Next-Next-Gen Notes
Object-Oriented Maths

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Format: $characteristic\big((subjects),(dependencies)\big) \iff \big(conditions(dependencies)\big) \wedge \big(conditions(subjects)\big)$
Note: All weaker objects automatically induces notions inherited from stronger objects.
TODO assign free variables as parameters
TODO define || abs cross-product and other missing refs
TODO distinguish new condition vs implied proposition - separate propositions into new line thms
TODO silent link expressions! - e.g. $backslashsilentPLPL_X$

1 Logic and Set Theory

1.1 Logical Truths and Operators

$$truth[t][] := t = \begin{cases} T \\ F \end{cases} \tag{1}$$

$$statement[s][] := correctSyntaxSemantics[s][] \tag{2}$$

$$proposition[s,t][] := \big(statement[s][]\big), \big(truth[t][]\big). \tag{3}$$

$$operatorOR[\vee][x,y] := \big(truth[x][]\big), \big(truth[y][]\big), \left(truth[x \vee y][] = \begin{cases} F & x=F, y=F \\ T & x=F, y=T \\ T & x=T, y=F \\ T & x=T, y=T \end{cases} \right). \tag{4}$$

$$operatorAND[\wedge][x,y] := \big(truth[x][]\big), \big(truth[y][]\big), \left(truth[x \wedge y][] = \begin{cases} F & x=F, y=F \\ F & x=F, y=T \\ F & x=T, y=F \\ T & x=T, y=T \end{cases} \right). \tag{5}$$

$$operatorNOT[\neg][x] := \big(truth[x][]\big), \left(truth[\neg x][] = \begin{cases} T & x=F \\ F & x=T \end{cases} \right). \tag{6}$$

$$\begin{aligned} booleanAlgebra[\{T,F\},\wedge,\vee,\neg][] := {}& {}^{POS-LCom}\big((x \wedge y = y \wedge x),(x \vee y = y \vee x)\big) \ \#\ \text{Commutative}, \\ & {}^{POS-LDis}\Big(\big(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\big),\big(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\big)\Big) \ \#\ \text{Distributive}, \\ & {}^{POS-LIdn}\big((x \wedge T = x),(x \vee F = x)\big) \ \#\ \text{Identity}, \\ & {}^{POS-LCmp}\big((x \wedge \neg x = F),(x \vee \neg x = T)\big) \ \#\ \text{Complement}. \end{aligned} \tag{7}$$

$$operatorXOR[\veebar][x,y] := \big(truth[x][]\big), \big(truth[y][]\big), \left(truth[x \veebar y][] = \begin{cases} F & x=F, y=F \\ T & x=F, y=T \\ T & x=T, y=F \\ F & x=T, y=T \end{cases} \right). \tag{8}$$

$$operatorIF[\Longrightarrow][x,y] := \big(truth[x][]\big), \big(truth[y][]\big), \left(truth[x \Longrightarrow y][] = (\neg x) \vee y = \begin{cases} T & x=F, y=F \\ T & x=F, y=T \\ F & x=T, y=F \\ T & x=T, y=T \end{cases} \right). \tag{9}$$

$$\begin{matrix} THM-LExp-1 \\ POS-LCmp \end{matrix}(F = x \wedge \neg x) \Longrightarrow \begin{matrix} THM-LExp-2 \\ THM-LExp-1 \end{matrix}(x), \begin{matrix} THM-LExp-3 \\ THM-LExp-1 \end{matrix}(\neg x), \begin{matrix} THM-LExp-4 \\ THM-LExp-2 \end{matrix}(x \vee y), \begin{matrix} THM-LExp-5 \\ THM-LExp-4 \\ THM-LExp-3 \end{matrix}(y). \begin{matrix} THM-LExp \\ THM-LExp-1 \\ THM-LExp-2 \\ THM-LExp-3 \\ THM-LExp-4 \\ THM-LExp-5 \end{matrix}(F \Longrightarrow y)$$

$\#$ The Principle of Explosion, anything follows from a false (F) premise (10)

$$operatorOIF[\Longleftarrow][x,y] := \big(truth[x][]\big), \big(truth[y][]\big), \left(truth[x \Longleftarrow y][] = \begin{cases} T & x=F, y=F \\ F & x=F, y=T \\ T & x=T, y=F \\ T & x=T, y=T \end{cases} \right). \tag{11}$$

$$operatorIIF[\Longleftrightarrow][x,y] := \big(truth[x][]\big), \big(truth[y][]\big), \left(truth[x \Longleftrightarrow y][] = \begin{cases} T & x=F, y=F \\ F & x=F, y=T \\ F & x=T, y=F \\ T & x=T, y=T \end{cases} \right). \tag{12}$$

