

Next-Next-Gen Notes

Object-Oriented Maths

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Note: Operators (op)s preserve type; Relations (rel)s return truths; todo rename default funcs

1 Logic and Set Theory

1.1 D: Logical Truths and Operators

undefined terms: $:=, =, (_), , , \cdot,$

$$\text{truth}[t] := \left(t = \begin{cases} T \\ F \end{cases} \right) \quad (1)$$

$$\text{operatorOR}[\vee][x, y] := {}_1(\text{truth}[x]), {}_1(\text{truth}[y]), {}_1 \left(\text{truth}[x \vee y] = \begin{cases} F & x = F, y = F \\ T & x = F, y = T \\ T & x = T, y = F \\ T & x = T, y = T \end{cases} \right) \cdot {}_1 \quad (2)$$

$$\text{operatorAND}[\wedge][x, y] := {}_1(\text{truth}[x]), {}_1(\text{truth}[y]), {}_1 \left(\text{truth}[x \wedge y] = \begin{cases} F & x = F, y = F \\ F & x = F, y = T \\ F & x = T, y = F \\ T & x = T, y = T \end{cases} \right) \cdot {}_1 \quad (3)$$

$$\text{operatorNOT}[\neg][x] := {}_1(\text{truth}[x]), {}_1 \left(\text{truth}[\neg x] = \begin{cases} T & x = F \\ F & x = T \end{cases} \right) \cdot {}_1 \quad (4)$$

$$\text{operatorXOR}[\underline{\vee}][x, y] := {}_1(\text{truth}[x]), {}_1(\text{truth}[y]), {}_1 \left(\text{truth}[x \underline{\vee} y] = \begin{cases} F & x = F, y = F \\ T & x = F, y = T \\ T & x = T, y = F \\ F & x = T, y = T \end{cases} \right) \cdot {}_1 \quad (5)$$

$$\text{operatorIF}[\implies][x, y] := {}_1(\text{truth}[x]), {}_1(\text{truth}[y]), {}_1 \left(\text{truth}[x \implies y] = (\neg x) \vee y = \begin{cases} T & x = F, y = F \\ T & x = F, y = T \\ F & x = T, y = F \\ T & x = T, y = T \end{cases} \right) \cdot {}_1 \quad (6)$$

a counterexample cannot follow from a false precedence, thus the conditional cannot be false

$$\text{operatorOIF}[\Leftarrow][x, y] := {}_1(\text{truth}[x][\Box], {}_1(\text{truth}[y][\Box]), {}_1 \left(\text{truth}[x \Leftarrow y][\Box] = (\neg y) \vee x = \begin{cases} T & x=F, y=F \\ F & x=F, y=T \\ T & x=T, y=F \\ T & x=T, y=T \end{cases} \right) {}_1 \right) \quad (7)$$

$$\text{operatorIIF}[\Leftrightarrow][x, y] := {}_1(\text{truth}[x][\Box], {}_1(\text{truth}[y][\Box]), {}_1 \left(\text{truth}[x \Leftrightarrow y][\Box] = (x \Rightarrow y) \wedge (y \Rightarrow x) = \begin{cases} T & x=F, y=F \\ F & x=F, y=T \\ F & x=T, y=F \\ T & x=T, y=T \end{cases} \right) {}_1 \right) \quad (8)$$

1.2 P: Boolean Algebra

$$\begin{aligned} \text{booleanAlgebra}[(\top, \perp, \otimes, \oplus, \ominus)][\Box] := & {}_1 \text{POS-LCom}((x \otimes y = y \otimes x), {}_1(x \oplus y = y \oplus x)) \# \text{Commutative}, {}_1 \\ & \text{POS-LDis}((x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)), {}_1(x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z))) \# \text{Distributive}, {}_1 \\ & \text{POS-LIdn}((x \otimes \top = x), {}_1(x \oplus \perp = x)) \# \text{Identity}, {}_1 \\ & \text{POS-LCmp}((x \otimes (\ominus x) = \perp), {}_1(x \oplus (\ominus x) = \top)) \# \text{Complement}, {}_1 \\ & \# \text{Note: I sometimes get too lazy to refer to POS-LCom.} \end{aligned} \quad (9)$$

