

Next-Next-Gen Notes

Object-Oriented Maths

JP Guzman

October 9, 2017

Format: $characteristic((subjects), (dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$
 TODO only metric space for limit points? or keep topological space (technically, metric is stronger than topology and topology was never used in the conditions)
 TODO define $\|$ abs cross-product and other missing refs
 TODO define $**args$ for comparison callbacks, predicate args, norms and or placeholders
 TODO link thms?

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s, (RegEx)) \iff well_formedString(s, ()) \quad (1)$$

$$proposition((p, t), ()) \iff \left(statement(p, ()) \wedge \begin{aligned} &(t = eval(p)) \wedge \\ &(t = true \vee t = false) \end{aligned} \right) \quad (2)$$

$$operator\left(o, \left((p)_{n \in \mathbb{N}}\right)\right) \iff proposition\left(o\left((p)_{n \in \mathbb{N}}\right), ()\right) \quad (3)$$

$$operator(\neg, (p_1)) \iff \left(proposition((p_1, true), ()) \implies ((\neg p_1, false), ()) \right) \wedge \left(proposition((p_1, false), ()) \implies ((\neg p_1, true), ()) \right) \\ \# \text{ an operator takes in propositions and returns a proposition} \quad (4)$$

$$operator(\neg) \iff \text{NOT} ; operator(\vee) \iff \text{OR} ; operator(\wedge) \iff \text{AND} ; operator(\vee) \iff \text{XOR} \\ operator(\implies) \iff \text{IF} ; operator(\iff) \iff \text{OIF} ; operator(\iff) \iff \text{IFF} \quad (5)$$

$$proposition((false \implies true), true, ()) \wedge proposition((false \implies false), true, ()) \\ \# \text{ truths based on a false premise is not false; ex falso quodlibet principle} \quad (6)$$

$$(\text{THM}) : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \quad (7)$$

$$predicate(P, (V)) \iff \forall_{v \in V} \left(proposition\left((P(v), t), ()\right) \right) \quad (8)$$

$$0thOrderLogic(P, ()) \iff proposition((P, t), ()) \\ \# \text{ individual proposition} \quad (9)$$

$$1stOrderLogic(P, (V)) \iff \left(\forall_{v \in V} \left(0thOrderLogic(v, ()) \right) \right) \wedge \left(\forall_{v \in V} \left(proposition \left((P(v), t), () \right) \right) \right)$$

propositions defined over a set of the lower order logical statements (10)

$$quantifier(q, (p, V)) \iff \left(predicate(p, (V)) \right) \wedge \left(proposition \left((q(p), t), () \right) \right)$$

a quantifier takes in a predicate and returns a proposition (11)

$$quantifier(\forall, (p, V)) \iff proposition \left(\left(\wedge_{v \in V} (p(v)), t \right), () \right)$$

universal quantifier (12)

$$quantifier(\exists, (p, V)) \iff proposition \left(\left(\vee_{v \in V} (p(v)), t \right), () \right)$$

existential quantifier (13)

$$quantifier(\exists!, (p, V)) \iff \exists_{x \in V} \left(P(x) \wedge \neg \left(\exists_{y \in V \setminus \{x\}} (P(y)) \right) \right)$$

uniqueness quantifier (14)

$$(THM) : \forall_x p(x) \iff \neg \exists_x \neg p(x)$$

De Morgan's law (15)

$$(THM) : \forall_x \exists_y p(x, y) = \forall_x \neg \forall_y \neg p(x, y) \neq \exists_y \forall_x p(x, y) = \neg \forall_y \neg (\forall_x p(x, y)) = \neg \forall_y \exists_x \neg p(x, y)$$

different quantifiers are not interchangeable (16)

$$===== \text{ N O T } = \text{ U P D A T E D } =====$$

(17)

$$\text{proof} = \text{truths derived from a finite number of axioms and deductions}$$

(18)

$$\text{elementary arithmetics} = \text{system with substitutions, and some notion of addition, multiplication, and prime numbers for encoding metamathematics}$$

(19)

$$\text{Gödel theorem} \implies \text{axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions}$$

(20)

$$sequenceSet((A)_{\mathbb{N}}, (A)) \iff (Amapinputn)((A)_{\mathbb{N}} = \{A(1), A(2), A(3), \dots\})$$

(21)

$$\text{TODO: define union, intersection, complement, etc.}$$

(22)

$$===== \text{ N O T } = \text{ U P D A T E D } =====$$

(23)

1.1 Axiomatic Set Theory

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (24)$$

$$\text{ZFC set theory} = \text{standard form of axiomatic set theory} \quad (25)$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \quad (26)$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \quad (27)$$

$$\in \text{ basis} \implies \{x, y\} = \{y, x\} \wedge \{x\} = \{x, x\} \quad (28)$$

$$\in \text{ and sets works following the 9 ZFC axioms:} \quad (29)$$

$$\forall_x \forall_y (x \in y \vee \neg(x \in y)) \# \text{ E: } \in \text{ is only a proposition on sets} \quad (30)$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \# \text{ E: existence of empty set} \quad (31)$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \# \text{ C: pair set construction} \quad (32)$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \# \text{ C: union set construction} \quad (33)$$

$$x = \{\{a\}, \{b\}\} \# \text{ from the pair set axiom} \quad (34)$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \quad (35)$$

$$\forall_x \exists!_y R(x, y) \# \text{ functional relation } R \quad (36)$$

$$\begin{aligned} \exists_i \forall_x \exists!_y R(x, y) \implies y \in i \# \text{ C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \mid P(y)\} \# \text{ Restricted Comprehension } \not\Rightarrow \{y \mid P(y)\} \# \text{ Universal Comprehension} \end{aligned} \quad (37)$$

$$\forall_{x \in m} P(x) = \forall_x (x \in m \implies P(x)) \# \text{ ignores out of scope } \neq \forall_x (x \in m \wedge P(x)) \# \text{ restricts entirety} \quad (38)$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} (n \subseteq m \implies n \subseteq \mathcal{P}(m)) \# \text{ C: existence of power set} \quad (39)$$

$$\exists_I (\emptyset \in I \wedge \forall_{x \in I} (\{x\} \in I)) \# \text{ I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}; I \cong \mathbb{N} \implies \mathbb{N} \text{ is a set} \quad (40)$$

$$\forall_x ((\emptyset \notin x \wedge x \cap x' = \emptyset) \implies \exists_y (\text{set of each } \mathbf{e} \in x)) \# \text{ C: axiom of choice} \quad (41)$$

$$\forall_x x \neq \emptyset \implies x \notin x \# \text{ F: axiom of foundation covers further paradoxes} \quad (42)$$

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (43)$$

1.2 Classification of sets

$$\begin{aligned} \text{space}((\text{set}, \text{structure}), ()) &\iff \text{structure}(\text{set}) \\ \# \text{ a space a set equipped with some structure} \\ \# \text{ various spaces can be studied through structure preserving maps between those spaces} \end{aligned} \quad (44)$$

$$\begin{aligned} \text{map}(\phi, (A, B)) &\iff \left(\forall_{a \in A} \exists!_{b \in B} (\phi(a, b)) \right) \vee \\ &\quad \left(\forall_{a \in A} \exists!_{b \in B} (b = \phi(a)) \right) \\ \# \text{ maps elements of a set to elements of another set} \end{aligned} \quad (45)$$

$$\text{domain}(A, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (46)$$

$$\text{codomain}(B, (\phi, A, B)) \iff (\text{map}(\phi, (A, B))) \quad (47)$$

$$\begin{aligned} \text{image}(B, (A, q, M, N)) &\iff \left(\text{map}(q, (M, N)) \wedge A \subseteq M \right) \wedge \\ &\quad \left(B = \{n \in N \mid \exists_{a \in A} (q(a) = n)\} \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \text{preimage}(A, (B, q, M, N)) &\iff \left(\text{map}(q, (M, N)) \wedge B \subseteq N \right) \wedge \\ &\quad \left(A = \{m \in M \mid \exists_{b \in B} (b = q(m))\} \right) \end{aligned} \quad (49)$$

$$\begin{aligned} \text{injection}(q, (M, N)) &\iff \left(\text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{u, v \in M} (q(u) = q(v) \implies u = v) \\ \# \text{ every } m \text{ has at most 1 image} \end{aligned} \quad (50)$$

$$\begin{aligned} \text{surjection}(q, (M, N)) &\iff \left(\text{map}(q, (M, N)) \right) \wedge \\ &\quad \forall_{n \in N} \exists_{m \in M} (n = q(m)) \\ \# \text{ every } n \text{ has at least 1 preimage} \end{aligned} \quad (51)$$

$$\begin{aligned} \text{bijection}(q, (M, N)) &\iff \left(\text{injection}(q, (M, N)) \right) \wedge \\ &\quad \left(\text{surjection}(q, (M, N)) \right) \\ \# \text{ every unique } m \text{ corresponds to a unique } n \end{aligned} \quad (52)$$

$$\text{isomorphicSets}((A, B), ()) \iff \exists_{\phi} (\text{bijection}(\phi, (A, B))) \quad (53)$$

$$\text{infiniteSet}(S, ()) \iff \exists_{T \subseteq S} (\text{isomorphicSets}((T, S), ())) \quad (54)$$

$$\text{finiteSet}(S, ()) \iff \left(\neg \text{infiniteSet}(S, ()) \right) \vee (|S| \in \mathbb{N}) \quad (55)$$

$$\text{countablyInfinite}(S, ()) \iff \left(\text{infiniteSet}(S, ()) \right) \wedge \left(\text{isomorphicSets}((S, \mathbb{N}), ()) \right) \quad (56)$$

$$\text{uncountablyInfinite}(S, ()) \iff (\text{infiniteSet}(S, ())) \wedge (\neg \text{isomorphicSets}((S, \mathbb{N}), ())) \quad (57)$$

$$\begin{aligned} \text{inverseMap}(q^{-1}, (q, M, N)) &\iff (\text{bijection}(q, (M, N))) \wedge \\ &\quad (\text{map}(q^{-1}, (N, M))) \wedge \\ &\quad \left(\forall_{n \in N} \exists!_{m \in M} (q(m) = n \implies q^{-1}(n) = m) \right) \end{aligned} \quad (58)$$

$$\begin{aligned} \text{mapComposition}(\phi \circ \psi, (\phi, \psi, A, B, C)) &\iff \text{map}(\psi, (A, B)) \wedge \text{map}(\phi, (B, C)) \wedge \\ &\quad \forall_{a \in A} (\phi \circ \psi(a) = \phi(\psi(a))) \end{aligned} \quad (59)$$

