

Next-Next-Gen Notes

Object-Oriented Maths

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Format: $characteristic((subjects), (dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$

Note: All weaker objects automatically induces notions inherited from stronger objects.

TODO assign free variables as parameters

TODO define || abs cross-product and other missing refs

TODO distinguish new condition vs implied proposition

TODO link thms?

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s, (RegEx)) \iff well-formedString(s, ()) \quad (1)$$

$$proposition((p, t), ()) \iff \left(statement(p, ()) \wedge \begin{aligned} &(t = eval(p)) \wedge \\ &(t = true \vee t = false) \end{aligned} \right) \quad (2)$$

$$operator\left(o, \left((p)_{n \in \mathbb{N}}\right)\right) \iff proposition\left(o\left((p)_{n \in \mathbb{N}}\right), ()\right) \quad (3)$$

$$operator(\neg, (p_1)) \iff \left(proposition((p_1, true), ()) \implies ((\neg p_1, false), ()) \right) \wedge \left(proposition((p_1, false), ()) \implies ((\neg p_1, true), ()) \right) \\ \# \text{ an operator takes in propositions and returns a proposition} \quad (4)$$

$$operator(\neg) \iff \mathbf{NOT} ; operator(\vee) \iff \mathbf{OR} ; operator(\wedge) \iff \mathbf{AND} ; operator(\vee) \iff \mathbf{XOR} \\ operator(\implies) \iff \mathbf{IF} ; operator(\iff) \iff \mathbf{OIF} ; operator(\iff) \iff \mathbf{IFF} \quad (5)$$

$$proposition((false \implies true), true, ()) \wedge proposition((false \implies false), true, ()) \\ \# \text{ truths based on a false premise is not false; ex falso quodlibet principle} \quad (6)$$

$$(\text{THM}) : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \quad (7)$$

$$predicate(P, (V)) \iff \forall_{v \in V} \left(proposition\left(P(v), t, ()\right) \right) \quad (8)$$

$$0thOrderLogic(P, ()) \iff proposition((P, t), ()) \\ \# \text{ individual proposition} \quad (9)$$

$$1stOrderLogic(P, (V)) \iff \left(\forall_{v \in V} \left(0thOrderLogic(v, ()) \right) \right) \wedge \left(\forall_{v \in V} \left(proposition \left((P(v), t), () \right) \right) \right)$$

propositions defined over a set of the lower order logical statements (10)

$$quantifier(q, (p, V)) \iff \left(predicate(p, (V)) \right) \wedge \left(proposition \left((q(p), t), () \right) \right)$$

a quantifier takes in a predicate and returns a proposition (11)

$$quantifier(\forall, (p, V)) \iff proposition \left(\left(\wedge_{v \in V} (p(v)), t \right), () \right)$$

universal quantifier (12)

$$quantifier(\exists, (p, V)) \iff proposition \left(\left(\vee_{v \in V} (p(v)), t \right), () \right)$$

existential quantifier (13)

$$quantifier(\exists!, (p, V)) \iff \exists_{x \in V} \left(P(x) \wedge \neg \left(\exists_{y \in V \setminus \{x\}} (P(y)) \right) \right)$$

uniqueness quantifier (14)

$$(THM) : \forall_x p(x) \iff \neg \exists_x \neg p(x)$$

De Morgan's law (15)

$$(THM) : \forall_x \exists_y p(x, y) = \forall_x \neg \forall_y \neg p(x, y) \neq \exists_y \forall_x p(x, y) = \neg \forall_y \neg (\forall_x p(x, y)) = \neg \forall_y \exists_x \neg p(x, y)$$

different quantifiers are not interchangeable (16)

$$===== NOT = UPDATED =====$$

(17)

$$\text{proof} = \text{truths derived from a finite number of axioms and deductions}$$

(18)

$$\text{elementary arithmetics} = \text{system with substitutions, and some notion of addition, multiplication, and prime numbers for encoding metamathematics}$$

(19)

$$\text{Gödel theorem} \implies \text{axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions}$$

(20)

$$sequenceSet((A)_{\mathbb{N}}, (A)) \iff (Amapinputn)((A)_{\mathbb{N}} = \{A(1), A(2), A(3), \dots\})$$

(21)

$$\text{TODO: define union, intersection, complement, etc.}$$

(22)

$$===== NOT = UPDATED =====$$

(23)

1.1 Axiomatic Set Theory

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (24)$$

$$\text{ZFC set theory} = \text{usual form of axiomatic set theory} \quad (25)$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \quad (26)$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \quad (27)$$

$$\in \text{ basis} \implies \{x, y\} = \{y, x\} \wedge \{x\} = \{x, x\} \quad (28)$$

$$\in \text{ and sets works following the 9 ZFC axioms:} \quad (29)$$

$$\forall_x \forall_y (x \in y \vee \neg(x \in y)) \# \text{ E: } \in \text{ is only a proposition on sets} \quad (30)$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \# \text{ E: existence of empty set} \quad (31)$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \# \text{ C: pair set construction} \quad (32)$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \# \text{ C: union set construction} \quad (33)$$

$$x = \{\{a\}, \{b\}\} \# \text{ from the pair set axiom} \quad (34)$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \quad (35)$$

$$\forall_x \exists!_y R(x, y) \# \text{ functional relation } R \quad (36)$$

$$\begin{aligned} \exists_i \forall_x \exists!_y R(x, y) \implies y \in i \# \text{ C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \mid P(y)\} \# \text{ Restricted Comprehension } \not\Rightarrow \{y \mid P(y)\} \# \text{ Universal Comprehension} \end{aligned} \quad (37)$$

$$\forall_{x \in m} P(x) = \forall_x (x \in m \implies P(x)) \# \text{ ignores out of scope } \neq \forall_x (x \in m \wedge P(x)) \# \text{ restricts entirety} \quad (38)$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} (n \subseteq m \implies n \subseteq \mathcal{P}(m)) \# \text{ C: existence of power set} \quad (39)$$

$$\exists_I (\emptyset \in I \wedge \forall_{x \in I} (\{x\} \in I)) \# \text{ I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}; I \cong \mathbb{N} \implies \mathbb{N} \text{ is a set} \quad (40)$$

$$\forall_x ((\emptyset \notin x \wedge x \cap x' = \emptyset) \implies \exists_y (\text{set of each } \mathbf{e} \in x)) \# \text{ C: axiom of choice} \quad (41)$$

$$\forall_x x \neq \emptyset \implies x \notin x \# \text{ F: axiom of foundation covers further paradoxes} \quad (42)$$

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (43)$$

1.2 Classification of sets

$$\begin{aligned} \text{space}((\text{set}, \text{structure}), ()) &\iff \text{structure}(\text{set}) \\ \# \text{ a space a set equipped with some structure} \\ \# \text{ various spaces can be studied through structure preserving maps between those spaces} \end{aligned} \quad (44)$$

$$\begin{aligned} \text{map}(\phi, (A, B)) &\iff \left(\forall_{a \in A} \exists!_{b \in B} (\phi(a, b)) \right) \vee \\ &\quad \left(\forall_{a \in A} \exists!_{b \in B} (b = \phi(a)) \right) \\ \# \text{ maps elements of a set to elements of another set} \end{aligned} \quad (45)$$

$$\text{domain}(A, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (46)$$

$$\text{codomain}(B, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (47)$$

$$\begin{aligned} \text{image}(B, (A, q, M, N)) &\iff \left(\text{map}(q, (M, N)) \wedge A \subseteq M \right) \wedge \\ &\quad \left(B = \{n \in N \mid \exists_{a \in A} (q(a) = n)\} \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \text{preimage}(A, (B, q, M, N)) &\iff \left(\text{map}(q, (M, N)) \wedge B \subseteq N \right) \wedge \\ &\quad \left(A = \{m \in M \mid \exists_{b \in B} (b = q(m))\} \right) \end{aligned} \quad (49)$$

$$\begin{aligned} \text{injection}(q, (M, N)) &\iff \left(\text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{u, v \in M} (q(u) = q(v) \implies u = v) \\ \# \text{ every } m \text{ has at most 1 image} \end{aligned} \quad (50)$$

$$\begin{aligned} \text{surjection}(q, (M, N)) &\iff \left(\text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{n \in N} \exists_{m \in M} (n = q(m)) \\ \# \text{ every } n \text{ has at least 1 preimage} \end{aligned} \quad (51)$$

$$\begin{aligned} \text{bijection}(q, (M, N)) &\iff \left(\text{injection}(q, (M, N)) \right) \wedge \\ &\quad \left(\text{surjection}(q, (M, N)) \right) \\ \# \text{ every unique } m \text{ corresponds to a unique } n \end{aligned} \quad (52)$$

$$\text{isomorphicSets}((A, B), ()) \iff \exists_{\phi} (\text{bijection}(\phi, (A, B))) \quad (53)$$

$$\text{infiniteSet}(S, ()) \iff \exists_{T \subseteq S} (\text{isomorphicSets}((T, S), ())) \quad (54)$$

$$\text{finiteSet}(S, ()) \iff \left(\neg \text{infiniteSet}(S, ()) \right) \vee (|S| \in \mathbb{N}) \quad (55)$$

$$\text{countablyInfinite}(S, ()) \iff \left(\text{infiniteSet}(S, ()) \right) \wedge \left(\text{isomorphicSets}((S, \mathbb{N}), ())) \quad (56)$$

$$\text{uncountablyInfinite}(S, ()) \iff (\text{infiniteSet}(S, ())) \wedge (\neg \text{isomorphicSets}((S, \mathbb{N}), ())) \quad (57)$$

$$\begin{aligned} \text{inverseMap}(q^{-1}, (q, M, N)) &\iff (\text{bijection}(q, (M, N))) \wedge \\ &\quad (\text{map}(q^{-1}, (N, M))) \wedge \\ &\quad \left(\forall_{n \in N} \exists!_{m \in M} (q(m) = n \implies q^{-1}(n) = m) \right) \end{aligned} \quad (58)$$

$$\begin{aligned} \text{mapComposition}(\phi \circ \psi, (\phi, \psi, A, B, C)) &\iff \text{map}(\psi, (A, B)) \wedge \text{map}(\phi, (B, C)) \wedge \\ &\quad \forall_{a \in A} (\phi \circ \psi(a) = \phi(\psi(a))) \end{aligned} \quad (59)$$

$$\begin{aligned} \text{equivalenceRelation}(\sim (\$1, \$2), (M)) &\iff (\forall_{m \in M} (m \sim m)) \wedge \\ &\quad (\forall_{m, n \in M} (m \sim n \implies n \sim m)) \wedge \\ &\quad (\forall_{m, n, p \in M} (m \sim n \wedge n \sim p \implies m \sim p)) \\ &\quad \# \text{ behaves as equivalences should} \end{aligned} \quad (60)$$

$$\begin{aligned} \text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) &\iff [m]_{\sim} = \{n \in M \mid n \sim m\} \\ &\quad \# \text{ set of elements satisfying the equivalence relation with } m \end{aligned} \quad (61)$$

$$\begin{aligned} (\text{THM}) : a \in [m]_{\sim} &\implies [a]_{\sim} = [m]_{\sim} ; [m]_{\sim} = [n]_{\sim} \vee [m]_{\sim} \cap [n]_{\sim} = \emptyset \\ &\quad \# \text{ equivalence class properties} \end{aligned} \quad (62)$$

$$\begin{aligned} \text{quotientSet}(M/\sim, (M, \sim)) &\iff M/\sim = \{\text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) \in \mathcal{P}(M) \mid m \in M\} \\ &\quad \# \text{ set of all equivalence classes} \end{aligned} \quad (63)$$

$$\begin{aligned} (\text{THM}) : (M, \sim, +) &\implies (\text{quotientSet}(M/\sim, (M, \sim)), +_{\sim}) \iff \forall_{[r], [s] \in M/\sim} \forall_{a \in [r]} \forall_{b \in [s]} ([r] +_{\sim} [s] = [a + b]) \\ &\quad \# \text{ a quotient set can inherit the operations on the original set if it is well-defined} \end{aligned} \quad (64)$$

$$\begin{aligned} (\text{THM}) : \text{axiom of choice} &\implies \forall_{[m]_{\sim} \in M/\sim} \exists_r (r \in [m]_{\sim}) \\ &\quad \# \text{ well-defined maps may be defined in terms of chosen representative elements } r \end{aligned} \quad (65)$$

1.3 Construction of number sets

$$\text{===== N O T = U P D A T E D =====} \quad (66)$$

$$\text{axiom of infinity} \implies \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\} \cong \mathbb{N} \quad (67)$$

$$\mathbb{N}^* = \mathbb{N} \setminus \{0\} \quad (68)$$

$$\text{addition} = \text{successor map: } \mathbb{N} \rightarrow \mathbb{N} = S(n) = \{n\} \# \text{ adds a layer of brackets} \quad (69)$$

$$\text{subtraction} = \text{predecessor map: } \mathbb{N}^* \rightarrow \mathbb{N} = P(n) = m \mid m \in n \# \text{ removes a layer of brackets} \quad (70)$$

$$S^0 = id ; n \in \mathbb{N}^* \implies S^n = S \circ S^{P(n)} \quad (71)$$

$$\mathbf{addition} = + : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} = +(m, n) = m + n = S^n(m) \quad (72)$$

$$S^x = id = S^0 \implies x = \mathbf{additive\ identity} = 0 \quad (73)$$

$$S^n(x) = 0 \implies x = \mathbf{additive\ inverse} \notin \mathbb{N} \# \text{ git gud smh -_-} \quad (74)$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim, \text{ s.t.: } (m, n) \sim (p, q) \iff m + q = p + n \# \text{ span } \mathbb{Z} \text{ using differences then group equal differences} \quad (75)$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \rightarrow [(n, 0)] \# \mathbb{N} \text{ embedded in } \mathbb{Z} \quad (76)$$

$$+_Z = [(m +_{\mathbb{N}} p, n +_{\mathbb{N}} q)] \# \text{ well-defined and consistent} \quad (77)$$

$$\mathbf{multiplication} \dots M^x = id \implies x = \mathbf{multiplicative\ identity} = 1 \dots \mathbf{multiplicative\ inverse} \notin \mathbb{N} \quad (78)$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^*) / \sim, \text{ s.t.: } (x, y) \sim (u, v) \iff x \cdot v = u \cdot y \quad (79)$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \rightarrow [(q, 1)] ; \dots \{x \mid x^2 = 2\} \notin \mathbb{Q} \quad (80)$$

$$\mathbb{R} = \mathbf{almost\ homomorphisms\ on\ } \mathbb{Z} / \sim \# \text{ http://blog.sigfpe.com/2006/05/defining-reals.html} \quad (81)$$

$$\text{===== N O T = U P D A T E D =====} \quad (82)$$

1.4 Topology

$$\text{topology}(\mathcal{O}, (M)) \iff (\mathcal{O} \subseteq \mathcal{P}(M)) \wedge (\emptyset, M \in \mathcal{O}) \wedge$$

$$\left((F \in \mathcal{O} \wedge |F| < |\mathbb{N}|) \implies \cap F \in \mathcal{O} \right) \wedge (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O})$$

topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.

arbitrary unions of open sets always result in an open set

open sets do not contain their boundaries and infinite intersections of open sets may approach and

induce boundaries resulting in a closed set (83)

$$\text{topologicalSpace}((M, \mathcal{O}), ()) \iff \text{topology}(\mathcal{O}, (M)) \quad (84)$$

$$\text{open}(S, (M, \mathcal{O})) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge (S \subseteq M) \wedge (S \in \mathcal{O})$$

an open set do not contains its own boundaries (85)

$$\begin{aligned} \text{closed}(S, (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\quad (S \subseteq M) \wedge (S \in \mathcal{P}(M) \setminus \mathcal{O}) \\ \# \text{ a closed set contains the boundaries an open set} \end{aligned} \quad (86)$$

$$\text{clopen}(S, (M, \mathcal{O})) \iff \left(\text{closed}(S, (M, \mathcal{O})) \right) \wedge \left(\text{open}(S, (M, \mathcal{O})) \right) \quad (87)$$

$$\begin{aligned} \text{neighborhood}(U, (a, \mathcal{O})) &\iff (a \in U \in \mathcal{O}) \\ \# \text{ another name for open set containing } a \end{aligned} \quad (88)$$

$$\begin{aligned} M = \{a, b, c, d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} \implies \\ \left(\text{open}(X, (M, \mathcal{O})) \iff X = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} \right) \wedge \\ \left(\text{closed}(Y, (M, \mathcal{O})) \iff Y = \{\emptyset, \{a, b, d\}, \{c, d\}, \{a, b\}, \{d\}, M\} \right) \wedge \\ \left(\text{clopen}(Z, (M, \mathcal{O})) \iff Z = \{\emptyset, \{a, b\}, \{c, d\}, M\} \right) \end{aligned} \quad (89)$$

$$\text{chaoticTopology}(M) = \{0, M\} ; \text{discreteTopology} = \mathcal{P}(M) \quad (90)$$

