Next-Next-Gen Notes
Object-Oriented Maths

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Format: $characteristic((subjects),(dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$
TODO should really define union intersection complement etc

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s,()) \iff well\text{-}formedString(s,()) \tag{1}$$

$$\begin{aligned} proposition((p,t),()) \iff \Big(statement(p,())\Big) \wedge \\ (t = eval(p)) \wedge \\ (t = true \veebar t = false) \end{aligned} \tag{2}$$

$$operator\Big(o,\Big((p)_{n\in\mathbb{N}}\Big)\Big) \iff proposition\Big(o\Big((p)_{n\in\mathbb{N}}\Big),()\Big) \tag{3}$$

$$\begin{aligned} operator(\neg,(p_1)) \iff \Big(proposition((p_1,true),()) \implies ((\neg p_1, false),())\Big) \wedge \\ \Big(proposition((p_1,false),()) \implies ((\neg p_1, true),())\Big) \\ \#\ \text{an operator takes in propositions and returns a proposition} \end{aligned} \tag{4}$$

$$\begin{aligned} operator(\neg) \iff \textbf{NOT} \ ; operator(\vee) \iff \textbf{OR} \ ; operator(\wedge) \iff \textbf{AND} \ ; operator(\veebar) \iff \textbf{XOR} \\ operator(\implies) \iff \textbf{IF} \ ; operator(\impliedby) \iff \textbf{OIF} \ ; operator(\iff) \iff \textbf{IFF} \end{aligned} \tag{5}$$

$$\begin{aligned} proposition((false \implies true), true, ()) \wedge proposition((false \implies false), true, ()) \\ \#\ \text{truths based on a false premise is not false; ex falso quodlibet principle} \end{aligned} \tag{6}$$

$$(\text{THM}) : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \tag{7}$$

$$predicate(P,(V)) \iff \forall_{v\in V}\Big(proposition\Big((P(v),t),()\Big)\Big) \tag{8}$$

$$\begin{aligned} 0thOrderLogic(P,()) \iff proposition((P,t),()) \\ \#\ \text{individual proposition} \end{aligned} \tag{9}$$

$$1stOrderLogic(P,(V)) \iff \Big(\forall_{v\in V}\Big(0thOrderLogic(v,())\Big)\Big) \wedge$$

$$\left(\forall_{v\in V}\left(\textit{proposition}\Big(\big(P(v),t\big),()\Big)\right)\right)$$

$$\text{\# propositions defined over a set of the lower order logical statements} \tag{10}$$

$$\textit{quantifier}\big(q,(p,V)\big) \iff \Big(\textit{predicate}\big(p,(V)\big)\Big)\wedge$$
$$\Big(\textit{proposition}\Big(\big(q(p),t\big),()\Big)\Big)$$
$$\text{\# a quantifier takes in a predicate and returns a proposition} \tag{11}$$

$$\textit{quantifier}\big(\forall,(p,V)\big) \iff \textit{proposition}\Big(\Big(\wedge_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\text{\# universal quantifier} \tag{12}$$

$$\textit{quantifier}\big(\exists,(p,V)\big) \iff \textit{proposition}\Big(\Big(\vee_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\text{\# existential quantifier} \tag{13}$$

$$\textit{quantifier}\big(\exists!,(p,V)\big) \iff \exists_{x\in V}\Big(P(x)\wedge\neg\Big(\exists_{y\in V\setminus\{x\}}\big(P(y)\big)\Big)\Big)$$
$$\text{\# uniqueness quantifier} \tag{14}$$

$$(\text{THM}):\forall_x p(x) \iff \neg\exists_x\neg p(x)$$
$$\text{\# De Morgan's law} \tag{15}$$

$$(\text{THM}):\forall_x\exists_y p(x,y)=\forall_x\neg\forall_y\neg p(x,y)\neq\exists_y\forall_x p(x,y)=\neg\forall_y\neg\big(\forall_x p(x,y)\big)=\neg\forall_y\exists_x\neg p(x,y)$$
$$\text{\# different quantifiers are not interchangeable} \tag{16}$$

$$============\ \textbf{N O T}=\textbf{U P D A T E D}\ ============ \tag{17}$$

$$\textbf{proof} = \textbf{truths derived from a finite number of axioms and deductions} \tag{18}$$

$$\textbf{elementary arithmetics} = \textbf{system with substitutions, and some notion of}$$
$$\textbf{addition, multiplication, and prime nuumbers for encoding metamathematics} \tag{19}$$

$$\textbf{Gödel theorem} \implies \textbf{axiomatic systems equivalent in power to elementary mathematics}$$
$$\textbf{either has unprovable statements or has contradictions} \tag{20}$$

$$============\ \textbf{N O T}=\textbf{U P D A T E D}\ ============ \tag{21}$$

1.1 Axiomatic Set Theory

$$============\ \textbf{N O T}=\textbf{U P D A T E D}\ ============ \tag{22}$$

$$\textbf{ZFC set theory} = \textbf{standard form of axiomatic set theory} \tag{23}$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \tag{24}$$

$$(A = B) = A \subseteq B \land B \subseteq A \tag{25}$$

$$\in \textbf{basis} \implies \{x, y\} = \{y, x\} \land \{x\} = \{x, x\} \tag{26}$$

$$\in \textbf{and sets works following the 9 ZFC axioms:} \tag{27}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \text{ \# E: } \in \text{ is only a proposition on sets} \tag{28}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \text{ \# E: existence of empty set} \tag{29}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \lor u = y \text{ \# C: pair set construction} \tag{30}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \land y \in x \implies y \in u) \text{ \# C: union set construction} \tag{31}$$

$$x = \{\{a\}, \{b\}\} \text{ \# from the pair set axiom} \tag{32}$$

$$u = \cup x = \cup\{\{a\}, \{b\}\} = \{a, b\} \tag{33}$$

$$\forall_x \exists!_y R(x, y) \text{ \# functional relation } R \tag{34}$$

$$\begin{gathered}\exists_i \forall_x \exists!_y R(x, y) \implies y \in i \text{ \# C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \mid P(y)\} \text{ \# Restricted Comprehension } \not\Longrightarrow \{y \mid P(y)\} \text{ \# Universal Comprehension}\end{gathered} \tag{35}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \implies P(x)\big) \text{ \# ignores out of scope} \neq \forall_x \big(x \in m \land P(x)\big) \text{ \# restricts entirety} \tag{36}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \implies n \subseteq \mathcal{P}(m)\big) \text{ \# C: existence of power set} \tag{37}$$

$$\exists_I \Big(\emptyset \in I \land \forall_{x \in I} \big(\{x\} \in I\big)\Big) \text{ \# I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \implies \mathbb{N} \textbf{ is a set} \tag{38}$$

$$\forall_x \Big(\big(\emptyset \notin x \land x \cap x' = \emptyset\big) \implies \exists_y \big(\textbf{set of each e} \in x\big)\Big) \text{ \# C: axiom of choice} \tag{39}$$

$$\forall_x x \neq \emptyset \implies x \notin x \text{ \# F: axiom of foundation covers further paradoxes} \tag{40}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{41}$$

