Next-Next-Gen Notes
Object-Oriented Maths

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Format: $characteristic((subjects),(dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s,()) \iff well\text{-}formedString(s,()) \tag{1}$$

$$\begin{aligned} proposition((p,t),()) \iff \Big(statement(p,())\Big)\wedge \\ (t = eval(p))\wedge \\ (t = true \veebar t = false) \end{aligned} \tag{2}$$

$$operator\Big(o,\Big((p)_{n\in\mathbb{N}}\Big)\Big) \iff proposition\Big(o\Big((p)_{n\in\mathbb{N}}\Big),()\Big) \tag{3}$$

$$\begin{aligned} operator(\neg,(p_1)) \iff \Big(proposition((p_1,true),()) \implies ((\neg p_1, false),())\Big)\wedge \\ \Big(proposition((p_1,false),()) \implies ((\neg p_1, true),())\Big) \\ \#\text{ an operator takes in propositions and returns a proposition} \end{aligned} \tag{4}$$

$$\begin{aligned} operator(\neg) \iff \textbf{NOT} \text{ ; } operator(\vee) \iff \textbf{OR} \text{ ; } operator(\wedge) \iff \textbf{AND} \text{ ; } operator(\veebar) \iff \textbf{XOR} \\ operator(\implies) \iff \textbf{IF} \text{ ; } operator(\impliedby) \iff \textbf{OIF} \text{ ; } operator(\iff) \iff \textbf{IFF} \end{aligned} \tag{5}$$

$$\begin{aligned} proposition((false \implies true),true,()) \wedge proposition((false \implies false),true,()) \\ \#\text{ truths based on a false premise is not false; ex falso quodlibet principle} \end{aligned} \tag{6}$$

$$\text{(THM)} : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \tag{7}$$

$$predicate(P,(V)) \iff \forall_{v\in V}\Big(proposition\Big((P(v),t),()\Big)\Big) \tag{8}$$

$$\begin{aligned} 0thOrderLogic(P,()) \iff proposition((P,t),()) \\ \#\text{ individual proposition} \end{aligned} \tag{9}$$

$$\begin{aligned} 1stOrderLogic(P,(V)) \iff \Big(\forall_{v\in V}\Big(0thOrderLogic(v,())\Big)\Big)\wedge \\ \Big(\forall_{v\in V}\Big(proposition\Big((P(v),t),()\Big)\Big)\Big) \end{aligned}$$

$$\# \text{ propositions defined over a set of the lower order logical statements} \tag{10}$$

$$quantifier\big(q,(p,V)\big) \iff \Big(predicate\big(p,(V)\big)\Big)\wedge \Big(proposition\Big(\big(q(p),t\big),()\Big)\Big)$$
$$\# \text{ a quantifier takes in a predicate and returns a proposition} \tag{11}$$

$$quantifier\big(\forall,(p,V)\big) \iff proposition\Big(\Big(\wedge_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\# \text{ universal quantifier} \tag{12}$$

$$quantifier\big(\exists,(p,V)\big) \iff proposition\Big(\Big(\vee_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\# \text{ existential quantifier} \tag{13}$$

$$quantifier\big(\exists!,(p,V)\big) \iff \exists_{x\in V}\Big(P(x)\wedge\neg\Big(\exists_{y\in V\setminus\{x\}}\big(P(y)\big)\Big)\Big)$$
$$\# \text{ uniqueness quantifier} \tag{14}$$

$$(\text{THM}) : \forall_x p(x) \iff \neg\exists_x\neg p(x)$$
$$\# \text{ De Morgan's law} \tag{15}$$

$$(\text{THM}) : \forall_x\exists_y p(x,y) = \forall_x\neg\forall_y\neg p(x,y) \neq \exists_y\forall_x p(x,y) = \neg\forall_y\neg\big(\forall_x p(x,y)\big) = \neg\forall_y\exists_x\neg p(x,y)$$
$$\# \text{ different quantifiers are not interchangeable} \tag{16}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{17}$$

$$\textbf{proof = truths derived from a finite number of axioms and deductions} \tag{18}$$

$$\textbf{elementary arithmetics = system with substitutions, and some notion of addition, multiplication, and prime nuumbers for encoding metamathematics} \tag{19}$$

$$\textbf{Gödel theorem} \implies \textbf{axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions} \tag{20}$$

$$sequenceSet((A)_{\mathbb{N}},(A)) \iff (Amapinputn)((A)_{\mathbb{N}} = \{A(1),A(2),A(3),\ldots\}) \tag{21}$$

$$\textbf{TODO: define union, intersection, complement, etc.} \tag{22}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{23}$$

1.1 Axiomatic Set Theory

$$============ \textbf{N O T = U P D A T E D} ============ \tag{24}$$

$$\textbf{ZFC set theory} = \textbf{standard form of axiomatic set theory} \tag{25}$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \tag{26}$$

$$(A = B) = A \subseteq B \land B \subseteq A \tag{27}$$

$$\in \textbf{basis} \implies \{x, y\} = \{y, x\} \land \{x\} = \{x, x\} \tag{28}$$

$$\in \textbf{and sets works following the 9 ZFC axioms:} \tag{29}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \ \# \text{ E: } \in \text{ is only a proposition on sets} \tag{30}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \ \# \text{ E: existence of empty set} \tag{31}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \lor u = y \ \# \text{ C: pair set construction} \tag{32}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \land y \in x \implies y \in u) \ \# \text{ C: union set construction} \tag{33}$$

$$x = \{\{a\}, \{b\}\} \ \# \text{ from the pair set axiom} \tag{34}$$

$$u = \cup x = \cup \{\{a\}, \{b\}\} = \{a, b\} \tag{35}$$

$$\forall_x \exists!_y R(x, y) \ \# \text{ functional relation } R \tag{36}$$

$$\begin{gathered} \exists_i \forall_x \exists!_y R(x, y) \implies y \in i \ \# \text{ C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \mid P(y)\} \ \# \text{ Restricted Comprehension } \not\Longrightarrow \{y \mid P(y)\} \ \# \text{ Universal Comprehension} \end{gathered} \tag{37}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \implies P(x)\big) \ \# \text{ ignores out of scope} \neq \forall_x \big(x \in m \land P(x)\big) \ \# \text{ restricts entirety} \tag{38}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \implies n \subseteq \mathcal{P}(m)\big) \ \# \text{ C: existence of power set} \tag{39}$$

$$\exists_I \Big(\emptyset \in I \land \forall_{x \in I} \big(\{x\} \in I\big)\Big) \ \# \text{ I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \implies \mathbb{N} \textbf{ is a set} \tag{40}$$

$$\forall_x \Big(\big(\emptyset \notin x \land x \cap x' = \emptyset\big) \implies \exists_y (\textbf{set of each e} \in x)\Big) \ \# \text{ C: axiom of choice} \tag{41}$$

$$\forall_x x \neq \emptyset \implies x \notin x \ \# \text{ F: axiom of foundation covers further paradoxes} \tag{42}$$