$$\begin{aligned} & \text{INS-LBAI}(\text{booleanAlgebra}[T, F, \wedge, \vee, \neg][\Box]) \\ & \# \text{Proven by way of cases or truth tables} \end{aligned} \quad (10)$$

$$\begin{aligned} & \text{THM-Dual-1} \left(\text{booleanAlgebra}[(T, F, \wedge, \vee, \neg)][\Box] \Leftrightarrow {}_1 \right. \\ & \quad ((x \wedge y = y \wedge x), {}_1(x \vee y = y \vee x)) \# \text{Commutative}, {}_1 \\ & \quad ((x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)), {}_1(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z))) \# \text{Distributive}, {}_1 \\ & \quad ((x \wedge T = x), {}_1(x \vee F = x)) \# \text{Identity}, {}_1 \\ & \quad ((x \wedge \neg x = F), {}_1(x \vee \neg x = T)) \# \text{Complement}, {}_1 \Leftrightarrow {}_2 \\ & \quad ((x \vee y = y \vee x), {}_2(x \wedge y = y \wedge x)) \# \text{Reordered Commutative}, {}_2 \\ & \quad ((x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)), {}_2(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z))) \# \text{Reordered Distributive}, {}_2 \\ & \quad ((x \vee F = x), {}_2(x \wedge T = x)) \# \text{Reordered Identity}, {}_2 \\ & \quad ((x \vee \neg x = T), {}_2(x \wedge \neg x = F)) \# \text{Reordered Complement}, {}_2 \Leftrightarrow \\ & \quad \text{booleanAlgebra}[(F, T, \vee, \wedge, \neg)][\Box] \\ & \quad \text{THM-Dual-2} \text{THM-Dual-1}(\text{booleanAlgebra}[(T, F, \wedge, \vee, \neg)][\Box] \Leftrightarrow \text{booleanAlgebra}[(F, T, \vee, \wedge, \neg)][\Box]) \\ & \quad \text{THM-Dual} \text{THM-Dual-2} \text{INS-LBAI} \text{POS-LIdn} \text{booleanAlgebra}[(F, T, \vee, \wedge, \neg)][\Box] () \\ & \# \text{Boolean Algebra Duality follows from the swap symmetry of } (\wedge, T) \text{ and } (\vee, F) \text{ within the axioms} \end{aligned} \quad (11)$$

$$\begin{aligned} & \text{THM-LUNt-1}((x \vee y = T = x \vee z) \wedge (x \wedge y = F = x \wedge z)) \Rightarrow {}_1 \\ & \quad \text{THM-LUNt-2} \text{INS-LBAI} \text{POS-LIdn} (y = y \wedge T), {}_1 \\ & \quad \text{THM-LUNt-3} \text{THM-LUNt-1} (y \wedge T = y \wedge (x \vee z)), {}_1 \end{aligned}$$

$$\begin{aligned}
& \text{THM-LUNt-4} \text{ POS-LDis } (y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z))_{,1} \\
& \text{THM-LUNt-5} \text{ POS-LCom } ((y \wedge x) \vee (y \wedge z) = (x \wedge z) \vee (y \wedge z))_{,1} \\
& \text{THM-LUNt-4} \text{ THM-LUNt-6 } ((x \wedge z) \vee (y \wedge z) = z \wedge (x \vee y))_{,1} \\
& \text{THM-LUNt-7} \text{ THM-LUNt-1 } (z \wedge (x \vee y) = z \wedge T)_{,1} \\
& \text{THM-LUNt-8} \text{ POS-LIdn } (z \wedge T = z)_{,1} \\
& \text{THM-LUNt} \left(((x \vee y = T = x \vee z) \wedge (x \wedge y = F = x \wedge z)) \implies (y = z) \right) \\
& \text{THM-LUNt-1} \\
& \text{THM-LUNt-2} \\
& \text{THM-LUNt-3} \\
& \text{THM-LUNt-4} \\
& \text{THM-LUNt-5} \\
& \text{THM-LUNt-6} \\
& \text{THM-LUNt-7} \\
& \text{THM-LUNt-8}
\end{aligned}$$