1.2 Classification of sets

$$\begin{gathered}\textit{space}\big((\textit{set}, \textit{structure}), ()\big) \iff \textit{structure}(\textit{set}) \\ \text{\# a space a set equipped with some structure} \\ \text{\# various spaces can be studied through structure preserving maps between those spaces}\end{gathered} \tag{42}$$

$$\mathit{map}(\phi,(A,B)) \iff \Big(\forall_{a\in A}\exists!_{b\in B}\big(\phi(a,b)\big)\Big)\vee \Big(\forall_{a\in A}\exists!_{b\in B}\big(b=\phi(a)\big)\Big) \quad \text{\# maps elements of a set to elements of another set} \tag{43}$$

$$\mathit{domain}(A,(\phi,A,B)) \iff \Big(\mathit{map}(\phi,(A,B))\Big) \tag{44}$$

$$\mathit{codomain}(B,(\phi,A,B)) \iff \Big(\mathit{map}(\phi,(A,B))\Big) \tag{45}$$

$$\mathit{image}(B,(A,q,M,N)) \iff \Big(\mathit{map}(q,(M,N)) \wedge A\subseteq M\Big)\wedge \Big(B=\{n\in N \mid \exists_{a\in A}\big(q(a)=n\big)\}\Big) \tag{46}$$

$$\mathit{preimage}(A,(B,q,M,N)) \iff \Big(\mathit{map}(q,(M,N)) \wedge B\subseteq N\Big)\wedge \Big(A=\{m\in M \mid \exists_{b\in B}\big(b=q(m)\big)\}\Big) \tag{47}$$

$$\mathit{injection}(q,(M,N)) \iff \Big(\mathit{map}(q,(M,N))\Big)\wedge \forall_{u,v\in M}\big(q(u)=q(v) \implies u=v\big) \quad \text{\# every } m \text{ has at most 1 image} \tag{48}$$

$$\mathit{surjection}(q,(M,N)) \iff \Big(\mathit{map}(q,(M,N))\Big)\wedge \forall_{n\in N}\exists_{m\in M}\big(n=q(m)\big) \quad \text{\# every } n \text{ has at least 1 preimage} \tag{49}$$

$$\mathit{bijection}(q,(M,N)) \iff \Big(\mathit{injection}(q,(M,N))\Big)\wedge \Big(\mathit{surjection}(q,(M,N))\Big) \quad \text{\# every unique } m \text{ corresponds to a unique } n \tag{50}$$

$$\mathit{isomorphicSets}((A,B),()) \iff \exists_{\phi}\Big(\mathit{bijection}(\phi,(A,B))\Big) \tag{51}$$

$$\mathit{infiniteSet}(S,()) \iff \exists_{T\subset S}\Big(\mathit{isomorphicSets}((T,S),())\Big) \tag{52}$$

$$\mathit{finiteSet}(S,()) \iff \Big(\neg \mathit{infiniteSet}(S,())\Big)\vee (|S|\in\mathbb{N}) \tag{53}$$

$$\mathit{countablyInfinite}(S,()) \iff \Big(\mathit{infiniteSet}(S,())\Big)\wedge\Big(\mathit{isomorphicSets}((S,\mathbb{N}),())\Big) \tag{54}$$

$$\mathit{uncountablyInfinite}(S,()) \iff \Big(\mathit{infiniteSet}(S,())\Big)\wedge\Big(\neg \mathit{isomorphicSets}((S,\mathbb{N}),())\Big) \tag{55}$$

$$\mathit{inverseMap}\Big(q^{-1},(q,M,N)\Big) \iff \Big(\mathit{bijection}(q,(M,N))\Big)\wedge$$

$$\left(map\left(q^{-1},(N,M)\right)\right)\wedge$$
$$\left(\forall_{n\in N}\exists!_{m\in M}\left(q(m)=n\implies q^{-1}(n)=m\right)\right) \tag{56}$$

$$mapComposition\big(\phi\circ\psi,(\phi,\psi,A,B,C)\big)\iff map\big(\psi,(A,B)\big)\wedge map\big(\phi,(B,C)\big)\wedge$$
$$\forall_{a\in A}\Big(\phi\circ\psi(a)=\phi\big(\psi(a)\big)\Big) \tag{57}$$

$$equivalenceRelation\big(\sim,(M)\big)\iff\big(\forall_{m\in M}(m\sim m)\big)\wedge$$
$$\big(\forall_{m,n\in M}(m\sim n\implies n\sim m)\big)\wedge$$
$$\big(\forall_{m,n,p\in M}(m\sim n\wedge n\sim p\implies m\sim p)\big)$$
$$\#\text{ behaves as equivalences should} \tag{58}$$

$$equivalenceClass\big([m],(m,M,\sim)\big)\iff[m]=\{n\in M\mid n\sim m\}$$
$$\#\text{ set of elements satisfying the equivalence relation with } m \tag{59}$$

$$\text{(THM)}: a\in[m]\implies[a]=[m]\ ;\ [m]=[n]\veebar[m]\cap[n]=\emptyset$$
$$\#\text{ equivalence class properties} \tag{60}$$

$$quotientSet\big(M/\sim,(M,\sim)\big)\iff M/\sim=\{[m]\in\mathcal{P}(M)\mid m\in M\}$$
$$\#\text{ set of all equivalence classes} \tag{61}$$

$$\text{(THM)}: \textbf{axiom of choice}\implies\forall_{[m]\in M/\sim}\exists_r\big(r\in[m]\big)$$
$$\#\text{ well-defined maps may be defined in terms of chosen representative elements } r \tag{62}$$

