

# Next-Next-Gen Notes

## Object-Oriented Maths

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Format:  $characteristic((subjects), (dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$

Note: All weaker objects automatically induces notions inherited from stronger objects.

TODO define  $\parallel$  abs cross-product and other missing refs

TODO define  $**args$  for comparison callbacks, predicate args, norms and or placeholders

TODO link thms?

## 1 Mathematical Analysis

### 1.0.1 Formal Logic

$$statement(s, (RegEx)) \iff well-formedString(s, ()) \quad (1)$$

$$proposition((p, t), ()) \iff \left( statement(p, ()) \wedge \right. \\ \left. (t = eval(p)) \wedge \right. \\ \left. (t = true \vee t = false) \right) \quad (2)$$

$$operator\left(o, \left((p)_{n \in \mathbb{N}}\right)\right) \iff proposition\left(o\left((p)_{n \in \mathbb{N}}\right), ()\right) \quad (3)$$

$$operator(\neg, (p_1)) \iff \left( proposition((p_1, true), ()) \implies ((\neg p_1, false), ()) \right) \wedge \\ \left( proposition((p_1, false), ()) \implies ((\neg p_1, true), ()) \right) \\ \# \text{ an operator takes in propositions and returns a proposition} \quad (4)$$

$$operator(\neg) \iff \mathbf{NOT} ; operator(\vee) \iff \mathbf{OR} ; operator(\wedge) \iff \mathbf{AND} ; operator(\veebar) \iff \mathbf{XOR} \\ operator(\implies) \iff \mathbf{IF} ; operator(\impliedby) \iff \mathbf{OIF} ; operator(\iff) \iff \mathbf{IFF} \quad (5)$$

$$proposition((false \implies true), true, ()) \wedge proposition((false \implies false), true, ()) \\ \# \text{ truths based on a false premise is not false; ex falso quodlibet principle} \quad (6)$$

$$(\text{THM}) : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \quad (7)$$

$$predicate(P, (V)) \iff \forall_{v \in V} \left( proposition\left((P(v), t), ()\right) \right) \quad (8)$$

$$0thOrderLogic(P, ()) \iff proposition((P, t), ()) \\ \# \text{ individual proposition} \quad (9)$$

$$1stOrderLogic(P, (V)) \iff \left( \forall_{v \in V} \left( 0thOrderLogic(v, ()) \right) \right) \wedge$$

$$\left( \forall_{v \in V} \left( \text{proposition} \left( (P(v), t), () \right) \right) \right)$$

# propositions defined over a set of the lower order logical statements (10)

$$\text{quantifier}(q, (p, V)) \iff \left( \text{predicate}(p, (V)) \right) \wedge \left( \text{proposition} \left( (q(p), t), () \right) \right)$$

# a quantifier takes in a predicate and returns a proposition (11)

$$\text{quantifier}(\forall, (p, V)) \iff \text{proposition} \left( \left( \bigwedge_{v \in V} (p(v)), t \right), () \right)$$

# universal quantifier (12)

$$\text{quantifier}(\exists, (p, V)) \iff \text{proposition} \left( \left( \bigvee_{v \in V} (p(v)), t \right), () \right)$$

# existential quantifier (13)

$$\text{quantifier}(\exists!, (p, V)) \iff \exists_{x \in V} \left( P(x) \wedge \neg \left( \exists_{y \in V \setminus \{x\}} (P(y)) \right) \right)$$

# uniqueness quantifier (14)

$$(\text{THM}) : \forall_x p(x) \iff \neg \exists_x \neg p(x)$$

# De Morgan's law (15)

$$(\text{THM}) : \forall_x \exists_y p(x, y) = \forall_x \neg \forall_y \neg p(x, y) \neq \exists_y \forall_x p(x, y) = \neg \forall_y \neg (\forall_x p(x, y)) = \neg \forall_y \exists_x \neg p(x, y)$$

# different quantifiers are not interchangeable (16)

$$\text{===== N O T = U P D A T E D =====}$$

(17)

$$\text{proof} = \text{truths derived from a finite number of axioms and deductions}$$

(18)

$$\text{elementary arithmetics} = \text{system with substitutions, and some notion of addition, multiplication, and prime numbers for encoding metamathematics}$$

(19)

$$\text{Gödel theorem} \implies \text{axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions}$$

(20)

$$\text{sequenceSet}((A)_{\mathbb{N}}, (A)) \iff (\text{Amapinputn})((A)_{\mathbb{N}} = \{A(1), A(2), A(3), \dots\})$$

(21)

$$\text{TODO: define union, intersection, complement, etc.}$$

(22)

$$\text{===== N O T = U P D A T E D =====}$$

(23)

## 1.1 Axiomatic Set Theory

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (24)$$

$$\text{ZFC set theory} = \text{standard form of axiomatic set theory} \quad (25)$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \quad (26)$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \quad (27)$$

$$\in \text{ basis} \implies \{x, y\} = \{y, x\} \wedge \{x\} = \{x, x\} \quad (28)$$

$$\in \text{ and sets works following the 9 ZFC axioms:} \quad (29)$$

$$\forall_x \forall_y (x \in y \vee \neg(x \in y)) \# \text{ E: } \in \text{ is only a proposition on sets} \quad (30)$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \# \text{ E: existence of empty set} \quad (31)$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \# \text{ C: pair set construction} \quad (32)$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \# \text{ C: union set construction} \quad (33)$$

$$x = \{\{a\}, \{b\}\} \# \text{ from the pair set axiom} \quad (34)$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \quad (35)$$

$$\forall_x \exists!_y R(x, y) \# \text{ functional relation } R \quad (36)$$

$$\begin{aligned} \exists_i \forall_x \exists!_y R(x, y) \implies y \in i \# \text{ C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \mid P(y)\} \# \text{ Restricted Comprehension } \not\Rightarrow \{y \mid P(y)\} \# \text{ Universal Comprehension} \end{aligned} \quad (37)$$

$$\forall_{x \in m} P(x) = \forall_x (x \in m \implies P(x)) \# \text{ ignores out of scope } \neq \forall_x (x \in m \wedge P(x)) \# \text{ restricts entirety} \quad (38)$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} (n \subseteq m \implies n \subseteq \mathcal{P}(m)) \# \text{ C: existence of power set} \quad (39)$$

$$\exists_I (\emptyset \in I \wedge \forall_{x \in I} (\{x\} \in I)) \# \text{ I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}; I \cong \mathbb{N} \implies \mathbb{N} \text{ is a set} \quad (40)$$

$$\forall_x ((\emptyset \notin x \wedge x \cap x' = \emptyset) \implies \exists_y (\text{set of each } \mathbf{e} \in x)) \# \text{ C: axiom of choice} \quad (41)$$

$$\forall_x x \neq \emptyset \implies x \notin x \# \text{ F: axiom of foundation covers further paradoxes} \quad (42)$$

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (43)$$

## 1.2 Classification of sets

$$\begin{aligned} \text{space}((\text{set}, \text{structure}), ()) &\iff \text{structure}(\text{set}) \\ \# \text{ a space a set equipped with some structure} \\ \# \text{ various spaces can be studied through structure preserving maps between those spaces} \end{aligned} \quad (44)$$

$$\begin{aligned} \text{map}(\phi, (A, B)) &\iff \left( \forall_{a \in A} \exists!_{b \in B} (\phi(a, b)) \right) \vee \\ &\quad \left( \forall_{a \in A} \exists!_{b \in B} (b = \phi(a)) \right) \\ \# \text{ maps elements of a set to elements of another set} \end{aligned} \quad (45)$$

$$\text{domain}(A, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (46)$$

$$\text{codomain}(B, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (47)$$

$$\begin{aligned} \text{image}(B, (A, q, M, N)) &\iff \left( \text{map}(q, (M, N)) \wedge A \subseteq M \right) \wedge \\ &\quad \left( B = \{n \in N \mid \exists_{a \in A} (q(a) = n)\} \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \text{preimage}(A, (B, q, M, N)) &\iff \left( \text{map}(q, (M, N)) \wedge B \subseteq N \right) \wedge \\ &\quad \left( A = \{m \in M \mid \exists_{b \in B} (b = q(m))\} \right) \end{aligned} \quad (49)$$

$$\begin{aligned} \text{injection}(q, (M, N)) &\iff \left( \text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{u, v \in M} (q(u) = q(v) \implies u = v) \\ \# \text{ every } m \text{ has at most 1 image} \end{aligned} \quad (50)$$

$$\begin{aligned} \text{surjection}(q, (M, N)) &\iff \left( \text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{n \in N} \exists_{m \in M} (n = q(m)) \\ \# \text{ every } n \text{ has at least 1 preimage} \end{aligned} \quad (51)$$

$$\begin{aligned} \text{bijection}(q, (M, N)) &\iff \left( \text{injection}(q, (M, N)) \right) \wedge \\ &\quad \left( \text{surjection}(q, (M, N)) \right) \\ \# \text{ every unique } m \text{ corresponds to a unique } n \end{aligned} \quad (52)$$

$$\text{isomorphicSets}((A, B), ()) \iff \exists_{\phi} (\text{bijection}(\phi, (A, B))) \quad (53)$$

$$\text{infiniteSet}(S, ()) \iff \exists_{T \subseteq S} (\text{isomorphicSets}((T, S), ())) \quad (54)$$

$$\text{finiteSet}(S, ()) \iff \left( \neg \text{infiniteSet}(S, ()) \right) \vee (|S| \in \mathbb{N}) \quad (55)$$

$$\text{countablyInfinite}(S, ()) \iff \left( \text{infiniteSet}(S, ()) \right) \wedge \left( \text{isomorphicSets}((S, \mathbb{N}), ()) \right) \quad (56)$$

$$\text{uncountablyInfinite}(S, ()) \iff (\text{infiniteSet}(S, ())) \wedge (\neg \text{isomorphicSets}((S, \mathbb{N}), ())) \quad (57)$$

$$\begin{aligned} \text{inverseMap}(q^{-1}, (q, M, N)) &\iff (\text{bijection}(q, (M, N))) \wedge \\ &\quad (\text{map}(q^{-1}, (N, M))) \wedge \\ &\quad \left( \forall_{n \in N} \exists!_{m \in M} (q(m) = n \implies q^{-1}(n) = m) \right) \end{aligned} \quad (58)$$

$$\begin{aligned} \text{mapComposition}(\phi \circ \psi, (\phi, \psi, A, B, C)) &\iff \text{map}(\psi, (A, B)) \wedge \text{map}(\phi, (B, C)) \wedge \\ &\quad \forall_{a \in A} (\phi \circ \psi(a) = \phi(\psi(a))) \end{aligned} \quad (59)$$

$$\begin{aligned} \text{equivalenceRelation}(\sim (\$1, \$2), (M)) &\iff (\forall_{m \in M} (m \sim m)) \wedge \\ &\quad (\forall_{m, n \in M} (m \sim n \implies n \sim m)) \wedge \\ &\quad (\forall_{m, n, p \in M} (m \sim n \wedge n \sim p \implies m \sim p)) \\ &\quad \# \text{ behaves as equivalences should} \end{aligned} \quad (60)$$

$$\begin{aligned} \text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) &\iff [m]_{\sim} = \{n \in M \mid n \sim m\} \\ &\quad \# \text{ set of elements satisfying the equivalence relation with } m \end{aligned} \quad (61)$$

$$\begin{aligned} (\text{THM}) : a \in [m]_{\sim} &\implies [a]_{\sim} = [m]_{\sim} ; [m]_{\sim} = [n]_{\sim} \vee [m]_{\sim} \cap [n]_{\sim} = \emptyset \\ &\quad \# \text{ equivalence class properties} \end{aligned} \quad (62)$$

$$\begin{aligned} \text{quotientSet}(M/\sim, (M, \sim)) &\iff M/\sim = \{\text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) \in \mathcal{P}(M) \mid m \in M\} \\ &\quad \# \text{ set of all equivalence classes} \end{aligned} \quad (63)$$

$$\begin{aligned} (\text{THM}) : (M, \sim, +) &\implies (\text{quotientSet}(M/\sim, (M, \sim)), +_{\sim}) \iff \forall_{[r], [s] \in M/\sim} \forall_{a \in [r]} \forall_{b \in [s]} ([r] +_{\sim} [s] = [a + b]) \\ &\quad \# \text{ a quotient set can inherit the operations on the original set if it is well-defined} \end{aligned} \quad (64)$$

$$\begin{aligned} (\text{THM}) : \text{axiom of choice} &\implies \forall_{[m]_{\sim} \in M/\sim} \exists_r (r \in [m]_{\sim}) \\ &\quad \# \text{ well-defined maps may be defined in terms of chosen representative elements } r \end{aligned} \quad (65)$$

