Next-Next-Gen Notes
Object-Oriented Maths

JP Guzman

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Format: $characteristic((subjects),(dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$
Note: All weaker objects automatically induces notions inherited from stronger objects.
TODO define $\|$ abs cross-product and other missing refs
TODO define **args for comparison callbacks, predicate args, norms and or placeholders
TODO link thms?

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement\big(s,(RegEx)\big) \iff well\text{-}formedString\big(s,()\big) \tag{1}$$

$$proposition\big((p,t),()\big) \iff \Big(statement\big(p,()\big)\Big) \wedge$$
$$\big(t = eval(p)\big) \wedge$$
$$(t = true \veebar t = false) \tag{2}$$

$$operator\bigg(o,\Big((p)_{n\in\mathbb{N}}\Big)\bigg) \iff proposition\bigg(o\Big((p)_{n\in\mathbb{N}}\Big),()\bigg) \tag{3}$$

$$operator\big(\neg,(p_1)\big) \iff \Big(proposition\big((p_1,true),()\big) \implies \big((\neg p_1, false),()\big)\Big) \wedge$$
$$\Big(proposition\big((p_1,false),()\big) \implies \big((\neg p_1, true),()\big)\Big)$$
$$\#\text{ an operator takes in propositions and returns a proposition} \tag{4}$$

$$operator(\neg) \iff \mathbf{NOT} \; ; \; operator(\vee) \iff \mathbf{OR} \; ; \; operator(\wedge) \iff \mathbf{AND} \; ; \; operator(\veebar) \iff \mathbf{XOR}$$
$$operator(\implies) \iff \mathbf{IF} \; ; \; operator(\impliedby) \iff \mathbf{OIF} \; ; \; operator(\iff) \iff \mathbf{IFF} \tag{5}$$

$$proposition\big((false \implies true),true,()\big) \wedge proposition\big((false \implies false),true,()\big)$$
$$\#\text{ truths based on a false premise is not false; ex falso quodlibet principle} \tag{6}$$

$$(\text{THM}) : \big(a \implies b \implies c\big) \iff \big(a \implies (b \implies c)\big) \iff \big((a \wedge b) \implies c\big) \tag{7}$$

$$predicate\big(P,(V)\big) \iff \forall_{v\in V}\bigg(proposition\Big((P(v),t),()\Big)\bigg) \tag{8}$$

$$0thOrderLogic\big(P,()\big) \iff proposition\big((P,t),()\big)$$
$$\#\text{ individual proposition} \tag{9}$$

$$1stOrderLogic\big(P,(V)\big) \iff \bigg(\forall_{v\in V}\Big(0thOrderLogic\big(v,()\big)\Big)\bigg) \wedge$$

$$\left(\forall_{v \in V} \Big(proposition\Big(\big(P(v), t\big), ()\Big)\Big) \right)$$
$$\text{\# propositions defined over a set of the lower order logical statements} \qquad (10)$$

$$quantifier\big(q, (p, V)\big) \Longleftrightarrow \Big(predicate\big(p, (V)\big)\Big) \wedge$$
$$\Big(proposition\Big(\big(q(p), t\big), ()\Big)\Big)$$
$$\text{\# a quantifier takes in a predicate and returns a proposition} \qquad (11)$$

$$quantifier\big(\forall, (p, V)\big) \Longleftrightarrow proposition\Big(\Big(\wedge_{v \in V}\big(p(v)\big), t\Big), ()\Big)$$
$$\text{\# universal quantifier} \qquad (12)$$

$$quantifier\big(\exists, (p, V)\big) \Longleftrightarrow proposition\Big(\Big(\vee_{v \in V}\big(p(v)\big), t\Big), ()\Big)$$
$$\text{\# existential quantifier} \qquad (13)$$

$$quantifier\big(\exists!, (p, V)\big) \Longleftrightarrow \exists_{x \in V}\Big(P(x) \wedge \neg\Big(\exists_{y \in V \setminus \{x\}}\big(P(y)\big)\Big)\Big)$$
$$\text{\# uniqueness quantifier} \qquad (14)$$

$$\text{(THM)} : \forall_x p(x) \Longleftrightarrow \neg\exists_x \neg p(x)$$
$$\text{\# De Morgan's law} \qquad (15)$$

$$\text{(THM)} : \forall_x \exists_y p(x,y) = \forall_x \neg\forall_y \neg p(x,y) \neq \exists_y \forall_x p(x,y) = \neg\forall_y \neg\big(\forall_x p(x,y)\big) = \neg\forall_y \exists_x \neg p(x,y)$$
$$\text{\# different quantifiers are not interchangeable} \qquad (16)$$

$$\mathbf{=========== \ N \ O \ T = U \ P \ D \ A \ T \ E \ D \ ===========} \qquad (17)$$

$$\mathbf{proof = truths\ derived\ from\ a\ finite\ number\ of\ axioms\ and\ deductions} \qquad (18)$$

$$\mathbf{elementary\ arithmetics = system\ with\ substitutions,\ and\ some\ notion\ of}$$
$$\mathbf{addition,\ multiplication,\ and\ prime\ nuumbers\ for\ encoding\ metamathematics} \qquad (19)$$

$$\mathbf{G\ddot{o}del\ theorem \Longrightarrow axiomatic\ systems\ equivalent\ in\ power\ to\ elementary\ mathematics}$$
$$\mathbf{either\ has\ unprovable\ statements\ or\ has\ contradictions} \qquad (20)$$

$$sequenceSet((A)_{\mathbb{N}}, (A)) \Longleftrightarrow (A\,map\,input\,n)((A)_{\mathbb{N}} = \{A(1), A(2), A(3), \ldots\}) \qquad (21)$$

$$\mathbf{TODO:\ define\ union,\ intersection,\ complement,\ etc.} \qquad (22)$$

$$\mathbf{=========== \ N \ O \ T = U \ P \ D \ A \ T \ E \ D \ ===========} \qquad (23)$$

1.1 Axiomatic Set Theory

$$============ \text{ N O T } = \text{ U P D A T E D } ============ \tag{24}$$

$$\textbf{ZFC set theory} = \textbf{standard form of axiomatic set theory} \tag{25}$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \tag{26}$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \tag{27}$$

$$\in \textbf{ basis} \implies \{x, y\} = \{y, x\} \wedge \{x\} = \{x, x\} \tag{28}$$

$$\in \textbf{ and sets works following the 9 ZFC axioms:} \tag{29}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \ \# \ \text{E: } \in \text{ is only a proposition on sets} \tag{30}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \ \# \ \text{E: existence of empty set} \tag{31}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \ \# \ \text{C: pair set construction} \tag{32}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \ \# \ \text{C: union set construction} \tag{33}$$

$$x = \{\{a\}, \{b\}\} \ \# \ \text{from the pair set axiom} \tag{34}$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \tag{35}$$

$$\forall_x \exists!_y R(x, y) \ \# \ \text{functional relation } R \tag{36}$$

$$\exists_i \forall_x \exists!_y R(x, y) \implies y \in i \ \# \ \text{C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set}$$
$$\implies \{y \in m \,|\, P(y)\} \ \# \ \text{Restricted Comprehension} \implies\!\!\!\!/ \ \{y \,|\, P(y)\} \ \# \ \text{Universal Comprehension} \tag{37}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \implies P(x)\big) \ \# \ \text{ignores out of scope} \neq \forall_x \big(x \in m \wedge P(x)\big) \ \# \ \text{restricts entirety} \tag{38}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \implies n \subseteq \mathcal{P}(m)\big) \ \# \ \text{C: existence of power set} \tag{39}$$

$$\exists_I \Big(\emptyset \in I \wedge \forall_{x \in I} \big(\{x\} \in I\big)\Big) \ \# \ \text{I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \implies \mathbb{N} \textbf{ is a set} \tag{40}$$

$$\forall_x \Big((\emptyset \notin x \wedge x \cap x' = \emptyset) \implies \exists_y (\textbf{set of each } \mathbf{e} \in x)\Big) \ \# \ \text{C: axiom of choice} \tag{41}$$

$$\forall_x x \neq \emptyset \implies x \notin x \ \# \ \text{F: axiom of foundation covers further paradoxes} \tag{42}$$

$$============ \text{ N O T } = \text{ U P D A T E D } ============ \tag{43}$$

1.2 Classification of sets

$$space\big((set, structure), ()\big) \iff structure(set)$$
$$\text{\# a space a set equipped with some structure}$$
$$\text{\# various spaces can be studied through structure preserving maps between those spaces} \quad (44)$$

$$map\big(\phi, (A, B)\big) \iff \Big(\forall_{a \in A} \exists!_{b \in B}\big(\phi(a, b)\big)\Big) \vee$$
$$\Big(\forall_{a \in A} \exists!_{b \in B}\big(b = \phi(a)\big)\Big)$$
$$\text{\# maps elements of a set to elements of another set} \quad (45)$$

$$domain\big(A, (\phi, A, B)\big) \iff \Big(map\big(\phi, (A, B)\big)\Big) \quad (46)$$

$$codomain\big(B, (\phi, A, B)\big) \iff \Big(map\big(\phi, (A, B)\big)\Big) \quad (47)$$

$$image\big(B, (A, q, M, N)\big) \iff \Big(map\big(q, (M, N)\big) \wedge A \subseteq M\Big) \wedge$$
$$\Big(B = \{n \in N \mid \exists_{a \in A}\big(q(a) = n\big)\}\Big) \quad (48)$$

$$preimage\big(A, (B, q, M, N)\big) \iff \Big(map\big(q, (M, N)\big) \wedge B \subseteq N\Big) \wedge$$
$$\Big(A = \{m \in M \mid \exists_{b \in B}\big(b = q(m)\big)\}\Big) \quad (49)$$

$$injection\big(q, (M, N)\big) \iff \Big(map\big(q, (M, N)\big)\Big) \wedge$$
$$\forall_{u, v \in M}\big(q(u) = q(v) \implies u = v\big)$$
$$\text{\# every } m \text{ has at most 1 image} \quad (50)$$

$$surjection\big(q, (M, N)\big) \iff \Big(map\big(q, (M, N)\big)\Big) \wedge$$
$$\forall_{n \in N} \exists_{m \in M}\big(n = q(m)\big)$$
$$\text{\# every } n \text{ has at least 1 preimage} \quad (51)$$

$$bijection\big(q, (M, N)\big) \iff \Big(injection\big(q, (M, N)\big)\Big) \wedge$$
$$\Big(surjection\big(q, (M, N)\big)\Big)$$
$$\text{\# every unique } m \text{ corresponds to a unique } n \quad (52)$$

$$isomorphicSets\big((A, B), ()\big) \iff \exists_\phi\Big(bijection\big(\phi, (A, B)\big)\Big) \quad (53)$$

$$infiniteSet\big(S, ()\big) \iff \exists_{T \subset S}\Big(isomorphicSets\big((T, S), ()\big)\Big) \quad (54)$$

$$finiteSet\big(S, ()\big) \iff \Big(\neg infiniteSet\big(S, ()\big)\Big) \vee \big(|S| \in \mathbb{N}\big) \quad (55)$$

$$countablyInfinite\big(S, ()\big) \iff \Big(infiniteSet\big(S, ()\big)\Big) \wedge \Big(isomorphicSets\big((S, \mathbb{N}), ()\big)\Big) \quad (56)$$

$$uncountablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big) \wedge \Big(\neg isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \quad (57)$$

$$inverseMap\Big(q^{-1},(q,M,N)\Big) \iff \Big(bijection\big(q,(M,N)\big)\Big)\wedge$$
$$\Big(map\big(q^{-1},(N,M)\big)\Big)\wedge$$
$$\Big(\forall_{n\in N}\exists!_{m\in M}\big(q(m)=n \implies q^{-1}(n)=m\big)\Big) \quad (58)$$

$$mapComposition\big(\phi\circ\psi,(\phi,\psi,A,B,C)\big) \iff map\big(\psi,(A,B)\big)\wedge map\big(\phi,(B,C)\big)\wedge$$
$$\forall_{a\in A}\Big(\phi\circ\psi(a)=\phi\big(\psi(a)\big)\Big) \quad (59)$$

$$equivalenceRelation\Big(\sim(\$1,\$2),(M)\Big) \iff \big(\forall_{m\in M}(m\sim m)\big)\wedge$$
$$\big(\forall_{m,n\in M}(m\sim n \implies n\sim m)\big)\wedge$$
$$\big(\forall_{m,n,p\in M}(m\sim n \wedge n\sim p \implies m\sim p)\big)$$
$$\text{\# behaves as equivalences should} \quad (60)$$

$$equivalenceClass\big([m]_\sim,(m,M,\sim)\big) \iff [m]_\sim=\{n\in M\,|\,n\sim m\}$$
$$\text{\# set of elements satisfying the equivalence relation with } m \quad (61)$$

$$\text{(THM)} : a\in[m]_\sim \implies [a]_\sim=[m]_\sim \ ; \ [m]_\sim=[n]_\sim \veebar [m]_\sim\cap[n]_\sim=\emptyset$$
$$\text{\# equivalence class properties} \quad (62)$$

$$quotientSet\big(M/\sim,(M,\sim)\big) \iff M/\sim=\{equivalenceClass\big([m]_\sim,(m,M,\sim)\big)\in\mathcal{P}(M)\,|\,m\in M\}$$
$$\text{\# set of all equivalence classes} \quad (63)$$

$$\text{(THM)} : \bigg(\big(M,\sim,+\big) \implies \Big(quotientSet\big(M/\sim,(M,\sim)\big),+_\sim\Big)\bigg) \iff \forall_{[r],[s]\in M/\sim}\forall_{a\in[r]}\forall_{b\in[s]}\big([r]+_\sim[s]=[a+b]\big)$$
$$\text{\# a quotient set can inherit the operations on the original set if it is well-defined} \quad (64)$$

$$\text{(THM)} : \textbf{axiom of choice} \implies \forall_{[m]_\sim\in M/\sim}\exists_r\big(r\in[m]_\sim\big)$$
$$\text{\# well-defined maps may be defined in terms of chosen representative elements } r \quad (65)$$