1.3 Construction of number sets

$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{63}$$

$$\textbf{axiom of infinity}\implies\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\ldots\}\cong\mathbb{N} \tag{64}$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \tag{65}$$

$$\textbf{addition}=\textbf{successor map:}\ \mathbb{N}\to\mathbb{N}=S(n)=\{n\}\ \#\text{ adds a layer of brackets} \tag{66}$$

$$\textbf{subtraction}=\textbf{predecessor map:}\ \mathbb{N}^\star\to\mathbb{N}=P(n)=m\mid m\in n\ \#\text{ removes a layer of brackets} \tag{67}$$

$$S^0=id\ ;\ n\in\mathbb{N}^\star\implies S^n=S\circ S^{P(n)} \tag{68}$$

$$\textbf{addition}=+:\mathbb{N}\times\mathbb{N}\to\mathbb{N}=+(m,n)=m+n=S^n(m) \tag{69}$$

$$S^x=id=S^0\implies x=\textbf{additive identity}=0 \tag{70}$$

$$S^n(x)=0\implies x=\textbf{additive inverse}\notin\mathbb{N}\ \#\text{ git gud smh -\_-} \tag{71}$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim \textbf{, s.t.:}$$
$$(m, n) \sim (p, q) \iff m + q = p + n \text{ \# span } \mathbb{Z} \text{ using differences then group equal differences} \tag{72}$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \to [(n, 0)] \text{ \# } \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{73}$$

$$+_{\mathbb{Z}} = [(m +_{\mathbb{N}} p, n +_{\mathbb{N}} q)] \text{ \# well-defined and consistent} \tag{74}$$

$$\textbf{multiplication} \ldots M^x = id \implies x = \textbf{multiplicative identity} = 1 \ldots \textbf{multiplicative inverse} \notin \mathbb{N} \tag{75}$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^\star) / \sim \textbf{, s.t.: } (x, y) \sim (u, v) \iff x \cdot v = u \cdot y \tag{76}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \to [(q, 1)] \; ; \; \ldots \{x \mid x^2 = 2\} \notin \mathbb{Q} \tag{77}$$

$$\mathbb{R} = \textbf{almost homomorphisms on } \mathbb{Z} / \sim \text{ \# http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{78}$$

$$============ \textbf{ N O T = U P D A T E D } ============ \tag{79}$$

1.4 Topology

$$\begin{aligned} \textit{topology}\big(\mathcal{O}, (M)\big) \iff & \big(\mathcal{O} \subseteq \mathcal{P}(M)\big) \land \\ & \big(\emptyset, M \in \mathcal{O}\big) \land \\ & \Big(\big(F \in \mathcal{O} \land |F| < |\mathbb{N}|\big) \implies \cap F \in \mathcal{O}\Big) \land \\ & \big(C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O}\big) \\ & \text{\# topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.} \\ & \text{\# arbitrary unions of open sets always result in an open set} \\ & \text{\# open sets do not contain their boundaries and infinite intersections of open sets may approach and} \\ & \text{\# induce boundaries resulting in a closed set} \end{aligned} \tag{80}$$

$$\textit{topologicalSpace}\big((M, \mathcal{O}), ()\big) \iff \textit{topology}\big(\mathcal{O}, (M)\big) \tag{81}$$

$$\begin{aligned} \textit{open}\big(S, (M, \mathcal{O})\big) \iff & \Big(\textit{topologicalSpace}\big((M, \mathcal{O}), ()\big)\Big) \land \\ & (S \subseteq M) \land (S \in \mathcal{O}) \\ & \text{\# an open set do not contains its own boundaries} \end{aligned} \tag{82}$$

$$\begin{aligned} \textit{closed}\big(S, (M, \mathcal{O})\big) \iff & \Big(\textit{topologicalSpace}\big((M, \mathcal{O}), ()\big)\Big) \land \\ & (S \subseteq M) \land \big(S \in \mathcal{P}(M) \setminus \mathcal{O}\big) \\ & \text{\# a closed set contains the boundaries an open set} \end{aligned} \tag{83}$$

$$\textit{clopen}\big(S, (M, \mathcal{O})\big) \iff \Big(\textit{closed}\big(S, (M, \mathcal{O})\big)\Big) \land \Big(\textit{open}\big(S, (M, \mathcal{O})\big)\Big) \tag{84}$$

$$\begin{aligned} \textit{neighborhood}\big(U, (a, \mathcal{O})\big) \iff & (a \in U \in \mathcal{O}) \\ & \text{\# another name for open set containing } a \end{aligned} \tag{85}$$

$$M = \{a,b,c,d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\} \implies$$
$$\Big(open\big(X,(M,\mathcal{O})\big) \iff X = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\}\Big) \wedge$$
$$\Big(closed\big(Y,(M,\mathcal{O})\big) \iff Y = \{\emptyset, \{a,b,d\}, \{c,d\}, \{a,b\}, \{d\}, M\}\Big) \wedge$$
$$\Big(clopen\big(Z,(M,\mathcal{O})\big) \iff Z = \{\emptyset, \{a,b\}, \{c,d\}, M\}\Big) \quad (86)$$

$$chaoticTopology(M) = \{0, M\} \; ; \; discreteTopology = \mathcal{P}(M) \quad (87)$$

1.5 Induced topology

$$distance\big(d,(M)\big) \iff \Big(\forall_{x,y \in M}\Big(d(x,y) = d(y,x) \in \mathbb{R}_0^+\Big)\Big) \wedge$$
$$\Big(\forall_{x,y \in M}\big(d(x,y) = 0 \iff x = y\big)\Big) \wedge$$
$$\Big(\forall_{x,y,z}\Big(\big(d(x,z) \leq d(x,y) + d(y,z)\big)\Big)\Big)$$
$$\text{\# behaves as distances should} \quad (88)$$

$$metricSpace\big((M,d),()\big) \iff distance\big(d,(M)\big) \quad (89)$$

$$openBall\big(B,(r,p,M,d)\big) \iff \Big(metricSpace\big((M,d),()\big)\Big) \wedge$$
$$\big(r \in \mathbb{R}^+, p \in M\big) \wedge$$
$$\big(B = \{q \in M \mid d(p,q) < r\}\big) \quad (90)$$

$$metricTopology\big(\mathcal{O},(M,d)\big) \iff \Big(metricSpace\big((M,d),()\big)\Big) \wedge$$
$$\Big(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big) \wedge B \subseteq U\Big)\}\Big)$$
$$\text{\# every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (91)$$

