Next-Next-Gen Notes

Object-Oriented Maths

JP Guzman

October 3, 2017

Format: $characteristic((subjects),(dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$
TODO all distance must take two parameters
TODO define || abs and other missing refs

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s,()) \iff well\text{-}formedString(s,()) \tag{1}$$

$$\begin{aligned} proposition((p,t),()) \iff & \Big(statement(p,())\Big) \wedge \\ & (t = eval(p)) \wedge \\ & (t = true \veebar t = false) \end{aligned} \tag{2}$$

$$operator\Big(o,\Big((p)_{n\in\mathbb{N}}\Big)\Big) \iff proposition\Big(o\Big((p)_{n\in\mathbb{N}}\Big),()\Big) \tag{3}$$

$$\begin{aligned} operator(\neg,(p_1)) \iff & \Big(proposition((p_1,true),()) \implies ((\neg p_1,false),())\Big) \wedge \\ & \Big(proposition((p_1,false),()) \implies ((\neg p_1,true),())\Big) \\ & \# \text{ an operator takes in propositions and returns a proposition} \end{aligned} \tag{4}$$

$$\begin{aligned} operator(\neg) \iff \textbf{NOT} \text{ ; } operator(\vee) \iff \textbf{OR} \text{ ; } operator(\wedge) \iff \textbf{AND} \text{ ; } operator(\veebar) \iff \textbf{XOR} \\ operator(\implies) \iff \textbf{IF} \text{ ; } operator(\impliedby) \iff \textbf{OIF} \text{ ; } operator(\iff) \iff \textbf{IFF} \end{aligned} \tag{5}$$

$$\begin{aligned} proposition((false \implies true),true,()) \wedge proposition((false \implies false),true,()) \\ \# \text{ truths based on a false premise is not false; ex falso quodlibet principle} \end{aligned} \tag{6}$$

$$(\text{THM}) : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \tag{7}$$

$$predicate(P,(V)) \iff \forall_{v\in V}\Big(proposition\Big((P(v),t),()\Big)\Big) \tag{8}$$

$$\begin{aligned} 0thOrderLogic(P,()) \iff proposition((P,t),()) \\ \# \text{ individual proposition} \end{aligned} \tag{9}$$

$$1stOrderLogic(P,(V)) \iff \Big(\forall_{v\in V}\Big(0thOrderLogic(v,())\Big)\Big) \wedge$$

$$\left(\forall_{v\in V}\left(\mathit{proposition}\left(\left(P(v),t\right),()\right)\right)\right)$$

propositions defined over a set of the lower order logical statements (10)

$$\mathit{quantifier}(q,(p,V)) \iff \left(\mathit{predicate}(p,(V))\right)\wedge \left(\mathit{proposition}\left(\left(q(p),t\right),()\right)\right)$$

a quantifier takes in a predicate and returns a proposition (11)

$$\mathit{quantifier}(\forall,(p,V)) \iff \mathit{proposition}\left(\left(\wedge_{v\in V}\left(p(v)\right),t\right),()\right)$$

universal quantifier (12)

$$\mathit{quantifier}(\exists,(p,V)) \iff \mathit{proposition}\left(\left(\vee_{v\in V}\left(p(v)\right),t\right),()\right)$$

existential quantifier (13)

$$\mathit{quantifier}(\exists!,(p,V)) \iff \exists_{x\in V}\left(P(x)\wedge\neg\left(\exists_{y\in V\setminus\{x\}}\left(P(y)\right)\right)\right)$$

uniqueness quantifier (14)

$$\text{(THM)} : \forall_x p(x) \iff \neg\exists_x\neg p(x)$$

De Morgan's law (15)

$$\text{(THM)} : \forall_x\exists_y p(x,y)=\forall_x\neg\forall_y\neg p(x,y)\neq\exists_y\forall_x p(x,y)=\neg\forall_y\neg\left(\forall_x p(x,y)\right)=\neg\forall_y\exists_x\neg p(x,y)$$

different quantifiers are not interchangeable (16)

============ **N O T = U P D A T E D** ============ (17)

---

proof=truths derived from a finite number of axioms and deductions (18)

---

elementary arithmetics=system with substitutions, and some notion of addition, multiplication, and prime nuumbers for encoding metamathematics (19)

Gödel theorem $\implies$ axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions (20)

$$\mathit{sequenceSet}((A)_{\mathbb{N}},(A)) \iff (Amapinputn)((A)_{\mathbb{N}}=\{A(1),A(2),A(3),\ldots\}) \quad (21)$$

TODO: define union, intersection, complement, etc. (22)

============ **N O T = U P D A T E D** ============ (23)

1.1 Axiomatic Set Theory

$$\text{============ } \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} \text{ ============} \tag{24}$$

$$\textbf{ZFC set theory}=\textbf{standard form of axiomatic set theory} \tag{25}$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \tag{26}$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \tag{27}$$

$$\in \textbf{ basis} \implies \{x,y\} = \{y,x\} \wedge \{x\} = \{x,x\} \tag{28}$$

$$\in \textbf{ and sets works following the 9 ZFC axioms:} \tag{29}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \text{ \# E: } \in \text{ is only a proposition on sets} \tag{30}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \text{ \# E: existence of empty set} \tag{31}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \text{ \# C: pair set construction} \tag{32}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \text{ \# C: union set construction} \tag{33}$$

$$x = \{\{a\},\{b\}\} \text{ \# from the pair set axiom} \tag{34}$$

$$u = \cup x = \cup \{\{a\},\{b\}\} = \{a,b\} \tag{35}$$

$$\forall_x \exists !_y R(x,y) \text{ \# functional relation } R \tag{36}$$

$$\begin{gathered} \exists_i \forall_x \exists !_y R(x,y) \implies y \in i \text{ \# C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \,|\, P(y)\} \text{ \# Restricted Comprehension} \not\Longrightarrow \{y \,|\, P(y)\} \text{ \# Universal Comprehension} \end{gathered} \tag{37}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \implies P(x)\big) \text{ \# ignores out of scope} \neq \forall_x \big(x \in m \wedge P(x)\big) \text{ \# restricts entirety} \tag{38}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \implies n \subseteq \mathcal{P}(m)\big) \text{ \# C: existence of power set} \tag{39}$$

$$\exists_I \Big(\emptyset \in I \wedge \forall_{x \in I} \big(\{x\} \in I\big)\Big) \text{ \# I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \implies \mathbb{N} \textbf{ is a set} \tag{40}$$

$$\forall_x \Big(\big(\emptyset \notin x \wedge x \cap x' = \emptyset\big) \implies \exists_y (\textbf{set of each e} \in x)\Big) \text{ \# C: axiom of choice} \tag{41}$$

$$\forall_x x \neq \emptyset \implies x \notin x \text{ \# F: axiom of foundation covers further paradoxes} \tag{42}$$

$$\text{============ } \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D} \text{ ============} \tag{43}$$

1.2 Classification of sets

$$space\big((set, structure), ()\big) \iff structure(set)$$
a space a set equipped with some structure
various spaces can be studied through structure preserving maps between those spaces (44)

$$map\big(\phi, (A,B)\big) \iff \Big(\forall_{a\in A}\exists!_{b\in B}\big(\phi(a,b)\big)\Big) \vee \Big(\forall_{a\in A}\exists!_{b\in B}\big(b=\phi(a)\big)\Big)$$
maps elements of a set to elements of another set (45)

$$domain\big(A, (\phi, A, B)\big) \iff \Big(map\big(\phi, (A,B)\big)\Big) \quad (46)$$

$$codomain\big(B, (\phi, A, B)\big) \iff \Big(map\big(\phi, (A,B)\big)\Big) \quad (47)$$

$$image\big(B, (A, q, M, N)\big) \iff \Big(map\big(q, (M,N)\big) \wedge A \subseteq M\Big) \wedge \Big(B = \{n \in N \mid \exists_{a\in A}\big(q(a)=n\big)\}\Big) \quad (48)$$

$$preimage\big(A, (B, q, M, N)\big) \iff \Big(map\big(q, (M,N)\big) \wedge B \subseteq N\Big) \wedge \Big(A = \{m \in M \mid \exists_{b\in B}\big(b=q(m)\big)\}\Big) \quad (49)$$