$$\begin{aligned} \text{equivalenceRelation}(\sim (\$1, \$2), (M)) &\iff (\forall_{m \in M} (m \sim m)) \wedge \\ &\quad (\forall_{m, n \in M} (m \sim n \implies n \sim m)) \wedge \\ &\quad (\forall_{m, n, p \in M} (m \sim n \wedge n \sim p \implies m \sim p)) \\ &\quad \# \text{ behaves as equivalences should} \end{aligned} \quad (60)$$

$$\begin{aligned} \text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) &\iff [m]_{\sim} = \{n \in M \mid n \sim m\} \\ &\quad \# \text{ set of elements satisfying the equivalence relation with } m \end{aligned} \quad (61)$$

$$\begin{aligned} (\text{THM}) : a \in [m]_{\sim} &\implies [a]_{\sim} = [m]_{\sim} ; [m]_{\sim} = [n]_{\sim} \vee [m]_{\sim} \cap [n]_{\sim} = \emptyset \\ &\quad \# \text{ equivalence class properties} \end{aligned} \quad (62)$$

$$\begin{aligned} \text{quotientSet}(M/\sim, (M, \sim)) &\iff M/\sim = \{\text{equivalenceClass}([m]_{\sim}, (m, M, \sim)) \in \mathcal{P}(M) \mid m \in M\} \\ &\quad \# \text{ set of all equivalence classes} \end{aligned} \quad (63)$$

$$\begin{aligned} (\text{THM}) : (M, \sim, +) &\implies (\text{quotientSet}(M/\sim, (M, \sim)), +_{\sim}) \iff \forall_{[r], [s] \in M/\sim} \forall_{a \in [r]} \forall_{b \in [s]} ([r] +_{\sim} [s] = [a + b]) \\ &\quad \# \text{ a quotient set can inherit the operations on the original set if it is well-defined} \end{aligned} \quad (64)$$

$$\begin{aligned} (\text{THM}) : \text{axiom of choice} &\implies \forall_{[m]_{\sim} \in M/\sim} \exists_r (r \in [m]_{\sim}) \\ &\quad \# \text{ well-defined maps may be defined in terms of chosen representative elements } r \end{aligned} \quad (65)$$

1.3 Construction of number sets

$$\text{===== N O T = U P D A T E D =====} \quad (66)$$

$$\text{axiom of infinity} \implies \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\} \cong \mathbb{N} \quad (67)$$

$$\mathbb{N}^* = \mathbb{N} \setminus \{0\} \quad (68)$$

$$\text{addition} = \text{successor map: } \mathbb{N} \rightarrow \mathbb{N} = S(n) = \{n\} \# \text{ adds a layer of brackets} \quad (69)$$

$$\text{subtraction} = \text{predecessor map: } \mathbb{N}^* \rightarrow \mathbb{N} = P(n) = m \mid m \in n \# \text{ removes a layer of brackets} \quad (70)$$

$$S^0 = id ; n \in \mathbb{N}^* \implies S^n = S \circ S^{P(n)} \quad (71)$$

$$\mathbf{addition} = + : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} = +(m, n) = m + n = S^n(m) \quad (72)$$

$$S^x = id = S^0 \implies x = \mathbf{additive\ identity} = 0 \quad (73)$$

$$S^n(x) = 0 \implies x = \mathbf{additive\ inverse} \notin \mathbb{N} \# \text{ git gud smh -_-} \quad (74)$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim, \mathbf{s.t.}: (m, n) \sim (p, q) \iff m + q = p + n \# \text{ span } \mathbb{Z} \text{ using differences then group equal differences} \quad (75)$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \rightarrow [(n, 0)] \# \mathbb{N} \text{ embedded in } \mathbb{Z} \quad (76)$$

$$+_Z = [(m +_{\mathbb{N}} p, n +_{\mathbb{N}} q)] \# \text{ well-defined and consistent} \quad (77)$$

$$\mathbf{multiplication} \dots M^x = id \implies x = \mathbf{multiplicative\ identity} = 1 \dots \mathbf{multiplicative\ inverse} \notin \mathbb{N} \quad (78)$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^*) / \sim, \mathbf{s.t.}: (x, y) \sim (u, v) \iff x \cdot v = u \cdot y \quad (79)$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \rightarrow [(q, 1)] ; \dots \{x \mid x^2 = 2\} \notin \mathbb{Q} \quad (80)$$

$$\mathbb{R} = \mathbf{almost\ homomorphisms\ on\ } \mathbb{Z} / \sim \# \text{ http://blog.sigfpe.com/2006/05/defining-reals.html} \quad (81)$$

$$\text{===== N O T = U P D A T E D =====} \quad (82)$$

1.4 Topology

$$\text{topology}(\mathcal{O}, (M)) \iff (\mathcal{O} \subseteq \mathcal{P}(M)) \wedge (\emptyset, M \in \mathcal{O}) \wedge$$

$$\left((F \in \mathcal{O} \wedge |F| < |\mathbb{N}|) \implies \cap F \in \mathcal{O} \right) \wedge (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O})$$

topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.

arbitrary unions of open sets always result in an open set

open sets do not contain their boundaries and infinite intersections of open sets may approach and

induce boundaries resulting in a closed set (83)

$$\text{topologicalSpace}((M, \mathcal{O}), ()) \iff \text{topology}(\mathcal{O}, (M)) \quad (84)$$

$$\text{open}(S, (M, \mathcal{O})) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge (S \subseteq M) \wedge (S \in \mathcal{O})$$

an open set do not contains its own boundaries (85)

$$\begin{aligned} \text{closed}(S, (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\quad (S \subseteq M) \wedge (S \in \mathcal{P}(M) \setminus \mathcal{O}) \\ \# \text{ a closed set contains the boundaries an open set} \end{aligned} \quad (86)$$

$$\text{clopen}(S, (M, \mathcal{O})) \iff \left(\text{closed}(S, (M, \mathcal{O})) \right) \wedge \left(\text{open}(S, (M, \mathcal{O})) \right) \quad (87)$$

$$\begin{aligned} \text{neighborhood}(U, (a, \mathcal{O})) &\iff (a \in U \in \mathcal{O}) \\ \# \text{ another name for open set containing } a \end{aligned} \quad (88)$$

$$\begin{aligned} M = \{a, b, c, d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} &\implies \\ \left(\text{open}(X, (M, \mathcal{O})) \iff X = \{\emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, M\} \right) &\wedge \\ \left(\text{closed}(Y, (M, \mathcal{O})) \iff Y = \{\emptyset, \{a, b, d\}, \{c, d\}, \{a, b\}, \{d\}, M\} \right) &\wedge \\ \left(\text{clopen}(Z, (M, \mathcal{O})) \iff Z = \{\emptyset, \{a, b\}, \{c, d\}, M\} \right) \end{aligned} \quad (89)$$

$$\text{chaoticTopology}(M) = \{0, M\} ; \text{discreteTopology} = \mathcal{P}(M) \quad (90)$$

1.5 Induced topology

$$\begin{aligned} \text{metric}(d(\$1, \$2), (M)) &\iff \left(\text{map}\left(d, (M \times M, \mathbb{R}_0^+)\right) \right) \\ &\quad \left(\forall_{x, y \in M} (d(x, y) = d(y, x)) \right) \wedge \\ &\quad \left(\forall_{x, y \in M} (d(x, y) = 0 \iff x = y) \right) \wedge \\ &\quad \left(\forall_{x, y, z} (d(x, z) \leq d(x, y) + d(y, z)) \right) \\ \# \text{ behaves as distances should} \end{aligned} \quad (91)$$

$$\text{metricSpace}((M, d), ()) \iff \text{metric}(d, (M)) \quad (92)$$

$$\begin{aligned} \text{openBall}(B, (r, p, M, d)) &\iff \left(\text{metricSpace}((M, d), ()) \right) \wedge \\ &\quad (r \in \mathbb{R}^+, p \in M) \wedge \\ &\quad (B = \{q \in M \mid d(p, q) < r\}) \end{aligned} \quad (93)$$

$$\begin{aligned} \text{metricTopology}(\mathcal{O}, (M, d)) &\iff \left(\text{metricSpace}((M, d), ()) \right) \wedge \\ &\quad \left(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (\text{openBall}(B, (r, p, M, d)) \wedge B \subseteq U)\} \right) \\ \# \text{ every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \end{aligned} \quad (94)$$

$$\text{metricTopologicalSpace}((M, \mathcal{O}, d), ()) \iff \text{metricTopology}(\mathcal{O}, (M, d)) \quad (95)$$

$$\begin{aligned} \text{limitPoint}(p, (S, M, \mathcal{O}, d)) &\iff \left(\text{metricTopologicalSpace}((M, \mathcal{O}, d), ()) \right) \wedge (S \subseteq M) \wedge \\ &\quad \forall_{r \in \mathbb{R}^+} \left(\text{openBall}(B, (r, p, M, d)) \wedge B \cap S \neq \emptyset \right) \end{aligned}$$

$$\# \text{ every open ball centered at } p \text{ contains some intersection with } S \quad (96)$$

$$\begin{aligned} \text{interiorPoint}(p, (S, M, \mathcal{O}, d)) &\iff \left(\text{metricTopologySpace}((M, \mathcal{O}, d), ()) \right) \wedge (S \subseteq M) \wedge \\ &\quad \left(\exists_{r \in \mathbb{R}^+} \left(\text{openBall}(B, (r, p, M, d)) \wedge B \subseteq S \right) \right) \\ \# \text{ there is an open ball centered at } p \text{ that is fully enclosed in } S \end{aligned} \quad (97)$$

$$\text{closure}(\bar{S}, (S, M, \mathcal{O}, d)) \iff \bar{S} = S \cup \{ \text{limitPoint}(p, (S, M, \mathcal{O}, d)) \mid p \in M \} \quad (98)$$

$$\begin{aligned} \text{dense}(S, (M, \mathcal{O}, d)) &\iff (S \subseteq M) \wedge \left(\forall_{p \in M} \left(p \in \text{closure}(\bar{S}, (S, M, \mathcal{O}, d)) \right) \right) \\ \# \text{ every of point in } M \text{ is a point or a limit point of } S \end{aligned} \quad (99)$$

$$\text{eucD}(d, (n)) \iff \left(\forall_{i \in \mathbb{N} \wedge i \leq n} (x_i \in \mathbb{R}) \right) \wedge \left(d = \sqrt[2]{\sum_{i=1}^n x_i^2} \right) \quad (100)$$