1.2 Boolean Algebra Properties

$$\begin{matrix} THM-Dual-1 \\ POS-LCom \end{matrix}\Big(booleanAlgebra[\{T,F\}, \wedge, \vee, \neg][] \Longleftrightarrow \big((x \vee y = y \vee x), (x \wedge y = y \wedge x)\big) \ \# \text{ Reordered Commutative}, \Big(\big(x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\big), \big(x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\big)\Big) \ \# \text{ Reordered Distributive}, \big((x \vee F = x), (x \wedge T = x)\big) \ \# \text{ Reordered Identity}, \big((x \vee \neg x = T), (x \wedge \neg x = F)\big) \ \# \text{ Reordered Complement}. \Longleftrightarrow booleanAlgebra[\{F,T\}, \vee, \wedge, \neg][] \Big)$$

$$\begin{matrix} THM-Dual \\ THM-Dual-1 \end{matrix}\big(booleanAlgebra[\{T,F\}, \wedge, \vee, \neg][] \Longleftrightarrow booleanAlgebra[\{F,T\}, \vee, \wedge, \neg][] \big)$$

$\#$ Boolean Algebra Duality follows from the swap symmetry of $(\wedge, T)$ and $(\vee, F)$ within the axioms (13)

$$\overset{THM-LUNt-1}{}\big((x\vee y=T=x\vee z)\wedge(x\wedge y=F=x\wedge z)\big)\Longrightarrow$$
$$\overset{\substack{THM-LUNt-2\\POS-LIdn}}{}(y=y\wedge T),$$
$$\overset{\substack{THM-LUNt-3\\THM-LUNt-1}}{}\big(y\wedge T=y\wedge(x\vee z)\big),$$
$$\overset{\substack{THM-LUNt-4\\POS-LDis}}{}\big(y\wedge(x\vee z)=(y\wedge x)\vee(y\wedge z)\big),$$
$$\overset{\substack{THM-LUNt-5\\POS-LCom\\THM-LUNt-4}}{}\big((y\wedge x)\vee(y\wedge z)=(x\wedge z)\vee(y\wedge z)\big),$$
$$\overset{\substack{THM-LUNt-6\\POS-LCom\\POS-LDis}}{}\big((x\wedge z)\vee(y\wedge z)=z\wedge(x\vee y)\big),$$
$$\overset{\substack{THM-LUNt-7\\THM-LUNt-1}}{}\big(z\wedge(x\vee y)=z\wedge T\big),$$
$$\overset{\substack{THM-LUNt-8\\POS-LIdn}}{}(z\wedge T=z).$$
$$\overset{\substack{THM-LUNt\\THM-LUNt-1\\THM-LUNt-2\\THM-LUNt-3\\THM-LUNt-4\\THM-LUNt-5\\THM-LUNt-6\\THM-LUNt-7\\THM-LUNt-8}}{}\Big(\big((x\vee y=T=x\vee z)\wedge(x\wedge y=F=x\wedge z)\big)\Longrightarrow(y=z)\Big)$$

\# Uniqueness of Complements (14)

$$\overset{\substack{THM-LDom-1\\POS-LIdn}}{}\big(x\vee T=(x\vee T)\wedge T\big),$$
$$\overset{\substack{THM-LDom-2\\POS-LCmp}}{}\big((x\vee T)\wedge T=(x\vee T)\wedge(x\vee\neg x)\big),$$
$$\overset{\substack{THM-LDom-3\\POS-LDis}}{}\big((x\vee T)\wedge(x\vee\neg x)=x\vee(T\wedge\neg x)\big),$$
$$\overset{\substack{THM-LDom-4\\POS-LIdn}}{}\big(x\vee(T\wedge\neg x)=x\vee\neg x\big),$$
$$\overset{\substack{THM-LDom-5\\POS-LCmp}}{}(x\vee\neg x=T).$$
$$\overset{\substack{THM-LDom-6\\THM-LDom-1\\THM-LDom-2\\THM-LDom-3\\THM-LDom-4\\THM-LDom-5}}{}(x\vee T=T),$$
$$\overset{\substack{THM-LDom\\THM-LDom-6\\THM-Dual}}{}\big((x\vee T=T),(x\wedge F=F)\big).$$