1.5 Induced topology

$$\begin{aligned} \text{metric}(d(\$1, \$2), (M)) &\iff \left(\text{map} \left(d, \left(M \times M, \mathbb{R}_0^+ \right) \right) \right) \\ &\quad \left(\forall_{x, y \in M} (d(x, y) = d(y, x)) \right) \wedge \\ &\quad \left(\forall_{x, y \in M} (d(x, y) = 0 \iff x = y) \right) \wedge \\ &\quad \left(\forall_{x, y, z} \left(d(x, z) \leq d(x, y) + d(y, z) \right) \right) \\ \# \text{ behaves as distances should} \end{aligned} \quad (91)$$

$$\text{metricSpace}((M, d), ()) \iff \text{metric}(d, (M)) \quad (92)$$

$$\text{openBall}(B, (r, p, M, d)) \iff \left(\text{metricSpace}((M, d), ()) \right) \wedge (r \in \mathbb{R}^+, p \in M) \wedge (B = \{q \in M \mid d(p, q) < r\}) \quad (93)$$

$$\begin{aligned} \text{metricTopology}(\mathcal{O}, (M, d)) &\iff \left(\text{metricSpace}((M, d), ()) \right) \wedge \\ &\quad \left(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (\text{openBall}(B, (r, p, M, d)) \wedge B \subseteq U)\} \right) \\ \# \text{ every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \end{aligned} \quad (94)$$

$$\text{metricTopologicalSpace}((M, \mathcal{O}, d), ()) \iff \text{metricTopology}(\mathcal{O}, (M, d)) \quad (95)$$

$$\begin{aligned} \text{limitPoint}(p, (S, M, d)) &\iff (S \subseteq M) \wedge \forall_{r \in \mathbb{R}^+} \left(\text{openBall}(B, (r, p, M, d)) \cap S \neq \emptyset \right) \\ \# \text{ every open ball centered at } p \text{ contains some intersection with } S \end{aligned} \quad (96)$$

$$\text{interiorPoint}(p, (S, M, d)) \iff (S \subseteq M) \wedge \left(\exists_{r \in \mathbb{R}^+} \left(\text{openBall}(B, (r, p, M, d)) \subseteq S \right) \right)$$

$$\# \text{ there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (97)$$

$$\text{closure}(\bar{S}, (S, M, d)) \iff \bar{S} = S \cup \{\text{limitPoint}(p, (S, M, d)) \mid p \in M\} \quad (98)$$

$$\text{dense}(S, (M, d)) \iff (S \subseteq M) \wedge \left(\forall_{p \in M} \left(p \in \text{closure}(\bar{S}, (S, M, d)) \right) \right) \\ \# \text{ every of point in } M \text{ is a point or a limit point of } S \quad (99)$$

$$\text{eucD}(d, (n)) \iff (\forall_{i \in \mathbb{N} \wedge i \leq n} (x_i \in \mathbb{R})) \wedge \left(d = \sqrt[2]{\sum_{i=1}^n x_i^2} \right) \quad (100)$$

$$\text{metricTopology} \left(\text{euclideanTopology}, \left(\mathbb{R}^n, \text{eucD}(d, (n)) \right) \right) \\ \text{===== NOT UPDATED =====} \\ \mathbf{L1:} \forall_{p \in U = \emptyset} (\dots) \implies \forall_p \left((p \in \emptyset) \implies \dots \right) \implies \forall_p (\mathbf{False} \implies \dots) \implies \emptyset \in \mathcal{O}_{\text{euclidean}} \\ \mathbf{L2:} \forall_{p \in \mathbb{R}^n} B(r, p, \mathbb{R}^n, d) \subseteq \mathbb{R}^n \implies M \in \mathcal{O}_{\text{euclidean}} \\ \mathbf{L4:} C \subseteq \mathcal{O}_{\text{euclidean}} \implies \forall_{U \in C} \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (B_r(p) \subseteq U \subseteq \cup C) \implies \cup C \in \mathcal{O}_{\text{euclidean}} \\ \mathbf{L3:} U, V \in \mathcal{O}_{\text{euclidean}} \implies p \in U \cap V \implies p \in U \wedge p \in V \implies \\ \exists_{r \in \mathbb{R}^+} B(r, p, \mathbb{R}^n, d) \wedge \exists_{s \in \mathbb{R}^+} B(s, p, \mathbb{R}^n, d) \implies \\ B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \subseteq U \wedge B(\min(r, s), q, \mathbb{R}^n, d) \subseteq V \implies \\ B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \in U \cap V \implies U \cap V \in \mathcal{O}_{\text{euclidean}} \\ \# \text{ natural topology for } \mathbb{R}^d \\ \# \text{ could fail on infinite sets since } \min \text{ could approach } 0 \\ \text{===== NOT UPDATED =====} \quad (101)$$

$$\text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \iff \text{topology}(\mathcal{O}, (M)) \wedge (N \subseteq M) \wedge (\mathcal{O}|_N = \{U \cap N \mid U \in \mathcal{O}\}) \\ \# \text{ crops open sets outside } N \quad (102)$$

$$(\text{THM}) : \text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \wedge \text{topology}(\mathcal{O}|_N, (N)) \iff \\ \text{===== NOT UPDATED =====} \\ \mathbf{L1:} \emptyset \in \mathcal{O} \implies U = \emptyset \implies \emptyset \cap N = \emptyset \implies \emptyset \in \mathcal{O}|_N \\ \mathbf{L2:} M \in \mathcal{O} \implies U = M \implies M \cap N = N \implies N \in \mathcal{O}|_N \\ \mathbf{L3:} S, T \in \mathcal{O}|_N \implies \exists_{U \in \mathcal{O}} (S = U \cap N) \wedge \exists_{V \in \mathcal{O}} (T = V \cap N) \implies S \cap T = (U \cap N) \cap (V \cap N) \\ = (U \cap V) \cap N \wedge U \cap V \in \mathcal{O} \implies S \cap T \in \mathcal{O}|_N \\ \mathbf{L4:} \text{TODO: EXERCISE} \\ \text{===== NOT UPDATED =====} \quad (103)$$

$$\text{productTopology} \left(\mathcal{O}_{A \times B}, ((A, \mathcal{O}_A), (B, \mathcal{O}_B)) \right) \iff \left(\text{topology}(\mathcal{O}_A, (A)) \right) \wedge \left(\text{topology}(\mathcal{O}_B, (B)) \right) \wedge \\ (\mathcal{O}_{A \times B} = \{(a, b) \in A \times B \mid \exists_S (a \in S \in \mathcal{O}_A) \exists_T (b \in T \in \mathcal{O}_B)\}) \\ \# \text{ open in cross iff open in each} \quad (104)$$

1.6 Convergence

$$\text{sequence}(q, (M)) \iff \text{map}(q, (\mathbb{N}, M)) \quad (105)$$

$$\begin{aligned} \text{sequenceConvergesTo}((q, a), (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\text{sequence}(q, (M)) \right) \wedge (a \in M) \wedge \left(\forall U \in \mathcal{O} | a \in U \exists N \in \mathbb{N} \forall n > N (q(n) \in U) \right) \\ &\# \text{ each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \end{aligned} \quad (106)$$

(THM) : convergence generalizes to: the sequence $q: \mathbb{N} \rightarrow \mathbb{R}^d$ converges against $a \in \mathbb{R}^d$ in \mathcal{O}_S if:

$$\forall r > 0 \exists N \in \mathbb{N} \forall n > N (||q(n) - a|| < r) \# \text{ distance based convergence} \quad (107)$$

1.7 Continuity

$$\begin{aligned} \text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}_M), ()) \right) \wedge \\ &\left(\text{topologicalSpace}((N, \mathcal{O}_N), ()) \right) \wedge \left(\forall V \in \mathcal{O}_N \left(\text{preimage}(A, (V, \phi, M, N)) \in \mathcal{O}_M \right) \right) \\ &\# \text{ preimage of open sets are open} \end{aligned} \quad (108)$$

$$\begin{aligned} \text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left(\text{inverseMap}(\phi^{-1}, (\phi, M, N)) \right) \\ &\left(\text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \wedge \left(\text{continuous}(\phi^{-1}, (N, \mathcal{O}_N, M, \mathcal{O}_M)) \right) \\ &\# \text{ structure preserving maps in topology, ability to share topological properties} \end{aligned} \quad (109)$$

$$\begin{aligned} \text{isomorphicTopologicalSpace}((M, \mathcal{O}_M), (N, \mathcal{O}_N), ()) &\iff \\ &\exists \phi \left(\text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \end{aligned} \quad (110)$$

1.8 Separation

$$\begin{aligned} T0Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U \in \mathcal{O} \left((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U) \right) \right) \\ &\# \text{ each pair of points has a neighborhood s.t. one is inside and the other is outside} \end{aligned} \quad (111)$$

$$\begin{aligned} T1Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V \left((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V) \right) \right) \\ &\# \text{ every point has a neighborhood that does not contain another point} \end{aligned} \quad (112)$$

$$\begin{aligned} T2Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V (U \cap V = \emptyset) \right) \\ &\# \text{ every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \end{aligned} \quad (113)$$

$$(THM) : T2Separate \implies T1Separate \implies T0Separate \quad (114)$$

1.9 Compactness

$$\begin{aligned} openCover(C, (M, \mathcal{O})) &\iff \left(topologicalSpace((M, \mathcal{O}), ()) \right) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M) \\ &\# \text{ collection of open sets whose elements cover the entire space} \end{aligned} \quad (115)$$

$$\begin{aligned} finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) &\iff (\tilde{C} \subseteq C) \wedge (openCover(C, (M, \mathcal{O}))) \wedge \\ &\left(openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge (finiteSet(\tilde{C}, ())) \\ &\# \text{ finite subset of a cover that is also a cover} \end{aligned} \quad (116)$$

$$\begin{aligned} compact((M, \mathcal{O}), ()) &\iff \left(topologicalSpace((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall C \subseteq \mathcal{O} \left(openCover(C, (M, \mathcal{O})) \implies \exists \tilde{C} \subseteq C \left(finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) \right) \right) \right) \\ &\# \text{ every covering of the space is represented by a finite number of nhbhd} \end{aligned} \quad (117)$$

$$\begin{aligned} compactSubset(N, (M, \mathcal{O})) &\iff \left(compact((M, \mathcal{O}), ()) \right) \wedge \\ &\left(subsetTopology(\mathcal{O}|_N, (M, \mathcal{O}, N)) \right) \wedge \left(compact((N, \mathcal{O}|_N), ()) \right) \end{aligned} \quad (118)$$

$$\begin{aligned} bounded(N, (M, d)) &\iff \left(metricSpace((M, d), ()) \right) \wedge (N \subseteq M) \wedge \\ &\left(\exists r \in \mathbb{R}^+ \forall p, q \in N (d(p, q) < r) \right) \end{aligned} \quad (119)$$

$$\begin{aligned} &(THM) \text{ Heine-Borel thm.: } metricTopologicalSpace((M, \mathcal{O}_d, d), ()) \implies \\ &\forall S \subseteq M \left(\left(closed(S, (M, \mathcal{O}_d)) \wedge bounded(S, (M, \mathcal{O}_d)) \right) \iff compactSubset(S, (M, \mathcal{O}_d)) \right) \\ &\# \text{ when metric topologies are involved, compactness is equivalent to being closed and bounded} \end{aligned} \quad (120)$$

1.10 Paracompactness

$$\begin{aligned} openRefinement(\tilde{C}, (C, M, \mathcal{O})) &\iff \left(openCover(C, (M, \mathcal{O})) \right) \wedge \left(openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge \\ &\left(\forall \tilde{U} \in \tilde{C} \exists U \in C (\tilde{U} \subseteq U) \right) \\ &\# \text{ a refined cover can be constructed by removing the excess nhbhd} \end{aligned} \quad (121)$$

$$(THM) : finiteSubcover \implies openRefinement \quad (122)$$

$$\begin{aligned} locallyFinite(C, (M, \mathcal{O})) &\iff \left(openCover(C, (M, \mathcal{O})) \right) \wedge \\ &\forall p \in M \exists U \in \mathcal{O} | p \in U \left(finiteSet(\{U_c \in C | U \cap U_c \neq \emptyset\}, ()) \right) \\ &\# \text{ each point has a neighborhood that intersects with only finitely many sets in the cover} \end{aligned} \quad (123)$$

$$\begin{aligned} & \text{paracompact}((M, \mathcal{O}), ()) \iff \\ \forall_C \left(\text{openCover}(C, (M, \mathcal{O})) \implies \exists_{\tilde{C}} \left(\text{locallyFinite} \left(\text{openRefinement}(\tilde{C}, (C, M, \mathcal{O})), (M, \mathcal{O}) \right) \right) \right) \\ & \# \text{ every open cover has a locally finite open refinement} \end{aligned} \quad (124)$$

$$(\text{THM}) : \text{metricTopologicalSpace} \implies \text{paracompact} \quad (125)$$

$$\text{===== NOT UPDATED =====} \quad (126)$$

$$\begin{aligned} & \text{partitionOfUnitySubjCover}(\mathcal{F}, (C, M, \mathcal{O})) \iff \left(\text{locallyFinite}(C, (M, \mathcal{O})) \right) \wedge (f \in \mathcal{F}) \wedge \\ & \left(\text{continuous} \left(f, \left(M, \mathcal{O}, [0, 1], \text{subsetTopology}(\mathcal{O}|_{[0, 1]}, ([0, 1], \mathbb{R}, \text{euclideanTopology})) \right) \right) \right) \wedge \\ & \left(\exists_{U_f \in C} \forall_{p \in M} (f(p) \neq 0 \implies p \in U_f) \right) \wedge \\ & \left(\forall_{p \in M} \exists_{U \in \mathcal{O}} \forall_{p \in U} ((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\}) \right) \wedge \\ & \left(\text{locallyFinite}(C, M, \mathcal{O}) \implies \text{finiteSet}((f_U)_n, ()) \right) \wedge \\ & \left(\forall_{p \in M} \exists_{U \in \mathcal{O}} \forall_{p \in U} \left(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1 \right) \right) \\ & \# \text{ useful for defining integrals between overlapping neighborhoods} \end{aligned} \quad (127)$$

$$\begin{aligned} & T2Separate((M, \mathcal{O}), ()) \implies \left(\text{paracompact}((M, \mathcal{O}), ()) \right) \iff \\ & \forall_C \left(\text{openCover}(C, (M, \mathcal{O})) \implies \text{partitionOfUnitySOTCover}(\mathcal{F}, (C, M, \mathcal{O})) \right) \end{aligned} \quad (128)$$

$$\text{===== NOT UPDATED =====} \quad (129)$$

1.11 Connectedness and path-connectedness

$$\begin{aligned} & \text{connected}((M, \mathcal{O}), ()) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \left(\neg \exists_{A, B \in \mathcal{O} \setminus \emptyset} (A \cap B \neq \emptyset \wedge A \cup B = M) \right) \\ & \# \text{ if there is some covering of the space that does not intersect} \end{aligned} \quad (130)$$

$$\begin{aligned} & (\text{THM}) : \neg \text{connected} \left(\left(\mathbb{R} \setminus \{0\}, \text{subsetTopology}(\mathcal{O}_{\text{euclidean}}|_{\mathbb{R} \setminus \{0\}}, (\mathbb{R}, \text{euclideanTopology}, \mathbb{R} \setminus \{0\})) \right), () \right) \\ & \iff \left(A = (-\infty, 0) \in \mathcal{O}_{\text{euclidean}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \left(B = (0, \infty) \in \mathcal{O}_{\text{euclidean}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \\ & (A \cap B = \emptyset) \wedge (A \cup B = \mathbb{R} \setminus \{0\}) \end{aligned} \quad (131)$$

$$(\text{THM}) : \text{connected}((M, \mathcal{O}), ()) \iff \forall_{S \in \mathcal{O}} \left(\text{clopen}(S, (M, \mathcal{O})) \implies (S = \emptyset \vee S = M) \right) \quad (132)$$

$$\text{pathConnected}((M, \mathcal{O}), ()) \iff \left(\text{subsetTopology}(\mathcal{O}_{\text{euclidean}}|_{[0, 1]}, (\mathbb{R}, \text{euclideanTopology}, [0, 1])) \right) \wedge$$

$$\left(\forall_{p,q \in M} \exists_{\gamma} \left(\text{continuous} \left(\gamma, ([0,1], \mathcal{O}_{\text{euclidean}}|_{[0,1]}, M, \mathcal{O}) \right) \wedge \gamma(0)=p \wedge \gamma(1)=q \right) \right) \quad (133)$$

$$(\text{THM}) : \text{pathConnected} \implies \text{connected} \quad (134)$$

1.12 Homotopic curve and the fundamental group

$$\text{===== N O T = U P D A T E D =====} \quad (135)$$

$$\begin{aligned} \text{homotopic}(\sim, (\gamma, \delta, M, \mathcal{O})) &\iff (\text{map}(\gamma, ([0,1], M)) \wedge \text{map}(\delta, ([0,1], M))) \wedge \\ &\quad (\gamma(0)=\delta(0) \wedge \gamma(1)=\delta(1)) \wedge \\ (\exists_H \forall_{\lambda \in [0,1]} (\text{continuous}(H, ([0,1] \times [0,1], \mathcal{O}_{\text{euclidean}^2}|_{[0,1] \times [0,1]}, (M, \mathcal{O})) \wedge H(0, \lambda) = \gamma(\lambda) \wedge H(1, \lambda) = \delta(\lambda)))) \\ &\quad \# H \text{ is a continuous deformation of one curve into another} \end{aligned} \quad (136)$$