1.3 Construction of number sets

$$============ \textbf{N O T = U P D A T E D} ============ \quad (66)$$

$$\textbf{axiom of infinity} \implies \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\ldots\}\cong\mathbb{N} \quad (67)$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \quad (68)$$

$$\textbf{addition}=\textbf{successor map:} \ \mathbb{N}\to\mathbb{N}=S(n)=\{n\} \ \text{\# adds a layer of brackets} \quad (69)$$

$$\textbf{subtraction}=\textbf{predecessor map:} \ \mathbb{N}^\star\to\mathbb{N}=P(n)=m\,|\,m\in n \ \text{\# removes a layer of brackets} \quad (70)$$

$$S^0 = id \; ; \; n \in \mathbb{N}^\star \Longrightarrow S^n = S \circ S^{P(n)} \tag{71}$$

$$\mathbf{addition} = + : \mathbb{N} \times \mathbb{N} \to \mathbb{N} = +(m,n) = m + n = S^n(m) \tag{72}$$

$$S^x = id = S^0 \Longrightarrow x = \mathbf{additive\ identity} = 0 \tag{73}$$

$$S^n(x) = 0 \Longrightarrow x = \mathbf{additive\ inverse} \notin \mathbb{N} \; \# \text{ git gud smh -\_-} \tag{74}$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim, \mathbf{s.t.:}$$
$$(m,n) \sim (p,q) \Longleftrightarrow m + q = p + n \; \# \text{ span } \mathbb{Z} \text{ using differences then group equal differences} \tag{75}$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \to [(n,0)] \; \# \; \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{76}$$

$$+_\mathbb{Z} = [(m +_\mathbb{N} p, n +_\mathbb{N} q)] \; \# \text{ well-defined and consistent} \tag{77}$$

$$\mathbf{multiplication} \dots M^x = id \Longrightarrow x = \mathbf{multiplicative\ identity} = 1 \dots \mathbf{multiplicative\ inverse} \notin \mathbb{N} \tag{78}$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^\star)/\sim, \mathbf{s.t.:} \; (x,y) \sim (u,v) \Longleftrightarrow x \cdot v = u \cdot y \tag{79}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \to [(q,1)] \; ; \; \dots \{x \, | \, x^2 = 2\} \notin \mathbb{Q} \tag{80}$$

$$\mathbb{R} = \mathbf{almost\ homomorphisms\ on\ } \mathbb{Z}/\sim \; \# \text{ http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{81}$$

$$============= \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} ============= \tag{82}$$

1.4 Topology

$$topology\big(\mathcal{O},(M)\big) \Longleftrightarrow \big(\mathcal{O} \subseteq \mathcal{P}(M)\big) \wedge$$
$$\big(\emptyset, M \in \mathcal{O}\big) \wedge$$
$$\Big(\big(F \in \mathcal{O} \wedge |F| < |\mathbb{N}|\big) \Longrightarrow \cap F \in \mathcal{O}\Big) \wedge$$
$$\big(C \subseteq \mathcal{O} \Longrightarrow \cup C \in \mathcal{O}\big)$$
$$\# \text{ topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.}$$
$$\# \text{ arbitrary unions of open sets always result in an open set}$$
$$\# \text{ open sets do not contain their boundaries and infinite intersections of open sets may approach and}$$
$$\# \text{ induce boundaries resulting in a closed set} \tag{83}$$

$$topologicalSpace\big((M,\mathcal{O}),()\big) \Longleftrightarrow topology\big(\mathcal{O},(M)\big) \tag{84}$$

$$open\big(S,(M,\mathcal{O})\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$(S \subseteq M) \wedge (S \in \mathcal{O})$$
$$\# \text{ an open set do not contains its own boundaries} \tag{85}$$

$$closed\big(S,(M,\mathcal{O})\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$(S \subseteq M) \wedge \big(S \in \mathcal{P}(M) \setminus \mathcal{O}\big)$$
$$\#\text{ a closed set contains the boundaries an open set} \quad (86)$$

$$clopen\big(S,(M,\mathcal{O})\big) \Longleftrightarrow \Big(closed\big(S,(M,\mathcal{O})\big)\Big) \wedge \Big(open\big(S,(M,\mathcal{O})\big)\Big) \quad (87)$$

$$neighborhood\big(U,(a,\mathcal{O})\big) \Longleftrightarrow (a \in U \in \mathcal{O})$$
$$\#\text{ another name for open set containing } a \quad (88)$$

$$M = \{a,b,c,d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\} \Longrightarrow$$
$$\Big(open\big(X,(M,\mathcal{O})\big) \Longleftrightarrow X = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\}\Big) \wedge$$
$$\Big(closed\big(Y,(M,\mathcal{O})\big) \Longleftrightarrow Y = \{\emptyset, \{a,b,d\}, \{c,d\}, \{a,b\}, \{d\}, M\}\Big) \wedge$$
$$\Big(clopen\big(Z,(M,\mathcal{O})\big) \Longleftrightarrow Z = \{\emptyset, \{a,b\}, \{c,d\}, M\}\Big) \quad (89)$$

$$chaoticTopology(M) = \{0, M\} \; ; \; discreteTopology = \mathcal{P}(M) \quad (90)$$

1.5 Induced topology

$$metric\Big(d(\$1,\$2),(M)\Big) \Longleftrightarrow \Big(map\Big(d,\big(M \times M, \mathbb{R}_0^+\big)\Big)\Big)$$
$$\Big(\forall_{x,y \in M}\big(d(x,y) = d(y,x)\big)\Big) \wedge$$
$$\Big(\forall_{x,y \in M}\big(d(x,y) = 0 \Longleftrightarrow x = y\big)\Big) \wedge$$
$$\Big(\forall_{x,y,z}\Big(\big(d(x,z) \leq d(x,y) + d(y,z)\big)\Big)\Big)$$
$$\#\text{ behaves as distances should} \quad (91)$$

$$metricSpace\big((M,d),()\big) \Longleftrightarrow metric\big(d,(M)\big) \quad (92)$$

$$openBall\big(B,(r,p,M,d)\big) \Longleftrightarrow \Big(metricSpace\big((M,d),()\big)\Big) \wedge \big(r \in \mathbb{R}^+, p \in M\big) \wedge \big(B = \{q \in M \,|\, d(p,q) < r\}\big) \quad (93)$$

$$metricTopology\big(\mathcal{O},(M,d)\big) \Longleftrightarrow \Big(metricSpace\big((M,d),()\big)\Big) \wedge$$
$$\Big(\mathcal{O} = \{U \in \mathcal{P}(M) \,|\, \forall_{p \in U} \exists_{r \in \mathbb{R}^+}\big(openBall\big(B,(r,p,M,d)\big) \wedge B \subseteq U\big)\}\Big)$$
$$\#\text{ every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (94)$$

$$metricTopologicalSpace\big((M,\mathcal{O},d),()\big) \Longleftrightarrow metricTopology\big(\mathcal{O},(M,d)\big) \quad (95)$$

$$limitPoint\big(p,(S,M,d)\big) \Longleftrightarrow (S \subseteq M) \wedge \forall_{r \in \mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big) \cap S \neq \emptyset\Big)$$
$$\#\text{ every open ball centered at } p \text{ contains some intersection with } S \quad (96)$$

$$interiorPoint\big(p,(S,M,d)\big) \Longleftrightarrow (S \subseteq M) \wedge \Big(\exists_{r \in \mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big) \subseteq S\Big)\Big)$$

$$\# \text{ there is an open ball centered at } p \text{ that is fully enclosed in } S \tag{97}$$

$$closure\big(\bar{S},(S,M,d)\big) \Longleftrightarrow \bar{S}=S\cup\{limitPoint\big(p,(S,M,d)\big)\,|\,p\in M\} \tag{98}$$

$$dense\big(S,(M,d)\big) \Longleftrightarrow (S\subseteq M)\wedge\left(\forall_{p\in M}\Big(p\in closure\big(\bar{S},(S,M,d)\big)\Big)\right)$$
$$\# \text{ every of point in } M \text{ is a point or a limit point of } S \tag{99}$$

$$eucD\big(d,(n)\big) \Longleftrightarrow \big(\forall_{i\in\mathbb{N}\wedge i\leq n}(x_i\in\mathbb{R})\big)\wedge\left(d=\sqrt[2]{\sum_{i=1}^{n}x_i{}^2}\right) \tag{100}$$

$$metricTopology\left(standardTopology,\Big(\mathbb{R}^n,eucD\big(d,(n)\big)\Big)\right)$$
$$=========== \mathbf{N\ O\ T\ =\ U\ P\ D\ A\ T\ E\ D} ===========$$
$$\mathbf{L1:}\ \forall_{p\in U=\emptyset}(\ldots)\Longrightarrow\forall_p\Big((p\in\emptyset)\Longrightarrow\ldots\Big)\Longrightarrow\forall_p\big((\mathbf{False})\Longrightarrow\ldots\big)\Longrightarrow\emptyset\in\mathcal{O}_{standard}$$
$$\mathbf{L2:}\ \forall_{p\in\mathbb{R}^n}B(r,p,\mathbb{R}^n,d)\subseteq\mathbb{R}^n\Longrightarrow M\in\mathcal{O}_{standard}$$
$$\mathbf{L4:}\ C\subseteq\mathcal{O}_{standard}\Longrightarrow\forall_{U\in C}\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\big(B_r(p)\subseteq U\subseteq\cup C\big)\Longrightarrow\cup C\in\mathcal{O}_{standard}$$
$$\mathbf{L3:}\ U,V\in\mathcal{O}_{standard}\Longrightarrow p\in U\cap V\Longrightarrow p\in U\wedge p\in V\Longrightarrow$$
$$\exists_{r\in\mathbb{R}^+}B(r,p,\mathbb{R}^n,d)\wedge\exists_{s\in\mathbb{R}^+}B(s,p,\mathbb{R}^n,d)\Longrightarrow$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\subseteq U\wedge B\big(min(r,s),q,\mathbb{R}^n,d\big)\subseteq V\Longrightarrow$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\in U\cap V\Longrightarrow U\cap V\in\mathcal{O}_{standard}$$
$$\# \text{ natural topology for } \mathbb{R}^d$$
$$\# \text{ could fail on infinite sets since } min \text{ could approach } 0$$
$$=========== \mathbf{N\ O\ T\ =\ U\ P\ D\ A\ T\ E\ D} =========== \tag{101}$$

$$subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big) \Longleftrightarrow topology\big(\mathcal{O},(M)\big)\wedge(N\subseteq M)\wedge\big(\mathcal{O}|_N=\{U\cap N\,|\,U\in\mathcal{O}\}\big)$$
$$\# \text{ crops open sets outside } N \tag{102}$$

$$(\text{THM}):\ subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\wedge topology\big(\mathcal{O}|_N,(N)\big)\Longleftarrow$$
$$=========== \mathbf{N\ O\ T\ =\ U\ P\ D\ A\ T\ E\ D} ===========$$
$$\mathbf{L1:}\ \emptyset\in\mathcal{O}\Longrightarrow U=\emptyset\Longrightarrow\emptyset\cap N=\emptyset\Longrightarrow\emptyset\in\mathcal{O}|_N$$
$$\mathbf{L2:}\ M\in\mathcal{O}\Longrightarrow U=M\Longrightarrow M\cap N=N\Longrightarrow N\in\mathcal{O}|_N$$
$$\mathbf{L3:}\ S,T\in\mathcal{O}|_N\Longrightarrow\exists_{U\in\mathcal{O}}(S=U\cap N)\wedge\exists_{V\in\mathcal{O}}(T=V\cap N)\Longrightarrow S\cap T=(U\cap N)\cap(V\cap N)$$
$$=(U\cap V)\cap N\wedge U\cap V\in\mathcal{O}\Longrightarrow S\cap T\in\mathcal{O}|_N$$
$$\mathbf{L4:}\ TODO:EXERCISE$$
$$=========== \mathbf{N\ O\ T\ =\ U\ P\ D\ A\ T\ E\ D} =========== \tag{103}$$

$$productTopology\Big(\mathcal{O}_{A\times B},\big((A,\mathcal{O}_A),(B,\mathcal{O}_B)\big)\Big) \Longleftrightarrow \Big(topology\big(\mathcal{O}_A,(A)\big)\Big)\wedge\Big(topology\big(\mathcal{O}_B,(B)\big)\Big)\wedge$$
$$\big(\mathcal{O}_{A\times B}=\{(a,b)\in A\times B\,|\,\exists_S(a\in S\in\mathcal{O}_A)\exists_T(b\in T\in\mathcal{O}_B)\}\big)$$
$$\# \text{ open in cross iff open in each} \tag{104}$$