### 1.3 Construction of number sets

$$\text{===== N O T = U P D A T E D =====} \quad (66)$$

$$\text{axiom of infinity} \implies \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\} \cong \mathbb{N} \quad (67)$$

$$\mathbb{N}^* = \mathbb{N} \setminus \{0\} \quad (68)$$

$$\text{addition} = \text{successor map: } \mathbb{N} \rightarrow \mathbb{N} = S(n) = \{n\} \# \text{ adds a layer of brackets} \quad (69)$$

$$\text{subtraction} = \text{predecessor map: } \mathbb{N}^* \rightarrow \mathbb{N} = P(n) = m \mid m \in n \# \text{ removes a layer of brackets} \quad (70)$$

$$S^0 = id ; n \in \mathbb{N}^* \implies S^n = S \circ S^{P(n)} \quad (71)$$

$$\mathbf{addition} = + : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} = +(m, n) = m + n = S^n(m) \quad (72)$$

$$S^x = id = S^0 \implies x = \mathbf{additive\ identity} = 0 \quad (73)$$

$$S^n(x) = 0 \implies x = \mathbf{additive\ inverse} \notin \mathbb{N} \# \text{ git gud smh -_-} \quad (74)$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim, \text{ s.t.: } (m, n) \sim (p, q) \iff m + q = p + n \# \text{ span } \mathbb{Z} \text{ using differences then group equal differences} \quad (75)$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \rightarrow [(n, 0)] \# \mathbb{N} \text{ embedded in } \mathbb{Z} \quad (76)$$

$$+_Z = [(m +_N p, n +_N q)] \# \text{ well-defined and consistent} \quad (77)$$

$$\mathbf{multiplication} \dots M^x = id \implies x = \mathbf{multiplicative\ identity} = 1 \dots \mathbf{multiplicative\ inverse} \notin \mathbb{N} \quad (78)$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^*) / \sim, \text{ s.t.: } (x, y) \sim (u, v) \iff x \cdot v = u \cdot y \quad (79)$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \rightarrow [(q, 1)] ; \dots \{x \mid x^2 = 2\} \notin \mathbb{Q} \quad (80)$$

$$\mathbb{R} = \mathbf{almost\ homomorphisms\ on\ } \mathbb{Z} / \sim \# \text{ http://blog.sigfpe.com/2006/05/defining-reals.html} \quad (81)$$

$$\text{===== N O T = U P D A T E D =====} \quad (82)$$

## 1.4 Topology

$$\text{topology}(\mathcal{O}, (M)) \iff (\mathcal{O} \subseteq \mathcal{P}(M)) \wedge (\emptyset, M \in \mathcal{O}) \wedge$$

$$\left( (F \in \mathcal{O} \wedge |F| < |\mathbb{N}|) \implies \cap F \in \mathcal{O} \right) \wedge (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O})$$

# topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.

# arbitrary unions of open sets always result in an open set

# open sets do not contain their boundaries and infinite intersections of open sets may approach and

# induce boundaries resulting in a closed set (83)

$$\text{topologicalSpace}((M, \mathcal{O}), ()) \iff \text{topology}(\mathcal{O}, (M)) \quad (84)$$

$$\text{open}(S, (M, \mathcal{O})) \iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge (S \subseteq M) \wedge (S \in \mathcal{O})$$

# an open set do not contains its own boundaries (85)

$$\begin{aligned} \text{closed}(S, (M, \mathcal{O})) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\quad (S \subseteq M) \wedge (S \in \mathcal{P}(M) \setminus \mathcal{O}) \\ \# \text{ a closed set contains the boundaries an open set} \end{aligned} \quad (86)$$

$$\text{clopen}(S, (M, \mathcal{O})) \iff \left( \text{closed}(S, (M, \mathcal{O})) \right) \wedge \left( \text{open}(S, (M, \mathcal{O})) \right) \quad (87)$$

$$\begin{aligned} \text{neighborhood}(U, (a, \mathcal{O})) &\iff (a \in U \in \mathcal{O}) \\ \# \text{ another name for open set containing } a \end{aligned} \quad (88)$$

$$\begin{aligned} M = \{a, b, c, d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} &\implies \\ \left( \text{open}(X, (M, \mathcal{O})) \iff X = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} \right) &\wedge \\ \left( \text{closed}(Y, (M, \mathcal{O})) \iff Y = \{\emptyset, \{a, b, d\}, \{c, d\}, \{a, b\}, \{d\}, M\} \right) &\wedge \\ \left( \text{clopen}(Z, (M, \mathcal{O})) \iff Z = \{\emptyset, \{a, b\}, \{c, d\}, M\} \right) \end{aligned} \quad (89)$$

$$\text{chaoticTopology}(M) = \{0, M\} ; \text{discreteTopology} = \mathcal{P}(M) \quad (90)$$

## 1.5 Induced topology

$$\begin{aligned} \text{metric}(d(\$1, \$2), (M)) &\iff \left( \text{map} \left( d, \left( M \times M, \mathbb{R}_0^+ \right) \right) \right) \\ &\quad \left( \forall_{x, y \in M} (d(x, y) = d(y, x)) \right) \wedge \\ &\quad \left( \forall_{x, y \in M} (d(x, y) = 0 \iff x = y) \right) \wedge \\ &\quad \left( \forall_{x, y, z} \left( d(x, z) \leq d(x, y) + d(y, z) \right) \right) \\ \# \text{ behaves as distances should} \end{aligned} \quad (91)$$

$$\text{metricSpace}((M, d), ()) \iff \text{metric}(d, (M)) \quad (92)$$

$$\text{openBall}(B, (r, p, M, d)) \iff \left( \text{metricSpace}((M, d), ()) \right) \wedge (r \in \mathbb{R}^+, p \in M) \wedge (B = \{q \in M \mid d(p, q) < r\}) \quad (93)$$

$$\begin{aligned} \text{metricTopology}(\mathcal{O}, (M, d)) &\iff \left( \text{metricSpace}((M, d), ()) \right) \wedge \\ &\quad \left( \mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (\text{openBall}(B, (r, p, M, d)) \wedge B \subseteq U)\} \right) \\ \# \text{ every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \end{aligned} \quad (94)$$

$$\text{metricTopologicalSpace}((M, \mathcal{O}, d), ()) \iff \text{metricTopology}(\mathcal{O}, (M, d)) \quad (95)$$

$$\begin{aligned} \text{limitPoint}(p, (S, M, d)) &\iff (S \subseteq M) \wedge \forall_{r \in \mathbb{R}^+} \left( \text{openBall}(B, (r, p, M, d)) \cap S \neq \emptyset \right) \\ \# \text{ every open ball centered at } p &\text{ contains some intersection with } S \end{aligned} \quad (96)$$

$$\text{interiorPoint}(p, (S, M, d)) \iff (S \subseteq M) \wedge \left( \exists_{r \in \mathbb{R}^+} \left( \text{openBall}(B, (r, p, M, d)) \subseteq S \right) \right)$$

$$\# \text{ there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (97)$$

$$\text{closure}(\bar{S}, (S, M, d)) \iff \bar{S} = S \cup \{\text{limitPoint}(p, (S, M, d)) \mid p \in M\} \quad (98)$$

$$\text{dense}(S, (M, d)) \iff (S \subseteq M) \wedge \left( \forall_{p \in M} \left( p \in \text{closure}(\bar{S}, (S, M, d)) \right) \right) \\ \# \text{ every of point in } M \text{ is a point or a limit point of } S \quad (99)$$

$$\text{eucD}(d, (n)) \iff (\forall_{i \in \mathbb{N} \wedge i \leq n} (x_i \in \mathbb{R})) \wedge \left( d = \sqrt[2]{\sum_{i=1}^n x_i^2} \right) \quad (100)$$

$$\text{metricTopology} \left( \text{standardTopology}, \left( \mathbb{R}^n, \text{eucD}(d, (n)) \right) \right) \\ \text{===== N O T = U P D A T E D =====} \\ \text{L1: } \forall_{p \in U = \emptyset} (\dots) \implies \forall_p \left( (p \in \emptyset) \implies \dots \right) \implies \forall_p ((\text{False}) \implies \dots) \implies \emptyset \in \mathcal{O}_{\text{standard}} \\ \text{L2: } \forall_{p \in \mathbb{R}^n} B(r, p, \mathbb{R}^n, d) \subseteq \mathbb{R}^n \implies M \in \mathcal{O}_{\text{standard}} \\ \text{L4: } C \subseteq \mathcal{O}_{\text{standard}} \implies \forall_{U \in C} \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (B_r(p) \subseteq U \subseteq \cup C) \implies \cup C \in \mathcal{O}_{\text{standard}} \\ \text{L3: } U, V \in \mathcal{O}_{\text{standard}} \implies p \in U \cap V \implies p \in U \wedge p \in V \implies \\ \exists_{r \in \mathbb{R}^+} B(r, p, \mathbb{R}^n, d) \wedge \exists_{s \in \mathbb{R}^+} B(s, p, \mathbb{R}^n, d) \implies \\ B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \subseteq U \wedge B(\min(r, s), q, \mathbb{R}^n, d) \subseteq V \implies \\ B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \in U \cap V \implies U \cap V \in \mathcal{O}_{\text{standard}} \\ \# \text{ natural topology for } \mathbb{R}^d \\ \# \text{ could fail on infinite sets since } \min \text{ could approach } 0 \\ \text{===== N O T = U P D A T E D =====} \quad (101)$$

$$\text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \iff \text{topology}(\mathcal{O}, (M)) \wedge (N \subseteq M) \wedge (\mathcal{O}|_N = \{U \cap N \mid U \in \mathcal{O}\}) \\ \# \text{ crops open sets outside } N \quad (102)$$

$$(\text{THM}) : \text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \wedge \text{topology}(\mathcal{O}|_N, (N)) \iff \\ \text{===== N O T = U P D A T E D =====} \\ \text{L1: } \emptyset \in \mathcal{O} \implies U = \emptyset \implies \emptyset \cap N = \emptyset \implies \emptyset \in \mathcal{O}|_N \\ \text{L2: } M \in \mathcal{O} \implies U = M \implies M \cap N = N \implies N \in \mathcal{O}|_N \\ \text{L3: } S, T \in \mathcal{O}|_N \implies \exists_{U \in \mathcal{O}} (S = U \cap N) \wedge \exists_{V \in \mathcal{O}} (T = V \cap N) \implies S \cap T = (U \cap N) \cap (V \cap N) \\ = (U \cap V) \cap N \wedge U \cap V \in \mathcal{O} \implies S \cap T \in \mathcal{O}|_N \\ \text{L4: } \text{TODO: EXERCISE} \\ \text{===== N O T = U P D A T E D =====} \quad (103)$$

$$\text{productTopology} \left( \mathcal{O}_{A \times B}, ((A, \mathcal{O}_A), (B, \mathcal{O}_B)) \right) \iff \left( \text{topology}(\mathcal{O}_A, (A)) \right) \wedge \left( \text{topology}(\mathcal{O}_B, (B)) \right) \wedge \\ (\mathcal{O}_{A \times B} = \{(a, b) \in A \times B \mid \exists_S (a \in S \in \mathcal{O}_A) \exists_T (b \in T \in \mathcal{O}_B)\}) \\ \# \text{ open in cross iff open in each} \quad (104)$$



## 1.6 Convergence

$$\text{sequence}(q, (M)) \iff \text{map}(q, (\mathbb{N}, M)) \quad (105)$$

$$\begin{aligned} \text{sequenceConvergesTo}((q, a), (M, \mathcal{O})) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left( \text{sequence}(q, (M)) \right) \wedge (a \in M) \wedge \left( \forall U \in \mathcal{O} | a \in U \exists N \in \mathbb{N} \forall n > N (q(n) \in U) \right) \\ &\# \text{ each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \end{aligned} \quad (106)$$