$$metricTopologicalSpace\big((M,\mathcal{O},d),()\big) \iff metricTopology\big(\mathcal{O},(M,d)\big) \quad (92)$$

$$limitPoint\big(p,(S,M,\mathcal{O},d)\big) \iff \Big(metricTopologicalSpace\big((M,\mathcal{O},d),()\big)\Big) \wedge (S \subseteq M) \wedge$$
$$\forall_{r \in \mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big) \wedge B \cap S \neq \emptyset\Big)$$
$$\text{\# every open ball centered at } p \text{ contains some intersection with } S \quad (93)$$

$$interiorPoint\big(p,(S,M,\mathcal{O},d)\big) \iff \Big(metricTopologicalSpace\big((M,\mathcal{O},d),()\big)\Big) \wedge (S \subseteq M) \wedge$$
$$\Big(\exists_{r \in \mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big) \wedge B \subseteq S\Big)\Big)$$
$$\text{\# there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (94)$$

$$closure\big(\bar{S},(S,M,\mathcal{O},d)\big) \iff \bar{S} = S \cup \{p \in M \mid limitPoint\big(p,(S,M,\mathcal{O},d)\big)\} \quad (95)$$

$$dense(S,(M,\mathcal{O},d)) \iff (S \subseteq M) \land \Big(\forall_{p \in M}\Big(p \in closure(\bar{S},(S,M,\mathcal{O},d))\Big)\Big)$$
$$\text{\# every of point in } M \text{ is a point or a limit point of } S \quad (96)$$

$$eucD(d,(n)) \iff (\forall_{i \in \mathbb{N} \land i \leq n}(x_i \in \mathbb{R})) \land \left(d = \sqrt[2]{\sum_{i=1}^{n} {x_i}^2}\right) \quad (97)$$

$$metricTopology\Big(standardTopology,\Big(\mathbb{R}^n, eucD(d,(n))\Big)\Big)$$
$$\text{============ } \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} \text{ ============}$$
$$\mathbf{L1:}\ \forall_{p \in U = \emptyset}(\ldots) \implies \forall_p\Big((p \in \emptyset) \implies \ldots\Big) \implies \forall_p((\mathbf{False}) \implies \ldots) \implies \emptyset \in \mathcal{O}_{standard}$$
$$\mathbf{L2:}\ \forall_{p \in \mathbb{R}^n} B(r,p,\mathbb{R}^n,d) \subseteq \mathbb{R}^n \implies M \in \mathcal{O}_{standard}$$
$$\mathbf{L4:}\ C \subseteq \mathcal{O}_{standard} \implies \forall_{U \in C}\forall_{p \in U}\exists_{r \in \mathbb{R}^+}\big(B_r(p) \subseteq U \subseteq \cup C\big) \implies \cup C \in \mathcal{O}_{standard}$$
$$\mathbf{L3:}\ U,V \in \mathcal{O}_{standard} \implies p \in U \cap V \implies p \in U \land p \in V \implies$$
$$\exists_{r \in \mathbb{R}^+} B(r,p,\mathbb{R}^n,d) \land \exists_{s \in \mathbb{R}^+} B(s,p,\mathbb{R}^n,d) \implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big) \subseteq U \land B\big(min(r,s),q,\mathbb{R}^n,d\big) \subseteq V \implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big) \in U \cap V \implies U \cap V \in \mathcal{O}_{standard}$$
$$\text{\# could fail on infinite sets since } min \text{ could approach } 0$$
$$\text{============ } \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} \text{ ============} \quad (98)$$

$$subsetTopology(\mathcal{O}|_N,(M,\mathcal{O},N)) \iff topology(\mathcal{O},(M)) \land (N \subseteq M) \land \big(\mathcal{O}|_N = \{U \cap N \mid U \in \mathcal{O}\}\big)$$
$$\text{\# crops open sets outside } N \quad (99)$$

$$\text{(THM)} : subsetTopology(\mathcal{O}|_N,(M,\mathcal{O},N)) \land topology(\mathcal{O}|_N,(N)) \Longleftarrow$$
$$\text{============ } \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} \text{ ============}$$
$$\mathbf{L1:}\ \emptyset \in \mathcal{O} \implies U = \emptyset \implies \emptyset \cap N = \emptyset \implies \emptyset \in \mathcal{O}|_N$$
$$\mathbf{L2:}\ M \in \mathcal{O} \implies U = M \implies M \cap N = N \implies N \in \mathcal{O}|_N$$
$$\mathbf{L3:}\ S,T \in \mathcal{O}|_N \implies \exists_{U \in \mathcal{O}}(S = U \cap N) \land \exists_{V \in \mathcal{O}}(T = V \cap N) \implies S \cap T = (U \cap N) \cap (V \cap N)$$
$$= (U \cap V) \cap N \land U \cap V \in \mathcal{O} \implies S \cap T \in \mathcal{O}|_N$$
$$\mathbf{L4:}\ TODO : EXERCISE$$
$$\text{============ } \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} \text{ ============} \quad (100)$$

$$productTopology\Big(\mathcal{O}_{A \times B},((A,\mathcal{O}_A),(B,\mathcal{O}_B))\Big) \iff \Big(topology(\mathcal{O}_A,(A))\Big) \land \Big(topology(\mathcal{O}_B,(B))\Big) \land$$
$$(\mathcal{O}_{A \times B} = \{(a,b) \in A \times B \mid \exists_S(a \in S \in \mathcal{O}_A)\exists_T(b \in T \in \mathcal{O}_B)\})$$
$$\text{\# open in cross iff open in each} \quad (101)$$

1.6 Convergence

$$sequence(q,(M)) \iff map(q,(\mathbb{N},M)) \quad (102)$$

$$sequenceConvergesTo((q,a),(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \land$$
$$\Big(sequence(q,(M))\Big) \land (a \in M) \land \Big(\forall_{U \in \mathcal{O} | a \in U}\exists_{N \in \mathbb{N}}\forall_{n > N}\big(q(n) \in U\big)\Big)$$
$$\text{\# each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \quad (103)$$

(THM) : **convergence generalizes to: the sequence $q:\mathbb{N}\to\mathbb{R}^d$ converges against $a\in\mathbb{R}^d$ in $\mathcal{O}_S$ if:**

$$\forall_{r>0}\exists_{N\in\mathbb{N}}\forall_{n>N}\big(||q(n)-a||<\epsilon\big) \text{ \# distance based convergence} \tag{104}$$