\# Domination (15)

$$\overset{\substack{THM-LIdm-1\\POS-LIdn}}{}\big(x\vee x=(x\vee x)\wedge T\big),$$
$$\overset{\substack{THM-LIdm-2\\POS-LCmp}}{}\big((x\vee x)\wedge T=(x\vee x)\wedge(x\vee\neg x)\big),$$
$$\overset{\substack{THM-LIdm-3\\POS-LDis}}{}\big((x\vee x)\wedge(x\vee\neg x)=x\wedge(x\vee\neg x)\big),$$
$$\overset{\substack{THM-LIdm-4\\POS-LCmp}}{}\big(x\wedge(x\vee\neg x)=x\wedge T\big),$$
$$\overset{\substack{THM-LIdm-5\\POS-LIdn}}{}(x\wedge T=x),$$
$$\overset{\substack{THM-LIdm-6\\THM-LIdm-1\\THM-LIdm-2\\THM-LIdm-3\\THM-LIdm-4\\THM-LIdm-5}}{}(x\vee x=x),$$
$$\overset{\substack{THM-LIdm\\THM-LIdm-6\\THM-Dual}}{}\big((x\vee x=x),(x\wedge x=x)\big).$$

\# Idempotent (16)

$$\overset{\substack{THM-LInv-1\\POS-LIdn}}{}(\neg\neg x=\neg\neg x\vee F),$$
$$\overset{\substack{THM-LInv-2\\POS-LCmp}}{}\big(\neg\neg x\vee F=\neg\neg x\vee(x\wedge\neg x)\big),$$
$$\overset{\substack{THM-LInv-3\\POS-LDis}}{}\big(\neg\neg x\vee(x\wedge\neg x)=(\neg\neg x\vee x)\wedge(\neg\neg x\vee\neg x)\big),$$
$$\overset{\substack{THM-LInv-4\\POS-LCmp}}{}\big((\neg\neg x\vee x)\wedge(\neg\neg x\vee\neg x)=(\neg\neg x\vee x)\wedge T\big),$$
$$\overset{\substack{THM-LInv-5\\POS-LCmp}}{}\big((\neg\neg x\vee x)\wedge T=(\neg\neg x\vee x)\wedge(x\vee\neg x)\big),$$
$$\overset{\substack{THM-LInv-6\\POS-LDis}}{}\big((\neg\neg x\vee x)\wedge(x\vee\neg x)=x\vee(\neg\neg x\wedge\neg x)\big),$$

$$\overset{THM-LInv-7}{POS-LCmp}\big(x\vee(\neg\neg x\wedge\neg x)=x\vee F\big),$$
$$\overset{THM-LInv-8}{POS-LIdn}(x\vee F=x),$$
$$\overset{\substack{THM-LInv\\THM-LInv-1\\THM-LInv-2\\THM-LInv-3\\THM-LInv-4\\THM-LInv-5\\THM-LInv-6\\THM-LInv-7\\THM-LInv-8}}{}(\neg\neg x=x).$$

\# Involution (17)

$$\overset{THM-LAbs-1}{POS-LIdn}\big(x\vee(x\wedge y)=(x\wedge T)\vee(x\wedge y)\big),$$
$$\overset{THM-LAbs-2}{POS-LDis}\big((x\wedge T)\vee(x\wedge y)=x\wedge(T\vee y)\big),$$
$$\overset{THM-LAbs-3}{THM-LDom}\big(x\wedge(T\vee y)=x\wedge T\big),$$
$$\overset{THM-LAbs-4}{THM-LIdn}(x\wedge T=x),$$
$$\overset{\substack{THM-LAbs-5\\THM-LAbs-1\\THM-LAbs-2\\THM-LAbs-3\\THM-LAbs-4}}{}\big(x\vee(x\wedge y)=x\big),$$
$$\overset{\substack{THM-LAbs\\THM-LAbs-5\\THM-Dual}}{}\Big(\big(x\vee(x\wedge y)=x\big),\big(x\wedge(x\vee y)=x\big)\Big).$$

\# Absorption (18)