$$\text{homotopic}(\sim) \implies \text{equivalenceRelation}(\sim) \quad (137)$$

$$\text{loopSpace}(\mathcal{L}_p, (p, M, \mathcal{O})) \iff \mathcal{L}_p = \{\text{map}(\gamma, ([0,1], M)) \mid \text{continuous}(\gamma) \wedge \gamma(0)=\gamma(1)\} \quad (138)$$

$$\begin{aligned} \text{concatination}(\star, (p, \gamma, \delta)) &\iff (\gamma, \delta \in \text{loopSpace}(\mathcal{L}_p)) \wedge \\ (\forall_{\lambda \in [0,1]} ((\gamma \star \delta)(\lambda) &= \begin{cases} \gamma(2\lambda) & 0 \leq \lambda < 0.5 \\ \delta(2\lambda - 1) & 0.5 \leq \lambda \leq 1 \end{cases})) \end{aligned} \quad (139)$$

$$\begin{aligned} \text{group}((G, \bullet), ()) &\iff (\text{map}(\bullet, (G \times G, G))) \wedge \\ &\quad (\forall_{a,b \in G} (a \bullet b \in G)) \\ &\quad (\forall_{a,b,c \in G} ((a \bullet b) \bullet c = a \bullet (b \bullet c))) \\ &\quad (\exists_e \forall_{a \in G} (e \bullet a = a = a \bullet e)) \wedge \\ &\quad (\forall_{a \in G} \exists_{a^{-1}} (a \bullet a^{-1} = e = a^{-1} \bullet a)) \\ &\quad \# \text{ characterizes symmetry of a set structure} \end{aligned} \quad (140)$$

$$\text{isomorphic}(\cong, (X, \odot), (Y, \ominus)) \iff \exists_f \forall_{a,b \in X} (\text{bijection}(f, (X, Y)) \wedge f(a \odot b) = f(a) \ominus f(b)) \quad (141)$$

$$\begin{aligned} \text{fundamentalGroup}((\pi_{1,p}, \bullet), (p, M, \mathcal{O})) &\iff (\pi_{1,p} = \mathcal{L}_p / \sim) \wedge \\ &\quad (\text{map}(\bullet, (\pi_{1,p} \times \pi_{1,p}, \pi_{1,p}))) \wedge \\ &\quad (\forall_{A,B \in \pi_{1,p}} ([A] \bullet [B] = [A \star B])) \wedge \\ &\quad (\text{group}((\pi_{1,p}, \bullet), ())) \\ &\quad \# \text{ an equivalence class of all loops induced from the homotopic equivalence relation} \end{aligned} \quad (142)$$

$$\text{fundamentalGroup}_1 \not\cong \text{fundamentalGroup}_2 \implies \text{topologicalSpace}_1 \not\cong \text{topologicalSpace}_2 \quad (143)$$

$$\text{there exists no known list of topological properties that can imply homeomorphisms} \quad (144)$$

$$\text{CONTINUE @ Lecture 6: manifolds} \quad (145)$$

$$\text{===== N O T = U P D A T E D =====} \quad (146)$$

1.13 Measure theory

$$\begin{aligned}
& \text{sigmaAlgebra}(\sigma, (M)) \iff (M \neq \emptyset) \wedge (\sigma \subseteq \mathcal{P}(M)) \wedge \\
& \quad (M \in \sigma) \wedge \left(\forall A \in \sigma (M \setminus A \in \sigma) \right) \wedge \\
& \quad \left(\left(A \subseteq \sigma \wedge \neg \text{uncountablyInfinite}(A, ()) \right) \implies \cup A \in \sigma \right) \\
& \# \text{ behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu
\end{aligned} \tag{147}$$

$$\text{measurableSpace}((M, \sigma), ()) \iff \text{sigmaAlgebra}(\sigma, (M)) \tag{148}$$

$$\text{measurableSet}(A, (M, \sigma)) \iff \left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge (A \in \sigma) \tag{149}$$

$$\begin{aligned}
& \text{measure}(\mu, (M, \sigma)) \iff \left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge \left(\text{map} \left(\mu, \left(\sigma, \left(\mathbb{R}^+ \right)_0 \right) \right) \right) \wedge (\mu(\emptyset) = 0) \wedge \\
& \quad \left(\left((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} \forall j \in \mathbb{N} \setminus \{i\} (A_i \cap A_j = \emptyset) \right) \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) = \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \\
& \# \text{ enforces meaningful concepts of measures such as precise additivity}
\end{aligned} \tag{150}$$

$$\begin{aligned}
& (\text{THM}) : \text{measure}(\mu, (M, \sigma)) \implies \\
& \quad \left(\forall A, B \in \sigma (A \subseteq B \implies \mu(A) \leq \mu(B)) \right) \wedge \\
& \quad \left((A)_{\mathbb{N}} \subseteq \sigma \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) \leq \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \wedge \\
& \quad \left(((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (B_i \subseteq B_{i+1}) \wedge B = \cup (B)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(B_n)) = \mu(B) \right) \wedge \\
& \quad \left(((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (C_{i+1} \subseteq C_i) \wedge C = \cap (C)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(C_n)) = \mu(C) \right) \\
& \# \text{ immediate implications of the measurable set } A \in \sigma \text{ axioms and the measure } \mu \text{ axioms}
\end{aligned} \tag{151}$$

$$\text{measureSpace}((M, \sigma, \mu), ()) \iff \text{measure}(\mu, (M, \sigma)) \tag{152}$$

$$\begin{aligned}
& \text{finiteMeasure}(\mu, (M, \sigma)) \iff \left(\text{measure}(\mu, (M, \sigma)) \right) \wedge \\
& \quad \left(\exists (A)_{\mathbb{N}} \subseteq \sigma \left(\cup ((A)_{\mathbb{N}}) = M \wedge \forall n \in \mathbb{N} (\mu(A_n) < \infty) \right) \right)
\end{aligned} \tag{153}$$

$$\begin{aligned}
& \text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) \iff \left(G = \{ \sigma \subseteq \mathcal{P}(M) \mid \text{sigmaAlgebra}(\sigma, (M)) \} \right) \wedge (\sigma(\zeta) = \cap G) \\
& \# \text{ smallest } \sigma\text{-algebra containing the generating set } \zeta
\end{aligned} \tag{154}$$

$$(\text{THM}) : \exists \zeta \subseteq M \left(\text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) = \text{sigmaAlgebra}(\sigma, (M)) \right) \tag{155}$$

$$\begin{aligned}
& \text{borelSigmaAlgebra}(\sigma(\mathcal{O}), (M, \mathcal{O})) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\
& \quad \left(\text{generatedSigmaAlgebra}(\sigma(\mathcal{O}), (\mathcal{O}, M)) \right) \\
& \# \sigma\text{-algebra induced by a topology}
\end{aligned} \tag{156}$$

$$euclideanSigma(\sigma_s, ()) \iff \left(borelSigmaAlgebra \left(\sigma_s, \left(\mathbb{R}^d, euclideanTopology \right) \right) \right) \quad (157)$$

$$\begin{aligned} lebesgueMeasure(\lambda, ()) \iff & \left(measure \left(\lambda, \left(\mathbb{R}^d, euclideanSigma \right) \right) \right) \wedge \\ & \left(\lambda \left(\times_{i=1}^d ([a_i, b_i]) \right) = \sum_{i=1}^d \left(\sqrt[d]{(a_i - b_i)^2} \right) \right) \\ & \# \text{ natural measure for } \mathbb{R}^d \end{aligned} \quad (158)$$

$$\begin{aligned} measurableMap(f, (M, \sigma_M, N, \sigma_N)) \iff & \left(measurableSpace((M, \sigma_M), ()) \right) \wedge \\ & \left(measurableSpace((N, \sigma_N), ()) \right) \wedge \left(\forall B \in \sigma_N \left(preimage(A, (B, f, M, N)) \in \sigma_M \right) \right) \\ & \# \text{ preimage of measurable sets are measurable} \end{aligned} \quad (159)$$

$$\begin{aligned} pushForwardMeasure(f \star \lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \iff & \left(measureSpace((M, \sigma_M, \mu_M), ()) \right) \wedge \\ & \left(measurableSpace((N, \sigma_N), ()) \right) \wedge \left(measurableMap(f, (M, \sigma_M, N, \sigma_N)) \right) \wedge \\ & \left(\forall B \in N \left(f \star \lambda_M(B) = \mu_M \left(preimage(A, (B, f, M, N)) \right) \right) \right) \wedge \left(measure(f \star \lambda_M, (N, \sigma_N)) \right) \\ & \# \text{ natural construction of a measure based primarily on measurable map} \end{aligned} \quad (160)$$

$$nullSet(A, (M, \sigma, \mu)) \iff \left(measureSpace((M, \sigma, \mu), ()) \right) \wedge (A \in \sigma) \wedge (\mu(A) = 0) \quad (161)$$

$$\begin{aligned} almostEverywhere(p, (M, \sigma, \mu)) \iff & \left(measureSpace((M, \sigma, \mu), ()) \right) \wedge \left(predicate(p, (M)) \right) \wedge \\ & \left(\exists A \in \sigma \left(nullSet(A, (M, \sigma, \mu)) \implies \forall n \in M \setminus A (p(n)) \right) \right) \\ & \# \text{ the predicate holds true for all points except the points in the null set} \\ & \# \text{ in terms of measure, almost nothing is not equivalent to nothing} \end{aligned} \quad (162)$$

1.14 Lebesgue integration

$$simpleTopology(\mathcal{O}_{simple}, ()) \iff \mathcal{O}_{simple} = subsetTopology \left(\mathcal{O}|_{\mathbb{R}_0^+}, \left(\mathbb{R}, euclideanTopology, \mathbb{R}_0^+ \right) \right) \quad (163)$$

$$simpleSigma(\sigma_{simple}, ()) \iff borelSigmaAlgebra \left(\sigma_{simple}, \left(\mathbb{R}_0^+, simpleTopology \right) \right) \quad (164)$$

$$simpleFunction(s, (M, \sigma)) \iff \left(measurableMap \left(s, \left(M, \sigma, \mathbb{R}_0^+, simpleSigma \right) \right) \right) \wedge$$

$$\left(\text{finiteSet} \left(\text{image} \left(B, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right)$$

if the map takes on finitely many values on \mathbb{R}_0^+

(165)

$$\text{characteristicFunction}(X_A, (A, M)) \iff (A \subseteq M) \wedge \left(\text{map}(X_A, (M, \mathbb{R})) \right) \wedge$$

$$\left(\forall_{m \in M} \left(X_A(m) = \begin{cases} 1 & m \in A \\ 0 & m \notin A \end{cases} \right) \right)$$

(166)

$$(\text{THM}) : \text{simpleFunction}(s, (M, \sigma_M)) \implies$$

$$\left(\text{finiteSet} \left(\text{image} \left(Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right) \wedge$$

$$\left(\text{characteristicFunction}(X_A, (A, M)) \right) \wedge \left(\forall_{m \in M} \left(s(m) = \sum_{z \in Z} \left(z \cdot X_{\text{preimage}(A, (\{z\}, s, M, \mathbb{R}_0^+))}(m) \right) \right) \right)$$

(167)

$$\text{exEuclideanSigma}(\overline{\sigma}_s, ()) \iff \overline{\sigma}_s = \{A \subseteq \overline{\mathbb{R}} \mid A \cap R \in \text{euclideanSigma}\}$$

ignores $\pm\infty$ to preserve the points in the domain of the measurable map

(168)

$$\text{nonNegIntegrable}(f, (M, \sigma)) \iff \left(\text{measurableMap} \left(f, (M, \sigma, \overline{\mathbb{R}}, \text{exEuclideanSigma}) \right) \right) \wedge$$

$$\left(\forall_{m \in M} (f(m) \geq 0) \right)$$

(169)

$$\text{nonNegIntegral} \left(\int_M (f d\mu), (f, M, \sigma, \mu) \right) \iff \left(\text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge$$

$$\left(\text{measureSpace} \left((\overline{\mathbb{R}}, \text{exEuclideanSigma}, \text{lebesgueMeasure}), () \right) \right) \wedge$$

$$\left(\text{nonNegIntegrable}(f, (M, \sigma)) \right) \wedge \left(\int_M (f d\mu) = \sup \left(\left\{ \sum_{z \in Z} \left(z \cdot \mu \left(\text{preimage} \left(A, (\{z\}, s, M, \mathbb{R}_0^+) \right) \right) \right) \right\} \right) \mid \right.$$

$$\left. \forall_{m \in M} (s(m) \leq f(m)) \wedge \text{simpleFunction}(s, (M, \sigma)) \wedge \text{finiteSet} \left(\text{image} \left(Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right\})$$

lebesgue measure on z reduces to z

(170)

$$\text{explicitIntegral} \iff \int (f(x) \mu(dx)) = \int (f d\mu)$$

alternative notation for lebesgue integrals

(171)

$$(\text{THM}) : \text{nonNegIntegral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies$$

$$\begin{aligned}
\text{(THM) Markov inequality: } & \left(\forall_{z \in \mathbb{R}_0^+} \left(\int (f d\mu) \geq z \cdot \mu \left(\text{preimage} \left(A, ([z, \infty), f, M, \overline{\mathbb{R}}) \right) \right) \right) \right) \wedge \\
& \left(\text{almostEverywhere}(f = g, (M, \sigma, \mu)) \implies \int (f d\mu) = \int (g d\mu) \right) \\
& \left(\int (f d\mu) = 0 \implies \text{almostEverywhere}(f = 0, (M, \sigma, \mu)) \right) \wedge \\
& \left(\int (f d\mu) \leq \infty \implies \text{almostEverywhere}(f < \infty, (M, \sigma, \mu)) \right)
\end{aligned} \tag{172}$$

$$\begin{aligned}
\text{(THM) Mono. conv.: } & \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{R}, \text{exEuclideanSigma}) \right) \wedge 0 \leq f_{n-1} \leq f_n \} \right) \wedge \\
& \left(\text{map} \left(f, (M, \overline{\mathbb{R}}) \right) \right) \wedge \left(\forall_{m \in M} \left(f(m) = \sup(f_n(m) \mid f_n \in (f)_{\mathbb{N}}) \right) \right) \implies \left(\lim_{n \rightarrow \infty} \left(\int_M (f_n d\mu) \right) = \int_M (f d\mu) \right) \\
& \# \text{ lengths now depend on } M, \sigma \text{ and limits can be pulled in or out of an integral}
\end{aligned} \tag{173}$$

$$\begin{aligned}
\text{(THM) : } & \text{nonNegIntegral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\
& \left(\forall_{\alpha \in \mathbb{R}_0^+} \left(\int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \right) \\
& \# \text{ integral acts linearly and commutes finite summations}
\end{aligned} \tag{174}$$

$$\begin{aligned}
\text{(THM) : } & \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{R}, \text{exEuclideanSigma}) \right) \wedge 0 \leq f_n \} \right) \implies \\
& \left(\int \left(\left(\sum_{n=1}^{\infty} (f_n) \right) d\mu \right) = \sum_{n=1}^{\infty} \left(\int (f_n d\mu) \right) \right) \\
& \# \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n \rightarrow \infty} \sum_{i=1}^n f_n \text{ since } f_n \geq 0 \text{ and it commutes with integral from monotone conv.}
\end{aligned} \tag{175}$$

$$\begin{aligned}
\text{integrable}(f, (M, \sigma)) & \iff \left(\text{measurableMap} \left(f, (M, \sigma, \overline{\mathbb{R}}, \text{exEuclideanSigma}) \right) \right) \wedge \\
& \left(\forall_{m \in M} \left(f(m) = \max(f(m), 0) - \max(0, -f(m)) \right) \right) \wedge \\
& \left(\text{measureSpace}(M, \sigma, \mu) \implies \left(\int (\max(f(m), 0) d\mu) < \infty \wedge \int (\max(0, -f(m)) d\mu) < \infty \right) \right) \\
& \# \text{ extra condition prevents the occurrence of the indeterminate } \infty - \infty
\end{aligned} \tag{176}$$

$$\begin{aligned}
\text{integral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) & \iff \left(\text{nonNegIntegral} \left(\int (f^+ d\mu), (\max(f, 0), M, \sigma, \mu) \right) \right) \wedge \\
& \left(\text{nonNegIntegral} \left(\int (f^- d\mu), (\max(0, -f), M, \sigma, \mu) \right) \right) \wedge \left(\text{integrable}(f, (M, \sigma)) \right) \wedge
\end{aligned}$$

$$\left(\int (f d\mu) = \int (f^+ d\mu) - \int (f^- d\mu) \right)$$

arbitrary integral in terms of nonnegative integrals
(177)

$$(\text{THM}) : \left(\text{map}(f, (M, \mathbb{C})) \right) \implies \left(\int (f d\mu) = \int (\text{Re}(f) d\mu) - \int (\text{Im}(f) d\mu) \right)$$

(178)

$$\begin{aligned} (\text{THM}) : \text{integral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{integral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\ \left(\text{almostEverywhere}(f \leq g, (M, \sigma, \mu)) \implies \int (f d\mu) \leq \int (g d\mu) \right) \wedge \\ \left(\forall_{m \in M} (f(m), g(m), \alpha \in \mathbb{R}) \implies \int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \end{aligned}$$