$$\begin{aligned} &\text{metricTopology} \left(\text{standardTopology}, \left(\mathbb{R}^n, \text{eucD}(d, (n)) \right) \right) \\ &===== \text{ N O T = U P D A T E D } ===== \\ \text{L1: } \forall_{p \in U = \emptyset} (\dots) &\implies \forall_p \left((p \in \emptyset) \implies \dots \right) \implies \forall_p ((\text{False}) \implies \dots) \implies \emptyset \in \mathcal{O}_{\text{standard}} \\ \text{L2: } \forall_{p \in \mathbb{R}^n} B(r, p, \mathbb{R}^n, d) &\subseteq \mathbb{R}^n \implies M \in \mathcal{O}_{\text{standard}} \\ \text{L4: } C \subseteq \mathcal{O}_{\text{standard}} &\implies \forall_{U \in C} \forall_{p \in U} \exists_{r \in \mathbb{R}^+} (B_r(p) \subseteq U \subseteq \cup C) \implies \cup C \in \mathcal{O}_{\text{standard}} \\ \text{L3: } U, V \in \mathcal{O}_{\text{standard}} &\implies p \in U \cap V \implies p \in U \wedge p \in V \implies \\ &\quad \exists_{r \in \mathbb{R}^+} B(r, p, \mathbb{R}^n, d) \wedge \exists_{s \in \mathbb{R}^+} B(s, p, \mathbb{R}^n, d) \implies \\ &\quad B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \subseteq U \wedge B(\min(r, s), q, \mathbb{R}^n, d) \subseteq V \implies \\ &\quad B(\min(r, s), p, \mathbb{R}^n, \text{eucD}) \in U \cap V \implies U \cap V \in \mathcal{O}_{\text{standard}} \\ &\quad \# \text{ natural topology for } \mathbb{R}^d \\ &\quad \# \text{ could fail on infinite sets since } \min \text{ could approach } 0 \\ &===== \text{ N O T = U P D A T E D } ===== \end{aligned} \quad (101)$$

$$\begin{aligned} \text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) &\iff \text{topology}(\mathcal{O}, (M)) \wedge (N \subseteq M) \wedge (\mathcal{O}|_N = \{U \cap N \mid U \in \mathcal{O}\}) \\ \# \text{ crops open sets outside } N \end{aligned} \quad (102)$$

$$\begin{aligned} &(\text{THM}) : \text{subsetTopology}(\mathcal{O}|_N, (M, \mathcal{O}, N)) \wedge \text{topology}(\mathcal{O}|_N, (N)) \iff \\ &===== \text{ N O T = U P D A T E D } ===== \\ \text{L1: } \emptyset \in \mathcal{O} &\implies U = \emptyset \implies \emptyset \cap N = \emptyset \implies \emptyset \in \mathcal{O}|_N \\ \text{L2: } M \in \mathcal{O} &\implies U = M \implies M \cap N = N \implies N \in \mathcal{O}|_N \\ \text{L3: } S, T \in \mathcal{O}|_N &\implies \exists_{U \in \mathcal{O}} (S = U \cap N) \wedge \exists_{V \in \mathcal{O}} (T = V \cap N) \implies S \cap T = (U \cap N) \cap (V \cap N) \\ &\quad = (U \cap V) \cap N \wedge U \cap V \in \mathcal{O} \implies S \cap T \in \mathcal{O}|_N \\ \text{L4: } &\text{TODO: EXERCISE} \\ &===== \text{ N O T = U P D A T E D } ===== \end{aligned} \quad (103)$$

$$\begin{aligned} \text{productTopology}(\mathcal{O}_{A \times B}, ((A, \mathcal{O}_A), (B, \mathcal{O}_B))) &\iff \left(\text{topology}(\mathcal{O}_A, (A)) \right) \wedge \left(\text{topology}(\mathcal{O}_B, (B)) \right) \wedge \\ &\quad (\mathcal{O}_{A \times B} = \{ (a, b) \in A \times B \mid \exists_S (a \in S \in \mathcal{O}_A) \exists_T (b \in T \in \mathcal{O}_B) \}) \\ \# \text{ open in cross iff open in each} \end{aligned} \quad (104)$$

1.6 Convergence

$$\text{sequence}(q, (M)) \iff \text{map}(q, (\mathbb{N}, M)) \quad (105)$$

$$\begin{aligned} \text{sequenceConvergesTo}((q, a), (M, \mathcal{O})) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\text{sequence}(q, (M)) \right) \wedge (a \in M) \wedge \left(\forall U \in \mathcal{O} | a \in U \exists N \in \mathbb{N} \forall n > N (q(n) \in U) \right) \\ &\# \text{ each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \end{aligned} \quad (106)$$

(THM) : convergence generalizes to: the sequence $q: \mathbb{N} \rightarrow \mathbb{R}^d$ converges against $a \in \mathbb{R}^d$ in \mathcal{O}_S if:

$$\forall r > 0 \exists N \in \mathbb{N} \forall n > N (||q(n) - a|| < r) \# \text{ distance based convergence} \quad (107)$$

1.7 Continuity

$$\begin{aligned} \text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}_M), ()) \right) \wedge \\ &\left(\text{topologicalSpace}((N, \mathcal{O}_N), ()) \right) \wedge \left(\forall V \in \mathcal{O}_N \left(\text{preimage}(A, (V, \phi, M, N)) \in \mathcal{O}_M \right) \right) \\ &\# \text{ preimage of open sets are open} \end{aligned} \quad (108)$$

$$\begin{aligned} \text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) &\iff \left(\text{inverseMap}(\phi^{-1}, (\phi, M, N)) \right) \\ &\left(\text{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \wedge \left(\text{continuous}(\phi^{-1}, (N, \mathcal{O}_N, M, \mathcal{O}_M)) \right) \\ &\# \text{ structure preserving maps in topology, ability to share topological properties} \end{aligned} \quad (109)$$

$$\begin{aligned} \text{isomorphicTopologicalSpace}((M, \mathcal{O}_M), (N, \mathcal{O}_N), ()) &\iff \\ &\exists \phi \left(\text{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \right) \end{aligned} \quad (110)$$

1.8 Separation

$$\begin{aligned} T0Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U \in \mathcal{O} \left((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U) \right) \right) \\ &\# \text{ each pair of points has a neighborhood s.t. one is inside and the other is outside} \end{aligned} \quad (111)$$

$$\begin{aligned} T1Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V \left((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V) \right) \right) \\ &\# \text{ every point has a neighborhood that does not contain another point} \end{aligned} \quad (112)$$

$$\begin{aligned} T2Separate((M, \mathcal{O}), ()) &\iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall x, y \in M \wedge x \neq y \exists U, V \in \mathcal{O} \wedge U \neq V (U \cap V = \emptyset) \right) \\ &\# \text{ every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \end{aligned} \quad (113)$$

$$(THM) : T2Separate \implies T1Separate \implies T0Separate \quad (114)$$

1.9 Compactness

$$\begin{aligned} openCover(C, (M, \mathcal{O})) &\iff \left(topologicalSpace((M, \mathcal{O}), ()) \right) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M) \\ &\# \text{ collection of open sets whose elements cover the entire space} \end{aligned} \quad (115)$$

$$\begin{aligned} finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) &\iff (\tilde{C} \subseteq C) \wedge (openCover(C, (M, \mathcal{O}))) \wedge \\ &\left(openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge (finiteSet(\tilde{C}, ())) \\ &\# \text{ finite subset of a cover that is also a cover} \end{aligned} \quad (116)$$

$$\begin{aligned} compact((M, \mathcal{O}), ()) &\iff \left(topologicalSpace((M, \mathcal{O}), ()) \right) \wedge \\ &\left(\forall C \subseteq \mathcal{O} \left(openCover(C, (M, \mathcal{O})) \implies \exists \tilde{C} \subseteq C \left(finiteSubcover(\tilde{C}, (C, M, \mathcal{O})) \right) \right) \right) \\ &\# \text{ every covering of the space is represented by a finite number of nhbds} \end{aligned} \quad (117)$$

$$compactSubset(N, (M, \mathcal{O}_d, d)) \iff \left(compact((M, \mathcal{O}), ()) \right) \wedge \left(subsetTopology(\mathcal{O}|_N, (M, \mathcal{O}, N)) \right) \quad (118)$$

$$\begin{aligned} bounded(N, (M, d)) &\iff \left(metricSpace((M, d), ()) \right) \wedge (N \subseteq M) \wedge \\ &\left(\exists r \in \mathbb{R}^+ \forall p, q \in n (d(p, q) < r) \right) \end{aligned} \quad (119)$$

$$\begin{aligned} (THM) \text{ HeineBorel: } &metricTopologicalSpace((M, \mathcal{O}_d, d), ()) \implies \\ &\forall S \in \mathcal{P}(M) \left(\left(closed(S, (M, \mathcal{O}_d)) \wedge bounded(S, (M, \mathcal{O}_d)) \right) \iff compactSubset(S, (M, \mathcal{O}_d)) \right) \\ &\# \text{ when metric topologies are involved, compactness is equivalent to being closed and bounded} \end{aligned} \quad (120)$$

1.10 Paracompactness

$$\begin{aligned} openRefinement(\tilde{C}, (C, M, \mathcal{O})) &\iff \left(openCover(C, (M, \mathcal{O})) \right) \wedge \left(openCover(\tilde{C}, (M, \mathcal{O})) \right) \wedge \\ &\left(\forall \tilde{U} \in \tilde{C} \exists U \in C (\tilde{U} \subseteq U) \right) \\ &\# \text{ a refined cover can be constructed by removing the excess nhbds and points that lie outside the space} \end{aligned} \quad (121)$$

$$(THM) : finiteSubcover \implies openRefinement \quad (122)$$

$$\begin{aligned} locallyFinite(C, (M, \mathcal{O})) &\iff \left(openCover(C, (M, \mathcal{O})) \right) \wedge \\ &\forall p \in M \exists U \in \mathcal{O} |_{p \in U} \left(finiteSet(\{U_c \in C | U \cap U_c \neq \emptyset\}, ()) \right) \\ &\# \text{ each point has a neighborhood that intersects with only finitely many sets in the cover} \end{aligned} \quad (123)$$

$$paracompact((M, \mathcal{O}), ()) \iff$$

$$\forall_C \left(\text{openCover}(C, (M, \mathcal{O})) \implies \exists_{\tilde{C}} \left(\text{locallyFinite} \left(\text{openRefinement}(\tilde{C}, (C, M, \mathcal{O})), (M, \mathcal{O}) \right) \right) \right)$$

every open cover has a locally finite open refinement (124)

$$(\text{THM}) : \text{metricTopologicalSpace} \implies \text{paracompact} \quad (125)$$