Uniqueness of Complements (12)

$$\begin{aligned}
& \text{THM-LDom-1} \text{ INS-LBAI } (x \vee T = (x \vee T) \wedge T) \\
& \text{THM-LDom-2} \text{ POS-LCmp } ((x \vee T) \wedge T = (x \vee T) \wedge (x \vee \neg x)) \\
& \text{THM-LDom-3} \text{ POS-LDis } ((x \vee T) \wedge (x \vee \neg x) = x \vee (T \wedge \neg x)) \\
& \text{THM-LDom-4} \text{ POS-LIdn } (x \vee (T \wedge \neg x) = x \vee \neg x) \\
& \text{THM-LDom-5} \text{ POS-LCmp } (x \vee \neg x = T) \\
& \text{THM-LDom-6} \text{ THM-LDom-1 } (x \vee T = T) \\
& \text{THM-LDom-2} \\
& \text{THM-LDom-3} \\
& \text{THM-LDom-4} \\
& \text{THM-LDom-5} \\
& \text{THM-LDom-6} \text{ THM-LDom-6 } ((x \vee T = T), (x \wedge F = F)) \\
& \text{THM-Dual}
\end{aligned}$$

Domination (13)

$$\begin{aligned}
& \text{THM-LIdm-1} \text{ INS-LBAI } (x \vee x = (x \vee x) \wedge T) \\
& \text{THM-LIdm-2} \text{ POS-LCmp } ((x \vee x) \wedge T = (x \vee x) \wedge (x \vee \neg x)) \\
& \text{THM-LIdm-3} \text{ POS-LDis } ((x \vee x) \wedge (x \vee \neg x) = x \wedge (x \vee \neg x)) \\
& \text{THM-LIdm-4} \text{ POS-LCmp } (x \wedge (x \vee \neg x) = x \wedge T) \\
& \text{THM-LIdm-5} \text{ POS-LIdn } (x \wedge T = x) \\
& \text{THM-LIdm-6} \text{ THM-LIdm-1 } (x \vee x = x) \\
& \text{THM-LIdm-2} \\
& \text{THM-LIdm-3} \\
& \text{THM-LIdm-4} \\
& \text{THM-LIdm-5} \\
& \text{THM-LIdm-6} \text{ THM-LIdm-6 } ((x \vee x = x), (x \wedge x = x)) \\
& \text{THM-Dual}
\end{aligned}$$

Idempotent (14)

$$\begin{aligned}
& \text{THM-LInv-1} \text{ INS-LBAI } (\neg x = \neg x \vee F) \\
& \text{THM-LInv-2} \text{ POS-LCmp } (\neg x \vee F = \neg x \vee (x \wedge \neg x)) \\
& \text{THM-LInv-3} \text{ POS-LDis } (\neg x \vee (x \wedge \neg x) = (\neg x \vee x) \wedge (\neg x \vee \neg x)) \\
& \text{THM-LInv-4} \text{ POS-LCmp } ((\neg x \vee x) \wedge (\neg x \vee \neg x) = (\neg x \vee x) \wedge T) \\
& \text{THM-LInv-5} \text{ POS-LCmp } ((\neg x \vee x) \wedge T = (\neg x \vee x) \wedge (x \vee \neg x)) \\
& \text{THM-LInv-6} \text{ POS-LDis } ((\neg x \vee x) \wedge (x \vee \neg x) = x \vee (\neg x \wedge \neg x)) \\
& \text{THM-LInv-7} \text{ POS-LCmp } (x \vee (\neg x \wedge \neg x) = x \vee F) \\
& \text{THM-LInv-8} \text{ POS-LIdn } (x \vee F = x)
\end{aligned}$$

$THM-LInv$
 $THM-LInv-1$
 $THM-LInv-2$
 $THM-LInv-3$
 $THM-LInv-4$
 $THM-LInv-5$
 $THM-LInv-6$
 $THM-LInv-7$
 $THM-LInv-8$

Involution (15)

$THM-LAbs-1$
 $INS-LBAI$
 $POS-LIdn$

$THM-LAbs-2$
 $POS-LDis$

$THM-LAbs-3$
 $THM-LDom$

$THM-LAbs-4$
 $POS-LIdn$

$THM-LAbs-5$
 $THM-LAbs-1$
 $THM-LAbs-2$
 $THM-LAbs-3$
 $THM-LAbs-4$

$THM-LAbs$
 $THM-LAbs-5$
 $THM-Dual$

Absorption (16)

