(C, (M, \mathcal{O})) \implies \exists \tilde{C} \subseteq C \left(finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) \right) \right) \right) \\ &\# \text{ every covering of the space is represented by a finite number of nbhds} \end{aligned} \quad (117)$$

$$\begin{aligned} compactSubset(N, (M, \mathcal{O})) &\iff \left(compact((M, \mathcal{O}), ()) \right) \wedge \\ &\left(subsetTopology(\mathcal{O}|_N, (M, \mathcal{O}, N)) \right) \wedge \left(compact((N, \mathcal{O}|_N), ()) \right) \end{aligned} \quad (118)$$

$$\begin{aligned} bounded(N, (M, d)) &\iff \left(metricSpace((M, d), ()) \right) \wedge (N \subseteq M) \wedge \\ &\left(\exists r \in \mathbb{R}^+ \forall p, q \in N (d(p, q) < r) \right) \end{aligned} \quad (119)$$

$$\begin{aligned} &(THM) \text{ Heine-Borel thm.: } metricTopologicalSpace((M, \mathcal{O}_d, d), ()) \implies \\ &\forall S \subseteq M \left(\left(closed(S, (M, \mathcal{O}_d)) \wedge bounded(S, (M, \mathcal{O}_d)) \right) \iff compactSubset(S, (M, \mathcal{O}_d)) \right) \\ &\# \text{ when metric topologies are involved, compactness is equivalent to being closed and bounded} \end{aligned} \quad (120)$$

1.10 Paracompactness

$$\begin{aligned} openRefinement(\tilde{C}, (C, M, \mathcal{O})) &\iff \left(openCover(C, (M, \mathcal{O})) \right) \wedge \left(openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge \\ &\left(\forall \tilde{U} \in \tilde{C} \exists U \in C (\tilde{U} \subseteq U) \right) \\ &\# \text{ a refined cover can be constructed by removing the excess nbhds and points that lie outside the space} \end{aligned} \quad (121)$$

$$(THM) : finiteSubcover \implies openRefinement \quad (122)$$

$$\begin{aligned} locallyFinite(C, (M, \mathcal{O})) &\iff \left(openCover(C, (M, \mathcal{O})) \right) \wedge \\ &\forall p \in M \exists U \in \mathcal{O} | p \in U \left(finiteSet(\{U_c \in C | U \cap U_c \neq \emptyset\}, ()) \right) \\ &\# \text{ each point has a neighborhood that intersects with only finitely many sets in the cover} \end{aligned} \quad (123)$$

$$\begin{aligned} & \text{paracompact}((M, \mathcal{O}), ()) \iff \\ \forall_C \left(\text{openCover}(C, (M, \mathcal{O})) \implies \exists_{\tilde{C}} \left(\text{locallyFinite} \left(\text{openRefinement}(\tilde{C}, (C, M, \mathcal{O})), (M, \mathcal{O}) \right) \right) \right) \\ & \# \text{ every open cover has a locally finite open refinement} \end{aligned} \quad (124)$$

$$(\text{THM}) : \text{metricTopologicalSpace} \implies \text{paracompact} \quad (125)$$

$$\text{===== NOT UPDATED =====} \quad (126)$$

$$\begin{aligned} & \text{partitionOfUnitySubjCover}(\mathcal{F}, (C, M, \mathcal{O})) \iff \left(\text{locallyFinite}(C, (M, \mathcal{O})) \right) \wedge (f \in \mathcal{F}) \wedge \\ & \left(\text{continuous} \left(f, \left(M, \mathcal{O}, [0, 1], \text{subsetTopology}(\mathcal{O}|_{[0, 1]}, ([0, 1], \mathbb{R}, \text{euclideanTopology})) \right) \right) \right) \wedge \\ & \left(\exists_{U_f \in C} \forall_{p \in M} (f(p) \neq 0 \implies p \in U_f) \right) \wedge \\ & \left(\forall_{p \in M} \exists_{U \in \mathcal{O}} \exists_{p \in U} ((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\}) \right) \wedge \\ & \left(\text{locallyFinite}(C, M, \mathcal{O}) \implies \text{finiteSet}((f_U)_n, ()) \right) \wedge \\ & \left(\forall_{p \in M} \exists_{U \in \mathcal{O}} \exists_{p \in U} \left(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1 \right) \right) \\ & \# \text{ useful for defining integrals between overlapping neighborhoods} \end{aligned} \quad (127)$$

$$\begin{aligned} & T2Separate((M, \mathcal{O}), ()) \implies \left(\text{paracompact}((M, \mathcal{O}), ()) \right) \iff \\ & \forall_C \left(\text{openCover}(C, (M, \mathcal{O})) \implies \text{partitionOfUnitySOTCover}(\mathcal{F}, (C, M, \mathcal{O})) \right) \end{aligned} \quad (128)$$

$$\text{===== NOT UPDATED =====} \quad (129)$$

1.11 Connectedness and path-connectedness

$$\begin{aligned} & \text{connected}((M, \mathcal{O}), ()) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \left(\neg \exists_{A, B \in \mathcal{O} \setminus \emptyset} (A \cap B \neq \emptyset \wedge A \cup B = M) \right) \\ & \# \text{ if there is some covering of the space that does not intersect} \end{aligned} \quad (130)$$

$$\begin{aligned} & (\text{THM}) : \neg \text{connected} \left(\left(\mathbb{R} \setminus \{0\}, \text{subsetTopology}(\mathcal{O}_{\text{euclidean}}|_{\mathbb{R} \setminus \{0\}}, (\mathbb{R}, \text{euclideanTopology}, \mathbb{R} \setminus \{0\})) \right), () \right) \\ & \iff \left(A = (-\infty, 0) \in \mathcal{O}_{\text{euclidean}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \left(B = (0, \infty) \in \mathcal{O}_{\text{euclidean}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \\ & (A \cap B = \emptyset) \wedge (A \cup B = \mathbb{R} \setminus \{0\}) \end{aligned} \quad (131)$$

$$(\text{THM}) : \text{connected}((M, \mathcal{O}), ()) \iff \forall_{S \in \mathcal{O}} \left(\text{clopen}(S, (M, \mathcal{O})) \implies (S = \emptyset \vee S = M) \right) \quad (132)$$

$$\text{pathConnected}((M, \mathcal{O}), ()) \iff \left(\text{subsetTopology}(\mathcal{O}_{\text{euclidean}}|_{[0, 1]}, (\mathbb{R}, \text{euclideanTopology}, [0, 1])) \right) \wedge$$

$$\left(\forall_{p,q \in M} \exists_{\gamma} \left(\text{continuous} \left(\gamma, ([0,1], \mathcal{O}_{\text{euclidean}}|_{[0,1]}, M, \mathcal{O}) \right) \wedge \gamma(0)=p \wedge \gamma(1)=q \right) \right) \quad (133)$$

$$(\text{THM}) : \text{pathConnected} \implies \text{connected} \quad (134)$$

1.12 Homotopic curve and the fundamental group

$$\text{===== NOT UPDATED =====} \quad (135)$$

$$\begin{aligned} \text{homotopic}(\sim, (\gamma, \delta, M, \mathcal{O})) &\iff (\text{map}(\gamma, ([0,1], M)) \wedge \text{map}(\delta, ([0,1], M))) \wedge \\ &\quad (\gamma(0)=\delta(0) \wedge \gamma(1)=\delta(1)) \wedge \\ (\exists_H \forall_{\lambda \in [0,1]} (\text{continuous}(H, ([0,1] \times [0,1], \mathcal{O}_{\text{euclidean}^2}|_{[0,1] \times [0,1]}, (M, \mathcal{O})) \wedge H(0, \lambda) = \gamma(\lambda) \wedge H(1, \lambda) = \delta(\lambda))) & \\ \# H \text{ is a continuous deformation of one curve into another} & \end{aligned} \quad (136)$$

$$\text{homotopic}(\sim) \implies \text{equivalenceRelation}(\sim) \quad (137)$$

$$\text{loopSpace}(\mathcal{L}_p, (p, M, \mathcal{O})) \iff \mathcal{L}_p = \{\text{map}(\gamma, ([0,1], M)) \mid \text{continuous}(\gamma) \wedge \gamma(0)=\gamma(1)\} \quad (138)$$

$$\begin{aligned} \text{concatination}(\star, (p, \gamma, \delta)) &\iff (\gamma, \delta \in \text{loopSpace}(\mathcal{L}_p)) \wedge \\ (\forall_{\lambda \in [0,1]} ((\gamma \star \delta)(\lambda) = \begin{cases} \gamma(2\lambda) & 0 \leq \lambda < 0.5 \\ \delta(2\lambda - 1) & 0.5 \leq \lambda \leq 1 \end{cases})) & \end{aligned} \quad (139)$$

$$\begin{aligned} \text{group}((G, \bullet), ()) &\iff (\text{map}(\bullet, (G \times G, G))) \wedge \\ &\quad (\forall_{a,b \in G} (a \bullet b \in G)) \\ &\quad (\forall_{a,b,c \in G} ((a \bullet b) \bullet c = a \bullet (b \bullet c))) \\ &\quad (\exists_e \forall_{a \in G} (e \bullet a = a = a \bullet e)) \wedge \\ &\quad (\forall_{a \in G} \exists_{a^{-1}} (a \bullet a^{-1} = e = a^{-1} \bullet a)) \\ \# \text{ characterizes symmetry of a set structure} & \end{aligned} \quad (140)$$

$$\text{isomorphic}(\cong, (X, \odot), (Y, \ominus)) \iff \exists_f \forall_{a,b \in X} (\text{bijection}(f, (X, Y)) \wedge f(a \odot b) = f(a) \ominus f(b)) \quad (141)$$

$$\begin{aligned} \text{fundamentalGroup}((\pi_{1,p}, \bullet), (p, M, \mathcal{O})) &\iff (\pi_{1,p} = \mathcal{L}_p / \sim) \wedge \\ &\quad (\text{map}(\bullet, (\pi_{1,p} \times \pi_{1,p}, \pi_{1,p}))) \wedge \\ &\quad (\forall_{A,B \in \pi_{1,p}} ([A] \bullet [B] = [A \star B])) \wedge \\ &\quad (\text{group}((\pi_{1,p}, \bullet), ())) \\ \# \text{ an equivalence class of all loops induced from the homotopic equivalence relation} & \end{aligned} \quad (142)$$

$$\text{fundamentalGroup}_1 \not\cong \text{fundamentalGroup}_2 \implies \text{topologicalSpace}_1 \not\cong \text{topologicalSpace}_2 \quad (143)$$

$$\text{there exists no known list of topological properties that can imply homeomorphisms} \quad (144)$$

$$\text{CONTINUE @ Lecture 6: manifolds} \quad (145)$$

$$\text{===== NOT UPDATED =====} \quad (146)$$

1.13 Measure theory

$$\begin{aligned}
\text{sigmaAlgebra}(\sigma, (M)) &\iff (M \neq \emptyset) \wedge (\sigma \subseteq \mathcal{P}(M)) \wedge \\
&\quad (M \in \sigma) \wedge \left(\forall A \in \sigma (M \setminus A \in \sigma) \right) \wedge \\
&\quad \left(\left(A \subseteq \sigma \wedge \neg \text{uncountablyInfinite}(A, ()) \right) \implies \cup A \in \sigma \right) \\
\# \text{ behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu &\quad (147)
\end{aligned}$$

$$\text{measurableSpace}((M, \sigma), ()) \iff \text{sigmaAlgebra}(\sigma, (M)) \quad (148)$$

$$\text{measurableSet}(A, (M, \sigma)) \iff \left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge (A \in \sigma) \quad (149)$$

$$\begin{aligned}
\text{measure}(\mu, (M, \sigma)) &\iff \left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge \left(\text{map} \left(\mu, \left(\sigma, \left(\mathbb{R}^+ \right)_0 \right) \right) \right) \wedge (\mu(\emptyset) = 0) \wedge \\
&\quad \left(\left((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} \forall j \in \mathbb{N} \setminus \{i\} (A_i \cap A_j = \emptyset) \right) \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) = \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \\
\# \text{ enforces meaningful concepts of measures such as precise additivity} &\quad (150)
\end{aligned}$$

$$\begin{aligned}
&(\text{THM}) : \text{measure}(\mu, (M, \sigma)) \implies \\
&\quad \left(\forall A, B \in \sigma (A \subseteq B \implies \mu(A) \leq \mu(B)) \right) \wedge \\
&\quad \left((A)_{\mathbb{N}} \subseteq \sigma \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) \leq \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \wedge \\
&\quad \left(((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (B_i \subseteq B_{i+1}) \wedge B = \cup (B)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(B_n)) = \mu(B) \right) \wedge \\
&\quad \left(((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (C_{i+1} \subseteq C_i) \wedge C = \cap (C)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(C_n)) = \mu(C) \right) \\
\# \text{ immediate implications of the measurable set } A \in \sigma \text{ axioms and the measure } \mu \text{ axioms} &\quad (151)
\end{aligned}$$

$$\text{measureSpace}((M, \sigma, \mu), ()) \iff \text{measure}(\mu, (M, \sigma)) \quad (152)$$

$$\begin{aligned}
\text{finiteMeasure}(\mu, (M, \sigma)) &\iff \left(\text{measure}(\mu, (M, \sigma)) \right) \wedge \\
&\quad \left(\exists (A)_{\mathbb{N}} \subseteq \sigma \left(\cup (A)_{\mathbb{N}} = M \wedge \forall n \in \mathbb{N} (\mu(A_n) < \infty) \right) \right) \\
&\quad (153)
\end{aligned}$$

$$\begin{aligned}
\text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) &\iff \left(G = \{ \sigma \subseteq \mathcal{P}(M) \mid \text{sigmaAlgebra}(\sigma, (M)) \} \right) \wedge (\sigma(\zeta) = \cap G) \\
\# \text{ smallest } \sigma\text{-algebra containing the generating set } \zeta &\quad (154)
\end{aligned}$$

$$(\text{THM}) : \exists \zeta \subseteq M \left(\text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) = \text{sigmaAlgebra}(\sigma, (M)) \right) \quad (155)$$

$$\begin{aligned}
\text{borelSigmaAlgebra}(\sigma(\mathcal{O}), (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\
&\quad \left(\text{generatedSigmaAlgebra}(\sigma(\mathcal{O}), (\mathcal{O}, M)) \right) \\
\# \sigma\text{-algebra induced by a topology} &\quad (156)
\end{aligned}$$

$$euclideanSigma(\sigma_s, ()) \iff \left(borelSigmaAlgebra \left(\sigma_s, \left(\mathbb{R}^d, euclideanTopology \right) \right) \right) \quad (157)$$