(THM) : convergence generalizes to: the sequence  $q: \mathbb{N} \rightarrow \mathbb{R}^d$  converges against  $a \in \mathbb{R}^d$  in  $\mathcal{O}_S$  if:

$$\forall r > 0 \exists N \in \mathbb{N} \forall n > N (||q(n) - a|| < r) \# \text{ distance based convergence} \quad (107)$$

## 1.7 Continuity

$$\begin{aligned} \text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}_M), ()) \right) \wedge \\ &\left( \text{topologicalSpace}((N, \mathcal{O}_N), ()) \right) \wedge \left( \forall V \in \mathcal{O}_N \left( \text{preimage}(A, (V, \phi, M, N)) \in \mathcal{O}_M \right) \right) \\ &\# \text{ preimage of open sets are open} \end{aligned} \quad (108)$$

$$\begin{aligned} \text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left( \text{inverseMap}(\phi^{-1}, (\phi, M, N)) \right) \\ &\left( \text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \wedge \left( \text{continuous}(\phi^{-1}, (N, \mathcal{O}_N, M, \mathcal{O}_M)) \right) \\ &\# \text{ structure preserving maps in topology, ability to share topological properties} \end{aligned} \quad (109)$$

$$\begin{aligned} \text{isomorphicTopologicalSpace}((M, \mathcal{O}_M), (N, \mathcal{O}_N), ()) &\iff \\ &\exists \phi \left( \text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \end{aligned} \quad (110)$$

## 1.8 Separation

$$\begin{aligned} T0Separate((M, \mathcal{O}), ()) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left( \forall x, y \in M \wedge x \neq y \exists U \in \mathcal{O} \left( (x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U) \right) \right) \\ &\# \text{ each pair of points has a neighborhood s.t. one is inside and the other is outside} \end{aligned} \quad (111)$$

$$\begin{aligned} T1Separate((M, \mathcal{O}), ()) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left( \forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V \left( (x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V) \right) \right) \\ &\# \text{ every point has a neighborhood that does not contain another point} \end{aligned} \quad (112)$$

$$\begin{aligned} T2Separate((M, \mathcal{O}), ()) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left( \forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V (U \cap V = \emptyset) \right) \\ &\# \text{ every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \end{aligned} \quad (113)$$

$$(THM) : T2Separate \implies T1Separate \implies T0Separate \quad (114)$$

## 1.9 Compactness

$$\begin{aligned} openCover(C, (M, \mathcal{O})) &\iff \left( topologicalSpace((M, \mathcal{O}), ()) \right) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M) \\ &\# \text{ collection of open sets whose elements cover the entire space} \end{aligned} \quad (115)$$

$$\begin{aligned} finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) &\iff (\tilde{C} \subseteq C) \wedge (openCover(C, (M, \mathcal{O}))) \wedge \\ &\left( openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge (finiteSet(\tilde{C}, ())) \\ &\# \text{ finite subset of a cover that is also a cover} \end{aligned} \quad (116)$$

$$\begin{aligned} compact((M, \mathcal{O}), ()) &\iff \left( topologicalSpace((M, \mathcal{O}), ()) \right) \wedge \\ &\left( \forall C \subseteq \mathcal{O} \left( openCover(C, (M, \mathcal{O})) \implies \exists \tilde{C} \subseteq C \left( finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) \right) \right) \right) \\ &\# \text{ every covering of the space is represented by a finite number of nhbhd} \end{aligned} \quad (117)$$

$$\begin{aligned} compactSubset(N, (M, \mathcal{O})) &\iff \left( compact((M, \mathcal{O}), ()) \right) \wedge \\ &\left( subsetTopology(\mathcal{O}|_N, (M, \mathcal{O}, N)) \right) \wedge \left( compact((N, \mathcal{O}|_N), ()) \right) \end{aligned} \quad (118)$$

$$\begin{aligned} bounded(N, (M, d)) &\iff \left( metricSpace((M, d), ()) \right) \wedge (N \subseteq M) \wedge \\ &\left( \exists r \in \mathbb{R}^+ \forall p, q \in N (d(p, q) < r) \right) \end{aligned} \quad (119)$$

$$\begin{aligned} (THM) \text{ Heine-Borel thm.: } &metricTopologicalSpace((M, \mathcal{O}_d, d), ()) \implies \\ &\forall S \subseteq M \left( \left( closed(S, (M, \mathcal{O}_d)) \wedge bounded(S, (M, \mathcal{O}_d)) \right) \iff compactSubset(S, (M, \mathcal{O}_d)) \right) \\ &\# \text{ when metric topologies are involved, compactness is equivalent to being closed and bounded} \end{aligned} \quad (120)$$

## 1.10 Paracompactness

$$\begin{aligned} openRefinement(\tilde{C}, (C, M, \mathcal{O})) &\iff \left( openCover(C, (M, \mathcal{O})) \right) \wedge \left( openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge \\ &\left( \forall \tilde{U} \in \tilde{C} \exists U \in C (\tilde{U} \subseteq U) \right) \\ &\# \text{ a refined cover can be constructed by removing the excess nhbhd} \end{aligned} \quad (121)$$

$$(THM) : finiteSubcover \implies openRefinement \quad (122)$$

$$\begin{aligned} locallyFinite(C, (M, \mathcal{O})) &\iff \left( openCover(C, (M, \mathcal{O})) \right) \wedge \\ &\forall p \in M \exists U \in \mathcal{O} | p \in U \left( finiteSet(\{U_c \in C | U \cap U_c \neq \emptyset\}, ()) \right) \\ &\# \text{ each point has a neighborhood that intersects with only finitely many sets in the cover} \end{aligned} \quad (123)$$

$$\begin{aligned} & \text{paracompact}((M, \mathcal{O}), ()) \iff \\ \forall_C \left( \text{openCover}(C, (M, \mathcal{O})) \implies \exists_{\tilde{C}} \left( \text{locallyFinite} \left( \text{openRefinement}(\tilde{C}, (C, M, \mathcal{O})), (M, \mathcal{O}) \right) \right) \right) \\ & \# \text{ every open cover has a locally finite open refinement} \end{aligned} \quad (124)$$

$$(\text{THM}) : \text{metricTopologicalSpace} \implies \text{paracompact} \quad (125)$$

$$\text{===== N O T = U P D A T E D =====} \quad (126)$$

$$\begin{aligned} & \text{partitionOfUnitySubjCover}(\mathcal{F}, (C, M, \mathcal{O})) \iff \left( \text{locallyFinite}(C, (M, \mathcal{O})) \right) \wedge (f \in \mathcal{F}) \wedge \\ & \left( \text{continuous} \left( f, \left( M, \mathcal{O}, [0, 1], \text{subsetTopology}(\mathcal{O}|_{[0, 1]}, ([0, 1], \mathbb{R}, \text{standardTopology})) \right) \right) \right) \wedge \\ & \left( \exists_{U_f \in C} \forall_{p \in M} (f(p) \neq 0 \implies p \in U_f) \right) \wedge \\ & \left( \forall_{p \in M} \exists_{U \in \mathcal{O}} |_{p \in U} ((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\}) \right) \wedge \\ & \left( \text{locallyFinite}(C, M, \mathcal{O}) \implies \text{finiteSet}((f_U)_n, ()) \right) \wedge \\ & \left( \forall_{p \in M} \exists_{U \in \mathcal{O}} |_{p \in U} \left( \sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1 \right) \right) \\ & \# \text{ useful for defining integrals between overlapping neighborhoods} \end{aligned} \quad (127)$$

$$\begin{aligned} & T2Separate((M, \mathcal{O}), ()) \implies \left( \text{paracompact}((M, \mathcal{O}), ()) \right) \iff \\ & \forall_C \left( \text{openCover}(C, (M, \mathcal{O})) \implies \text{partitionOfUnitySOTCover}(\mathcal{F}, (C, M, \mathcal{O})) \right) \end{aligned} \quad (128)$$

$$\text{===== N O T = U P D A T E D =====} \quad (129)$$

### 1.11 Connectedness and path-connectedness

$$\begin{aligned} & \text{connected}((M, \mathcal{O}), ()) \iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \left( \neg \exists_{A, B \in \mathcal{O} \setminus \emptyset} (A \cap B \neq \emptyset \wedge A \cup B = M) \right) \\ & \# \text{ if there is some covering of the space that does not intersect} \end{aligned} \quad (130)$$

$$\begin{aligned} & (\text{THM}) : \neg \text{connected} \left( \left( \mathbb{R} \setminus \{0\}, \text{subsetTopology}(\mathcal{O}_{\text{standard}}|_{\mathbb{R} \setminus \{0\}}, (\mathbb{R}, \text{standardTopology}, \mathbb{R} \setminus \{0\})) \right), () \right) \\ & \iff \left( A = (-\infty, 0) \in \mathcal{O}_{\text{standard}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \left( B = (0, \infty) \in \mathcal{O}_{\text{standard}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \\ & (A \cap B = \emptyset) \wedge (A \cup B = \mathbb{R} \setminus \{0\}) \end{aligned} \quad (131)$$

$$(\text{THM}) : \text{connected}((M, \mathcal{O}), ()) \iff \forall_{S \in \mathcal{O}} \left( \text{clopen}(S, (M, \mathcal{O})) \implies (S = \emptyset \vee S = M) \right) \quad (132)$$

$$\text{pathConnected}((M, \mathcal{O}), ()) \iff \left( \text{subsetTopology}(\mathcal{O}_{\text{standard}}|_{[0, 1]}, (\mathbb{R}, \text{standardTopology}, [0, 1])) \right) \wedge$$

$$\left( \forall_{p,q \in M} \exists_{\gamma} \left( \text{continuous} \left( \gamma, ([0,1], \mathcal{O}_{\text{standard}}|_{[0,1]}, M, \mathcal{O}) \right) \wedge \gamma(0)=p \wedge \gamma(1)=q \right) \right) \quad (133)$$

$$(\text{THM}) : \text{pathConnected} \implies \text{connected} \quad (134)$$

## 1.12 Homotopic curve and the fundamental group

$$\text{===== NOT UPDATED =====} \quad (135)$$

$$\begin{aligned} \text{homotopic}(\sim, (\gamma, \delta, M, \mathcal{O})) &\iff (\text{map}(\gamma, ([0,1], M)) \wedge \text{map}(\delta, ([0,1], M))) \wedge \\ &\quad (\gamma(0)=\delta(0) \wedge \gamma(1)=\delta(1)) \wedge \\ (\exists_H \forall_{\lambda \in [0,1]} (\text{continuous}(H, ([0,1] \times [0,1], \mathcal{O}_{\text{standard}^2}|_{[0,1] \times [0,1]}), (M, \mathcal{O})) \wedge H(0, \lambda) = \gamma(\lambda) \wedge H(1, \lambda) = \delta(\lambda))) \\ &\quad \# H \text{ is a continuous deformation of one curve into another} \end{aligned} \quad (136)$$

$$\text{homotopic}(\sim) \implies \text{equivalenceRelation}(\sim) \quad (137)$$

$$\text{loopSpace}(\mathcal{L}_p, (p, M, \mathcal{O})) \iff \mathcal{L}_p = \{ \text{map}(\gamma, ([0,1], M)) \mid \text{continuous}(\gamma) \wedge \gamma(0)=\gamma(1) \} \quad (138)$$

$$\begin{aligned} \text{concatination}(\star, (p, \gamma, \delta)) &\iff (\gamma, \delta \in \text{loopSpace}(\mathcal{L}_p)) \wedge \\ (\forall_{\lambda \in [0,1]} ((\gamma \star \delta)(\lambda) &= \begin{cases} \gamma(2\lambda) & 0 \leq \lambda < 0.5 \\ \delta(2\lambda - 1) & 0.5 \leq \lambda \leq 1 \end{cases})) \end{aligned} \quad (139)$$

$$\begin{aligned} \text{group}((G, \bullet), ()) &\iff (\text{map}(\bullet, (G \times G, G))) \wedge \\ &\quad (\forall_{a,b \in G} (a \bullet b \in G)) \\ &\quad (\forall_{a,b,c \in G} ((a \bullet b) \bullet c = a \bullet (b \bullet c))) \\ &\quad (\exists_e \forall_{a \in G} (e \bullet a = a = a \bullet e)) \wedge \\ &\quad (\forall_{a \in G} \exists_{a^{-1}} (a \bullet a^{-1} = e = a^{-1} \bullet a)) \\ &\quad \# \text{ characterizes symmetry of a set structure} \end{aligned} \quad (140)$$