$$injection\big(q, (M,N)\big) \iff \Big(map\big(q, (M,N)\big)\Big) \wedge \forall_{u,v\in M}\big(q(u)=q(v) \implies u=v\big)$$
every m has at most 1 image (50)

$$surjection\big(q, (M,N)\big) \iff \Big(map\big(q, (M,N)\big)\Big) \wedge \forall_{n\in N}\exists_{m\in M}\big(n=q(m)\big)$$
every n has at least 1 preimage (51)

$$bijection\big(q, (M,N)\big) \iff \Big(injection\big(q, (M,N)\big)\Big) \wedge \Big(surjection\big(q, (M,N)\big)\Big)$$
every unique m corresponds to a unique n (52)

$$isomorphicSets\big((A,B), ()\big) \iff \exists_{\phi}\Big(bijection\big(\phi, (A,B)\big)\Big) \quad (53)$$

$$infiniteSet\big(S, ()\big) \iff \exists_{T\subset S}\Big(isomorphicSets\big((T,S), ()\big)\Big) \quad (54)$$

$$finiteSet\big(S, ()\big) \iff \Big(\neg infiniteSet\big(S, ()\big)\Big) \vee \big(|S| \in \mathbb{N}\big) \quad (55)$$

$$countablyInfinite\big(S, ()\big) \iff \Big(infiniteSet\big(S, ()\big)\Big) \wedge \Big(isomorphicSets\big((S, \mathbb{N}), ()\big)\Big) \quad (56)$$

$$uncountablyInfinite(S,()) \iff \Big(infiniteSet(S,())\Big) \wedge \Big(\neg isomorphicSets((S,\mathbb{N}),())\Big) \tag{57}$$

$$\begin{aligned} inverseMap\Big(q^{-1},(q,M,N)\Big) \iff \Big(bijection(q,(M,N))\Big) \wedge \\ \Big(map\Big(q^{-1},(N,M)\Big)\Big) \wedge \\ \Big(\forall_{n\in N}\exists!_{m\in M}\Big(q(m)=n \implies q^{-1}(n)=m\Big)\Big) \end{aligned} \tag{58}$$

$$\begin{aligned} mapComposition(\phi\circ\psi,(\phi,\psi,A,B,C)) \iff map(\psi,(A,B)) \wedge map(\phi,(B,C)) \wedge \\ \forall_{a\in A}\Big(\phi\circ\psi(a)=\phi(\psi(a))\Big) \end{aligned} \tag{59}$$

$$\begin{aligned} equivalenceRelation(\sim,(M)) \iff (\forall_{m\in M}(m\sim m)) \wedge \\ (\forall_{m,n\in M}(m\sim n \implies n\sim m)) \wedge \\ (\forall_{m,n,p\in M}(m\sim n \wedge n\sim p \implies m\sim p)) \\ \#\text{ behaves as equivalences should} \end{aligned} \tag{60}$$

$$\begin{aligned} equivalenceClass([m]_\sim,(m,M,\sim)) \iff [m]_\sim=\{n\in M\,|\,n\sim m\} \\ \#\text{ set of elements satisfying the equivalence relation with } m \end{aligned} \tag{61}$$

$$\begin{aligned} \text{(THM)}: a\in[m]_\sim \implies [a]_\sim=[m]_\sim\ ;\ [m]_\sim=[n]_\sim \veebar [m]_\sim\cap[n]_\sim=\emptyset \\ \#\text{ equivalence class properties} \end{aligned} \tag{62}$$

$$\begin{aligned} quotientSet(M/\sim,(M,\sim)) \iff M/\sim=\{equivalenceClass([m]_\sim,(m,M,\sim))\in\mathcal{P}(M)\,|\,m\in M\} \\ \#\text{ set of all equivalence classes} \end{aligned} \tag{63}$$

$$\begin{aligned} \text{(THM)}: \textbf{axiom of choice} \implies \forall_{[m]_\sim\in M/\sim}\exists_r\Big(r\in[m]_\sim\Big) \\ \#\text{ well-defined maps may be defined in terms of chosen representative elements } r \end{aligned} \tag{64}$$

1.3 Construction of number sets

$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{65}$$

$$\textbf{axiom of infinity} \implies \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\ldots\}\cong\mathbb{N} \tag{66}$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \tag{67}$$

$$\textbf{addition}=\textbf{successor map: } \mathbb{N}\to\mathbb{N}=S(n)=\{n\}\ \#\text{ adds a layer of brackets} \tag{68}$$

$$\textbf{subtraction}=\textbf{predecessor map: } \mathbb{N}^\star\to\mathbb{N}=P(n)=m\,|\,m\in n\ \#\text{ removes a layer of brackets} \tag{69}$$

$$S^0=id\ ;\ n\in\mathbb{N}^\star \implies S^n=S\circ S^{P(n)} \tag{70}$$

$$\textbf{addition}=+:\mathbb{N}\times\mathbb{N}\to\mathbb{N}=+(m,n)=m+n=S^n(m) \tag{71}$$

$$S^x = id = S^0 \implies x = \textbf{additive identity} = 0 \tag{72}$$

$$S^n(x) = 0 \implies x = \textbf{additive inverse} \notin \mathbb{N} \text{ \# git gud smh -\_-} \tag{73}$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim \textbf{, s.t.:}$$
$$(m,n) \sim (p,q) \iff m+q = p+n \text{ \# span } \mathbb{Z} \text{ using differences then group equal differences} \tag{74}$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \to [(n,0)] \text{ \# } \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{75}$$

$$+_{\mathbb{Z}} = [(m +_{\mathbb{N}} p, n +_{\mathbb{N}} q)] \text{ \# well-defined and consistent} \tag{76}$$

$$\textbf{multiplication} \ldots M^x = id \implies x = \textbf{multiplicative identity} = 1 \ldots \textbf{multiplicative inverse} \notin \mathbb{N} \tag{77}$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^\star) / \sim \textbf{, s.t.: } (x,y) \sim (u,v) \iff x \cdot v = u \cdot y \tag{78}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \to [(q,1)] \; ; \; \ldots \{x \mid x^2 = 2\} \notin \mathbb{Q} \tag{79}$$

$$\mathbb{R} = \textbf{almost homomorphisms on } \mathbb{Z} / \sim \text{ \# http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{80}$$

$$============ \textbf{ N O T = U P D A T E D } ============ \tag{81}$$

1.4 Topology

$$\begin{aligned} topology(\mathcal{O},(M)) \iff & (\mathcal{O} \subseteq \mathcal{P}(M)) \land \\ & (\emptyset, M \in \mathcal{O}) \land \\ & \Big((F \in \mathcal{O} \land |F| < |\mathbb{N}|) \implies \cap F \in \mathcal{O} \Big) \land \\ & (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O}) \end{aligned}$$
topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.
arbitrary unions of open sets always result in an open set
open sets do not contain their boundaries and infinite intersections of open sets may approach and
induce boundaries resulting in a closed set (82)

$$topologicalSpace((M,\mathcal{O}),()) \iff topology(\mathcal{O},(M)) \tag{83}$$

$$\begin{aligned} open(S,(M,\mathcal{O})) \iff & \Big(topologicalSpace((M,\mathcal{O}),()) \Big) \land \\ & (S \subseteq M) \land (S \in \mathcal{O}) \end{aligned}$$
an open set do not contains its own boundaries (84)

$$\begin{aligned} closed(S,(M,\mathcal{O})) \iff & \Big(topologicalSpace((M,\mathcal{O}),()) \Big) \land \\ & (S \subseteq M) \land \big(S \in \mathcal{P}(M) \setminus \mathcal{O}\big) \end{aligned}$$
a closed set contains the boundaries an open set (85)

$$clopen(S,(M,\mathcal{O})) \iff \Big(closed(S,(M,\mathcal{O}))\Big) \wedge \Big(open(S,(M,\mathcal{O}))\Big) \quad (86)$$

$$neighborhood(U,(a,\mathcal{O})) \iff (a \in U \in \mathcal{O})$$
$$\text{\# another name for open set containing } a \quad (87)$$

$$M = \{a,b,c,d\} \wedge \mathcal{O} = \{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},M\} \implies$$
$$\Big(open(X,(M,\mathcal{O})) \iff X = \{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},M\}\Big) \wedge$$
$$\Big(closed(Y,(M,\mathcal{O})) \iff Y = \{\emptyset,\{a,b,d\},\{c,d\},\{a,b\},\{d\},M\}\Big) \wedge$$
$$\Big(clopen(Z,(M,\mathcal{O})) \iff Z = \{\emptyset,\{a,b\},\{c,d\},M\}\Big) \quad (88)$$

$$chaoticTopology(M) = \{0,M\} \; ; \; discreteTopology = \mathcal{P}(M) \quad (89)$$