$$\begin{aligned} lebesgueMeasure(\lambda, ()) \iff & \left(measure \left(\lambda, \left(\mathbb{R}^d, euclideanSigma \right) \right) \right) \wedge \\ & \left(\lambda \left(\times_{i=1}^d ([a_i, b_i]) \right) = \sum_{i=1}^d \left(\sqrt[d]{(a_i - b_i)^2} \right) \right) \\ & \# \text{ natural measure for } \mathbb{R}^d \end{aligned} \quad (158)$$

$$\begin{aligned} measurableMap(f, (M, \sigma_M, N, \sigma_N)) \iff & \left(measurableSpace((M, \sigma_M), ()) \right) \wedge \\ & \left(measurableSpace((N, \sigma_N), ()) \right) \wedge \left(\forall B \in \sigma_N \left(preimage(A, (B, f, M, N)) \in \sigma_M \right) \right) \\ & \# \text{ preimage of measurable sets are measurable} \end{aligned} \quad (159)$$

$$\begin{aligned} pushForwardMeasure(f \star \lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \iff & \left(measureSpace((M, \sigma_M, \mu_M), ()) \right) \wedge \\ & \left(measurableSpace((N, \sigma_N), ()) \right) \wedge \left(measurableMap(f, (M, \sigma_M, N, \sigma_N)) \right) \wedge \\ & \left(\forall B \in N \left(f \star \lambda_M(B) = \mu_M \left(preimage(A, (B, f, M, N)) \right) \right) \right) \wedge \left(measure(f \star \lambda_M, (N, \sigma_N)) \right) \\ & \# \text{ natural construction of a measure based primarily on measurable map} \end{aligned} \quad (160)$$

$$nullSet(A, (M, \sigma, \mu)) \iff \left(measureSpace((M, \sigma, \mu), ()) \right) \wedge (A \in \sigma) \wedge (\mu(A) = 0) \quad (161)$$

$$\begin{aligned} almostEverywhere(p, (M, \sigma, \mu)) \iff & \left(measureSpace((M, \sigma, \mu), ()) \right) \wedge \left(predicate(p, (M)) \right) \wedge \\ & \left(\exists A \in \sigma \left(nullSet(A, (M, \sigma, \mu)) \implies \forall n \in M \setminus A (p(n)) \right) \right) \\ & \# \text{ the predicate holds true for all points except the points in the null set} \\ & \# \text{ in terms of measure, almost nothing is not equivalent to nothing} \end{aligned} \quad (162)$$

1.14 Lebesgue integration

$$simpleTopology(\mathcal{O}_{simple}, ()) \iff \mathcal{O}_{simple} = subsetTopology \left(\mathcal{O}|_{\mathbb{R}_0^+}, \left(\mathbb{R}, euclideanTopology, \mathbb{R}_0^+ \right) \right) \quad (163)$$

$$simpleSigma(\sigma_{simple}, ()) \iff borelSigmaAlgebra \left(\sigma_{simple}, \left(\mathbb{R}_0^+, simpleTopology \right) \right) \quad (164)$$

$$simpleFunction(s, (M, \sigma)) \iff \left(measurableMap \left(s, \left(M, \sigma, \mathbb{R}_0^+, simpleSigma \right) \right) \right) \wedge$$

$$\left(\text{finiteSet} \left(\text{image} \left(B, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right)$$

if the map takes on finitely many values on \mathbb{R}_0^+

(165)

$$\text{characteristicFunction}(X_A, (A, M)) \iff (A \subseteq M) \wedge \left(\text{map}(X_A, (M, \mathbb{R})) \right) \wedge$$

$$\left(\forall_{m \in M} \left(X_A(m) = \begin{cases} 1 & m \in A \\ 0 & m \notin A \end{cases} \right) \right)$$

(166)

$$(\text{THM}) : \text{simpleFunction}(s, (M, \sigma_M)) \implies$$

$$\left(\text{finiteSet} \left(\text{image} \left(Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right) \wedge$$

$$\left(\text{characteristicFunction}(X_A, (A, M)) \right) \wedge \left(\forall_{m \in M} \left(s(m) = \sum_{z \in Z} \left(z \cdot X_{\text{preimage}(A, (\{z\}, s, M, \mathbb{R}_0^+))}(m) \right) \right) \right)$$

(167)

$$\text{exEuclideanSigma}(\overline{\sigma}_s, ()) \iff \overline{\sigma}_s = \{A \subseteq \overline{\mathbb{R}} \mid A \cap R \in \text{euclideanSigma}\}$$

ignores $\pm\infty$ to preserve the points in the domain of the measurable map

(168)

$$\text{nonNegIntegrable}(f, (M, \sigma)) \iff \left(\text{measurableMap} \left(f, (M, \sigma, \overline{\mathbb{R}}, \text{exEuclideanSigma}) \right) \right) \wedge$$

$$\left(\forall_{m \in M} (f(m) \geq 0) \right)$$

(169)

$$\text{nonNegIntegral} \left(\int_M (f d\mu), (f, M, \sigma, \mu) \right) \iff \left(\text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge$$

$$\left(\text{measureSpace} \left((\overline{\mathbb{R}}, \text{exEuclideanSigma}, \text{lebesgueMeasure}), () \right) \right) \wedge$$

$$\left(\text{nonNegIntegrable}(f, (M, \sigma)) \right) \wedge \left(\int_M (f d\mu) = \sup \left(\left\{ \sum_{z \in Z} \left(z \cdot \mu \left(\text{preimage} \left(A, (\{z\}, s, M, \mathbb{R}_0^+) \right) \right) \right) \right\} \right) \mid \right.$$

$$\left. \forall_{m \in M} (s(m) \leq f(m)) \wedge \text{simpleFunction}(s, (M, \sigma)) \wedge \text{finiteSet} \left(\text{image} \left(Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right\})$$

lebesgue measure on z reduces to z

(170)

$$\text{explicitIntegral} \iff \int (f(x)\mu(dx)) = \int (f d\mu)$$

alternative notation for lebesgue integrals

(171)

$$(\text{THM}) : \text{nonNegIntegral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies$$

$$\begin{aligned}
\text{(THM) Markov inequality: } & \left(\forall_{z \in \mathbb{R}_0^+} \left(\int (f d\mu) \geq z \cdot \mu \left(\text{preimage} \left(A, ([z, \infty), f, M, \overline{\mathbb{R}}) \right) \right) \right) \right) \wedge \\
& \left(\text{almostEverywhere}(f = g, (M, \sigma, \mu)) \implies \int (f d\mu) = \int (g d\mu) \right) \\
& \left(\int (f d\mu) = 0 \implies \text{almostEverywhere}(f = 0, (M, \sigma, \mu)) \right) \wedge \\
& \left(\int (f d\mu) \leq \infty \implies \text{almostEverywhere}(f < \infty, (M, \sigma, \mu)) \right) \\
& (172)
\end{aligned}$$

$$\begin{aligned}
\text{(THM) Mono. conv.: } & \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{R}, \text{exEuclideanSigma}) \right) \wedge 0 \leq f_{n-1} \leq f_n \} \right) \wedge \\
& \left(\text{map} \left(f, (M, \overline{\mathbb{R}}) \right) \right) \wedge \left(\forall_{m \in M} \left(f(m) = \sup(f_n(m) \mid f_n \in (f)_{\mathbb{N}}) \right) \right) \implies \left(\lim_{n \rightarrow \infty} \left(\int_M (f_n d\mu) \right) = \int_M (f d\mu) \right) \\
& \# \text{ lengths now depend on } M, \sigma \text{ and limits can be pulled in or out of an integral} \\
& (173)
\end{aligned}$$

$$\begin{aligned}
\text{(THM) : } & \text{nonNegIntegral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\
& \left(\forall_{\alpha \in \mathbb{R}_0^+} \left(\int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \right) \\
& \# \text{ integral acts linearly and commutes finite summations} \\
& (174)
\end{aligned}$$

$$\begin{aligned}
\text{(THM) : } & \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{R}, \text{exEuclideanSigma}) \right) \wedge 0 \leq f_n \} \right) \implies \\
& \left(\int \left(\left(\sum_{n=1}^{\infty} (f_n) \right) d\mu \right) = \sum_{n=1}^{\infty} \left(\int (f_n d\mu) \right) \right) \\
& \# \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n \rightarrow \infty} \sum_{i=1}^n f_n \text{ since } f_n \geq 0 \text{ and it commutes with integral from monotone conv.} \\
& (175)
\end{aligned}$$

$$\begin{aligned}
\text{integrable}(f, (M, \sigma)) & \iff \left(\text{measurableMap} \left(f, (M, \sigma, \overline{\mathbb{R}}, \text{exEuclideanSigma}) \right) \right) \wedge \\
& \left(\forall_{m \in M} \left(f(m) = \max(f(m), 0) - \max(0, -f(m)) \right) \right) \wedge \\
& \left(\text{measureSpace}(M, \sigma, \mu) \implies \left(\int (\max(f(m), 0) d\mu) < \infty \wedge \int (\max(0, -f(m)) d\mu) < \infty \right) \right) \\
& \# \text{ extra condition prevents the occurrence of the indeterminate } \infty - \infty \\
& (176)
\end{aligned}$$

$$\begin{aligned}
\text{integral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) & \iff \left(\text{nonNegIntegral} \left(\int (f^+ d\mu), (\max(f, 0), M, \sigma, \mu) \right) \right) \wedge \\
& \left(\text{nonNegIntegral} \left(\int (f^- d\mu), (\max(0, -f), M, \sigma, \mu) \right) \right) \wedge \left(\text{integrable}(f, (M, \sigma)) \right) \wedge
\end{aligned}$$

$$\left(\int (f d\mu) = \int (f^+ d\mu) - \int (f^- d\mu) \right)$$

arbitrary integral in terms of nonnegative integrals
(177)

$$(\text{THM}) : \left(\text{map}(f, (M, \mathbb{C})) \right) \implies \left(\int (f d\mu) = \int (\text{Re}(f) d\mu) - \int (\text{Im}(f) d\mu) \right)$$

(178)

$$\begin{aligned} (\text{THM}) : \text{integral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{integral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\ \left(\text{almostEverywhere}(f \leq g, (M, \sigma, \mu)) \implies \int (f d\mu) \leq \int (g d\mu) \right) \wedge \\ \left(\forall_{m \in M} (f(m), g(m), \alpha \in \mathbb{R}) \implies \int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \end{aligned}$$

(179)

$$\begin{aligned} (\text{THM}) \text{ Dominant convergence: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{\mathbb{R}}, \text{exEuclideanSigma}) \right) \} \right) \wedge \\ \left(\text{map}(f, (M, \overline{\mathbb{R}})) \right) \wedge \left(\text{almostEverywhere} \left(f(m) = \lim_{n \rightarrow \infty} (f_n(m)), (M, \sigma, \mu) \right) \right) \wedge \\ \left(\text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \right) \wedge \left(\left| \int (g d\mu) \right| < \infty \right) \wedge \left(\text{almostEverywhere}(|f_n| \leq g, (M, \sigma, \mu)) \right) \\ \# \text{ if all } f_n(m) \text{ are bounded by some integrable } |g(m)| \implies \\ \# \text{ then all } f_n(m) \text{ including } f \text{ satisfy bounded and integrable properties} \\ \left(\forall_{\phi \in \{f\} \cup (f)_{\mathbb{N}}} \left(\text{integrable}(\phi, (M, \sigma)) \right) \right) \wedge \left(\lim_{n \rightarrow \infty} \left(\int (|f_n - f| d\mu) = 0 \right) \right) \wedge \left(\lim_{n \rightarrow \infty} \left(\int (f_n d\mu) \right) = \int (f d\mu) \right) \end{aligned}$$

(180)

1.15 Vector space and structures

$$\begin{aligned} \text{vectorSpace}((V, +, \cdot), ()) \iff \left(\text{map}(+, (V \times V, V)) \right) \wedge \left(\text{map}(\cdot, (\mathbb{R} \times V, V)) \right) \wedge \\ \left(\forall_{v, w \in V} (v + w = w + v) \right) \wedge \\ \left(\forall_{v, w, x \in V} ((v + w) + x = v + (w + x)) \right) \wedge \\ \left(\exists \mathbf{0} \in V \forall v \in V (v + \mathbf{0} = v) \right) \wedge \\ \left(\forall v \in V \exists -v \in V (v + (-v) = \mathbf{0}) \right) \wedge \\ \left(\forall_{a, b \in \mathbb{R}} \forall v \in V (a(b \cdot v) = (ab) \cdot v) \right) \wedge \\ \left(\exists 1 \in \mathbb{R} \forall v \in V (1 \cdot v = v) \right) \wedge \\ \left(\forall_{a, b \in \mathbb{R}} \forall v \in V ((a + b) \cdot v = a \cdot v + b \cdot v) \right) \wedge \\ \left(\forall_{a \in \mathbb{R}} \forall_{v, w \in V} (a \cdot (v + w) = a \cdot v + a \cdot w) \right) \\ \# \text{ behaves similar as vectors should i.e., additive, scalable, linear distributive} \end{aligned}$$