(179)

$$\begin{aligned} (\text{THM}) \text{ Dominant convergence: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{\mathbb{R}}, \text{exEuclideanSigma}) \right) \} \right) \wedge \\ \left(\text{map}(f, (M, \overline{\mathbb{R}})) \right) \wedge \left(\text{almostEverywhere} \left(f(m) = \lim_{n \rightarrow \infty} (f_n(m)), (M, \sigma, \mu) \right) \right) \wedge \\ \left(\text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \right) \wedge \left(\left| \int (g d\mu) \right| < \infty \right) \wedge \left(\text{almostEverywhere}(|f_n| \leq g, (M, \sigma, \mu)) \right) \\ \# \text{ if all } f_n(m) \text{ are bounded by some integrable } |g(m)| \implies \\ \# \text{ then all } f_n(m) \text{ including } f \text{ satisfy bounded and integrable properties} \\ \left(\forall_{\phi \in \{f\} \cup (f)_{\mathbb{N}}} \left(\text{integrable}(\phi, (M, \sigma)) \right) \right) \wedge \left(\lim_{n \rightarrow \infty} \left(\int (|f_n - f| d\mu) = 0 \right) \right) \wedge \left(\lim_{n \rightarrow \infty} \left(\int (f_n d\mu) \right) = \int (f d\mu) \right) \end{aligned}$$

(180)

1.15 Vector space and structures

$$\begin{aligned} \text{vectorSpace}((V, +, \cdot), ()) \iff \left(\text{map}(+, (V \times V, V)) \right) \wedge \left(\text{map}(\cdot, (\mathbb{R} \times V, V)) \right) \wedge \\ \left(\forall_{v, w \in V} (v + w = w + v) \right) \wedge \\ \left(\forall_{v, w, x \in V} ((v + w) + x = v + (w + x)) \right) \wedge \\ \left(\exists \mathbf{0} \in V \forall v \in V (v + \mathbf{0} = v) \right) \wedge \\ \left(\forall v \in V \exists -v \in V (v + (-v) = \mathbf{0}) \right) \wedge \\ \left(\forall_{a, b \in \mathbb{R}} \forall v \in V (a(b \cdot v) = (ab) \cdot v) \right) \wedge \\ \left(\exists 1 \in \mathbb{R} \forall v \in V (1 \cdot v = v) \right) \wedge \\ \left(\forall_{a, b \in \mathbb{R}} \forall v \in V ((a + b) \cdot v = a \cdot v + b \cdot v) \right) \wedge \\ \left(\forall_{a \in \mathbb{R}} \forall_{v, w \in V} (a \cdot (v + w) = a \cdot v + a \cdot w) \right) \\ \# \text{ behaves similar as vectors should i.e., additive, scalable, linear distributive} \end{aligned}$$

(181)

$$\begin{aligned} \text{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \iff \left(\text{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\text{map}(\langle \$1, \$2 \rangle, (V \times V, \mathbb{R})) \right) \wedge \\ \left(\forall_{v, w \in V} (\langle v, w \rangle = \langle w, v \rangle) \right) \wedge \end{aligned}$$

$$\begin{aligned}
& \left(\forall_{v,w,x \in V} \forall_{a,b \in \mathbb{R}} \left(\langle av + bw, x \rangle = a \langle v, x \rangle + b \langle w, x \rangle \right) \right) \wedge \\
& \left(\forall_{v \in V} \left(\langle v, v \rangle \geq 0 \right) \wedge \left(\forall_{v \in V} \left(\langle v, v \rangle = 0 \iff v = \mathbf{0} \right) \right) \right) \\
& \# \text{ the sesquilinear or l.s linear map inner product provides info. on distance and orthogonality}
\end{aligned} \tag{182}$$

$$\textit{innerProductSpace} \left((V, +, \cdot, \langle \$1, \$2 \rangle), () \right) \iff \textit{innerProduct} \left(\langle \$1, \$2 \rangle, (V, +, \cdot) \right) \tag{183}$$

$$\begin{aligned}
\textit{vectorNorm}(\| \$1 \|, (V, +, \cdot)) & \iff \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\textit{map} \left(\| \$1 \|, (V, \mathbb{R}_0^+) \right) \right) \wedge \\
& \left(\forall_{v \in V} \left(\|v\| = 0 \iff v = \mathbf{0} \right) \right) \wedge \\
& \left(\forall_{v \in V} \forall_{s \in \mathbb{R}} \left(\|sv\| = |s| \|v\| \right) \right) \wedge \\
& \left(\forall_{v,w \in V} \left(\|v+w\| \leq \|v\| + \|w\| \right) \right) \\
& \# \text{ magnitude of a point in a vector space}
\end{aligned} \tag{184}$$

$$\textit{normedVectorSpace} \left((V, +, \cdot, \| \$1 \|), () \right) \iff \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\textit{vectorNorm}(\| \$1 \|, (V, +, \cdot)) \right) \tag{185}$$

$$\begin{aligned}
\textit{vectorMetric} \left(d(\$1, \$2), (V, +, \cdot) \right) & \iff \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \\
& \left(\textit{metric} \left(d(\$1, \$2), (V) \right) \vee \left(\textit{map} \left(d, (V \times V, \mathbb{R}_0^+) \right) \right) \right) \\
& \left(\forall_{x,y \in V} \left(d(x, y) = d(y, x) \right) \right) \wedge \\
& \left(\forall_{x,y \in V} \left(d(x, y) = 0 \iff x = y \right) \right) \wedge \\
& \left(\forall_{x,y,z \in V} \left(d(x, z) \leq d(x, y) + d(y, z) \right) \right) \\
& \# \text{ behaves as distances should}
\end{aligned} \tag{186}$$

$$\begin{aligned}
\textit{metricVectorSpace} \left((V, +, \cdot, d(\$1, \$2)), () \right) & \iff \left(\textit{vectorSpace}((V, +, \cdot), ()) \right) \wedge \\
& \left(\textit{vectorMetric} \left(d(\$1, \$2), (V, +, \cdot) \right) \right)
\end{aligned} \tag{187}$$

$$\begin{aligned}
\textit{innerProductNorm} \left(\| \$1 \|, (V, +, \cdot, \langle \$1, \$2 \rangle) \right) & \iff \left(\textit{innerProductSpace} \left((V, +, \cdot, \langle \$1, \$2 \rangle), () \right) \right) \wedge \\
& \left(\forall_{v \in V} \left(\|v\| = \sqrt[2]{\langle v, v \rangle} \right) \implies \textit{vectorNorm}(\| \$1 \|, (V, +, \cdot)) \right)
\end{aligned} \tag{188}$$

$$\begin{aligned}
\textit{normInnerProduct} \left(\langle \$1, \$2 \rangle, (V, +, \cdot, \| \$1 \|) \right) & \iff \left(\textit{normedVectorSpace} \left((V, +, \cdot, \| \$1 \|), () \right) \right) \wedge \\
& \left(\forall_{u,v \in V} \left(2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2 \right) \right) \wedge \\
& \left(\forall_{v,w \in V} \left(\langle v, w \rangle = \frac{\|v+w\|^2 - \|v-w\|^2}{4} \right) \implies \textit{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \right)
\end{aligned} \tag{189}$$

$$\textit{normMetric} \left(d(\$1, \$2), (V, +, \cdot, \| \$1 \|) \right) \iff \left(\textit{normedVectorSpace} \left((V, +, \cdot, \| \$1 \|), () \right) \right) \wedge$$

$$\left(\forall_{v,w \in V} (d(v,w) = ||v-w||) \implies \text{vectorMetric}(d(\$1,\$2), (V, +, \cdot)) \right) \quad (190)$$

$$\begin{aligned} \text{metricNorm}\left(||\$1||, (V, +, \cdot, d(\$1,\$2))\right) &\iff \left(\text{metricVectorSpace}\left((V, +, \cdot, d(\$1,\$2)), ()\right) \right) \wedge \\ &\left(\forall_{u,v,w \in V} \forall_{s \in \mathbb{R}} \left(d(s(u+w), s(v+w)) = |s|d(u,v) \right) \right) \wedge \\ &\left(\forall_{v \in V} (||v|| = d(v, \mathbf{0})) \implies \text{vectorNorm}(|\$1|, (V, +, \cdot)) \right) \end{aligned} \quad (191)$$

$$\begin{aligned} \text{orthogonal}\left((v,w), (V, +, \cdot, \langle \$1, \$2 \rangle)\right) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \wedge \\ &(v, w \in V) \wedge (\langle v, w \rangle = 0) \\ &\# \text{ the inner product also provides info. on orthogonality} \end{aligned} \quad (192)$$

$$\begin{aligned} \text{normal}\left(v, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \wedge (v \in V) \wedge (\langle v, v \rangle = 1) \\ &\# \text{ the vector has unit length} \end{aligned} \quad (193)$$

$$\text{(THM) Cauchy-Schwarz inequality: } \forall_{v,w \in V} (\langle v, w \rangle \leq ||v|| ||w||) \quad (194)$$

$$\text{basis}((b)_n, (V, +, \cdot, \cdot)) \iff \left(\text{vectorSpace}((V, +, \cdot, \cdot), ()) \right) \wedge \left(\forall_{v \in V} \exists_{(a)_n \in \mathbb{R}^n} \left(v = \sum_{i=1}^n (a_i b_i) \right) \right) \quad (195)$$

$$\begin{aligned} \text{orthonormalBasis}((b)_n, (V, +, \cdot, \langle \$1, \$2 \rangle)) &\iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \wedge \\ &\left(\text{basis}((b)_n, (V, +, \cdot, \cdot)) \right) \wedge \left(\forall_{v \in (b)_n} \left(\text{normal}(v, (V, +, \cdot, \langle \$1, \$2 \rangle)) \right) \right) \wedge \\ &\left(\forall_{v \in (b)_n} \forall_{w \in (b)_n \setminus \{v\}} \left(\text{orthogonal}((v,w), (V, +, \cdot, \langle \$1, \$2 \rangle)) \right) \right) \end{aligned} \quad (196)$$

1.16 Subvector space

$$\text{subspace}((U, \circ), (V, \circ)) \iff \left(\text{space}((V, \circ), ()) \right) \wedge (U \subseteq V) \wedge \left(\text{space}((U, \circ), ()) \right) \quad (197)$$

$$\begin{aligned} \text{subspaceSum}(U+W, (U, W, V, +)) &\iff \left(\text{subspace}((U, +), (V, +)) \right) \wedge \left(\text{subspace}((W, +), (V, +)) \right) \wedge \\ &(U+W = \{u+w \mid u \in U \wedge w \in W\}) \end{aligned} \quad (198)$$

$$\text{subspaceDirectSum}(U \oplus W, (U, W, V, +)) \iff (U \cap W = \emptyset) \wedge \left(\text{subspaceSum}(U \oplus W, (U, W, V, +)) \right) \quad (199)$$

$$\begin{aligned} \text{orthogonalComplement}(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)) &\iff \\ &\left(\text{subspace}\left((W, +, \cdot, \langle \$1, \$2 \rangle), \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \right) \right) \right) \wedge \end{aligned}$$

$$\left(W^\perp = \left\{ v \in V \mid w \in W \wedge \text{orthogonal}\left((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \right\} \right) \quad (200)$$

$$\begin{aligned} & \text{orthogonalDecomposition}\left(\left(W, W^\perp\right), (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right) \iff \\ & \left(\text{orthogonalComplement}\left(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right) \wedge \left(\text{subspaceDirectSum}\left(V, \left(W, W^\perp, V, +\right)\right)\right) \end{aligned} \quad (201)$$

$$\text{(THM) if } V \text{ is finite dimensional, then every vector has an orthogonal decomposition:} \quad (202)$$

1.17 Banach and Hilbert Space

$$\begin{aligned} & \text{cauchy}\left((s)_{\mathbb{N}}, (V, d(\$1, \$2))\right) \iff \left(\text{metricSpace}\left((V, d(\$1, \$2)), ()\right)\right) \wedge ((s)_{\mathbb{N}} \subseteq V) \\ & \quad \left(\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N (d(s_m, s_n) < \epsilon)\right) \\ & \quad \# \text{ distances between some tail-end point gets arbitrarily small} \end{aligned} \quad (203)$$

$$\begin{aligned} & \text{complete}\left((V, d(\$1, \$2)), ()\right) \iff \left(\forall (s)_{\mathbb{N}} \subseteq V \exists s \in V \left(\text{cauchy}\left((s)_{\mathbb{N}}, (V, d(\$1, \$2))\right) \implies \lim_{n \rightarrow \infty} (d(s, s_n)) = 0\right)\right) \\ & \quad \# \text{ or converges within the induced topological space} \\ & \quad \# \text{ in complete spaces, the weaker notion of cauchy is enforced to be equivalent to convergence} \end{aligned} \quad (204)$$

$$\begin{aligned} & \text{banachSpace}\left((V, +, \cdot, \|\$1\|), ()\right) \iff \left(\text{normMetric}\left(d(\$1, \$2), (V, \|\$1\|)\right)\right) \wedge \left(\text{complete}\left(V, d(\$1, \$2)\right), ()\right) \\ & \quad \# \text{ a complete normed vector space} \end{aligned} \quad (205)$$

$$\begin{aligned} & \text{hilbertSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \iff \left(\text{innerProductNorm}\left(\|\$1\|, (V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right) \wedge \\ & \quad \left(\text{normMetric}\left(d(\$1, \$2), (V, \|\$1\|)\right)\right) \wedge \left(\text{complete}\left(V, d(\$1, \$2)\right), ()\right) \\ & \quad \# \text{ a complete inner product space} \end{aligned} \quad (206)$$

$$\text{(THM) : } \text{hilbertSpace} \implies \text{banachSpace} \quad (207)$$

$$\begin{aligned} & \text{separable}\left((V, d), ()\right) \iff \left(\exists S \subseteq V \left(\text{dense}(S, (V, d)) \wedge \text{countablyInfinite}(S, ())\right)\right) \\ & \quad \# \text{ needs only a countable subset to approximate any element in the entire space} \end{aligned} \quad (208)$$

$$\begin{aligned} & \text{(THM) : } \text{hilbertSpace}\left(\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right), ()\right) \implies \\ & \left(\exists (b)_{\mathbb{N}} \subseteq V \left(\text{orthonormalBasis}\left((b)_{\mathbb{N}}, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \wedge \text{countablyInfinite}\left((b)_{\mathbb{N}}, ()\right)\right)\right) \iff \\ & \quad \text{separable}\left(\left(V, \sqrt{\langle \$1 - \$2, \$1 - \$2 \rangle}\right), ()\right) \\ & \quad \# \text{ separability in hilbert spaces is equivalent to the existence of a countable orthonormal basis} \end{aligned} \quad (209)$$

1.18 Matrices, Operators, and Functionals

$$\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)) \iff \left(\text{map}(L, (V, W)) \right) \wedge \left(\text{vectorSpace}((V, +_V, \cdot_V), ()) \right) \wedge \left(\text{vectorSpace}((W, +_W, \cdot_W), ()) \right) \wedge \left(\forall_{v_1, v_2 \in V} \forall_{s_1, s_2 \in \mathbb{R}} (L(s_1 \cdot_V v_1 +_V s_2 \cdot_V v_2) = s_1 \cdot_W L(v_1) +_W s_2 \cdot_W L(v_2)) \right) \quad (210)$$

$$\text{matrix}(L, (n, m)) \iff \left(\text{linearOperator}(L, (\mathbb{R}^m, +_m, \cdot_m, \mathbb{R}^n, +_n, \cdot_n)) \right) \\ \# \text{ rows=dimensions, cols=vectors} \quad (211)$$

$$\text{eigenvector}(v, (L, V, +, \cdot)) \iff \left(\text{linearOperator}(L, (V, +, \cdot, V, +, \cdot)) \right) \wedge \left(\exists_{\lambda \in \mathbb{R}} (L(v) = \lambda v) \right) \quad (212)$$

$$\text{eigenvalue}(\lambda, (v, L, V, +, \cdot)) \iff \left(\text{eigenvector}(v, (L, V, +, \cdot)) \right) \quad (213)$$

$$\text{identityOperator}(I, (A)) \iff \left(\text{matrix}(A, (n, n)) \right) \wedge (AI = IA = A) \quad (214)$$

$$\text{inverseOperator}(A^{-1}, (A)) \iff \left(A^{-1}A = AA^{-1} = I \right) \\ \# \text{ gauss-jordan elimination: } E[A|I] = [I|E] = [I|A^{-1}] \quad (215)$$

$$\text{CONTHETODOABSTRACTALGEB} \quad (216)$$

$$(\text{THM}) : (AB)^{-1}(AB) = I = B^{-1}A^{-1}AB \quad (217)$$

$$\text{transposeOperator}(A^T, (A)) \iff \left((A^T)_{m,n} = (A)_{n,m} \right) \vee \text{adjoint}(A^T, (A)) \quad (218)$$

$$\text{symmetricOperator}(A, ()) \iff \left(A = \text{transposeOperator}(A^T, (A)) \right) \vee \left(\text{selfAdjoint}(A, ()) \right) \quad (219)$$

$$(\text{THM}) : (AB)^T = B^T A^T \wedge (A^T)^{-1} = (A^{-1})^T \quad (220)$$

$$\text{triangularOperator}(A, ()) \iff \left(\text{matrix}(A, (n, n)) \right) \wedge \left(\forall_{x < n} \forall_{0 < i < x} (A_{i,i} = 0) \right) \quad (221)$$