1.7 Continuity

$$\mathit{continuous}\Big(\phi,\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big)\Big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}_M),()\big)\Big)\wedge \Big(\mathit{topologicalSpace}\big((N,\mathcal{O}_N),()\big)\Big)\wedge\Big(\mathit{map}\big(\phi,(M,N)\big)\Big)\wedge\Big(\forall_{V\in\mathcal{O}_N}\Big(\mathit{preimage}\big(A,(V,\phi,M,N)\big)\wedge A\in\mathcal{O}_M\Big)\Big)$$

$$\text{\# preimage of open sets are open} \tag{105}$$

$$\mathit{homeomorphism}\Big(\phi,\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big)\Big) \iff \Big(\mathit{inverseMap}\Big(\phi^{-1},(\phi,M,N)\Big)\Big)$$

$$\Big(\mathit{continuous}\Big(\phi,\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big)\Big)\Big)\wedge\Big(\mathit{continuous}\Big(\phi^{-1},\big((N,\mathcal{O}_N),(M,\mathcal{O}_M)\big)\Big)\Big)$$

$$\text{\# structure preserving maps in topology, ability to share topological properties} \tag{106}$$

$$\mathit{isomorphicTopologicalSpace}\Big(\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big),()\Big) \iff \exists_\phi\Big(\mathit{homeomorphism}\Big(\phi,\big((M,\mathcal{O}_M),(N,\mathcal{O}_N)\big)\Big)\Big) \tag{107}$$

1.8 Separation

$$\mathit{T0Separate}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge\Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U\in\mathcal{O}}\Big(\big(x\in U\wedge y\notin U\big)\vee\big(y\in U\wedge x\notin U\big)\Big)\Big)$$

$$\text{\# each pair of points has a neighborhood s.t. one is inside and the other is outside} \tag{108}$$

$$\mathit{T1Separate}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge\Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U,V\in\mathcal{O}\wedge U\neq V}\Big(\big(x\in U\wedge y\notin U\big)\wedge\big(y\in V\wedge x\notin V\big)\Big)\Big)$$

$$\text{\# every point has a neighborhood that does not contain another point} \tag{109}$$

$$\mathit{T2Separate}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge\Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U,V\in\mathcal{O}\wedge U\neq V}\big(U\cap V=\emptyset\big)\Big)$$

$$\text{\# every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \tag{110}$$

$$\text{(THM)}:\mathit{T2Separate}\implies\mathit{T1Separate}\implies\mathit{T0Separate} \tag{111}$$

1.9 Compactness

$$\mathit{openCover}\big(C,(M,\mathcal{O})\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge(C\subseteq\mathcal{O})\wedge(\cup C=M)$$

$$\text{\# collection of open sets whose elements cover the entire space} \tag{112}$$

$$\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(\widetilde{C}\subseteq C\Big)\wedge\Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big)\wedge$$
$$\Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big)\wedge\Big(\mathit{finiteSet}\Big(\widetilde{C},()\Big)\Big)$$
$$\text{\# finite subset of a cover that is also a cover} \tag{113}$$

$$\mathit{compact}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\left(\forall_{C\subseteq\mathcal{O}}\left(\mathit{openCover}(C,(M,\mathcal{O})) \implies \exists_{\widetilde{C}\subseteq C}\left(\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big)\right)\right)\right)$$
$$\text{\# every covering of the space is represented by a finite number of nhbhds} \tag{114}$$

$$\mathit{compactSubset}(N,(M,\mathcal{O}_d,d)) \iff \Big(\mathit{compact}\big((M,\mathcal{O}),()\big)\Big)\wedge\Big(\mathit{subsetTopology}(\mathcal{O}|_N,(M,\mathcal{O},N))\Big) \tag{115}$$

$$\mathit{bounded}\big(N,(M,d)\big) \iff \Big(\mathit{metricSpace}\big((M,d),()\big)\Big)\wedge(N\subseteq M)\wedge$$
$$\Big(\exists_{r\in\mathbb{R}^+}\forall_{p,q\in n}\big(d(p,q)<r\big)\Big) \tag{116}$$

$$\text{(THM) HeineBorel: } \mathit{metricTopologicalSpace}\big((M,\mathcal{O}_d,d),()\big) \implies$$
$$\forall_{S\in\mathcal{P}(M)}\bigg(\Big(\mathit{closed}\big(S,(M,\mathcal{O}_d)\big)\wedge\mathit{bounded}\big(S,(M,\mathcal{O}_d)\big)\Big) \iff \mathit{compactSubset}\big(S,(M,\mathcal{O}_d)\big)\bigg)$$
$$\text{\# when metric topologies are involved, compactness is equivalent to being closed and bounded} \tag{117}$$

1.10 Paracompactness

$$\mathit{openRefinement}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big)\wedge\Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big)\wedge$$
$$\Big(\forall_{\widetilde{U}\in\widetilde{C}}\exists_{U\in C}\Big(\widetilde{U}\subseteq U\Big)\Big)$$
$$\text{\# a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space} \tag{118}$$

$$\text{(THM) : } \mathit{finiteSubcover} \implies \mathit{openRefinement} \tag{119}$$

$$\mathit{locallyFinite}(C,(M,\mathcal{O})) \iff \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big)\wedge$$
$$\forall_{p\in M}\exists_{U\in\mathcal{O}|p\in U}\Big(\mathit{finiteSet}(\{U_c\in C|U\cap U_c\neq\emptyset\},())\Big)$$
$$\text{\# each point has a neighborhood that intersects with only finitely many sets in the cover} \tag{120}$$

$$\mathit{paracompact}\big((M,\mathcal{O}),()\big) \iff$$
$$\forall_C\left(\mathit{openCover}(C,(M,\mathcal{O})) \implies \exists_{\widetilde{C}}\left(\mathit{locallyFinite}\left(\mathit{openRefinement}\Big(\widetilde{C},(C,M,\mathcal{O})\Big),(M,\mathcal{O})\right)\right)\right)$$
$$\text{\# every open cover has a locally finite open refinement} \tag{121}$$

$$\text{(THM) : } \mathit{metricTopologicalSpace} \implies \mathit{paracompact} \tag{122}$$

$$============ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} ============ \tag{123}$$