(181)

$$\begin{aligned} \text{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \iff \left(\text{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\text{map}(\langle \$1, \$2 \rangle, (V \times V, \mathbb{R})) \right) \wedge \\ \left(\forall_{v, w \in V} (\langle v, w \rangle = \langle w, v \rangle) \right) \wedge \end{aligned}$$

$$\begin{aligned}
& \left(\forall_{v,w,x \in V} \forall_{a,b \in \mathbb{R}} (\langle av + bw, x \rangle = a\langle v, x \rangle + b\langle w, x \rangle) \right) \wedge \\
& \left(\forall_{v \in V} (\langle v, v \rangle \geq 0) \right) \wedge \left(\forall_{v \in V} (\langle v, v \rangle = 0 \iff v = \mathbf{0}) \right) \\
& \# \text{ the sesquilinear or l.5 linear map inner product provides info. on distance and orthogonality}
\end{aligned} \tag{182}$$

$$\textit{innerProductSpace}((V, +, \cdot, \langle \$1, \$2 \rangle), ()) \iff \textit{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \tag{183}$$

$$\begin{aligned}
\textit{vectorNorm}(\| \$1 \|, (V, +, \cdot)) \iff & \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\textit{map} \left(\| \$1 \|, (V, \mathbb{R}_0^+) \right) \right) \wedge \\
& \left(\forall_{v \in V} (\|v\| = 0 \iff v = \mathbf{0}) \right) \wedge \\
& \left(\forall_{v \in V} \forall_{s \in \mathbb{R}} (\|sv\| = |s| \|v\|) \right) \wedge \\
& \left(\forall_{v,w \in V} (\|v+w\| \leq \|v\| + \|w\|) \right) \\
& \# \text{ magnitude of a point in a vector space}
\end{aligned} \tag{184}$$

$$\textit{normedVectorSpace}((V, +, \cdot, \| \$1 \|), ()) \iff \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\textit{vectorNorm}(\| \$1 \|, (V, +, \cdot)) \right) \tag{185}$$

$$\begin{aligned}
\textit{vectorMetric}(d(\$1, \$2), (V, +, \cdot)) \iff & \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \\
& \left(\textit{metric}(d(\$1, \$2), (V)) \vee \left(\textit{map} \left(d, (V \times V, \mathbb{R}_0^+) \right) \right) \right) \\
& \left(\forall_{x,y \in V} (d(x, y) = d(y, x)) \right) \wedge \\
& \left(\forall_{x,y \in V} (d(x, y) = 0 \iff x = y) \right) \wedge \\
& \left(\forall_{x,y,z \in V} (d(x, z) \leq d(x, y) + d(y, z)) \right) \\
& \# \text{ behaves as distances should}
\end{aligned} \tag{186}$$

$$\begin{aligned}
\textit{metricVectorSpace}((V, +, \cdot, d(\$1, \$2)), ()) \iff & \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \\
& \left(\textit{vectorMetric}(d(\$1, \$2), (V, +, \cdot)) \right)
\end{aligned} \tag{187}$$

$$\begin{aligned}
\textit{innerProductNorm}(\| \$1 \|, (V, +, \cdot, \langle \$1, \$2 \rangle)) \iff & \left(\textit{innerProductSpace}((V, +, \cdot, \langle \$1, \$2 \rangle), ()) \right) \wedge \\
& \left(\forall_{v \in V} (\|v\| = \sqrt[2]{\langle v, v \rangle}) \implies \textit{vectorNorm}(\| \$1 \|, (V, +, \cdot)) \right)
\end{aligned} \tag{188}$$

$$\begin{aligned}
\textit{normInnerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot, \| \$1 \|)) \iff & \left(\textit{normedVectorSpace}((V, +, \cdot, \| \$1 \|), ()) \right) \wedge \\
& \left(\forall_{u,v \in V} (2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2) \right) \wedge \\
& \left(\forall_{v,w \in V} \left(\langle v, w \rangle = \frac{\|v+w\|^2 - \|v-w\|^2}{4} \right) \implies \textit{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \right)
\end{aligned} \tag{189}$$

$$\textit{normMetric}(d(\$1, \$2), (V, +, \cdot, \| \$1 \|)) \iff \left(\textit{normedVectorSpace}((V, +, \cdot, \| \$1 \|), ()) \right) \wedge$$

$$\left(\forall_{v,w \in V} (d(v,w) = ||v-w||) \implies \text{vectorMetric}(d(\$1,\$2), (V, +, \cdot)) \right) \quad (190)$$

$$\begin{aligned} \text{metricNorm}\left(||\$1||, (V, +, \cdot, d(\$1,\$2))\right) &\iff \left(\text{metricVectorSpace}\left((V, +, \cdot, d(\$1,\$2)), ()\right) \right) \wedge \\ &\left(\forall_{u,v,w \in V} \forall_{s \in \mathbb{R}} \left(d(s(u+w), s(v+w)) = |s|d(u,v) \right) \right) \wedge \\ &\left(\forall_{v \in V} (||v|| = d(v, \mathbf{0})) \implies \text{vectorNorm}(|\$1|, (V, +, \cdot)) \right) \end{aligned} \quad (191)$$

$$\begin{aligned} \text{orthogonal}\left((v,w), (V, +, \cdot, \langle \$1, \$2 \rangle)\right) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \wedge \\ &(v, w \in V) \wedge (\langle v, w \rangle = 0) \\ &\# \text{ the inner product also provides info. on orthogonality} \end{aligned} \quad (192)$$

$$\begin{aligned} \text{normal}\left(v, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \wedge (v \in V) \wedge (\langle v, v \rangle = 1) \\ &\# \text{ the vector has unit length} \end{aligned} \quad (193)$$

$$\text{(THM) Cauchy-Schwarz inequality: } \forall_{v,w \in V} (\langle v, w \rangle \leq ||v|| ||w||) \quad (194)$$

$$\text{basis}((b)_n, (V, +, \cdot)) \iff \left(\text{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\forall_{v \in V} \exists_{(a)_n \in \mathbb{R}^n} \left(v = \sum_{i=1}^n (a_i b_i) \right) \right) \quad (195)$$

$$\begin{aligned} \text{orthonormalBasis}((b)_n, (V, +, \cdot, \langle \$1, \$2 \rangle)) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \wedge \\ &\left(\text{basis}((b)_n, (V, +, \cdot)) \right) \wedge \left(\forall_{v \in (b)_n} \left(\text{normal}(v, (V, +, \cdot, \langle \$1, \$2 \rangle)) \right) \right) \wedge \\ &\left(\forall_{v \in (b)_n} \forall_{w \in (b)_n \setminus \{v\}} \left(\text{orthogonal}((v,w), (V, +, \cdot, \langle \$1, \$2 \rangle)) \right) \right) \end{aligned} \quad (196)$$

1.16 Subvector space

$$\text{subspace}((U, \circ), (V, \circ)) \iff \left(\text{space}((V, \circ), ()) \right) \wedge (U \subseteq V) \wedge \left(\text{space}((U, \circ), ()) \right) \quad (197)$$

$$\begin{aligned} \text{subspaceSum}(U+W, (U, W, V, +)) &\iff \left(\text{subspace}((U, +), (V, +)) \right) \wedge \left(\text{subspace}((W, +), (V, +)) \right) \wedge \\ &(U+W = \{u+w \mid u \in U \wedge w \in W\}) \end{aligned} \quad (198)$$

$$\text{subspaceDirectSum}(U \oplus W, (U, W, V, +)) \iff (U \cap W = \emptyset) \wedge \left(\text{subspaceSum}(U \oplus W, (U, W, V, +)) \right) \quad (199)$$

$$\begin{aligned} \text{orthogonalComplement}(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)) &\iff \\ &\left(\text{subspace}\left((W, +, \cdot, \langle \$1, \$2 \rangle), \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \right) \right) \wedge \end{aligned}$$

$$\left(W^\perp = \left\{ v \in V \mid w \in W \wedge \text{orthogonal}\left((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \right\} \right) \quad (200)$$

$$\begin{aligned} & \text{orthogonalDecomposition}\left(\left(W, W^\perp\right), (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right) \iff \\ & \left(\text{orthogonalComplement}\left(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right) \wedge \left(\text{subspaceDirectSum}\left(V, \left(W, W^\perp, V, +\right)\right)\right) \end{aligned} \quad (201)$$

$$(\text{THM}) \text{ if } V \text{ is finite dimensional, then every vector has an orthogonal decomposition:} \quad (202)$$

1.17 Banach and Hilbert Space

$$\begin{aligned} & \text{cauchy}\left((s)_\mathbb{N}, (V, d(\$1, \$2))\right) \iff \left(\text{metricSpace}\left((V, d(\$1, \$2)), ()\right)\right) \wedge ((s)_\mathbb{N} \subseteq V) \\ & \quad \left(\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N (d(s_m, s_n) < \epsilon)\right) \\ & \quad \# \text{ distances between some tail-end point gets arbitrarily small} \end{aligned} \quad (203)$$

$$\begin{aligned} & \text{complete}\left((V, d(\$1, \$2)), ()\right) \iff \left(\forall (s)_\mathbb{N} \subseteq V \exists s \in V \left(\text{cauchy}\left((s)_\mathbb{N}, (V, d(\$1, \$2))\right) \implies \lim_{n \rightarrow \infty} (d(s, s_n)) = 0\right)\right) \\ & \quad \# \text{ or converges within the induced topological space} \\ & \quad \# \text{ in complete spaces, the weaker notion of cauchy is enforced to be equivalent to convergence} \end{aligned} \quad (204)$$

$$\begin{aligned} & \text{banachSpace}\left((V, +, \cdot, \|\$1\|), ()\right) \iff \left(\text{normMetric}\left(d(\$1, \$2), (V, \|\$1\|)\right)\right) \wedge \left(\text{complete}\left(V, d(\$1, \$2)\right), ()\right) \\ & \quad \# \text{ a complete normed vector space} \end{aligned} \quad (205)$$

$$\begin{aligned} & \text{hilbertSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \iff \left(\text{innerProductNorm}\left(\|\$1\|, (V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right) \wedge \\ & \quad \left(\text{normMetric}\left(d(\$1, \$2), (V, \|\$1\|)\right)\right) \wedge \left(\text{complete}\left(V, d(\$1, \$2)\right), ()\right) \\ & \quad \# \text{ a complete inner product space} \end{aligned} \quad (206)$$

$$(\text{THM}) : \text{hilbertSpace} \implies \text{banachSpace} \quad (207)$$

$$\begin{aligned} & \text{separable}\left((V, d), ()\right) \iff \left(\exists S \subseteq V \left(\text{dense}(S, (V, d)) \wedge \text{countablyInfinite}(S, ())\right)\right) \\ & \quad \# \text{ needs only a countable subset to approximate any element in the entire space} \end{aligned} \quad (208)$$

$$\begin{aligned} & (\text{THM}) : \text{hilbertSpace}\left(\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right), ()\right) \implies \\ & \left(\exists (b)_\mathbb{N} \subseteq V \left(\text{orthonormalBasis}\left((b)_\mathbb{N}, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \wedge \text{countablyInfinite}\left((b)_\mathbb{N}, ()\right)\right)\right) \iff \\ & \quad \text{separable}\left(\left(V, \sqrt{\langle \$1 - \$2, \$1 - \$2 \rangle}\right), ()\right) \\ & \quad \# \text{ separability in hilbert spaces is equivalent to the existence of a countable orthonormal basis} \end{aligned} \quad (209)$$

1.18 Abstract algebra

(210)

$$\text{watR}(GL_n(\mathbb{R}), ()) \iff GL_n(\mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid \det(M) \neq 0\} \quad (211)$$

$$\begin{aligned} \text{group}((G, \bullet), ()) &\iff (\text{map}(\bullet, (G \times G, G))) \wedge \\ &\quad (\forall_{a,b \in G} (a \bullet b \in G)) \wedge \\ &\quad (\forall_{a,b,c \in G} ((a \bullet b) \bullet c = a \bullet (b \bullet c))) \wedge \\ &\quad (\exists_e \forall_{a \in G} (e \bullet a = a = a \bullet e)) \wedge \\ &\quad (\forall_{a \in G} \exists_{a^{-1}} (a \bullet a^{-1} = e = a^{-1} \bullet a)) \\ \# \text{ characterizes symmetry of a set structure} \end{aligned} \quad (212)$$

$$0 \quad (213)$$

$$\text{dfn queue: abelian, symmetric group,} \quad (214)$$

1.19 Matrices, Operators, and Functionals

$$\begin{aligned} \text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)) &\iff (\text{map}(L, (V, W))) \wedge (\text{vectorSpace}((V, +_V, \cdot_V), ())) \wedge \\ &(\text{vectorSpace}((W, +_W, \cdot_W), ())) \wedge (\forall_{v_1, v_2 \in V} \forall_{s_1, s_2 \in \mathbb{R}} (L(s_1 \cdot_V v_1 +_V s_2 \cdot_V v_2) = s_1 \cdot_W L(v_1) +_W s_2 \cdot_W L(v_2))) \end{aligned} \quad (215)$$

$$\begin{aligned} \text{matrix}(L, (n, m)) &\iff (\text{linearOperator}(L, (\mathbb{R}^m, +_m, \cdot_m, \mathbb{R}^n, +_n, \cdot_n))) \\ \# \text{ rows=dimensions, cols=vectors} \end{aligned} \quad (216)$$

$$\text{eigenvector}(v, (L, V, +, \cdot)) \iff (\text{linearOperator}(L, (V, +, \cdot, V, +, \cdot))) \wedge (\exists_{\lambda \in \mathbb{R}} (L(v) = \lambda v)) \quad (217)$$

$$\text{eigenvalue}(\lambda, (v, L, V, +, \cdot)) \iff (\text{eigenvector}(v, (L, V, +, \cdot))) \quad (218)$$

$$\text{identityOperator}(I, (A)) \iff (\text{matrix}(A, (n, n))) \wedge (AI = IA = A) \quad (219)$$

$$\begin{aligned} \text{inverseOperator}(A^{-1}, (A)) &\iff (A^{-1}A = AA^{-1} = I) \\ \# \text{ gauss-jordan elimination: } E[A|I] &= [I|E] = [I|A^{-1}] \end{aligned} \quad (220)$$

$$\text{CONTHERTODOABSTRACTALGEB} \quad (221)$$

$$(\text{THM}) : (AB)^{-1}(AB) = I = B^{-1}A^{-1}AB \quad (222)$$

$$\text{transposeOperator}(A^T, (A)) \iff \left((A^T)_{m,n} = (A)_{n,m} \right) \vee \text{adjoint}(A^T, (A)) \quad (223)$$

$$\text{symmetricOperator}(A, ()) \iff \left(A = \text{transposeOperator}(A^T, (A)) \right) \vee \left(\text{selfAdjoint}(A, ()) \right) \quad (224)$$

$$(\text{THM}) : (AB)^T = B^T A^T \wedge (A^T)^{-1} = (A^{-1})^T \quad (225)$$

$$\text{triangularOperator}(A, ()) \iff (\text{matrix}(A, (n, n))) \wedge (\forall_{x < n} \forall_{0 < i < x} (A_{i,i} = 0)) \quad (226)$$

$$\begin{aligned} \text{decomposeLU}(LU(A), (A)) \iff (\text{matrix}(A, (n, n))) \wedge (\exists_E (EA = \text{triangularOperator}(U, ()))) \wedge \\ (LU(A) = E^{-1}U = A) \\ \# \text{ lower triangle are all 0; useful for solving linear equations} \end{aligned} \quad (227)$$

$$\begin{aligned} \text{Img}(\text{Img}(A), (A)) \iff (\text{matrix}(A, (n, m))) \wedge (\text{Img}(A) = \{Av \in \mathbb{R}^n \mid v \in \mathbb{R}^m\}) \\ \# \text{ the column space; not always a subspace since } A \text{ can map to a set not containing } \mathbf{0} \end{aligned} \quad (228)$$

$$\begin{aligned} \text{Ker}(\text{Ker}(A), (A)) \iff (\text{matrix}(A, (n, m))) \wedge (\text{Ker}(A) = \{v \in \mathbb{R}^m \mid Av = \mathbf{0} \in \mathbb{R}^n\}) \\ \# \text{ the null or solution space; always a subspace due to linearity } Av + Aw = \mathbf{0} = A(v + w) \end{aligned} \quad (229)$$