$$\text{decomposeLU}(LU(A), (A)) \iff \left(\text{matrix}(A, (n, n)) \right) \wedge \left(\exists_E (EA = \text{triangularOperator}(U, ())) \right) \wedge \left(LU(A) = E^{-1}U = A \right) \\ \# \text{ lower triangle are all 0; useful for solving linear equations} \quad (222)$$

$$\text{Img}(\text{Img}(A), (A)) \iff \left(\text{matrix}(A, (n, m)) \right) \wedge (\text{Img}(A) = \{Av \in \mathbb{R}^n \mid v \in \mathbb{R}^m\}) \\ \# \text{ the column space; not always a subspace since } A \text{ can map to a set not containing } \mathbf{0} \quad (223)$$

$$\text{Ker}(\text{Ker}(A), (A)) \iff \left(\text{matrix}(A, (n, m)) \right) \wedge (\text{Ker}(A) = \{v \in \mathbb{R}^m \mid Av = \mathbf{0} \in \mathbb{R}^n\}) \\ \# \text{ the null or solution space; always a subspace due to linearity } Av + Aw = \mathbf{0} = A(v + w) \quad (224)$$

$$\text{(THM) general linear solution: } (Ax_p = b) \wedge (x_n \in \text{Ker}(A)) \implies (Ax_p + Ax_n = b + 0 = A(x_p + x_n) = b) \quad (225)$$

$$\text{independentOperator}(A, ()) \iff (\text{matrix}(A, (n, m))) \wedge (\neg \exists_{v \in \mathbb{R}^m \setminus \mathbf{0}_m} (Av = 0) \iff \text{Ker}(A) = \{\mathbf{0}_m\})$$

also equivalent to invertible operator (226)

$$\text{dimensionality}(N, (A)) \iff (\text{matrix}(A, (n, m))) \wedge (N = \inf(\{|(b)_n| \mid \text{basis}((b)_n, (A))\})) \quad (227)$$

$$\text{rank}(r, (A)) \iff (\text{matrix}(A, (n, m))) \wedge (\text{dimensionality}(r, (A))) \quad (228)$$

$$\text{(THM) : } (\text{matrix}(A, (n, m))) \implies (\text{dimensionality}(\text{Ker}(A)) = n - \text{rank}(r, (A)))$$

number of free variables (229)

$$\text{transposeNorm}(\|x\|, ()) \iff (\|x\| = \sqrt{x^T x}) \quad (230)$$

$$\text{(THM) : } P = P^T = P^2 \quad (231)$$

$$\begin{aligned} \text{orthogonalVectors}((x, y), ()) &\iff (\|x\|^2 + \|y\|^2 = \|x + y\|^2) \iff \\ & (x^T x + y^T y = (x + y)^T (x + y) = x^T x + y^T y + x^T y + y^T x) \iff \\ \left(0 = \frac{x^T x + y^T y - (x^T x + y^T y)}{2} = \frac{x^T y + y^T x}{2} = x^T y\right) &\iff \left(0 = \sum_i (x_i y_i) \vee \int (x(u) y(u) du)\right) \\ &\# \text{ vector and functional orthogonality} \end{aligned} \quad (232)$$

$$\text{orthogonalOperator}(Q, (V, +, \cdot, \langle \$1, \$2 \rangle)) \iff \left(\text{orthonormalBasis}\left(Q^T, (V, +, \cdot, \langle \$1^T, \$2 \rangle)\right) \right) \vee (Q^T Q = I) \quad (233)$$

$$\text{(THM) : } \text{orthogonalOperator}(Q, (V, +, \cdot, \langle \$1, \$2 \rangle)) \implies (Q^T Q Q^{-1} = I Q^{-1} = Q^T = Q^{-1}) \quad (234)$$

$$\begin{aligned} \text{orthogonalProjection}(P_A b, (A, b)) &\iff (\text{matrix}(A, (n, m))) \wedge (\text{matrix}(b, (m, 1))) \wedge \\ & \left(\exists_{c \in \mathbb{R}^m} (A^T (b - P_A b) = 0 = A^T (b - Ac)) \iff \right. \\ A^T b = A^T Ac &\iff c = (A^T A)^{-1} A^T b \iff P_A b = Ac = \left(A (A^T A)^{-1} A^T \right) b \\ &\# A, A^T \text{ may not necessarily be invertible} \end{aligned} \quad (235)$$

$$\text{(THM) : } \text{independentOperator}(A, ()) \implies \text{independentOperator}(A^T A, ()) \quad (236)$$

$$\begin{aligned} \text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|)) &\iff (\text{normedVectorSpace}((V, +, \cdot, \|\$1\|), ())) \wedge \\ (X = \{v \in V \mid \|v\| = 1 \wedge \text{eigenvector}(v, (A, V, +, \cdot))\}) &\end{aligned} \quad (237)$$

$$\begin{aligned} \text{det}(\text{det}(A), (A, V, +, \cdot, \|\$1\|)) &\iff (\text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|))) \wedge \\ &(\text{det}(A) = \prod_{x \in X} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)))) \\ \# \text{ DEFINE; exterior algebra wedge product area??} \end{aligned} \quad (238)$$

$$\begin{aligned} \text{tr}(\text{tr}(A), (A, V, +, \cdot, \|\$1\|)) &\iff (\text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|))) \wedge \\ &(\text{tr}(A) = \sum_{x \in X} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)))) \\ \# \text{ DEFINE} \end{aligned} \quad (239)$$

$$(\text{THM}) : \text{independentOperator}(A, ()) \iff \text{det}(A) \neq 0 \quad (240)$$

$$(\text{THM}) : A = A^T = A^2 \implies \text{Tr}(A) = \text{dimensionality}(N, (A)) \# \text{ counts dimensions} \quad (241)$$

$$\begin{aligned} (\text{normalOperator}(A, ())) &\iff A^T A = A A^T \\ \# \text{ DEFINE} \end{aligned} \quad (242)$$

$$\text{diagonalOperator}(A, ()) \iff (\text{normalOperator}(A, ())) \wedge (\text{triangularOperator}(A, ())) \quad (243)$$

$$\begin{aligned} \text{characteristicEquation}((A - \lambda I)x = 0, (A)) &\iff (Ax = \lambda x \implies Ax - \lambda x = (A - \lambda I)x = 0) \wedge \\ &(x \neq \mathbf{0} \implies \text{eigenvalue}(0, (x, A - \lambda I) \implies \prod_{\lambda_i \in \Lambda} = 0 = \text{det}(A - \lambda I))) \\ \# \text{ characterizes eigenvalues} \end{aligned} \quad (244)$$

$$\begin{aligned} \text{eigenDecomposition}(S \Lambda S^{-1}, (A, V, +, \cdot, \|\$1\|)) &\iff (S \subseteq (\text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|)))^T) \wedge \\ &(\text{diagonalOperator}(\Lambda, ()) \{1\}^n = (\lambda)_n = \{\lambda \in \mathbb{R} \mid s \in S^T \wedge \text{eigenvalue}(\lambda, s, A, V)\}) \\ &(\text{independentOperator}(S, ())) \wedge (\exists_{S^{-1}} (AS = S\Lambda \implies A = S\Lambda S^{-1})) \end{aligned} \quad (245)$$

$$(\text{THM}) : \text{eigenDecomposition}(S \Lambda S^{-1}, (A, V, +, \cdot, \|\$1\|)) \implies A^2 = (A)(A) = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1} \quad (246)$$

$$\begin{aligned} (\text{THM}) : \text{spectralDecomposition}(Q \Lambda Q^T, (A, V, +, \cdot, \|\$1\|)) &\iff (\text{symmetricOperator}(A, ())) \implies \\ (\exists_Q (\text{eigenDecomposition}(Q \Lambda Q^{-1}, (A, V, +, \cdot, \|\$1\|)) \wedge \text{orthogonalOperator}(Q, (V, +, \cdot, \|\$1\|))) \wedge (\lambda)_n \in \mathbb{R}^n) \\ \# \text{ if symmetric and eigenvalues are real, then there exists orthonormal eigenbasis} \end{aligned} \quad (247)$$

$$\begin{aligned} \text{hermitianAdjoint}(A^H, (A)) &\iff (A^H = \overline{A}^T) \iff (\langle A, A \rangle = \overline{A}^T A \in \mathbb{R}) \\ \# \text{ complex analog to adjoint} \end{aligned} \quad (248)$$

$$\begin{aligned} \text{hermitianOperator}(A, ()) &\iff A = A^H \\ \# \text{ complex analog to symmetric operator} \end{aligned} \quad (249)$$

$$\begin{aligned} \text{unitaryOperator}(Q^H Q, (Q)) &\iff Q^H Q = I \\ \# \text{ complex analog to orthogonal operator} \end{aligned} \quad (250)$$

$$\begin{aligned} \text{positiveDefiniteOperator}(A, (V, +, \cdot, \|\$1\|)) &\iff (\forall_{x \in V \setminus \{\mathbf{0}\}} (x^T A x > 0)) \vee \\ &(\forall_{x \in \text{eigenvectors}(X, (A, V, +, \cdot, \|\$1\|))} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)) \implies \lambda > 0)) \end{aligned}$$

acts like a positive scalar where any vector only scales and cannot reflect against its perpendicular axis (251)

$$(THM) : \text{positiveDefiniteOperator}(A^T A) \Leftarrow \forall_{x \in V \setminus \{0\}} (x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 > 0) \quad (252)$$

$$\begin{aligned} \text{semiPositiveDefiniteOperator}(A, (V, +, \cdot, \|\cdot\|_1)) &\Longleftrightarrow (\forall_{x \in V \setminus \{0\}} (x^T A x \geq 0)) \vee \\ &(\forall_{x \in \text{eigenvectors}(X, (A, V, +, \|\cdot\|_1))} (\text{eigenvalue}(\lambda, (x, A, V, +, \cdot)) \implies \lambda \geq 0)) \\ &\# \text{ acts like a nonnegative scalar} \end{aligned} \quad (253)$$

$$(THM) : \text{symmetricOperator}(A^T A) \Leftarrow (A^T A = (A^T A)^T = A^T A^{TT} = A^T A) \quad (254)$$

$$\text{similarOperators}((A, B), ()) \Longleftrightarrow (\text{matrix}(A, (n, n))) \wedge (\text{matrix}(B, (n, n))) \wedge (\exists_M (B = M^{-1} A M)) \quad (255)$$

$$\begin{aligned} (THM) : (\text{similarOperators}((A, B), ()) \wedge Ax = \lambda x) &\implies (\exists_M (M^{-1} A x = \lambda M^{-1} x = M^{-1} A M M^{-1} x = B M^{-1} x)) \\ &\# \text{ similar operators have the same eigenvalues but } M^{-1} \text{ shifted eigenvectors} \end{aligned} \quad (256)$$

$$\begin{aligned} \text{singularValueDecomposition}(Q \Sigma R^T, (A, V, +, \cdot, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2)) &\Longleftrightarrow (\text{orthogonalOperator}(R, (V, +, \cdot, \|\cdot\|_2))) \wedge \\ &(\text{orthogonalOperator}(Q, (\text{Img}(A), +, \cdot, \|\cdot\|_2))) \wedge (\text{semiPositiveDefiniteOperator}(\Sigma, (V, +, \cdot, \|\cdot\|_1))) \wedge \\ &(AR = Q \Sigma) \wedge (A = Q \Sigma R^{-1} = Q \Sigma R^T) \wedge (\text{symmetricOperator}(A^T A)) \wedge (\text{symmetricOperator}(AA^T)) \wedge \\ &(A^T A = R \Sigma^T Q^T Q \Sigma R^T = R \Sigma^T \Sigma R^T) \wedge (\text{spectralDecomposition}(R(\Sigma^T \Sigma)R^T, (A^T A, V, +, \cdot, \|\cdot\|_1))) \wedge \\ &(AA^T = Q \Sigma R^T R \Sigma^T Q^T = Q \Sigma \Sigma^T Q^T) \wedge (\text{spectralDecomposition}(Q(\Sigma \Sigma^T)Q^T, (AA^T, V, +, \cdot, \|\cdot\|_1))) \wedge \\ &(\text{diagonalOperator}(\Sigma^T \Sigma) \implies \text{normalOperator}(\Sigma^T \Sigma) = \Sigma \Sigma^T = \Sigma_{\sigma^2}) \wedge (\Sigma = \Sigma_{\sqrt{\sigma^2}} = \Sigma_{|\sigma|}) \\ &(THM) \text{ based on the spectral theorem:} \end{aligned} \quad (257)$$

$$\begin{aligned} \text{leftInverseOperator}(A_L^{-1}, (A)) &\Longleftrightarrow (\text{matrix}(A, (n, m))) \wedge (\text{rank}(A) = n < m) \wedge \\ &(A_L^{-1} A = I = ((A^T A)^{-1} A^T) A) \end{aligned} \quad (258)$$

$$\begin{aligned} \text{rightInverseOperator}(A_R^{-1}, (A)) &\Longleftrightarrow (\text{matrix}(A, (n, m))) \wedge (\text{rank}(A) = m < n) \wedge \\ &(A A_R^{-1} = I = A(A^T(AA^T)^{-1})) \end{aligned} \quad (259)$$

1.19 Functional analysis

$$\begin{aligned} \text{denseMap}(L, (D, H, +, \cdot, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2)) &\Longleftrightarrow (D \subseteq H) \wedge (\text{linearOperator}(L, (D, +, \cdot, H, +, \cdot))) \wedge \\ &\left(\text{innerProductTopology}(\mathcal{O}, (H, +, \cdot, \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2)) \right) \wedge \left(\text{dense}(D, (H, \mathcal{O}, d(\cdot, \cdot))) \right) \end{aligned} \quad (260)$$

$$\begin{aligned} \text{mapNorm}(\|L\|, (L, V, +_V, \cdot_V, \|\cdot\|_V, W, +_W, \cdot_W, \|\cdot\|_W)) &\Longleftrightarrow \\ &(\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W))) \wedge \\ &\left(\text{normedVectorSpace}((V, +_V, \cdot_V, \|\cdot\|_V), ()) \right) \wedge \left(\text{normedVectorSpace}((W, +_W, \cdot_W, \|\cdot\|_W), ()) \right) \wedge \\ &\left(\|L\| = \sup \left(\left\{ \frac{\|L f\|_W}{\|f\|_V} \mid f \in V \right\} \right) = \sup \left(\{ \|L f\|_W \mid f \in V \wedge \|f\|_V = 1 \} \right) \right) \end{aligned} \quad (261)$$

$$\begin{aligned} & \text{boundedMap}\left(L, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)\right) \iff \\ & \left(\text{mapNorm}\left(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)\right) < \infty\right) \end{aligned} \quad (262)$$

$$\begin{aligned} & \neg\text{boundedMap}\left(L, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)\right) \iff \\ & (U \subset V) \wedge \left(\infty = \text{mapNorm}\left(\|L\|_U, (L, U, +_U, \cdot_U, \|\$1\|_U, W, +_W, \cdot_W, \|\$1\|_W)\right) \leq \|L\|\right) \end{aligned} \quad (263)$$

$$\begin{aligned} & \text{extensionMap}\left(\widehat{L}, (L, V, D, W)\right) \iff (D \subseteq V) \wedge \left(\text{linearOperator}\left(L, (D, +_D, \cdot_D, W, +_W, \cdot_W)\right)\right) \wedge \\ & \left(\text{linearOperator}\left(\widehat{L}, (V, +_V, \cdot_V, W, +_W, \cdot_W)\right)\right) \wedge \left(\forall d \in D \left(\widehat{L}(d) = L(d)\right)\right) \end{aligned} \quad (264)$$

$$\begin{aligned} & \text{adjoint}\left(L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)\right) \iff \left(\text{hilbertSpace}\left((V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V), ()\right)\right) \wedge \\ & \left(\text{hilbertSpace}\left((W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W), ()\right)\right) \wedge \left(\text{linearOperator}\left(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)\right)\right) \wedge \\ & \left(\forall v \in V \forall w \in W \left(\langle Lv, w \rangle_W = \langle v, L^T w \rangle_V\right) \vee \left((Lv)^T w = v^T L^T w\right)\right) \\ & \# \text{ target operator that acts similar to the domain operator} \end{aligned} \quad (265)$$

$$\begin{aligned} & \text{selfAdjoint}\left(L, (V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)\right) \iff \\ & L = \text{adjoint}\left(L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)\right) \\ & \# \text{ also a generalization of symmetric matrices} \end{aligned} \quad (266)$$

$$\begin{aligned} & \text{compactMap}\left(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)\right) \iff \left(\text{boundedMap}\left(L, (V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)\right)\right) \wedge \\ & \left(\forall v \in V \left(\text{openBall}\left(B, (1.0, v, V, d_V(\$1, \$2))\right)\right) \implies \\ & \text{compactSubset}\left(\text{closure}\left(\overline{L(B)}, \text{image}(L(B), (B, L, V, W)), W, d_W(\$1, \$2)\right), (W, \mathcal{O}_W)\right)\right) \end{aligned} \quad (267)$$

$$\begin{aligned} & \text{(THM) Spectral thm.:} \\ & \left(\text{selfAdjoint}\left(L, (V, +, \cdot, \langle \$1, \$2 \rangle, V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right) \wedge \left(\text{compactMap}(L, (V, +, \cdot, V, +, \cdot))\right) \implies \\ & \left(\exists_{(e)_{\mathbb{N}} \subseteq V} \left(\text{orthonormalBasis}\left((e)_{\mathbb{N}}, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \wedge \forall e_n \in (e)_{\mathbb{N}} \left(\text{eigenvector}(e_n, (L, V, +, \cdot))\right)\right)\right) \implies \\ & \left(\exists_{(\lambda)_{\mathbb{N}} \subseteq \mathbb{R}^n} \forall e_n \in (e)_{\mathbb{N}} \exists \lambda_n \in (\lambda)_{\mathbb{N}} \left(\text{eigenvalue}(\lambda_n, (e_n, L, V, +, \cdot)) \wedge \lim_{n \rightarrow \infty} (\lambda_n = 0) \wedge L = \sum_{n=1}^{\infty} (\lambda_n e_n e_n^T)\right)\right) \\ & \# \text{ DEFINE} \end{aligned} \quad (268)$$