1.5 Induced topology

$$distance(d,(M)) \iff \Big(\forall_{x,y \in M}\Big(d(x,y) = d(y,x) \in \mathbb{R}_0^+\Big)\Big) \wedge$$
$$\Big(\forall_{x,y \in M}\big(d(x,y) = 0 \iff x = y\big)\Big) \wedge$$
$$\Big(\forall_{x,y,z}\Big(\big(d(x,z) \leq d(x,y) + d(y,z)\big)\Big)\Big)$$
$$\text{\# behaves as distances should} \quad (90)$$

$$metricSpace((M,d),()) \iff distance(d,(M)) \quad (91)$$

$$openBall(B,(r,p,M,d)) \iff \Big(metricSpace((M,d),())\Big) \wedge$$
$$(r \in \mathbb{R}^+, p \in M) \wedge$$
$$(B = \{q \in M \mid d(p,q) < r\}) \quad (92)$$

$$metricTopology(\mathcal{O},(M,d)) \iff \Big(metricSpace((M,d),())\Big) \wedge$$
$$\Big(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+} \Big(openBall(B,(r,p,M,d)) \wedge B \subseteq U\Big)\}\Big)$$
$$\text{\# every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (93)$$

$$metricTopologicalSpace((M,\mathcal{O},d),()) \iff metricTopology(\mathcal{O},(M,d)) \quad (94)$$

$$limitPoint(p,(S,M,\mathcal{O},d)) \iff \Big(metricTopologicalSpace((M,\mathcal{O},d),())\Big) \wedge (S \subseteq M) \wedge$$
$$\forall_{r \in \mathbb{R}^+}\Big(openBall(B,(r,p,M,d)) \wedge B \cap S \neq \emptyset\Big)$$
$$\text{\# every open ball centered at } p \text{ contains some intersection with } S \quad (95)$$

$$interiorPoint(p,(S,M,\mathcal{O},d)) \iff \Big(metricTopologicalSpace((M,\mathcal{O},d),())\Big) \wedge (S \subseteq M) \wedge$$
$$\Big(\exists_{r \in \mathbb{R}^+}\Big(openBall(B,(r,p,M,d)) \wedge B \subseteq S\Big)\Big)$$
$$\text{\# there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (96)$$

$$closure\big(\bar{S},(S,M,\mathcal{O},d)\big) \iff \bar{S}=S\cup\{p\in M \mid limitPoint\big(p,(S,M,\mathcal{O},d)\big)\} \tag{97}$$

$$dense\big(S,(M,\mathcal{O},d)\big) \iff (S\subseteq M)\wedge\Big(\forall_{p\in M}\Big(p\in closure\big(\bar{S},(S,M,\mathcal{O},d)\big)\Big)\Big)$$
$$\text{\# every of point in } M \text{ is a point or a limit point of } S \tag{98}$$

$$eucD\big(d,(n)\big) \iff \big(\forall_{i\in\mathbb{N}\wedge i\leq n}(x_i\in\mathbb{R})\big)\wedge\left(d=\sqrt[2]{\sum_{i=1}^{n}{x_i}^2}\right) \tag{99}$$

$$metricTopology\Big(standardTopology,\Big(\mathbb{R}^n,eucD\big(d,(n)\big)\Big)\Big)$$
$$=============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ =============$$
$$\mathbf{L1:}\ \forall_{p\in U=\emptyset}(\ldots)\implies\forall_p\Big(\big(p\in\emptyset\big)\implies\ldots\Big)\implies\forall_p\big((\mathbf{False})\implies\ldots\big)\implies\emptyset\in\mathcal{O}_{standard}$$
$$\mathbf{L2:}\ \forall_{p\in\mathbb{R}^n}B(r,p,\mathbb{R}^n,d)\subseteq\mathbb{R}^n\implies M\in\mathcal{O}_{standard}$$
$$\mathbf{L4:}\ C\subseteq\mathcal{O}_{standard}\implies\forall_{U\in C}\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\big(B_r(p)\subseteq U\subseteq\cup C\big)\implies\cup C\in\mathcal{O}_{standard}$$
$$\mathbf{L3:}\ U,V\in\mathcal{O}_{standard}\implies p\in U\cap V\implies p\in U\wedge p\in V\implies$$
$$\exists_{r\in\mathbb{R}^+}B(r,p,\mathbb{R}^n,d)\wedge\exists_{s\in\mathbb{R}^+}B(s,p,\mathbb{R}^n,d)\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\subseteq U\wedge B\big(min(r,s),q,\mathbb{R}^n,d\big)\subseteq V\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\in U\cap V\implies U\cap V\in\mathcal{O}_{standard}$$
$$\text{\# natural topology for } \mathbb{R}^d$$
$$\text{\# could fail on infinite sets since } min \text{ could approach } 0$$
$$=============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============= \tag{100}$$

$$subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\iff topology\big(\mathcal{O},(M)\big)\wedge(N\subseteq M)\wedge\big(\mathcal{O}|_N=\{U\cap N\mid U\in\mathcal{O}\}\big)$$
$$\text{\# crops open sets outside } N \tag{101}$$

$$(\text{THM}):\ subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\wedge topology\big(\mathcal{O}|_N,(N)\big)\impliedby$$
$$=============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ =============$$
$$\mathbf{L1:}\ \emptyset\in\mathcal{O}\implies U=\emptyset\implies\emptyset\cap N=\emptyset\implies\emptyset\in\mathcal{O}|_N$$
$$\mathbf{L2:}\ M\in\mathcal{O}\implies U=M\implies M\cap N=N\implies N\in\mathcal{O}|_N$$
$$\mathbf{L3:}\ S,T\in\mathcal{O}|_N\implies\exists_{U\in\mathcal{O}}(S=U\cap N)\wedge\exists_{V\in\mathcal{O}}(T=V\cap N)\implies S\cap T=(U\cap N)\cap(V\cap N)$$
$$=(U\cap V)\cap N\wedge U\cap V\in\mathcal{O}\implies S\cap T\in\mathcal{O}|_N$$
$$\mathbf{L4:}\ TODO:EXERCISE$$
$$=============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============= \tag{102}$$

$$productTopology\Big(\mathcal{O}_{A\times B},\big((A,\mathcal{O}_A),(B,\mathcal{O}_B)\big)\Big)\iff\Big(topology\big(\mathcal{O}_A,(A)\big)\Big)\wedge\Big(topology\big(\mathcal{O}_B,(B)\big)\Big)\wedge$$
$$\big(\mathcal{O}_{A\times B}=\{(a,b)\in A\times B\mid\exists_S(a\in S\in\mathcal{O}_A)\exists_T(b\in T\in\mathcal{O}_B)\}\big)$$
$$\text{\# open in cross iff open in each} \tag{103}$$

1.6 Convergence

$$sequence\big(q,(M)\big)\iff map\big(q,(\mathbb{N},M)\big) \tag{104}$$

$$sequenceConvergesTo((q,a),(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(sequence(q,(M))\Big) \wedge (a \in M) \wedge \Big(\forall_{U \in \mathcal{O} | a \in U} \exists_{N \in \mathbb{N}} \forall_{n > N} (q(n) \in U)\Big)$$
$$\text{\# each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \quad (105)$$