$$(\text{THM}) \text{ general linear solution: } (Ax_p = b) \wedge (x_n \in \text{Ker}(A)) \implies (Ax_p + Ax_n = b + 0 = A(x_p + x_n) = b) \quad (230)$$

$$\begin{aligned} \text{independentOperator}(A, ()) \iff (\text{matrix}(A, (n, m))) \wedge (\neg \exists_{v \in \mathbb{R}^m \setminus \mathbf{0}_m} (Av = 0) \iff \text{Ker}(A) = \{\mathbf{0}_m\}) \\ \# \text{ also equivalent to invertible operator} \end{aligned} \quad (231)$$

$$\text{dimensionality}(N, (A)) \iff (\text{matrix}(A, (n, m))) \wedge (N = \inf(\{||b|| \mid \text{basis}((b)_n, (A))\})) \quad (232)$$

$$\text{rank}(r, (A)) \iff (\text{matrix}(A, (n, m))) \wedge (\text{dimensionality}(r, (A))) \quad (233)$$

$$\begin{aligned} (\text{THM}) : (\text{matrix}(A, (n, m))) \implies (\text{dimensionality}(\text{Ker}(A)) = n - \text{rank}(r, (A))) \\ \# \text{ number of free variables} \end{aligned} \quad (234)$$

$$\text{transposeNorm}(|x|, ()) \iff (||x|| = \sqrt{x^T x}) \quad (235)$$

$$(\text{THM}) : P = P^T = P^2 \quad (236)$$

$$\begin{aligned} \text{orthogonalVectors}((x, y), ()) \iff (||x||^2 + ||y||^2 = ||x + y||^2) \iff \\ (x^T x + y^T y = (x + y)^T (x + y) = x^T x + y^T y + x^T y + y^T x) \iff \\ \left(0 = \frac{x^T x + y^T y - (x^T x + y^T y)}{2} = \frac{x^T y + y^T x}{2} = x^T y\right) \iff \left(0 = \sum_i (x_i y_i) \vee \int (x(u) y(u) du)\right) \\ \# \text{ vector and functional orthogonality} \end{aligned} \quad (237)$$

$$\text{orthogonalOperator}(Q, (V, +, \cdot, \langle \$1, \$2 \rangle)) \iff \left(\text{orthonormalBasis} \left(Q^T, (V, +, \cdot, \langle \$1^T, \$2 \rangle) \right) \right) \vee (Q^T Q = I) \quad (238)$$

$$(\text{THM}) : \text{orthogonalOperator}(Q, (V, +, \cdot, \langle \$1, \$2 \rangle)) \implies (Q^T Q Q^{-1} = I Q^{-1} = Q^T = Q^{-1}) \quad (239)$$

$$\begin{aligned} \text{orthogonalProjection}(P_A b, (A, b)) &\iff (\text{matrix}(A, (n, m))) \wedge (\text{matrix}(b, (m, 1))) \wedge \\ &\left(\exists c \in \mathbb{R}^m (A^T (b - P_A b) = 0 = A^T (b - A c)) \right) \iff \\ A^T b &= A^T A c \iff c = (A^T A)^{-1} A^T b \iff P_A b = A c = \left(A (A^T A)^{-1} A^T \right) b \\ &\# A, A^T \text{ may not necessarily be invertible} \end{aligned} \quad (240)$$

$$(\text{THM}) : \text{independentOperator}(A, ()) \implies \text{independentOperator}(A^T A, ()) \quad (241)$$

$$\begin{aligned} \text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|)) &\iff (\text{normedVectorSpace}((V, +, \cdot, \|\$1\|), ())) \wedge \\ (X = \{v \in V \mid \|v\| = 1 \wedge \text{eigenvector}(v, (A, V, +, \cdot))\}) \end{aligned} \quad (242)$$

$$\begin{aligned} \text{det}(\text{det}(A), (A, V, +, \cdot, \|\$1\|)) &\iff (\text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|))) \wedge \\ (\text{det}(A) = \prod_{x \in X} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)))) \\ &\# \text{ DEFINE; exterior algebra wedge product area??} \end{aligned} \quad (243)$$

$$\begin{aligned} \text{tr}(\text{tr}(A), (A, V, +, \cdot, \|\$1\|)) &\iff (\text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|))) \wedge \\ (\text{tr}(A) = \sum_{x \in X} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)))) \\ &\# \text{ DEFINE} \end{aligned} \quad (244)$$

$$(\text{THM}) : \text{independentOperator}(A, ()) \iff \text{det}(A) \neq 0 \quad (245)$$

$$(\text{THM}) : A = A^T = A^2 \implies \text{Tr}(A) = \text{dimensionality}(N, (A)) \# \text{ counts dimensions} \quad (246)$$

$$\begin{aligned} (\text{normalOperator}(A, ())) &\iff A^T A = A A^T \\ &\# \text{ DEFINE} \end{aligned} \quad (247)$$

$$\text{diagonalOperator}(A, ()) \iff (\text{normalOperator}(A, ())) \wedge (\text{triangularOperator}(A, ())) \quad (248)$$

$$\begin{aligned} \text{characteristicEquation}((A - \lambda I)x = 0, (A)) &\iff (Ax = \lambda x \implies Ax - \lambda x = (A - \lambda I)x = 0) \wedge \\ (x \neq \mathbf{0} \implies \text{eigenvalue}(0, (x, A - \lambda I)) \implies \prod_{\lambda_i \in \Lambda} = 0 = \text{det}(A - \lambda I)) \\ &\# \text{ characterizes eigenvalues} \end{aligned} \quad (249)$$

$$\begin{aligned} \text{eigenDecomposition}(S \Lambda S^{-1}, (A, V, +, \cdot, \|\$1\|)) &\iff (S \subseteq (\text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|))^T) \wedge \\ (\text{diagonalOperator}(\Lambda, ()) \{1\}^n = (\lambda)_n = \{\lambda \in \mathbb{R} \mid s \in S^T \wedge \text{eigenvalue}(\lambda, s, A, V)\}) \\ (\text{independentOperator}(S, ())) \wedge (\exists_{S^{-1}} (AS = \Lambda \implies A = S \Lambda S^{-1})) \end{aligned} \quad (250)$$

$$(\text{THM}) : \text{eigenDecomposition}(S\Lambda S^{-1}, (A, V, +, \cdot, \|\$1\|)) \implies A^2 = (A)(A) = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1} \quad (251)$$

$$(\text{THM}) : \text{spectralDecomposition}(Q\Lambda Q^T, (A, V, +, \cdot, \|\$1\|)) \iff (\text{symmetricOperator}(A, ())) \implies$$

$$(\exists_Q (\text{eigenDecomposition}(Q\Lambda Q^{-1}, (A, V, +, \cdot, \$1^T \$1)) \wedge \text{orthogonalOperator}(Q, (V, +, \cdot, \$1^T \$2)) \wedge (\lambda)_n \in \mathbb{R}^n))$$

if symmetric and eigenvalues are real, then there exists orthonormal eigenbasis (252)

$$\text{hermitianAdjoint}(A^H, (A)) \iff (A^H = \overline{A}^T) \iff (\langle A, A \rangle = \overline{A}^T A \in \mathbb{R})$$

complex analog to adjoint (253)

$$\text{hermitianOperator}(A, ()) \iff A = A^H$$

complex analog to symmetric operator (254)

$$\text{unitaryOperator}(Q^H Q, (Q)) \iff Q^H Q = I$$

complex analog to orthogonal operator (255)

$$\text{positiveDefiniteOperator}(A, (V, +, \cdot, \|\$1\|)) \iff (\forall_{x \in V \setminus \{0\}} (x^T A x > 0)) \vee$$

$$(\forall_{x \in \text{eigenvectors}(X, (A, V, +, \$1^T \$1))} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)) \implies \lambda > 0))$$

acts like a positive scalar where any vector only scales and cannot reflect against its perpendicular axis (256)

$$(\text{THM}) : \text{positiveDefiniteOperator}(A^T A) \iff \forall_{x \in V \setminus \{0\}} (x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 > 0) \quad (257)$$

$$\text{semiPositiveDefiniteOperator}(A, (V, +, \cdot, \|\$1\|)) \iff (\forall_{x \in V \setminus \{0\}} (x^T A x \geq 0)) \vee$$

$$(\forall_{x \in \text{eigenvectors}(X, (A, V, +, \$1^T \$1))} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)) \implies \lambda \geq 0))$$

acts like a nonnegative scalar (258)

$$(\text{THM}) : \text{symmetricOperator}(A^T A) \iff (A^T A = (A^T A)^T = A^T A^{TT} = A^T A) \quad (259)$$

$$\text{similarOperators}((A, B), ()) \iff (\text{matrix}(A, (n, n))) \wedge (\text{matrix}(B, (n, n))) \wedge (\exists_M (B = M^{-1} A M)) \quad (260)$$

$$(\text{THM}) : (\text{similarOperators}((A, B), ()) \wedge Ax = \lambda x) \implies (\exists_M (M^{-1} A x = \lambda M^{-1} x = M^{-1} A M M^{-1} x = B M^{-1} x))$$

similar operators have the same eigenvalues but M^{-1} shifted eigenvectors (261)

$$\text{singularValueDecomposition}(Q\Sigma R^T, (A, V, +, \cdot, (\$1, \$2))) \iff (\text{orthogonalOperator}(R, (V, +, \cdot, \$1^T \$2))) \wedge$$

$$(\text{orthogonalOperator}(Q, (\text{Img}(A), +, \cdot, \$1^T \$2))) \wedge (\text{semiPositiveDefiniteOperator}(\Sigma, (V, +, \cdot, \$1^T \$1))) \wedge$$

$$(AR = Q\Sigma) \wedge (A = Q\Sigma R^{-1} = Q\Sigma R^T) \wedge (\text{symmetricOperator}(A^T A)) \wedge (\text{symmetricOperator}(AA^T)) \wedge$$

$$(A^T A = R\Sigma^T Q^T Q\Sigma R^T = R\Sigma^T \Sigma R^T) \wedge (\text{spectralDecomposition}(R(\Sigma^T \Sigma)R^T, (A^T A, V, +, \cdot, \$1^T \$1))) \wedge$$

$$(AA^T = Q\Sigma R^T R\Sigma^T Q^T = Q\Sigma \Sigma^T Q^T) \wedge (\text{spectralDecomposition}(Q(\Sigma \Sigma^T)Q^T, (AA^T, V, +, \cdot, \$1^T \$1))) \wedge$$

$$(\text{diagonalOperator}(\Sigma^T \Sigma) \implies \text{normalOperator}(\Sigma^T \Sigma) = \Sigma \Sigma^T = \Sigma_{\sigma^2}) \wedge (\Sigma = \Sigma_{\sqrt{\sigma^2}} = \Sigma_{|\sigma|})$$

(THM) based on the spectral theorem: (262)

$$\text{leftInverseOperator}(A_L^{-1}, (A)) \iff (\text{matrix}(A, (n, m))) \wedge (\text{rank}(A) = n < m) \wedge$$

$$(A_L^{-1} A = I = ((A^T A)^{-1} A^T) A) \quad (263)$$

$$\begin{aligned} \text{rightInverseOperator}(A_R^{-1}, (A)) &\iff (\text{matrix}(A, (n, m))) \wedge (\text{rank}(A) = m < n) \wedge \\ & (AA_R^{-1} = I = A(A^T(AA^T)^{-1})) \end{aligned} \quad (264)$$

1.20 Functional analysis

$$\begin{aligned} \text{denseMap}(L, (D, H, +, \cdot, \langle \$1, \$2 \rangle)) &\iff (D \subseteq H) \wedge (\text{linearOperator}(L, (D, +, \cdot, H, +, \cdot))) \wedge \\ & (\text{innerProductTopology}(\mathcal{O}, (H, +, \cdot, \langle \$1, \$2 \rangle))) \wedge (\text{dense}(D, (H, \mathcal{O}, d(\$1, \$2)))) \end{aligned} \quad (265)$$

$$\begin{aligned} \text{mapNorm}(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) &\iff \\ & (\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W))) \wedge \\ & (\text{normedVectorSpace}((V, +_V, \cdot_V, \|\$1\|_V), ())) \wedge (\text{normedVectorSpace}((W, +_W, \cdot_W, \|\$1\|_W), ())) \wedge \\ & \left(\|L\| = \sup \left(\left\{ \frac{\|Lf\|_W}{\|f\|_V} \mid f \in V \right\} \right) = \sup \left(\{ \|Lf\|_W \mid f \in V \wedge \|f\| = 1 \} \right) \right) \end{aligned} \quad (266)$$

$$\begin{aligned} \text{boundedMap}(L, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) &\iff \\ & (\text{mapNorm}(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) < \infty) \end{aligned} \quad (267)$$

$$\begin{aligned} \neg \text{boundedMap}(L, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) &\iff \\ (U \subset V) \wedge \left(\infty = \text{mapNorm}(\|L\|_U, (L, U, +_U, \cdot_U, \|\$1\|_U, W, +_W, \cdot_W, \|\$1\|_W)) \leq \|L\| \right) \end{aligned} \quad (268)$$

$$\begin{aligned} \text{extensionMap}(\widehat{L}, (L, V, D, W)) &\iff (D \subseteq V) \wedge (\text{linearOperator}(L, (D, +_D, \cdot_D, W, +_W, \cdot_W))) \wedge \\ & (\text{linearOperator}(\widehat{L}, (V, +_V, \cdot_V, W, +_W, \cdot_W))) \wedge \left(\forall d \in D \left(\widehat{L}(d) = L(d) \right) \right) \end{aligned} \quad (269)$$

$$\begin{aligned} \text{adjoint}(L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)) &\iff (\text{hilbertSpace}((V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V), ())) \wedge \\ & (\text{hilbertSpace}((W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W), ())) \wedge (\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W))) \wedge \\ & \left(\forall v \in V \forall w \in W \left((\langle Lv, w \rangle_W = \langle v, L^T w \rangle_V) \vee ((Lv)^T w = v^T L^T w) \right) \right) \\ & \# \text{ target operator that acts similar to the domain operator} \end{aligned} \quad (270)$$

$$\begin{aligned} \text{selfAdjoint}(L, (V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)) &\iff \\ L = \text{adjoint}(L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)) \\ & \# \text{ also a generalization of symmetric matrices} \end{aligned} \quad (271)$$

$$\text{compactMap}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)) \iff (\text{boundedMap}(L, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W))) \wedge$$

$$\left(\forall_{v \in V} \left(\text{openBall} \left(B, (1.0, v, V, d_V(\$1, \$2)) \right) \implies \right. \right. \\ \left. \left. \text{compactSubset} \left(\text{closure} \left(\overline{L(B)}, \text{image}(L(B), (B, L, V, W)), W, d_W(\$1, \$2) \right), (W, \mathcal{O}_W) \right) \right) \right) \quad (272)$$