$$============ \textbf{N O T} = \textbf{U P D A T E D} ============ \tag{43}$$

1.2 Classification of sets

$$\textit{space}\big((\textit{set}, \textit{structure}), ()\big) \iff \textit{structure}(\textit{set})$$
$\#$ a space a set equipped with some structure

$$\# \text{ various spaces can be studied through structure preserving maps between those spaces} \tag{44}$$

$$\begin{gathered} map\big(\phi,(A,B)\big) \iff \Big(\forall_{a\in A}\exists!_{b\in B}\big(\phi(a,b)\big)\Big)\lor \\ \Big(\forall_{a\in A}\exists!_{b\in B}\big(b=\phi(a)\big)\Big) \\ \# \text{ maps elements of a set to elements of another set} \end{gathered} \tag{45}$$

$$domain\big(A,(\phi,A,B)\big) \iff \Big(map\big(\phi,(A,B)\big)\Big) \tag{46}$$

$$codomain\big(B,(\phi,A,B)\big) \iff \Big(map\big(\phi,(A,B)\big)\Big) \tag{47}$$

$$\begin{gathered} image\big(B,(A,q,M,N)\big) \iff \Big(map\big(q,(M,N)\big)\land A\subseteq M\Big)\land \\ \Big(B=\{n\in N \mid \exists_{a\in A}\big(q(a)=n\big)\}\Big) \end{gathered} \tag{48}$$

$$\begin{gathered} preimage\big(A,(B,q,M,N)\big) \iff \Big(map\big(q,(M,N)\big)\land B\subseteq N\Big)\land \\ \Big(A=\{m\in M \mid \exists_{b\in B}\big(b=q(m)\big)\}\Big) \end{gathered} \tag{49}$$

$$\begin{gathered} injection\big(q,(M,N)\big) \iff \Big(map\big(q,(M,N)\big)\Big)\land \\ \forall_{u,v\in M}\big(q(u)=q(v) \implies u=v\big) \\ \# \text{ every } m \text{ has at most 1 image} \end{gathered} \tag{50}$$

$$\begin{gathered} surjection\big(q,(M,N)\big) \iff \Big(map\big(q,(M,N)\big)\Big)\land \\ \forall_{n\in N}\exists_{m\in M}\big(n=q(m)\big) \\ \# \text{ every } n \text{ has at least 1 preimage} \end{gathered} \tag{51}$$

$$\begin{gathered} bijection\big(q,(M,N)\big) \iff \Big(injection\big(q,(M,N)\big)\Big)\land \\ \Big(surjection\big(q,(M,N)\big)\Big) \\ \# \text{ every unique } m \text{ corresponds to a unique } n \end{gathered} \tag{52}$$

$$isomorphicSets\big((A,B),()\big) \iff \exists_\phi\Big(bijection\big(\phi,(A,B)\big)\Big) \tag{53}$$

$$infiniteSet\big(S,()\big) \iff \exists_{T\subset S}\Big(isomorphicSets\big((T,S),()\big)\Big) \tag{54}$$

$$finiteSet\big(S,()\big) \iff \Big(\neg infiniteSet\big(S,()\big)\Big)\lor\big(|S|\in\mathbb{N}\big) \tag{55}$$

$$countablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big)\land\Big(isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \tag{56}$$

$$uncountablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big)\land\Big(\neg isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \tag{57}$$

$$inverseMap\Big(q^{-1},(q,M,N)\Big) \iff \Big(bijection(q,(M,N))\Big)\wedge \Big(map\Big(q^{-1},(N,M)\Big)\Big)\wedge \Big(\forall_{n\in N}\exists!_{m\in M}\Big(q(m)=n \implies q^{-1}(n)=m\Big)\Big) \tag{58}$$

$$mapComposition\big(\phi\circ\psi,(\phi,\psi,A,B,C)\big) \iff map\big(\psi,(A,B)\big)\wedge map\big(\phi,(B,C)\big)\wedge \forall_{a\in A}\Big(\phi\circ\psi(a)=\phi\big(\psi(a)\big)\Big) \tag{59}$$

$$equivalenceRelation\big(\sim,(M)\big) \iff \big(\forall_{m\in M}(m\sim m)\big)\wedge \big(\forall_{m,n\in M}(m\sim n \implies n\sim m)\big)\wedge \big(\forall_{m,n,p\in M}(m\sim n\wedge n\sim p \implies m\sim p)\big) \text{ \# behaves as equivalences should} \tag{60}$$

$$equivalenceClass\big([m],(m,M,\sim)\big) \iff [m]=\{n\in M \mid n\sim m\} \text{ \# set of elements satisfying the equivalence relation with } m \tag{61}$$

$$\text{(THM)} : a\in[m] \implies [a]=[m] \;;\; [m]=[n] \veebar [m]\cap[n]=\emptyset \text{ \# equivalence class properties} \tag{62}$$

$$quotientSet\big(M/\sim,(M,\sim)\big) \iff M/\sim=\{[m]\in\mathcal{P}(M) \mid m\in M\} \text{ \# set of all equivalence classes} \tag{63}$$

$$\text{(THM)} : \textbf{axiom of choice} \implies \forall_{[m]\in M/\sim}\exists_r\big(r\in[m]\big) \text{ \# well-defined maps may be defined in terms of chosen representative elements } r \tag{64}$$

1.3 Construction of number sets

$$============ \textbf{N O T = U P D A T E D} ============ \tag{65}$$

$$\textbf{axiom of infinity} \implies \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\dots\}\cong\mathbb{N} \tag{66}$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \tag{67}$$

$$\textbf{addition} = \textbf{successor map:}\ \mathbb{N}\to\mathbb{N}=S(n)=\{n\} \text{ \# adds a layer of brackets} \tag{68}$$

$$\textbf{subtraction} = \textbf{predecessor map:}\ \mathbb{N}^\star\to\mathbb{N}=P(n)=m \mid m\in n \text{ \# removes a layer of brackets} \tag{69}$$

$$S^0=id \;;\; n\in\mathbb{N}^\star \implies S^n=S\circ S^{P(n)} \tag{70}$$

$$\textbf{addition} = + : \mathbb{N}\times\mathbb{N}\to\mathbb{N}=+(m,n)=m+n=S^n(m) \tag{71}$$

$$S^x=id=S^0 \implies x=\textbf{additive identity}=0 \tag{72}$$

$$S^n(x) = 0 \implies x = \textbf{additive inverse} \notin \mathbb{N} \ \# \text{ git gud smh -\_-} \tag{73}$$

$$\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim \textbf{, s.t.:}$$
$$(m,n) \sim (p,q) \iff m+q = p+n \ \# \text{ span } \mathbb{Z} \text{ using differences then group equal differences} \tag{74}$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \to [(n,0)] \ \# \ \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{75}$$

$$+_{\mathbb{Z}} = [(m +_{\mathbb{N}} p, n +_{\mathbb{N}} q)] \ \# \text{ well-defined and consistent} \tag{76}$$

$$\textbf{multiplication} \ldots M^x = id \implies x = \textbf{multiplicative identity} = 1 \ldots \textbf{multiplicative inverse} \notin \mathbb{N} \tag{77}$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^{\star})/\sim \textbf{, s.t.:} \ (x,y) \sim (u,v) \iff x \cdot v = u \cdot y \tag{78}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \to [(q,1)] \ ; \ \ldots \{x \mid x^2 = 2\} \notin \mathbb{Q} \tag{79}$$

$$\mathbb{R} = \textbf{almost homomorphisms on } \mathbb{Z}/\sim \ \# \text{ http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{80}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{81}$$