1.6 Convergence

$$sequence\big(q,(M)\big) \Longleftrightarrow map\big(q,(\mathbb{N},M)\big) \quad (105)$$

$$sequenceConvergesTo\big((q,a),(M,\mathcal{O})\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$

$$\Big(sequence\big(q,(M)\big)\Big) \wedge (a \in M) \wedge \Big(\forall_{U \in \mathcal{O}|a \in U} \exists_{N \in \mathbb{N}} \forall_{n > N}\big(q(n) \in U\big)\Big)$$

$$\text{\# each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \quad (106)$$

(THM) : **convergence generalizes to: the sequence** $q : \mathbb{N} \to \mathbb{R}^d$ **converges against** $a \in \mathbb{R}^d$ **in** $\mathcal{O}_S$ **if:**

$$\forall_{r>0} \exists_{N \in \mathbb{N}} \forall_{n>N}\big(||q(n) - a|| < \epsilon\big) \text{ \# distance based convergence} \quad (107)$$

1.7 Continuity

$$continuous\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}_M),()\big)\Big) \wedge$$

$$\Big(topologicalSpace\big((N,\mathcal{O}_N),()\big)\Big) \wedge \Big(\forall_{V \in \mathcal{O}_N}\big(preimage\big(A,(V,\phi,M,N)\big) \in \mathcal{O}_M\big)\Big)$$

$$\text{\# preimage of open sets are open} \quad (108)$$

$$homeomorphism\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big) \Longleftrightarrow \Big(inverseMap\big(\phi^{-1},(\phi,M,N)\big)\Big)$$

$$\Big(continuous\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big)\Big) \wedge \Big(continuous\big(\phi^{-1},(N,\mathcal{O}_N,M,\mathcal{O}_M)\big)\Big)$$

$$\text{\# structure preserving maps in topology, ability to share topological properties} \quad (109)$$

$$isomorphicTopologicalSpace\Big(\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big),()\Big) \Longleftrightarrow$$

$$\exists_{\phi}\Big(homeomorphism\big(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)\big)\Big) \quad (110)$$

1.8 Separation

$$T0Separate\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$

$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U \in \mathcal{O}}\big((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U)\big)\Big)$$

$$\text{\# each pair of points has a neighborhood s.t. one is inside and the other is outside} \quad (111)$$

$$T1Separate\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$

$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V}\big((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V)\big)\Big)$$

$$\text{\# every point has a neighborhood that does not contain another point} \quad (112)$$

$$T2Separate\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$

$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V}\big(U \cap V = \emptyset\big)\Big)$$

$$\text{\# every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \quad (113)$$

$$(\text{THM}) : T2Separate \implies T1Separate \implies T0Separate \quad (114)$$

1.9 Compactness

$$openCover\big(C,(M,\mathcal{O})\big) \iff \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M)$$
$$\# \text{ collection of open sets whose elements cover the entire space} \quad (115)$$

$$finiteSubcover\big(\widetilde{C},(C,M,\mathcal{O})\big) \iff \big(\widetilde{C} \subseteq C\big) \wedge \Big(openCover\big(C,(M,\mathcal{O})\big)\Big) \wedge$$
$$\Big(openCover\big(\widetilde{C},(M,\mathcal{O})\big)\Big) \wedge \Big(finiteSet\big(\widetilde{C},()\big)\Big)$$
$$\# \text{ finite subset of a cover that is also a cover} \quad (116)$$

$$compact\big((M,\mathcal{O}),()\big) \iff \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$\Bigg(\forall_{C \subseteq \mathcal{O}}\bigg(openCover\big(C,(M,\mathcal{O})\big) \implies \exists_{\widetilde{C} \subseteq C}\Big(finiteSubcover\big(\widetilde{C},(C,M,\mathcal{O})\big)\Big)\bigg)\Bigg)$$
$$\# \text{ every covering of the space is represented by a finite number of nhbhds} \quad (117)$$

$$compactSubset\big(N,(M,\mathcal{O})\big) \iff \Big(compact\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$\Big(subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\Big) \wedge \Big(compact\big((N,\mathcal{O}|_N),()\big)\Big) \quad (118)$$

$$bounded\big(N,(M,d)\big) \iff \Big(metricSpace\big((M,d),()\big)\Big) \wedge (N \subseteq M) \wedge$$
$$\Big(\exists_{r \in \mathbb{R}^+}\forall_{p,q \in n}\big(d(p,q) < r\big)\Big) \quad (119)$$

$$(\text{THM}) \text{ Heine-Borel thm.: } metricTopologicalSpace\big((M,\mathcal{O}_d,d),()\big) \implies$$
$$\forall_{S \subseteq M}\bigg(\Big(closed\big(S,(M,\mathcal{O}_d)\big) \wedge bounded\big(S,(M,\mathcal{O}_d)\big)\Big) \iff compactSubset\big(S,(M,\mathcal{O}_d)\big)\bigg)$$
$$\# \text{ when metric topologies are involved, compactness is equivalent to being closed and bounded} \quad (120)$$

1.10 Paracompactness

$$openRefinement\big(\widetilde{C},(C,M,\mathcal{O})\big) \iff \Big(openCover\big(C,(M,\mathcal{O})\big)\Big) \wedge \Big(openCover\big(\widetilde{C},(M,\mathcal{O})\big)\Big) \wedge$$
$$\Big(\forall_{\widetilde{U} \in \widetilde{C}}\exists_{U \in C}\big(\widetilde{U} \subseteq U\big)\Big)$$
$$\# \text{ a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space} \quad (121)$$

$$(\text{THM}) : finiteSubcover \implies openRefinement \quad (122)$$

$$locallyFinite\big(C,(M,\mathcal{O})\big) \iff \Big(openCover\big(C,(M,\mathcal{O})\big)\Big) \wedge$$
$$\forall_{p \in M}\exists_{U \in \mathcal{O}|p \in U}\Big(finiteSet\big(\{U_c \in C | U \cap U_c \neq \emptyset\},()\big)\Big)$$
$$\# \text{ each point has a neighborhood that intersects with only finitely many sets in the cover} \quad (123)$$

$$paracompact\big((M,\mathcal{O}),()\big) \Longleftrightarrow$$

$$\forall_C \Bigg(openCover\big(C,(M,\mathcal{O})\big) \Longrightarrow \exists_{\widetilde{C}} \bigg(locallyFinite\Big(openRefinement\big(\widetilde{C},(C,M,\mathcal{O})\big),(M,\mathcal{O})\Big) \bigg) \Bigg)$$

$\#$ every open cover has a locally finite open refinement $\quad$ (124)

$$(\text{THM}) : metricTopologicalSpace \Longrightarrow paracompact \quad (125)$$

$$============ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} ============ \quad (126)$$

$$partitionOfUnitySubjCover\big(\mathcal{F},(C,M,\mathcal{O})\big) \Longleftrightarrow \Big(locallyFinite\big(C,(M,\mathcal{O})\big)\Big) \wedge (f \in \mathcal{F}) \wedge$$

$$\Bigg(continuous\bigg(f, \Big(M,\mathcal{O},[0,1],subsetTopology\big(\mathcal{O}|_{[0,1]},([0,1],\mathbb{R},standardTopology)\big)\Big)\bigg)\Bigg) \wedge$$

$$\Big(\exists_{U_f \in C}\forall_{p \in M}\big(f(p) \neq 0 \Longrightarrow p \in U_f\big)\Big) \wedge$$

$$\Big(\forall_{p \in M}\exists_{U \in \mathcal{O}|p \in U}\big((f_U)_n = \{f \in \mathcal{F}|p \in M \wedge f(p) \neq 0\}\big)\Big) \wedge$$

$$\Big(locallyFinite(C,M,\mathcal{O}) \Longrightarrow finiteSet\big((f_U)_n,()\big)\Big) \wedge$$

$$\Bigg(\forall_{p \in M}\exists_{U \in \mathcal{O}|p \in U}\bigg(\sum_{i=1}^{|(f_U)_n|}(f_U)_i(p) = 1\bigg)\Bigg)$$

$\#$ useful for defining integrals between overlapping neighborhoods $\quad$ (127)

$$T2Separate\big((M,\mathcal{O}),()\big) \Longrightarrow \Big(paracompact\big((M,\mathcal{O}),()\big)\Big) \Longleftrightarrow$$

$$\forall_C \Big(openCover\big(C,(M,\mathcal{O})\big) \Longrightarrow partitionOfUnitySOTCover\big(\mathcal{F},(C,M,\mathcal{O})\big)\Big) \quad (128)$$

$$============ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} ============ \quad (129)$$

1.11 Connectedness and path-connectedness

$$connected\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge \Big(\neg\exists_{A,B \in \mathcal{O}\setminus\emptyset}\big(A \cap B \neq \emptyset \wedge A \cup B = M\big)\Big)$$

$\#$ if there is some covering of the space that does not intersect $\quad$ (130)

$$(\text{THM}) : \neg connected\Bigg(\bigg(\mathbb{R}\setminus\{0\}, subsetTopology\Big(\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}},(\mathbb{R},standardTopology,\mathbb{R}\setminus\{0\})\Big)\bigg),()\Bigg)$$

$$\Longleftarrow \Big(A = (-\infty,0) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge \Big(B = (0,\infty) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge$$

$$\big(A \cap B = \emptyset\big) \wedge \big(A \cup B = \mathbb{R}\setminus\{0\}\big) \quad (131)$$

$$(\text{THM}) : connected\big((M,\mathcal{O}),()\big) \Longleftrightarrow \forall_{S \in \mathcal{O}}\Big(clopen\big(S,(M,\mathcal{O}) \Longrightarrow (S = \emptyset \vee S = M)\big)\Big) \quad (132)$$

$$pathConnected\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(subsetTopology\big(\mathcal{O}_{standard}|_{[0,1]},(\mathbb{R},standardTopology,[0,1])\big)\Big) \wedge$$

$$\left(\forall_{p,q\in M}\exists_\gamma\left(continuous\left(\gamma,\left([0,1],\mathcal{O}_{standard}|_{[0,1]},M,\mathcal{O}\right)\right)\wedge\gamma(0)\!=\!p\wedge\gamma(1)\!=\!q\right)\right) \tag{133}$$

$$(\text{THM}):pathConnected\Longrightarrow connected \tag{134}$$

1.12 Homotopic curve and the fundamental group

$$============\mathbf{N\ O\ T}=\mathbf{U\ P\ D\ A\ T\ E\ D}============ \tag{135}$$

$$homotopic(\sim,(\gamma,\delta,M,\mathcal{O}))\Longleftrightarrow(map(\gamma,([0,1],M))\wedge map(\delta,([0,1],M)))\wedge$$
$$(\gamma(0)\!=\!\delta(0)\wedge\gamma(1)\!=\!\delta(1))\wedge$$
$$(\exists_H\forall_{\lambda\in[0,1]}(continuous(H,(([0,1]\times[0,1],\mathcal{O}_{standard^2}|_{[0,1]\times[0,1]}),(M,\mathcal{O}))\wedge H(0,\lambda)\!=\!\gamma(\lambda)\wedge H(1,\lambda)\!=\!\delta(\lambda))))$$
$$\#\ H\text{ is a continuous deformation of one curve into another} \tag{136}$$