$THM-LAsc-1$

$THM-LAsc-2$
 $THM-LAsc-1$

$THM-LAsc-3$
 $THM-LAbs$

$THM-LAsc-4$
 $THM-LAsc-1$

$THM-LAsc-5$
 $INS-LBAI$
 $POS-LDis$

$THM-LAsc-6$
 $THM-LAbs$

$THM-LAsc-7$
 $THM-LAbs$

$THM-LAsc-8$
 $THM-LAbs$

$THM-LAsc-9$
 $THM-LAsc-2$
 $THM-LAsc-3$
 $THM-LAsc-4$
 $THM-LAsc-5$
 $THM-LAsc-6$
 $THM-LAsc-7$
 $THM-LAsc-8$

$THM-LAsc-10$
 $THM-LAsc-1$

$THM-LAsc-11$
 $INS-LBAI$
 $POS-LDis$

$THM-LAsc-12$
 $INS-LBAI$
 $POS-LCmp$

$THM-LAsc-13$
 $INS-LBAI$
 $POS-LIdn$

$THM-LAsc-14$
 $THM-LAsc-1$

$THM-LAsc-15$
 $INS-LBAI$
 $POS-LDis$

$THM-LAsc-16$
 $INS-LBAI$
 $POS-LDis$

$THM-LAsc-17$
 $INS-LBAI$
 $POS-LCmp$

$THM-LAsc-18$
 $INS-LBAI$
 $POS-LIdn$

$THM-LAsc-19$
 $INS-LBAI$
 $POS-LDis$

$$\begin{aligned}
& \begin{array}{l} \text{THM-LAsc-20} \\ \text{THM-LAsc-10} \\ \text{THM-LAsc-11} \\ \text{THM-LAsc-12} \\ \text{THM-LAsc-13} \\ \text{THM-LAsc-14} \\ \text{THM-LAsc-15} \\ \text{THM-LAsc-16} \\ \text{THM-LAsc-17} \\ \text{THM-LAsc-18} \\ \text{THM-LAsc-19} \end{array} (\neg x \wedge A = \neg x \wedge (y \vee z) = \neg x \wedge B),_1 \\
& \begin{array}{l} \text{THM-LAsc-21} \\ \text{INS-LBAI} \\ \text{POS-LDis} \end{array} (A = A \wedge T),_1 \\
& \begin{array}{l} \text{THM-LAsc-22} \\ \text{INS-LBAI} \\ \text{POS-LCmp} \end{array} (A \wedge T = A \wedge (x \vee \neg x)),_1 \\
& \begin{array}{l} \text{THM-LAsc-23} \\ \text{INS-LBAI} \\ \text{POS-LDis} \end{array} (A \wedge (x \vee \neg x) = (x \wedge A) \vee (\neg x \wedge A)),_1 \\
& \begin{array}{l} \text{THM-LAsc-24} \\ \text{THM-LAsc-9} \end{array} ((x \wedge A) \vee (\neg x \wedge A) = (x \wedge B) \vee (\neg x \wedge A)),_1 \\
& \begin{array}{l} \text{THM-LAsc-25} \\ \text{THM-LAsc-20} \end{array} ((x \wedge B) \vee (\neg x \wedge A) = (x \wedge B) \vee (\neg x \wedge B)),_1 \\
& \begin{array}{l} \text{THM-LAsc-26} \\ \text{INS-LBAI} \\ \text{POS-LDis} \end{array} ((x \wedge B) \vee (\neg x \wedge B) = B \wedge (x \vee \neg x)),_1 \\
& \begin{array}{l} \text{THM-LAsc-27} \\ \text{INS-LBAI} \\ \text{POS-LCmp} \end{array} (B \wedge (x \vee \neg x) = B \wedge T),_1 \\
& \begin{array}{l} \text{THM-LAsc-27} \\ \text{INS-LBAI} \\ \text{POS-LIdn} \end{array} (B \wedge T = B),_1 \\
& \begin{array}{l} \text{THM-LAsc-28} \\ \text{THM-LAsc-21} \\ \text{THM-LAsc-22} \\ \text{THM-LAsc-23} \\ \text{THM-LAsc-24} \\ \text{THM-LAsc-25} \\ \text{THM-LAsc-26} \\ \text{THM-LAsc-27} \end{array} (A = B),_1 \\
& \begin{array}{l} \text{THM-LAsc-29} \\ \text{THM-LAsc-28} \\ \text{THM-LAsc-1} \end{array} (x \vee (y \vee z) = (x \vee y) \vee z),_1 \\
& \begin{array}{l} \text{THM-LAsc} \\ \text{THM-LAsc-29} \\ \text{THM-Dual} \end{array} ((x \vee (y \vee z) = (x \vee y) \vee z), (x \wedge (y \wedge z) = (x \wedge y) \wedge z))
\end{aligned}$$