$$\text{===== N O T = U P D A T E D =====} \quad (126)$$

$$\begin{aligned} \text{partitionOfUnitySubjCover}(\mathcal{F}, (C, M, \mathcal{O})) &\iff \left(\text{locallyFinite}(C, (M, \mathcal{O})) \right) \wedge (f \in \mathcal{F}) \wedge \\ &\left(\text{continuous} \left(f, \left(M, \mathcal{O}, [0, 1], \text{subsetTopology}(\mathcal{O}|_{[0, 1]}, ([0, 1], \mathbb{R}, \text{standardTopology})) \right) \right) \right) \wedge \\ &\left(\exists_{U_f \in C} \forall_{p \in M} (f(p) \neq 0 \implies p \in U_f) \right) \wedge \\ &\left(\forall_{p \in M} \exists_{U \in \mathcal{O}} |p \in U \left((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\} \right) \right) \wedge \\ &\left(\text{locallyFinite}(C, M, \mathcal{O}) \implies \text{finiteSet}((f_U)_n, ()) \right) \wedge \\ &\left(\forall_{p \in M} \exists_{U \in \mathcal{O}} |p \in U \left(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1 \right) \right) \end{aligned}$$

useful for defining integrals between overlapping neighborhoods (127)

$$\begin{aligned} T2Separate((M, \mathcal{O}), ()) &\implies \left(\text{paracompact}((M, \mathcal{O}), ()) \right) \iff \\ \forall_C \left(\text{openCover}(C, (M, \mathcal{O})) &\implies \text{partitionOfUnitySOTCover}(\mathcal{F}, (C, M, \mathcal{O})) \right) \end{aligned} \quad (128)$$

$$\text{===== N O T = U P D A T E D =====} \quad (129)$$

1.11 Connectedness and path-connectedness

$$\text{connected}((M, \mathcal{O}), ()) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \left(\neg \exists_{A, B \in \mathcal{O} \setminus \emptyset} (A \cap B \neq \emptyset \wedge A \cup B = M) \right)$$

if there is some covering of the space that does not intersect (130)

$$\begin{aligned} (\text{THM}) : \neg \text{connected} &\left(\left(\mathbb{R} \setminus \{0\}, \text{subsetTopology}(\mathcal{O}_{\text{standard}}|_{\mathbb{R} \setminus \{0\}}, (\mathbb{R}, \text{standardTopology}, \mathbb{R} \setminus \{0\})) \right), () \right) \\ &\iff \left(A = (-\infty, 0) \in \mathcal{O}_{\text{standard}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \left(B = (0, \infty) \in \mathcal{O}_{\text{standard}}|_{\mathbb{R} \setminus \{0\}} \right) \wedge \\ &\quad (A \cap B = \emptyset) \wedge (A \cup B = \mathbb{R} \setminus \{0\}) \end{aligned} \quad (131)$$

$$(\text{THM}) : \text{connected}((M, \mathcal{O}), ()) \iff \forall_{S \in \mathcal{O}} \left(\text{clopen}(S, (M, \mathcal{O})) \implies (S = \emptyset \vee S = M) \right) \quad (132)$$

$$\begin{aligned} \text{pathConnected}((M, \mathcal{O}), ()) &\iff \left(\text{subsetTopology}(\mathcal{O}_{\text{standard}}|_{[0, 1]}, (\mathbb{R}, \text{standardTopology}, [0, 1])) \right) \wedge \\ &\left(\forall_{p, q \in M} \exists_{\gamma} \left(\text{continuous} \left(\gamma, ([0, 1], \mathcal{O}_{\text{standard}}|_{[0, 1]}, M, \mathcal{O}) \right) \wedge \gamma(0) = p \wedge \gamma(1) = q \right) \right) \end{aligned} \quad (133)$$

$$(THM) : \text{pathConnected} \implies \text{connected} \quad (134)$$

1.12 Homotopic curve and the fundamental group

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (135)$$

$$\begin{aligned} \text{homotopic}(\sim, (\gamma, \delta, M, \mathcal{O})) &\iff (\text{map}(\gamma, ([0, 1], M)) \wedge \text{map}(\delta, ([0, 1], M))) \wedge \\ &\quad (\gamma(0) = \delta(0) \wedge \gamma(1) = \delta(1)) \wedge \\ (\exists_H \forall_{\lambda \in [0, 1]} (\text{continuous}(H, ([0, 1] \times [0, 1], \mathcal{O}_{\text{standard}^2|_{[0, 1] \times [0, 1]}}, (M, \mathcal{O}))) \wedge H(0, \lambda) = \gamma(\lambda) \wedge H(1, \lambda) = \delta(\lambda))) \\ &\quad \# H \text{ is a continuous deformation of one curve into another} \end{aligned} \quad (136)$$

$$\text{homotopic}(\sim) \implies \text{equivalenceRelation}(\sim) \quad (137)$$

$$\text{loopSpace}(\mathcal{L}_p, (p, M, \mathcal{O})) \iff \mathcal{L}_p = \{\text{map}(\gamma, ([0, 1], M)) \mid \text{continuous}(\gamma) \wedge \gamma(0) = \gamma(1)\} \quad (138)$$

$$\begin{aligned} \text{concatination}(\star, (p, \gamma, \delta)) &\iff (\gamma, \delta \in \text{loopSpace}(\mathcal{L}_p)) \wedge \\ (\forall_{\lambda \in [0, 1]} ((\gamma \star \delta)(\lambda) &= \begin{cases} \gamma(2\lambda) & 0 \leq \lambda < 0.5 \\ \delta(2\lambda - 1) & 0.5 \leq \lambda \leq 1 \end{cases})) \end{aligned} \quad (139)$$

$$\begin{aligned} \text{group}((G, \bullet), ()) &\iff (\text{map}(\bullet, (G \times G, G))) \wedge \\ &\quad (\forall_{a, b \in G} (a \bullet b \in G)) \\ &\quad (\forall_{a, b, c \in G} ((a \bullet b) \bullet c = a \bullet (b \bullet c))) \\ &\quad (\exists_e \forall_{a \in G} (e \bullet a = a = a \bullet e)) \wedge \\ &\quad (\forall_{a \in G} \exists_{a^{-1}} (a \bullet a^{-1} = e = a^{-1} \bullet a)) \\ &\quad \# \text{ characterizes symmetry of a set structure} \end{aligned} \quad (140)$$

$$\text{isomorphic}(\cong, (X, \odot), (Y, \ominus)) \iff \exists_f \forall_{a, b \in X} (\text{bijection}(f, (X, Y)) \wedge f(a \odot b) = f(a) \ominus f(b)) \quad (141)$$

$$\begin{aligned} \text{fundamentalGroup}((\pi_{1,p}, \bullet), (p, M, \mathcal{O})) &\iff (\pi_{1,p} = \mathcal{L}_p / \sim) \wedge \\ &\quad (\text{map}(\bullet, (\pi_{1,p} \times \pi_{1,p}, \pi_{1,p}))) \wedge \\ &\quad (\forall_{A, B \in \pi_{1,p}} ([A] \bullet [B] = [A \star B])) \wedge \\ &\quad (\text{group}((\pi_{1,p}, \bullet), ())) \\ &\quad \# \text{ an equivalence class of all loops induced from the homotopic equivalence relation} \end{aligned} \quad (142)$$

$$\text{fundamentalGroup}_1 \not\cong \text{fundamentalGroup}_2 \implies \text{topologicalSpace}_1 \not\cong \text{topologicalSpace}_2 \quad (143)$$

$$\text{there exists no known list of topological properties that can imply homeomorphisms} \quad (144)$$

$$\text{CONTINUE @ Lecture 6: manifolds} \quad (145)$$

$$===== \text{ N O T } = \text{ U P D A T E D } ===== \quad (146)$$

1.13 Measure theory

$$\begin{aligned}
& \text{sigmaAlgebra}(\sigma, (M)) \iff (M \neq \emptyset) \wedge (\sigma \subseteq \mathcal{P}(M)) \wedge \\
& \quad (M \in \sigma) \wedge \left(\forall A \in \sigma (M \setminus A \in \sigma) \right) \wedge \\
& \quad \left(\left(A \subseteq \sigma \wedge \neg \text{uncountablyInfinite}(A, ()) \right) \implies \cup A \in \sigma \right) \\
& \# \text{ behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu
\end{aligned} \tag{147}$$

$$\text{measurableSpace}((M, \sigma), ()) \iff \text{sigmaAlgebra}(\sigma, (M)) \tag{148}$$

$$\text{measurableSet}(A, (M, \sigma)) \iff \left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge (A \in \sigma) \tag{149}$$

$$\begin{aligned}
& \text{measure}(\mu, (M, \sigma)) \iff \left(\text{measurableSpace}((M, \sigma), ()) \right) \wedge \left(\text{map} \left(\mu, \left(\sigma, \left(\mathbb{R}^+ \right)_0 \right) \right) \right) \wedge (\mu(\emptyset) = 0) \wedge \\
& \quad \left(\left((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} \forall j \in \mathbb{N} \setminus \{i\} (A_i \cap A_j = \emptyset) \right) \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) = \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \\
& \# \text{ enforces meaningful concepts of measures such as precise additivity}
\end{aligned} \tag{150}$$

$$\begin{aligned}
& (\text{THM}) : \text{measure}(\mu, (M, \sigma)) \implies \\
& \quad \left(\forall A, B \in \sigma (A \subseteq B \implies \mu(A) \leq \mu(B)) \right) \wedge \\
& \quad \left((A)_{\mathbb{N}} \subseteq \sigma \implies \mu(\cup_{i \in \mathbb{N}} (A_i)) \leq \sum_{i \in \mathbb{N}} (\mu(A_i)) \right) \wedge \\
& \quad \left(((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (B_i \subseteq B_{i+1}) \wedge B = \cup (B)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(B_n)) = \mu(B) \right) \wedge \\
& \quad \left(((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall i \in \mathbb{N} (C_{i+1} \subseteq C_i) \wedge C = \cap (C)_{\mathbb{N}}) \implies \lim_{n \rightarrow \infty} (\mu(C_n)) = \mu(C) \right) \\
& \# \text{ immediate implications of the measurable set } A \in \sigma \text{ axioms and the measure } \mu \text{ axioms}
\end{aligned} \tag{151}$$

$$\text{measureSpace}((M, \sigma, \mu), ()) \iff \text{measure}(\mu, (M, \sigma)) \tag{152}$$

$$\begin{aligned}
& \text{finiteMeasure}(\mu, (M, \sigma)) \iff \left(\text{measure}(\mu, (M, \sigma)) \right) \wedge \\
& \quad \left(\exists (A)_{\mathbb{N}} \subseteq \sigma \left(\cup (A)_{\mathbb{N}} = M \wedge \forall n \in \mathbb{N} (\mu(A_n) < \infty) \right) \right)
\end{aligned} \tag{153}$$