$$partitionOfUnitySubjCover\big(\mathcal{F}, (C, M, \mathcal{O})\big) \iff \Big(locallyFinite\big(C, (M, \mathcal{O})\big)\Big) \land (f \in \mathcal{F}) \land$$
$$\Bigg(continuous\bigg(f, \bigg((M, \mathcal{O}), \Big([0,1], subsetTopology\Big(\mathcal{O}|_{[0,1]}, \big([0,1], \mathbb{R}, standardTopology\big)\Big)\Big)\bigg)\bigg)\Bigg) \land$$
$$\Big(\exists_{U_f \in C} \forall_{p \in M} \big(f(p) \neq 0 \implies p \in U_f\big)\Big) \land$$
$$\Big(\forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U} \big((f_U)_n = \{f \in \mathcal{F} | p \in M \land f(p) \neq 0\}\big)\Big) \land$$
$$\Big(locallyFinite(C, M, \mathcal{O}) \implies finiteSet\big((f_U)_n, ()\big)\Big) \land$$
$$\left(\forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U} \left(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1\right)\right)$$
$$\#\ \text{useful for defining integrals between overlapping neighborhoods} \tag{124}$$

$$T2Separate\big((M, \mathcal{O}), ()\big) \implies \Big(paracompact\big((M, \mathcal{O}), ()\big)\Big) \iff$$
$$\forall_C \Big(openCover\big(C, (M, \mathcal{O})\big) \implies partitionOfUnitySOTCover\big(\mathcal{F}, (C, M, \mathcal{O})\big)\Big) \tag{125}$$

$$============ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} ============ \tag{126}$$

1.11 Connectedness and path-connectedness

$$connected\big((M, \mathcal{O}), ()\big) \iff \Big(topologicalSpace\big((M, \mathcal{O}), ()\big)\Big) \land \Big(\neg \exists_{A, B \in \mathcal{O} \setminus \emptyset} \big(A \cap B \neq \emptyset \land A \cup B = M\big)\Big)$$
$$\#\ \text{if there is some covering of the space that does not intersect} \tag{127}$$

$$(\text{THM}) : \neg connected\bigg(\bigg(\mathbb{R} \setminus \{0\}, subsetTopology\Big(\mathcal{O}_{standard}|_{\mathbb{R} \setminus \{0\}}, \big(\mathbb{R}, standardTopology, \mathbb{R} \setminus \{0\}\big)\Big)\bigg), ()\bigg)$$
$$\impliedby \Big(A = (-\infty, 0) \in \mathcal{O}_{standard}|_{\mathbb{R} \setminus \{0\}}\Big) \land \Big(B = (0, \infty) \in \mathcal{O}_{standard}|_{\mathbb{R} \setminus \{0\}}\Big) \land$$
$$\big(A \cap B = \emptyset\big) \land \big(A \cup B = \mathbb{R} \setminus \{0\}\big) \tag{128}$$

$$(\text{THM}) : connected\big((M, \mathcal{O}), ()\big) \iff \forall_{S \in \mathcal{O}} \Big(clopen\Big(S, (M, \mathcal{O}) \implies \big(S = \emptyset \lor S = M\big)\Big)\Big) \tag{129}$$

$$pathConnected\big((M, \mathcal{O}), ()\big) \iff \Big(subsetTopology\Big(\mathcal{O}_{standard}|_{[0,1]}, \big(\mathbb{R}, standardTopology, [0,1]\big)\Big)\Big) \land$$
$$\Bigg(\forall_{p, q \in M} \exists_\gamma \bigg(continuous\bigg(\gamma, \Big(\big([0,1], \mathcal{O}_{standard}|_{[0,1]}\big), (M, \mathcal{O})\Big)\bigg) \land \gamma(0) = p \land \gamma(1) = q\bigg)\Bigg) \tag{130}$$

$$(\text{THM}) : pathConnected \implies connected \tag{131}$$

1.12 Homotopic curve and the fundamental group

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{132}$$

$$\begin{aligned} &homotopic(\sim, (\gamma, \delta, M, \mathcal{O})) \iff (map(\gamma, ([0,1], M)) \land map(\delta, ([0,1], M)))\land \\ &(\gamma(0) = \delta(0) \land \gamma(1) = \delta(1))\land \\ &(\exists_H \forall_{\lambda\in[0,1]}(continuous(H, (([0,1]\times[0,1], \mathcal{O}_{standard^2}|_{[0,1]\times[0,1]}), (M, \mathcal{O})) \land H(0,\lambda) = \gamma(\lambda) \land H(1,\lambda) = \delta(\lambda)))) \\ &\#\ H \text{ is a continuous deformation of one curve into another} \end{aligned} \tag{133}$$

$$homotopic(\sim) \implies equivalenceRelation(\sim) \tag{134}$$

$$loopSpace(\mathcal{L}_p, (p, M, \mathcal{O})) \iff \mathcal{L}_p = \{map(\gamma, ([0,1], M)) | continuous(\gamma) \land \gamma(0) = \gamma(1)\} \tag{135}$$

$$\begin{aligned} &concatination(\star, (p, \gamma, \delta)) \iff (\gamma, \delta \in loopSpace(\mathcal{L}_p))\land \\ &(\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda) = \begin{cases} \gamma(2\lambda) & 0 \le \lambda < 0.5 \\ \delta(2\lambda - 1) & 0.5 \le \lambda \le 1 \end{cases})) \end{aligned} \tag{136}$$

$$\begin{aligned} &group((G, \bullet), ()) \iff (map(\bullet, (G\times G, G)))\land \\ &(\forall_{a,b\in G}(a \bullet b \in G)) \\ &(\forall_{a,b,c\in G}((a\bullet b)\bullet C = a \bullet (b \bullet c))) \\ &(\exists_{\boldsymbol{e}}\forall_{a\in G}(\boldsymbol{e}\bullet a = a = a \bullet \boldsymbol{e}))\land \\ &(\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1} = \boldsymbol{e} = a^{-1}\bullet a)) \\ &\#\ \text{characterizes symmetry of a set structure} \end{aligned} \tag{137}$$

$$isomorphic(\cong, (X, \odot), (Y, \ominus))) \iff \exists_f \forall_{a,b\in X}(bijection(f, (X, Y)) \land f(a\odot b) = f(a) \ominus f(b)) \tag{138}$$

$$\begin{aligned} &fundamentalGroup((\pi_{1,p}, \bullet), (p, M, \mathcal{O})) \iff (\pi_{1,p} = \mathcal{L}_p / \sim)\land \\ &(map(\bullet, (\pi_{1,p}\times\pi_{1,p}, \pi_{1,p})))\land \\ &(\forall_{A,B\in\pi_{1,p}}([A]\bullet[B] = [A\star B]))\land \\ &(group((\pi_{1,p}, \bullet), ())) \\ &\#\ \text{an equivalence class of all loops induced from the homotopic equivalence relation} \end{aligned} \tag{139}$$