$$homotopic(\sim)\Longrightarrow equivalenceRelation(\sim) \tag{137}$$

$$loopSpace(\mathcal{L}_p,(p,M,\mathcal{O}))\Longleftrightarrow\mathcal{L}_p=\{map(\gamma,([0,1],M))|continuous(\gamma)\wedge\gamma(0)\!=\!\gamma(1)\}) \tag{138}$$

$$concatination(\star,(p,\gamma,\delta))\Longleftrightarrow(\gamma,\delta\in loopSpace(\mathcal{L}_p))\wedge$$
$$(\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda)=\begin{cases}\gamma(2\lambda)&0\le\lambda<0.5\\\delta(2\lambda-1)&0.5\le\lambda\le1\end{cases})) \tag{139}$$

$$group((G,\bullet),())\Longleftrightarrow(map(\bullet,(G\times G,G)))\wedge$$
$$(\forall_{a,b\in G}(a\bullet b\in G))$$
$$(\forall_{a,b,c\in G}((a\bullet b)\bullet C=a\bullet(b\bullet c)))$$
$$(\exists_e\forall_{a\in G}(e\bullet a=a=a\bullet e))\wedge$$
$$(\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1}=e=a^{-1}\bullet a))$$
$$\#\text{ characterizes symmetry of a set structure} \tag{140}$$

$$isomorphic(\cong,(X,\odot),(Y,\ominus)))\Longleftrightarrow\exists_f\forall_{a,b\in X}(bijection(f,(X,Y))\wedge f(a\odot b)=f(a)\ominus f(b)) \tag{141}$$

$$fundamentalGroup((\pi_{1,p},\bullet),(p,M,\mathcal{O}))\Longleftrightarrow(\pi_{1,p}=\mathcal{L}_p/\sim)\wedge$$
$$(map(\bullet,(\pi_{1,p}\times\pi_{1,p},\pi_{1,p})))\wedge$$
$$(\forall_{A,B\in\pi_{1,p}}([A]\bullet[B]=[A\star B]))\wedge$$
$$(group((\pi_{1,p},\bullet),()))$$
$$\#\text{ an equivalence class of all loops induced from the homotopic equivalence relation} \tag{142}$$

$$fundamentalGroup_1\not\cong fundamentalGroup_2\Longrightarrow topologicalSpace_1\not\cong topologicalSpace_2 \tag{143}$$

$$\textbf{there exists no known list of topological properties that can imply homeomorphisms} \tag{144}$$

$$\textbf{CONTINUE @ Lecture 6: manifolds} \tag{145}$$

$$============\mathbf{N\ O\ T}=\mathbf{U\ P\ D\ A\ T\ E\ D}============ \tag{146}$$

1.13 Measure theory

$$sigmaAlgebra\big(\sigma,(M)\big) \Longleftrightarrow (M \neq \emptyset) \wedge \big(\sigma \subseteq \mathcal{P}(M)\big) \wedge$$
$$(M \in \sigma) \wedge \Big(\forall_{A \in \sigma}\big(M \setminus A \in \sigma\big)\Big) \wedge$$
$$\bigg(\Big(A \subseteq \sigma \wedge \neg uncountablyInfinite\big(A,()\big)\Big) \Longrightarrow \cup A \in \sigma\bigg)$$

$\#$ behaves as measurable sets should; provides the sufficient structure for defining a measure μ $\qquad(147)$

$$measurableSpace\big((M,\sigma),()\big) \Longleftrightarrow sigmaAlgebra\big(\sigma,(M)\big) \qquad(148)$$

$$measurableSet\big(A,(M,\sigma)\big) \Longleftrightarrow \Big(measurableSpace\big((M,\sigma),()\big)\Big) \wedge (A \in \sigma) \qquad(149)$$

$$measure\big(\mu,(M,\sigma)\big) \Longleftrightarrow \Big(measurableSpace\big((M,\sigma),()\big)\Big) \wedge \left(map\left(\mu,\left(\sigma,\left(\overline{\mathbb{R}}\right)_0^+\right)\right)\right) \wedge \Big(\mu(\emptyset)=0\Big) \wedge$$
$$\left(\Big((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i \in \mathbb{N}}\forall_{j \in \mathbb{N}\setminus\{i\}}\big(A_i \cap A_j = \emptyset\big)\Big) \Longrightarrow \mu\big(\cup_{i \in \mathbb{N}}(A_i)\big) = \sum_{i \in \mathbb{N}}\big(\mu(A_i)\big)\right)$$

$\#$ enforces meaningful concepts of measures such as precise additivity $\qquad(150)$

$$(\text{THM}): measure\big(\mu,(M,\sigma)\big) \Longrightarrow$$
$$\Big(\forall_{A,B \in \sigma}\big(A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)\big)\Big) \wedge$$
$$\left((A)_{\mathbb{N}} \subseteq \sigma \Longrightarrow \mu\big(\cup_{i \in \mathbb{N}}(A_i)\big) \leq \sum_{i \in \mathbb{N}}\big(\mu(A_i)\big)\right) \wedge$$
$$\left(\Big((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i \in \mathbb{N}}(B_i \subseteq B_{i+1}) \wedge B = \cup(B)_{\mathbb{N}}\Big) \Longrightarrow \lim_{n \to \infty}\big(\mu(B_n)\big) = \mu(B)\right) \wedge$$
$$\left(\Big((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i \in \mathbb{N}}(C_{i+1} \subseteq C_i) \wedge C = \cap(C)_{\mathbb{N}}\Big) \Longrightarrow \lim_{n \to \infty}\big(\mu(C_n)\big) = \mu(C)\right)$$

$\#$ immediate implications of the measurable set $A \in \sigma$ axioms and the measure μ axioms $\qquad(151)$

$$measureSpace\big((M,\sigma,\mu),()\big) \Longleftrightarrow measure\big(\mu,(M,\sigma)\big) \qquad(152)$$

$$finiteMeasure\big(\mu,(M,\sigma)\big) \Longleftrightarrow \Big(measure\big(\mu,(M,\sigma)\big)\Big) \wedge$$
$$\left(\exists_{(A)_{\mathbb{N}} \subseteq \sigma}\Big(\cup\big((A)_{\mathbb{N}}\big) = M \wedge \forall_{n \in \mathbb{N}}\big(\mu(A_n) < \infty\big)\Big)\right) \qquad(153)$$

$$generatedSigmaAlgebra\big(\sigma(\zeta),(\zeta,M)\big) \Longleftrightarrow \Big(G = \{\sigma \subseteq \mathcal{P}(M) \,|\, sigmaAlgebra\big(\sigma,(M)\big)\}\Big) \wedge \big(\sigma(\zeta) = \cap G\big)$$

$\#$ smallest σ-algebra containing the generating set ζ $\qquad(154)$

$$(\text{THM}): \exists_{\zeta \subseteq M}\Big(generatedSigmaAlgebra\big(\sigma(\zeta),(\zeta,M)\big) = sigmaAlgebra\big(\sigma,(M)\big)\Big) \qquad(155)$$

$$borelSigmaAlgebra\big(\sigma(\mathcal{O}),(M,\mathcal{O})\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$\Big(generatedSigmaAlgebra\big(\sigma(\mathcal{O}),(\mathcal{O},M)\big)\Big)$$

$\#$ σ-algebra induced by a topology $\qquad(156)$

$$standardSigma(\sigma_s, ()) \Longleftrightarrow \left(borelSigmaAlgebra\left(\sigma_s, \left(\mathbb{R}^d, standardTopology\right)\right)\right) \tag{157}$$

$$lebesgueMeasure(\lambda, ()) \Longleftrightarrow \left(measure\left(\lambda, \left(\mathbb{R}^d, standardSigma\right)\right)\right) \wedge$$
$$\left(\lambda\left(\times_{i=1}^d ([a_i, b_i))\right) = \sum_{i=1}^d \left(\sqrt[2]{(a_i - b_i)^2}\right)\right)$$
$$\text{\# natural measure for } \mathbb{R}^d \tag{158}$$

$$measurableMap(f, (M, \sigma_M, N, \sigma_N)) \Longleftrightarrow \left(measurableSpace((M, \sigma_M), ())\right) \wedge$$
$$\left(measurableSpace((N, \sigma_N), ())\right) \wedge \left(\forall_{B \in \sigma_N}\left(preimage(A, (B, f, M, N)) \in \sigma_M\right)\right)$$
$$\text{\# preimage of measurable sets are measurable} \tag{159}$$

$$pushForwardMeasure(f \star \lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \Longleftrightarrow \left(measureSpace((M, \sigma_M, \mu_M), ())\right) \wedge$$
$$\left(measurableSpace((N, \sigma_N), ())\right) \wedge \left(measurableMap(f, (M, \sigma_M, N, \sigma_N))\right) \wedge$$
$$\left(\forall_{B \in N}\left(f \star \lambda_M(B) = \mu_M\left(preimage(A, (B, f, M, N))\right)\right)\right) \wedge \left(measure(f \star \lambda_M, (N, \sigma_N))\right)$$
$$\text{\# natural construction of a measure based primarily on measurable map} \tag{160}$$

$$nullSet(A, (M, \sigma, \mu)) \Longleftrightarrow \left(measureSpace((M, \sigma, \mu), ())\right) \wedge (A \in \sigma) \wedge \left(\mu(A) = 0\right) \tag{161}$$

$$almostEverywhere(p, (M, \sigma, \mu)) \Longleftrightarrow \left(measureSpace((M, \sigma, \mu), ())\right) \wedge \left(predicate(p, (M))\right) \wedge$$
$$\left(\exists_{A \in \sigma}\left(nullSet(A, (M, \sigma, \mu)) \Longrightarrow \forall_{n \in M \setminus A}(p(n))\right)\right)$$
$$\text{\# the predicate holds true for all points except the points in the null set} \tag{162}$$

1.14   Lebesque integration

$$simpleTopology\left(\mathcal{O}_{simple}, ()\right) \Longleftrightarrow \mathcal{O}_{simple} = subsetTopology\left(\mathcal{O}|_{\mathbb{R}_0^+}, \left(\mathbb{R}, standardTopology, \mathbb{R}_0^+\right)\right) \tag{163}$$

$$simpleSigma\left(\sigma_{simple}, ()\right) \Longleftrightarrow borelSigmaAlgebra\left(\sigma_{simple}, \left(\mathbb{R}_0^+, simpleTopology\right)\right) \tag{164}$$

$$simpleFunction\left(s, (M, \sigma)\right) \Longleftrightarrow \left(measurableMap\left(s, \left(M, \sigma, \mathbb{R}_0^+, simpleSigma\right)\right)\right) \wedge$$
$$\left(finiteSet\left(image\left(B, \left(M, s, M, \mathbb{R}_0^+\right)\right), ()\right)\right)$$
$$\text{\# if the map takes on finitely many values on } \mathbb{R}_0^+ \tag{165}$$

$$characteristicFunction\big(X_A,(A,M)\big) \Longleftrightarrow (A \subseteq M) \wedge \Big(map\big(X_A,(M,\mathbb{R})\big)\Big) \wedge$$

$$\left(\forall_{m \in M}\left(X_A(m) = \begin{cases} 1 & m \in A \\ 0 & m \notin A \end{cases}\right)\right) \quad (166)$$

$$(\text{THM}) : simpleFunction\big(s,(M,\sigma_M)\big) \Longrightarrow$$

$$\left(finiteSet\Big(image\Big(Z,\big(M,s,M,\mathbb{R}_0^+\big)\Big),()\Big)\right) \wedge$$

$$\Big(characteristicFunction\big(X_A,(A,M)\big)\Big) \wedge \left(\forall_{m \in M}\left(s(m) = \sum_{z \in Z}\Big(z \cdot X_{preimage\big(A,(\{z\},s,M,\mathbb{R}_0^+)\big)}(m)\Big)\right)\right) \quad (167)$$

$$exStandardSigma\big(\overline{\sigma_s},()\big) \Longleftrightarrow \overline{\sigma_s} = \{A \subseteq \overline{\mathbb{R}} \,|\, A \cap R \in standardSigma\}$$

$$\# \text{ ignores } \pm\infty \text{ to preserve the points in the domain of the measurable map} \quad (168)$$

$$nonNegIntegrable\big(f,(M,\sigma)\big) \Longleftrightarrow \left(measurableMap\Big(f,\big(M,\sigma,\overline{\mathbb{R}},exStandardSigma\big)\Big)\right) \wedge$$

$$\Big(\forall_{m \in M}\big(f(m) \geq 0\big)\Big) \quad (169)$$

$$nonNegIntegral\left(\int_M (fd\mu),(f,M,\sigma,\mu)\right) \Longleftrightarrow \Big(measureSpace\big((M,\sigma,\mu),()\big)\Big) \wedge$$