1.4 Topology

$$\begin{aligned} \mathit{topology}(\mathcal{O},(M)) \iff & (\mathcal{O} \subseteq \mathcal{P}(M)) \wedge \\ & (\emptyset, M \in \mathcal{O}) \wedge \\ & \Big((F \in \mathcal{O} \wedge |F| < |\mathbb{N}|) \implies \cap F \in \mathcal{O}\Big) \wedge \\ & (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O}) \end{aligned}$$
\# topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.
\# arbitrary unions of open sets always result in an open set
\# open sets do not contain their boundaries and infinite intersections of open sets may approach and
\# induce boundaries resulting in a closed set (82)

$$\mathit{topologicalSpace}((M,\mathcal{O}),()) \iff \mathit{topology}(\mathcal{O},(M)) \tag{83}$$

$$\begin{aligned} \mathit{open}(S,(M,\mathcal{O})) \iff & \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big) \wedge \\ & (S \subseteq M) \wedge (S \in \mathcal{O}) \end{aligned}$$
\# an open set do not contains its own boundaries (84)

$$\begin{aligned} \mathit{closed}(S,(M,\mathcal{O})) \iff & \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big) \wedge \\ & (S \subseteq M) \wedge \big(S \in \mathcal{P}(M) \setminus \mathcal{O}\big) \end{aligned}$$
\# a closed set contains the boundaries an open set (85)

$$\mathit{clopen}(S,(M,\mathcal{O})) \iff \Big(\mathit{closed}(S,(M,\mathcal{O}))\Big) \wedge \Big(\mathit{open}(S,(M,\mathcal{O}))\Big) \tag{86}$$

$$\mathit{neighborhood}(U,(a,\mathcal{O})) \iff (a \in U \in \mathcal{O})$$

$$\text{\# another name for open set containing } a \quad (87)$$

$$M = \{a,b,c,d\} \wedge \mathcal{O} = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\} \implies \Big(\mathit{open}(X,(M,\mathcal{O})) \iff X = \{\emptyset, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, M\}\Big) \wedge \Big(\mathit{closed}(Y,(M,\mathcal{O})) \iff Y = \{\emptyset, \{a,b,d\}, \{c,d\}, \{a,b\}, \{d\}, M\}\Big) \wedge \Big(\mathit{clopen}(Z,(M,\mathcal{O})) \iff Z = \{\emptyset, \{a,b\}, \{c,d\}, M\}\Big) \quad (88)$$

$$\mathit{chaoticTopology}(M) = \{0, M\} \; ; \; \mathit{discreteTopology} = \mathcal{P}(M) \quad (89)$$

1.5 Induced topology

$$\mathit{distance}(d,(M)) \iff \Big(\forall_{x,y\in M}\Big(d(x,y) = d(y,x) \in \mathbb{R}_0^+\Big)\Big) \wedge \Big(\forall_{x,y\in M}\big(d(x,y) = 0 \iff x = y\big)\Big) \wedge \Big(\forall_{x,y,z}\Big(\big(d(x,z) \leq d(x,y) + d(y,z)\big)\Big)\Big)$$

$$\text{\# behaves as distances should} \quad (90)$$

$$\mathit{metricSpace}((M,d),()) \iff \mathit{distance}(d,(M)) \quad (91)$$

$$\mathit{openBall}(B,(r,p,M,d)) \iff \Big(\mathit{metricSpace}((M,d),())\Big) \wedge (r \in \mathbb{R}^+, p \in M) \wedge (B = \{q \in M \mid d(p,q) < r\}) \quad (92)$$

$$\mathit{metricTopology}(\mathcal{O},(M,d)) \iff \Big(\mathit{metricSpace}((M,d),())\Big) \wedge \Big(\mathcal{O} = \{U \in \mathcal{P}(M) \mid \forall_{p\in U}\exists_{r\in\mathbb{R}^+}\Big(\mathit{openBall}(B,(r,p,M,d)) \wedge B \subseteq U\Big)\}\Big)$$

$$\text{\# every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (93)$$

$$\mathit{metricTopologicalSpace}((M,\mathcal{O},d),()) \iff \mathit{metricTopology}(\mathcal{O},(M,d)) \quad (94)$$

$$\mathit{limitPoint}(p,(S,M,\mathcal{O},d)) \iff \Big(\mathit{metricTopologicalSpace}((M,\mathcal{O},d),())\Big) \wedge (S \subseteq M) \wedge \forall_{r\in\mathbb{R}^+}\Big(\mathit{openBall}(B,(r,p,M,d)) \wedge B \cap S \neq \emptyset\Big)$$

$$\text{\# every open ball centered at } p \text{ contains some intersection with } S \quad (95)$$

$$\mathit{interiorPoint}(p,(S,M,\mathcal{O},d)) \iff \Big(\mathit{metricTopologicalSpace}((M,\mathcal{O},d),())\Big) \wedge (S \subseteq M) \wedge \Big(\exists_{r\in\mathbb{R}^+}\Big(\mathit{openBall}(B,(r,p,M,d)) \wedge B \subseteq S\Big)\Big)$$

$$\text{\# there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (96)$$

$$\mathit{closure}(\bar{S},(S,M,\mathcal{O},d)) \iff \bar{S} = S \cup \{p \in M \mid \mathit{limitPoint}(p,(S,M,\mathcal{O},d))\} \quad (97)$$

$$dense\big(S,(M,\mathcal{O},d)\big) \iff (S \subseteq M) \wedge \Big(\forall_{p\in M}\Big(p \in closure\big(\bar{S},(S,M,\mathcal{O},d)\big)\Big)\Big)$$
$$\text{\# every of point in } M \text{ is a point or a limit point of } S \tag{98}$$

$$eucD\big(d,(n)\big) \iff \big(\forall_{i\in\mathbb{N}\wedge i\leq n}(x_i \in \mathbb{R})\big) \wedge \left(d = \sqrt[2]{\sum_{i=1}^{n} {x_i}^2}\right) \tag{99}$$