(THM) : **convergence generalizes to: the sequence** $q : \mathbb{N} \to \mathbb{R}^d$ **converges against** $a \in \mathbb{R}^d$ **in** $\mathcal{O}_S$ **if:**

$$\forall_{r>0} \exists_{N \in \mathbb{N}} \forall_{n > N} \big(||q(n) - a|| < \epsilon\big) \ \text{\# distance based convergence} \quad (106)$$

1.7 Continuity

$$continuous(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)) \iff \Big(topologicalSpace((M,\mathcal{O}_M),())\Big) \wedge$$
$$\Big(topologicalSpace((N,\mathcal{O}_N),())\Big) \wedge \Big(\forall_{V \in \mathcal{O}_N} \Big(preimage(A,(V,\phi,M,N)) \in \mathcal{O}_M\Big)\Big)$$
$$\text{\# preimage of open sets are open} \quad (107)$$

$$homeomorphism(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)) \iff \Big(inverseMap\Big(\phi^{-1},(\phi,M,N)\Big)\Big)$$
$$\Big(continuous(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N))\Big) \wedge \Big(continuous\Big(\phi^{-1},(N,\mathcal{O}_N,M,\mathcal{O}_M)\Big)\Big)$$
$$\text{\# structure preserving maps in topology, ability to share topological properties} \quad (108)$$

$$isomorphicTopologicalSpace\Big(((M,\mathcal{O}_M),(N,\mathcal{O}_N)),()\Big) \iff$$
$$\exists_\phi \Big(homeomorphism(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N))\Big) \quad (109)$$

1.8 Separation

$$T0Separate((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U \in \mathcal{O}} \Big((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U)\Big)\Big)$$
$$\text{\# each pair of points has a neighborhood s.t. one is inside and the other is outside} \quad (110)$$

$$T1Separate((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V} \Big((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V)\Big)\Big)$$
$$\text{\# every point has a neighborhood that does not contain another point} \quad (111)$$

$$T2Separate((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V} (U \cap V = \emptyset)\Big)$$
$$\text{\# every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \quad (112)$$

$$\text{(THM)} : T2Separate \Longrightarrow T1Separate \Longrightarrow T0Separate \quad (113)$$

1.9 Compactness

$$openCover(C,(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M)$$

collection of open sets whose elements cover the entire space (114)

$$finiteSubcover\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(\widetilde{C} \subseteq C\Big) \wedge \Big(openCover(C,(M,\mathcal{O}))\Big) \wedge \Big(openCover\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big) \wedge \Big(finiteSet\Big(\widetilde{C},()\Big)\Big)$$

finite subset of a cover that is also a cover (115)

$$compact((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge \Bigg(\forall_{C \subseteq \mathcal{O}}\Bigg(openCover(C,(M,\mathcal{O})) \implies \exists_{\widetilde{C} \subseteq C}\Big(finiteSubcover\Big(\widetilde{C},(C,M,\mathcal{O})\Big)\Big)\Bigg)\Bigg)$$

every covering of the space is represented by a finite number of nhbhds (116)

$$compactSubset(N,(M,\mathcal{O}_d,d)) \iff \Big(compact((M,\mathcal{O}),())\Big) \wedge \Big(subsetTopology(\mathcal{O}|_N,(M,\mathcal{O},N))\Big) \quad (117)$$

$$bounded(N,(M,d)) \iff \Big(metricSpace((M,d),())\Big) \wedge (N \subseteq M) \wedge \Big(\exists_{r \in \mathbb{R}^+} \forall_{p,q \in n}\big(d(p,q) < r\big)\Big) \quad (118)$$

(THM) HeineBorel: $metricTopologicalSpace((M,\mathcal{O}_d,d),()) \implies$

$$\forall_{S \in \mathcal{P}(M)}\Big(\Big(closed(S,(M,\mathcal{O}_d)) \wedge bounded(S,(M,\mathcal{O}_d))\Big) \iff compactSubset(S,(M,\mathcal{O}_d))\Big)$$

when metric topologies are involved, compactness is equivalent to being closed and bounded (119)

1.10 Paracompactness

$$openRefinement\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(openCover(C,(M,\mathcal{O}))\Big) \wedge \Big(openCover\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big) \wedge \Big(\forall_{\widetilde{U} \in \widetilde{C}} \exists_{U \in C}\Big(\widetilde{U} \subseteq U\Big)\Big)$$

a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space (120)

(THM) : $finiteSubcover \implies openRefinement$ (121)

$$locallyFinite(C,(M,\mathcal{O})) \iff \Big(openCover(C,(M,\mathcal{O}))\Big) \wedge \forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U}\Big(finiteSet(\{U_c \in C | U \cap U_c \neq \emptyset\},())\Big)$$

each point has a neighborhood that intersects with only finitely many sets in the cover (122)

$$paracompact((M,\mathcal{O}),()) \iff \forall_C\Bigg(openCover(C,(M,\mathcal{O})) \implies \exists_{\widetilde{C}}\Bigg(locallyFinite\Big(openRefinement\Big(\widetilde{C},(C,M,\mathcal{O})\Big),(M,\mathcal{O})\Big)\Bigg)\Bigg)$$

every open cover has a locally finite open refinement (123)

$$(\text{THM}) : \textit{metricTopologicalSpace} \Longrightarrow \textit{paracompact} \tag{124}$$

$$= = = = = = = = = = = = \textbf{ N O T = U P D A T E D } = = = = = = = = = = = = \tag{125}$$

$$\begin{aligned}
&partitionOfUnitySubjCover\big(\mathcal{F},(C,M,\mathcal{O})\big) \Longleftrightarrow \Big(locallyFinite\big(C,(M,\mathcal{O})\big)\Big) \wedge (f \in \mathcal{F}) \wedge \\
&\Bigg(continuous\bigg(f,\bigg(M,\mathcal{O},[0,1],subsetTopology\Big(\mathcal{O}|_{[0,1]},\big([0,1],\mathbb{R},standardTopology\big)\Big)\bigg)\bigg)\Bigg) \wedge \\
&\Big(\exists_{U_f \in C}\forall_{p\in M}\big(f(p)\neq 0 \Longrightarrow p \in U_f\big)\Big) \wedge \\
&\Big(\forall_{p\in M}\exists_{U\in\mathcal{O}|p\in U}\big((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\}\big)\Big) \wedge \\
&\Big(locallyFinite(C,M,\mathcal{O}) \Longrightarrow finiteSet\big((f_U)_n,()\big)\Big) \wedge \\
&\left(\forall_{p\in M}\exists_{U\in\mathcal{O}|p\in U}\left(\sum_{i=1}^{|(f_U)_n|}(f_U)_i(p) = 1\right)\right) \\
&\#\ \text{useful for defining integrals between overlapping neighborhoods}
\end{aligned} \tag{126}$$

$$\begin{aligned}
&T2Separate\big((M,\mathcal{O}),()\big) \Longrightarrow \Big(paracompact\big((M,\mathcal{O}),()\big)\Big) \Longleftrightarrow \\
&\forall_C\Big(openCover\big(C,(M,\mathcal{O})\big) \Longrightarrow partitionOfUnitySOTCover\big(\mathcal{F},(C,M,\mathcal{O})\big)\Big)
\end{aligned} \tag{127}$$

$$= = = = = = = = = = = = \textbf{ N O T = U P D A T E D } = = = = = = = = = = = = \tag{128}$$