$$\left(measureSpace\Big(\big(\overline{\mathbb{R}},exStandardSigma,lebesgueMeasure\big),()\Big)\right) \wedge$$

$$\Big(nonNegIntegrable\big(f,(M,\sigma)\big)\Big) \wedge \big(\int_M (fd\mu) = sup(\{\sum_{z \in Z}\Big(z \cdot \mu\Big(preimage\Big(A,\big(\{z\},s,M,\mathbb{R}_0^+\big)\Big)\Big)\Big) \,|\,$$

$$\forall_{m \in M}(s(m) \leq f(m)) \wedge simpleFunction\big(s,(M,\sigma)\big) \wedge finiteSet\Big(image\Big(Z,\big(M,s,M,\mathbb{R}_0^+\big)\Big),()\Big)\}))$$

$$\# \text{ lebesgue measure on } z \text{ reduces to } z \quad (170)$$

$$explicitIntegral \Longleftrightarrow \int\big(f(x)\mu(dx)\big) = \int(fd\mu)$$

$$\# \text{ alternative notation for lebesgue integrals} \quad (171)$$

$$(\text{THM}) : nonNegIntegral\left(\int(fd\mu),(f,M,\sigma,\mu)\right) \wedge nonNegIntegral\left(\int(gd\mu),(g,M,\sigma,\mu)\right) \Longrightarrow$$

$$(\text{THM}) \text{ Markov inequality:} \left(\forall_{z \in \mathbb{R}_0^+}\left(\int(fd\mu) \geq z \cdot \mu\Big(preimage\Big(A,\big([z,\infty),f,M,\overline{\mathbb{R}}\big)\Big)\Big)\right)\right) \wedge$$

$$\left(almostEverywhere\big(f=g,(M,\sigma,\mu)\big) \Longrightarrow \int(fd\mu) = \int(gd\mu)\right)$$

$$\left(\int(fd\mu) = 0 \Longrightarrow almostEverywhere\big(f=0,(M,\sigma,\mu)\big)\right) \wedge$$

$$\left(\int(fd\mu) \leq \infty \Longrightarrow almostEverywhere\big(f<\infty,(M,\sigma,\mu)\big)\right) \quad (172)$$

(THM) Mono. conv.: $\left((f)_\mathbb{N}=\{f_n\,|\wedge measurableMap\Big(f_n,\Big(M,\sigma,\overline{R},exStandardSigma\Big)\Big)\wedge 0\leq f_{n-1}\leq f_n\}\right)\wedge$

$\left(map\Big(f,\Big(M,\overline{\mathbb{R}}\Big)\Big)\right)\wedge\left(\forall_{m\in M}\Big(f(m)=sup(f_n(m)|f_n\in(f)_\mathbb{N})\Big)\right)\implies\left(\lim_{n\to\infty}\left(\int_M(f_n d\mu)\right)=\int_M(f d\mu)\right)$

lengths now depend on M,σ and limits can be pulled in or out of an integral (173)

(THM) : $nonNegIntegral\left(\int(f d\mu),(f,M,\sigma,\mu)\right)\wedge nonNegIntegral\left(\int(g d\mu),(g,M,\sigma,\mu)\right)\implies$

$\left(\forall_{\alpha\in\mathbb{R}_0^+}\left(\int((f+\alpha g)d\mu)=\int(f d\mu)+\alpha\int(g d\mu)\right)\right)$

integral acts linearly and commutes finite summations (174)

(THM) : $\left((f)_\mathbb{N}=\{f_n\,|\wedge measurableMap\Big(f_n,\Big(M,\sigma,\overline{R},exStandardSigma\Big)\Big)\wedge 0\leq f_n\}\right)\implies$

$\left(\int\left(\left(\sum_{n=1}^\infty(f_n)\right)d\mu\right)=\sum_{n=1}^\infty\left(\int(f_n d\mu)\right)\right)$

$\sum_{n=1}^\infty f_n$ can be treated as $\lim_{n\to\infty}\sum_{i=1}^n f_n$ since $f_n\geq 0$ and it commutes with integral from monotone conv. (175)

$integrable\big(f,(M,\sigma)\big)\iff\left(measurableMap\Big(f,\Big(M,\sigma,\overline{\mathbb{R}},exStandardSigma\Big)\Big)\right)\wedge$

$\left(\forall_{m\in M}\Big(f(m)=max\big(f(m),0\big)-max\big(0,-f(m)\big)\Big)\right)\wedge$

$\left(measureSpace(M,\sigma,\mu)\implies\left(\int\Big(max\big(f(m),0\big)d\mu\Big)<\infty\wedge\int\Big(max\big(0,-f(m)\big)d\mu\Big)<\infty\right)\right)$

extra condition prevents the occurrence of the indeterminate $\infty-\infty$ (176)

$integral\left(\int(f d\mu),(f,M,\sigma,\mu)\right)\iff\left(nonNegIntegral\left(\int(f^+ d\mu),\big(max(f,0),M,\sigma,\mu\big)\right)\right)\wedge$

$\left(nonNegIntegral\left(\int(f^- d\mu),\big(max(0,-f),M,\sigma,\mu\big)\right)\right)\wedge\Big(integrable\big(f,(M,\sigma)\big)\Big)\wedge$

$\left(\int(f d\mu)=\int(f^+ d\mu)-\int(f^- d\mu)\right)$

arbitrary integral in terms of nonnegative integrals (177)

(THM) : $\Big(map(f,(M,\mathbb{C}))\Big)\implies\left(\int(f d\mu)=\int\big(Re(f)d\mu\big)-\int\big(Im(f)d\mu\big)\right)$ (178)

(THM) : $integral\left(\int(f d\mu),(f,M,\sigma,\mu)\right)\wedge integral\left(\int(g d\mu),(g,M,\sigma,\mu)\right)\implies$

$\left(almostEverywhere(f\leq g,(M,\sigma,\mu))\implies\int(f d\mu)\leq\int(g d\mu)\right)\wedge$

$\left(\forall_{m\in M}(f(m),g(m),\alpha\in\mathbb{R})\implies\int((f+\alpha g)d\mu)=\int(f d\mu)+\alpha\int(g d\mu)\right)$ (179)

(THM) Dominant convergence: $\left((f)_\mathbb{N}=\{f_n\,|\wedge measurableMap\Big(f_n,\Big(M,\sigma,\overline{R},exStandardSigma\Big)\Big)\}\right)\wedge$

$$\Big(map(f,(M,\overline{\mathbb{R}}))\Big)\wedge\left(almostEverywhere\Big(f(m)=\lim_{n\to\infty}(f_n(m)),(M,\sigma,\mu)\Big)\right)\wedge$$

$$\left(nonNegIntegral\Big(\int(gd\mu),(g,M,\sigma,\mu)\Big)\right)\wedge\left(\Big|\int(gd\mu)\Big|<\infty\right)\wedge\Big(almostEverywhere(|f_n|\le g,(M,\sigma,\mu))\Big)$$

$\#$ if all $f_n(m)$ are bounded by some integrable $|g(m)|\implies$

$\#$ then all $f_n(m)$ including f satisfy bounded and integrable properties

$$\left(\forall_{\phi\in\{f\}\cup(f)_\mathbb{N}}\Big(integrable(\phi,(M,\sigma))\Big)\right)\wedge\left(\lim_{n\to\infty}\Big(\int(|f_n-f|d\mu)=0\Big)\right)\wedge\left(\lim_{n\to\infty}\Big(\int(f_nd\mu)\Big)=\int(fd\mu)\right)\quad(180)$$

1.15 Vector space and structures

$$vectorSpace((V,+,\cdot),())\iff\Big(map(+,(V\times V,V))\Big)\wedge\Big(map(\cdot,(\mathbb{R}\times V,V))\Big)\wedge$$
$$\Big(\forall_{v,w\in v}(v+w=w+v)\Big)\wedge$$
$$\Big(\forall_{v,w,x\in v}((v+w)+x=v+(w+x))\Big)\wedge$$
$$\Big(\exists_{\boldsymbol{0}\in V}\forall_{v\in V}(v+\boldsymbol{0}=v)\Big)\wedge$$
$$\Big(\forall_{v\in V}\exists_{-v\in V}\big(v+(-v)=\boldsymbol{0}\big)\Big)\wedge$$
$$\Big(\forall_{a,b\in\mathbb{R}}\forall_{v\in V}\big(a(b\cdot v)=(ab)\cdot v\big)\Big)\wedge$$
$$\Big(\exists_{1\in\mathbb{R}}\forall_{v\in V}(1\cdot v=v)\Big)\wedge$$
$$\Big(\forall_{a,b\in\mathbb{R}}\forall_{v\in V}\big((a+b)\cdot v=a\cdot v+b\cdot v\big)\Big)\wedge$$
$$\Big(\forall_{a\in\mathbb{R}}\forall_{v,w\in V}\big(a\cdot(v+w)=a\cdot v+a\cdot w\big)\Big)$$
$\#$ behaves similar as vectors should i.e., additive, scalable, linear distributive $\quad(181)$

$$innerProduct(\langle\$1,\$2\rangle,(V,+,\cdot))\iff\Big(vectorSpace((V,+,\cdot),())\Big)\wedge\Big(map(\langle\$1,\$2\rangle,(V\times V,\mathbb{R}))\Big)\wedge$$
$$\Big(\forall_{v,w\in V}\big(\langle v,w\rangle=\langle w,v\rangle\big)\Big)\wedge$$
$$\Big(\forall_{v,w,x\in V}\forall_{a,b\in\mathbb{R}}\big(\langle av+bw,x\rangle=a\langle v,x\rangle+b\langle w,x\rangle\big)\Big)\wedge$$
$$\Big(\forall_{v\in V}\big(\langle v,v\rangle\ge0\big)\Big)\wedge\Big(\forall_{v\in V}\big(\langle v,v\rangle\big)=0\iff v=\boldsymbol{0}\Big)$$
$\#$ the sesquilinear or 1.5 linear map inner product provides info. on distance and orthogonality $\quad(182)$

$$innerProductSpace\Big((V,+,\cdot,\langle\$1,\$2\rangle),()\Big)\iff innerProduct\big(\langle\$1,\$2\rangle,(V,+,\cdot)\big)\quad(183)$$

$$vectorNorm\big(||\$1||,(V,+,\cdot)\big)\iff\Big(vectorSpace((V,+,\cdot),())\Big)\wedge\left(map\Big(||\$1||,\Big(V,\mathbb{R}_0^+\Big)\Big)\right)\wedge$$
$$\Big(\forall_{v\in V}\big(||v||=0\iff v=\boldsymbol{0}\big)\Big)\wedge$$
$$\Big(\forall_{v\in V}\forall_{s\in\mathbb{R}}\big(||sv||=|s|||v||\big)\Big)\wedge$$
$$\Big(\forall_{v,w\in V}\big(||v+w||\le||v||+||w||\big)\Big)$$
$\#$ magnitude of a point in a vector space $\quad(184)$

$$normedVectorSpace\Big((V,+,\cdot,||\$1||),()\Big) \Longleftrightarrow \Big(vectorSpace((V,+,\cdot),())\Big) \wedge \Big(vectorNorm(||\$1||,(V,+,\cdot))\Big) \tag{185}$$

$$vectorMetric\Big(d(\$1,\$2),(V,+,\cdot)\Big) \Longleftrightarrow \Big(vectorSpace((V,+,\cdot),())\Big) \wedge$$
$$\left(metric\Big(d(\$1,\$2),(V)\Big) \vee \Big(map\Big(d,\Big(V \times V,\mathbb{R}_0^+\Big)\Big)\Big)\right)$$
$$\Big(\forall_{x,y \in V}\big(d(x,y)=d(y,x)\big)\Big) \wedge$$
$$\Big(\forall_{x,y \in V}\big(d(x,y)=0 \Longleftrightarrow x=y\big)\Big) \wedge$$
$$\left(\forall_{x,y,z \in V}\Big(\big(d(x,z) \leq d(x,y)+d(y,z)\big)\Big)\right)\Big)$$
$$\#\text{ behaves as distances should} \tag{186}$$

$$metricVectorSpace\left(\Big(V,+,\cdot,d(\$1,\$2)\Big),()\right) \Longleftrightarrow \Big(vectorSpace((V,+,\cdot),())\Big) \wedge$$
$$\Big(vectorMetric\Big(d(\$1,\$2),(V,+,\cdot)\Big)\Big) \tag{187}$$