$$\begin{aligned}
& \text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) \iff \left(G = \{ \sigma \subseteq \mathcal{P}(M) \mid \text{sigmaAlgebra}(\sigma, (M)) \} \right) \wedge (\sigma(\zeta) = \cap G) \\
& \# \text{ smallest } \sigma\text{-algebra containing the generating set } \zeta
\end{aligned} \tag{154}$$

$$(\text{THM}) : \exists \zeta \subseteq M \left(\text{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) = \text{sigmaAlgebra}(\sigma, (M)) \right) \tag{155}$$

$$\begin{aligned}
& \text{borelSigmaAlgebra}(\sigma(\mathcal{O}), (M, \mathcal{O})) \iff \left(\text{topologicalSpace}((M, \mathcal{O}), ()) \right) \wedge \\
& \quad \left(\text{generatedSigmaAlgebra}(\sigma(\mathcal{O}), (\mathcal{O}, M)) \right) \\
& \# \sigma\text{-algebra induced by a topology}
\end{aligned} \tag{156}$$

$$\text{standardSigma}(\sigma_s, ()) \iff \left(\text{borelSigmaAlgebra} \left(\sigma_s, \left(\mathbb{R}^d, \text{standardTopology} \right) \right) \right) \quad (157)$$

$$\begin{aligned} \text{lebesgueMeasure}(\lambda, ()) \iff & \left(\text{measure} \left(\lambda, \left(\mathbb{R}^d, \text{standardSigma} \right) \right) \right) \wedge \\ & \left(\lambda \left(\times_{i=1}^d ([a_i, b_i]) \right) = \sum_{i=1}^d \left(\sqrt[2]{(a_i - b_i)^2} \right) \right) \\ & \# \text{ natural measure for } \mathbb{R}^d \end{aligned} \quad (158)$$

$$\begin{aligned} \text{measurableMap}(f, (M, \sigma_M, N, \sigma_N)) \iff & \left(\text{measurableSpace}((M, \sigma_M), ()) \right) \wedge \\ & \left(\text{measurableSpace}((N, \sigma_N), ()) \right) \wedge \left(\forall B \in \sigma_N \left(\text{preimage}(A, (B, f, M, N)) \in \sigma_M \right) \right) \\ & \# \text{ preimage of measurable sets are measurable} \end{aligned} \quad (159)$$

$$\begin{aligned} \text{pushForwardMeasure}(f \star \lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \iff & \left(\text{measureSpace}((M, \sigma_M, \mu_M), ()) \right) \wedge \\ & \left(\text{measurableSpace}((N, \sigma_N), ()) \right) \wedge \left(\text{measurableMap}(f, (M, \sigma_M, N, \sigma_N)) \right) \wedge \\ & \left(\forall B \in \sigma_N \left(f \star \lambda_M(B) = \mu_M \left(\text{preimage}(A, (B, f, M, N)) \right) \right) \right) \wedge \left(\text{measure}(f \star \lambda_M, (N, \sigma_N)) \right) \\ & \# \text{ natural construction of a measure based primarily on measurable map} \end{aligned} \quad (160)$$

$$\text{nullSet}(A, (M, \sigma, \mu)) \iff \left(\text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge (A \in \sigma) \wedge (\mu(A) = 0) \quad (161)$$

$$\begin{aligned} \text{almostEverywhere}(p, (M, \sigma, \mu)) \iff & \left(\text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge \left(\text{predicate}(p, (M)) \right) \wedge \\ & \left(\exists A \in \sigma \left(\text{nullSet}(A, (M, \sigma, \mu)) \implies \forall n \in M \setminus A (p(n)) \right) \right) \\ & \# \text{ the predicate holds true for all points except the points in the null set} \end{aligned} \quad (162)$$

1.14 Lebesgue integration

$$\text{simpleTopology}(\mathcal{O}_{\text{simple}}, ()) \iff \mathcal{O}_{\text{simple}} = \text{subsetTopology} \left(\mathcal{O}|_{\mathbb{R}_0^+}, \left(\mathbb{R}, \text{standardTopology}, \mathbb{R}_0^+ \right) \right) \quad (163)$$

$$\text{simpleSigma}(\sigma_{\text{simple}}, ()) \iff \text{borelSigmaAlgebra} \left(\sigma_{\text{simple}}, \left(\mathbb{R}_0^+, \text{simpleTopology} \right) \right) \quad (164)$$

$$\begin{aligned} \text{simpleFunction}(s, (M, \sigma)) \iff & \left(\text{measurableMap} \left(s, \left(M, \sigma, \mathbb{R}_0^+, \text{simpleSigma} \right) \right) \right) \wedge \\ & \left(\text{finiteSet} \left(\text{image} \left(B, \left(M, s, M, \mathbb{R}_0^+ \right) \right), () \right) \right) \\ & \# \text{ if the map takes on finitely many values on } \mathbb{R}_0^+ \end{aligned} \quad (165)$$

$$\begin{aligned} \text{characteristicFunction}(X_A, (A, M)) &\iff (A \subseteq M) \wedge \left(\text{map}(X_A, (M, \mathbb{R})) \right) \wedge \\ &\left(\forall_{m \in M} \left(X_A(m) = \begin{cases} 1 & m \in A \\ 0 & m \notin A \end{cases} \right) \right) \end{aligned} \quad (166)$$

$$\begin{aligned} (\text{THM}) : \text{simpleFunction}(s, (M, \sigma_M)) &\implies \\ &\left(\text{finiteSet} \left(\text{image} \left(Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right) \wedge \\ &\left(\text{characteristicFunction}(X_A, (A, M)) \right) \wedge \left(\forall_{m \in M} \left(s(m) = \sum_{z \in Z} \left(z \cdot X_{\text{preimage} \left(A, (\{z\}, s, M, \mathbb{R}_0^+) \right)}(m) \right) \right) \right) \end{aligned} \quad (167)$$

$$\begin{aligned} \text{exStandardSigma}(\overline{\sigma_s}, ()) &\iff \overline{\sigma_s} = \{A \subseteq \mathbb{R} \mid A \cap R \in \text{standardSigma}\} \\ \# \text{ ignores } \pm\infty \text{ to preserve the points in the domain of the measurable map} \end{aligned} \quad (168)$$

$$\begin{aligned} \text{nonNegIntegrable}(f, (M, \sigma)) &\iff \left(\text{measurableMap} \left(f, (M, \sigma, \mathbb{R}, \text{exStandardSigma}) \right) \right) \wedge \\ &\left(\forall_{m \in M} (f(m) \geq 0) \right) \end{aligned} \quad (169)$$

$$\begin{aligned} \text{nonNegIntegral} \left(\int_M (f d\mu), (f, M, \sigma, \mu) \right) &\iff \left(\text{measureSpace}((M, \sigma, \mu), ()) \right) \wedge \\ &\left(\text{measureSpace} \left((\mathbb{R}, \text{exStandardSigma}, \text{lebesgueMeasure}), () \right) \right) \wedge \\ &\left(\text{nonNegIntegrable}(f, (M, \sigma)) \right) \wedge \left(\int_M (f d\mu) = \sup \left(\left\{ \sum_{z \in Z} \left(z \cdot \mu \left(\text{preimage} \left(A, (\{z\}, s, M, \mathbb{R}_0^+) \right) \right) \right) \right\} \right) \mid \right. \\ &\left. \forall_{m \in M} (s(m) \leq f(m)) \wedge \text{simpleFunction}(s, (M, \sigma)) \wedge \text{finiteSet} \left(\text{image} \left(Z, (M, s, M, \mathbb{R}_0^+) \right), () \right) \right\}) \\ &\# \text{ lebesgue measure on } z \text{ reduces to } z \end{aligned} \quad (170)$$

$$\begin{aligned} \text{explicitIntegral} &\iff \int (f(x) \mu(dx)) = \int (f d\mu) \\ \# \text{ alternative notation for lebesgue integrals} \end{aligned} \quad (171)$$

$$\begin{aligned} (\text{THM}) : \text{nonNegIntegral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) &\wedge \text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\ (\text{THM}) \text{ Markov inequality: } &\left(\forall_{z \in \mathbb{R}_0^+} \left(\int (f d\mu) \geq z \cdot \mu \left(\text{preimage} \left(A, ([z, \infty), f, M, \mathbb{R}] \right) \right) \right) \right) \wedge \\ &\left(\text{almostEverywhere}(f = g, (M, \sigma, \mu)) \implies \int (f d\mu) = \int (g d\mu) \right) \\ &\left(\int (f d\mu) = 0 \implies \text{almostEverywhere}(f = 0, (M, \sigma, \mu)) \right) \wedge \\ &\left(\int (f d\mu) \leq \infty \implies \text{almostEverywhere}(f < \infty, (M, \sigma, \mu)) \right) \end{aligned} \quad (172)$$

$$\begin{aligned}
(\text{THM}) \text{ Mono. conv.: } & \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{R}, \text{exStandardSigma}) \right) \wedge 0 \leq f_{n-1} \leq f_n \} \right) \wedge \\
& \left(\text{map} \left(f, (M, \overline{\mathbb{R}}) \right) \right) \wedge \left(\forall_{m \in M} \left(f(m) = \sup(f_n(m) \mid f_n \in (f)_{\mathbb{N}}) \right) \right) \implies \left(\lim_{n \rightarrow \infty} \left(\int_M (f_n d\mu) \right) = \int_M (f d\mu) \right) \\
& \# \text{ lengths now depend on } M, \sigma \text{ and limits can be pulled in or out of an integral (173)}
\end{aligned}$$

$$\begin{aligned}
(\text{THM}) : & \text{nonNegIntegral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{nonNegIntegral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\
& \left(\forall_{\alpha \in \mathbb{R}_0^+} \left(\int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \right) \\
& \# \text{ integral acts linearly and commutes finite summations (174)}
\end{aligned}$$

$$\begin{aligned}
(\text{THM}) : & \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap} \left(f_n, (M, \sigma, \overline{R}, \text{exStandardSigma}) \right) \wedge 0 \leq f_n\} \right) \implies \\
& \left(\int \left(\left(\sum_{n=1}^{\infty} (f_n) \right) d\mu \right) = \sum_{n=1}^{\infty} \left(\int (f_n d\mu) \right) \right) \\
& \# \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n \rightarrow \infty} \sum_{i=1}^n f_n \text{ since } f_n \geq 0 \text{ and it commutes with integral from monotone conv. (175)}
\end{aligned}$$

$$\begin{aligned}
\text{integrable}(f, (M, \sigma)) & \iff \left(\text{measurableMap} \left(f, (M, \sigma, \overline{\mathbb{R}}, \text{exStandardSigma}) \right) \right) \wedge \\
& \left(\forall_{m \in M} \left(f(m) = \max(f(m), 0) - \max(0, -f(m)) \right) \right) \wedge \\
& \left(\text{measureSpace}(M, \sigma, \mu) \implies \left(\int (\max(f(m), 0) d\mu) < \infty \wedge \int (\max(0, -f(m)) d\mu) < \infty \right) \right) \\
& \# \text{ extra condition prevents the occurrence of the indeterminate } \infty - \infty \text{ (176)}
\end{aligned}$$