1.20 Function spaces

$$\text{curLp}(\mathcal{L}^p, (p, M, \sigma, \mu)) \iff (p \in \mathbb{R}) \wedge (1 \leq p < \infty) \wedge$$

$$\left(\mathcal{L}^p = \{ \text{map}(f, (M, \mathbb{R})) \mid \text{measurableMap}(f, (M, \sigma, \mathbb{R}, \text{euclideanSigma})) \wedge \int (|f|^p d\mu) < \infty \} \right) \quad (269)$$

$$\begin{aligned} \text{vecLp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) &\iff \left(\text{curLp}(\mathcal{L}^p, (p, M, \sigma, \mu)) \right) \wedge \left(\forall f, g \in \mathcal{L}^p \forall m \in M ((f + g)(m) = f(m) + g(m)) \right) \wedge \\ &\left(\forall f \in \mathcal{L}^p \forall s \in \mathbb{R} \forall m \in M ((s \cdot f)(m) = (s)f(m)) \right) \wedge \left(\text{vectorSpace}((\mathcal{L}^p, +, \cdot, ())) \right) \end{aligned} \quad (270)$$

$$\begin{aligned} \text{integralNorm}(\lambda \mathbb{1} \lambda, (+, \cdot, p, M, \sigma, \mu)) &\iff \left(\text{vecLp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \left(\text{map} \left(\lambda \mathbb{1} \lambda, (\mathcal{L}^p, \mathbb{R}_0^+) \right) \right) \wedge \\ &\left(\forall f \in \mathcal{L}^p \left(0 \leq \lambda f \lambda = \left(\int (|f|^p d\mu) \right)^{1/p} \right) \right) \end{aligned} \quad (271)$$

$$\begin{aligned} (\text{THM}) : \text{integralNorm}(\lambda \mathbb{1} \lambda, (+, \cdot, p, M, \sigma, \mu)) &\implies \\ \left(\forall f \in \mathcal{L}^p \left(\lambda f \lambda = 0 \implies \text{almostEverywhere}(f = \mathbf{0}, (M, \sigma, \mu)) \right) \right) & \\ \# \text{ not an expected property from a norm} & \quad (272) \end{aligned}$$

$$\begin{aligned} \text{Lp}(\mathcal{L}^p, ((+, \cdot, p, M, \sigma, \mu))) &\iff \left(\text{integralNorm}(\lambda \mathbb{1} \lambda, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \\ \left(L^p = \text{quotientSet} \left(\mathcal{L}^p / \sim, \left(\mathcal{L}^p, (\lambda \mathbb{1} + (-\mathbb{2}) \lambda = 0) \right) \right) \right) & \\ \# \text{ functions in } L^p \text{ that have finite integrals above and below the x-axis} & \quad (273) \end{aligned}$$

$$(\text{THM}) : \text{banachSpace} \left(\left(\text{Lp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)), +, \cdot, \lambda \mathbb{1} \lambda \right), () \right) \quad (274)$$

$$(\text{THM}) : \text{hilbertSpace} \left(\left(\text{Lp}(\mathcal{L}^p, (+, \cdot, 2, M, \sigma, \mu)), +, \cdot, \frac{\lambda \mathbb{1} + \mathbb{2} \lambda^2 - \lambda \mathbb{1} - \mathbb{2} \lambda^2}{4} \right), () \right) \quad (275)$$

$$\begin{aligned} \text{curL}(\mathcal{L}, (V, +_V, \cdot_V, \|\mathbb{1}\|_V, W, +_W, \cdot_W, \|\mathbb{1}\|_W)) &\iff \left(\text{banachSpace}((W, +_W, \cdot_W, \|\mathbb{1}\|_W), ()) \right) \wedge \\ \left(\text{normedVectorSpace}((V, +_V, \cdot_V, \|\mathbb{1}\|_V), ()) \right) &\wedge \\ \left(\mathcal{L} = \{ f \mid \text{boundedMap}(f, (V, +_V, \cdot_V, \|\mathbb{1}\|_V, W, +_W, \cdot_W, \|\mathbb{1}\|_W)) \} \right) & \end{aligned} \quad (276)$$

$$(\text{THM}) : \text{banachSpace} \left(\left(\text{curL}(\mathcal{L}, (V, +_V, \cdot_V, \|\mathbb{1}\|_V, W, +_W, \cdot_W, \|\mathbb{1}\|_W)), +, \cdot, \text{mapNorm} \right), () \right) \quad (277)$$

$$(\text{THM}) : \|L\| \geq \frac{\|Lf\|}{\|f\|} \quad \# \text{ from choosing an arbitrary element in the mapNorm sup} \quad (278)$$

$$\begin{aligned} (\text{THM}) : \left(\text{cauchy}((f)_{\mathbb{N}}, (\mathcal{L}, +, \cdot, \text{mapNorm})) \implies \text{cauchy}((f_n v)_{\mathbb{N}}, (W, +_W, \cdot_W, \|\mathbb{1}\|_W)) \right) &\iff \\ \left(\forall \epsilon' > 0 \forall v \in V (\|f_n v - f_m v\|_W = \|(f_n - f_m)v\|_W \leq \|f_n - f_m\| \cdot \|v\|_V < \epsilon' \cdot \|v\|_V = \epsilon') \right) & \\ \# \text{ a cauchy sequence of operators maps to a cauchy sequence of targets} & \quad (279) \end{aligned}$$

$$\begin{aligned}
(\text{THM}) \text{ BLT thm.: } & \left(\left(\text{dense}(D, (V, \mathcal{O}, d_V)) \wedge \text{boundedMap}(A, (D, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) \right) \right) \implies \\
& \left(\exists!_{\hat{A}} \left(\text{extensionMap}(\hat{A}, (A, V, D, W)) \right) \wedge \|\hat{A}\| = \|A\| \right) \iff \\
& \left(\forall_{v \in V} \exists_{(v)_{\mathbb{N}} \subseteq D} \left(\lim_{n \rightarrow \infty} (v_n = v) \right) \right) \wedge \left(\hat{A}v = \lim_{n \rightarrow \infty} (Av_n) \right) \quad (280)
\end{aligned}$$

2 Probability Theory

2.1 Definitions

$$\text{randomExperiment}(E, (\Omega)) \iff \Omega = \{\omega \mid \text{experiment} = E \rightarrow \text{outcome} = \omega\} \quad (281)$$

$$\text{probabilitySpace}((\Omega, \mathcal{F}, P), ()) \iff \text{measureSpace}((\Omega, \mathcal{F}, P), ()) \wedge (P(\Omega) = 1) \quad (282)$$

$$\begin{aligned}
& \text{event}(F, (\Omega, \mathcal{F}, P)) \iff \left(\text{probabilitySpace}((\Omega, \mathcal{F}, P), ()) \right) \wedge (F \in \mathcal{F}) \\
& \# F \text{ can represent both singleton outcomes and outcome combinations and } \mathcal{F} \text{ can represent} \\
& \# \text{ a countable event that contains outcomes with even number of coin tosses before the first head} \\
& \# \mathcal{P}(\mathbb{R}) \text{ sets are not considered because definite uniform measures diverge everywhere} \\
& \# \mathcal{P}(\mathbb{N}) \text{ sets can be assigned a meaningful convergent measure e.g., } \forall_{k \in \mathbb{R}^+} \forall_{f \in F} P(\{f\}) = k^{-f} \quad (283)
\end{aligned}$$

$$\begin{aligned}
(\text{THM}) : & \left(\text{probabilitySpace}((\Omega, \mathcal{F}, P), ()) \wedge F, A, B \in \mathcal{F} \right) \implies \\
& \left(F^C \cup F = \Omega \wedge F^C \cap F = \emptyset \implies P(F^C) + P(F) = 1 \implies P(F^C) = 1 - P(F) \right) \wedge \\
& \left(P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - \left(1 - P(A^C \cup B^C) \right) = \right. \\
& \quad \left. P(A) + P(B) - 1 + P(A^C) + P(B^C) - P(A^C \cap B^C) = \right. \\
& \quad \left. P(A) + P(B) - 1 + 1 - P(A) + 1 - P(B) - \left(1 - P(A \cup B) \right) = P(A \cup B) \right) \wedge \\
& \left(P\left(\bigcup_{i=1}^n (A_i) \right) = \sum_{k=1}^n \left((-1)^{k-1} \sum_{I \subseteq \mathbb{N}_1^n \wedge |I|=k} \left(P\left(\bigcap_{i \in I} (A_i) \right) \right) \right) \right) \quad (284)
\end{aligned}$$

$$\begin{aligned}
(\text{THM}) : & \left(\text{measureSpace}((\Omega, \mathcal{F}, P), ()) \wedge (A)_{\mathbb{N}}, (B)_{\mathbb{N}} \subseteq \mathcal{F} \wedge A, B \in \mathcal{F} \right) \implies \\
& \text{CL285} \left(B_n = A_n \setminus \bigcup_{i=1}^{n-1} (A_i) \right) \wedge \text{DL285}_{\text{CL285}} \left(\forall_{i \in \mathbb{N}} \forall_{j \in \mathbb{N} \setminus \{i\}} (B_i \cap B_j = \emptyset) \right) \wedge \text{EL285}_{\text{CL285}} \left(\bigcup_{i \in \mathbb{N}} (A_i) = \bigcup_{i \in \mathbb{N}} (B_i) \right) \wedge \\
& \text{1IL285}_{\text{DL285}_{\text{measure}}} \left(P\left(\bigcup_{i \in \mathbb{N}} (B_i) \right) = \sum_{i \in \mathbb{N}} (P(B_i)) \right) \wedge \text{2IL285}_{\text{limit}} \left(\sum_{i \in \mathbb{N}} (P(B_i)) = \lim_{m \rightarrow \infty} \left(\sum_{i=1}^m (P(B_i)) \right) \right) \wedge \\
& \text{3IL285}_{\text{DL285}_{\text{measure}}} \left(\lim_{m \rightarrow \infty} \left(\sum_{i=1}^m (P(B_i)) \right) = \lim_{m \rightarrow \infty} \left(P\left(\bigcup_{i=1}^m (B_i) \right) \right) \right) \wedge
\end{aligned}$$

$$\begin{aligned}
& \textcolor{blue}{4IL285} \textcolor{blue}{EL285} \left(\lim_{m \rightarrow \infty} \left(P \left(\bigcup_{i=1}^m (B_i) \right) \right) = \lim_{m \rightarrow \infty} \left(P \left(\bigcup_{i=1}^m (A_i) \right) \right) \right) \wedge \\
& \textcolor{teal}{MSCont} \textcolor{blue}{EL285} \textcolor{blue}{1IL285} \textcolor{blue}{2IL285} \textcolor{blue}{3IL285} \textcolor{blue}{4IL285} \left(P \left(\bigcup_{i \in \mathbb{N}} (A_i) \right) = \lim_{m \rightarrow \infty} \left(P \left(\bigcup_{i=1}^m (A_i) \right) \right) \right) \wedge \\
& \textcolor{teal}{MSConvL} \textcolor{blue}{MSCont} \left(\forall_{j \in \mathbb{N}} (A_j \subseteq A_{j+1}) \implies P \left(\bigcup_{i \in \mathbb{N}} (A_i) \right) = \lim_{m \rightarrow \infty} (P(A_m)) \right) \wedge \\
& \textcolor{teal}{MSConvU} \textcolor{blue}{MSConvL} \textcolor{blue}{DeMorgans} \left(\forall_{j \in \mathbb{N}} (A_{j+1} \subseteq A_j) \implies P \left(\bigcap_{i \in \mathbb{N}} (A_i) \right) = \lim_{m \rightarrow \infty} (P(A_m)) \right) \wedge \\
& \textcolor{teal}{MSSetOrder} \textcolor{blue}{measure} (A \subseteq B \implies P(A) \leq P(B)) \wedge \textcolor{teal}{MSSetBound} \textcolor{blue}{measure} \left(\bigcup_{i \in \mathbb{N}} (A_i) \leq \sum_{i \in \mathbb{N}} (P(A_i)) \right) \quad (285)
\end{aligned}$$

2.2 Random variables

$$\begin{aligned}
& \textcolor{teal}{randomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{probabilitySpace}(\Omega, \mathcal{F}, P)) \wedge (\textcolor{blue}{map}(X, (\Omega, \mathbb{R}))) \wedge \\
& \quad (\textcolor{blue}{measurableMap}(X, (\Omega, \mathcal{F}, \mathbb{R}, \textcolor{blue}{euclideanSigma})))
\end{aligned}$$

maps elementary outcomes to a numeric value and the measurable space to the real line and borel sets (286)

$$\begin{aligned}
& \textcolor{teal}{PL}(P_X, (X, \Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\
& \quad \left(\forall_{B \in \sigma_S} \left(P_X(B) = P(\{\omega \in \Omega \mid X(\omega) \in B\}) = (P \circ X^{-1})(B) = P(X \in B) \right) \right)
\end{aligned}$$

probability of borel set events occuring and equips probabilities to numeric valued borel sets (287)

$$(\text{THM}) : \textcolor{blue}{probabilitySpace}(\mathbb{R}, \textcolor{blue}{euclideanSigma}, P_X) \quad (288)$$

$$\begin{aligned}
& \textcolor{teal}{generatedSigmaAlgebra}(\sigma(\mathcal{M}), (\mathcal{M}, S)) \iff (\mathcal{M} \subseteq \mathcal{P}(S)) \\
& \quad (\textcolor{blue}{sigmaAlgebra}(\sigma(\mathcal{M}), (S)) = \bigcap \{ \mathcal{H} \mid \mathcal{M} \subseteq \textcolor{blue}{sigmaAlgebra}(\mathcal{H}, S) \})
\end{aligned}$$

the smallest sigma algebra containing the generating sets (289)

$$\textcolor{teal}{piSystem}(\mathcal{G}, (\Omega)) \iff (\mathcal{G} \subseteq \mathcal{P}(\Omega)) \wedge (\forall_{A, B \in \mathcal{G}} (A \cap B \in \mathcal{G})) \quad (290)$$

$$(\text{THM}) \text{ pi measure extension: } \left(\textcolor{blue}{piSystem}(\mathcal{G}, (\Omega)) \wedge \textcolor{blue}{probabilitySpace}(\Omega, \sigma(\mathcal{G}), \lambda) \wedge \right.$$

$$\left. \textcolor{blue}{probabilitySpace}(\Omega, \sigma(\mathcal{G}), \mu) \wedge \exists_{(S)_{\mathbb{N}} \subseteq \Omega} \left(\bigcup_{i \in \mathbb{N}} ((S)_i) = \Omega \wedge \lambda(\Omega) < \infty \right) \right) \implies$$

$$\left(\forall_{G \in \mathcal{G}} (\lambda(G) = \mu(G)) \implies \forall_{F \in \sigma(\mathcal{G})} (\lambda(F) = \mu(F)) \right)$$