$$\text{isomorphic}(\cong, (X, \odot), (Y, \ominus)) \iff \exists_f \forall_{a,b \in X} (\text{bijection}(f, (X, Y)) \wedge f(a \odot b) = f(a) \ominus f(b)) \quad (141)$$

$$\begin{aligned} \text{fundamentalGroup}((\pi_{1,p}, \bullet), (p, M, \mathcal{O})) &\iff (\pi_{1,p} = \mathcal{L}_p / \sim) \wedge \\ &\quad (\text{map}(\bullet, (\pi_{1,p} \times \pi_{1,p}, \pi_{1,p}))) \wedge \\ &\quad (\forall_{A,B \in \pi_{1,p}} ([A] \bullet [B] = [A \star B])) \wedge \\ &\quad (\text{group}((\pi_{1,p}, \bullet), ())) \\ &\quad \# \text{ an equivalence class of all loops induced from the homotopic equivalence relation} \end{aligned} \quad (142)$$

$$\text{fundamentalGroup}_1 \not\cong \text{fundamentalGroup}_2 \implies \text{topologicalSpace}_1 \not\cong \text{topologicalSpace}_2 \quad (143)$$

$$\text{there exists no known list of topological properties that can imply homeomorphisms} \quad (144)$$

$$\text{CONTINUE @ Lecture 6: manifolds} \quad (145)$$

$$\text{===== NOT UPDATED =====} \quad (146)$$

### 1.13 Measure theory

$$\begin{aligned}
\text{sigmaAlgebra}(\sigma, (M)) &\iff (M \neq \emptyset) \wedge (\sigma \subseteq \mathcal{P}(M)) \wedge \\
&\quad (M \in \sigma) \wedge \left( \forall A \in \sigma (M \setminus A \in \sigma) \right) \wedge \\
&\quad \left( \left( A \subseteq \sigma \wedge \neg \text{uncountablyInfinite}(A, ()) \right) \implies \cup A \in \sigma \right) \\
\# \text{ behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu &\quad (147)
\end{aligned}$$

$$\text{measurableSpace}((M, \sigma), ()) \iff \text{sigmaAlgebra}(\sigma, (M)) \quad (148)$$

$$\text{measurableSet}(A, (M, \sigma)) \iff \left( \text{measurableSpace}((M, \sigma), ()) \right) \wedge (A \in \sigma) \quad (149)$$

$$\begin{aligned}
\text{measure}(\mu, (M, \sigma)) &\iff \left( \text{measurableSpace}((M, \sigma), ()) \right) \wedge \left( \text{map} \left( \mu, \left( \sigma, \left( \mathbb{R}^+ \right)_0 \right) \right) \right) \wedge (\mu(\emptyset) = 0) \wedge \\
&\quad \left( \left( (A)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} \forall j \in \mathbb{N} \setminus \{i\} (A_i \cap A_j = \emptyset) \right) \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) = \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \\
\# \text{ enforces meaningful concepts of measures such as precise additivity} &\quad (150)
\end{aligned}$$

$$\begin{aligned}
&(\text{THM}) : \text{measure}(\mu, (M, \sigma)) \implies \\
&\quad \left( \forall A, B \in \sigma (A \subseteq B \implies \mu(A) \leq \mu(B)) \right) \wedge \\
&\quad \left( (A)_{\mathbb{N}} \subseteq \sigma \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) \leq \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \wedge \\
&\quad \left( ((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (B_i \subseteq B_{i+1}) \wedge B = \cup (B)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(B_n)) = \mu(B) \right) \wedge \\
&\quad \left( ((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (C_{i+1} \subseteq C_i) \wedge C = \cap (C)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(C_n)) = \mu(C) \right) \\
\# \text{ immediate implications of the measurable set } A \in \sigma \text{ axioms and the measure } \mu \text{ axioms} &\quad (151)
\end{aligned}$$

$$\text{measureSpace}((M, \sigma, \mu), ()) \iff \text{measure}(\mu, (M, \sigma)) \quad (152)$$

$$\begin{aligned}
\text{finiteMeasure}(\mu, (M, \sigma)) &\iff \left( \text{measure}(\mu, (M, \sigma)) \right) \wedge \\
&\quad \left( \exists (A)_{\mathbb{N}} \subseteq \sigma \left( \cup (A)_{\mathbb{N}} = M \wedge \forall n \in \mathbb{N} (\mu(A_n) < \infty) \right) \right) \\
&\quad (153)
\end{aligned}$$

$$\begin{aligned}
\text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) &\iff \left( G = \{ \sigma \subseteq \mathcal{P}(M) \mid \text{sigmaAlgebra}(\sigma, (M)) \} \right) \wedge (\sigma(\zeta) = \cap G) \\
\# \text{ smallest } \sigma\text{-algebra containing the generating set } \zeta &\quad (154)
\end{aligned}$$

$$(\text{THM}) : \exists \zeta \subseteq M \left( \text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) = \text{sigmaAlgebra}(\sigma, (M)) \right) \quad (155)$$

$$\begin{aligned}
\text{borelSigmaAlgebra}(\sigma(\mathcal{O}), (M, \mathcal{O})) &\iff \left( \text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\
&\quad \left( \text{generatedSigmaAlgebra}(\sigma(\mathcal{O}), (\mathcal{O}, M)) \right) \\
\# \sigma\text{-algebra induced by a topology} &\quad (156)
\end{aligned}$$

$$\text{standardSigma}(\sigma_s, ()) \iff \left( \text{borelSigmaAlgebra} \left( \sigma_s, \left( \mathbb{R}^d, \text{standardTopology} \right) \right) \right) \quad (157)$$

$$\begin{aligned} \text{lebesgueMeasure}(\lambda, ()) \iff & \left( \text{measure} \left( \lambda, \left( \mathbb{R}^d, \text{standardSigma} \right) \right) \right) \wedge \\ & \left( \lambda \left( \times_{i=1}^d ([a_i, b_i]) \right) = \sum_{i=1}^d \left( \sqrt[2]{(a_i - b_i)^2} \right) \right) \\ & \# \text{ natural measure for } \mathbb{R}^d \end{aligned} \quad (158)$$

$$\begin{aligned} \text{measurableMap}(f, (M, \sigma_M, N, \sigma_N)) \iff & \left( \text{measurableSpace}((M, \sigma_M), ()) \right) \wedge \\ & \left( \text{measurableSpace}((N, \sigma_N), ()) \right) \wedge \left( \forall B \in \sigma_N \left( \text{preimage}(A, (B, f, M, N)) \in \sigma_M \right) \right) \\ & \# \text{ preimage of measurable sets are measurable} \end{aligned} \quad (159)$$

$$\begin{aligned} \text{pushForwardMeasure}(f \star \lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \iff & \left( \text{measureSpace}((M, \sigma_M, \mu_M), ()) \right) \wedge \\ & \left( \text{measurableSpace}((N, \sigma_N), ()) \right) \wedge \left( \text{measurableMap}(f, (M, \sigma_M, N, \sigma_N)) \right) \wedge \\ & \left( \forall B \in \sigma_N \left( f \star \lambda_M(B) = \mu_M \left( \text{preimage}(A, (B, f, M, N)) \right) \right) \right) \wedge \left( \text{measure}(f \star \lambda_M, (N, \sigma_N)) \right) \\ & \# \text{ natural construction of a measure based primarily on measurable map} \end{aligned} \quad (160)$$

$$\text{nullSet}(A, (M, \sigma, \mu)) \iff \left( \text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge (A \in \sigma) \wedge (\mu(A) = 0) \quad (161)$$

$$\begin{aligned} \text{almostEverywhere}(p, (M, \sigma, \mu)) \iff & \left( \text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge \left( \text{predicate}(p, (M)) \right) \wedge \\ & \left( \exists A \in \sigma \left( \text{nullSet}(A, (M, \sigma, \mu)) \implies \forall n \in M \setminus A (p(n)) \right) \right) \\ & \# \text{ the predicate holds true for all points except the points in the null set} \end{aligned} \quad (162)$$

## 1.14 Lebesgue integration

$$\text{simpleTopology}(\mathcal{O}_{\text{simple}}, ()) \iff \mathcal{O}_{\text{simple}} = \text{subsetTopology} \left( \mathcal{O}|_{\mathbb{R}_0^+}, \left( \mathbb{R}, \text{standardTopology}, \mathbb{R}_0^+ \right) \right) \quad (163)$$

$$\text{simpleSigma}(\sigma_{\text{simple}}, ()) \iff \text{borelSigmaAlgebra} \left( \sigma_{\text{simple}}, \left( \mathbb{R}_0^+, \text{simpleTopology} \right) \right) \quad (164)$$

$$\begin{aligned} \text{simpleFunction}(s, (M, \sigma)) \iff & \left( \text{measurableMap} \left( s, \left( M, \sigma, \mathbb{R}_0^+, \text{simpleSigma} \right) \right) \right) \wedge \\ & \left( \text{finiteSet} \left( \text{image} \left( B, \left( M, s, M, \mathbb{R}_0^+ \right) \right), () \right) \right) \\ & \# \text{ if the map takes on finitely many values on } \mathbb{R}_0^+ \end{aligned} \quad (165)$$

$$\begin{aligned} \text{characteristicFunction}(X_A, (A, M)) &\iff (A \subseteq M) \wedge \left( \text{map}(X_A, (M, \mathbb{R})) \right) \wedge \\ &\left( \forall_{m \in M} \left( X_A(m) = \begin{cases} 1 & m \in A \\ 0 & m \notin A \end{cases} \right) \right) \end{aligned} \quad (166)$$


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$$\begin{aligned} (\text{THM}) : \text{simpleFunction}(s, (M, \sigma_M)) &\implies \\ &\left( \text{finiteSet} \left( \text{image} \left( Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right) \wedge \\ &\left( \text{characteristicFunction}(X_A, (A, M)) \right) \wedge \left( \forall_{m \in M} \left( s(m) = \sum_{z \in Z} \left( z \cdot X_{\text{preimage} \left( A, (\{z\}, s, M, \mathbb{R}_0^+) \right)}(m) \right) \right) \right) \end{aligned} \quad (167)$$


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$$\begin{aligned} \text{exStandardSigma}(\overline{\sigma_s}, ()) &\iff \overline{\sigma_s} = \{A \subseteq \mathbb{R} \mid A \cap R \in \text{standardSigma}\} \\ \# \text{ ignores } \pm\infty \text{ to preserve the points in the domain of the measurable map} \end{aligned} \quad (168)$$


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$$\begin{aligned} \text{nonNegIntegrable}(f, (M, \sigma)) &\iff \left( \text{measurableMap} \left( f, (M, \sigma, \mathbb{R}, \text{exStandardSigma}) \right) \right) \wedge \\ &\left( \forall_{m \in M} (f(m) \geq 0) \right) \end{aligned} \quad (169)$$


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$$\begin{aligned} \text{nonNegIntegral} \left( \int_M (f d\mu), (f, M, \sigma, \mu) \right) &\iff \left( \text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge \\ &\left( \text{measureSpace} \left( (\mathbb{R}, \text{exStandardSigma}, \text{lebesgueMeasure}), () \right) \right) \wedge \\ &\left( \text{nonNegIntegrable}(f, (M, \sigma)) \right) \wedge \left( \int_M (f d\mu) = \sup \left( \left\{ \sum_{z \in Z} \left( z \cdot \mu \left( \text{preimage} \left( A, (\{z\}, s, M, \mathbb{R}_0^+) \right) \right) \right) \right\} \right) \mid \right. \\ &\left. \forall_{m \in M} (s(m) \leq f(m)) \wedge \text{simpleFunction}(s, (M, \sigma)) \wedge \text{finiteSet} \left( \text{image} \left( Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right\}) \\ &\# \text{ lebesgue measure on } z \text{ reduces to } z \end{aligned} \quad (170)$$


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$$\begin{aligned} \text{explicitIntegral} &\iff \int (f(x) \mu(dx)) = \int (f d\mu) \\ \# \text{ alternative notation for lebesgue integrals} \end{aligned} \quad (171)$$