$$fundamentalGroup_1 \ncong fundamentalGroup_2 \implies topologicalSpace_1 \ncong topologicalSpace_2 \tag{140}$$

$$\textbf{there exists no known list of topological properties that can imply homeomorphisms} \tag{141}$$

$$\textbf{CONTINUE @ Lecture 6: manifolds} \tag{142}$$

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{143}$$

1.13 Measure theory

$$\tag{144}$$

$$\sigma\text{-}algebra(\sigma, (M)) \iff (\sigma \subseteq \mathcal{P}(M))\land$$

$$(M \in \sigma)\wedge$$
$$(\forall_{A\in\sigma}(M \setminus A \in \sigma))\wedge$$
$$((A)_{\mathbb{N}} \subseteq \sigma \implies \cup((A)_{\mathbb{N}}) \in \sigma) \tag{145}$$

$$measurableSpace((M,\sigma),()) \iff \sigma\text{-}algebra(\sigma,(M) \tag{146}$$

$$measurableSet(A,(M,\sigma)) \iff A \in \sigma \tag{147}$$

$$measure(\mu,(M,\sigma)) \iff (map(\mu,(\sigma,\overline{\mathbb{R}}_0^+)))\wedge$$
$$(\mu(\emptyset) = 0)\wedge$$
$$((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}\forall_{j\in\mathbb{N}\setminus\{i\}}(A_i \cap A_j = \emptyset) \implies \mu(\cup_{i\in\mathbb{N}}(A_i)) = \sum_{i\in\mathbb{N}}(\mu(A_i))) \tag{148}$$

$$measureSpace((M,\sigma,\mu),()) \tag{149}$$

$$measure \implies$$
$$\forall_{A,B\in\sigma}(A \subseteq B \implies \mu(A) \leq \mu(B))$$
$$(A)_{\mathbb{N}} \subseteq \sigma \implies \mu\cup \leq \sum \mu$$
$$A_1 \subseteq A_2 ... = A \implies \lim_{n\to\infty}(\mu(A_n)) = \mu(\cup A_n) = \mu(A)$$
$$...A_2 \subseteq A_1 = A \tag{150}$$

2 Statistics

2.1 Overview

$$randomExperiment(X,(\Omega)) \iff \forall_{\omega\in\Omega}(outcome(\omega,(X))) \tag{151}$$

$$sampleSpace(\Omega,(X)) \iff \Omega = \{\omega | outcome(\omega,(X))\} \tag{152}$$

$$event(A,(\Omega)) \implies A \subseteq \Omega \text{ \# that is of interest} \tag{153}$$

$$eventOccured(A,(\omega,\Omega)) \iff \omega \in A, \Omega \wedge event(A,(\Omega)) \tag{154}$$

$$algebra(\mathcal{F}_0,(\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega))\wedge$$
$$(\Omega \in \mathcal{F}_0)\wedge$$
$$(\forall_{A\in\mathcal{F}_0}(A^C \in \mathcal{F}_0))\wedge$$
$$(\forall_{A,B\in\mathcal{F}_0}(A \cup B \in \mathcal{F}_0))$$
$$\text{\# but this is unable to capture some countable events} \tag{155}$$

$$\sigma\text{-}algebra(\mathcal{F},(\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega))\wedge$$
$$(\Omega \in \mathcal{F})\wedge$$
$$(\forall_{A\in\mathcal{F}}(A^C \in \mathcal{F}))\wedge$$
$$(\forall_{F\subseteq\mathcal{F}}(\neg uncountablyInfinite(F,()) \implies \cup F \in \mathcal{F})) \tag{156}$$

3 Statistical Learning Theory

3.1 Overview

$$ \tag{157} $$

$$curve - fitting/explaining \neq prediction \tag{158}$$

$$ill - definedproblem + solutionspaceconstraints \implies well - definedproblem \tag{159}$$

$$x \text{ \# input ; } y \text{ \# output} \tag{160}$$

$$S_n = \{(x_1, y_1), \ldots, (x_n, y_n)\} \text{ \# training set} \tag{161}$$

$$f_S(x) \sim y \text{ \# solution} \tag{162}$$

$$each(x, y) \in p(x, y) \text{ \# training data } x, y \text{ is a sample from an unknown distribution } p \tag{163}$$

$$V(f(x), y) = d(f(x), y) \text{ \# loss function} \tag{164}$$

$$I[f] = \int_{X \times Y} V(f(x), y)p(x, y)dxdy \text{ \# expected error} \tag{165}$$

$$I_n[f] = \frac{1}{n} \sum_{i=1}^{n} V(f(x_i), y_i) \text{ \# empirical error} \tag{166}$$

$$probabilisticConvergence(X, ()) \iff \forall_{\epsilon>0} \lim_{n \to \infty} Pxn - x \leq \epsilon = 0 \tag{167}$$

$$I - Ingeneralizationerror \tag{168}$$

$$well - posed := exists, unique, stable; elseill - posed \tag{169}$$

3.2 Background maths

$$\begin{aligned} vectorSpace(V, (+, *)) \iff (u, v, w \in V), (c, d \in \mathbb{R} \in F) \wedge \\ (u + v, c * u = c(u) = cu \in V) \wedge \\ (u + v = v + u) \wedge \\ ((u + v) + w = u + (v + w)) \wedge \\ (\exists_{\boldsymbol{0}}(u + \boldsymbol{0} = u)) \wedge \\ (\exists_{-u}(u + (-u) = \boldsymbol{0})) \wedge \\ ((1)u = u) \\ ((cd)u = c(du)) \wedge \\ ((c + d)u = cu + du) \wedge \text{ \# linearity} \\ (c(u + v) = cu + cv) \wedge \text{ \# linearity} \\ \text{\# behaves similar to vectors} \end{aligned} \tag{170}$$

$$innerProduct(\langle\cdot,\cdot\rangle,(V)) \iff (u,v,w \in V),(c \in \mathbb{R} \in F)\wedge$$
$$(\langle v,w\rangle = \langle w,v\rangle)\wedge$$
$$(\langle cv,w\rangle = c\langle v,w\rangle)\wedge$$
$$(\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle)\wedge \ \text{\# linearity}$$
$$(\langle u,u\rangle \geq 0 \in \mathbb{R}_0^+)\wedge \ \text{\# metric inducing}$$
$$(\langle u,u\rangle = 0 \iff u = \boldsymbol{0}) \tag{171}$$