PL in terms of a simpler generating pi system (291)

$$(\text{THM}) : (\textcolor{blue}{piSystem}(\{(-\infty, x] \mid x \in \mathbb{R}\}, (\mathbb{R}))) \wedge (\textcolor{blue}{euclideanSigma} = \sigma(\{(-\infty, x] \mid x \in \mathbb{R}\})) \quad (292)$$

$$\begin{aligned}
& \textcolor{teal}{CDF}(F_X, (X, \Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\
& \left(\forall_{x \in \mathbb{R}} \left(P(\{\omega \in \Omega \mid X(\omega) \in (-\infty, x]\}) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) = P(X \leq x) = F_X(x) \right) \right) \\
& \quad \# \text{ PL of the semi infinite pi system on the real numbers} \\
& \# \text{ specifies PL following pi measure extension theorem but simpler than definitions on complex borel sets} \quad (293)
\end{aligned}$$

$$\begin{aligned}
& (\text{THM}) : \textcolor{teal}{CDF}(F_X, (X, \Omega, \mathcal{F}, P)) \implies \left(\lim_{x \rightarrow -\infty} (F_X(x)) = 0 \right) \wedge \left(\lim_{x \rightarrow \infty} (F_X(x)) = 1 \right) \wedge \\
& \left(\forall_{x_1, x_2 \in \mathbb{R}} (x_1 \leq x_2 \implies F_X(x_1) \leq F_X(x_2)) \right) \wedge \left((e)_{\mathbb{N}} \subseteq \mathbb{R}_0^+ \right) \wedge \left(\lim_{n \rightarrow \infty} (e_n) = 0 \right) \wedge \\
& \left(\forall_{x \in \mathbb{R}} \left(\lim_{\epsilon \rightarrow 0^+} (F(x + \epsilon)) = \lim_{n \rightarrow \infty} (F(x + e_n)) = \lim_{n \rightarrow \infty} (P(X \leq x + e_n)) = \lim_{n \rightarrow \infty} (P(\{\omega \in \Omega \mid X(\omega) \leq x + e_n\})) = \right. \right. \\
& \quad \left. \left. P \left(\bigcap_{n=1}^{\infty} (\{\omega \in \Omega \mid X(\omega) \leq x + e_n\}) \right) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}) = F_X(x) \right) \right) \\
& \quad \# \text{ depends on the nested decreasing subsets induced by the limit from right} \quad (294)
\end{aligned}$$

$$\begin{aligned}
& \textcolor{teal}{PMF}(H_X, (X, \Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\
& \left(\forall_{x \in \mathbb{R}} \left(H_X(x) = P(\{\omega \in \Omega \mid X(\omega) = x\}) = P(X = x) \right) \right) \\
& \quad \# \text{ type of probability law} \quad (295)
\end{aligned}$$

$$0000000000000000 \textcolor{teal}{SPLITSECTIONSFIXPMFTORELATETOPL} \quad (296)$$

$$\begin{aligned}
& \textcolor{teal}{indicatorRandomVariable}(I_A, (\Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{randomVariable}(I_A, (\Omega, \mathcal{F}, P))) \wedge \\
& \left(\forall_{A \in \mathcal{F}} \forall_{\omega \in \Omega} (I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}) \right) \quad (297)
\end{aligned}$$

$$(\text{THM}) : \text{measures on } \mathbf{R} = \text{discrete, continous, and singular components} \quad (298)$$

$$\begin{aligned}
& \textcolor{teal}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\
& (\exists_{E \subseteq \mathbb{R}} (\textcolor{blue}{countablyInfinite}(E) \wedge P_X(E) = 1)) \wedge (\cup((e)_{\mathbb{N}}) = E) \wedge (\forall_{i \in \mathbb{N}} (e_i \in E)) \quad (299)
\end{aligned}$$

$$\begin{aligned}
& (\text{THM}) : (\textcolor{blue}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \implies \\
& (1 = P(E) = \sum_{i \in \mathbb{N}} (P_X(\{e_i\})) = \sum_{i \in \mathbb{N}} (P(X = e_i))) \wedge (\forall_{B \in \sigma_S} (P_X(B) = \sum_{x \in E \cap B} (P(X = x)))) \quad (300)
\end{aligned}$$

$$\begin{aligned}
& \textcolor{teal}{bernoulliRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge (E = \{0, 1\}) \wedge \\
& (p \in \mathbb{R}) \wedge (P_X = P(X = x) = \begin{cases} 1 & x = 1 \\ 0 & x = 0 \end{cases}) \quad (301)
\end{aligned}$$

$$\begin{aligned}
& \textcolor{teal}{uniformRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\
& (n = |\textcolor{blue}{finiteSet}(E)|) \wedge (\forall_{i \in \mathbb{N} \wedge i \leq n} (P_X(\{e_i\}) = P(X = e_i) = \frac{1}{n})) \quad (302)
\end{aligned}$$

$$\textcolor{teal}{geometricRandomVariable}(X, (\Omega, \mathcal{F}, P)) \iff (\textcolor{blue}{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge$$

$$(\text{countablyInfinite}(E)) \wedge (p \in \mathbb{R}) \wedge (\forall_{i \in \mathbb{N}} (P_X(\{e_i\}) = P(X = e_i) = (1-p)^{i-1}p)) \quad (303)$$

$$\begin{aligned} \text{binomialRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (n = |\text{finiteSet}(E)|) \wedge (p \in \mathbb{R}) \wedge (\forall_{i \in \mathbb{N}} (P_X(\{e_i\}) = P(X = e_i) = \binom{n}{i} p^i (1-p)^{n-i})) \end{aligned} \quad (304)$$

$$\begin{aligned} \text{poissonRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{discreteRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\text{countablyInfinite}(E)) \wedge (\lambda \in \mathbb{R}^+) \wedge (\forall_{i \in \mathbb{N}} (P_X(\{e_i\}) = P(X = e_i) = \frac{e^{-\lambda} \lambda^i}{i!})) \end{aligned} \quad (305)$$

$$\begin{aligned} \text{absolutelyContinuous}((f, g), (M, \sigma)) &\iff (\text{measure}(f, (M, \sigma))) \wedge (\text{measure}(g, (M, \sigma))) \wedge \\ (\forall_{A \in \sigma} (g(A) = 0 \implies f(A) = 0)) \end{aligned} \quad (306)$$

$$\begin{aligned} \text{(THM) Radon-Nikodym: } &(\text{measurableSpace}((M, \sigma), ())) \wedge (\text{finiteMeasure}(\mu, (M, \sigma))) \wedge \\ &(\text{finiteMeasure}(\nu, (M, \sigma))) \wedge (\text{absolutelyContinuous}((\nu, \mu), (M, \sigma))) \implies \\ &(\exists_{\text{map}(f, (M, \mathbb{R}^+))} \forall_{A \in \sigma} (\nu(A) = \int_A (f d\mu))) \\ &\# \text{ connects } P_X = F_X = \int (f_x dx) \end{aligned} \quad (307)$$

$$\begin{aligned} \text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\text{absolutelyContinuous}((P_X, \text{lebesgueMeasure}), (\mathbb{R}, \text{euclideanSigma}))) \\ &\# \text{ the probabilities lie on nonzero lebesgue measure sets} \end{aligned} \quad (308)$$

$$\begin{aligned} \text{contUniformRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (a, b \in \mathbb{R}) \wedge (a < b) \wedge (P_X = F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}) \end{aligned} \quad (309)$$

$$\begin{aligned} \text{exponentialRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\lambda \in \mathbb{R}^+) \wedge (P_X = F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}) \end{aligned} \quad (310)$$

$$\text{memorylessRandomVariable}(X, ()) \iff (\forall_{\omega \in \Omega} (X(\omega) \geq 0)) \wedge (\forall_{s, t \in \mathbb{R}_0^+} (P(X > s) = P(X > s+t | x > t))) \quad (311)$$

$$\begin{aligned} \text{gaussianRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\mu \in \mathbb{R}) \wedge (\sigma \in \mathbb{R}^+) \wedge (P_X = F_X(x) = \int (\frac{\exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sigma\sqrt{2\pi}} dx)) \end{aligned} \quad (312)$$

$$\text{(THM) : DEFINE gaussian is stable and is an attractor} \quad (313)$$

$$\begin{aligned} \text{simplifiedCauchyRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{continuousRandomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (P_X = F_X(x) = \int (\frac{1}{\pi(1+x^2)} dx)) \end{aligned} \quad (314)$$

$$\begin{aligned} \text{singularRandomVariable}(X, (\Omega, \mathcal{F}, P)) &\iff (\text{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ (\forall_{x \in \mathbb{R}} (P_X(\{x\}) = 0)) \wedge (\exists_{F \in \text{euclideanSigma}} (P_X(F) = 1 \wedge \text{lebesgueMeasure}(F) = 0)) \end{aligned}$$

an example is uniform measure on the Cantor set (315)

$$\begin{aligned} & \text{preimageSigma}(\sigma(X), (X, \Omega, \mathcal{F}, P)) \iff (\text{randomVariable}(X, (\Omega, \mathcal{F}, P))) \wedge \\ & (\sigma(X) = \{A \subseteq \Omega \mid B \in \text{euclideanSigma} \wedge \text{preimage}(A, (B, X, \Omega, \mathbb{R}))\}) \wedge (\text{subSigmaAlgebra}(\sigma(X), (\mathcal{F}, \Omega))) \\ & \# X(\omega) \in B \text{ determines all events } A \text{ that occurs} \end{aligned} \quad (316)$$

$$\begin{aligned} & n\text{RandomVariable}((X)_n, (\Omega, \mathcal{F}, P)) \iff (\forall_{i \in \mathbb{N} \wedge i \leq n} (\text{randomVariable}(X_i, (\Omega, \mathcal{F}, P)))) \wedge \\ & (\text{measurableMap}((X)_n, (\Omega, \mathcal{F}, \mathbb{R}^n, \text{euclideanSigma}^n))) \\ & \# \text{ if the preimage of a measurable set of } n \text{ dimensional vectors is an event} \end{aligned} \quad (317)$$

$$\begin{aligned} (\text{THM}) : \text{PL}(P_{X,Y}, ((X,Y), \Omega, \mathcal{F}, P)) & \iff (\forall_{B \in \sigma_{S^2}} (P_{X,Y}(B) = P(\{\omega \in \Omega \mid (X(\omega), Y(\omega)) \in B\}))) \\ & \# \text{ joint PL} \end{aligned} \quad (318)$$

$$\begin{aligned} (\text{THM}) : \text{CDF}(F_{X,Y}, ((X,Y), \Omega, \mathcal{F}, P)) & \iff \\ \forall_{x,y \in \mathbb{R}} (F_{X,Y}(x,y) = P(\{\omega \in \Omega \mid X(\omega) \leq x\} \cap \{\omega \in \Omega \mid Y(\omega) \leq y\})) & = P(X \leq x, Y \leq y) \\ & \# \text{ joint CDF} \end{aligned} \quad (319)$$

$$\begin{aligned} (\text{THM}) : \text{CDF}(F_{X,Y}, ((X,Y), \Omega, \mathcal{F}, P)) & \iff (\lim_{\substack{x \rightarrow -\infty \\ y \rightarrow -\infty}} (F_{X,Y}(x,y)) = 0) \wedge (\lim_{\substack{x \rightarrow \infty \\ y \rightarrow \infty}} (F_{X,Y}(x,y)) = 1) \wedge \\ & (\forall_{x_1, x_2, y_1, y_2 \in \mathbb{R}} ((x_1 \leq x_2 \wedge y_1 \leq y_2) \implies (F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)))) \wedge \\ & (\forall_{x,y \in \mathbb{R}} (\lim_{\substack{\epsilon_x \rightarrow 0^+ \\ \epsilon_y \rightarrow 0^+}} (F(x + \epsilon_x, y + \epsilon_y)) = F(x + y))) \\ & (\forall_{x \in \mathbb{R}} (\lim_{y \rightarrow \infty} (F_{X,Y}(x, y)) = F_X(x))) \wedge (\forall_{y \in \mathbb{R}} (\lim_{x \rightarrow \infty} (F_{X,Y}(x, y)) = F_Y(y))) \\ & \# \text{ limit evaluation order or trajectory should not matter} \end{aligned} \quad (320)$$

2.3 CONDITIONALS Probability Theory

(321)

$$(\text{THM}) : (\text{cantor set} \cong \mathcal{P}(\mathbb{N}) \wedge (\mathbb{R}, \text{euclideanSigma}, \text{lebesgueMeasure})) \implies P(\text{cantor set}) = 0 \# :O \quad (322)$$

$$\begin{aligned} & \text{conditionalProbability}(P(A|B), (A, B, \Omega, \mathcal{F}, P)) \iff (\text{probabilitySpace}(\Omega, \mathcal{F}, P)) \wedge (A, B \in \mathcal{F}) \wedge \\ & (P(B) > 0) \wedge \left(P(A|B) = \frac{P(A \cap B)}{P(B)} \vee P(B)P(A|B) = P(A \cap B) \right) \\ & \# \text{ calculates } P(A) \text{ for the subset spanned by } B \\ & \# \text{ conditioning on 0 probability sets leads to paradoxes} \end{aligned} \quad (323)$$

$$(\text{THM}) : (\text{probabilitySpace}(\Omega, \mathcal{F}, P) \wedge P(B) > 0) \implies \forall_{F \in \mathcal{F}} (P'(F) = P(F|B)) \wedge \text{probabilitySpace}(\Omega, \mathcal{F}, P') \quad (324)$$

$$\begin{aligned} & \text{independentEvents}((A, B), (\Omega, \mathcal{F}, P)) \iff (A, B \in \mathcal{F}) \wedge (P(A \cap B) = P(A)P(B)) \\ & \# \text{ depends on the } P, \text{ not only on } A, B \end{aligned} \quad (325)$$

$$\text{setPartition}((X)_{\mathbb{N}}, (Y)) \iff \left(\bigcup_{i \in \mathbb{N}} (X_i) = Y \right) \wedge \left(\forall_{i \in \mathbb{N}} \forall_{j \in \mathbb{N} \setminus \{i\}} (X_i \cap X_j = \emptyset) \right) \quad (326)$$

$$\begin{aligned}
(\text{THM}) : & \left(\text{probabilitySpace}(\Omega, \mathcal{F}, P) \wedge \{A\} \cup (B)_{\mathbb{N}} \subseteq \mathcal{F} \wedge \text{setPartition}((B)_{\mathbb{N}}, (\Omega)) \right) \implies \\
& \left(P(A) = \sum_{i \in \mathbb{N}} \left(P(A|B_i) P(B_i) \right) \right) \wedge \\
& \left(\forall_{i \in \mathbb{N}} \left(P(A|B_i) P(B_i) = P(A) P(B_i|A) = \left(\sum_{j \in \mathbb{N}} \left(P(B_j|A) \right) \right) P(B_i|A) \right) \right) \wedge \\
& \left(P \left(\bigcap_{i \in \mathbb{N}} (B_i) \right) = P(B_1) \prod_{i=2}^{\infty} \left(P \left(B_i \mid \bigcap_{j=1}^{i-1} (B_j) \right) \right) \right)
\end{aligned}$$

from the subspace definition of conditional probability and algebraic manipulations (327)

$$\begin{aligned}
\text{finIndEvents}((A)_{\mathbb{N}_k}, (\Omega, \mathcal{F}, P)) & \iff (\text{probabilitySpace}(\Omega, \mathcal{F}, P)) \wedge (k \in \mathbb{N}) \wedge \\
& (A_{\mathbb{N}_k} \subseteq \mathcal{F}) \wedge \left(\forall_{I_0 \in \mathcal{P}(\mathbb{N}_k) \setminus \emptyset} \left(P \left(\bigcap_{i \in I_0} (A_i) \right) = \prod_{i \in I_0} (P(A_i)) \right) \right) \\
& \# \text{ every combination of subsets must be independent} \quad (328)
\end{aligned}$$

$$\begin{aligned}
& \text{infIndEvents}((A)_I, (\Omega, \mathcal{F}, P)) \iff \\
& \left(\forall_{I_F \subseteq I} \left(\text{finiteSet}(I_F) \implies \text{finIndEvents}((A)_{I_F}, (\Omega, \mathcal{F}, P)) \right) \right) \quad (329)
\end{aligned}$$

$$\text{subSigmaAlgebra}(\mathcal{B}, (\mathcal{F}, \Omega)) \iff (\text{sigmaAlgebra}(\mathcal{F}, (\Omega))) \wedge (\text{sigmaAlgebra}(\mathcal{B}, (\Omega))) \wedge (\mathcal{B} \subseteq \mathcal{A}) \quad (330)$$

$$\begin{aligned}
\text{independentSigmaAlgebras}((\mathcal{A}, \mathcal{B}), (\Omega, \mathcal{F}, P)) & \iff (\text{probabilitySpace}(\Omega, \mathcal{F}, P)) \wedge \\
& (\text{subSigmaAlgebra}(\mathcal{A}, (\mathcal{F}, \Omega))) \wedge (\text{subSigmaAlgebra}(\mathcal{B}, (\mathcal{F}, \Omega))) \wedge \\
& \left(\forall_{A \in \mathcal{A}} \forall_{B \in \mathcal{B}} (\text{independentEvents}((A, B), (\Omega, \mathcal{F}, P))) \right) \quad (331)
\end{aligned}$$

$$\begin{aligned}
\text{infIndSigmaAlgebras}((\mathcal{A})_I, (\Omega, \mathcal{F}, P)) & \iff \left(\forall_{i \in I} (\text{subSigmaAlgebra}(\mathcal{A}_i), (\mathcal{F}, \Omega)) \right) \wedge \\
& (\forall_{i \in I} (F_i \in \mathcal{A}_i)) \wedge (\text{infIndEvents}((F)_I, (\Omega, \mathcal{F}, P))) \quad (332)
\end{aligned}$$

$$\text{infinitelyOften}(\{A_n \text{ i-o}\}, ()) \iff \left(B_n = \bigcup_{i=n}^{\infty} (A_i) \in \mathcal{F} \right) \wedge \left(\{A_n \text{ i-o}\} = \bigcap_{n \in \mathbb{N}} (B_n) = \bigcap_{n \in \mathbb{N}} \bigcup_{i=n}^{\infty} (A_i) \in \mathcal{F} \right)$$

the event that infinitely many A_n 's will occur

B_n occur if some event within the n th-tail-end event $A_i | i \geq n$ occur, which follows from \cup