$$innerProductNorm\Big(||\$1||,(V,+,\cdot,\langle\$1,\$2\rangle)\Big) \Longleftrightarrow \Big(innerProductSpace\big((V,+,\cdot,\langle\$1,\$2\rangle),()\big)\Big) \wedge$$
$$\left(\forall_{v \in V}\Big(||v||=\sqrt[2]{\langle v,v\rangle}\Big) \Longrightarrow vectorNorm(||\$1||,(V,+,\cdot))\right) \tag{188}$$

$$normInnerProduct\Big(\langle\$1,\$2\rangle,(V,+,\cdot,||\$1||)\Big) \Longleftrightarrow \Big(normedVectorSpace\big((V,+,\cdot,||\$1||),()\big)\Big) \wedge$$
$$\Big(\forall_{u,v \in V}\big(2||u||^2+2||v||^2=||u+v||^2+||u-v||^2\big)\Big) \wedge$$
$$\left(\forall_{v,w \in V}\left(\langle v,w\rangle=\frac{||v+w||^2-||v-w||^2}{4}\right) \Longrightarrow innerProduct(\langle\$1,\$2\rangle,(V,+,\cdot))\right) \tag{189}$$

$$normMetric\Big(d(\$1,\$2),(V,+,\cdot,||\$1||)\Big) \Longleftrightarrow \Big(normedVectorSpace\big((V,+,\cdot,||\$1||),()\big)\Big) \wedge$$
$$\left(\forall_{v,w \in V}\big(d(v,w)=||v-w||\big) \Longrightarrow vectorMetric\Big(d(\$1,\$2),(V,+,\cdot)\Big)\right) \tag{190}$$

$$metricNorm\left(||\$1||,\Big(V,+,\cdot,d(\$1,\$2)\Big)\right) \Longleftrightarrow \left(metricVectorSpace\Big(\big(V,+,\cdot,d(\$1,\$2)\big),()\Big)\right) \wedge$$
$$\Big(\forall_{u,v,w \in V}\forall_{s \in \mathbb{R}}\big(d(s(u+w),s(v+w))=|s|d(u,v)\big)\Big) \wedge$$
$$\Big(\forall_{v \in V}\big(||v||=d(v,\boldsymbol{0})\big) \Longrightarrow vectorNorm(||\$1||,(V,+,\cdot))\Big) \tag{191}$$

$$orthogonal\Big((v,w),(V,+,\cdot,\langle\$1,\$2\rangle)\Big) \Longleftrightarrow \Big(innerProductSpace\big((V,+,\cdot,\langle\$1,\$2\rangle),()\big)\Big) \wedge$$
$$(v,w \in V) \wedge \big(\langle v,w\rangle=0\big)$$
$$\#\text{ the inner product also provides info. on orthogonality} \tag{192}$$

$$normal\Big(v,(V,+,\cdot,\langle\$1,\$2\rangle)\Big) \Longleftrightarrow \Big(innerProductSpace\big((V,+,\cdot,\langle\$1,\$2\rangle),()\big)\Big) \wedge (v \in V) \wedge \big(\langle v,v\rangle=1\big)$$

$$\# \text{ the vector has unit length} \tag{193}$$

$$\text{(THM) Cauchy-Schwarz inequality: } \forall_{v,w \in V} \big(\langle v, w \rangle \leq ||v|| \, ||w|| \big) \tag{194}$$

$$basis\big((b)_n, (V, +, \cdot,)\big) \iff \Big(vectorSpace\big((V, +, \cdot), ()\big) \Big) \wedge \left(\forall_{v \in V} \exists_{(a)_n \in \mathbb{R}^n} \left(v = \sum_{i=1}^{n} (a_i b_i) \right) \right) \tag{195}$$

$$orthonormalBasis\Big((b)_n, \big(V, +, \cdot, \langle \$1, \$2 \rangle\big)\Big) \iff \Big(innerProductSpace\big((V, +, \cdot, \langle \$1, \$2 \rangle), ()\big) \Big) \wedge$$
$$\Big(basis\big((b)_n, (V, +, \cdot)\big) \Big) \wedge \left(\forall_{v \in (b)_n} \Big(normal\big(v, (V, +, \cdot, \langle \$1, \$2 \rangle)\big) \Big) \right) \wedge$$
$$\left(\forall_{v \in (b)_n} \forall_{w \in (b)_n \setminus \{v\}} \Big(orthogonal\big((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle)\big) \Big) \right) \tag{196}$$

1.16 Subvector space

$$subspace\big((U, \circ), (V, \circ)\big) \iff \Big(space\big((V, \circ), ()\big) \Big) \wedge (U \subseteq V) \wedge \Big(space\big((U, \circ), ()\big) \Big) \tag{197}$$

$$subspaceSum\big(U + W, (U, W, V, +)\big) \iff \Big(subspace\big((U, +), (V, +)\big) \Big) \wedge \Big(subspace\big((W, +), (V, +)\big) \Big) \wedge$$
$$\big(U + W = \{u + w \,|\, u \in U \wedge w \in W\}\big) \tag{198}$$

$$subspaceDirectSum\big(U \oplus W, (U, W, V, +)\big) \iff (U \cap W = \emptyset) \wedge \Big(subspaceSum\big(U \oplus W, (U, W, V, +)\big) \Big) \tag{199}$$

$$orthogonalComplement\Big(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)\Big) \iff$$
$$\left(subspace\left((W, +, \cdot, \langle \$1, \$2 \rangle), \Big(innerProductSpace\big((V, +, \cdot, \langle \$1, \$2 \rangle), ()\big) \Big) \right) \right) \wedge$$
$$\left(W^\perp = \left\{ v \in V \,|\, w \in W \wedge orthogonal\big((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle)\big) \right\} \right) \tag{200}$$

$$orthogonalDecomposition\Big(\big(W, W^\perp\big), (W, V, +, \cdot, \langle \$1, \$2 \rangle) \Big) \iff$$
$$\Big(orthogonalComplement\big(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)\big) \Big) \wedge \Big(subspaceDirectSum\big(V, \big(W, W^\perp, V, +\big)\big) \Big) \tag{201}$$

$$\text{(THM) if } V \text{ is finite dimensional, then every vector has an orthogonal decomposition:} \tag{202}$$

1.17 Banach and Hilbert Space

$$cauchy\left((s)_\mathbb{N}, \Big(V, d(\$1, \$2)\Big) \right) \iff \left(metricSpace\left(\Big(V, d(\$1, \$2)\Big), () \right) \right) \wedge \big((s)_\mathbb{N} \subseteq V\big)$$
$$\Big(\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{m, n \geq N} \big(d(s_m, s_n) < \epsilon \big) \Big)$$

$$complete\Big(\big((V,d(\$1,\$2)),()\big)\Big) \iff \left(\forall_{(s)_{\mathbb{N}}\subseteq V}\exists_{s\in V}\left(cauchy\Big((s)_{\mathbb{N}},\big(V,d(\$1,\$2)\big)\Big)\implies \lim_{n\to\infty}\big(d(s,s_n)\big)=0\right)\right)$$

$$banachSpace\Big((V,+,\cdot,||\$1||),()\Big) \iff \Big(normMetric\big(d(\$1,\$2),(V,||\$1||)\big)\Big) \wedge \Big(complete\big(V,d(\$1,\$2)\big),()\Big)$$

$$hilbertSpace\Big((V,+,\cdot,\langle\$1,\$2\rangle),()\Big) \iff \Big(innerProductNorm\big(||\$1||,(V,+,\cdot,\langle\$1,\$2\rangle)\big)\Big) \wedge$$
$$\Big(normMetric\big(d(\$1,\$2),(V,||\$1||)\big)\Big) \wedge \Big(complete\big(V,d(\$1,\$2)\big),()\Big)$$

$$(\text{THM}) : hilbertSpace \implies banachSpace \quad (207)$$

$$separable\big((V,d),()\big) \iff \Big(\exists_{S\subseteq V}\big(dense(S,(V,d)) \wedge countablyInfinite(S,())\big)\Big)$$

$$(\text{THM}) : hilbertSpace\Big(\big(((V,+,\cdot,\langle\$1,\$2\rangle),()),()\big)\Big) \implies$$
$$\Big(\exists_{(b)_{\mathbb{N}}\subseteq V}\big(orthonormalBasis\big((b)_{\mathbb{N}},(V,+,\cdot,\langle\$1,\$2\rangle)\big) \wedge countablyInfinite((b)_{\mathbb{N}},())\big)\Big) \iff$$
$$separable\Big(\big((V,\sqrt{\langle\$1-\$2,\$1-\$2\rangle}),()\big)\Big)$$

1.18   Matrices, Operators, and Functionals

$$linearOperator\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big) \iff \Big(map\big(L,(V,W)\big)\Big) \wedge \Big(vectorSpace\big((V,+_V,\cdot_V),()\big)\Big) \wedge$$
$$\Big(vectorSpace\big((W,+_W,\cdot_W),()\big)\Big) \wedge \Big(\forall_{v_1,v_2\in V}\forall_{s_1,s_2\in\mathbb{R}}\big(L(s_1\cdot_V v_1 +_V s_2\cdot_V v_2)=s_1\cdot_W L(v_1)+_W s_2\cdot_W L(v_2)\big)\Big)$$
$$(210)$$

$$denseMap\Big(L,(D,H,+,\cdot,\langle\$1,\$2\rangle)\Big) \iff (D\subseteq H) \wedge \Big(linearOperator\big(L,(D,+,\cdot,H,+,\cdot)\big)\Big) \wedge$$
$$\Big(innerProductTopology\big(\mathcal{O},(H,+,\cdot,\langle\$1,\$2\rangle)\big)\Big) \wedge \Big(dense\big(D,(H,\mathcal{O},d(\$1,\$2))\big)\Big)$$
$$(211)$$

$$mapNorm\Big(||L||,(L,V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\Big) \iff$$
$$\Big(linearOperator\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big) \wedge$$

$$\Big(normedVectorSpace\big((V,+_V,\cdot_V,||\$1||_V),()\big)\Big) \wedge \Big(normedVectorSpace\big((W,+_W,\cdot_W,||\$1||_W),()\big)\Big) \wedge$$

$$\left(||L|| = sup\left(\left\{\frac{||Lf||_W}{||f||_V} \mid f \in V\right\}\right) = sup\big(\{||Lf||_W \mid f \in V \wedge ||f|| = 1\}\big)\right)$$

(212)

$$boundedMap\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big) \Longleftrightarrow$$

$$\Big(mapNorm\big(||L||,(L,V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\big) < \infty\Big)$$

(213)

$$\neg boundedMap\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big) \Longleftarrow$$

$$(U \subset V) \wedge \Big(\infty = mapNorm\big(||L||_U,(L,U,+_U,\cdot_U,||\$1||_U,W,+_W,\cdot_W,||\$1||_W)\big) \le ||L||\Big)$$

(214)

$$extensionMap\Big(\widehat{L},(L,V,D,W)\Big) \Longleftrightarrow (D \subseteq V) \wedge \Big(linearOperator\big(L,(D,+_D,\cdot_D,W,+_W,\cdot_W)\big)\Big) \wedge$$

$$\Big(linearOperator\big(\widehat{L},(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big) \wedge \Big(\forall_{d \in D}\big(\widehat{L}(d) = L(d)\big)\Big)$$

(215)

$$adjoint\Big(L^T,(L,V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W)\Big) \Longleftrightarrow \Big(hilbertSpace\big((V,+_V,\cdot_V,\langle\$1,\$2\rangle_V),()\big)\Big) \wedge$$

$$\Big(hilbertSpace\big((W,+_W,\cdot_W,\langle\$1,\$2\rangle_W),()\big)\Big) \wedge \Big(linearOperator\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big) \wedge$$

$$\left(\forall_{v \in V}\forall_{w \in W}\Big(\big(\langle Lv,w\rangle_W = \langle v,L^T w\rangle_V\big) \vee \big((Lv)^T w = v^T L^T w\big)\Big)\right)$$

target operator that acts similar to the domain operator

(216)

$$selfAdjoint\Big(L,\big(V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big) \Longleftrightarrow$$

$$L = adjoint\Big(L^T,\big(L,V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big)$$

also a generalization of symmetric matrices

(217)

$$matrix\big(L,(n,m)\big) \Longleftrightarrow \Big(linearOperator\big(L,(\mathbb{R}^m,+_m,\cdot_m,\mathbb{R}^n,+_n,\cdot_n)\big)\Big)$$

rows=dimensions, cols=vectors

(218)

$$eigenvector\big(v,(L,V,+,\cdot)\big) \Longleftrightarrow \Big(linearOperator\big(L,(V,+,\cdot,V,+,\cdot)\big)\Big) \wedge \Big(\exists_{\lambda \in \mathbb{R}}\big(L(v) = \lambda v\big)\Big)$$