Associative (17)

$$\begin{aligned}
& \begin{array}{l} \text{THM-LDMr-1} \\ \text{INS-LBAI} \\ \text{POS-LDis} \end{array} ((x \vee y) \vee (\neg x \wedge \neg y) = ((x \vee y) \vee \neg x) \wedge ((x \vee y) \vee \neg y)) \\
& \begin{array}{l} \text{THM-LDMr-2} \\ \text{POS-LCom} \\ \text{THM-LAsc} \end{array} (((x \vee y) \vee \neg x) \wedge ((x \vee y) \vee \neg y) = ((x \vee \neg x) \vee y) \wedge ((\neg y \vee y) \vee x)) \\
& \begin{array}{l} \text{THM-LDMr-3} \\ \text{INS-LBAI} \\ \text{POS-LCmp} \end{array} (((x \vee \neg x) \vee y) \wedge ((\neg y \vee y) \vee x) = (T \vee y) \wedge (T \vee x)) \\
& \begin{array}{l} \text{THM-LDMr-4} \\ \text{THM-LDom} \end{array} ((T \vee y) \wedge (T \vee x) = T \wedge T) \\
& \begin{array}{l} \text{THM-LDMr-5} \\ \text{THM-LIdm} \end{array} (T \wedge T = T) \\
& \begin{array}{l} \text{THM-LDMr-6} \\ \text{THM-LDMr-1} \\ \text{THM-LDMr-2} \\ \text{THM-LDMr-3} \\ \text{THM-LDMr-4} \\ \text{THM-LDMr-5} \end{array} ((x \vee y) \vee (\neg x \wedge \neg y) = T) \\
& \begin{array}{l} \text{THM-LDMr-7} \\ \text{INS-LBAI} \\ \text{POS-LDis} \end{array} ((x \vee y) \wedge (\neg x \wedge \neg y) = (x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y)) \\
& \begin{array}{l} \text{THM-LDMr-8} \\ \text{POS-LCom} \\ \text{THM-LAsc} \end{array} ((x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y) = ((x \wedge \neg x) \wedge \neg y) \vee ((y \wedge \neg y) \wedge \neg x)) \\
& \begin{array}{l} \text{THM-LDMr-9} \\ \text{INS-LBAI} \\ \text{POS-LCmp} \end{array} (((x \wedge \neg x) \wedge \neg y) \vee ((y \wedge \neg y) \wedge \neg x) = (F \wedge \neg y) \vee (F \wedge \neg x)) \\
& \begin{array}{l} \text{THM-LDMr-10} \\ \text{THM-LDom} \end{array} ((F \wedge \neg y) \vee (F \wedge \neg x) = F \vee F) \\
& \begin{array}{l} \text{THM-LDMr-11} \\ \text{THM-LIdm} \end{array} (F \vee F = F) \\
& \begin{array}{l} \text{THM-LDMr-12} \\ \text{THM-LDMr-7} \\ \text{THM-LDMr-8} \\ \text{THM-LDMr-9} \\ \text{THM-LDMr-10} \\ \text{THM-LDMr-11} \end{array} ((x \vee y) \wedge (\neg x \wedge \neg y) = F) \\
& \begin{array}{l} \text{THM-LDMr-13} \\ \text{THM-LDMr-6} \\ \text{THM-LDMr-12} \\ \text{POS-LCmp} \end{array} (((x \vee y) \vee (\neg x \wedge \neg y) = T = (x \vee y) \vee \neg(x \vee y)), ((x \vee y) \wedge (\neg x \wedge \neg y) = F = (x \vee y) \wedge \neg(x \vee y)))
\end{aligned}$$