$$\begin{aligned}
\text{integral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) & \iff \left(\text{nonNegIntegral} \left(\int (f^+ d\mu), (\max(f, 0), M, \sigma, \mu) \right) \right) \wedge \\
& \left(\text{nonNegIntegral} \left(\int (f^- d\mu), (\max(0, -f), M, \sigma, \mu) \right) \right) \wedge \left(\text{integrable}(f, (M, \sigma)) \right) \wedge \\
& \left(\int (f d\mu) = \int (f^+ d\mu) - \int (f^- d\mu) \right) \\
& \# \text{ arbitrary integral in terms of nonnegative integrals (177)}
\end{aligned}$$

$$(\text{THM}) : \left(\text{map}(f, (M, \mathbb{C})) \right) \implies \left(\int (f d\mu) = \int (\text{Re}(f) d\mu) - \int (\text{Im}(f) d\mu) \right) \quad (178)$$

$$\begin{aligned}
(\text{THM}) : & \text{integral} \left(\int (f d\mu), (f, M, \sigma, \mu) \right) \wedge \text{integral} \left(\int (g d\mu), (g, M, \sigma, \mu) \right) \implies \\
& \left(\text{almostEverywhere}(f \leq g, (M, \sigma, \mu)) \implies \int (f d\mu) \leq \int (g d\mu) \right) \wedge \\
& \left(\forall_{m \in M} (f(m), g(m), \alpha \in \mathbb{R}) \implies \int ((f + \alpha g) d\mu) = \int (f d\mu) + \alpha \int (g d\mu) \right) \quad (179)
\end{aligned}$$

$$\begin{aligned}
& \text{(THM) Dominant convergence: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \text{measurableMap}\left(f_n, (M, \sigma, \overline{R}, \text{exStandardSigma})\right) \} \right) \wedge \\
& \quad \left(\text{map}(f, (M, \overline{\mathbb{R}})) \right) \wedge \left(\text{almostEverywhere}\left(f(m) = \lim_{n \rightarrow \infty} (f_n(m)), (M, \sigma, \mu)\right) \right) \wedge \\
& \quad \left(\text{nonNegIntegral}\left(\int (gd\mu), (g, M, \sigma, \mu)\right) \right) \wedge \left(\left| \int (gd\mu) \right| < \infty \right) \wedge \left(\text{almostEverywhere}(|f_n| \leq g, (M, \sigma, \mu)) \right) \\
& \quad \# \text{ if all } f_n(m) \text{ are bounded by some integrable } |g(m)| \implies \\
& \quad \# \text{ then all } f_n(m) \text{ including } f \text{ satisfy bounded and integrable properties} \\
& \quad \left(\forall_{\phi \in \{f\} \cup (f)_{\mathbb{N}}} \left(\text{integrable}(\phi, (M, \sigma)) \right) \right) \wedge \left(\lim_{n \rightarrow \infty} \left(\int (|f_n - f| d\mu) = 0 \right) \right) \wedge \left(\lim_{n \rightarrow \infty} \left(\int (f_n d\mu) \right) = \int (f d\mu) \right) \quad (180)
\end{aligned}$$

1.15 Function spaces

$$\begin{aligned}
& \text{curLp}(\mathcal{L}^p, (p, M, \sigma, \mu)) \iff (p \in \mathbb{R}) \wedge (1 \leq p < \infty) \wedge \\
& \quad \left(\mathcal{L}^p = \{ \text{map}(f, (M, \mathbb{R})) \mid \text{measurableMap}(f, (M, \sigma, \mathbb{R}, \text{standardSigma})) \wedge \int (|f|^p d\mu) < \infty \} \right) \quad (181)
\end{aligned}$$

$$\begin{aligned}
& \text{vectorSpace}((V, +, \cdot, ())) \iff \left(\text{map}(+, (V \times V, V)) \right) \wedge \left(\text{map}(\cdot, (\mathbb{R} \times V, V)) \right) \wedge \\
& \quad (\forall_{v, w \in V} (v + w = w + v)) \wedge \\
& \quad (\forall_{v, w, x \in V} ((v + w) + x = v + (w + x))) \wedge \\
& \quad (\exists \mathbf{0} \in V \forall_{v \in V} (v + \mathbf{0} = v)) \wedge \\
& \quad (\forall_{v \in V} \exists_{-v \in V} (v + (-v) = \mathbf{0})) \wedge \\
& \quad (\forall_{a, b \in \mathbb{R}} \forall_{v \in V} (a(b \cdot v) = (ab) \cdot v)) \wedge \\
& \quad (\exists 1 \in \mathbb{R} \forall_{v \in V} (1 \cdot v = v)) \wedge \\
& \quad (\forall_{a, b \in \mathbb{R}} \forall_{v \in V} ((a + b) \cdot v = a \cdot v + b \cdot v)) \wedge \\
& \quad (\forall_{a \in \mathbb{R}} \forall_{v, w \in V} (a \cdot (v + w) = a \cdot v + a \cdot w)) \\
& \quad \# \text{ behaves similar as vectors should i.e., additive, scalable, linear distributive} \quad (182)
\end{aligned}$$

$$\begin{aligned}
& \text{vecLp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \iff \left(\text{curLp}(\mathcal{L}^p, (p, M, \sigma, \mu)) \right) \wedge \left(\forall_{f, g \in \mathcal{L}^p} \forall_{m \in M} ((f + g)(m) = f(m) + g(m)) \right) \wedge \\
& \quad \left(\forall_{f \in \mathcal{L}^p} \forall_{s \in \mathbb{R}} \forall_{m \in M} ((s \cdot f)(m) = (s)f(m)) \right) \wedge \left(\text{vectorSpace}((\mathcal{L}^p, +, \cdot, ())) \right) \quad (183)
\end{aligned}$$

$$\begin{aligned}
& \text{norm}(\|\cdot\|, (V, +, \cdot)) \iff \left(\text{vectorSpace}((V, +, \cdot, ())) \right) \wedge \left(\text{map}(\|\cdot\|, (V, \mathbb{R}_0^+)) \right) \wedge \\
& \quad (\forall_{v \in V} (\|v\| = 0 \iff v = \mathbf{0})) \wedge \\
& \quad (\forall_{v \in V} \forall_{s \in \mathbb{R}} (\|sv\| = |s| \|v\|)) \wedge \\
& \quad (\forall_{v, w \in V} (\|v + w\| \leq \|v\| + \|w\|)) \\
& \quad \# \text{ magnitude of a point in a vector space} \quad (184)
\end{aligned}$$

$$\text{seminormLp}(\|\cdot\|, (+, \cdot, p, M, \sigma, \mu)) \iff \left(\text{vecLp}(\mathcal{L}^p, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \left(\text{map}(\|\cdot\|, (\mathcal{L}^p, \mathbb{R})) \right) \wedge$$

$$\begin{aligned} & \left(\forall_{f \in \mathcal{L}^p} \left(0 \leq \int f \, d\mu = \left(\int |f|^p \, d\mu \right)^{1/p} \right) \right) \wedge \\ & \left(\forall_{f \in \mathcal{L}^p} \forall_{s \in \mathbb{R}} \left(\int s \cdot f \, d\mu = (|s|) \int f \, d\mu \right) \right) \wedge \\ & \left(\forall_{f, g \in \mathcal{L}^p} \left(\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu \right) \right) \end{aligned} \quad (185)$$

$$\begin{aligned} & (\text{THM}) : \text{seminorm}Lp(\int \mathbb{1} \, d\mu, (+, \cdot, p, M, \sigma, \mu)) \implies \\ & \left(\forall_{f \in \mathcal{L}^p} \left(\int f \, d\mu = 0 \implies \text{almostEverywhere}(f = \mathbf{0}, (M, \sigma, \mu)) \right) \right) \\ & \# \text{ not an expected property from a norm} \end{aligned} \quad (186)$$

$$\begin{aligned} & Lp\left(L^p, ((+, \cdot, p, M, \sigma, \mu))\right) \iff \left(\text{seminorm}Lp(\int \mathbb{1} \, d\mu, (+, \cdot, p, M, \sigma, \mu)) \right) \wedge \\ & \left(L^p = \text{quotientSet}\left(\mathcal{L}^p / \sim, \left(\mathcal{L}^p, \int \mathbb{1} + (-\mathbb{2}) \int \mathbb{1} = 0 \right) \right) \right) \\ & \# \text{ functions in } L^p \text{ that have finite integrals above and below the x-axis} \end{aligned} \quad (187)$$

$$(\text{THM}) \int \mathbb{1} \, d\mu, +, \cdot \text{ can be inherited into } L^p, \text{ thus it can be called a normed vector space:} \quad (188)$$

$$(\text{THM}) L^{p=2} \text{ is complete or contains all its limit points w.r.t. to its norm:} \quad (189)$$

2 Statistical Learning Theory

2.1 Functional Analysis

$$\begin{aligned} & \text{innerProduct}(\langle \mathbb{1}, \mathbb{2} \rangle, (V, +, \cdot)) \iff \left(\text{vectorSpace}((V, +, \cdot), ()) \right) \wedge \left(\text{map}(\langle \mathbb{1}, \mathbb{2} \rangle, (V \times V, \mathbb{R})) \right) \wedge \\ & \left(\forall_{v, w \in V} (\langle v, w \rangle = \langle w, v \rangle) \right) \wedge \\ & \left(\forall_{v, w, x \in V} \forall_{a, b \in \mathbb{R}} (\langle av + bw, x \rangle = a \langle v, x \rangle + b \langle w, x \rangle) \right) \wedge \\ & \left(\forall_{v \in V} (\langle v, v \rangle \geq 0) \right) \wedge \left(\forall_{v \in V} (\langle v, v \rangle = 0 \iff v = \mathbf{0}) \right) \\ & \# \text{ the sesquilinear or 1.5 linear map inner product provides info. on distance and orthogonality} \end{aligned} \quad (190)$$

$$\text{innerProductSpace}\left((V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle), ()\right) \iff \text{innerProduct}(\langle \mathbb{1}, \mathbb{2} \rangle, (V, +, \cdot)) \quad (191)$$

$$\begin{aligned} & \text{orthogonal}\left((v, w), (V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle)\right) \iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle), ()\right) \right) \wedge \\ & (v, w \in V) \wedge (\langle v, w \rangle = 0) \\ & \# \text{ the inner product also provides info. on orthogonality} \end{aligned} \quad (192)$$