$$metricTopology\Big(standardTopology,\Big(\mathbb{R}^n, eucD\big(d,(n)\big)\Big)\Big)$$
$$============= \textbf{N O T = U P D A T E D} =============$$
$$\textbf{L1: } \forall_{p\in U=\emptyset}(\ldots) \implies \forall_p\Big(\big(p \in \emptyset\big) \implies \ldots\Big) \implies \forall_p\big((\textbf{False}) \implies \ldots\big) \implies \emptyset \in \mathcal{O}_{standard}$$
$$\textbf{L2: } \forall_{p\in\mathbb{R}^n} B(r,p,\mathbb{R}^n,d) \subseteq \mathbb{R}^n \implies M \in \mathcal{O}_{standard}$$
$$\textbf{L4: } C \subseteq \mathcal{O}_{standard} \implies \forall_{U\in C}\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\big(B_r(p) \subseteq U \subseteq \cup C\big) \implies \cup C \in \mathcal{O}_{standard}$$
$$\textbf{L3: } U,V \in \mathcal{O}_{standard} \implies p \in U \cap V \implies p \in U \wedge p \in V \implies$$
$$\exists_{r\in\mathbb{R}^+} B(r,p,\mathbb{R}^n,d) \wedge \exists_{s\in\mathbb{R}^+} B(s,p,\mathbb{R}^n,d) \implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big) \subseteq U \wedge B\big(min(r,s),q,\mathbb{R}^n,d\big) \subseteq V \implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big) \in U \cap V \implies U \cap V \in \mathcal{O}_{standard}$$
$$\text{\# natural topology for } \mathbb{R}^d$$
$$\text{\# could fail on infinite sets since } min \text{ could approach } 0$$
$$============= \textbf{N O T = U P D A T E D} ============= \tag{100}$$

$$subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big) \iff topology\big(\mathcal{O},(M)\big) \wedge (N \subseteq M) \wedge \big(\mathcal{O}|_N = \{U \cap N \mid U \in \mathcal{O}\}\big)$$
$$\text{\# crops open sets outside } N \tag{101}$$

$$\text{(THM)}: subsetTopology\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big) \wedge topology\big(\mathcal{O}|_N,(N)\big) \impliedby$$
$$============= \textbf{N O T = U P D A T E D} =============$$
$$\textbf{L1: } \emptyset \in \mathcal{O} \implies U = \emptyset \implies \emptyset \cap N = \emptyset \implies \emptyset \in \mathcal{O}|_N$$
$$\textbf{L2: } M \in \mathcal{O} \implies U = M \implies M \cap N = N \implies N \in \mathcal{O}|_N$$
$$\textbf{L3: } S,T \in \mathcal{O}|_N \implies \exists_{U\in\mathcal{O}}(S = U \cap N) \wedge \exists_{V\in\mathcal{O}}(T = V \cap N) \implies S \cap T = (U \cap N) \cap (V \cap N)$$
$$= (U \cap V) \cap N \wedge U \cap V \in \mathcal{O} \implies S \cap T \in \mathcal{O}|_N$$
$$\textbf{L4: } TODO : EXERCISE$$
$$============= \textbf{N O T = U P D A T E D} ============= \tag{102}$$

$$productTopology\Big(\mathcal{O}_{A\times B},\big((A,\mathcal{O}_A),(B,\mathcal{O}_B)\big)\Big) \iff \Big(topology\big(\mathcal{O}_A,(A)\big)\Big) \wedge \Big(topology\big(\mathcal{O}_B,(B)\big)\Big) \wedge$$
$$\big(\mathcal{O}_{A\times B} = \{(a,b) \in A \times B \mid \exists_S(a \in S \in \mathcal{O}_A)\exists_T(b \in T \in \mathcal{O}_B)\}\big)$$
$$\text{\# open in cross iff open in each} \tag{103}$$

1.6 Convergence

$$sequence\big(q,(M)\big) \iff map\big(q,(\mathbb{N},M)\big) \tag{104}$$

$$sequenceConvergesTo\big((q,a),(M,\mathcal{O})\big) \iff \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$\Big(sequence\big(q,(M)\big)\Big) \wedge (a \in M) \wedge \Big(\forall_{U\in\mathcal{O}|a\in U}\exists_{N\in\mathbb{N}}\forall_{n>N}\big(q(n) \in U\big)\Big)$$
$$\text{\# each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \tag{105}$$

(THM) : **convergence generalizes to: the sequence** $q : \mathbb{N} \to \mathbb{R}^d$ **converges against** $a \in \mathbb{R}^d$ **in** $\mathcal{O}_S$ **if:**

$$\forall_{r>0} \exists_{N \in \mathbb{N}} \forall_{n>N} \big(||q(n) - a|| < \epsilon \big) \text{ \# distance based convergence} \tag{106}$$

1.7 Continuity

$$\mathit{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \iff \Big(\mathit{topologicalSpace}((M, \mathcal{O}_M), ()) \Big) \wedge \Big(\mathit{topologicalSpace}((N, \mathcal{O}_N), ()) \Big) \wedge \Big(\forall_{V \in \mathcal{O}_N} \Big(\mathit{preimage}(A, (V, \phi, M, N)) \in \mathcal{O}_M \Big) \Big) \text{ \# preimage of open sets are open} \tag{107}$$

$$\mathit{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \iff \Big(\mathit{inverseMap}\big(\phi^{-1}, (\phi, M, N)\big) \Big) \Big(\mathit{continuous}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \Big) \wedge \Big(\mathit{continuous}\big(\phi^{-1}, (N, \mathcal{O}_N, M, \mathcal{O}_M)\big) \Big) \text{ \# structure preserving maps in topology, ability to share topological properties} \tag{108}$$

$$\mathit{isomorphicTopologicalSpace}\Big(((M, \mathcal{O}_M), (N, \mathcal{O}_N)), () \Big) \iff \exists_\phi \Big(\mathit{homeomorphism}(\phi, (M, \mathcal{O}_M, N, \mathcal{O}_N)) \Big) \tag{109}$$

1.8 Separation

$$\mathit{T0Separate}((M, \mathcal{O}), ()) \iff \Big(\mathit{topologicalSpace}((M, \mathcal{O}), ()) \Big) \wedge \Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U \in \mathcal{O}} \Big((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U) \Big) \Big) \text{ \# each pair of points has a neighborhood s.t. one is inside and the other is outside} \tag{110}$$

$$\mathit{T1Separate}((M, \mathcal{O}), ()) \iff \Big(\mathit{topologicalSpace}((M, \mathcal{O}), ()) \Big) \wedge \Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V} \Big((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V) \Big) \Big) \text{ \# every point has a neighborhood that does not contain another point} \tag{111}$$

$$\mathit{T2Separate}((M, \mathcal{O}), ()) \iff \Big(\mathit{topologicalSpace}((M, \mathcal{O}), ()) \Big) \wedge \Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V} (U \cap V = \emptyset) \Big) \text{ \# every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \tag{112}$$

$$\text{(THM)} : \mathit{T2Separate} \implies \mathit{T1Separate} \implies \mathit{T0Separate} \tag{113}$$