$$\begin{array}{c}
\textcolor{blue}{THM-LDMr-14} \\
\textcolor{blue}{THM-LDMr-13} \\
\textcolor{blue}{THM-LUNt}
\end{array}
(\neg x \wedge \neg y = \neg(x \vee y))$$

$$\begin{array}{c}
\textcolor{blue}{THM-LDMr} \\
\textcolor{blue}{THM-LDMr-14} \\
\textcolor{blue}{THM-Dual}
\end{array}
\left((\neg x \wedge \neg y = \neg(x \vee y)), (\neg x \vee \neg y = \neg(x \wedge y)) \right)$$

Boolean De Morgan's Laws (18)

$$\begin{array}{c}
\textcolor{blue}{THM-CtrP-1} \\
\textcolor{blue}{operatorIF}
\end{array}
(x \implies y = (\neg x) \vee y)$$

$$\begin{array}{c}
\textcolor{blue}{THM-CtrP-2} \\
\textcolor{blue}{POS-LCom} \\
\textcolor{blue}{THM-LInv}
\end{array}
((\neg x) \vee y = ((\neg y) \vee (\neg x)))$$

$$\begin{array}{c}
\textcolor{blue}{THM-CtrP-3} \\
\textcolor{blue}{operatorIF}
\end{array}
((\neg y) \vee (\neg x) = (\neg y) \implies (\neg x))$$

$$\begin{array}{c}
\textcolor{blue}{THM-CtrP} \\
\textcolor{blue}{THM-CtrP-1} \\
\textcolor{blue}{THM-CtrP-2} \\
\textcolor{blue}{THM-CtrP-3}
\end{array}
(x \implies y = (\neg y) \implies (\neg x))$$

Contrapositive Law (19)

MISC IMPLICATION LAWS:

$$\begin{array}{l}
(T \implies x = x) \\
(F \implies x = T) \\
(x \implies T = T) \\
(x \implies F = \neg x) \\
((x \vee y) \implies z) = (x \implies z) \wedge (y \implies z) \\
(x \implies (y \wedge z)) = (x \implies y) \wedge (x \implies z) \\
(x \iff y = x \implies y \wedge \neg x \implies \neg y)
\end{array}
\tag{20}$$

1.3 Predicates, Sets, Tuples

undefined terms: $arg_(_)$, set , \in , $\{_ \}$,

$$\textcolor{teal}{predicate}[P][_] := \textcolor{blue}{truth}[P(v_{free})][_] \tag{21}$$

$$\begin{array}{c}
\textcolor{teal}{universalQuantifier}[\forall][Q, P] := {}_1(\textcolor{blue}{predicate}[Q][_] , {}_1(\textcolor{blue}{predicate}[P][_] , \\
(\forall_{arg_x(Q(x))}(P(x)) = Q(y_{free}) \implies P(y_{free})) . {}_1
\end{array}
\tag{22}$$

$$\textcolor{teal}{existentialQuantifier}[\exists][Q, P] := (\exists_{arg_x(Q(x))}(P(x)) = \neg \forall_{arg_x(Q(x))}(\neg P(x))) \tag{23}$$

$$\textcolor{teal}{uniquenessQuantifier}[\exists!][Q, P] := (\exists!_{arg_x(Q(x))}(P(x)) = \exists_{arg_x(Q(x))}(P(x) \wedge \neg \exists_{arg_y(Q(y))}(P(y) \wedge \neg(y = x)))) \tag{24}$$

$$\textcolor{teal}{relationSetEq}[=][X, Y] := (\forall_{arg_z(z \in X \vee z \in Y)}(z \in X \wedge z \in Y)) \tag{25}$$

$$\textcolor{teal}{operatorIntersection}[\bigcap][X] := (z \in \bigcap(X) \iff \forall_{x \in X}(z \in x)) \tag{26}$$

$$\textcolor{teal}{operatorUnion}[\bigcup][X] := (z \in \bigcup(X) \iff \exists_{x \in X}(z \in x)) \tag{27}$$

$$\textcolor{teal}{orderedPair}[< x, y >][_] == < x, y > == < a, b > \text{ iff } x = a \text{ and } y = b == \{\{x\}, \{x, y\}\} \tag{28}$$