$$\begin{aligned} & \text{normal}\left(v, (V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle)\right) \iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle), ()\right) \right) \wedge (v \in V) \wedge (\langle v, v \rangle = 1) \\ & \# \text{ the vector has unit length} \end{aligned} \quad (193)$$

$$\text{innerProductNorm}\left(\|\mathbb{1}\|, (V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle)\right) \iff \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \mathbb{1}, \mathbb{2} \rangle), ()\right) \right) \wedge$$

$$\left(\text{norm}(\|\$1\|, (V, +, \cdot))\right) \wedge \left(\forall_{v \in V} \left(\|v\| = \langle v, v \rangle^{1/2}\right)\right) \quad (194)$$

$$\text{(THM) Cauchy-Schwarz inequality: } \forall_{v, w \in V} (\langle v, w \rangle \leq \|v\| \|w\|) \quad (195)$$

$$\text{normedVectorSpace}\left((V, +, \cdot, \|\$1\|), ()\right) \iff \left(\text{vectorSpace}\left((V, +, \cdot), ()\right)\right) \wedge \left(\text{norm}(\|\$1\|, (V, +, \cdot))\right) \quad (196)$$

$$\begin{aligned} \text{normMetric}\left(d(\$1, \$2), (V, \|\$1\|)\right) \iff & \left(\text{normedVectorSpace}\left((V, +, \cdot, \|\$1\|), ()\right)\right) \wedge \\ & \left(\text{metric}(\|\$1 - \$2\|, (V))\right) \end{aligned} \quad (197)$$

$$\text{(THM)} : \text{innerProduct} \implies \text{norm} \implies \text{metric} \quad (198)$$

$$\begin{aligned} \text{(THM)} : & \left(\text{metric} \wedge \forall_{u, v, w \in V} \forall_{s \in \mathbb{R}} \left(d(s(u+w), s(v+w)) = |s|d(u, v)\right) \wedge \forall_{v \in V} (\|v\| = d(v, \mathbf{0}))\right) \implies \\ & \left(\text{norm} \wedge \forall_{u, v \in V} \left(2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2\right) \wedge \forall_{v, w \in V} \left(\langle v, w \rangle = \frac{\|v+w\|^2 - \|v-w\|^2}{4}\right)\right) \implies \\ & \quad (\text{innerProduct}) \end{aligned} \quad (199)$$

$$\begin{aligned} \text{basis}((b)_n, (V, +, \cdot, \cdot)) \iff & \left(\text{vectorSpace}\left((V, +, \cdot), ()\right)\right) \wedge \\ & \left(\forall_{v \in V} \exists_{(a)_n \in \mathbb{R}^n} \left(v = \sum_{i=1}^n (a_i b_i)\right)\right) \end{aligned} \quad (200)$$

$$\begin{aligned} \text{orthonormalBasis}((b)_n, (V, +, \cdot, \langle \$1, \$2 \rangle)) \iff & \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right)\right) \wedge \\ & \left(\text{basis}((b)_n, (V, +, \cdot))\right) \wedge \left(\forall_{v \in (b)_n} \left(\text{normal}\left(v, (V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right)\right) \wedge \\ & \left(\forall_{v \in (b)_n} \forall_{w \in (b)_n \setminus \{v\}} \left(\text{orthogonal}\left((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle)\right)\right)\right) \end{aligned} \quad (201)$$

$$\text{subspace}((U, \circ), (V, \circ)) \iff \left(\text{space}((V, \circ), ())\right) \wedge (U \subseteq V) \wedge \left(\text{space}((U, \circ), ())\right) \quad (202)$$

$$\begin{aligned} \text{subspaceSum}(U+W, (U, W, V, +)) \iff & \left(\text{subspace}((U, +), (V, +))\right) \wedge \left(\text{subspace}((W, +), (V, +))\right) \wedge \\ & (U+W = \{u+w \mid u \in U \wedge w \in W\}) \end{aligned} \quad (203)$$

$$\text{subspaceDirectSum}(U \oplus W, (U, W, V, +)) \iff (U \cap W = \emptyset) \wedge \left(\text{subspaceSum}(U \oplus W, (U, W, V, +))\right) \quad (204)$$

$$\begin{aligned} \text{orthogonalComplement}\left(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right) \iff \\ \left(\text{subspace}\left((W, +, \cdot, \langle \$1, \$2 \rangle), \left(\text{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right)\right)\right)\right) \wedge \end{aligned}$$

$$\left(W^\perp = \{v \in V \mid w \in W \wedge \text{orthogonal}((v, w), (V, +, \cdot, \langle \$1, \$2 \rangle))\} \right) \quad (205)$$

$$\begin{aligned} & \text{orthogonalDecomposition}\left((W, W^\perp), (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right) \iff \\ & \left(\text{orthogonalComplement}\left(W^\perp, (W, V, +, \cdot, \langle \$1, \$2 \rangle)\right) \right) \wedge \left(\text{subspaceDirectSum}\left(V, (W, W^\perp, V, +)\right) \right) \end{aligned} \quad (206)$$

$$(\text{THM}) \text{ if } V \text{ is finite dimensional, then every vector has an orthogonal decomposition:} \quad (207)$$

2.2 Banach and Hilbert Space

$$\begin{aligned} & \text{cauchy}((s)_\mathbb{N}, (V, d(\$1, \$2))) \iff (\text{metricSpace}((V, d(\$1, \$2)), ()) \wedge ((s)_\mathbb{N} \subseteq V) \\ & \quad (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N (d(s_m, s_n) < \epsilon)) \\ & \quad \# \text{ distances between some tail-end point gets arbitrarily small} \end{aligned} \quad (208)$$

$$\begin{aligned} & \text{complete}((V, d(\$1, \$2)), ()) \iff (\forall (s)_\mathbb{N} \subseteq V \exists s \in V (\text{cauchy}((s)_\mathbb{N}, (V, d(\$1, \$2))) \implies \lim_{n \rightarrow \infty} (d(s, s_n)) = 0)) \\ & \quad \# \text{ or converges within the induced topological space} \\ & \quad \# \text{ in complete spaces, the weaker notion of cauchy is enforced to be equivalent to convergence} \end{aligned} \quad (209)$$

$$\begin{aligned} & \text{banachSpace}((V, +, \cdot, \|\$1\|), ()) \iff (\text{normMetric}(d(\$1, \$2), (V, \|\$1\|))) \wedge (\text{complete}(V, d(\$1, \$2)), ()) \\ & \quad \# \text{ a complete normed vector space} \end{aligned} \quad (210)$$

$$\begin{aligned} & \text{hilbertSpace}((V, +, \cdot, \langle \$1, \$2 \rangle), ()) \iff (\text{innerProductNorm}(\|\$1\|, (V, +, \cdot, \langle \$1, \$2 \rangle))) \wedge \\ & \quad (\text{normMetric}(d(\$1, \$2), (V, \|\$1\|))) \wedge (\text{complete}(V, d(\$1, \$2)), ()) \\ & \quad \# \text{ a complete inner product space} \end{aligned} \quad (211)$$

$$(\text{THM}) : \text{hilbertSpace} \implies \text{banachSpace} \quad (212)$$

$$\begin{aligned} & \text{innerProductMetric}(d(\$1, \$2), (V, +, \cdot, \langle \$1, \$2 \rangle)) \iff \\ & (\text{innerProductNorm}(\|\$1\|, (V, +, \cdot, \langle \$1, \$2 \rangle))) \wedge (\text{normMetric}(d(\$1, \$2), (V, \|\$1\|))) \end{aligned} \quad (213)$$

$$\begin{aligned} & \text{innerProductTopology}(\mathcal{O}, (V, +, \cdot, \langle \$1, \$2 \rangle)) \iff \\ & (\text{innerProductMetric}(d(\$1, \$2), (V, +, \cdot, \langle \$1, \$2 \rangle))) \wedge (\text{metricTopology}(\mathcal{O}, (V, d(\$1, \$2)))) \end{aligned} \quad (214)$$

$$\begin{aligned} & \text{separable}((V, \mathcal{O}, d), ()) \iff (\exists S \subseteq V (\text{dense}(S, (V, \mathcal{O}, d)) \wedge \text{countablyInfinite}(S, ()))) \\ & \quad \# \text{ only a countable subset needed to approximate any element in the entire space} \end{aligned} \quad (215)$$

$$\begin{aligned} & (\text{THM}) : \text{hilbertSpace}\left(\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right), ()\right) \implies \\ & \left(\left(\exists (b)_\mathbb{N} \subseteq V \left(\text{orthonormalBasis}\left((b)_\mathbb{N}, (V, +, \cdot, \langle \$1, \$2 \rangle)\right) \wedge \text{countablyInfinite}((b)_\mathbb{N}, ()) \right) \right) \iff \right. \\ & \quad \left. \left(\text{separable}\left(\left(V, \text{innerProductTopology}\left(\mathcal{O}, (V, +, \cdot, \langle \$1, \$2 \rangle)\right), d(\$1, \$2)\right), ()\right) \right) \right) \\ & \quad \# \text{ separability in hilbert spaces is equivalent to the existence of a countable orthonormal basis} \end{aligned} \quad (216)$$