(THM) Spectral thm.:

$$\left(\text{selfAdjoint} \left(L, (V, +, \cdot, \langle \$1, \$2 \rangle, V, +, \cdot, \langle \$1, \$2 \rangle) \right) \right) \wedge \left(\text{compactMap} \left(L, (V, +, \cdot, V, +, \cdot) \right) \right) \implies \\ \left(\exists_{(e)_{\mathbb{N}} \subseteq V} \left(\text{orthonormalBasis} \left((e)_{\mathbb{N}}, (V, +, \cdot, \langle \$1, \$2 \rangle) \right) \wedge \forall_{e_n \in (e)_{\mathbb{N}}} \left(\text{eigenvector}(e_n, (L, V, +, \cdot)) \right) \right) \right) \implies \\ \left(\exists_{(\lambda)_{\mathbb{N}} \subseteq \mathbb{R}^n} \forall_{e_n \in (e)_{\mathbb{N}}} \exists_{\lambda_n \in (\lambda)_{\mathbb{N}}} \left(\text{eigenvalue}(\lambda_n, (e_n, L, V, +, \cdot)) \wedge \lim_{n \rightarrow \infty} (\lambda_n = 0) \wedge L = \sum_{n=1}^{\infty} (\lambda_n e_n e_n^T) \right) \right) \\ \# \text{ DEFINE} \quad (273)$$

1.21 Function spaces

$$\text{curLp}(\mathcal{L}^p, (p, M, \sigma, \mu)) \iff (p \in \mathbb{R}) \wedge (1 \leq p < \infty) \wedge \\ \left(\mathcal{L}^p = \{ \text{map}(f, (M, \mathbb{R})) \mid \text{measurableMap}(f, (M, \sigma, \mathbb{R}, \text{euclideanSigma})) \wedge \int (|f|^p d\mu) < \infty \} \right) \quad (274)$$

$$\text{vecLp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \iff \left(\text{curLp}(\mathcal{L}^p, (p, M, \sigma, \mu)) \right) \wedge \left(\forall_{f, g \in \mathcal{L}^p} \forall_{m \in M} ((f + g)(m) = f(m) + g(m)) \right) \wedge \\ \left(\forall_{f \in \mathcal{L}^p} \forall_{s \in \mathbb{R}} \forall_{m \in M} ((s \cdot f)(m) = (s)f(m)) \right) \wedge \left(\text{vectorSpace}((\mathcal{L}^p, +, \cdot, ())) \right) \quad (275)$$

$$\text{integralNorm}(\|\cdot\|, (+, \cdot, p, M, \sigma, \mu)) \iff \left(\text{vecLp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \left(\text{map} \left(\|\cdot\|, \left(\mathcal{L}^p, \mathbb{R}_0^+ \right) \right) \right) \wedge \\ \left(\forall_{f \in \mathcal{L}^p} \left(0 \leq \|\cdot\| f = \left(\int (|f|^p d\mu) \right)^{1/p} \right) \right) \quad (276)$$

$$(\text{THM}) : \text{integralNorm}(\|\cdot\|, (+, \cdot, p, M, \sigma, \mu)) \implies \\ \left(\forall_{f \in \mathcal{L}^p} \left(\|\cdot\| f = 0 \implies \text{almostEverywhere}(f = \mathbf{0}, (M, \sigma, \mu)) \right) \right) \\ \# \text{ not an expected property from a norm} \quad (277)$$

$$\text{Lp}(\mathcal{L}^p, ((+, \cdot, p, M, \sigma, \mu))) \iff \left(\text{integralNorm}(\|\cdot\|, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \\ \left(L^p = \text{quotientSet} \left(\mathcal{L}^p / \sim, \left(\mathcal{L}^p, (\|\cdot\| + (-\$2)) \|\cdot\| = 0 \right) \right) \right) \\ \# \text{ functions in } L^p \text{ that have finite integrals above and below the x-axis} \quad (278)$$

$$(\text{THM}) : \text{banachSpace} \left(\left(\text{Lp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)), +, \cdot, \|\cdot\| \right), (,) \right) \quad (279)$$

$$(THM) : \text{hilbertSpace} \left(\left(Lp(L^p, (+, \cdot, 2, M, \sigma, \mu)), +, \cdot, \frac{\lambda \lambda \$1 + \$2 \lambda^2 - \lambda \lambda \$1 - \$2 \lambda^2}{4} \right), () \right) \quad (280)$$

$$\begin{aligned} \text{curL} \left(\mathcal{L}, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right) &\iff \left(\text{banachSpace} \left((W, +_W, \cdot_W, \|\$1\|_W), () \right) \right) \wedge \\ &\left(\text{normedVectorSpace} \left((V, +_V, \cdot_V, \|\$1\|_V), () \right) \right) \wedge \\ &\left(\mathcal{L} = \{f \mid \text{boundedMap} \left(f, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right)\} \right) \end{aligned} \quad (281)$$

$$(THM) : \text{banachSpace} \left(\left(\text{curL} \left(\mathcal{L}, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right), +, \cdot, \text{mapNorm} \right), () \right) \quad (282)$$

$$(THM) : \|L\| \geq \frac{\|Lf\|}{\|f\|} \# \text{ from choosing an arbitrary element in the mapNorm sup} \quad (283)$$

$$\begin{aligned} (THM) : \left(\text{cauchy}((f)_{\mathbb{N}}, (\mathcal{L}, +, \cdot, \text{mapNorm})) \implies \text{cauchy}((f_n v)_{\mathbb{N}}, (W, +_W, \cdot_W, \|\$1\|_W)) \right) &\iff \\ \left(\forall \epsilon' > 0 \forall v \in V (\|f_n v - f_m v\|_W = \|(f_n - f_m)v\|_W \leq \|f_n - f_m\| \cdot \|v\|_V < \epsilon \cdot \|v\|_V = \epsilon') \right) & \\ \# \text{ a cauchy sequence of operators maps to a cauchy sequence of targets} \end{aligned} \quad (284)$$

$$\begin{aligned} (THM) \text{ BLT thm.: } \left(\left(\text{dense}(D, (V, \mathcal{O}, d_V)) \wedge \text{boundedMap} \left(A, (D, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right) \right) \implies \right. & \\ \left(\exists!_{\hat{A}} \left(\text{extensionMap} \left(\hat{A}, (A, V, D, W) \right) \right) \wedge \|\hat{A}\| = \|A\| \right) &\iff \\ \left(\forall v \in V \exists (v_n)_{n \in \mathbb{N}} \subseteq D \left(\lim_{n \rightarrow \infty} (v_n = v) \right) \right) \wedge \left(\hat{A}v = \lim_{n \rightarrow \infty} (Av_n) \right) & \end{aligned} \quad (285)$$

2 Probability Theory

2.1 Definitions

$$\text{randomExperiment}(E, (\Omega)) \iff \Omega = \{\omega \mid \text{experiment} = E \rightarrow \text{outcome} = \omega\} \quad (286)$$

$$\text{probabilitySpace}((\Omega, \mathcal{F}, P), ()) \iff \text{measureSpace}((\Omega, \mathcal{F}, P), ()) \wedge (P(\Omega) = 1) \quad (287)$$

$$\begin{aligned} \text{event}(F, (\Omega, \mathcal{F}, P)) &\iff \left(\text{probabilitySpace}((\Omega, \mathcal{F}, P), ()) \right) \wedge (F \in \mathcal{F}) \\ \# F &\text{ can represent both singleton outcomes and outcome combinations and } \mathcal{F} \text{ can represent} \\ \# \text{ a countable event that contains outcomes with even number of coin tosses before the first head} \\ \# \mathcal{P}(\mathbb{R}) &\text{ sets are not considered because definite uniform measures diverge everywhere} \\ \# \mathcal{P}(\mathbb{N}) &\text{ sets can be assigned a meaningful convergent measure e.g., } \forall k \in \mathbb{R}^+ \forall f \in F P(\{f\}) = k^{-f} \end{aligned} \quad (288)$$

$$\begin{aligned} (THM) : \left(\text{probabilitySpace}((\Omega, \mathcal{F}, P), ()) \wedge F, A, B \in \mathcal{F} \right) &\implies \\ \left(F^C \cup F = \Omega \wedge F^C \cap F = \emptyset \implies P(F^C) + P(F) = 1 \implies P(F^C) = 1 - P(F) \right) &\wedge \end{aligned}$$

$$\begin{aligned}
& \left(P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - \left(1 - P(A^C \cup B^C) \right) = \right. \\
& \quad P(A) + P(B) - 1 + P(A^C) + P(B^C) - P(A^C \cap B^C) = \\
& \quad \left. P(A) + P(B) - 1 + 1 - P(A) + 1 - P(B) - \left(1 - P(A \cup B) \right) = P(A \cup B) \right) \wedge \\
& \quad \left(P\left(\bigcup_{i=1}^n (A_i)\right) = \sum_{k=1}^n \left((-1)^{k-1} \sum_{I \subset \mathbb{N}_1^n \wedge |I|=k} \left(P\left(\bigcap_{i \in I} (A_i)\right) \right) \right) \right) \quad (289)
\end{aligned}$$

$$\begin{aligned}
& (\text{THM}) : \left(\text{measureSpace}((\Omega, \mathcal{F}, P), ()) \wedge (A)_{\mathbb{N}}, (B)_{\mathbb{N}} \subseteq \mathcal{F} \wedge A, B \in \mathcal{F} \right) \implies \\
& \text{CL285} \left(B_n = A_n \setminus \bigcup_{i=1}^{n-1} (A_i) \right) \wedge \text{DL285} \left(\forall i \in \mathbb{N} \forall j \in \mathbb{N} \setminus \{i\} (B_i \cap B_j = \emptyset) \right) \wedge \text{EL285} \left(\bigcup_{i \in \mathbb{N}} (A_i) = \bigcup_{i \in \mathbb{N}} (B_i) \right) \wedge \\
& \text{1IL285} \left(P\left(\bigcup_{i \in \mathbb{N}} (B_i)\right) = \sum_{i \in \mathbb{N}} (P(B_i)) \right) \wedge \text{2IL285} \left(\sum_{i \in \mathbb{N}} (P(B_i)) = \lim_{m \rightarrow \infty} \left(\sum_{i=1}^m (P(B_i)) \right) \right) \wedge \\
& \text{3IL285} \left(\lim_{m \rightarrow \infty} \left(\sum_{i=1}^m (P(B_i)) \right) = \lim_{m \rightarrow \infty} \left(P\left(\bigcup_{i=1}^m (B_i)\right) \right) \right) \wedge \\
& \text{4IL285} \left(\lim_{m \rightarrow \infty} \left(P\left(\bigcup_{i=1}^m (B_i)\right) \right) = \lim_{m \rightarrow \infty} \left(P\left(\bigcup_{i=1}^m (A_i)\right) \right) \right) \wedge \\
& \text{MSCont} \left(P\left(\bigcup_{i \in \mathbb{N}} (A_i)\right) = \lim_{m \rightarrow \infty} \left(P\left(\bigcup_{i=1}^m (A_i)\right) \right) \right) \wedge \\
& \text{MSConvL} \left(\forall j \in \mathbb{N} (A_j \subseteq A_{j+1}) \implies P\left(\bigcup_{i \in \mathbb{N}} (A_i)\right) = \lim_{m \rightarrow \infty} (P(A_m)) \right) \wedge \\
& \text{MSConvU} \left(\forall j \in \mathbb{N} (A_{j+1} \subseteq A_j) \implies P\left(\bigcap_{i \in \mathbb{N}} (A_i)\right) = \lim_{m \rightarrow \infty} (P(A_m)) \right) \wedge \\
& \text{MSSetOrder} (A \subseteq B \implies P(A) \leq P(B)) \wedge \text{MSSetBound} \left(\bigcup_{i \in \mathbb{N}} (A_i) \leq \sum_{i \in \mathbb{N}} (P(A_i)) \right) \quad (290)
\end{aligned}$$

2.2 Conditional probability

$$\begin{aligned}
& \text{conditionalProbability} \left(P(A|B), (A, B, \Omega, \mathcal{F}, P) \right) \iff (\text{probabilitySpace}(\Omega, \mathcal{F}, P)) \wedge (A, B \in \mathcal{F}) \wedge \\
& (P(B) > 0) \wedge \left(P(A|B) = \frac{P(A \cap B)}{P(B)} \vee P(B)P(A|B) = P(A \cap B) \right) \\
& \quad \# \text{ calculates } P(A) \text{ on the subset spanned by } B \\
& \quad \# \text{ conditioning on a 0 probability set B leads to paradoxes} \quad (291)
\end{aligned}$$

$$(\text{THM}) : (\text{probabilitySpace}(\Omega, \mathcal{F}, P) \wedge P(B) > 0) \implies \forall F \in \mathcal{F} (P'(F) = P(F|B)) \wedge \text{probabilitySpace}(\Omega, \mathcal{F}, P') \quad (292)$$

$$\text{setPartition}((X)_{\mathbb{N}}, (Y)) \iff \left(\bigcup_{i \in \mathbb{N}} (X_i) = Y \right) \wedge \left(\forall_{i \in \mathbb{N}} \forall_{j \in \mathbb{N} \setminus \{i\}} (X_i \cap X_j = \emptyset) \right) \quad (293)$$

$$\begin{aligned} (\text{THM}) : & \left(\text{probabilitySpace}(\Omega, \mathcal{F}, P) \wedge \{A\} \cup (B)_{\mathbb{N}} \subseteq \mathcal{F} \wedge \text{setPartition}((B)_{\mathbb{N}}, (\Omega)) \right) \implies \\ & \left(P(A) = \sum_{i \in \mathbb{N}} \left(P(A|B_i) P(B_i) \right) \right) \wedge \\ & \left(\forall_{i \in \mathbb{N}} \left(P(A|B_i) P(B_i) = P(A) P(B_i|A) = \left(\sum_{j \in \mathbb{N}} \left(P(B_j|A) \right) \right) P(B_i|A) \right) \right) \wedge \\ & \left(P \left(\bigcap_{i \in \mathbb{N}} (B_i) \right) = P(B_1) \prod_{i=2}^{\infty} \left(P \left(B_i \mid \bigcap_{j=1}^{i-1} (B_j) \right) \right) \right) \end{aligned}$$

from the subspace definition of conditional probability and algebraic manipulations (294)

$$\text{infinitelyOften}(\{A_n \text{ i-o}\}, ()) \iff \left(B_n = \bigcup_{i=n}^{\infty} (A_i) \in \mathcal{F} \right) \wedge \left(\{A_n \text{ i-o}\} = \bigcap_{n \in \mathbb{N}} (B_n) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=n}^{\infty} (A_i) \in \mathcal{F} \right)$$

the event that infinitely many A_n 's will occur

B_n occur if some event within the n th-tail-end event $A_i | i \geq n$ occur, which follows from \cup