(219)

$$eigenvalue\big(\lambda,(v,L,V,+,\cdot)\big) \Longleftrightarrow \forall_{s \in \mathbb{R}\setminus\{0\}}\Big(eigenvector\big(sv,(L,V,+,\cdot)\big)\Big)$$

(220)

$$identityOperator\big(I, (A)\big) \Longleftrightarrow \Big(matrix\big(A, (n,n)\big)\Big) \wedge (AI = IA = A)$$
(221)

$$inverseOperator\Big(A^{-1}, (A)\Big) \Longleftrightarrow \Big(A^{-1}A = identityOperator\big(I, (A)\big)\Big)$$
gauss-jordan elimination: $E[A|I] = [I|E] = [I|A^{-1}]$
(222)

$$(\text{THM}) : (AB)^{-1}(AB) = I = B^{-1}A^{-1}AB$$
(223)

$$transposeOperator\Big(A^T, (A)\Big) \Longleftrightarrow \Big(\big(A^T\big)_{m,n} = (A)_{n,m}\Big) \vee adjoint\Big(A^T, (A)\Big)$$
(224)

$$symmetricOperator\big(A, ()\big) \Longleftrightarrow \Big(A = transposeOperator\big(A^T, (A)\big)\Big) \vee \Big(selfAdjoint\big(A, ()\big)\Big)$$
(225)

$$(\text{THM}) : (AB)^T = B^T A^T \wedge \Big(A^T\Big)^{-1} = \Big(A^{-1}\Big)^T$$
(226)

$$(\text{THM}) : symmetricOperator\Big(A^T A, ()\Big) \Longleftarrow \Big(A^T A = \big(A^T A\big)^T = A^T A^{TT} = A^T A\Big)$$
(227)

$$decomposeLU\big(LU(A), (A)\big) \Longleftrightarrow \Big(matrix\big(A, (n,n)\big)\Big) \wedge \big(\exists_E (EA = U)\big) \wedge \Big(\forall_{x<n} \forall_{0<i<x} \big(U_{i,i} = 0\big)\Big) \wedge$$
$$\Big(LU(A) = E^{-1}U = A\Big)$$
lower triangle are all 0; useful for solving linear equations
(228)

$$Img\big(Img(A), (A)\big) \Longleftrightarrow \Big(matrix\big(A, (n,m)\big)\Big) \wedge \Big(Img(A) = \{Av \in \mathbb{R}^n \,|\, v \in \mathbb{R}^m\}\Big)$$
the column space; not always a subspace since A can map to a set not containing $\boldsymbol{0}$
(229)

$$Ker\big(Ker(A), (A)\big) \Longleftrightarrow \Big(matrix\big(A, (n,m)\big)\Big) \wedge \Big(Ker(A) = \{v \in \mathbb{R}^m \,|\, Av = \boldsymbol{0} \in \mathbb{R}^n\}\Big)$$
the null or solution space; always a subspace due to linearity $Av + Aw = \boldsymbol{0} = A(v+w)$
(230)

$$(\text{THM}) \text{ general linear solution: } \big(Ax_p = b\big) \wedge \big(x_n \in Ker(A)\big) \Longrightarrow \Big(Ax_p + Ax_n = b + 0 = A\big(x_p + x_n\big) = b\Big)$$
(231)

$$independentOperator\big(A, ()\big) \Longleftrightarrow \Big(matrix\big(A, (n,m)\big)\Big) \wedge \Big(\neg \exists_{v \in \mathbb{R}^m \setminus \boldsymbol{o}_m}(Av = 0) \Longleftrightarrow Ker(A) = \{\boldsymbol{0}_m\}\Big)$$
(232)

$$dimensonality\big(N,(A)\big) \iff \Big(matrix\big(A,(n,m)\big)\Big) \wedge \Big(N = inf\big(\{|(b)_n| \,|\, basis((b)_n,(A))\}\big)\Big)$$
(233)

$$rank\big(r,(A)\big) \iff \Big(matrix\big(A,(n,m)\big)\Big) \wedge \Big(dimensonality\big(r,(A)\big)\Big)$$
(234)

$$(\text{THM}) : \Big(matrix\big(A,(n,m)\big)\Big) \implies \Big(dimensionality\big(Ker(A)\big) = n - rank\big(r,(A)\big)\Big)$$
\# number of free variables
(235)

$$transposeNorm\big(||x||,()\big) \iff \Big(||x|| = \sqrt{x^T x}\Big)$$
(236)

$$transposeOrthogonality\big((x,y),()\big) \iff \Big(||x||^2 + ||y||^2 = ||x+y||^2\Big) \iff$$
$$\Big(x^T x + y^T y = (x+y)^T(x+y) = x^T x + y^T y + x^T y = y^T x\Big) \iff$$
$$\left(0 = \frac{x^T x + y^T y - (x^T x + y^T y)}{2} = \frac{x^T y + y^T x}{2} = x^T y\right)$$
(237)

$$orthogonalProjection\big(P_A b,(A,b)\big) \iff \Big(matrix\big(A,(n,m)\big)\Big) \wedge \Big(matrix\big(b,(m,1)\big)\Big) \wedge$$
$$\left(\exists_{c \in \mathbb{R}^m}\Big(A^T(b - P_A b) = 0 = A^T(b - Ac)\Big) \iff\right.$$
$$\left. A^T b = A^T A c \iff c = \Big(A^T A\Big)^{-1} A^T b \iff P_A b = Ac = \Big(A\Big(A^T A\Big)^{-1} A^T\Big)b\right)$$
\# A, A^T may not necessarily be invertible
(238)

$$(\text{THM}) : P = P^T = P^2$$
(239)

$$(\text{THM}) : independent(A) \implies invertible(A^T A)$$
(240)

$$getEigenvectors(X,(A,V,+,\cdot,d(\$1,\$2))) \iff (metricVectorSpace((V,+,\cdot,d(\$1,\$2)),())) \wedge$$
$$(X = \{v \in V \,|\, ||v|| = 1 \wedge eigenvector(v,(A,V,+,\cdot))\})$$
(241)

$$getEigenvalues(Y,(A,V,+,\cdot,d)) \iff (getEigenvectors(X,(A,V,+,\cdot,d(\$1,\$2)))) \wedge (Y = \{\lambda \in \mathbb{R} \,|\, x \in X \wedge eigenvalue\})$$
(242)

$$CONTHEREFIXMULTIPLICITYIGNORED$$
(243)

$$det(det(A),(A)) \iff (det(A) = CONTHERE)$$
(244)

$$invertibleOperator() \Longleftrightarrow CONTHERE \tag{245}$$

$$============ \textbf{N O T = U P D A T E D} ============$$
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$$============ \textbf{N O T = U P D A T E D} ============ \tag{246}$$

$$det(I)=1 \; ; \; rowexchange*=-1 \; ; \; rowoperations*=1 \; ;$$
$$det(\{\{\{k(a+x)\},\{k(b+y)\}\},\{\{c\},\{d\}\}\})=k(det(\{\{\{a\},\{b\}\},\{\{c\},\{d\}\}\})+det(\{\{\{x\},\{y\}\},\{\{c\},\{d\}\}\}))$$
$$\Longrightarrow \det(LU(A))=\prod_i (d_i) \; \# \text{ product of diagonals in upper triangular A, area of col parallelepiped} \tag{247}$$

$$Tr(A)=\sum_{i=1}^{n}(A_{i,i})$$
$$A=A^T=A^2 \Longrightarrow Tr(A)=dim(A) \; \# \text{ counts dimensions} \tag{248}$$

$$Tr(A)=\sum(\lambda)$$
$$det(A)=\prod(\lambda) \; \# \text{ where } \lambda \text{ are the eigenvalues of A} \tag{249}$$

$$S=independent(eigenvectors(A)) \Longrightarrow AS=S\Lambda \; ; \; \Lambda=diagonalized_{\lambda}$$
$$S-1AS=\Lambda$$
$$A=S\Lambda S^{-1}$$
$$\# \text{ eigendecomposition} \tag{250}$$

$$Ax=\lambda x$$
$$Ax-\lambda x=(A-\lambda I)x=0 \wedge (x \neq \boldsymbol{0}) \Longrightarrow eigenvalue(A-\lambda I)=0 \Longrightarrow \Pi(\lambda)=0=det(A-\lambda I)$$
$$det(A-\lambda I)=0 \wedge algebraEval \Longrightarrow \lambda$$
$$(A-\lambda I)x=0 \wedge elim \Longrightarrow x$$
$$\Longrightarrow (x,\lambda) \tag{251}$$

$$contlecture22 \tag{252}$$

$$============ \textbf{N O T = U P D A T E D} ============$$
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$$============ \textbf{N O T = U P D A T E D} ============$$

$$========= \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} =========$$
$$(253)$$

$$compactMap\Big(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)\Big) \iff \Big(boundedMap\Big(L, \big(V, +_V, \cdot_V, ||\$1||_V, W, +_W, \cdot_W, ||\$1||_W\big)\Big)\Big) \wedge$$
$$\Big(\forall_{v \in V}\Big(openBall\Big(B, \big(1.0, v, V, d_V(\$1, \$2)\big)\Big)\Big) \implies$$
$$compactSubset\Big(closure\Big(\overline{L(B)}, image\big(L(B), (B, L, V, W)\big)\Big), W, d_W(\$1, \$2)\Big), (W, \mathcal{O}_W)\Big)\Big)\Big)$$
$$(254)$$

(THM) Spectral thm.:

$$\Big(selfAdjoint\Big(L, \big(V, +, \cdot, \langle\$1, \$2\rangle, V, +, \cdot, \langle\$1, \$2\rangle\big)\Big)\Big) \wedge \Big(compactMap\big(L, (V, +, \cdot, V, +, \cdot)\big)\Big) \implies$$
$$\Big(\exists_{(e)_\mathbb{N} \subseteq V}\Big(orthonormalBasis\Big((e)_\mathbb{N}, \big(V, +, \cdot, \langle\$1, \$2\rangle\big)\Big) \wedge \forall_{e_n \in (e)_\mathbb{N}}\Big(eigenvector\big(e_n, (L, V, +, \cdot)\big)\Big)\Big)\Big) \implies$$
$$\Big(\exists_{(\lambda)_\mathbb{N} \subseteq \mathbb{R}^n} \forall_{e_n \in (e)_\mathbb{N}} \exists_{\lambda_n \in (\lambda)_\mathbb{N}}\Big(eigenvalue\big(\lambda_n, (e_n, L, V, +, \cdot)\big) \wedge \lim_{n \to \infty}(\lambda_n = 0) \wedge L = \sum_{n=1}^{\infty}\Big(\lambda_n e_n e_n{}^T\Big)\Big)\Big)\Big)$$

\# TODO intuition
$$(255)$$

1.19   Function spaces

$$curLp\big(\mathcal{L}^p, (p, M, \sigma, \mu)\big) \iff (p \in \mathbb{R}) \wedge (1 \le p < \infty) \wedge$$
$$\Big(\mathcal{L}^p = \{map\big(f, (M, \mathbb{R})\big) \mid measurableMap\big(f, (M, \sigma, \mathbb{R}, standardSigma)\big) \wedge \int \big(|f|^p d\mu\big) < \infty\}\Big) \quad (256)$$

$$vecLp\big(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)\big) \iff \Big(curLp\big(\mathcal{L}^p, (p, M, \sigma, \mu)\big)\Big) \wedge \Big(\forall_{f,g \in \mathcal{L}^p} \forall_{m \in M}\big((f+g)(m) = f(m) + g(m)\big)\Big) \wedge$$
$$\Big(\forall_{f \in \mathcal{L}^p} \forall_{s \in \mathbb{R}} \forall_{m \in M}\big((s \cdot f)(m) = (s)f(m)\big)\Big) \wedge \Big(vectorSpace\big((\mathcal{L}^p, +, \cdot), ()\big)\Big) \quad (257)$$

$$integralNorm\big(\wr\wr\$1\wr\wr, (+, \cdot, p, M, \sigma, \mu)\big) \iff \Big(vecLp\big(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)\big)\Big) \wedge \Big(map\Big(\wr\wr\$1\wr\wr, \big(\mathcal{L}^p, \mathbb{R}_0^+\big)\Big)\Big) \wedge$$
$$\Big(\forall_{f \in \mathcal{L}^p}\Big(0 \le \wr\wr f\wr\wr = \Big(\int \big(|f|^p d\mu\big)\Big)^{1/p}\Big)\Big) \quad (258)$$