1.11 Connectedness and path-connectedness

$$\begin{aligned}
&connected\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge \Big(\neg\exists_{A,B\in\mathcal{O}\setminus\emptyset}\big(A\cap B \neq \emptyset \wedge A \cup B = M\big)\Big) \\
&\#\ \text{if there is some covering of the space that does not intersect}
\end{aligned} \tag{129}$$

$$\begin{aligned}
&(\text{THM}) : \neg connected\Bigg(\bigg(\mathbb{R}\setminus\{0\}, subsetTopology\Big(\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}},\big(\mathbb{R},standardTopology,\mathbb{R}\setminus\{0\}\big)\Big)\bigg),()\Bigg) \\
&\Longleftarrow \Big(A = (-\infty,0) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge \Big(B = (0,\infty) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge \\
&\big(A \cap B = \emptyset\big) \wedge \big(A \cup B = \mathbb{R}\setminus\{0\}\big)
\end{aligned} \tag{130}$$

$$(\text{THM}) : connected\big((M,\mathcal{O}),()\big) \Longleftrightarrow \forall_{S\in\mathcal{O}}\Big(clopen\Big(S,(M,\mathcal{O}) \Longrightarrow \big(S = \emptyset \vee S = M\big)\Big)\Big) \tag{131}$$

$$\begin{aligned}
&pathConnected\big((M,\mathcal{O}),()\big) \Longleftrightarrow \Big(subsetTopology\Big(\mathcal{O}_{standard}|_{[0,1]},\big(\mathbb{R},standardTopology,[0,1]\big)\Big)\Big) \wedge \\
&\Bigg(\forall_{p,q\in M}\exists_\gamma\bigg(continuous\Big(\gamma,\big([0,1],\mathcal{O}_{standard}|_{[0,1]},M,\mathcal{O}\big)\Big) \wedge \gamma(0) = p \wedge \gamma(1) = q\bigg)\Bigg)
\end{aligned} \tag{132}$$

$$(\text{THM}) : pathConnected \Longrightarrow connected \tag{133}$$

1.12 Homotopic curve and the fundamental group

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{134}$$

$$\begin{aligned}&homotopic(\sim,(\gamma,\delta,M,\mathcal{O})) \iff (map(\gamma,([0,1],M))\wedge map(\delta,([0,1],M)))\wedge\\&(\gamma(0)=\delta(0)\wedge\gamma(1)=\delta(1))\wedge\\&(\exists_H\forall_{\lambda\in[0,1]}(continuous(H,(([0,1]\times[0,1],\mathcal{O}_{standard^2}|_{[0,1]\times[0,1]}),(M,\mathcal{O}))\wedge H(0,\lambda)=\gamma(\lambda)\wedge H(1,\lambda)=\delta(\lambda))))\\&\#\ H\text{ is a continuous deformation of one curve into another}\end{aligned} \tag{135}$$

$$homotopic(\sim)\implies equivalenceRelation(\sim) \tag{136}$$

$$loopSpace(\mathcal{L}_p,(p,M,\mathcal{O}))\iff\mathcal{L}_p=\{map(\gamma,([0,1],M))|continuous(\gamma)\wedge\gamma(0)=\gamma(1)\} \tag{137}$$

$$\begin{aligned}&concatination(\star,(p,\gamma,\delta))\iff(\gamma,\delta\in loopSpace(\mathcal{L}_p))\wedge\\&(\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda)=\begin{cases}\gamma(2\lambda) & 0\le\lambda<0.5\\ \delta(2\lambda-1) & 0.5\le\lambda\le1\end{cases}))\end{aligned} \tag{138}$$

$$\begin{aligned}&group((G,\bullet),())\iff(map(\bullet,(G\times G,G)))\wedge\\&(\forall_{a,b\in G}(a\bullet b\in G))\\&(\forall_{a,b,c\in G}((a\bullet b)\bullet C=a\bullet(b\bullet c)))\\&(\exists_{\boldsymbol{e}}\forall_{a\in G}(\boldsymbol{e}\bullet a=a=a\bullet\boldsymbol{e}))\wedge\\&(\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1}=\boldsymbol{e}=a^{-1}\bullet a))\\&\#\ \text{characterizes symmetry of a set structure}\end{aligned} \tag{139}$$

$$isomorphic(\cong,(X,\odot),(Y,\ominus)))\iff\exists_f\forall_{a,b\in X}(bijection(f,(X,Y))\wedge f(a\odot b)=f(a)\ominus f(b)) \tag{140}$$

$$\begin{aligned}&fundamentalGroup((\pi_{1,p},\bullet),(p,M,\mathcal{O}))\iff(\pi_{1,p}=\mathcal{L}_p/\sim)\wedge\\&(map(\bullet,(\pi_{1,p}\times\pi_{1,p},\pi_{1,p})))\wedge\\&(\forall_{A,B\in\pi_{1,p}}([A]\bullet[B]=[A\star B]))\wedge\\&(group((\pi_{1,p},\bullet),()))\\&\#\ \text{an equivalence class of all loops induced from the homotopic equivalence relation}\end{aligned} \tag{141}$$

$$fundamentalGroup_1\ncong fundamentalGroup_2\implies topologicalSpace_1\ncong topologicalSpace_2 \tag{142}$$

$$\textbf{there exists no known list of topological properties that can imply homeomorphisms} \tag{143}$$

$$\textbf{CONTINUE @ Lecture 6: manifolds} \tag{144}$$

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{145}$$

1.13 Measure theory

$$\begin{aligned}&sigmaAlgebra(\sigma,(M))\iff(M\neq\emptyset)\wedge(\sigma\subseteq\mathcal{P}(M))\wedge\\&(M\in\sigma)\wedge\Big(\forall_{A\in\sigma}(M\setminus A\in\sigma)\Big)\wedge\\&\bigg(\Big(A\subseteq\sigma\wedge\neg uncountablyInfinite(A,())\Big)\implies\cup A\in\sigma\bigg)\end{aligned}$$

$$\text{\# behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu \tag{146}$$

$$\mathit{measurableSpace}\big((M,\sigma),()\big) \Longleftrightarrow \mathit{sigmaAlgebra}\big(\sigma,(M)\big) \tag{147}$$

$$\mathit{measurableSet}\big(A,(M,\sigma)\big) \Longleftrightarrow \Big(\mathit{measurableSpace}\big((M,\sigma),()\big)\Big) \wedge (A \in \sigma) \tag{148}$$

$$\mathit{measure}\big(\mu,(M,\sigma)\big) \Longleftrightarrow \Big(\mathit{measurableSpace}\big((M,\sigma),()\big)\Big) \wedge \left(\mathit{map}\left(\mu,\left(\sigma,\left(\overline{\mathbb{R}}\right)_0^+\right)\right)\right) \wedge \Big(\mu(\emptyset)=0\Big) \wedge$$
$$\left(\Big((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}} \forall_{j\in\mathbb{N}\setminus\{i\}} \big(A_i \cap A_j = \emptyset\big)\Big) \Longrightarrow \mu\big(\cup_{i\in\mathbb{N}}(A_i)\big) = \sum_{i\in\mathbb{N}} \big(\mu(A_i)\big)\right)$$
$$\text{\# enforces meaningful concepts of measures such as precise additivity} \tag{149}$$

$$(\text{THM}) : \mathit{measure}\big(\mu,(M,\sigma)\big) \Longrightarrow$$
$$\Big(\forall_{A,B\in\sigma}\big(A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)\big)\Big) \wedge$$
$$\left((A)_{\mathbb{N}} \subseteq \sigma \Longrightarrow \mu\big(\cup_{i\in\mathbb{N}}(A_i)\big) \leq \sum_{i\in\mathbb{N}} \big(\mu(A_i)\big)\right) \wedge$$
$$\Big(\big((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}(B_i \subseteq B_{i+1}) \wedge B = \cup(B)_{\mathbb{N}}\big) \Longrightarrow \lim_{n\to\infty}\big(\mu(B_n)\big) = \mu(B)\Big) \wedge$$
$$\Big(\big((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}(C_{i+1} \subseteq C_i) \wedge C = \cap(C)_{\mathbb{N}}\big) \Longrightarrow \lim_{n\to\infty}\big(\mu(C_n)\big) = \mu(C)\Big)$$
$$\text{\# immediate implications of the measurable set } A \in \sigma \text{ axioms and the measure } \mu \text{ axioms} \tag{150}$$

$$\mathit{measureSpace}\big((M,\sigma,\mu),()\big) \Longleftrightarrow \mathit{measure}\big(\mu,(M,\sigma)\big) \tag{151}$$

$$\mathit{finiteMeasure}\big(\mu,(M,\sigma)\big) \Longleftrightarrow \Big(\mathit{measure}\big(\mu,(M,\sigma)\big)\Big) \wedge$$
$$\Big(\exists_{(A)_{\mathbb{N}} \subseteq \sigma}\Big(\cup\big((A)_{\mathbb{N}}\big) = M \wedge \forall_{n\in\mathbb{N}}\big(\mu(A_n) < \infty\big)\Big)\Big) \tag{152}$$