1.9 Compactness

$$\mathit{openCover}(C, (M, \mathcal{O})) \iff \Big(\mathit{topologicalSpace}((M, \mathcal{O}), ()) \Big) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M)$$

$$\# \text{ collection of open sets whose elements cover the entire space} \tag{114}$$

$$\mathit{finiteSubcover}\Big(\widetilde{C}, (C, M, \mathcal{O})\Big) \iff \Big(\widetilde{C} \subseteq C\Big) \wedge \Big(\mathit{openCover}\big(C, (M, \mathcal{O})\big)\Big) \wedge \Big(\mathit{openCover}\Big(\widetilde{C}, (M, \mathcal{O})\Big)\Big) \wedge \Big(\mathit{finiteSet}\Big(\widetilde{C}, ()\Big)\Big)$$
$$\# \text{ finite subset of a cover that is also a cover} \tag{115}$$

$$\mathit{compact}\big((M, \mathcal{O}), ()\big) \iff \Big(\mathit{topologicalSpace}\big((M, \mathcal{O}), ()\big)\Big) \wedge \left(\forall_{C \subseteq \mathcal{O}} \left(\mathit{openCover}\big(C, (M, \mathcal{O})\big) \implies \exists_{\widetilde{C} \subseteq C} \left(\mathit{finiteSubcover}\Big(\widetilde{C}, (C, M, \mathcal{O})\Big)\right)\right)\right)$$
$$\# \text{ every covering of the space is represented by a finite number of nhbhds} \tag{116}$$

$$\mathit{compactSubset}\big(N, (M, \mathcal{O}_d, d)\big) \iff \Big(\mathit{compact}\big((M, \mathcal{O}), ()\big)\Big) \wedge \Big(\mathit{subsetTopology}\big(\mathcal{O}|_N, (M, \mathcal{O}, N)\big)\Big) \tag{117}$$

$$\mathit{bounded}\big(N, (M, d)\big) \iff \Big(\mathit{metricSpace}\big((M, d), ()\big)\Big) \wedge (N \subseteq M) \wedge \Big(\exists_{r \in \mathbb{R}^+} \forall_{p,q \in n} \big(d(p, q) < r\big)\Big) \tag{118}$$

$$\text{(THM) HeineBorel: } \mathit{metricTopologicalSpace}\big((M, \mathcal{O}_d, d), ()\big) \implies \forall_{S \in \mathcal{P}(M)} \bigg(\Big(\mathit{closed}\big(S, (M, \mathcal{O}_d)\big) \wedge \mathit{bounded}\big(S, (M, \mathcal{O}_d)\big)\Big) \iff \mathit{compactSubset}\big(S, (M, \mathcal{O}_d)\big)\bigg)$$
$$\# \text{ when metric topologies are involved, compactness is equivalent to being closed and bounded} \tag{119}$$

1.10 Paracompactness

$$\mathit{openRefinement}\Big(\widetilde{C}, (C, M, \mathcal{O})\Big) \iff \Big(\mathit{openCover}\big(C, (M, \mathcal{O})\big)\Big) \wedge \Big(\mathit{openCover}\Big(\widetilde{C}, (M, \mathcal{O})\Big)\Big) \wedge \Big(\forall_{\widetilde{U} \in \widetilde{C}} \exists_{U \in C} \Big(\widetilde{U} \subseteq U\Big)\Big)$$
$$\# \text{ a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space} \tag{120}$$

$$\text{(THM) : } \mathit{finiteSubcover} \implies \mathit{openRefinement} \tag{121}$$

$$\mathit{locallyFinite}\big(C, (M, \mathcal{O})\big) \iff \Big(\mathit{openCover}\big(C, (M, \mathcal{O})\big)\Big) \wedge \forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U} \Big(\mathit{finiteSet}\big(\{U_c \in C | U \cap U_c \neq \emptyset\}, ()\big)\Big)$$
$$\# \text{ each point has a neighborhood that intersects with only finitely many sets in the cover} \tag{122}$$

$$\mathit{paracompact}\big((M, \mathcal{O}), ()\big) \iff \forall_C \left(\mathit{openCover}\big(C, (M, \mathcal{O})\big) \implies \exists_{\widetilde{C}} \left(\mathit{locallyFinite}\Big(\mathit{openRefinement}\Big(\widetilde{C}, (C, M, \mathcal{O})\Big), (M, \mathcal{O})\Big)\right)\right)$$
$$\# \text{ every open cover has a locally finite open refinement} \tag{123}$$

$$\text{(THM) : } \mathit{metricTopologicalSpace} \implies \mathit{paracompact} \tag{124}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{125}$$

$$partitionOfUnitySubjCover(\mathcal{F},(C,M,\mathcal{O})) \iff \Big(locallyFinite(C,(M,\mathcal{O}))\Big) \land (f \in \mathcal{F}) \land$$
$$\left(continuous\left(f,\left(M,\mathcal{O},[0,1],subsetTopology\left(\mathcal{O}|_{[0,1]},([0,1],\mathbb{R},standardTopology)\right)\right)\right)\right) \land$$
$$\Big(\exists_{U_f \in C} \forall_{p \in M} \big(f(p) \neq 0 \implies p \in U_f\big)\Big) \land$$
$$\Big(\forall_{p \in M} \exists_{U \in \mathcal{O}|p \in U} \big((f_U)_n = \{f \in \mathcal{F} | p \in M \land f(p) \neq 0\}\big)\Big) \land$$
$$\Big(locallyFinite(C,M,\mathcal{O}) \implies finiteSet\big((f_U)_n,()\big)\Big) \land$$
$$\left(\forall_{p \in M} \exists_{U \in \mathcal{O}|p \in U} \left(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1\right)\right)$$
$$\text{\# useful for defining integrals between overlapping neighborhoods} \tag{126}$$

$$T2Separate((M,\mathcal{O}),()) \implies \Big(paracompact((M,\mathcal{O}),())\Big) \iff$$
$$\forall_C \Big(openCover(C,(M,\mathcal{O})) \implies partitionOfUnitySOTCover(\mathcal{F},(C,M,\mathcal{O}))\Big) \tag{127}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{128}$$

1.11 Connectedness and path-connectedness

$$connected((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \land \Big(\neg \exists_{A,B \in \mathcal{O} \setminus \emptyset} (A \cap B \neq \emptyset \land A \cup B = M)\Big)$$
$$\text{\# if there is some covering of the space that does not intersect} \tag{129}$$

$$(\text{THM}): \neg connected\left(\left(\mathbb{R} \setminus \{0\}, subsetTopology\left(\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}},(\mathbb{R},standardTopology,\mathbb{R}\setminus\{0\})\right)\right),()\right)$$
$$\Longleftarrow \Big(A = (-\infty,0) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \land \Big(B = (0,\infty) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \land$$
$$(A \cap B = \emptyset) \land (A \cup B = \mathbb{R} \setminus \{0\}) \tag{130}$$

$$(\text{THM}): connected((M,\mathcal{O}),()) \iff \forall_{S \in \mathcal{O}} \Big(clopen\big(S,(M,\mathcal{O}) \implies (S = \emptyset \lor S = M)\big)\Big) \tag{131}$$

$$pathConnected((M,\mathcal{O}),()) \iff \Big(subsetTopology\big(\mathcal{O}_{standard}|_{[0,1]},(\mathbb{R},standardTopology,[0,1])\big)\Big) \land$$
$$\left(\forall_{p,q \in M} \exists_\gamma \left(continuous\left(\gamma,\left([0,1],\mathcal{O}_{standard}|_{[0,1]},M,\mathcal{O}\right)\right) \land \gamma(0) = p \land \gamma(1) = q\right)\right) \tag{132}$$