(THM) : $integralNorm\big(\wr\wr\$1\wr\wr, (+, \cdot, p, M, \sigma, \mu)\big) \implies$
$$\Big(\forall_{f \in \mathcal{L}^p}\Big(\wr\wr f\wr\wr = 0 \implies almostEverywhere\big(f = \boldsymbol{0}, (M, \sigma, \mu)\big)\Big)\Big)$$

\# not an expected property from a norm   (259)

$$Lp\Big(L^p, \big((+, \cdot, p, M, \sigma, \mu)\big)\Big) \iff \Big(integralNorm\big(\wr\wr\$1\wr\wr, (+, \cdot, p, M, \sigma, \mu)\big)\Big) \wedge$$
$$\Big(L^p = quotientSet\Big(\mathcal{L}^p/\sim, \Big(\mathcal{L}^p, \big(\wr\wr\$1 + (-\$2)\wr\wr = 0\big)\Big)\Big)\Big)\Big)$$

$$\text{\# functions in } L^p \text{ that have finite integrals above and below the x-axis} \tag{260}$$

$$\text{(THM)} : banachSpace\Big(\Big(Lp\big(L^p, (+, \cdot, p, M, \sigma, \mu)\big), +, \cdot, \wr\wr\$1\wr\wr\Big), ()\Big) \tag{261}$$

$$\text{(THM)} : hilbertSpace\left(\left(Lp\big(L^p, (+, \cdot, 2, M, \sigma, \mu)\big), +, \cdot, \frac{\wr\wr\$1+\$2\wr\wr^2 - \wr\wr\$1-\$2\wr\wr^2}{4}\right), ()\right) \tag{262}$$

$$curL\Big(\mathcal{L}, \big(V, +_V, \cdot_V, ||\$1||_V, W, +_W, \cdot_W, ||\$1||_W\big)\Big) \Longleftrightarrow \Big(banachSpace\big((W, +_W, \cdot_W, ||\$1||_W), ()\big)\Big) \wedge$$
$$\Big(normedVectorSpace\big((V, +_V, \cdot_V, ||\$1||_V), ()\big)\Big) \wedge$$
$$\Big(\mathcal{L} = \{f \mid boundedMap\big(f, (V, +_V, \cdot_V, ||\$1||_V, W, +_W, \cdot_W, ||\$1||_W)\big)\}\Big) \tag{263}$$

$$\text{(THM)} : banachSpace\left(\left(curL\Big(\mathcal{L}, \big(V, +_V, \cdot_V, ||\$1||_V, W, +_W, \cdot_W, ||\$1||_W\big)\Big), +, \cdot, mapNorm\right), ()\right) \tag{264}$$

$$\text{(THM)} : ||L|| \geq \frac{||Lf||}{||f||} \quad \text{\# from choosing an arbitrary element in the mapNorm sup} \tag{265}$$

$$\text{(THM)} : \Big(cauchy\big((f)_{\mathbb{N}}, (\mathcal{L}, +, \cdot, mapNorm)\big)\Longrightarrow cauchy\Big((f_n v)_{\mathbb{N}}, \big(W, +_W, \cdot_W, ||\$1||_W\big)\Big)\Big) \Longleftarrow$$
$$\Big(\forall_{\epsilon'>0}\forall_{v\in V}\big(||f_n v - f_m v||_W = ||(f_n - f_m)v||_W \leq ||f_n - f_m|| \cdot ||v||_V < \epsilon \cdot ||v||_V = \epsilon'\big)\Big)$$
$$\text{\# a cauchy sequence of operators maps to a cauchy sequence of targets} \tag{266}$$

$$\text{(THM) BLT thm.:} \left(\Big(dense\big(D, (V, \mathcal{O}, d_V)\big) \wedge boundedMap\Big(A, \big(D, +_V, \cdot_V, ||\$1||_V, W, +_W, \cdot_W, ||\$1||_W\big)\Big)\Big) \Longrightarrow\right.$$
$$\left.\Big(\exists!_{\widehat{A}}\Big(extensionMap\big(\widehat{A}, (A, V, D, W)\big)\Big) \wedge ||\widehat{A}|| = ||A||\Big)\right) \Longleftarrow$$
$$\Big(\forall_{v\in V}\exists_{(v)_{\mathbb{N}}\subseteq D}\big(\lim_{n\to\infty}(v_n = v)\big)\Big) \wedge \Big(\widehat{A}v = \lim_{n\to\infty}(Av_n)\Big) \tag{267}$$

1.20   Probability Theory

$$0 \tag{268}$$

1.21   Underview

$$\tag{269}$$

$$curve - fitting/explaining \neq prediction \tag{270}$$

$$ill - definedproblem + solutionspaceconstraints \Longrightarrow well - definedproblem \tag{271}$$

$$x \ \# \ \text{input} \ ; \ y \ \# \ \text{output} \tag{272}$$

$$S_n = \{(x_1, y_1), \ldots, (x_n, y_n)\} \ \# \ \text{training set} \tag{273}$$

$$f_S(x) \sim y \ \# \ \text{solution} \tag{274}$$

$$each(x, y) \in p(x, y) \ \# \ \text{training data } x, y \text{ is a sample from an unknown distribution } p \tag{275}$$

$$V(f(x), y) = d(f(x), y) \ \# \ \text{loss function} \tag{276}$$

$$I[f] = \int_{X \times Y} V(f(x), y) p(x, y) dx dy \ \# \ \text{expected error} \tag{277}$$

$$I_n[f] = \frac{1}{n} \sum_{i=1}^{n} V(f(x_i), y_i) \ \# \ \text{empirical error} \tag{278}$$

$$probabilisticConvergence(X, ()) \iff \forall_{\epsilon > 0} \lim_{n \to \infty} P x n - x \leq \epsilon = 0 \tag{279}$$

$$I - I n \, generalization \, error \tag{280}$$

$$well - posed := exists, unique, stable; else ill - posed \tag{281}$$

2 Machine Learning

2.0.1 Overview

$$X \ \# \ \text{input} \ ; \ Y \ \# \ \text{output} \ ; \ S(X, Y) \ \# \ \text{dataset} \tag{282}$$

$$\textbf{learned parameters} = \textbf{parameters to be fixed by training with the dataset} \tag{283}$$

$$\textbf{hyperparameters} = \textbf{parameters that depends on a dataset} \tag{284}$$

$$\textbf{validation} = \textbf{partitions dataset into training and testing partitions, then evaluates the}$$
$$\textbf{accuracy of the parameters learned from the training partition in predicting the}$$
$$\textbf{outputs of the testing partition} \ \# \ \text{useful for fixing hyperparameters} \tag{285}$$

$$\textbf{cross-validation} = \textbf{average accuracy of validation for different choices of testing partition} \tag{286}$$

$$\textbf{L1} = \textbf{scales linearly} \ ; \ \textbf{L2} = \textbf{scales quadratically} \tag{287}$$

$$d = \textbf{distance} = \textbf{quantifies the the similarity between data points} \tag{288}$$

$$d_{L1}(A, B) = \sum_p |A_p - B_p| \ \# \ \text{Manhattan distance} \tag{289}$$

$$d_{L2}(A,B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \tag{290}$$

$$\textbf{kNN classifier} = \textbf{classifier based on } k \textbf{ nearest data points} \tag{291}$$

$$s = \textbf{class score} = \textbf{quantifies bias towards a particular class} \tag{292}$$

$$s_{linear} = f_{c \times 1}(x_{n \times 1}, W_{c \times n}, b_{c \times 1}) = W_{c \times n} x_{n \times 1} + b_{c \times 1} \text{ \# linear score function} \tag{293}$$

$$l = \textbf{loss} = \textbf{quantifies the errors by the learned parameters} \tag{294}$$

$$l = \frac{1}{|c_i|} \sum_{c_i} l_i \text{ \# average loss for all classes} \tag{295}$$

$$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \text{ \# SVM hinge class loss function:}$$
$$\text{\# ignores incorrect classes with lower scores including a non-zero margin} \tag{296}$$

$$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$
$$\text{\# lower scores correspond to lower exponentiated-normalized probabilities} \tag{297}$$

$$R = \textbf{regularization} = \textbf{optimizes the choice of learned parameters to minimize test error} \tag{298}$$

$$\lambda \text{ \# regularization strength hyperparameter} \tag{299}$$

$$R_{L1}(W) = \sum_{W_i} |W_i| \text{ \# L1 regularization} \tag{300}$$

$$R_{L2}(W) = \sum_{W_i} W_i{}^2 \text{ \# L2 regularization} \tag{301}$$

$$L' = L + \lambda R(W) \text{ \# weight regularization} \tag{302}$$

$$\nabla_W L = \frac{\overrightarrow{\partial}}{\partial W_i} L = \textbf{loss gradient w.r.t. weights} \tag{303}$$

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L} \text{ \# loss gradient w.r.t. input weight in terms of external and local gradients} \tag{304}$$

$$s = \textbf{forward API} \; ; \; \frac{\partial L_L}{\partial W_I} = \textbf{backward API} \tag{305}$$

$$W_{t+1} = W_t - \nabla_{W_t} L \text{ \# weight update loss minimization} \tag{306}$$

> **TODO:Research on Activation functions, Weight Initialization, Batch Normalization** $\qquad$ (307)
>
> $review5meanvardiscussion/hyperparameteroptimization/babysittinglearning$ $\qquad$ (308)

TODO loss L or l ??

3 Glossary

chaoticTopology
discreteTopology
topology
topologicalSpace
open
closed
clopen
neighborhood
chaoticTopology
discreteTopology
metric
metricSpace
openBall
metricTopology
metricTopologicalSpace
limitPoint
interiorPoint
closure
dense
eucD
standardTopology
subsetTopology
productTopology
metric
metricSpace
openBall
metricTopology
metricTopologicalSpace
limitPoint
interiorPoint
closure
dense
eucD
standardTopology
subsetTopology
productTopology
sequence
sequenceConvergesTo
sequence
sequenceConvergesTo
continuous
homeomorphism
isomorphicTopologicalSpace
continuous
homeomorphism
isomorphicTopologicalSpace
T0Separate
T1Separate
T2Separate
T0Separate
T1Separate
T2Separate
openCover
finiteSubcover

compact
compactSubset
bounded
openCover
finiteSubcover
compact
compactSubset
bounded
openRefinement
locallyFinite
paracompact
openRefinement
locallyFinite
paracompact
connected
pathConnected
connected
pathConnected
sigmaAlgebra
measurableSpace
measurableSet
measure
measureSpace
finiteMeasure
generatedSigmaAlgebra
borelSigmaAlgebra
standardSigma
lebesgueMeasure
measurableMap
pushForwardMeasure
nullSet
almostEverywhere
sigmaAlgebra
measurableSpace
measurableSet
measure
measureSpace
finiteMeasure
generatedSigmaAlgebra
borelSigmaAlgebra
standardSigma
lebesgueMeasure
measurableMap
pushForwardMeasure
nullSet
almostEverywhere
simpleTopology
simpleSigma
simpleFunction
characteristicFunction
exStandardSigma
nonNegIntegrable
nonNegIntegral
explicitIntegral

integrable
integral
simpleTopology
simpleSigma
simpleFunction
characteristicFunction
exStandardSigma
nonNegIntegrable
nonNegIntegral
explicitIntegral
integrable
integral
vectorSpace
innerProduct
innerProductSpace
vectorNorm
normedVectorSpace
vectorMetric
metricVectorSpace
innerProductNorm
normInnerProduct
normMetric
metricNorm
orthogonal
normal
basis
orthonormalBasis
vectorSpace
innerProduct
innerProductSpace
vectorNorm
normedVectorSpace
vectorMetric
metricVectorSpace
innerProductNorm
normInnerProduct
normMetric
metricNorm
orthogonal
normal
basis
orthonormalBasis
subspace
subspaceSum
subspaceDirectSum
orthogonalComplement
orthogonalDecomposition
subspace
subspaceSum
subspaceDirectSum
orthogonalComplement
orthogonalDecomposition
cauchy
complete

banachSpace
hilbertSpace
separable
cauchy
complete
banachSpace
hilbertSpace
separable
linearOperator
denseMap
mapNorm
boundedMap
extensionMap
adjoint
selfAdjoint
matrix
eigenvector
eigenvalue
identityOperator
inverseOperator
transposeOperator
symmetricOperator
decomposeLU
Img
Ker
independentOperator
dimensonality
rank
transposeNorm
transposeOrthogonality
orthogonalProjection
getEigenvectors
getEigenvalues
det
invertibleOperator
compactMap
linearOperator
denseMap
mapNorm
boundedMap
extensionMap
adjoint
selfAdjoint
matrix
eigenvector
eigenvalue
identityOperator
inverseOperator
transposeOperator
symmetricOperator
decomposeLU
Img
Ker
independentOperator