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$$\begin{aligned} (\text{THM}) : \text{nonNegIntegral} \left( \int (f d\mu), (f, M, \sigma, \mu) \right) &\wedge \text{nonNegIntegral} \left( \int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\ (\text{THM}) \text{ Markov inequality: } &\left( \forall_{z \in \mathbb{R}_0^+} \left( \int (f d\mu) \geq z \cdot \mu \left( \text{preimage} \left( A, ([z, \infty), f, M, \mathbb{R}] \right) \right) \right) \right) \wedge \\ &\left( \text{almostEverywhere}(f = g, (M, \sigma, \mu)) \implies \int (f d\mu) = \int (g d\mu) \right) \\ &\left( \int (f d\mu) = 0 \implies \text{almostEverywhere}(f = 0, (M, \sigma, \mu)) \right) \wedge \\ &\left( \int (f d\mu) \leq \infty \implies \text{almostEverywhere}(f < \infty, (M, \sigma, \mu)) \right) \end{aligned} \quad (172)$$


---

$$\begin{aligned}
\text{(THM) Mono. conv.: } & \left( (f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left( f_n, (M, \sigma, \overline{R}, \text{exStandardSigma}) \right) \wedge 0 \leq f_{n-1} \leq f_n \} \right) \wedge \\
& \left( \text{map} \left( f, (M, \overline{\mathbb{R}}) \right) \right) \wedge \left( \forall_{m \in M} \left( f(m) = \sup(f_n(m) \mid f_n \in (f)_{\mathbb{N}}) \right) \right) \implies \left( \lim_{n \rightarrow \infty} \left( \int_M (f_n d\mu) \right) = \int_M (f d\mu) \right) \\
& \# \text{ lengths now depend on } M, \sigma \text{ and limits can be pulled in or out of an integral (173)}
\end{aligned}$$


---

$$\begin{aligned}
\text{(THM) : } & \text{nonNegIntegral} \left( \int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{nonNegIntegral} \left( \int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\
& \left( \forall_{\alpha \in \mathbb{R}_0^+} \left( \int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \right) \\
& \# \text{ integral acts linearly and commutes finite summations (174)}
\end{aligned}$$


---

$$\begin{aligned}
\text{(THM) : } & \left( (f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left( f_n, (M, \sigma, \overline{R}, \text{exStandardSigma}) \right) \wedge 0 \leq f_n\} \right) \implies \\
& \left( \int \left( \left( \sum_{n=1}^{\infty} (f_n) \right) d\mu \right) = \sum_{n=1}^{\infty} \left( \int (f_n d\mu) \right) \right) \\
& \# \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n \rightarrow \infty} \sum_{i=1}^n f_n \text{ since } f_n \geq 0 \text{ and it commutes with integral from monotone conv. (175)}
\end{aligned}$$


---

$$\begin{aligned}
& \text{integrable}(f, (M, \sigma)) \iff \left( \text{measurableMap} \left( f, (M, \sigma, \overline{\mathbb{R}}, \text{exStandardSigma}) \right) \right) \wedge \\
& \left( \forall_{m \in M} \left( f(m) = \max(f(m), 0) - \max(0, -f(m)) \right) \right) \wedge \\
& \left( \text{measureSpace}(M, \sigma, \mu) \implies \left( \int (\max(f(m), 0) d\mu) < \infty \wedge \int (\max(0, -f(m)) d\mu) < \infty \right) \right) \\
& \# \text{ extra condition prevents the occurrence of the indeterminate } \infty - \infty \text{ (176)}
\end{aligned}$$


---

$$\begin{aligned}
& \text{integral} \left( \int (f d\mu), (f, M, \sigma, \mu) \right) \iff \left( \text{nonNegIntegral} \left( \int (f^+ d\mu), (\max(f, 0), M, \sigma, \mu) \right) \right) \wedge \\
& \left( \text{nonNegIntegral} \left( \int (f^- d\mu), (\max(0, -f), M, \sigma, \mu) \right) \right) \wedge \left( \text{integrable}(f, (M, \sigma)) \right) \wedge \\
& \left( \int (f d\mu) = \int (f^+ d\mu) - \int (f^- d\mu) \right) \\
& \# \text{ arbitrary integral in terms of nonnegative integrals (177)}
\end{aligned}$$


---

$$\text{(THM) : } \left( \text{map}(f, (M, \mathbb{C})) \right) \implies \left( \int (f d\mu) = \int (\text{Re}(f) d\mu) - \int (\text{Im}(f) d\mu) \right) \text{ (178)}$$


---

$$\begin{aligned}
\text{(THM) : } & \text{integral} \left( \int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{integral} \left( \int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\
& \left( \text{almostEverywhere}(f \leq g, (M, \sigma, \mu)) \implies \int (f d\mu) \leq \int (g d\mu) \right) \wedge \\
& \left( \forall_{m \in M} (f(m), g(m), \alpha \in \mathbb{R}) \implies \int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \text{ (179)}
\end{aligned}$$


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$$\begin{aligned}
& \text{(THM) Dominant convergence: } \left( (f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left( f_n, (M, \sigma, \overline{R}, \text{exStandardSigma}) \right) \} \right) \wedge \\
& \quad \left( \text{map}(f, (M, \overline{R})) \right) \wedge \left( \text{almostEverywhere} \left( f(m) = \lim_{n \rightarrow \infty} (f_n(m)), (M, \sigma, \mu) \right) \right) \wedge \\
& \quad \left( \text{nonNegIntegral} \left( \int (gd\mu), (g, M, \sigma, \mu) \right) \right) \wedge \left( \left| \int (gd\mu) \right| < \infty \right) \wedge \left( \text{almostEverywhere}(|f_n| \leq g, (M, \sigma, \mu)) \right) \\
& \quad \# \text{ if all } f_n(m) \text{ are bounded by some integrable } |g(m)| \implies \\
& \quad \# \text{ then all } f_n(m) \text{ including } f \text{ satisfy bounded and integrable properties} \\
& \quad \left( \forall_{\phi \in \{f\} \cup (f)_{\mathbb{N}}} \left( \text{integrable}(\phi, (M, \sigma)) \right) \right) \wedge \left( \lim_{n \rightarrow \infty} \left( \int (|f_n - f| d\mu) = 0 \right) \right) \wedge \left( \lim_{n \rightarrow \infty} \left( \int (f_n d\mu) \right) = \int (f d\mu) \right) \quad (180)
\end{aligned}$$

## 1.15 Vector space and structures

$$\begin{aligned}
& \text{vectorSpace}((V, +, \cdot), ()) \iff \left( \text{map}(+, (V \times V, V)) \right) \wedge \left( \text{map}(\cdot, (\mathbb{R} \times V, V)) \right) \wedge \\
& \quad (\forall_{v, w \in V} (v + w = w + v)) \wedge \\
& \quad (\forall_{v, w, x \in V} ((v + w) + x = v + (w + x))) \wedge \\
& \quad (\exists \mathbf{0} \in V \forall_{v \in V} (v + \mathbf{0} = v)) \wedge \\
& \quad (\forall_{v \in V} \exists_{-v \in V} (v + (-v) = \mathbf{0})) \wedge \\
& \quad (\forall_{a, b \in \mathbb{R}} \forall_{v \in V} (a(b \cdot v) = (ab) \cdot v)) \wedge \\
& \quad (\exists 1 \in \mathbb{R} \forall_{v \in V} (1 \cdot v = v)) \wedge \\
& \quad (\forall_{a, b \in \mathbb{R}} \forall_{v \in V} ((a + b) \cdot v = a \cdot v + b \cdot v)) \wedge \\
& \quad (\forall_{a \in \mathbb{R}} \forall_{v, w \in V} (a \cdot (v + w) = a \cdot v + a \cdot w)) \\
& \quad \# \text{ behaves similar as vectors should i.e., additive, scalable, linear distributive} \quad (181)
\end{aligned}$$

$$\begin{aligned}
& \text{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \iff \left( \text{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left( \text{map}(\langle \$1, \$2 \rangle, (V \times V, \mathbb{R})) \right) \wedge \\
& \quad (\forall_{v, w \in V} (\langle v, w \rangle = \langle w, v \rangle)) \wedge \\
& \quad (\forall_{v, w, x \in V} \forall_{a, b \in \mathbb{R}} (\langle av + bw, x \rangle = a \langle v, x \rangle + b \langle w, x \rangle)) \wedge \\
& \quad (\forall_{v \in V} (\langle v, v \rangle \geq 0)) \wedge (\forall_{v \in V} (\langle v, v \rangle = 0 \iff v = \mathbf{0})) \\
& \quad \# \text{ the sesquilinear or 1.5 linear map inner product provides info. on distance and orthogonality} \quad (182)
\end{aligned}$$

$$\text{innerProductSpace}((V, +, \cdot, \langle \$1, \$2 \rangle), ()) \iff \text{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \quad (183)$$

$$\begin{aligned}
& \text{vectorNorm}(\| \$1 \|, (V, +, \cdot)) \iff \left( \text{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left( \text{map}(\| \$1 \|, (V, \mathbb{R}_0^+)) \right) \wedge \\
& \quad (\forall_{v \in V} (\|v\| = 0 \iff v = \mathbf{0})) \wedge \\
& \quad (\forall_{v \in V} \forall_{s \in \mathbb{R}} (\|sv\| = |s| \|v\|)) \wedge \\
& \quad (\forall_{v, w \in V} (\|v + w\| \leq \|v\| + \|w\|)) \\
& \quad \# \text{ magnitude of a point in a vector space} \quad (184)
\end{aligned}$$

$$\text{normedVectorSpace}\left((V, +, \cdot, \|\$1\|), ()\right) \iff \left(\text{vectorSpace}\left((V, +, \cdot), ()\right)\right) \wedge \left(\text{vectorNorm}\left(\|\$1\|, (V, +, \cdot)\right)\right) \quad (185)$$

$$\begin{aligned} \text{vectorMetric}\left(d(\$1, \$2), (V, +, \cdot)\right) &\iff \left(\text{vectorSpace}\left((V, +, \cdot), ()\right)\right) \wedge \\ &\left(\text{metric}\left(d(\$1, \$2), (V)\right) \vee \left(\text{map}\left(d, \left(V \times V, \mathbb{R}_0^+\right)\right)\right)\right) \\ &\left(\forall_{x, y \in V} (d(x, y) = d(y, x))\right) \wedge \\ &\left(\forall_{x, y \in V} (d(x, y) = 0 \iff x = y)\right) \wedge \\ &\left(\forall_{x, y, z \in V} \left(d(x, z) \leq d(x, y) + d(y, z)\right)\right) \\ &\# \text{ behaves as distances should} \end{aligned} \quad (186)$$

$$\begin{aligned} \text{metricVectorSpace}\left((V, +, \cdot, d(\$1, \$2)), ()\right) &\iff \left(\text{vectorSpace}\left((V, +, \cdot), ()\right)\right) \wedge \\ &\left(\text{vectorMetric}\left(d(\$1, \$2), (V, +, \cdot)\right)\right) \end{aligned} \quad (187)$$

$$\begin{aligned} \text{innerProductNorm}\left(\|\$1\|, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right)\right) \wedge \\ &\left(\forall_{v \in V} \left(\|v\| = \sqrt[3]{\langle v, v \rangle}\right) \implies \text{vectorNorm}\left(\|\$1\|, (V, +, \cdot)\right)\right) \end{aligned} \quad (188)$$

$$\begin{aligned} \text{normInnerProduct}\left(\langle \$1, \$2 \rangle, (V, +, \cdot, \|\$1\|)\right) &\iff \left(\text{normedVectorSpace}\left((V, +, \cdot, \|\$1\|), ()\right)\right) \wedge \\ &\left(\forall_{u, v \in V} \left(2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2\right)\right) \wedge \\ &\left(\forall_{v, w \in V} \left(\langle v, w \rangle = \frac{\|v+w\|^2 - \|v-w\|^2}{4}\right) \implies \text{innerProduct}\left(\langle \$1, \$2 \rangle, (V, +, \cdot)\right)\right) \end{aligned} \quad (189)$$

$$\begin{aligned} \text{normMetric}\left(d(\$1, \$2), (V, +, \cdot, \|\$1\|)\right) &\iff \left(\text{normedVectorSpace}\left((V, +, \cdot, \|\$1\|), ()\right)\right) \wedge \\ &\left(\forall_{v, w \in V} (d(v, w) = \|v - w\|) \implies \text{vectorMetric}\left(d(\$1, \$2), (V, +, \cdot)\right)\right) \end{aligned} \quad (190)$$