$\{A_n \text{ i-o}\}$ occur if every tail-end event B_n occur for all n , which follows from \cap

similarly, $\{A_n \text{ i-o}\}$ occur, for all values of n , the n th-tail-end event occur (333)

$$(\text{THM}) \text{ BCL 1: } \left(\text{Cond300} \left(\sum_{n \in \mathbb{N}} (P(A_n)) < \infty \right) \implies (P(\{A_n \text{ i-o}\}) = 0) \right) \Leftarrow$$

$$\begin{aligned}
& \stackrel{\substack{1IL300 \\ \text{infinitelyOften} \\ MSContU}}{\left(P\left(\bigcap_{n \in \mathbb{N}} (B_n)\right) = \lim_{n \rightarrow \infty} (P(B_n)) = \lim_{n \rightarrow \infty} \left(P\left(\bigcup_{i=n}^{\infty} (A_i)\right) \right) \right)} \wedge \\
& \stackrel{\substack{2IL300 \\ MSSetBound}}{\left(\lim_{n \rightarrow \infty} \left(P\left(\bigcup_{i=n}^{\infty} (A_i)\right) \right) \leq \lim_{n \rightarrow \infty} \left(\sum_{i=n}^{\infty} (P(A_i)) \right) \right)} \wedge \\
& \stackrel{\substack{3IL300 \\ Cond300}}{\left(\lim_{n \rightarrow \infty} \left(\sum_{i=n}^{\infty} (P(A_i)) \right) = 0 \right)} \wedge \stackrel{\substack{Impl300 \\ 1IL300 \\ 2IL300 \\ 3IL300}}{\left(0 \leq P(\{A_n \text{ i-o}\}) \leq 0 \right)} \quad (334)
\end{aligned}$$

$$(\text{THM}) : \stackrel{\text{logp}}{\left(\forall_{x \in [0,1]} (\log(1-x) \leq -x) \right)} \quad (335)$$

$$\begin{aligned}
& (\text{THM}) : \stackrel{\text{sump}}{\left(\left(\stackrel{1Cond302}{\left(\forall_{i \in \mathbb{N}} (p_i \in [0,1]) \right)} \wedge \stackrel{2Cond302}{\left(\sum_{i \in \mathbb{N}} (p_i) = \infty \right)} \right) \Rightarrow \prod_{i \in \mathbb{N}} (1-p_i) = 0 \right)} \Leftarrow \\
& \stackrel{1IL302}{\left(\prod_{i \in \mathbb{N}} (1-p_i) = \exp \left(\log \left(\prod_{i \in \mathbb{N}} (1-p_i) \right) \right) = \exp \left(\log \left(\lim_{n \rightarrow \infty} \left(\prod_{i=1}^n (1-p_i) \right) \right) \right) \right)} \wedge \\
& \stackrel{\substack{2IL302 \\ \text{logp}}}{\left(\exp \left(\log \left(\lim_{n \rightarrow \infty} \left(\prod_{i=1}^n (1-p_i) \right) \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n (\log(1-p_i)) \right) \right) \leq \exp \left(\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n (-p_i) \right) \right) \right)} \wedge \\
& \stackrel{\substack{3IL302 \\ 2Cond302}}{\left(\exp \left(\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n (-p_i) \right) \right) = \exp(-\infty) = 0 \right)} \wedge \stackrel{\substack{Impl302 \\ 1Cond302 \\ 1IL302 \\ 2IL302 \\ 3IL302}}{\left(0 \leq \prod_{i \in \mathbb{N}} (1-p_i) \leq 0 \right)} \quad (336)
\end{aligned}$$

$$\begin{aligned}
& (\text{THM}) \text{ BCL 2: } \left(\left(\stackrel{1Cond303}{\left(\sum_{n \in \mathbb{N}} (P(A_n)) = \infty \right)} \wedge \stackrel{2Cond303}{\left(\text{infIndEvents}((A)_{\mathbb{N}}) \right)} \right) \Rightarrow P(\{A_n \text{ i-o}\}) = 1 \right) \\
& \Leftarrow \stackrel{\substack{1IL303 \\ MSSetBound}}{\left(1 - P(\{A_n \text{ i-o}\}) = P(\{A_n \text{ i-o}\}^C) = P\left(\bigcup_{n \in \mathbb{N}} (B_n^C)\right) \leq \sum_{n \in \mathbb{N}} (P(B_n^C)) \right)} \wedge \\
& \stackrel{\substack{2IL303 \\ \text{DeMorgans} \\ 2Cond303}}{\left(\sum_{n \in \mathbb{N}} (P(B_n^C)) = \sum_{n \in \mathbb{N}} \left(P\left(\bigcap_{i=n}^{\infty} (A_i^C)\right) \right) = \sum_{n=1}^{\infty} \left(\prod_{i=n}^{\infty} (P(A_i^C)) \right) = \sum_{n=1}^{\infty} \left(\prod_{i=n}^{\infty} (1-P(A_i)) \right) \right)} \wedge \\
& \stackrel{\substack{3IL303 \\ 1Cond303 \\ \text{sump}}}{\left(\sum_{n=1}^{\infty} \left(\prod_{i=n}^{\infty} (1-P(A_i)) \right) = \sum_{n=1}^{\infty} (0) = 0 \right)} \wedge \stackrel{\substack{Impl303 \\ 1IL303 \\ 2IL303 \\ 3IL303}}{\left(0 \leq 1 - P(\{A_n \text{ i-o}\}) \leq 0 \iff P(\{A_n \text{ i-o}\}) = 1 \right)} \quad (337)
\end{aligned}$$

$$\text{TODOFIXUPSECTIONINGANDFORMATTING} \quad (338)$$

$$\text{===== N O T = U P D A T E D =====} \quad (339)$$

$$S^n = (x, y)^n \subset Z \# \text{ sample set consists of } n \text{ input-output pairs} \quad (340)$$

$$S^n \implies \text{map}(f_{S^n}, (X, Y)) \# \text{ learned predictor function} \quad (341)$$

$$V \# \text{ loss function} \quad (342)$$

$$I_n[f] = \frac{1}{n} \sum_i (V(f(x_i), y_i)) \# \text{ empirical predictor error} \quad (343)$$

$$I[f] = \int_Z (V(f(x_i), y_i) d\mu(x_i, y_i)) \# \text{ expected predictor error} \quad (344)$$

$$f_\star \# \text{ optimal or lowest expected error hypothesis} \quad (345)$$

$$\lim_{n \rightarrow \infty} (I[f_n]) = I[f_\star] \# \text{ consistency: expected error of learned approaches best hypothesis} \quad (346)$$

$$\lim_{n \rightarrow \infty} (I_n[f_n]) = I[f_n] \# \text{ generalization: empirical error of learned hypothesis approximates expected error} \quad (347)$$

$$|I_n[f_n] - I[f_n]| < \epsilon(n, \delta) \text{ with P } 1 - \delta? \# \text{ generalization error: measure performance of learning algorithm}$$

$$\forall_{\epsilon > 0} \left(\lim_{n \rightarrow \infty} (P(\{|I_n[f_n] - I[f_n]| \geq \epsilon\})) = 0 \right)$$

$$\# \quad (348)$$

$$X \# \text{ random variable ; } \mu \# \text{ probability measure} \quad (349)$$

$$\text{measureSpace}(X, F, P) \quad (350)$$

$$IID(A, (X, P)) \iff (A \in F \subseteq X) \wedge P_{a_1, a_2, \dots} (a_1 = t_1, a_2 = t_2, \dots) = \prod_i (P_{a_i} (a_i = t_i))$$

$$\# \text{ outcomes are independent and equally likely} \quad (351)$$

$$E[X] = \int_{Range} (x d(P(x))) \quad (352)$$

$$0 \quad (353)$$

2.4 Underview

$$(354)$$

$$\text{curve-fitting/explaining} \neq \text{prediction} \quad (355)$$

$$\text{ill-defined problem} + \text{solutionspace constraints} \implies \text{well-defined problem} \quad (356)$$

$$x \# \text{ input ; } y \# \text{ output} \quad (357)$$

$$S_n = \{(x_1, y_1), \dots, (x_n, y_n)\} \# \text{ training set} \quad (358)$$

$$f_S(x) \sim y \text{ \# solution} \quad (359)$$

$$each(x, y) \in p(x, y) \text{ \# training data } x, y \text{ is a sample from an unknown distribution } p \quad (360)$$

$$V(f(x), y) = d(f(x), y) \text{ \# loss function} \quad (361)$$

$$I[f] = \int_{X \times Y} V(f(x), y) p(x, y) dx dy \text{ \# expected error} \quad (362)$$

$$I_n[f] = \frac{1}{n} \sum_{i=1}^n V(f(x_i), y_i) \text{ \# empirical error} \quad (363)$$

$$probabilisticConvergence(X, ()) \iff \forall \epsilon > 0 \lim_{n \rightarrow \infty} P\{x_n - x \leq \epsilon\} = 0 \quad (364)$$

$$I - Ingeneralizationerror \quad (365)$$

$$well - posed := exists, unique, stable; else ill - posed \quad (366)$$

3 Machine Learning

3.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X, Y) \text{ \# dataset} \quad (367)$$

$$\text{learned parameters} = \text{parameters to be fixed by training with the dataset} \quad (368)$$

$$\text{hyperparameters} = \text{parameters that depends on a dataset} \quad (369)$$

$$\text{validation} = \text{partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition} \text{ \# useful for fixing hyperparameters} \quad (370)$$

$$\text{cross-validation} = \text{average accuracy of validation for different choices of testing partition} \quad (371)$$

$$\mathbf{L1} = \text{scales linearly ; } \mathbf{L2} = \text{scales quadratically} \quad (372)$$

$$d = \text{distance} = \text{quantifies the the similarity between data points} \quad (373)$$

$$d_{L1}(A, B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance} \quad (374)$$

$$d_{L2}(A, B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \quad (375)$$

kNN classifier = classifier based on k nearest data points (376)

s = class score = quantifies bias towards a particular class (377)

$s_{linear} = f_{c \times 1}(x_{n \times 1}, W_{c \times n}, b_{c \times 1}) = W_{c \times n} x_{n \times 1} + b_{c \times 1}$ # linear score function (378)

l = loss = quantifies the errors by the learned parameters (379)

$l = \frac{1}{|c_i|} \sum_{c_i} l_i$ # average loss for all classes (380)

$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1)$ # SVM hinge class loss function:
ignores incorrect classes with lower scores including a non-zero margin (381)

$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right)$ # Softmax class loss function
lower scores correspond to lower exponentiated-normalized probabilities (382)

R = regularization = optimizes the choice of learned parameters to minimize test error (383)

λ # regularization strength hyperparameter (384)

$R_{L1}(W) = \sum_{W_i} |W_i|$ # L1 regularization (385)

$R_{L2}(W) = \sum_{W_i} W_i^2$ # L2 regularization (386)

$L' = L + \lambda R(W)$ # weight regularization (387)

$\nabla_W L = \frac{\partial}{\partial W_i} L$ = **loss gradient w.r.t. weights** (388)

$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L}$ # loss gradient w.r.t. input weight in terms of external and local gradients (389)

s = forward API ; $\frac{\partial L_L}{\partial W_I}$ = backward API (390)

$W_{t+1} = W_t - \nabla_{W_t} L$ # weight update loss minimization (391)

TODO: Research on Activation functions, Weight Initialization, Batch Normalization (392)

review5meanvardiscussion/hyperparameteroptimization/babysittinglearning (393)

TODO loss L or l ??

4 Glossary

chaoticTopology	openRefinement	normedVectorSpace	orthogonalVectors
discreteTopology	locallyFinite	vectorMetric	orthogonalOperator
topology	paracompact	metricVectorSpace	orthogonalProjection
topologicalSpace	openRefinement	innerProductNorm	eigenvectors
open	locallyFinite	normInnerProduct	det
closed	paracompact	normMetric	tr
clopen	connected	metricNorm	diagonalOperator
neighborhood	pathConnected	orthogonal	characteristicEquation
chaoticTopology	connected	normal	eigenDecomposition
discreteTopology	pathConnected	basis	spectralDecomposition
metric	sigmaAlgebra	orthonormalBasis	hermitianAdjoint
metricSpace	measurableSpace	vectorSpace	hermitianOperator
openBall	measurableSet	innerProduct	unitaryOperator
metricTopology	measure	innerProductSpace	positiveDefiniteOperator
metricTopologicalSpace	measureSpace	vectorNorm	semiPositiveDefiniteOperator
limitPoint	finiteMeasure	normedVectorSpace	similarOperators
interiorPoint	generatedSigmaAlgebra	vectorMetric	similarOperators
closure	borelSigmaAlgebra	metricVectorSpace	singularValueDecomposition
dense	euclideanSigma	innerProductNorm	linearOperator
eucD	lebesgueMeasure	normInnerProduct	matrix
euclideanTopology	measurableMap	normMetric	eigenvector
subsetTopology	pushForwardMeasure	metricNorm	eigenvalue
productTopology	nullSet	orthogonal	identityOperator
metric	almostEverywhere	normal	inverseOperator
metricSpace	sigmaAlgebra	basis	transposeOperator
openBall	measurableSpace	orthonormalBasis	symmetricOperator
metricTopology	measurableSet	subspace	triangularOperator
metricTopologicalSpace	measure	subspaceSum	decomposeLU
limitPoint	measureSpace	subspaceDirectSum	Img
interiorPoint	finiteMeasure	orthogonalComplement	Ker
closure	generatedSigmaAlgebra	orthogonalDecomposition	independentOperator
dense	borelSigmaAlgebra	subspace	dimensionality
eucD	euclideanSigma	subspaceSum	rank
euclideanTopology	lebesgueMeasure	subspaceDirectSum	transposeNorm
subsetTopology	measurableMap	orthogonalComplement	orthogonalVectors
productTopology	pushForwardMeasure	orthogonalDecomposition	orthogonalOperator
sequence	nullSet	cauchy	orthogonalProjection
sequenceConvergesTo	almostEverywhere	complete	eigenvectors
sequence	simpleTopology	banachSpace	det
sequenceConvergesTo	simpleSigma	hilbertSpace	tr
continuous	simpleFunction	separable	diagonalOperator
homeomorphism	characteristicFunction	cauchy	characteristicEquation
isomorphicTopologicalSpace	exEuclideanSigma	complete	eigenDecomposition
continuous	nonNegIntegrable	banachSpace	spectralDecomposition
homeomorphism	nonNegIntegral	hilbertSpace	hermitianAdjoint
isomorphicTopologicalSpace	explicitIntegral	separable	hermitianOperator
T0Separate	integrable	linearOperator	unitaryOperator
T1Separate	integral	matrix	positiveDefiniteOperator
T2Separate	simpleTopology	eigenvector	semiPositiveDefiniteOperator
T0Separate	simpleSigma	eigenvalue	similarOperators
T1Separate	simpleFunction	identityOperator	similarOperators
T2Separate	characteristicFunction	inverseOperator	singularValueDecomposition
openCover	exEuclideanSigma	transposeOperator	denseMap
finiteSubcover	nonNegIntegrable	symmetricOperator	mapNorm
compact	nonNegIntegral	triangularOperator	boundedMap
compactSubset	explicitIntegral	decomposeLU	extensionMap
bounded	integrable	Img	adjoint
openCover	integral	Ker	selfAdjoint
finiteSubcover	vectorSpace	independentOperator	compactMap
compact	innerProduct	dimensionality	denseMap
compactSubset	innerProductSpace	rank	mapNorm
bounded	vectorNorm	transposeNorm	boundedMap

extensionMap	2IL285	discreteRandomVariable	1Cond303
adjoint	3IL285	bernoulliRandomVariable	2Cond303
selfAdjoint	4IL285	uniformRandomVariable	1IL303
compactMap	MSCont	geometricRandomVariable	2IL303
curLp	MSConvL	binomialRandomVariable	3IL303
vecLp	MSConvU	poissonRandomVariable	Impl303
integralNorm	MSSetOrder	absolutelyContinuous	conditionalProbability
Lp	MSSetBound	continuousRandomVariable	independentEvents
curL	randomVariable	contUniformRandomVariable	setPartition
curLp	PL	exponentialRandomVariable	finIndEvents
vecLp	generatedSigmaAlgebra	memorylessRandomVariable	infIndEvents
integralNorm	piSystem	gaussianRandomVariable	subSigmaAlgebra
Lp	CDF	simplifiedCauchyRandomVariable	independentSigmaAlgebras
curL	PMF	singularRandomVariable	infIndSigmaAlgebras
randomExperiment	indicatorRandomVariable	preimageSigma	infinitelyOften
probabilitySpace	discreteRandomVariable	nRandomVariable	Cond300
measureSpace	bernoulliRandomVariable	conditionalProbability	1IL300
event	uniformRandomVariable	independentEvents	2IL300
CL285	geometricRandomVariable	setPartition	3IL300
DL285	binomialRandomVariable	finIndEvents	Impl300
EL285	poissonRandomVariable	infIndEvents	logp
1IL285	absolutelyContinuous	subSigmaAlgebra	sump
2IL285	continuousRandomVariable	independentSigmaAlgebras	1Cond302
3IL285	contUniformRandomVariable	infIndSigmaAlgebras	2Cond302
4IL285	exponentialRandomVariable	infinitelyOften	1IL302
MSCont	memorylessRandomVariable	Cond300	2IL302
MSConvL	gaussianRandomVariable	1IL300	3IL302
MSConvU	simplifiedCauchyRandomVariable	2IL300	Impl302
MSSetOrder	singularRandomVariable	3IL300	1Cond303
MSSetBound	preimageSigma	Impl300	2Cond303
randomExperiment	nRandomVariable	logp	1IL303
probabilitySpace	randomVariable	sump	2IL303
measureSpace	PL	1Cond302	3IL303
event	generatedSigmaAlgebra	2Cond302	Impl303
CL285	piSystem	1IL302	
DL285	CDF	2IL302	
EL285	PMF	3IL302	
1IL285	indicatorRandomVariable	Impl302	