$$(\text{THM}): pathConnected \implies connected \tag{133}$$

1.12 Homotopic curve and the fundamental group

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \quad (134)$$

$$\begin{aligned} homotopic(\sim, (\gamma, \delta, M, \mathcal{O})) \iff (map(\gamma, ([0,1], M)) \wedge map(\delta, ([0,1], M)))\wedge \\ (\gamma(0) = \delta(0) \wedge \gamma(1) = \delta(1))\wedge \\ (\exists_H \forall_{\lambda\in[0,1]}(continuous(H, (([0,1]\times[0,1], \mathcal{O}_{standard^2}|_{[0,1]\times[0,1]}), (M, \mathcal{O})) \wedge H(0,\lambda) = \gamma(\lambda) \wedge H(1,\lambda) = \delta(\lambda)))) \\ \#\ H \text{ is a continuous deformation of one curve into another} \end{aligned} \quad (135)$$

$$homotopic(\sim) \implies equivalenceRelation(\sim) \quad (136)$$

$$loopSpace(\mathcal{L}_p, (p, M, \mathcal{O})) \iff \mathcal{L}_p = \{map(\gamma, ([0,1], M)) | continuous(\gamma) \wedge \gamma(0) = \gamma(1)\} \quad (137)$$

$$\begin{aligned} concatination(\star, (p, \gamma, \delta)) \iff (\gamma, \delta \in loopSpace(\mathcal{L}_p))\wedge \\ (\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda) = \begin{cases} \gamma(2\lambda) & 0 \leq \lambda < 0.5 \\ \delta(2\lambda - 1) & 0.5 \leq \lambda \leq 1 \end{cases})) \end{aligned} \quad (138)$$

$$\begin{aligned} group((G, \bullet), ()) \iff (map(\bullet, (G\times G, G)))\wedge \\ (\forall_{a,b\in G}(a\bullet b \in G)) \\ (\forall_{a,b,c\in G}((a\bullet b)\bullet C = a\bullet(b\bullet c))) \\ (\exists_{\boldsymbol{e}}\forall_{a\in G}(\boldsymbol{e}\bullet a = a = a\bullet \boldsymbol{e}))\wedge \\ (\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1} = \boldsymbol{e} = a^{-1}\bullet a)) \\ \#\ \text{characterizes symmetry of a set structure} \end{aligned} \quad (139)$$

$$isomorphic(\cong, (X, \odot), (Y, \ominus))) \iff \exists_f \forall_{a,b\in X}(bijection(f, (X, Y)) \wedge f(a\odot b) = f(a)\ominus f(b)) \quad (140)$$

$$\begin{aligned} fundamentalGroup((\pi_{1,p}, \bullet), (p, M, \mathcal{O})) \iff (\pi_{1,p} = \mathcal{L}_p/\sim)\wedge \\ (map(\bullet, (\pi_{1,p}\times\pi_{1,p}, \pi_{1,p})))\wedge \\ (\forall_{A,B\in\pi_{1,p}}([A]\bullet[B] = [A\star B]))\wedge \\ (group((\pi_{1,p}, \bullet), ())) \\ \#\ \text{an equivalence class of all loops induced from the homotopic equivalence relation} \end{aligned} \quad (141)$$

$$fundamentalGroup_1 \not\cong fundamentalGroup_2 \implies topologicalSpace_1 \not\cong topologicalSpace_2 \quad (142)$$

there exists no known list of topological properties that can imply homeomorphisms (143)

CONTINUE @ Lecture 6: manifolds (144)

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \quad (145)$$

1.13 Measure theory

$$\begin{aligned} sigmaAlgebra(\sigma, (M)) \iff (M \neq \emptyset) \wedge (\sigma \subseteq \mathcal{P}(M))\wedge \\ (M\in\sigma) \wedge \Big(\forall_{A\in\sigma}(M\setminus A\in\sigma)\Big)\wedge \end{aligned}$$

$$\left(\left(A \subseteq \sigma \land \neg \mathit{uncountablyInfinite}(A, ())\right) \implies \cup A \in \sigma\right)$$

behaves as measurable sets should; provides the sufficient structure for defining a measure μ (146)

$$\mathit{measurableSpace}((M, \sigma), ()) \iff \mathit{sigmaAlgebra}(\sigma, (M)) \tag{147}$$

$$\mathit{measurableSet}(A, (M, \sigma)) \iff \left(\mathit{measurableSpace}((M, \sigma), ())\right) \land (A \in \sigma) \tag{148}$$

$$\mathit{measure}(\mu, (M, \sigma)) \iff \left(\mathit{measurableSpace}((M, \sigma), ())\right) \land \left(\mathit{map}\left(\mu, \left(\sigma, \left(\overline{\mathbb{R}}\right)_0^+\right)\right)\right) \land \left(\mu(\emptyset) = 0\right) \land$$
$$\left(\left((A)_{\mathbb{N}} \subseteq \sigma \land \forall_{i \in \mathbb{N}} \forall_{j \in \mathbb{N} \setminus \{i\}} \left(A_i \cap A_j = \emptyset\right)\right) \implies \mu\left(\cup_{i \in \mathbb{N}}(A_i)\right) = \sum_{i \in \mathbb{N}} \left(\mu(A_i)\right)\right)$$

enforces meaningful concepts of measures such as precise additivity (149)

$$(\text{THM}) : \mathit{measure}(\mu, (M, \sigma)) \implies$$
$$\left(\forall_{A, B \in \sigma} \left(A \subseteq B \implies \mu(A) \leq \mu(B)\right)\right) \land$$
$$\left((A)_{\mathbb{N}} \subseteq \sigma \implies \mu\left(\cup_{i \in \mathbb{N}}(A_i)\right) \leq \sum_{i \in \mathbb{N}} \left(\mu(A_i)\right)\right) \land$$
$$\left(\left((B)_{\mathbb{N}} \subseteq \sigma \land \forall_{i \in \mathbb{N}}(B_i \subseteq B_{i+1}) \land B = \cup(B)_{\mathbb{N}}\right) \implies \lim_{n \to \infty} \left(\mu(B_n)\right) = \mu(B)\right) \land$$
$$\left(\left((C)_{\mathbb{N}} \subseteq \sigma \land \forall_{i \in \mathbb{N}}(C_{i+1} \subseteq C_i) \land C = \cap(C)_{\mathbb{N}}\right) \implies \lim_{n \to \infty} \left(\mu(C_n)\right) = \mu(C)\right)$$

immediate implications of the measurable set $A \in \sigma$ axioms and the measure μ axioms (150)

$$\mathit{measureSpace}((M, \sigma, \mu), ()) \iff \mathit{measure}(\mu, (M, \sigma)) \tag{151}$$

$$\mathit{finiteMeasure}(\mu, (M, \sigma)) \iff \left(\mathit{measure}(\mu, (M, \sigma))\right) \land$$
$$\left(\exists_{(A)_{\mathbb{N}} \subseteq \sigma} \left(\cup\left((A)_{\mathbb{N}}\right) = M \land \forall_{n \in \mathbb{N}} \left(\mu(A_n) < \infty\right)\right)\right) \tag{152}$$

$$\mathit{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) \iff \left(G = \{\sigma \subseteq \mathcal{P}(M) \mid \mathit{sigmaAlgebra}(\sigma, (M))\}\right) \land \left(\sigma(\zeta) = \cap G\right)$$

smallest σ-algebra containing the generating set ζ (153)

$$(\text{THM}) : \exists_{\zeta \subseteq M} \left(\mathit{generatedSigmaAlgebra}(\sigma(\zeta), (\zeta, M)) = \mathit{sigmaAlgebra}(\sigma, (M))\right) \tag{154}$$

$$\mathit{borelSigmaAlgebra}(\sigma(\mathcal{O}), (M, \mathcal{O})) \iff \left(\mathit{topologicalSpace}((M, \mathcal{O}), ())\right) \land$$
$$\left(\mathit{generatedSigmaAlgebra}(\sigma(\mathcal{O}), (\mathcal{O}, M))\right)$$