$$\mathit{generatedSigmaAlgebra}\big(\sigma(\zeta),(\zeta,M)\big) \Longleftrightarrow \Big(G = \{\sigma \subseteq \mathcal{P}(M) \,|\, \mathit{sigmaAlgebra}\big(\sigma,(M)\big)\}\Big) \wedge \big(\sigma(\zeta) = \cap G\big)$$
$$\text{\# smallest } \sigma\text{-algebra containing the generating set } \zeta \tag{153}$$

$$(\text{THM}) : \exists_{\zeta \subseteq M}\Big(\mathit{generatedSigmaAlgebra}\big(\sigma(\zeta),(\zeta,M)\big) = \mathit{sigmaAlgebra}\big(\sigma,(M)\big)\Big) \tag{154}$$

$$\mathit{borelSigmaAlgebra}\big(\sigma(\mathcal{O}),(M,\mathcal{O})\big) \Longleftrightarrow \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$\Big(\mathit{generatedSigmaAlgebra}\big(\sigma(\mathcal{O}),(\mathcal{O},M)\big)\Big)$$
$$\text{\# } \sigma\text{-algebra induced by a topology} \tag{155}$$

$$\mathit{standardSigma}\big(\sigma_s,()\big) \Longleftrightarrow \left(\mathit{borelSigmaAlgebra}\left(\sigma_s,\left(\mathbb{R}^d,\mathit{standardTopology}\right)\right)\right) \tag{156}$$

$$lebesgueMeasure(\lambda,()) \iff \left(measure\left(\lambda,\left(\mathbb{R}^d,standardSigma\right)\right)\right)\wedge \left(\lambda\left(\times_{i=1}^{d}([a_i,b_i))\right)=\sum_{i=1}^{d}\left(\sqrt[2]{(a_i-b_i)^2}\right)\right)$$

\# natural measure for $\mathbb{R}^d$ (157)

$$measurableMap(f,(M,\sigma_M,N,\sigma_N)) \iff \left(measurableSpace((M,\sigma_M),())\right)\wedge \left(measurableSpace((N,\sigma_N),())\right)\wedge\left(\forall_{B\in\sigma_N}\left(preimage(A,(B,f,M,N))\in\sigma_M\right)\right)$$

\# preimage of measurable sets are measurable (158)

$$pushForwardMeasure(f\star\lambda_M,(f,M,\sigma_M,\mu_M,N,\sigma_N)) \iff \left(measureSpace((M,\sigma_M,\mu_M),())\right)\wedge \left(measurableSpace((N,\sigma_N),())\right)\wedge\left(measurableMap(f,(M,\sigma_M,N,\sigma_N))\right)\wedge \left(\forall_{B\in N}\left(f\star\lambda_M(B)=\mu_M\left(preimage(A,(B,f,M,N))\right)\right)\right)\wedge\left(measure(f\star\lambda_M,(N,\sigma_N))\right)$$

\# natural construction of a measure based primarily on measurable map (159)

$$nullSet(A,(M,\sigma,\mu)) \iff \left(measureSpace((M,\sigma,\mu),())\right)\wedge(A\in\sigma)\wedge\left(\mu(A)=0\right) \tag{160}$$

$$almostEverywhere(p,(M,\sigma,\mu)) \iff \left(measureSpace((M,\sigma,\mu),())\right)\wedge\left(predicate(p,(M))\right)\wedge \left(\exists_{A\in\sigma}\left(nullSet(A,(M,\sigma,\mu))\implies\forall_{n\in M\setminus A}(p(n))\right)\right)$$

\# the predicate holds true for all points except the points in the null set (161)

1.14 Lebesque integration

$$simpleTopology(\mathcal{O}_{simple},()) \iff \mathcal{O}_{simple}=subsetTopology\left(\mathcal{O}|_{\mathbb{R}_0^+},\left(\mathbb{R},standardTopology,\mathbb{R}_0^+\right)\right) \tag{162}$$

$$simpleSigma(\sigma_{simple},()) \iff borelSigmaAlgebra\left(\sigma_{simple},\left(\mathbb{R}_0^+,simpleTopology\right)\right) \tag{163}$$

$$simpleFunction(s,(M,\sigma)) \iff \left(measurableMap\left(s,\left(M,\sigma,\mathbb{R}_0^+,simpleSigma\right)\right)\right)\wedge \left(finiteSet\left(image\left(B,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right)$$

\# if the map takes on finitely many values on $\mathbb{R}_0^+$ (164)

$$characteristicFunction(X_A,(A,M)) \iff (A\subseteq M)\wedge\left(map(X_A,(M,\mathbb{R}))\right)\wedge \left(\forall_{m\in M}\left(X_A(m)=\begin{cases}1 & m\in A\\ 0 & m\notin A\end{cases}\right)\right) \tag{165}$$

$$(\text{THM}) : \mathit{simpleFunction}(s,(M,\sigma_M)) \implies \left(\mathit{finiteSet}\left(\mathit{image}\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right) \wedge$$
$$\left(\mathit{characteristicFunction}(X_A,(A,M))\right) \wedge \left(\forall_{m\in M}\left(s(m) = \sum_{z\in Z}\left(z\cdot X_{\mathit{preimage}\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)}(m)\right)\right)\right) \tag{166}$$

$$\mathit{exStandardSigma}(\overline{\sigma_s},()) \iff \overline{\sigma_s} = \{A \subseteq \overline{\mathbb{R}} \mid A\cap R \in \mathit{standardSigma}\}$$
$$\text{\# ignores } \pm\infty \text{ to preserve the points in the domain of the measurable map} \tag{167}$$

$$\mathit{nonNegIntegrable}(f,(M,\sigma)) \iff \left(\mathit{measurableMap}\left(f,\left(M,\sigma,\overline{\mathbb{R}},\mathit{exStandardSigma}\right)\right)\right) \wedge$$
$$\left(\forall_{m\in M}\left(f(m) \geq 0\right)\right) \tag{168}$$

$$\mathit{nonNegIntegral}\left(\int_M (f d\mu),(f,M,\sigma,\mu)\right) \iff \left(\mathit{measureSpace}((M,\sigma,\mu),())\right) \wedge$$
$$\left(\mathit{measureSpace}\left(\left(\overline{\mathbb{R}},\mathit{exStandardSigma},\mathit{lebesgueMeasure}\right),()\right)\right) \wedge$$
$$\left(\mathit{nonNegIntegrable}(f,(M,\sigma))\right) \wedge \left(\int_M (f d\mu) = \mathit{sup}\left(\left\{\sum_{z\in Z}\left(z\cdot\mu\left(\mathit{preimage}\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)\right)\right) \mid\right.\right.$$
$$\left.\left.\forall_{m\in M}(s(m) \leq f(m)) \wedge \mathit{simpleFunction}(s,(M,\sigma)) \wedge \mathit{finiteSet}\left(\mathit{image}\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right\}\right)\right)$$
$$\text{\# lebesgue measure on } z \text{ reduces to } z \tag{169}$$

$$\mathit{explicitIntegral} \iff \int (f(x)\mu(dx)) = \int (f d\mu)$$
$$\text{\# alternative notation for lebesgue integrals} \tag{170}$$

$$(\text{THM}) : \mathit{nonNegIntegral}\left(\int (f d\mu),(f,M,\sigma,\mu)\right) \wedge \mathit{nonNegIntegral}\left(\int (g d\mu),(g,M,\sigma,\mu)\right) \implies$$
$$(\text{THM}) \text{ Markov inequality: } \left(\forall_{z\in\mathbb{R}_0^+}\left(\int (f d\mu) \geq z\cdot\mu\left(\mathit{preimage}\left(A,\left([z,\infty),f,M,\overline{\mathbb{R}}\right)\right)\right)\right)\right) \wedge$$
$$\left(\mathit{almostEverywhere}(f=g,(M,\sigma,\mu)) \implies \int (f d\mu) = \int (g d\mu)\right)$$
$$\left(\int (f d\mu) = 0 \implies \mathit{almostEverywhere}(f=0,(M,\sigma,\mu))\right) \wedge$$
$$\left(\int (f d\mu) \leq \infty \implies \mathit{almostEverywhere}(f<\infty,(M,\sigma,\mu))\right) \tag{171}$$