$\{A_n \text{ i-o}\}$ occur if every tail-end event B_n occur for all n , which follows from \cap

similarly, $\{A_n \text{ i-o}\}$ occur, for all values of n , the n th-tail-end event occur (295)

$$\begin{aligned} (\text{THM}) \text{ BCL 1: } & \left(\text{Cond300} \left(\sum_{n \in \mathbb{N}} (P(A_n)) < \infty \right) \implies (P(\{A_n \text{ i-o}\}) = 0) \right) \Leftarrow \\ & \text{1IL300} \text{ infinitelyOften } \text{MSContU} \left(P \left(\bigcap_{n \in \mathbb{N}} (B_n) \right) = \lim_{n \rightarrow \infty} (P(B_n)) = \lim_{n \rightarrow \infty} \left(P \left(\bigcup_{i=n}^{\infty} (A_i) \right) \right) \right) \wedge \\ & \text{2IL300} \text{ MSSetBount} \left(\lim_{n \rightarrow \infty} \left(P \left(\bigcup_{i=n}^{\infty} (A_i) \right) \right) \leq \lim_{n \rightarrow \infty} \left(\sum_{i=n}^{\infty} (P(A)_i) \right) \right) \wedge \\ & \text{3IL300} \text{ Cond300} \left(\lim_{n \rightarrow \infty} \left(\sum_{i=n}^{\infty} (P(A)_i) \right) = 0 \right) \wedge \text{Impl300} \left(0 \leq P(\{A_n \text{ i-o}\}) \leq 0 \right) \quad (296) \end{aligned}$$

$$(\text{THM}) : \text{logp} \left(\forall_{x \in [0,1]} (\log(1-x) \leq -x) \right) \quad (297)$$

$$(\text{THM}) : \text{sump} \left(\left(\text{1Cond302} \left(\forall_{i \in \mathbb{N}} (p_i \in [0,1]) \right) \wedge \text{2Cond302} \left(\sum_{i \in \mathbb{N}} (p_i) = \infty \right) \right) \implies \prod_{i \in \mathbb{N}} (1-p_i) = 0 \right) \Leftarrow$$

$$\begin{aligned}
& \stackrel{1IL302}{\left(\prod_{i \in \mathbb{N}} (1 - p_i) = \exp \left(\log \left(\prod_{i \in \mathbb{N}} (1 - p_i) \right) \right) = \exp \left(\log \left(\lim_{n \rightarrow \infty} \left(\prod_{i=1}^n (1 - p_i) \right) \right) \right) \right)} \wedge \\
& \stackrel{2IL302 \text{ logp}}{\left(\exp \left(\log \left(\lim_{n \rightarrow \infty} \left(\prod_{i=1}^n (1 - p_i) \right) \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n (\log(1 - p_i)) \right) \right) \leq \exp \left(\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n (-p_i) \right) \right) \right)} \wedge \\
& \stackrel{3IL302 \text{ 2Cond302}}{\left(\exp \left(\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n (-p_i) \right) \right) = \exp(-\infty) = 0 \right)} \wedge \stackrel{Impl302 \text{ 1Cond302 1IL302 2IL302 3IL302}}{\left(0 \leq \prod_{i \in \mathbb{N}} (1 - p_i) \leq 0 \right)} \quad (298)
\end{aligned}$$

$$\begin{aligned}
\text{(THM) BCL 2: } & \left(\left(\stackrel{1Cond303}{\sum_{n \in \mathbb{N}} (P(A_n)) = \infty} \right) \wedge \stackrel{2Cond303}{\left(\inf IndEvents((A)_{\mathbb{N}}) \right)} \right) \implies P(\{A_n \text{ i-o}\}) = 1 \\
& \Leftarrow \stackrel{1IL303 \text{ MSSetBound}}{\left(1 - P(\{A_n \text{ i-o}\}) = P(\{A_n \text{ i-o}\}^C) = P \left(\bigcup_{n \in \mathbb{N}} (B_n^C) \right) \leq \sum_{n \in \mathbb{N}} (P(B_n^C)) \right)} \wedge \\
& \stackrel{2IL303 \text{ DeMorgans 2Cond303}}{\left(\sum_{n \in \mathbb{N}} (P(B_n^C)) = \sum_{n \in \mathbb{N}} \left(P \left(\bigcap_{i=n}^{\infty} (A_i^C) \right) \right) = \sum_{n=1}^{\infty} \left(\prod_{i=n}^{\infty} (P(A_i^C)) \right) = \sum_{n=1}^{\infty} \left(\prod_{i=n}^{\infty} (1 - P(A_i)) \right) \right)} \wedge \\
& \stackrel{3IL303 \text{ 1Cond303 sump}}{\left(\sum_{n=1}^{\infty} \left(\prod_{i=n}^{\infty} (1 - P(A_i)) \right) = \sum_{n=1}^{\infty} (0) = 0 \right)} \wedge \stackrel{Impl303 \text{ 1IL303 2IL303 3IL303}}{\left(0 \leq 1 - P(\{A_n \text{ i-o}\}) \leq 0 \iff P(\{A_n \text{ i-o}\}) = 1 \right)} \quad (299)
\end{aligned}$$

2.3 Random variables

$$\begin{aligned}
& \text{randomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\text{probabilitySpace}(\Omega, \mathcal{F}, P)) \wedge (\text{map}(X, (\Omega, \mathbb{R}))) \wedge \\
& \quad (\text{measurableMap}(X, (\Omega, \mathcal{F}, \mathbb{R}, \text{euclideanSigma}))) \\
& \# \text{ maps elementary outcomes to an observable numeric value and the measurable sets to measurable sets} \quad (300)
\end{aligned}$$

$$\begin{aligned}
& PL(P_X, (X, \Omega, \mathcal{F}, P)) \iff (\text{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\
& \quad \left(\forall B \in \sigma_S \left(P_X(B) = P(\{\omega \in \Omega \mid X(\omega) \in B\}) = (P \circ X^{-1})(B) = P(X \in B) \right) \right) \\
& \# \text{ probability of borel set events occuring and equips probabilities to numeric valued borel sets} \quad (301)
\end{aligned}$$

$$\text{(THM) : } \text{probabilitySpace}(\mathbb{R}, \text{euclideanSigma}, P_X) \quad (302)$$

$$\begin{aligned}
& \text{generatedSigmaAlgebra}(\sigma(\mathcal{M}), (\mathcal{M}, S)) \iff (\mathcal{M} \subseteq \mathcal{P}(S)) \\
& \quad (\text{sigmaAlgebra}(\sigma(\mathcal{M}), (S)) = \bigcap \{ \mathcal{H} \mid \mathcal{M} \subseteq \text{sigmaAlgebra}(\mathcal{H}, S) \}) \\
& \# \text{ the smallest sigma algebra containing the generating sets} \quad (303)
\end{aligned}$$

$$\text{piSystem}(\mathcal{G}, (\Omega)) \iff (\mathcal{G} \subseteq \mathcal{P}(\Omega)) \wedge (\forall A, B \in \mathcal{G} (A \cap B \in \mathcal{G})) \quad (304)$$

$$\begin{aligned}
& \text{(THM) pi measure extension: } \left(\textcolor{blue}{piSystem}(\mathcal{G}, (\Omega)) \wedge \textcolor{blue}{probabilitySpace}(\Omega, \sigma(\mathcal{G}), \lambda) \wedge \right. \\
& \quad \left. \textcolor{blue}{probabilitySpace}(\Omega, \sigma(\mathcal{G}), \mu) \wedge \exists_{(S)_{\mathbb{N}} \subseteq \Omega} \left(\bigcup ((S)_{\mathbb{N}}) = \Omega \wedge \lambda(\Omega) < \infty \right) \right) \implies \\
& \quad \left(\forall_{G \in \mathcal{G}} (\lambda(G) = \mu(G)) \implies \forall_{F \in \sigma(\mathcal{G})} (\lambda(F) = \mu(F)) \right) \\
& \quad \# \text{ PL in terms of a simpler generating pi system} \quad (305)
\end{aligned}$$

$$\text{(THM) : } \left(\textcolor{blue}{piSystem}(\{(-\infty, x] \mid x \in \mathbb{R}\}, (\mathbb{R})) \right) \wedge \left(\textcolor{blue}{euclideanSigma} = \sigma(\{(-\infty, x] \mid x \in \mathbb{R}\}) \right) \quad (306)$$

$$\begin{aligned}
& \textcolor{teal}{CDF}(F_X, (X, \Omega, \mathcal{F}, P)) \iff \left(\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\
& \quad \left(\forall_{x \in \mathbb{R}} \left(P(\{\omega \in \Omega \mid X(\omega) \in (-\infty, x]\}) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) = P(X \leq x) = F_X(x) \right) \right) \\
& \quad \# \text{ PL of the semi infinite pi system on the real numbers} \\
& \quad \# \text{ specifies PL following pi measure extension theorem but simpler than definitions on complex borel sets} \quad (307)
\end{aligned}$$

$$\begin{aligned}
& \text{(THM) : } \textcolor{blue}{CDF}(F_X, (X, \Omega, \mathcal{F}, P)) \implies \left(\lim_{x \rightarrow -\infty} (F_X(x)) = 0 \right) \wedge \left(\lim_{x \rightarrow \infty} (F_X(x)) = 1 \right) \wedge \\
& \quad \left(\forall_{x_1, x_2 \in \mathbb{R}} (x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)) \right) \wedge \left((e)_{\mathbb{N}} \subseteq \mathbb{R}_0^+ \right) \wedge \left(\lim_{n \rightarrow \infty} (e_n) = 0 \right) \wedge \\
& \quad \left(\forall_{x \in \mathbb{R}} \left(\lim_{\epsilon \rightarrow 0^+} (F(x + \epsilon)) = \lim_{n \rightarrow \infty} (F(x + e_n)) = \lim_{n \rightarrow \infty} (P(X \leq x + e_n)) = \lim_{n \rightarrow \infty} (P(\{\omega \in \Omega \mid X(\omega) \leq x + e_n\})) = \right. \right. \\
& \quad \left. \left. P \left(\bigcap_{n=1}^{\infty} (\{\omega \in \Omega \mid X(\omega) \leq x + e_n\}) \right) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) = F_X(x) \right) \right) \\
& \quad \# \text{ depends on the nested decreasing subsets induced by the limit from right} \quad (308)
\end{aligned}$$

2.4 Types of random variables

$$\text{(THM) : measures on } \mathbb{R} \text{ has only discrete, continuous, and singular components} \quad (309)$$

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (310)$$

$$\begin{aligned}
& \textcolor{teal}{PMF}(H_X, (X, \Omega, \mathcal{F}, P)) \iff \left(\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\
& \quad \left(\forall_{x \in \mathbb{R}} \left(P(\{\omega \in \Omega \mid X(\omega) = x\}) = P(X = x) = H_X(x) \right) \right) \\
& \quad \# \text{ complete probability decomposition of the probability law for discrete random variables} \quad (311)
\end{aligned}$$

$$\begin{aligned}
& \textcolor{teal}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff \left(\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\
& \quad \left(\exists_{E \subseteq \mathbb{R}} (\textcolor{teal}{countablyInfinite}(E) \wedge P_X(E) = 1) \right) \wedge \left(\bigcup ((e)_{\mathbb{N}}) = E \right) \wedge \left(\forall_{i \in \mathbb{N}} (e_i \in E) \right) \quad (312)
\end{aligned}$$

$$\begin{aligned}
& \text{(THM) : } \left(\textcolor{blue}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \implies \\
& \quad \left(1 = P_X(E) = \sum_{i \in \mathbb{N}} (P_X(\{e_i\})) = \sum_{i \in \mathbb{N}} (P(X = e_i)) \right) \wedge \left(\forall_{B \in \sigma_S} \left(P_X(B) = \sum_{x \in E \cap B} (P(X = x)) \right) \right) \quad (313)
\end{aligned}$$

$$\begin{aligned} \text{indicatorRandomVariable}(I_A, (\Omega, \mathcal{F}, P)) &\iff \left(\text{randomVariable}(I_A, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\left(\forall A \in \mathcal{F} \forall \omega \in \Omega \left(I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases} \right) \right) \end{aligned} \quad (314)$$

$$\begin{aligned} \text{bernoulliRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge (E = \{0, 1\}) \wedge \\ &(p \in \mathbb{R}) \wedge \left(P_X = P(X = x) = \begin{cases} 1 & x = 1 \\ 0 & x = 0 \end{cases} \right) \end{aligned} \quad (315)$$

$$\begin{aligned} \text{uniformRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &(n = |\text{finiteSet}(E)|) \wedge \left(\forall i \in \mathbb{N} \wedge i \leq n \left(P_X(\{e_i\}) = P(X = e_i) = \frac{1}{n} \right) \right) \end{aligned} \quad (316)$$

$$\begin{aligned} \text{geometricRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &(\text{countablyInfinite}(E)) \wedge (p \in \mathbb{R}) \wedge \left(\forall i \in \mathbb{N} \left(P_X(\{e_i\}) = P(X = e_i) = (1 - p)^{i-1} p \right) \right) \end{aligned} \quad (317)$$

$$\begin{aligned} \text{binomialRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &(n = |\text{finiteSet}(E)|) \wedge (p \in \mathbb{R}) \wedge \left(\forall i \in \mathbb{N} \left(P_X(\{e_i\}) = P(X = e_i) = \binom{n}{i} p^i (1 - p)^{n-i} \right) \right) \end{aligned} \quad (318)$$

$$\begin{aligned} \text{poissonRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &(\text{countablyInfinite}(E)) \wedge (\lambda \in \mathbb{R}^+) \wedge \left(\forall i \in \mathbb{N} \left(P_X(\{e_i\}) = P(X = e_i) = \frac{e^{-\lambda} \lambda^i}{i!} \right) \right) \end{aligned} \quad (319)$$

$$\begin{aligned} \text{absolutelyContinuous}((f, g), (M, \sigma)) &\iff \left(\text{measure}(f, (M, \sigma)) \right) \wedge \left(\text{measure}(g, (M, \sigma)) \right) \wedge \\ &\left(\forall A \in \sigma (g(A) = 0 \implies f(A) = 0) \right) \end{aligned} \quad (320)$$