$$\begin{aligned} \text{metricNorm}\left(\|\$1\|, (V, +, \cdot, d(\$1, \$2))\right) &\iff \left(\text{metricVectorSpace}\left((V, +, \cdot, d(\$1, \$2)), ()\right)\right) \wedge \\ &\left(\forall_{u, v, w \in V} \forall_{s \in \mathbb{R}} \left(d(s(u+w), s(v+w)) = |s|d(u, v)\right)\right) \wedge \\ &\left(\forall_{v \in V} (\|v\| = d(v, \mathbf{0})) \implies \text{vectorNorm}\left(\|\$1\|, (V, +, \cdot)\right)\right) \end{aligned} \quad (191)$$

$$\begin{aligned} \text{orthogonal}\left((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle)\right) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right)\right) \wedge \\ &(v, w \in V) \wedge (\langle v, w \rangle = 0) \\ &\# \text{ the inner product also provides info. on orthogonality} \end{aligned} \quad (192)$$

$$\text{normal}\left(v, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right)\right) \wedge (v \in V) \wedge (\langle v, v \rangle = 1)$$

$$\# \text{ the vector has unit length} \quad (193)$$

$$(\text{THM}) \text{ Cauchy-Schwarz inequality: } \forall_{v,w \in V} (\langle v, w \rangle \leq \|v\| \|w\|) \quad (194)$$

$$\text{basis}((b)_n, (V, +, \cdot, \cdot)) \iff \left( \text{vectorSpace}((V, +, \cdot, \cdot)) \right) \wedge \left( \forall_{v \in V} \exists_{(a)_n \in \mathbb{R}^n} \left( v = \sum_{i=1}^n (a_i b_i) \right) \right) \quad (195)$$

$$\begin{aligned} \text{orthonormalBasis}((b)_n, (V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) &\iff \left( \text{innerProductSpace}((V, +, \cdot, \cdot, \langle \$1, \$2 \rangle), ()) \right) \wedge \\ &\left( \text{basis}((b)_n, (V, +, \cdot, \cdot)) \right) \wedge \left( \forall_{v \in (b)_n} \left( \text{normal}(v, (V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) \right) \right) \wedge \\ &\left( \forall_{v \in (b)_n} \forall_{w \in (b)_n \setminus \{v\}} \left( \text{orthogonal}((v, w), (V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) \right) \right) \end{aligned} \quad (196)$$

### 1.16 Subvector space

$$\text{subspace}((U, \circ), (V, \circ)) \iff \left( \text{space}((V, \circ), ()) \right) \wedge (U \subseteq V) \wedge \left( \text{space}((U, \circ), ()) \right) \quad (197)$$

$$\begin{aligned} \text{subspaceSum}(U + W, (U, W, V, +)) &\iff \left( \text{subspace}((U, +), (V, +)) \right) \wedge \left( \text{subspace}((W, +), (V, +)) \right) \wedge \\ &(U + W = \{u + w \mid u \in U \wedge w \in W\}) \end{aligned} \quad (198)$$

$$\text{subspaceDirectSum}(U \oplus W, (U, W, V, +)) \iff (U \cap W = \emptyset) \wedge \left( \text{subspaceSum}(U \oplus W, (U, W, V, +)) \right) \quad (199)$$

$$\begin{aligned} \text{orthogonalComplement}(W^\perp, (W, V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) &\iff \\ \left( \text{subspace} \left( (W, +, \cdot, \cdot, \langle \$1, \$2 \rangle), \left( \text{innerProductSpace}((V, +, \cdot, \cdot, \langle \$1, \$2 \rangle), ()) \right) \right) \right) \wedge \\ \left( W^\perp = \left\{ v \in V \mid w \in W \wedge \text{orthogonal}((v, w), (V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) \right\} \right) \end{aligned} \quad (200)$$

$$\begin{aligned} \text{orthogonalDecomposition}((W, W^\perp), (W, V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) &\iff \\ \left( \text{orthogonalComplement}(W^\perp, (W, V, +, \cdot, \cdot, \langle \$1, \$2 \rangle)) \right) \wedge \left( \text{subspaceDirectSum}(V, (W, W^\perp, V, +)) \right) \end{aligned} \quad (201)$$

$$(\text{THM}) \text{ if } V \text{ is finite dimensional, then every vector has an orthogonal decomposition:} \quad (202)$$

### 1.17 Banach and Hilbert Space

$$\begin{aligned} \text{cauchy}((s)_\mathbb{N}, (V, d(\$1, \$2))) &\iff \left( \text{metricSpace}((V, d(\$1, \$2)), ()) \right) \wedge ((s)_\mathbb{N} \subseteq V) \\ &(\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{m, n \geq N} (d(s_m, s_n) < \epsilon)) \end{aligned}$$

# distances between some tail-end point gets arbitrarily small (203)

$$\text{complete}\left(\left(V, d(\$1, \$2)\right), ()\right) \iff \left(\forall_{(s)_{\mathbb{N}} \subseteq V} \exists_{s \in V} \left(\text{cauchy}\left((s)_{\mathbb{N}}, \left(V, d(\$1, \$2)\right)\right) \implies \lim_{n \rightarrow \infty} (d(s, s_n)) = 0\right)\right)$$

# or converges within the induced topological space

# in complete spaces, the weaker notion of cauchy is enforced to be equivalent to convergence (204)

$$\text{banachSpace}\left(\left(V, +, \cdot, \|\$1\|\right), ()\right) \iff \left(\text{normMetric}\left(d(\$1, \$2), \left(V, \|\$1\|\right)\right) \wedge \left(\text{complete}\left(V, d(\$1, \$2)\right), ()\right)\right)$$

# a complete normed vector space (205)

$$\text{hilbertSpace}\left(\left(V, +, \cdot, \langle \$1, \$2 \rangle\right), ()\right) \iff \left(\text{innerProductNorm}\left(\|\$1\|, \left(V, +, \cdot, \langle \$1, \$2 \rangle\right)\right) \wedge \left(\text{normMetric}\left(d(\$1, \$2), \left(V, \|\$1\|\right)\right) \wedge \left(\text{complete}\left(V, d(\$1, \$2)\right), ()\right)\right)$$

# a complete inner product space (206)

(THM) :  $\text{hilbertSpace} \implies \text{banachSpace}$  (207)

$$\text{separable}\left(\left(V, d\right), ()\right) \iff \left(\exists_{S \subseteq V} \left(\text{dense}\left(S, \left(V, d\right)\right) \wedge \text{countablyInfinite}\left(S, ()\right)\right)\right)$$

# needs only a countable subset to approximate any element in the entire space (208)

$$\begin{aligned} & \text{(THM)} : \text{hilbertSpace}\left(\left(\left(V, +, \cdot, \langle \$1, \$2 \rangle\right), ()\right), ()\right) \implies \\ & \left(\exists_{(b)_{\mathbb{N}} \subseteq V} \left(\text{orthonormalBasis}\left((b)_{\mathbb{N}}, \left(V, +, \cdot, \langle \$1, \$2 \rangle\right)\right) \wedge \text{countablyInfinite}\left((b)_{\mathbb{N}}, ()\right)\right) \iff \\ & \text{separable}\left(\left(V, \sqrt{\langle \$1 - \$2, \$1 - \$2 \rangle}\right), ()\right) \end{aligned}$$

# separability in hilbert spaces is equivalent to the existence of a countable orthonormal basis (209)

## 1.18 Matrices, Operators, and Functionals

$$\begin{aligned} & \text{linearOperator}\left(L, \left(V, +_V, \cdot_V, W, +_W, \cdot_W\right)\right) \iff \left(\text{map}\left(L, \left(V, W\right)\right) \wedge \left(\text{vectorSpace}\left(\left(V, +_V, \cdot_V\right), ()\right) \wedge \right. \right. \\ & \left. \left(\text{vectorSpace}\left(\left(W, +_W, \cdot_W\right), ()\right) \wedge \left(\forall_{v_1, v_2 \in V} \forall_{s_1, s_2 \in \mathbb{R}} \left(L(s_1 \cdot_V v_1 +_V s_2 \cdot_V v_2) = s_1 \cdot_W L(v_1) +_W s_2 \cdot_W L(v_2)\right)\right)\right)\right) \end{aligned} \quad (210)$$

$$\begin{aligned} & \text{denseMap}\left(L, \left(D, H, +, \cdot, \langle \$1, \$2 \rangle\right)\right) \iff \left(D \subseteq H\right) \wedge \left(\text{linearOperator}\left(L, \left(D, +, \cdot, H, +, \cdot\right)\right) \wedge \right. \\ & \left.\left(\text{innerProductTopology}\left(\mathcal{O}, \left(H, +, \cdot, \langle \$1, \$2 \rangle\right)\right) \wedge \left(\text{dense}\left(D, \left(H, \mathcal{O}, d(\$1, \$2)\right)\right)\right)\right) \end{aligned} \quad (211)$$

$$\begin{aligned} & \text{mapNorm}\left(\|L\|, \left(L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W\right)\right) \iff \\ & \left(\text{linearOperator}\left(L, \left(V, +_V, \cdot_V, W, +_W, \cdot_W\right)\right) \wedge \right. \\ & \left.\left(\text{normedVectorSpace}\left(\left(V, +_V, \cdot_V, \|\$1\|_V\right), ()\right) \wedge \left(\text{normedVectorSpace}\left(\left(W, +_W, \cdot_W, \|\$1\|_W\right), ()\right) \wedge \right.\right. \end{aligned}$$

$$\left( \|L\| = \sup \left( \left\{ \frac{\|Lf\|_W}{\|f\|_V} \mid f \in V \right\} \right) = \sup \left( \{ \|Lf\|_W \mid f \in V \wedge \|f\|_V = 1 \} \right) \right) \quad (212)$$

$$\begin{aligned} & \text{boundedMap} \left( L, (V, +_V, \cdot_V, \|1\|_V, W, +_W, \cdot_W, \|1\|_W) \right) \iff \\ & \left( \text{mapNorm} \left( \|L\|, (L, V, +_V, \cdot_V, \|1\|_V, W, +_W, \cdot_W, \|1\|_W) \right) < \infty \right) \end{aligned} \quad (213)$$

$$\begin{aligned} & \neg \text{boundedMap} \left( L, (V, +_V, \cdot_V, \|1\|_V, W, +_W, \cdot_W, \|1\|_W) \right) \iff \\ & (U \subset V) \wedge \left( \infty = \text{mapNorm} \left( \|L\|_U, (L, U, +_U, \cdot_U, \|1\|_U, W, +_W, \cdot_W, \|1\|_W) \right) \leq \|L\| \right) \end{aligned} \quad (214)$$

$$\begin{aligned} & \text{extensionMap} \left( \widehat{L}, (L, V, D, W) \right) \iff (D \subseteq V) \wedge \left( \text{linearOperator} \left( L, (D, +_D, \cdot_D, W, +_W, \cdot_W) \right) \right) \wedge \\ & \left( \text{linearOperator} \left( \widehat{L}, (V, +_V, \cdot_V, W, +_W, \cdot_W) \right) \right) \wedge \left( \forall d \in D \left( \widehat{L}(d) = L(d) \right) \right) \end{aligned} \quad (215)$$

$$\begin{aligned} & \text{adjoint} \left( L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W) \right) \iff \left( \text{hilbertSpace} \left( (V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V), () \right) \right) \wedge \\ & \left( \text{hilbertSpace} \left( (W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W), () \right) \right) \wedge \left( \text{linearOperator} \left( L, (V, +_V, \cdot_V, W, +_W, \cdot_W) \right) \right) \wedge \\ & \left( \forall v \in V \forall w \in W \left( \left( \langle Lv, w \rangle_W = \langle v, L^T w \rangle_V \right) \vee \left( (Lv)^T w = v^T L^T w \right) \right) \right) \\ & \# \text{ target operator that acts similar to the domain operator} \end{aligned} \quad (216)$$

$$\begin{aligned} & \text{selfAdjoint} \left( L, (V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W) \right) \iff \\ & L = \text{adjoint} \left( L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W) \right) \\ & \# \text{ also a generalization of symmetric matrices} \end{aligned} \quad (217)$$

$$\begin{aligned} & \text{matrix} (L, (n, m)) \iff \left( \text{linearOperator} \left( L, (\mathbb{R}^m, +_m, \cdot_m, \mathbb{R}^n, +_n, \cdot_n) \right) \right) \\ & \# \text{ rows=dimensions, cols=vectors} \end{aligned} \quad (218)$$