$$(\text{THM}) \text{ Mono. conv.: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \mathit{measurableMap}\left(f_n,\left(M,\sigma,\overline{R},\mathit{exStandardSigma}\right)\right) \wedge 0 \leq f_{n-1} \leq f_n\}\right) \wedge$$
$$\left(\mathit{map}\left(f,\left(M,\overline{\mathbb{R}}\right)\right)\right) \wedge \left(\forall_{m\in M}\left(f(m) = \mathit{sup}(f_n(m) | f_n \in (f)_{\mathbb{N}})\right)\right) \implies \left(\lim_{n\to\infty}\left(\int_M (f_n d\mu)\right) = \int_M (f d\mu)\right)$$
$$\text{\# lengths now depend on } M,\sigma \text{ and limits can be pulled in or out of an integral} \tag{172}$$

$$\text{(THM)}: \textit{nonNegIntegral}\left(\int (fd\mu),(f,M,\sigma,\mu)\right) \wedge \textit{nonNegIntegral}\left(\int (gd\mu),(g,M,\sigma,\mu)\right) \implies \left(\forall_{\alpha\in\mathbb{R}_0^+}\left(\int ((f+\alpha g)d\mu) = \int (fd\mu) + \alpha\int (gd\mu)\right)\right)$$
$$\text{\# integral acts linearly and commutes finite summations} \quad (173)$$

$$\text{(THM)}: \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \textit{measurableMap}\left(f_n,\left(M,\sigma,\overline{R},\textit{exStandardSigma}\right)\right) \wedge 0\le f_n\}\right) \implies \left(\int\left(\left(\sum_{n=1}^{\infty}(f_n)\right)d\mu\right) = \sum_{n=1}^{\infty}\left(\int (f_n d\mu)\right)\right)$$
$$\text{\# } \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n\to\infty}\sum_{i=1}^{n} f_n \text{ since } f_n \ge 0 \text{ and it commutes with integral from monotone conv.} \quad (174)$$

$$\textit{integrable}(f,(M,\sigma)) \iff \left(\textit{measurableMap}\left(f,\left(M,\sigma,\overline{\mathbb{R}},\textit{exStandardSigma}\right)\right)\right) \wedge \left(\forall_{m\in M}\left(f(m) = \textit{max}(f(m),0) - \textit{max}(0,-f(m))\right)\right) \wedge \left(\textit{measureSpace}(M,\sigma,\mu) \implies \left(\int\left(\textit{max}(f(m),0)d\mu\right) < \infty \wedge \int\left(\textit{max}(0,-f(m))d\mu\right) < \infty\right)\right)$$
$$\text{\# extra condition prevents the occurrence of the indeterminate } \infty-\infty \quad (175)$$

$$\textit{integral}\left(\int (fd\mu),(f,M,\sigma,\mu)\right) \iff \left(\textit{nonNegIntegral}\left(\int (f^+d\mu),(\textit{max}(f,0),M,\sigma,\mu)\right)\right) \wedge \left(\textit{nonNegIntegral}\left(\int (f^-d\mu),(\textit{max}(0,-f),M,\sigma,\mu)\right)\right) \wedge \left(\textit{integrable}(f,(M,\sigma))\right) \wedge \left(\int (fd\mu) = \int (f^+d\mu) - \int (f^-d\mu)\right)$$
$$\text{\# arbitrary integral in terms of nonnegative integrals} \quad (176)$$

$$\text{(THM)}: \left(\textit{map}(f,(M,\mathbb{C}))\right) \implies \left(\int (fd\mu) = \int (\textit{Re}(f)d\mu) - \int (\textit{Im}(f)d\mu)\right) \quad (177)$$

$$\text{(THM)}: \textit{integral}\left(\int (fd\mu),(f,M,\sigma,\mu)\right) \wedge \textit{integral}\left(\int (gd\mu),(g,M,\sigma,\mu)\right) \implies \left(\textit{almostEverywhere}(f\le g,(M,\sigma,\mu)) \implies \int (fd\mu) \le \int (gd\mu)\right) \wedge \left(\forall_{m\in M}(f(m),g(m),\alpha\in\mathbb{R}) \implies \int ((f+\alpha g)d\mu) = \int (fd\mu) + \alpha\int (gd\mu)\right) \quad (178)$$

$$\text{(THM) Dominant convergence: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \textit{measurableMap}\left(f_n,\left(M,\sigma,\overline{R},\textit{exStandardSigma}\right)\right)\}\right) \wedge \left(\textit{map}(f,(M,\overline{\mathbb{R}}))\right) \wedge \left(\textit{almostEverywhere}\left(f(m) = \lim_{n\to\infty}(f_n(m)),(M,\sigma,\mu)\right)\right) \wedge$$

$$\left(nonNegIntegral\left(\int(gd\mu),(g,M,\sigma,\mu)\right)\right)\wedge\left(\left|\int(gd\mu)\right|<\infty\right)\wedge\left(almostEverywhere(|f_n|\leq g,(M,\sigma,\mu))\right)$$

if all $f_n(m)$ are bounded by some integrable $|g(m)| \Longrightarrow$

then all $f_n(m)$ including f satisfy bounded and integrable properties

$$\left(\forall_{\phi\in\{f\}\cup(f)_{\mathbb{N}}}\left(integrable(\phi,(M,\sigma))\right)\right)\wedge\left(\lim_{n\to\infty}\left(\int(|f_n-f|d\mu)=0\right)\right)\wedge\left(\lim_{n\to\infty}\left(\int(f_nd\mu)\right)=\int(fd\mu)\right) \quad (179)$$

1.15 Function spaces

(180)

$$curLp(\mathcal{L}^p,(p,M,\sigma,\mu)) \iff (p\in\mathbb{R})\wedge(1\leq p<\infty)\wedge$$
$$(\mathcal{L}^p=\{map(f,(M,\mathbb{R}))\,|\,measurableMap(f,(M,\sigma,\mathbb{R},standardSigma))\wedge\int(|f|^pd\mu)<\infty\}) \quad (181)$$

$$vectorSpace(V,(+,\cdot)) \iff (map(+,(V\times V,V)))\wedge(map(\cdot,(\mathbb{R}\times V,V)))\wedge$$
$$(\forall_{v,w\in v}(v+w=w+v))\wedge$$
$$(\forall_{v,w,x\in v}((v+w)+x=v+(w+x)))\wedge$$
$$(\exists_{\boldsymbol{0}\in V}\forall_{v\in V}(v+\boldsymbol{0}=v))\wedge$$
$$(\forall_{v\in V}\exists_{-v\in V}(v+(-v)=\boldsymbol{0}))\wedge$$
$$(\forall_{a,b\in\mathbb{R}}\forall_{v\in V}(a(b\cdot v)=(ab)\cdot v))\wedge$$
$$(\exists_{1\in\mathbb{R}}\forall_{v\in V}(1\cdot v=v))\wedge$$
$$(\forall_{a,b\in\mathbb{R}}\forall_{v\in V}((a+b)\cdot v=a\cdot v+b\cdot v))\wedge$$
$$(\forall_{a\in\mathbb{R}}\forall_{v,w\in V}(a\cdot(v+w)=a\cdot v+a\cdot w))$$

behaves similar as vectors should i.e., additive, scalable, linear distributive (182)

$$vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu)) \iff (curLp(\mathcal{L}^p,(p,M,\sigma,\mu)))\wedge(\forall_{f,g\in\mathcal{L}^p}\forall_{m\in M}((f+g)(m)=f(m)+g(m)))\wedge$$
$$(\forall_{f\in\mathcal{L}^p}\forall_{s\in\mathbb{R}}\forall_{m\in M}((s\cdot f)(m)=(s)f(m)))\wedge(vectorSpace(\mathcal{L}^p,(+,\cdot))) \quad (183)$$