$$\begin{aligned} \text{(THM) Radon-Nikodym: } &\left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge \left(\text{finiteMeasure}(\mu, (M, \sigma)) \right) \wedge \\ &\left(\text{finiteMeasure}(\nu, (M, \sigma)) \right) \wedge \left(\text{absolutelyContinuous}((\nu, \mu), (M, \sigma)) \right) \implies \\ &\left(\exists_{\text{map}}(f, (M, \mathbb{R}^+)) \forall A \in \sigma \left(\nu(A) = \int_A (f d\mu) \right) \right) \\ &\# \text{ connects } P_X = F_X = \int (f_x dx) \end{aligned} \quad (321)$$

$$\begin{aligned} \text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{randomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\left(\text{absolutelyContinuous}((P_X, \text{lebesgueMeasure}), (\mathbb{R}, \text{euclideanSigma})) \right) \\ &\# \text{ the probabilities lie on nonzero lebesgue measure sets} \end{aligned} \quad (322)$$

$$\begin{aligned} \text{contUniformRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\quad (a, b \in \mathbb{R}) \wedge (a < b) \wedge \left(P_X = F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases} \right) \end{aligned} \quad (323)$$

$$\begin{aligned} \text{exponentialRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\quad (\lambda \in \mathbb{R}^+) \wedge \left(P_X = F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases} \right) \end{aligned} \quad (324)$$

$$\text{memorylessRandomVariable}(X, ()) \iff \left(\forall \omega \in \Omega (X(\omega) \geq 0) \right) \wedge \left(\forall s, t \in \mathbb{R}_0^+ (P(X > s) = P(X > s + t | x > t)) \right) \quad (325)$$

$$\begin{aligned} \text{gaussianRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\quad (\mu \in \mathbb{R}) \wedge (\sigma \in \mathbb{R}^+) \wedge \left(P_X = F_X(x) = \int \left(\frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}} dx \right) \right) \end{aligned} \quad (326)$$

$$(\text{THM}) : \text{DEFINE gaussian is stable and is an attractor} \quad (327)$$

$$\begin{aligned} \text{simplifiedCauchyRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\quad \left(P_X = F_X(x) = \int \left(\frac{1}{\pi(1+x^2)} dx \right) \right) \end{aligned} \quad (328)$$

$$\begin{aligned} \text{singularRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff \left(\text{randomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \\ &\quad \left(\forall x \in \mathbb{R} (P_X(\{x\}) = 0) \right) \wedge \left(\exists F \in \text{euclideanSigma} (P_X(F) = 1 \wedge \text{lebesgueMeasure}(F) = 0) \right) \\ &\quad \# \text{ an example is uniform measure on the Cantor set} \end{aligned} \quad (329)$$

$$(\text{THM}) : (\text{cantor set} \cong \mathcal{P}(\mathbb{N}) \wedge (\mathbb{R}, \text{euclideanSigma}, \text{lebesgueMeasure})) \implies P(\text{cantor set}) = 0 \# :O \quad (330)$$

$$\text{===== N O T = U P D A T E D =====} \quad (331)$$

2.5 Joint random variables

$$\begin{aligned} \text{jointRV}((X, Y), (\Omega, \mathcal{F}, P)) &\iff \left(\text{randomVariable}(X, (\Omega, \mathcal{F}, P)) \right) \wedge \left(\text{randomVariable}(Y, (\Omega, \mathcal{F}, P)) \right) \\ &\quad \left(\text{measurableMap}\left((X, Y), (\Omega, \mathcal{F}, \mathbb{R}^2, \sigma_S^2)\right) \right) \\ &\quad \# \text{ the preimage of a measurable set of n dimensional vectors is an event} \end{aligned} \quad (332)$$

$$\text{jointPL}(P_{X,Y}, ((X, Y), \Omega, \mathcal{F}, P)) \iff \left(\text{jointRV}((X, Y), (\Omega, \mathcal{F}, P)) \right) \wedge$$

$$\left(\forall_{(B_x, B_y) \in \sigma_{S^2}} \left(P_{X,Y}(B_x, B_y) = P(\{\omega \in \Omega \mid X(\omega) \in B_x\} \cap \{\omega \in \Omega \mid Y(\omega) \in B_y\}) = P(X \in B_x, Y \in B_y) \right) \right) \quad (333)$$

$$\begin{aligned} & \text{jointCDF}(F_{X,Y}, ((X,Y), \Omega, \mathcal{F}, P)) \iff \left(\text{jointRV}((X,Y), (\Omega, \mathcal{F}, P)) \right) \wedge \\ & \forall_{(x,y) \in \mathbb{R}^2} \left(F_{X,Y}(x,y) = P(\{\omega \in \Omega \mid X(\omega) \leq x\} \cap \{\omega \in \Omega \mid Y(\omega) \leq y\}) = P(X \leq x, Y \leq y) \right) \end{aligned} \quad (334)$$

$$\begin{aligned} \text{(THM)} : \text{jointCDF}(F_{X,Y}, ((X,Y), \Omega, \mathcal{F}, P)) & \iff \left(\lim_{\substack{x \rightarrow -\infty \\ y \rightarrow -\infty}} (F_{X,Y}(x,y)) = 0 \right) \wedge \left(\lim_{\substack{x \rightarrow \infty \\ y \rightarrow \infty}} (F_{X,Y}(x,y)) = 1 \right) \wedge \\ & \left(\forall_{(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2} \left((x_1 \leq x_2 \wedge y_1 \leq y_2) \implies (F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)) \right) \right) \wedge \\ & \left(\forall_{(x,y) \in \mathbb{R}^2} \left(\lim_{\substack{\epsilon_x \rightarrow 0^+ \\ \epsilon_y \rightarrow 0^+}} (F(x + \epsilon_x, y + \epsilon_y) - F(x, y)) = 0 \right) \right) \wedge \\ & \left(\forall_{x \in \mathbb{R}} \left(\lim_{y \rightarrow \infty} (F_{X,Y}(x, y)) = F_X(x) \right) \right) \wedge \left(\forall_{y \in \mathbb{R}} \left(\lim_{x \rightarrow \infty} (F_{X,Y}(x, y)) = F_Y(y) \right) \right) \\ & \# \text{ limit evaluation order or trajectory should not matter} \end{aligned} \quad (335)$$

$$\begin{aligned} & \text{jointPMF}(H_{X,Y}, ((X,Y), \Omega, \mathcal{F}, P)) \iff \left(\text{jointRV}((X,Y), (\Omega, \mathcal{F}, P)) \right) \wedge \\ & \left(\forall_{(x,y) \in \mathbb{R}^2} \left(H_{X,Y}(x,y) = P(\{\omega \in \Omega \mid X(\omega) = x\} \cap \{\omega \in \Omega \mid Y(\omega) = y\}) = P(X = x, Y = y) \right) \right) \end{aligned} \quad (336)$$

2.6 Independence

$$\begin{aligned} & \text{independentEvents}((A, B), (\Omega, \mathcal{F}, P)) \iff (A, B \in \mathcal{F}) \wedge (P(A \cap B) = P(A)P(B)) \\ & \# \text{ depends on } P, A, B \end{aligned} \quad (337)$$

$$\begin{aligned} & \text{finIndEvents}\left((A)_{i=1}^k, (\Omega, \mathcal{F}, P)\right) \iff \left(\text{probabilitySpace}(\Omega, \mathcal{F}, P) \right) \wedge \left(\forall_{i \in \mathbb{N} \wedge i \leq k} (A_i \in \mathcal{F}) \right) \wedge \\ & \left(\forall_{I_0 \subseteq (A)_{i=1}^k} \left(P\left(\bigcap_{i \in I_0} (A_i) \right) = \prod_{i \in I_0} (P(A_i)) \right) \right) \\ & \# \text{ every combination of events must be independent} \end{aligned} \quad (338)$$

$$\begin{aligned} & \text{arbIndEvents}((A)_I, (\Omega, \mathcal{F}, P)) \iff \left(\forall_{\text{finiteSet}(I_F) \subseteq I} \left(\text{finIndEvents}\left((A)_{I_F}, (\Omega, \mathcal{F}, P)\right) \right) \right) \\ & \# \text{ every finite subset is independent} \end{aligned} \quad (339)$$

$$\text{subSigmaAlgebra}(\mathcal{B}, (\mathcal{F}, \Omega)) \iff \left(\text{sigmaAlgebra}(\mathcal{F}, (\Omega)) \right) \wedge \left(\text{sigmaAlgebra}(\mathcal{B}, (\Omega)) \right) \wedge (\mathcal{B} \subseteq \mathcal{A}) \quad (340)$$

$$\begin{aligned} & \text{independentSigmaAlgebras}((\mathcal{A}, \mathcal{B}), (\Omega, \mathcal{F}, P)) \iff \left(\text{probabilitySpace}(\Omega, \mathcal{F}, P) \right) \wedge \\ & \left(\text{subSigmaAlgebra}(\mathcal{A}, (\mathcal{F}, \Omega)) \right) \wedge \left(\text{subSigmaAlgebra}(\mathcal{B}, (\mathcal{F}, \Omega)) \right) \wedge \\ & \left(\forall_{A \in \mathcal{A}} \forall_{B \in \mathcal{B}} \left(\text{independentEvents}((A, B), (\Omega, \mathcal{F}, P)) \right) \right) \end{aligned} \quad (341)$$

$$\begin{aligned} \text{finIndSigmaAlgebras}\left((\mathcal{A})_{i=1}^k, (\Omega, \mathcal{F}, P)\right) &\iff \left(\forall_{i \in \mathbb{N} \wedge i \leq k} \left(\text{subSigmaAlgebra}(\mathcal{A}_i), (\mathcal{F}, \Omega)\right)\right) \wedge \\ &\left(\forall_{i \in \mathbb{N} \wedge i \leq k} (A_i \in \mathcal{A}_i)\right) \wedge \left(\text{finIndEvents}\left((A)_{i=1}^k, (\Omega, \mathcal{F}, P)\right)\right) \end{aligned} \quad (342)$$

$$\text{arbIndSigmaAlgebras}\left((\mathcal{A})_I, (\Omega, \mathcal{F}, P)\right) \iff \left(\forall_{\text{finiteSet}(I_F) \subseteq I} \left(\text{finIndSigmaAlgebras}\left((\mathcal{A})_{I_F}, (\Omega, \mathcal{F}, P)\right)\right)\right) \quad (343)$$

$$\begin{aligned} \text{preimageSigma}\left(\sigma_{RV}(X), (X, \Omega, \mathcal{F}, P)\right) &\iff \left(\text{randomVariable}(X, (\Omega, \mathcal{F}, P))\right) \wedge \\ &\left(\sigma_{RV}(X) = \{\text{preimage}(A, (B, X, \Omega, \mathbb{R})) \mid B \in \text{euclideanSigma}\}\right) \end{aligned}$$

reduced sigma algebra generated by preimage of borel sets in X ; groups Ω subsets by borel preimages (344)

$$(\text{THM}) : \text{preimageSigma}\left(\sigma_{RV}(X), (X, \Omega, \mathcal{F}, P)\right) \implies \text{subSigmaAlgebra}\left(\sigma_{RV}(X), (\mathcal{F}, \Omega)\right) \quad (345)$$

$$\text{independentRVs}\left((X, Y), (\Omega, \mathcal{F}, P)\right) \iff \text{independentSigmaAlgebras}\left((\sigma_{RV}(X), \sigma_{RV}(Y)), (\Omega, \mathcal{F}, P)\right) \quad (346)$$

$$\begin{aligned} \text{finIndRVs}\left((X)_{i=1}^k, (\Omega, \mathcal{F}, P)\right) &\iff \left(\forall_{i \in \mathbb{N} \wedge i \leq k} \left(\text{randomVariable}(X_i), (\Omega, \mathcal{F}, P)\right)\right) \wedge \\ &\left(\forall_{i \in \mathbb{N} \wedge i \leq k} (\sigma_i = \sigma_{RV}(X_i))\right) \wedge \left(\text{finIndSigmaAlgebras}\left((\sigma)_{i=1}^k, (\Omega, \mathcal{F}, P)\right)\right) \end{aligned} \quad (347)$$

$$\text{arbIndRVs}\left((X)_I, (\Omega, \mathcal{F}, P)\right) \iff \left(\forall_{\text{finiteSet}(I_F) \subseteq I} \left(\text{finIndRVs}\left((X)_{I_F}, (\Omega, \mathcal{F}, P)\right)\right)\right) \quad (348)$$

$$(\text{THM}) : \text{finIndEvents}\left((A)_{i=1}^k, (\Omega, \mathcal{F}, P)\right) \implies P\left(\bigcap_{i=1}^k (A_i)\right) = \prod_{i=1}^k (P(A_i)) \quad (349)$$

$$\begin{aligned} (\text{THM}) : \text{independentRVs}\left((X, Y), (\Omega, \mathcal{F}, P)\right) &\iff \left(\forall_{x, y \in \mathbb{R}} (F_{X,Y}(x, y) = F_X(x)F_Y(y))\right) \wedge \\ &\left(\forall_{B_x, B_y \in \sigma_S} (P_{X,Y}(B_x, B_y) = P_X(B_x)P_Y(B_y))\right) \end{aligned} \quad (350)$$