σ-algebra induced by a topology (155)

$$\mathit{lebesgueMeasure}(\lambda, ()) \iff \left(\mathit{borelSigmaAlgebra}\left(\sigma(\mathit{standardTopology}), \left(\mathbb{R}^d, \mathit{standardTopology}\right)\right)\right) \land$$

$$\left(measure\left(\lambda, \left(\mathbb{R}^d, \sigma(standardTopology)\right)\right)\right) \wedge$$
$$\left(\lambda\left(\times_{i=1}^{d}([a_i, b_i))\right) = \sum_{i=1}^{d}\left(\sqrt[2]{(a_i - b_i)^2}\right)\right)$$
$$\text{\# natural measure for } \mathbb{R}^d \tag{156}$$

$$measurableMap(f, (M, \sigma_M, N, \sigma_N)) \iff \left(measurableSpace((M, \sigma_M), ())\right) \wedge$$
$$\left(measurableSpace((N, \sigma_N), ())\right) \wedge \left(\forall_{B \in \sigma_N}\left(preimage(A, (B, f, M, N)) \in \sigma_M\right)\right)$$
$$\text{\# preimage of measurable sets are measurable} \tag{157}$$

$$pushForwardMeasure(f \star \lambda_M, (f, M, \sigma_M, \mu_M, N, \sigma_N)) \iff \left(measureSpace((M, \sigma_M, \mu_M), ())\right) \wedge$$
$$\left(measurableSpace((N, \sigma_N), ())\right) \wedge \left(measurableMap(f, (M, \sigma_M, N, \sigma_N))\right) \wedge$$
$$\left(\forall_{B \in N}\left(f \star \lambda_M(B) = \mu_M\left(preimage(A, (B, f, M, N))\right)\right)\right) \wedge \left(measure(f \star \lambda_M, (N, \sigma_N))\right)$$
$$\text{\# natural construction of a measure based primarily on measurable map} \tag{158}$$

$$nullSet(A, (M, \sigma, \mu)) \iff \left(measureSpace((M, \sigma, \mu), ())\right) \wedge (A \in \sigma) \wedge \left(\mu(A) = 0\right) \tag{159}$$

$$almostEverywhere(p, (M, \sigma, \mu)) \iff \left(measureSpace((M, \sigma, \mu), ())\right) \wedge \left(predicate(p, (M))\right) \wedge$$
$$\left(\exists_{A \in \sigma}\left(nullSet(A, (M, \sigma, \mu)) \implies \forall_{m \in M \setminus A}(p(m))\right)\right)$$
$$\text{\# the predicate holds true for all points except the points in the null set} \tag{160}$$

1.14 Lebesque integration

$$0 \tag{161}$$

2 Statistics

2.1 Overview

$$randomExperiment(X, (\Omega)) \iff \forall_{\omega \in \Omega}(outcome(\omega, (X))) \tag{162}$$

$$sampleSpace(\Omega, (X)) \iff \Omega = \{\omega | outcome(\omega, (X))\} \tag{163}$$

$$event(A, (\Omega)) \implies A \subseteq \Omega \text{ \# that is of interest} \tag{164}$$

$$eventOccured(A, (\omega, \Omega)) \iff \omega \in A, \Omega \wedge event(A, (\Omega)) \tag{165}$$

$$algebra(\mathcal{F}_0, (\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega)) \wedge$$
$$(\Omega \in \mathcal{F}_0) \wedge$$

$$(\forall_{A\in\mathcal{F}_0}(A^C \in \mathcal{F}_0))\wedge$$
$$(\forall_{A,B\in\mathcal{F}_0}(A\cup B \in \mathcal{F}_0))$$
$$\# \text{ but this is unable to capture some countable events} \tag{166}$$

$$\sigma\text{-}algebra(\mathcal{F},(\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega))\wedge$$
$$(\Omega \in \mathcal{F})\wedge$$
$$(\forall_{A\in\mathcal{F}}(A^C \in \mathcal{F}))\wedge$$
$$(\forall_{F\subseteq\mathcal{F}}(\neg uncountablyInfinite(F,()) \implies \cup F \in \mathcal{F})) \tag{167}$$

3 Statistical Learning Theory

3.1 Overview

$$ \tag{168}$$

$$curve - fitting/explaining \neq prediction \tag{169}$$

$$ill - definedproblem + solutionspaceconstraints \implies well - definedproblem \tag{170}$$

$$x \ \# \text{ input ; } y \ \# \text{ output} \tag{171}$$

$$S_n = \{(x_1,y_1),\ldots,(x_n,y_n)\} \ \# \text{ training set} \tag{172}$$

$$f_S(x) \sim y \ \# \text{ solution} \tag{173}$$

$$each(x,y) \in p(x,y) \ \# \text{ training data } x, y \text{ is a sample from an unknown distribution } p \tag{174}$$

$$V(f(x),y) = d(f(x),y) \ \# \text{ loss function} \tag{175}$$

$$I[f] = \int_{X\times Y} V(f(x),y)p(x,y)dxdy \ \# \text{ expected error} \tag{176}$$

$$I_n[f] = \frac{1}{n}\sum_{i=1}^{n} V(f(x_i),y_i) \ \# \text{ empirical error} \tag{177}$$

$$probabilisticConvergence(X,()) \iff \forall_{\epsilon>0} \lim_{n\to\infty} Pxn - x \leq \epsilon = 0 \tag{178}$$

$$I - Ingeneralizationerror \tag{179}$$

$$well - posed := exists, unique, stable; elseill - posed \tag{180}$$

3.2 Background maths

$$\begin{aligned}
vectorSpace(V,(+,*)) \iff & (u,v,w \in V),(c,d \in \mathbb{R} \in F)\wedge \\
& (u+v, c*u = c(u) = cu \in V)\wedge \\
& (u+v = v+u)\wedge \\
& ((u+v)+w = u+(v+w))\wedge \\
& (\exists_{\boldsymbol{0}}(u+\boldsymbol{0} = u))\wedge \\
& (\exists_{-u}(u+(-u) = \boldsymbol{0}))\wedge \\
& ((1)u = u) \\
& ((cd)u = c(du))\wedge \\
& ((c+d)u = cu+du)\wedge \ \#\ \text{linearity} \\
& (c(u+v) = cu+cv)\wedge \ \#\ \text{linearity} \\
& \#\ \text{behaves similar to vectors}
\end{aligned} \tag{181}$$