2.3 Matrices, Operators, and Functionals

$$\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)) \iff (\text{map}(L, (V, W))) \wedge (\text{vectorSpace}((V, +_V, \cdot_V), ())) \wedge (\text{vectorSpace}((W, +_W, \cdot_W), ())) \wedge (\forall v_1, v_2 \in V \forall s_1, s_2 \in \mathbb{R} (L(s_1 \cdot_V v_1 +_V s_2 \cdot_V v_2) = s_1 \cdot_W L(v_1) +_W s_2 \cdot_W L(v_2))) \quad (217)$$

$$\text{denseMap}(L, (D, H, +, \cdot, \langle \$1, \$2 \rangle)) \iff (D \subseteq H) \wedge (\text{linearOperator}(L, (D, +, \cdot, H, +, \cdot))) \wedge (\text{innerProductTopology}(\mathcal{O}, (H, +, \cdot, \langle \$1, \$2 \rangle))) \wedge (\text{dense}(D, (H, \mathcal{O}, d(\$1, \$2)))) \quad (218)$$

$$\begin{aligned} \text{traceClass}(L, (H, +, \cdot, \langle \$1, \$2 \rangle)) &\iff (\text{hilbertSpace}((H, +, \cdot, \langle \$1, \$2 \rangle), ()), ())) \wedge \\ &(\text{linearOperator}(\text{Tr}(L), (L, H, +, \cdot, H, +, \cdot))) \wedge \\ &(\forall (b)_n \subseteq H (\text{orthonormalBasis}((b)_n, (H, +, \cdot, \langle \$1, \$2 \rangle)) \implies \text{Tr}(L) = \sum_n (\langle b_n, L(b_n) \rangle) < \infty)) \\ &\# \text{ does not depend on the choice of orthonormal basis} \\ &\# \text{ DEFINE} \end{aligned} \quad (219)$$

$$\begin{aligned} \text{mapNorm}(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) &\iff (\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W))) \wedge \\ &(\text{normedVectorSpace}((V, +_V, \cdot_V, \|\$1\|_V), ())) \wedge (\text{normedVectorSpace}((W, +_W, \cdot_W, \|\$1\|_W), ())) \wedge \\ &(\|L\| = \sup(\{\frac{\|Lf\|_W}{\|f\|_V} \mid f \in V\}) = \sup(\{\|Lf\|_W \mid f \in V \wedge \|f\| = 1\})) \end{aligned} \quad (220)$$

$$(\text{THM}) : \|Lf\| \leq \|L\| \|f\| \quad (221)$$

$$(\text{THM}) \text{ CONTINUE HERE 46:50:} \quad (222)$$

$$\begin{aligned} \text{boundedMap}(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) &\iff \\ (\text{mapNorm}(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W))) \wedge (\|L\| < \infty) \end{aligned} \quad (223)$$

$$\begin{aligned} \neg \text{boundedMap}(\|L\|, (L, V, +_V, \cdot_V, \|\$1\|_V, W, +_W, \cdot_W, \|\$1\|_W)) &\iff \\ (U \subset V) \wedge (\infty = \text{mapNorm}(\|L\|_U, (L, U, +_U, \cdot_U, \|\$1\|_U, W, +_W, \cdot_W, \|\$1\|_W))) \leq \|L\| \end{aligned} \quad (224)$$

$$0 \quad (225)$$

$$\begin{aligned} \text{adjoint}(L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)) &\iff (\text{hilbertSpace}((V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V), ())) \wedge \\ &(\text{hilbertSpace}((W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W), ())) \wedge (\text{linearOperator}(L, (V, +_V, \cdot_V, W, +_W, \cdot_W))) \wedge \\ &(\forall v \in V \forall w \in W ((\langle Lv, w \rangle_W = \langle v, L^T w \rangle_V) \vee ((Lv)^T w = v^T L^T w))) \\ &\# \text{ target operator that is equivalent to the domain operator} \end{aligned} \quad (226)$$

$$\begin{aligned} \text{selfAdjoint}(L, (V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)) &\iff \\ L = \text{adjoint}(L^T, (L, V, +_V, \cdot_V, \langle \$1, \$2 \rangle_V, W, +_W, \cdot_W, \langle \$1, \$2 \rangle_W)) \\ &\# \text{ also a generalization of symmetric matrices} \end{aligned} \quad (227)$$

$$\text{matrix}(L, (n, m)) \iff (\text{linearOperator}(L, (\mathbb{R}^n, +_n, \cdot_n, \mathbb{R}^m, +_m, \cdot_m))) \quad (228)$$

$$\text{eigenvector}(v, (L, V, +, \cdot)) \iff (\text{linearOperator}(L, (V, +, \cdot, V, +, \cdot))) \wedge (\exists \lambda \in \mathbb{R} (L(v) = \lambda v)) \quad (229)$$

$$\text{eigenvalue}(\lambda, (v, L, +, \cdot)) \iff \text{eigenvector}(v, (L, V, +, \cdot)) \quad (230)$$

$$\begin{aligned}
& \text{(THM) Spectral thm.: } (\text{matrix}(L, (n, n))) \wedge (\text{selfAdjoint}(L, (\mathbb{R}^n, +, \cdot, \langle \cdot, \cdot \rangle, \mathbb{R}^n, +, \cdot, \langle \cdot, \cdot \rangle))) \implies \\
& (E = \{v \in \mathbb{R}^n \mid \text{eigenvector}(v, (L, \mathbb{R}^n, +, \cdot))\}) \wedge (\text{orthonormalBasis}(E, (\mathbb{R}^n, +, \cdot, \langle \cdot, \cdot \rangle))) \\
& \# \text{ wow} \quad (231)
\end{aligned}$$

2.4 Underview

(232)

curve – fitting/explaining \neq *prediction* (233)

ill – defined problem + solution space constraints \implies *well – defined problem* (234)

x # input ; y # output (235)

$S_n = \{(x_1, y_1), \dots, (x_n, y_n)\}$ # training set (236)

$f_S(x) \sim y$ # solution (237)

each $(x, y) \in p(x, y)$ # training data x, y is a sample from an unknown distribution p (238)

$V(f(x), y) = d(f(x), y)$ # loss function (239)

$I[f] = \int_{X \times Y} V(f(x), y) p(x, y) dx dy$ # expected error (240)

$I_n[f] = \frac{1}{n} \sum_{i=1}^n V(f(x_i), y_i)$ # empirical error (241)

probabilisticConvergence $(X, ()) \iff \forall \epsilon > 0 \lim_{n \rightarrow \infty} P\|x_n - x\| \leq \epsilon = 0$ (242)

$I - I_n$ # generalization error (243)

well – posed := *exists, unique, stable*; else *ill – posed* (244)

3 Machine Learning

3.0.1 Overview

X # input ; Y # output ; $S(X, Y)$ # dataset (245)

learned parameters = parameters to be fixed by training with the dataset (246)

hyperparameters = parameters that depends on a dataset (247)

validation=partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition # useful for fixing hyperparameters (248)

cross-validation=average accuracy of validation for different choices of testing partition (249)

L1=scales linearly ; **L2**=scales quadratically (250)

d=distance=quantifies the the similarity between data points (251)

$$d_{L1}(A,B)=\sum_p |A_p - B_p| \text{ \# Manhattan distance} \quad (252)$$

$$d_{L2}(A,B)=\sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \quad (253)$$

kNN classifier=classifier based on k nearest data points (254)

s=class score=quantifies bias towards a particular class (255)

$$s_{linear} = f_{c \times 1}(x_{n \times 1}, W_{c \times n}, b_{c \times 1}) = W_{c \times n} x_{n \times 1} + b_{c \times 1} \text{ \# linear score function} \quad (256)$$

l=loss=quantifies the errors by the learned parameters (257)

$$l = \frac{1}{|c_i|} \sum_{c_i} l_i \text{ \# average loss for all classes} \quad (258)$$

$$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \text{ \# SVM hinge class loss function:}$$

ignores incorrect classes with lower scores including a non-zero margin (259)

$$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$

lower scores correspond to lower exponentiated-normalized probabilities (260)

R=regularization=optimizes the choice of learned parameters to minimize test error (261)

λ # regularization strength hyperparameter (262)

$$R_{L1}(W) = \sum_{W_i} |W_i| \text{ \# L1 regularization} \quad (263)$$

$$R_{L2}(W) = \sum_{W_i} W_i^2 \text{ \# L2 regularization} \quad (264)$$

$$L' = L + \lambda R(W) \# \text{ weight regularization} \quad (265)$$

$$\nabla_W L = \frac{\overrightarrow{\partial}}{\partial W_i} L = \text{loss gradient w.r.t. weights} \quad (266)$$

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L} \# \text{ loss gradient w.r.t. input weight in terms of external and local gradients} \quad (267)$$

$$s = \text{forward API} ; \frac{\partial L_L}{\partial W_I} = \text{backward API} \quad (268)$$

$$W_{t+1} = W_t - \nabla_{W_t} L \# \text{ weight update loss minimization} \quad (269)$$

$$\text{TODO: Research on Activation functions, Weight Initialization, Batch Normalization} \quad (270)$$

$$\text{review5meanvardiscussion/hyperparameteroptimization/babysittinglearning} \quad (271)$$

TODO loss L or l ??

4 Glossary

chaoticTopology	sequence	sigmaAlgebra	integrable
discreteTopology	sequenceConvergesTo	measurableSpace	integral
topology	sequence	measurableSet	simpleTopology
topologicalSpace	sequenceConvergesTo	measure	simpleSigma
open	continuous	measureSpace	simpleFunction
closed	homeomorphism	finiteMeasure	characteristicFunction
clopen	isomorphicTopologicalSpace	generatedSigmaAlgebra	exStandardSigma
neighborhood	continuous	borelSigmaAlgebra	nonNegIntegrable
chaoticTopology	homeomorphism	standardSigma	nonNegIntegral
discreteTopology	isomorphicTopologicalSpace	lebesgueMeasure	explicitIntegral
metric	T0Separate	measurableMap	integrable
metricSpace	T1Separate	pushForwardMeasure	integral
openBall	T2Separate	nullSet	curLp
metricTopology	T0Separate	almostEverywhere	vectorSpace
metricTopologicalSpace	T1Separate	sigmaAlgebra	vecLp
limitPoint	T2Separate	measurableSpace	norm
interiorPoint	openCover	measurableSet	seminormLp
closure	finiteSubcover	measure	Lp
dense	compact	measureSpace	curLp
eucD	compactSubset	finiteMeasure	vectorSpace
standardTopology	bounded	generatedSigmaAlgebra	vecLp
subsetTopology	openCover	borelSigmaAlgebra	norm
productTopology	finiteSubcover	standardSigma	seminormLp
metric	compact	lebesgueMeasure	Lp
metricSpace	compactSubset	measurableMap	innerProduct
openBall	bounded	pushForwardMeasure	innerProductSpace
metricTopology	openRefinement	nullSet	orthogonal
metricTopologicalSpace	locallyFinite	almostEverywhere	normal
limitPoint	paracompact	simpleTopology	innerProductNorm
interiorPoint	openRefinement	simpleSigma	normedVectorSpace
closure	locallyFinite	simpleFunction	normMetric
dense	paracompact	characteristicFunction	basis
eucD	connected	exStandardSigma	orthonormalBasis
standardTopology	pathConnected	nonNegIntegrable	subspace
subsetTopology	connected	nonNegIntegral	subspaceSum
productTopology	pathConnected	explicitIntegral	subspaceDirectSum

orthogonalComplement	subspaceDirectSum	hilbertSpace	eigenvalue
orthogonalDecomposition	orthogonalComplement	innerProductMetric	linearOperator
innerProduct	orthogonalDecomposition	innerProductTopology	denseMap
innerProductSpace	cauchy	separable	traceClass
orthogonal	complete	linearOperator	mapNorm
normal	banachSpace	denseMap	boundedMap
innerProductNorm	hilbertSpace	traceClass	adjoint
normedVectorSpace	innerProductMetric	mapNorm	selfAdjoint
normMetric	innerProductTopology	boundedMap	matrix
basis	separable	adjoint	eigenvector
orthonormalBasis	cauchy	selfAdjoint	eigenvalue
subspace	complete	matrix	
subspaceSum	banachSpace	eigenvector	