2.7 joint random variables shenanigans

$$\begin{aligned} \text{jointConditionalProbability}\left(P_{X|Y}(x|y), ((X, Y), \Omega, \mathcal{F}, P)\right) &\iff P_{X|Y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)} = \frac{P(X=x, Y=y)}{P(Y=y)} \\ \# \text{ conditions on the probability spanned by } Y; P_Y = P_{X,Y}(\mathbb{R}, B_y) &\text{ can be gained from PMF} \end{aligned} \quad (351)$$

$$\begin{aligned} \text{jointlyDiscreteRV}\left((X, Y), (\Omega, \mathcal{F}, P)\right) &\iff (\text{discreteRV}(X, (\Omega, \mathcal{F}, P))) \wedge (\text{discreteRV}(Y, (\Omega, \mathcal{F}, P))) \iff \\ &\text{existscountableblabla} \end{aligned} \quad (352)$$

$$\begin{aligned} (\text{THM}) : (\text{discreteRandomVariable}(X) \wedge \text{discreteRandomVariable}(Y)) &\implies \\ \left(\text{independentRVs}(X, Y) \implies \forall_{B_x, B_y \in \sigma_S} (P_{X,Y}(B_x, B_y) = P_X(B_x)P_Y(B_y))\right) &\implies \end{aligned}$$

$$\begin{aligned}
& \forall_{x,y \in \mathbb{R}} \left(P_X(\{x\})P_Y(\{y\}) = P(X=x)P(Y=y) \right) \implies \text{independentEvents}(\{X=x\}, \{Y=y\}) \Big) \wedge \\
& \left(\text{independentEvents}(\{X=x\}, \{Y=y\}) \implies \forall_{B_x, B_y \in \sigma} \left(P_{X,Y}(B_x, B_y) = \sum_{\substack{x \in B_x \\ y \in B_y}} (P_X(x)P_Y(y)) = \right. \right. \\
& \left. \left(\sum_{x \in B_x} (P(X=x)) \right) \left(\sum_{y \in B_y} (P(Y=y)) \right) = P_X(B_x)P_Y(B_y) \right) \implies \text{independentRVs}(X, Y) \Big) \\
& \# \text{ independence of discrete } X, Y \text{ is equivalent to independence of joint PMF} \quad (353)
\end{aligned}$$

$$\begin{aligned}
& \text{jointlyContinuousRV}((X, Y), (\Omega, \mathcal{F}, P)) \iff (\text{jointRV}((X, Y), (\Omega, \mathcal{F}, P))) \wedge \\
& (\text{absolutelyContinuous}((P_{X,Y}, \text{lebesgueMeasure}^2), (\mathbb{R}^2, \sigma_S^2))) \quad (354)
\end{aligned}$$

$$\begin{aligned}
& (\text{THM}) : \left((\text{continuousRV}(X, (\Omega, \mathcal{F}, P))) \wedge (\text{continuousRV}(Y, (\Omega, \mathcal{F}, P))) \right) \not\Rightarrow \\
& \text{jointlyContinuousRV}((X, Y), (\Omega, \mathcal{F}, P)) \Big) \Leftarrow (k \in \mathbb{R}) \wedge (Y = kX) \wedge (\text{maps probability to a line}) \quad (355)
\end{aligned}$$

$$\begin{aligned}
& (\text{THM}) : \left(\text{jointlyContinuousRV}((X, Y), (\Omega, \mathcal{F}, P)) \implies (\text{continuousRV}(X, (\Omega, \mathcal{F}, P))) \wedge \right. \\
& \left. (\text{continuousRV}(Y, (\Omega, \mathcal{F}, P))) \right) \Leftarrow (\forall_{x,y \in \mathbb{R}} \exists_{\text{map}(f_{X,Y}) \geq 0} (F_{X,Y}(x, y)) = \int_{-\infty}^x \int_{-\infty}^y (f_{X,Y}(s, t) dt ds)) \wedge \\
& \# \text{ from radon nikodym thm and semi infinite generating sets} \\
& (\forall_{x \in \mathbb{R}} (F_X(x) = \lim_{y \rightarrow \infty} (F_{X,Y}(x, y)) = \int_{-\infty}^x \int_{-\infty}^{\infty} (f_{X,Y}(s, t) dt ds))) = \int_{-\infty}^x (\int_{-\infty}^{\infty} (f_{X,Y}(s, t) dt) ds) = \int_{-\infty}^x (f(s) ds) \\
& \# \text{ from nonnegative integral} \quad (356)
\end{aligned}$$

$$\text{CONTINUE23} \quad (357)$$

$$\text{START10} - 715 \quad (358)$$

$$\text{conditionalIndependence}((A, B, C), (???)) \iff P(A \cap B | C) = P(A | C)P(B | C) \quad (359)$$

$$0 \quad (360)$$

3 Machine Learning

3.1 *terminology

$$(361)$$

$$\text{targetFunction}(f, (x_{\mathbb{N}}, y_{\mathbb{N}})) \iff (\text{map}(f, (x_{\mathbb{N}}, y_{\mathbb{N}}))) \wedge (\forall_{i \in \mathbb{N}} (f(x_i) = y_i)) \quad (362)$$

$$\begin{aligned}
& \text{hypothesis}(g, (f, x_{\mathbb{N}}, y_{\mathbb{N}})) \iff g \sim \text{targetFunction}(f, (x_{\mathbb{N}}, y_{\mathbb{N}})) \\
& \# \text{ TODO: formalize} \quad (363)
\end{aligned}$$

$$\text{hypothesisSpace}(\mathcal{H}, (f, x_{\mathbb{N}}, y_{\mathbb{N}})) \iff \forall_{h \in \mathcal{H}} (\text{hypothesis}(h, (f, x_{\mathbb{N}}, y_{\mathbb{N}})))$$

TODO: motivate (364)

$$\text{learningAlgorithm}(L, (\mathcal{H}, f, x_{\mathbb{N}}, y_{\mathbb{N}})) \iff (L(\mathcal{H}, f, x_{\mathbb{N}}, y_{\mathbb{N}}) = \text{hypothesis}(P, (f, x_{\mathbb{N}}, y_{\mathbb{N}})) \in \text{hypothesisSpace}(\mathcal{H}, (f, x_{\mathbb{N}}, y_{\mathbb{N}})))$$

(365)

$$\text{learningModel}(\text{learningAlgorithm}, \text{hypothesisSpace})$$

(366)

$$0$$

(367)

3.2 *axioms

(368)

(369)

TODO: make axioms (370)

4 Machine Learning ADV

4.0.1 OLD Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X, Y) \text{ \# dataset}$$

(371)

learned parameters = parameters to be fixed by training with the dataset

(372)

hyperparameters = parameters that depends on a dataset

(373)

validation = partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition # useful for fixing hyperparameters

(374)

cross-validation = average accuracy of validation for different choices of testing partition

(375)

L1 = scales linearly ; L2 = scales quadratically

(376)

d = distance = quantifies the the similarity between data points

(377)

$$d_{L1}(A, B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance}$$

(378)

$$d_{L2}(A, B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance}$$

(379)

kNN classifier = classifier based on k nearest data points (380)

s = class score = quantifies bias towards a particular class (381)

$s_{linear} = f_{c \times 1}(x_{n \times 1}, W_{c \times n}, b_{c \times 1}) = W_{c \times n} x_{n \times 1} + b_{c \times 1}$ # linear score function (382)

l = loss = quantifies the errors by the learned parameters (383)

$l = \frac{1}{|c_i|} \sum_{c_i} l_i$ # average loss for all classes (384)

$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1)$ # SVM hinge class loss function:
ignores incorrect classes with lower scores including a non-zero margin (385)

$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right)$ # Softmax class loss function
lower scores correspond to lower exponentiated-normalized probabilities (386)

R = regularization = optimizes the choice of learned parameters to minimize test error (387)

λ # regularization strength hyperparameter (388)

$R_{L1}(W) = \sum_{W_i} |W_i|$ # L1 regularization (389)

$R_{L2}(W) = \sum_{W_i} W_i^2$ # L2 regularization (390)

$L' = L + \lambda R(W)$ # weight regularization (391)

$\nabla_W L = \frac{\partial}{\partial W_i} L$ = **loss gradient w.r.t. weights** (392)

$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L}$ # loss gradient w.r.t. input weight in terms of external and local gradients (393)

s = forward API ; $\frac{\partial L_L}{\partial W_I}$ = backward API (394)

$W_{t+1} = W_t - \nabla_{W_t} L$ # weight update loss minimization (395)

TODO: Research on Activation functions, Weight Initialization, Batch Normalization (396)

review5meanvardiscussion/hyperparameteroptimization/babysittinglearning (397)

TODO loss L or l ??

5 Glossary

chaoticTopology	openRefinement	normedVectorSpace	independentOperator
discreteTopology	locallyFinite	vectorMetric	dimensionality
topology	paracompact	metricVectorSpace	rank
topologicalSpace	openRefinement	innerProductNorm	transposeNorm
open	locallyFinite	normInnerProduct	orthogonalVectors
closed	paracompact	normMetric	orthogonalOperator
clopen	connected	metricNorm	orthogonalProjection
neighborhood	pathConnected	orthogonal	eigenvectors
chaoticTopology	connected	normal	det
discreteTopology	pathConnected	basis	tr
metric	sigmaAlgebra	orthonormalBasis	diagonalOperator
metricSpace	measurableSpace	vectorSpace	characteristicEquation
openBall	measurableSet	innerProduct	eigenDecomposition
metricTopology	measure	innerProductSpace	spectralDecomposition
metricTopologicalSpace	measureSpace	vectorNorm	hermitianAdjoint
limitPoint	finiteMeasure	normedVectorSpace	hermitianOperator
interiorPoint	generatedSigmaAlgebra	vectorMetric	unitaryOperator
closure	borelSigmaAlgebra	metricVectorSpace	positiveDefiniteOperator
dense	euclideanSigma	innerProductNorm	semiPositiveDefiniteOperator
eucD	lebesgueMeasure	normInnerProduct	similarOperators
euclideanTopology	measurableMap	normMetric	similarOperators
subsetTopology	pushForwardMeasure	metricNorm	singularValueDecomposition
productTopology	nullSet	orthogonal	linearOperator
metric	almostEverywhere	normal	matrix
metricSpace	sigmaAlgebra	basis	eigenvector
openBall	measurableSpace	orthonormalBasis	eigenvalue
metricTopology	measurableSet	subspace	identityOperator
metricTopologicalSpace	measure	subspaceSum	inverseOperator
limitPoint	measureSpace	subspaceDirectSum	transposeOperator
interiorPoint	finiteMeasure	orthogonalComplement	symmetricOperator
closure	generatedSigmaAlgebra	orthogonalDecomposition	triangularOperator
dense	borelSigmaAlgebra	subspace	decomposeLU
eucD	euclideanSigma	subspaceSum	Img
euclideanTopology	lebesgueMeasure	subspaceDirectSum	Ker
subsetTopology	measurableMap	orthogonalComplement	independentOperator
productTopology	pushForwardMeasure	orthogonalDecomposition	dimensionality
sequence	nullSet	cauchy	rank
sequenceConvergesTo	almostEverywhere	complete	transposeNorm
sequence	simpleTopology	banachSpace	orthogonalVectors
sequenceConvergesTo	simpleSigma	hilbertSpace	orthogonalOperator
continuous	simpleFunction	separable	orthogonalProjection
homeomorphism	characteristicFunction	cauchy	eigenvectors
isomorphicTopologicalSpace	exEuclideanSigma	complete	det
continuous	nonNegIntegrable	banachSpace	tr
homeomorphism	nonNegIntegral	hilbertSpace	diagonalOperator
isomorphicTopologicalSpace	explicitIntegral	separable	characteristicEquation
T0Separate	integrable	watR	eigenDecomposition
T1Separate	integral	group	spectralDecomposition
T2Separate	simpleTopology	watR	hermitianAdjoint
T0Separate	simpleSigma	group	hermitianOperator
T1Separate	simpleFunction	linearOperator	unitaryOperator
T2Separate	characteristicFunction	matrix	positiveDefiniteOperator
openCover	exEuclideanSigma	eigenvector	semiPositiveDefiniteOperator
finiteSubcover	nonNegIntegrable	eigenvalue	similarOperators
compact	nonNegIntegral	identityOperator	similarOperators
compactSubset	explicitIntegral	inverseOperator	singularValueDecomposition
bounded	integrable	transposeOperator	denseMap
openCover	integral	symmetricOperator	mapNorm
finiteSubcover	vectorSpace	triangularOperator	boundedMap
compact	innerProduct	decomposeLU	extensionMap
compactSubset	innerProductSpace	Img	adjoint
bounded	vectorNorm	Ker	selfAdjoint

compactMap	MSConvU	randomVariable	jointPL
denseMap	MSSetOrder	PL	jointCDF
mapNorm	MSSetBound	generatedSigmaAlgebra	jointPMF
boundedMap	conditionalProbability	piSystem	independentEvents
extensionMap	setPartition	CDF	finIndEvents
adjoint	infinitelyOften	randomVariable	arbIndEvents
selfAdjoint	Cond300	PL	subSigmaAlgebra
compactMap	1IL300	generatedSigmaAlgebra	independentSigmaAlgebras
curLp	2IL300	piSystem	finIndSigmaAlgebras
vecLp	3IL300	CDF	arbIndSigmaAlgebras
integralNorm	Impl300	PMF	preimageSigma
Lp	logp	discreteRandomVariable	independentRVs
curL	sump	indicatorRandomVariable	finIndRVs
curLp	1Cond302	bernoulliRandomVariable	arbIndRVs
vecLp	2Cond302	uniformRandomVariable	independentEvents
integralNorm	1IL302	geometricRandomVariable	finIndEvents
Lp	2IL302	binomialRandomVariable	arbIndEvents
curL	3IL302	poissonRandomVariable	subSigmaAlgebra
randomExperiment	Impl302	absolutelyContinuous	independentSigmaAlgebras
probabilitySpace	1Cond303	continuousRandomVariable	finIndSigmaAlgebras
measureSpace	2Cond303	contUniformRandomVariable	arbIndSigmaAlgebras
event	1IL303	exponentialRandomVariable	preimageSigma
CL285	2IL303	memorylessRandomVariable	independentRVs
DL285	3IL303	gaussianRandomVariable	finIndRVs
EL285	Impl303	simplifiedCauchyRandomVariable	arbIndRVs
1IL285	conditionalProbability	singularRandomVariable	jointConditionalProbability
2IL285	setPartition	PMF	jointlyDiscreteRV
3IL285	infinitelyOften	discreteRandomVariable	jointlyContinuousRV
4IL285	Cond300	indicatorRandomVariable	conditionalIndependence
MSCont	1IL300	bernoulliRandomVariable	jointConditionalProbability
MSConvL	2IL300	uniformRandomVariable	jointlyDiscreteRV
MSConvU	3IL300	geometricRandomVariable	jointlyContinuousRV
MSSetOrder	Impl300	binomialRandomVariable	conditionalIndependence
MSSetBound	logp	poissonRandomVariable	targetFunction
randomExperiment	sump	absolutelyContinuous	hypothesis
probabilitySpace	1Cond302	continuousRandomVariable	hypothesisSpace
measureSpace	2Cond302	contUniformRandomVariable	learningAlgorithm
event	1IL302	exponentialRandomVariable	learningModel
CL285	2IL302	memorylessRandomVariable	targetFunction
DL285	3IL302	gaussianRandomVariable	hypothesis
EL285	Impl302	simplifiedCauchyRandomVariable	hypothesisSpace
1IL285	1Cond303	singularRandomVariable	learningAlgorithm
2IL285	2Cond303	jointRV	learningModel
3IL285	1IL303	jointPL	
4IL285	2IL303	jointCDF	
MSCont	3IL303	jointPMF	
MSConvL	Impl303	jointRV	