$$\text{eigenvector} (v, (L, V, +, \cdot)) \iff \left( \text{linearOperator} (L, (V, +, \cdot, V, +, \cdot)) \right) \wedge \left( \exists \lambda \in \mathbb{R} (L(v) = \lambda v) \right) \quad (219)$$

$$\text{eigenvalue} (\lambda, (v, L, V, +, \cdot)) \iff \forall s \in \mathbb{R} \setminus \{0\} \left( \text{eigenvector} (sv, (L, V, +, \cdot)) \right) \quad (220)$$

$$\text{identityOperator} (I, (A)) \iff \left( \text{matrix} (A, (n, n)) \right) \wedge (AI = IA = A) \quad (221)$$

$$\begin{aligned} & \text{inverseOperator} \left( A^{-1}, (A) \right) \iff \left( A^{-1}A = \text{identityOperator} (I, (A)) \right) \\ & \# \text{ gauss-jordan elimination: } E[A|I] = [I|E] = [I|A^{-1}] \end{aligned} \quad (222)$$

$$(\text{THM}) : (AB)^{-1}(AB) = I = B^{-1}A^{-1}AB \quad (223)$$

$$\text{transposeOperator}(A^T, (A)) \iff \left( (A^T)_{m,n} = (A)_{n,m} \right) \vee \text{adjoint}(A^T, (A)) \quad (224)$$

$$\text{symmetricOperator}(A, ()) \iff \left( A = \text{transposeOperator}(A^T, (A)) \right) \vee \left( \text{selfAdjoint}(A, ()) \right) \quad (225)$$

$$(\text{THM}) : (AB)^T = B^T A^T \wedge (A^T)^{-1} = (A^{-1})^T \quad (226)$$

$$(\text{THM}) : \text{symmetricOperator}(A^T A, ()) \iff \left( A^T A = (A^T A)^T = A^T A^{TT} = A^T A \right) \quad (227)$$

$$\begin{aligned} \text{decomposeLU}(LU(A), (A)) \iff & \left( \text{matrix}(A, (n, n)) \right) \wedge (\exists_E (EA = U)) \wedge \left( \forall_{x < n} \forall_{0 < i < x} (U_{i,i} = 0) \right) \wedge \\ & \left( LU(A) = E^{-1}U = A \right) \\ \# \text{ lower triangle are all 0; useful for solving linear equations} \end{aligned} \quad (228)$$

$$\begin{aligned} \text{Img}(\text{Img}(A), (A)) \iff & \left( \text{matrix}(A, (n, m)) \right) \wedge (\text{Img}(A) = \{Av \in \mathbb{R}^n \mid v \in \mathbb{R}^m\}) \\ \# \text{ the column space; not always a subspace since } A \text{ can map to a set not containing } \mathbf{0} \end{aligned} \quad (229)$$

$$\begin{aligned} \text{Ker}(\text{Ker}(A), (A)) \iff & \left( \text{matrix}(A, (n, m)) \right) \wedge (\text{Ker}(A) = \{v \in \mathbb{R}^m \mid Av = \mathbf{0} \in \mathbb{R}^n\}) \\ \# \text{ the null or solution space; always a subspace due to linearity } Av + Aw = \mathbf{0} = A(v + w) \end{aligned} \quad (230)$$

$$(\text{THM}) \text{ general linear solution: } (Ax_p = b) \wedge (x_n \in \text{Ker}(A)) \implies (Ax_p + Ax_n = b + 0 = A(x_p + x_n) = b) \quad (231)$$

$$\text{independentOperator}(A, ()) \iff \left( \text{matrix}(A, (n, m)) \right) \wedge \left( \neg \exists_{v \in \mathbb{R}^m \setminus \mathbf{0}_m} (Av = 0) \iff \text{Ker}(A) = \{\mathbf{0}_m\} \right) \quad (232)$$

$$\text{dimensionality}(N, (A)) \iff \left( \text{matrix}(A, (n, m)) \right) \wedge \left( N = \inf \left( \{ |(b)_n| \mid \text{basis}((b)_n, (A)) \} \right) \right) \quad (233)$$

$$\text{rank}(r, (A)) \iff \left( \text{matrix}(A, (n, m)) \right) \wedge \left( \text{dimensionality}(r, (A)) \right) \quad (234)$$

$$\begin{aligned} (\text{THM}) : \left( \text{matrix}(A, (n, m)) \right) \implies & \left( \text{dimensionality}(\text{Ker}(A)) = n - \text{rank}(r, (A)) \right) \\ \# \text{ number of free variables} \end{aligned} \quad (235)$$

$$\text{transposeNorm}(\|x\|, ()) \iff \left( \|x\| = \sqrt{x^T x} \right) \quad (236)$$

$$\begin{aligned} \text{transposeOrthogonality}((x, y), ()) \iff & \left( \|x\|^2 + \|y\|^2 = \|x + y\|^2 \right) \iff \\ & \left( x^T x + y^T y = (x + y)^T (x + y) = x^T x + y^T y + x^T y + y^T x \right) \iff \\ & \left( 0 = \frac{x^T x + y^T y - (x^T x + y^T y)}{2} = \frac{x^T y + y^T x}{2} = x^T y \right) \end{aligned} \quad (237)$$

$$\begin{aligned}
\text{orthogonalProjection}(P_A b, (A, b)) &\iff (\text{matrix}(A, (n, m))) \wedge (\text{matrix}(b, (m, 1))) \wedge \\
&\left( \exists_{c \in \mathbb{R}^m} (A^T(b - P_A b) = 0 = A^T(b - Ac)) \iff \right. \\
A^T b &= A^T A c \iff c = (A^T A)^{-1} A^T b \iff P_A b = A c = \left( A (A^T A)^{-1} A^T \right) b \\
&\# A, A^T \text{ may not necessarily be invertible} \quad (238)
\end{aligned}$$


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$$(THM) : P = P^T = P^2 \quad (239)$$


---

$$(THM) : \text{independent}(A) \implies \text{invertible}(A^T A) \quad (240)$$


---

$$\begin{aligned}
\text{eigenvectors}(X, (A, V, +, \cdot, ||\$1||)) &\iff (\text{normedVectorSpace}((V, +, \cdot, ||\$1||), ())) \wedge \\
(X = \{v \in V \mid ||v|| = 1 \wedge \text{eigenvector}(v, (A, V, +, \cdot))\}) &\quad (241)
\end{aligned}$$


---

$$\begin{aligned}
\text{det}(\text{det}(A), (A, V, +, \cdot, ||\$1||)) &\iff (\text{eigenvectors}(X, (A, V, +, \cdot, ||\$1||))) \wedge \\
(\text{det}(A) = \prod_{x \in X} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)))) & \\
\# \text{ DEFINE; exterior algebra wedge product area??} &\quad (242)
\end{aligned}$$


---

$$\begin{aligned}
\text{tr}(\text{tr}(A), (A, V, +, \cdot, ||\$1||)) &\iff (\text{eigenvectors}(X, (A, V, +, \cdot, ||\$1||))) \wedge \\
(\text{tr}(A) = \sum_{x \in X} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)))) & \\
\# \text{ DEFINE} &\quad (243)
\end{aligned}$$


---

$$(THM) : A = A^T = A^2 \implies \text{Tr}(A) = \text{dimensionality}(N, (A)) \# \text{ counts dimensions} \quad (244)$$


---

$$\text{invertibleOperator}(A, (V, +, \cdot, ||\$1||)) \iff (\text{det}(A) \neq 0) \vee (\text{independentOperator}(A, ())) \quad (245)$$


---

$$\text{TODOFIXINCOSISTENTPARAMETERS} \quad (246)$$


---

$$\text{CONSOLIDATEINVERTIBLEIDNEPENDENTDFNSCONTHHERE} \quad (247)$$


---

$$\begin{aligned}
\text{eigenDecomposition}(S \Lambda S^{-1}, (A, V, +, \cdot, ||\$1||)) &\iff (S \in (\text{eigenvectors}(X, (A, V, +, \cdot, ||\$1||))^n)^T) \wedge \\
(\exists_{\Lambda} \forall_{p \in \mathbb{N}} (\Lambda^p \{1\}^n = \{\lambda^p \mid \text{eigenvalue}(X X X)\})) &\wedge \\
(\text{invertibleOperator}(S, ())) \implies \exists_{S^{-1}} (AS = S \Lambda \implies A = S \Lambda S^{-1}) &\quad (248)
\end{aligned}$$


---

$$\begin{aligned}
Ax &= \lambda x \\
Ax - \lambda x &= (A - \lambda I)x = 0 \wedge (x \neq 0) \implies \text{eigenvalue}(A - \lambda I) = 0 \implies \Pi(\lambda) = 0 = \text{det}(A - \lambda I) \\
\text{det}(A - \lambda I) &= 0 \wedge \text{algebraEval} \implies \lambda \\
(A - \lambda I)x &= 0 \wedge \text{elim} \implies x \\
&\implies (x, \lambda) \quad (249)
\end{aligned}$$


---

$$\text{contlecture22} \quad (250)$$


---

$$\begin{aligned}
compactMap(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)) &\iff \left( boundedMap\left(L, (V, +_V, \cdot_V, \| \$1 \|_V, W, +_W, \cdot_W, \| \$1 \|_W)\right) \right) \wedge \\
&\left( \forall_{v \in V} \left( openBall\left(B, (1.0, v, V, d_V(\$1, \$2))\right) \implies \right. \right. \\
&\left. compactSubset\left(closure\left(\overline{L(B)}, image(L(B), (B, L, V, W)), W, d_W(\$1, \$2)\right), (W, \mathcal{O}_W)\right) \right) \right) \quad (251)
\end{aligned}$$

(THM) Spectral thm.:

$$\begin{aligned}
&\left( selfAdjoint\left(L, (V, +, \cdot, \langle \$1, \$2 \rangle, V, +, \cdot, \langle \$1, \$2 \rangle)\right) \right) \wedge \left( compactMap(L, (V, +, \cdot, V, +, \cdot)) \right) \implies \\
&\left( \exists_{(e)_{\mathbb{N}} \subseteq V} \left( orthonormalBasis\left((e)_{\mathbb{N}}, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \wedge \forall_{e_n \in (e)_{\mathbb{N}}} \left( eigenvector(e_n, (L, V, +, \cdot)) \right) \right) \right) \implies \\
&\left( \exists_{(\lambda)_{\mathbb{N}} \subseteq \mathbb{R}^n} \forall_{e_n \in (e)_{\mathbb{N}}} \exists_{\lambda_n \in (\lambda)_{\mathbb{N}}} \left( eigenvalue(\lambda_n, (e_n, L, V, +, \cdot)) \wedge \lim_{n \rightarrow \infty} (\lambda_n = 0) \wedge L = \sum_{n=1}^{\infty} (\lambda_n e_n e_n^T) \right) \right) \\
&\# \text{ TODO intuition} \quad (252)
\end{aligned}$$