$$seminorm(\wr\wr\cdot\wr\wr,(+,\cdot,p,M,\sigma,\mu)) \iff (vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu)))\wedge(map(\wr\wr\cdot\wr\wr,(\mathcal{L}^p,\mathbb{R})))\wedge$$
$$(\forall_{f\in\mathcal{L}^p}(0\leq\wr\wr f\wr\wr=(\int(|f|^pd\mu))^{1/p}))\wedge$$
$$(\forall_{f\in\mathcal{L}^p}\forall_{s\in\mathbb{R}}(\wr\wr s\cdot f\wr\wr=(|s|)\wr\wr f\wr\wr))\wedge$$
$$(\forall_{f,g\in\mathcal{L}^p}(\wr\wr f+g\wr\wr=\wr\wr f\wr\wr+\wr\wr g\wr\wr)) \quad (184)$$

$$(\text{THM}):seminorm(\wr\wr\cdot\wr\wr,(+,\cdot,p,M,\sigma,\mu)) \Longrightarrow$$
$$(\forall_{f\in\mathcal{L}^p}(\wr\wr f\wr\wr=0 \Longrightarrow almostEverywhere(f=\boldsymbol{0},(M,\sigma,\mu))))$$

not an expected property from a norm (185)

$$norm(||\cdot||,(+,\cdot,p,M,\sigma,\mu)) \iff (seminorm(||\cdot||,(+,\cdot,p,M,\sigma,\mu)))\wedge(||f||=0 \Longrightarrow f=\boldsymbol{0})$$

some of the parameters can be omitted for defining norms on general vector spaces (186)

$$Lp(L^p,((+,\cdot,p,M,\sigma,\mu))) \iff (vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu)))\wedge$$
$$(L^p=quotientSet(\mathcal{L}^p/\sim,(\mathcal{L}^p,almostEverywhere(=,(??M,??\sigma,??\mu)))))$$

functions in L^p have abs pow p and finite integral (187)

$$14 mins from end \quad (188)$$

2 Statistics

2.1 Overview

$$\# \text{ DELET THIS! gg indians} \tag{189}$$

$$randomExperiment(X,(\Omega)) \iff \forall_{\omega \in \Omega}(outcome(\omega,(X))) \tag{190}$$

$$sampleSpace(\Omega,(X)) \iff \Omega = \{\omega | outcome(\omega,(X))\} \tag{191}$$

$$event(A,(\Omega)) \implies A \subseteq \Omega \; \# \text{ that is of interest} \tag{192}$$

$$eventOccured(A,(\omega,\Omega)) \iff \omega \in A, \Omega \wedge event(A,(\Omega)) \tag{193}$$

$$\begin{aligned} algebra(\mathcal{F}_0,(\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega))\wedge \\ (\Omega \in \mathcal{F}_0)\wedge \\ (\forall_{A \in \mathcal{F}_0}(A^C \in \mathcal{F}_0))\wedge \\ (\forall_{A,B \in \mathcal{F}_0}(A \cup B \in \mathcal{F}_0)) \\ \# \text{ but this is unable to capture some countable events} \end{aligned} \tag{194}$$

$$\begin{aligned} \sigma\text{-}algebra(\mathcal{F},(\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega))\wedge \\ (\Omega \in \mathcal{F})\wedge \\ (\forall_{A \in \mathcal{F}}(A^C \in \mathcal{F}))\wedge \\ (\forall_{F \subseteq \mathcal{F}}(\neg uncountablyInfinite(F,()) \implies \cup F \in \mathcal{F})) \end{aligned} \tag{195}$$

3 Statistical Learning Theory

3.1 Overview

$$\tag{196}$$

$$curve-fitting/explaining \neq prediction \tag{197}$$

$$ill-definedproblem + solutionspaceconstraints \implies well-definedproblem \tag{198}$$

$$x \; \# \text{ input ; } y \; \# \text{ output} \tag{199}$$

$$S_n = \{(x_1,y_1),\ldots,(x_n,y_n)\} \; \# \text{ training set} \tag{200}$$

$$f_S(x) \sim y \; \# \text{ solution} \tag{201}$$

$$each(x,y) \in p(x,y) \; \# \text{ training data } x,y \text{ is a sample from an unknown distribution } p \tag{202}$$

$$V(f(x),y) = d(f(x),y) \; \# \text{ loss function} \tag{203}$$

$$I[f] = \int_{X \times Y} V(f(x),y)p(x,y)dxdy \; \# \text{ expected error} \tag{204}$$

$$I_n[f]=\frac{1}{n}\sum_{i=1}^{n}V(f(x_i),y_i) \text{ \# empirical error} \tag{205}$$

$$probabilisticConvergence(X,()) \Longleftrightarrow \forall_{\epsilon>0}\lim_{n\to\infty}Pxn-x\leq\epsilon=0 \tag{206}$$

$$I-Ingeneralizationerror \tag{207}$$

$$well-posed:=exists,unique,stable;elseill-posed \tag{208}$$

3.2 Background maths

$$\text{\# refer to 2 functional analysis and 1 probability pdfs} \tag{209}$$

4 Machine Learning

4.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X,Y) \text{ \# dataset} \tag{210}$$

$$\textbf{learned parameters}=\textbf{parameters to be fixed by training with the dataset} \tag{211}$$

$$\textbf{hyperparameters}=\textbf{parameters that depends on a dataset} \tag{212}$$

$$\textbf{validation}=\textbf{partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition} \text{ \# useful for fixing hyperparameters} \tag{213}$$

$$\textbf{cross-validation}=\textbf{average accuracy of validation for different choices of testing partition} \tag{214}$$

$$\textbf{L1}=\textbf{scales linearly} \text{ ; } \textbf{L2}=\textbf{scales quadratically} \tag{215}$$

$$d=\textbf{distance}=\textbf{quantifies the the similarity between data points} \tag{216}$$

$$d_{L1}(A,B)=\sum_p|A_p-B_p| \text{ \# Manhattan distance} \tag{217}$$

$$d_{L2}(A,B)=\sqrt{\sum_p(A_p-B_p)^2} \text{ \# Euclidean distance} \tag{218}$$

$$\textbf{kNN classifier}=\textbf{classifier based on } k \textbf{ nearest data points} \tag{219}$$

$$s=\textbf{class score}=\textbf{quantifies bias towards a particular class} \tag{220}$$

$$s_{linear} = f_{c\times 1}(x_{n\times 1}, W_{c\times n}, b_{c\times 1}) = W_{c\times n} x_{n\times 1} + b_{c\times 1} \text{ \# linear score function} \tag{221}$$

$$l = \textbf{loss} = \textbf{quantifies the errors by the learned parameters} \tag{222}$$

$$l = \frac{1}{|c_i|} \sum_{c_i} l_i \text{ \# average loss for all classes} \tag{223}$$

$$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \text{ \# SVM hinge class loss function:}$$
$$\text{\# ignores incorrect classes with lower scores including a non-zero margin} \tag{224}$$

$$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$
$$\text{\# lower scores correspond to lower exponentiated-normalized probabilities} \tag{225}$$

$$R = \textbf{regularization} = \textbf{optimizes the choice of learned parameters to minimize test error} \tag{226}$$

$$\lambda \text{ \# regularization strength hyperparameter} \tag{227}$$

$$R_{L1}(W) = \sum_{W_i} |W_i| \text{ \# L1 regularization} \tag{228}$$

$$R_{L2}(W) = \sum_{W_i} {W_i}^2 \text{ \# L2 regularization} \tag{229}$$

$$L' = L + \lambda R(W) \text{ \# weight regularization} \tag{230}$$

$$\nabla_W L = \overrightarrow{\frac{\partial}{\partial W_i}} L = \textbf{loss gradient w.r.t. weights} \tag{231}$$

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L} \text{ \# loss gradient w.r.t. input weight in terms of external and local gradients} \tag{232}$$

$$s = \textbf{forward API} \; ; \; \frac{\partial L_L}{\partial W_I} = \textbf{backward API} \tag{233}$$

$$W_{t+1} = W_t - \nabla_{W_t} L \text{ \# weight update loss minimization} \tag{234}$$

$$\textbf{TODO:Research on Activation functions, Weight Initialization, Batch Normalization} \tag{235}$$

$$review5meanvardiscussion/hyperparameteroptimization/babysittinglearning \tag{236}$$

TODO loss L or l ??