$$\begin{aligned}
innerProduct(\langle\cdot,\cdot\rangle,(V)) \iff & (u,v,w \in V),(c \in \mathbb{R} \in F)\wedge \\
& (\langle v,w\rangle = \langle w,v\rangle)\wedge \\
& (\langle cv,w\rangle = c\langle v,w\rangle)\wedge \\
& (\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle)\wedge \ \#\ \text{linearity} \\
& (\langle u,u\rangle \geq 0 \in \mathbb{R}_0^+)\wedge \ \#\ \text{metric inducing} \\
& (\langle u,u\rangle = 0 \iff u = \boldsymbol{0})
\end{aligned} \tag{182}$$

$$\begin{aligned}
innerProductNorm(||\cdot||,(V)) \iff & (v,w \in V),(r \in R)\wedge \\
& (||v|| = \sqrt{\langle v,v\rangle} \in \mathbb{R}_0^+)\wedge \\
& (||v|| = 0 \iff v = \boldsymbol{0})\wedge \\
& (||rv|| = |r|||v||)\wedge \\
& (||v+w|| \leq ||v|| + ||w||)\ \#\ \text{triangle inequality}
\end{aligned} \tag{183}$$

$$\begin{aligned}
normConvergences(v,(V,(v_n)_{n\in\mathbb{N}})) \iff & (\{v\} \cup (v_n)_{n\in\mathbb{N}} \subseteq V)\wedge \\
& (\lim_{n\to\infty} ||v - v_n|| = 0)
\end{aligned} \tag{184}$$

$$\begin{aligned}
& cauchySequence((v_n)_{n\in\mathbb{N}},(V)) \iff \\
& (\forall_{\epsilon>0}\exists_{n\in\mathbb{N}}\forall_{x,y>n}(||v_x - v_y|| < \epsilon))
\end{aligned} \tag{185}$$

$$normConvergences \implies cauchySequence\ \#\ \text{there might be holes in the space} \tag{186}$$

$$completeSpace(V,(innerProductNorm)) \iff (cauchySequence \iff normConvergences) \tag{187}$$

$$completion(R,(Q)) \iff R = QUcauchyUs = Qbar \tag{188}$$

$$\begin{aligned}
hilbertSpace(H,(+,*,\langle\cdot,\cdot\rangle)) \iff & (vectorSpace(H,(+,*)))\wedge \\
& (innerProduct(\langle\cdot,\cdot\rangle,(H)))\wedge \\
& completeSpace(H,(innerProductNorm))
\end{aligned} \tag{189}$$

$$separable(H,()) \iff \exists_{S\subseteq V}(countable(S,()) \wedge Sbar = V)\ \#\ \text{has a countable basis} \tag{190}$$

$$hilbertSpace \wedge seperable \iff \exists countableortho(gonal)normalbasisforspace, allnorm = 1, IP = 0 \tag{191}$$

$$x = \sum < x, v > v \text{ \# countable projection times v} \tag{192}$$

$$0000000000 \tag{193}$$

$$linearOperator(L, (V)) \iff (u, v \in V), (c, d \in \mathbb{R}) \wedge (L(cu + dv) = cL(u) + dL(v)) \tag{194}$$

$$adjoint(L^\dagger, (L, V)) \iff (\forall_{u,v \in V} < L(u), v >=< u, L^\dagger(v) >_\dagger) \tag{195}$$

$$selfAdjoint(L, ()) \iff L = L^\dagger \tag{196}$$

$$eigenvector(V) \iff Lv = kv \tag{197}$$

$$30mins \tag{198}$$

4 Machine Learning

4.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X, Y) \text{ \# dataset} \tag{199}$$

$$\textbf{learned parameters} = \textbf{parameters to be fixed by training with the dataset} \tag{200}$$

$$\textbf{hyperparameters} = \textbf{parameters that depends on a dataset} \tag{201}$$

$$\textbf{validation} = \textbf{partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition} \text{ \# useful for fixing hyperparameters} \tag{202}$$

$$\textbf{cross-validation} = \textbf{average accuracy of validation for different choices of testing partition} \tag{203}$$

$$\textbf{L1} = \textbf{scales linearly ; } \textbf{L2} = \textbf{scales quadratically} \tag{204}$$

$$d = \textbf{distance} = \textbf{quantifies the the similarity between data points} \tag{205}$$

$$d_{L1}(A, B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance} \tag{206}$$

$$d_{L2}(A, B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \tag{207}$$

$$\textbf{kNN classifier} = \textbf{classifier based on } k \textbf{ nearest data points} \tag{208}$$

$$s = \textbf{class score} = \textbf{quantifies bias towards a particular class} \tag{209}$$

$$s_{linear} = f_{c\times 1}(x_{n\times 1}, W_{c\times n}, b_{c\times 1}) = W_{c\times n}x_{n\times 1} + b_{c\times 1} \text{ \# linear score function} \tag{210}$$

$$l = \textbf{loss} = \textbf{quantifies the errors by the learned parameters} \tag{211}$$

$$l = \frac{1}{|c_i|}\sum_{c_i} l_i \text{ \# average loss for all classes} \tag{212}$$

$$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \text{ \# SVM hinge class loss function:}$$
$$\text{\# ignores incorrect classes with lower scores including a non-zero margin} \tag{213}$$

$$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$
$$\text{\# lower scores correspond to lower exponentiated-normalized probabilities} \tag{214}$$

$$R = \textbf{regularization} = \textbf{optimizes the choice of learned parameters to minimize test error} \tag{215}$$

$$\lambda \text{ \# regularization strength hyperparameter} \tag{216}$$

$$R_{L1}(W) = \sum_{W_i} |W_i| \text{ \# L1 regularization} \tag{217}$$

$$R_{L2}(W) = \sum_{W_i} {W_i}^2 \text{ \# L2 regularization} \tag{218}$$

$$L' = L + \lambda R(W) \text{ \# weight regularization} \tag{219}$$

$$\nabla_W L = \overrightarrow{\frac{\partial}{\partial W_i}} L = \textbf{loss gradient w.r.t. weights} \tag{220}$$

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I}\frac{\partial L_E}{\partial L_L} \text{ \# loss gradient w.r.t. input weight in terms of external and local gradients} \tag{221}$$

$$s = \textbf{forward API} \; ; \; \frac{\partial L_L}{\partial W_I} = \textbf{backward API} \tag{222}$$

$$W_{t+1} = W_t - \nabla_{W_t} L \text{ \# weight update loss minimization} \tag{223}$$

$$\textbf{TODO:Research on Activation functions, Weight Initialization, Batch Normalization} \tag{224}$$

$$review5meanvardiscussion/hyperparameteroptimization/babysittinglearning \tag{225}$$

TODO loss L or l ??