## 1.19 Function spaces

$$\begin{aligned}
&curLp(\mathcal{L}^p, (p, M, \sigma, \mu)) \iff (p \in \mathbb{R}) \wedge (1 \leq p < \infty) \wedge \\
&\left( \mathcal{L}^p = \{ map(f, (M, \mathbb{R})) \mid measurableMap(f, (M, \sigma, \mathbb{R}, standardSigma)) \wedge \int (|f|^p d\mu) < \infty \} \right) \quad (253)
\end{aligned}$$

$$\begin{aligned}
&vecLp(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \iff \left( curLp(\mathcal{L}^p, (p, M, \sigma, \mu)) \right) \wedge \left( \forall_{f, g \in \mathcal{L}^p} \forall_{m \in M} ((f + g)(m) = f(m) + g(m)) \right) \wedge \\
&\left( \forall_{f \in \mathcal{L}^p} \forall_{s \in \mathbb{R}} \forall_{m \in M} ((s \cdot f)(m) = (s)f(m)) \right) \wedge \left( vectorSpace((\mathcal{L}^p, +, \cdot, ())) \right) \quad (254)
\end{aligned}$$

$$\begin{aligned}
&integralNorm(\| \$1 \|, (+, \cdot, p, M, \sigma, \mu)) \iff \left( vecLp(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \left( map\left(\| \$1 \|, (\mathcal{L}^p, \mathbb{R}_0^+)\right) \right) \wedge \\
&\left( \forall_{f \in \mathcal{L}^p} \left( 0 \leq \| f \| = \left( \int (|f|^p d\mu) \right)^{1/p} \right) \right) \quad (255)
\end{aligned}$$

$$\begin{aligned}
&(\text{THM}) : integralNorm(\| \$1 \|, (+, \cdot, p, M, \sigma, \mu)) \implies \\
&\left( \forall_{f \in \mathcal{L}^p} \left( \| f \| = 0 \implies almostEverywhere(f = \mathbf{0}, (M, \sigma, \mu)) \right) \right) \\
&\# \text{ not an expected property from a norm} \quad (256)
\end{aligned}$$

$$\begin{aligned}
&Lp(L^p, ((+, \cdot, p, M, \sigma, \mu))) \iff \left( integralNorm(\| \$1 \|, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \\
&\left( L^p = quotientSet\left(\mathcal{L}^p / \sim, \left(\mathcal{L}^p, (\| \$1 + (-\$2) \| = 0)\right)\right) \right) \\
&\# \text{ functions in } L^p \text{ that have finite integrals above and below the x-axis} \quad (257)
\end{aligned}$$

$$(\text{THM}) : banachSpace\left(\left(Lp(L^p, (+, \cdot, p, M, \sigma, \mu)), +, \cdot, \| \$1 \|, ()\right)\right) \quad (258)$$



$$(THM) : \text{hilbertSpace} \left( \left( Lp(L^p, (+, \cdot, 2, M, \sigma, \mu)), +, \cdot, \frac{\lambda \lambda \$1 + \$2 \lambda^2 - \lambda \lambda \$1 - \$2 \lambda^2}{4} \right), () \right) \quad (259)$$

$$\begin{aligned} \text{curL} \left( \mathcal{L}, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right) &\iff \left( \text{banachSpace} \left( (W, +_W, \cdot_W, \|\$1\|_W), () \right) \right) \wedge \\ &\left( \text{normedVectorSpace} \left( (V, +_V, \cdot_V, \|\$1\|_V), () \right) \right) \wedge \\ &\left( \mathcal{L} = \{f \mid \text{boundedMap} \left( f, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right)\} \right) \end{aligned} \quad (260)$$

$$(THM) : \text{banachSpace} \left( \left( \text{curL} \left( \mathcal{L}, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right), +, \cdot, \text{mapNorm} \right), () \right) \quad (261)$$

$$(THM) : \|L\| \geq \frac{\|Lf\|}{\|f\|} \# \text{ from choosing an arbitrary element in the mapNorm sup} \quad (262)$$

$$\begin{aligned} (THM) : &\left( \text{cauchy}((f)_{\mathbb{N}}, (\mathcal{L}, +, \cdot, \text{mapNorm})) \implies \text{cauchy}((f_n v)_{\mathbb{N}}, (W, +_W, \cdot_W, \|\$1\|_W)) \right) \iff \\ &\left( \forall \epsilon' > 0 \forall v \in V (\|f_n v - f_m v\|_W = \|(f_n - f_m)v\|_W \leq \|f_n - f_m\| \cdot \|v\|_V < \epsilon \cdot \|v\|_V = \epsilon') \right) \\ &\# \text{ a cauchy sequence of operators maps to a cauchy sequence of targets} \end{aligned} \quad (263)$$

$$\begin{aligned} (THM) \text{ BLT thm.: } &\left( \left( \text{dense}(D, (V, \mathcal{O}, d_V)) \wedge \text{boundedMap} \left( A, (D, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W) \right) \right) \implies \right. \\ &\left. \left( \exists!_{\hat{A}} \left( \text{extensionMap} \left( \hat{A}, (A, V, D, W) \right) \right) \wedge \|\hat{A}\| = \|A\| \right) \right) \iff \\ &\left( \forall v \in V \exists (v)_{\mathbb{N}} \subseteq D \left( \lim_{n \rightarrow \infty} (v_n = v) \right) \right) \wedge \left( \hat{A}v = \lim_{n \rightarrow \infty} (Av_n) \right) \end{aligned} \quad (264)$$

## 1.20 Probability Theory

$$0 \quad (265)$$

## 1.21 Underview

$$(266)$$

$$\text{curve} - \text{fitting/explaining} \neq \text{prediction} \quad (267)$$

$$\text{ill-defined problem} + \text{solutionspace constraints} \implies \text{well-defined problem} \quad (268)$$

$$x \# \text{ input} ; y \# \text{ output} \quad (269)$$

$$S_n = \{(x_1, y_1), \dots, (x_n, y_n)\} \# \text{ training set} \quad (270)$$

$$f_S(x) \sim y \text{ \# solution} \quad (271)$$

$$each(x, y) \in p(x, y) \text{ \# training data } x, y \text{ is a sample from an unknown distribution } p \quad (272)$$

$$V(f(x), y) = d(f(x), y) \text{ \# loss function} \quad (273)$$

$$I[f] = \int_{X \times Y} V(f(x), y) p(x, y) dx dy \text{ \# expected error} \quad (274)$$

$$I_n[f] = \frac{1}{n} \sum_{i=1}^n V(f(x_i), y_i) \text{ \# empirical error} \quad (275)$$

$$probabilisticConvergence(X, ()) \iff \forall \epsilon > 0 \lim_{n \rightarrow \infty} P\|x_n - x\| \leq \epsilon = 0 \quad (276)$$

$$I - Ingeneralizationerror \quad (277)$$

$$well-posed := exists, unique, stable; else ill-posed \quad (278)$$

## 2 Machine Learning

### 2.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X, Y) \text{ \# dataset} \quad (279)$$

$$\text{learned parameters} = \text{parameters to be fixed by training with the dataset} \quad (280)$$

$$\text{hyperparameters} = \text{parameters that depends on a dataset} \quad (281)$$

$$\text{validation} = \text{partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition \# useful for fixing hyperparameters} \quad (282)$$

$$\text{cross-validation} = \text{average accuracy of validation for different choices of testing partition} \quad (283)$$

$$\mathbf{L1} = \text{scales linearly ; } \mathbf{L2} = \text{scales quadratically} \quad (284)$$

$$d = \text{distance} = \text{quantifies the similarity between data points} \quad (285)$$

$$d_{L1}(A, B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance} \quad (286)$$

$$d_{L2}(A, B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \quad (287)$$

$$\text{kNN classifier} = \text{classifier based on } k \text{ nearest data points} \quad (288)$$

---


$$s = \text{class score} = \text{quantifies bias towards a particular class} \quad (289)$$


---

$$s_{\text{linear}} = f_{c \times 1}(x_{n \times 1}, W_{c \times n}, b_{c \times 1}) = W_{c \times n} x_{n \times 1} + b_{c \times 1} \quad \# \text{ linear score function} \quad (290)$$


---

$$l = \text{loss} = \text{quantifies the errors by the learned parameters} \quad (291)$$


---

$$l = \frac{1}{|c_i|} \sum_{c_i} l_i \quad \# \text{ average loss for all classes} \quad (292)$$


---

$$l_{\text{SVM}_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \quad \# \text{ SVM hinge class loss function:}$$

$\#$  ignores incorrect classes with lower scores including a non-zero margin (293)

---

$$l_{\text{MLR}_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \quad \# \text{ Softmax class loss function}$$

$\#$  lower scores correspond to lower exponentiated-normalized probabilities (294)

---

$$R = \text{regularization} = \text{optimizes the choice of learned parameters to minimize test error} \quad (295)$$


---

$$\lambda \quad \# \text{ regularization strength hyperparameter} \quad (296)$$


---

$$R_{L1}(W) = \sum_{W_i} |W_i| \quad \# \text{ L1 regularization} \quad (297)$$


---

$$R_{L2}(W) = \sum_{W_i} W_i^2 \quad \# \text{ L2 regularization} \quad (298)$$


---

$$L' = L + \lambda R(W) \quad \# \text{ weight regularization} \quad (299)$$


---

$$\nabla_W L = \frac{\overrightarrow{\partial}}{\partial W_i} L = \text{loss gradient w.r.t. weights} \quad (300)$$


---

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L} \quad \# \text{ loss gradient w.r.t. input weight in terms of external and local gradients} \quad (301)$$


---

$$s = \text{forward API} ; \quad \frac{\partial L_L}{\partial W_I} = \text{backward API} \quad (302)$$


---

$$W_{t+1} = W_t - \nabla_{W_t} L \quad \# \text{ weight update loss minimization} \quad (303)$$


---

$$\text{TODO: Research on Activation functions, Weight Initialization, Batch Normalization} \quad (304)$$


---

$$\text{review5meanvardiscussion/hyperparameteroptimization/babysittinglearning} \quad (305)$$


---

TODO loss L or l ??

### 3 Glossary

chaoticTopology	compactSubset	vectorSpace	identityOperator
discreteTopology	bounded	innerProduct	inverseOperator
topology	openRefinement	innerProductSpace	transposeOperator
topologicalSpace	locallyFinite	vectorNorm	symmetricOperator
open	paracompact	normedVectorSpace	decomposeLU
closed	openRefinement	vectorMetric	Img
clopen	locallyFinite	metricVectorSpace	Ker
neighborhood	paracompact	innerProductNorm	independentOperator
chaoticTopology	connected	normInnerProduct	dimensionality
discreteTopology	pathConnected	normMetric	rank
metric	connected	metricNorm	transposeNorm
metricSpace	pathConnected	orthogonal	transposeOrthogonality
openBall	sigmaAlgebra	normal	orthogonalProjection
metricTopology	measurableSpace	basis	eigenvectors
metricTopologicalSpace	measurableSet	orthonormalBasis	det
limitPoint	measure	vectorSpace	tr
interiorPoint	measureSpace	innerProduct	invertibleOperator
closure	finiteMeasure	innerProductSpace	eigenDecomposition
dense	generatedSigmaAlgebra	vectorNorm	compactMap
eucD	borelSigmaAlgebra	normedVectorSpace	linearOperator
standardTopology	standardSigma	vectorMetric	denseMap
subsetTopology	lebesgueMeasure	metricVectorSpace	mapNorm
productTopology	measurableMap	innerProductNorm	boundedMap
metric	pushForwardMeasure	normInnerProduct	extensionMap
metricSpace	nullSet	normMetric	adjoint
openBall	almostEverywhere	metricNorm	selfAdjoint
metricTopology	sigmaAlgebra	orthogonal	matrix
metricTopologicalSpace	measurableSpace	normal	eigenvector
limitPoint	measurableSet	basis	eigenvalue
interiorPoint	measure	orthonormalBasis	identityOperator
closure	measureSpace	subspace	inverseOperator
dense	finiteMeasure	subspaceSum	transposeOperator
eucD	generatedSigmaAlgebra	subspaceDirectSum	symmetricOperator
standardTopology	borelSigmaAlgebra	orthogonalComplement	decomposeLU
subsetTopology	standardSigma	orthogonalDecomposition	Img
productTopology	lebesgueMeasure	subspace	Ker
sequence	measurableMap	subspaceSum	independentOperator
sequenceConvergesTo	pushForwardMeasure	subspaceDirectSum	dimensionality
sequence	nullSet	orthogonalComplement	rank
sequenceConvergesTo	almostEverywhere	orthogonalDecomposition	transposeNorm
continuous	simpleTopology	cauchy	transposeOrthogonality
homeomorphism	simpleSigma	complete	orthogonalProjection
isomorphicTopologicalSpace	simpleFunction	banachSpace	eigenvectors
continuous	characteristicFunction	hilbertSpace	det
homeomorphism	exStandardSigma	separable	tr
isomorphicTopologicalSpace	nonNegIntegrable	cauchy	invertibleOperator
T0Separate	nonNegIntegral	complete	eigenDecomposition
T1Separate	explicitIntegral	banachSpace	compactMap
T2Separate	integrable	hilbertSpace	curLp
T0Separate	integral	separable	vecLp
T1Separate	simpleTopology	linearOperator	integralNorm
T2Separate	simpleSigma	denseMap	Lp
openCover	simpleFunction	mapNorm	curL
finiteSubcover	characteristicFunction	boundedMap	curLp
compact	exStandardSigma	extensionMap	vecLp
compactSubset	nonNegIntegrable	adjoint	integralNorm
bounded	nonNegIntegral	selfAdjoint	Lp
openCover	explicitIntegral	matrix	curL
finiteSubcover	integrable	eigenvector	
compact	integral	eigenvalue	