$$innerProductNorm(||\cdot||,(V)) \iff (v,w \in V),(r \in R)\wedge$$
$$(||v|| = \sqrt{\langle v,v\rangle} \in \mathbb{R}_0^+)\wedge$$
$$(||v|| = 0 \iff v = \boldsymbol{0})\wedge$$
$$(||rv|| = |r|||v||)\wedge$$
$$(||v+w|| \leq ||v|| + ||w||) \ \text{\# triangle inequality} \tag{172}$$

$$normConvergences(v,(V,(v_n)_{n\in\mathbb{N}})) \iff (\{v\} \cup (v_n)_{n\in\mathbb{N}} \subseteq V)\wedge$$
$$(\lim_{n\to\infty} ||v - v_n|| = 0) \tag{173}$$

$$cauchySequence((v_n)_{n\in\mathbb{N}},(V)) \iff$$
$$(\forall_{\epsilon>0}\exists_{n\in\mathbb{N}}\forall_{x,y>n}(||v_x - v_y|| < \epsilon)) \tag{174}$$

$$normConvergences \implies cauchySequence \ \text{\# there might be holes in the space} \tag{175}$$

$$completeSpace(V,(innerProductNorm)) \iff (cauchySequence \iff normConvergences) \tag{176}$$

$$completion(R,(Q)) \iff R = QUcauchyUs = Qbar \tag{177}$$

$$hilbertSpace(H,(+,*,\langle\cdot,\cdot\rangle)) \iff (vectorSpace(H,(+,*)))\wedge$$
$$(innerProduct(\langle\cdot,\cdot\rangle,(H)))\wedge$$
$$completeSpace(H,(innerProductNorm)) \tag{178}$$

$$separable(H,()) \iff \exists_{S\subseteq V}(countable(S,()) \wedge Sbar = V) \ \text{\# has a countable basis} \tag{179}$$

$$hilbertSpace \wedge seperable \iff \exists countableortho(gonal)normalbasisforspace, allnorm = 1, IP = 0 \tag{180}$$

$$x = \sum < x,v > v \ \text{\# countable projection times v} \tag{181}$$

$$0000000000 \tag{182}$$

$$linearOperator(L,(V)) \iff (u,v \in V),(c,d \in \mathbb{R})\wedge$$
$$(L(cu+dv) = cL(u) + dL(v)) \tag{183}$$

$$adjoint(L^\dagger,(L,V)) \iff (\forall_{u,v\in V} < L(u),v >=< u,L^\dagger(v) >_\dagger) \tag{184}$$

$$selfAdjoint(L,()) \iff L = L^\dagger \tag{185}$$

$$eigenvector(V) \iff Lv = kv \tag{186}$$

$$30mins \tag{187}$$

4 Machine Learning

4.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X,Y) \text{ \# dataset} \tag{188}$$

$$\textbf{learned parameters} = \textbf{parameters to be fixed by training with the dataset} \tag{189}$$

$$\textbf{hyperparameters} = \textbf{parameters that depends on a dataset} \tag{190}$$

$$\begin{gathered}\textbf{validation} = \textbf{partitions dataset into training and testing partitions, then evaluates the} \\ \textbf{accuracy of the parameters learned from the training partition in predicting the} \\ \textbf{outputs of the testing partition} \text{ \# useful for fixing hyperparameters}\end{gathered} \tag{191}$$

$$\textbf{cross-validation} = \textbf{average accuracy of validation for different choices of testing partition} \tag{192}$$

$$\textbf{L1} = \textbf{scales linearly ; } \textbf{L2} = \textbf{scales quadratically} \tag{193}$$

$$d = \textbf{distance} = \textbf{quantifies the the similarity between data points} \tag{194}$$

$$d_{L1}(A,B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance} \tag{195}$$

$$d_{L2}(A,B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \tag{196}$$

$$\textbf{kNN classifier} = \textbf{classifier based on } k \textbf{ nearest data points} \tag{197}$$

$$s = \textbf{class score} = \textbf{quantifies bias towards a particular class} \tag{198}$$

$$s_{linear} = f_{c\times 1}(x_{n\times 1}, W_{c\times n}, b_{c\times 1}) = W_{c\times n}x_{n\times 1} + b_{c\times 1} \text{ \# linear score function} \tag{199}$$

$$l = \textbf{loss} = \textbf{quantifies the errors by the learned parameters} \tag{200}$$

$$l = \frac{1}{|c_i|}\sum_{c_i} l_i \text{ \# average loss for all classes} \tag{201}$$

$$\begin{gathered}l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \text{ \# SVM hinge class loss function:} \\ \text{\# ignores incorrect classes with lower scores including a non-zero margin}\end{gathered} \tag{202}$$

$$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$
$$\text{\# lower scores correspond to lower exponentiated-normalized probabilities} \tag{203}$$

$$R = \textbf{regularization} = \textbf{optimizes the choice of learned parameters to minimize test error} \tag{204}$$

$$\lambda \text{ \# regularization strength hyperparameter} \tag{205}$$

$$R_{L1}(W) = \sum_{W_i} |W_i| \text{ \# L1 regularization} \tag{206}$$

$$R_{L2}(W) = \sum_{W_i} {W_i}^2 \text{ \# L2 regularization} \tag{207}$$

$$L' = L + \lambda R(W) \text{ \# weight regularization} \tag{208}$$

$$\nabla_W L = \overrightarrow{\frac{\partial}{\partial W_i}} L = \textbf{loss gradient w.r.t. weights} \tag{209}$$

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I}\frac{\partial L_E}{\partial L_L} \text{ \# loss gradient w.r.t. input weight in terms of external and local gradients} \tag{210}$$

$$s = \textbf{forward API} \; ; \; \frac{\partial L_L}{\partial W_I} = \textbf{backward API} \tag{211}$$

$$W_{t+1} = W_t - \nabla_{W_t} L \text{ \# weight update loss minimization} \tag{212}$$

$$\textbf{TODO:Research on Activation functions, Weight Initialization, Batch Normalization} \tag{213}$$

$$review5meanvardiscussion/hyperparameteroptimization/babysittinglearning \tag{214}$$

TODO loss L or l ??