$$\overset{THM-LAsc-1}{}((A=x\vee(y\vee z)),(B=(x\vee y)\vee z))\Longrightarrow$$
$$\overset{THM-LAsc-2}{THM-LAsc-1}(x\wedge A=x\wedge(x\vee(y\vee z))),,$$
$$\overset{THM-LAsc-3}{THM-LAbs}(x\wedge(x\vee(y\vee z))=x),,$$
$$\overset{THM-LAsc-4}{THM-LAsc-1}(x\wedge B=x\wedge((x\vee y)\vee z)),,$$
$$\overset{THM-LAsc-5}{POS-LDis}(x\wedge((x\vee y)\vee z)=(x\wedge(x\vee y))\vee(x\wedge z)),,$$
$$\overset{THM-LAsc-6}{THM-LAbs}((x\wedge(x\vee y))\vee(x\wedge z)=x\vee(x\wedge z)),,$$
$$\overset{THM-LAsc-7}{THM-LAbs}(x\vee(x\wedge z)=x),,$$
$$\overset{THM-LAsc-8}{THM-LAbs}((x\wedge(x\vee y))\vee(x\wedge z)=x\vee(x\wedge z)),,$$
$$\overset{\substack{THM-LAsc-9\\THM-LAsc-2\\THM-LAsc-3\\THM-LAsc-4\\THM-LAsc-5\\THM-LAsc-6\\THM-LAsc-7\\THM-LAsc-8}}{}(x\wedge A=x=x\wedge B).,$$
$$\overset{THM-LAsc-10}{THM-LAsc-1}(\neg x\wedge A=\neg x\wedge(x\vee(y\vee z))),,$$
$$\overset{THM-LAsc-11}{POS-LDis}(\neg x\wedge(x\vee(y\vee z))=(\neg x\wedge x)\vee(\neg x\wedge(y+z))),,$$
$$\overset{THM-LAsc-12}{POS-LCmp}((\neg x\wedge x)\vee(\neg x\wedge(y\vee z))=F\vee(\neg x\wedge(y\vee z))),,$$
$$\overset{THM-LAsc-13}{POS-LIdn}(F\vee(\neg x\wedge(y+z))=\neg x\wedge(y\vee z)),,$$
$$\overset{THM-LAsc-14}{THM-LAsc-1}(\neg x\wedge B=\neg x\wedge((x\vee y)\vee z)),,$$
$$\overset{THM-LAsc-15}{POS-LDis}(\neg x\wedge((x\vee y)\vee z)=(\neg x\wedge(x\vee y))\vee(\neg x\wedge z)),,$$
$$\overset{THM-LAsc-16}{POS-LDis}((\neg x\wedge(x\vee y))\vee(\neg x\wedge z)=((\neg x\wedge x)\vee(\neg x\wedge y))\vee(\neg x\wedge z)),,$$
$$\overset{THM-LAsc-17}{POS-LCmp}(((\neg x\wedge x)\vee(\neg x\wedge y))\vee(\neg x\wedge z)=(F\vee(\neg x\wedge y))\vee(\neg x\wedge z)),,$$
$$\overset{THM-LAsc-18}{POS-LIdn}((F\vee(\neg x\wedge y))\vee(\neg x\wedge z)=(\neg x\wedge y)\vee(\neg x\wedge z)),,$$
$$\overset{THM-LAsc-19}{POS-LDis}((\neg x\wedge y)\vee(\neg x\wedge z)=\neg x\wedge(y\vee z)),,$$
$$\overset{\substack{THM-LAsc-20\\THM-LAsc-10\\THM-LAsc-11\\THM-LAsc-12\\THM-LAsc-13\\THM-LAsc-14\\THM-LAsc-15\\THM-LAsc-16\\THM-LAsc-17\\THM-LAsc-18\\THM-LAsc-19}}{}(\neg x\wedge A=\neg x\wedge(y\vee z)=\neg x\wedge B).,$$
$$\overset{THM-LAsc-21}{POS-LDis}(A=A\wedge T),,$$

$$\overset{THM-LAsc-22}{POS-LCmp}(A \wedge T = A \wedge (x \vee \neg x)),,$$

$$\overset{THM-LAsc-23}{POS-LDis}(A \wedge (x \vee \neg x) = (x \wedge A) \vee (\neg x \wedge A)),,$$

$$\overset{THM-LAsc-24}{THM-LAsc-9}((x \wedge A) \vee (\neg x \wedge A) = (x \wedge B) \vee (\neg x \wedge A)),,$$

$$\overset{THM-LAsc-25}{THM-LAsc-20}((x \wedge B) \vee (\neg x \wedge A) = (x \wedge B) \vee (\neg x \wedge B)),,$$

$$\overset{THM-LAsc-26}{POS-LDis}((x \wedge B) \vee (\neg x \wedge B) = B \wedge (x \vee \neg x)),,$$

$$\overset{THM-LAsc-27}{POS-LCmp}(B \wedge (x \vee \neg x) = B \wedge T),,$$

$$\overset{THM-LAsc-27}{POS-LIdn}(B \wedge T = B),,$$

$$\begin{matrix} THM-LAsc-28 \\ THM-LAsc-21 \\ THM-LAsc-22 \\ THM-LAsc-23 \\ THM-LAsc-24 \\ THM-LAsc-25 \\ THM-LAsc-26 \\ THM-LAsc-27 \end{matrix}(A = B),,$$

$$\overset{THM-LAsc-29}{\begin{matrix} THM-LAsc-28 \\ THM-LAsc-1 \end{matrix}}(x \vee (y \vee z) = (x \vee y) \vee z).,$$

$$\overset{THM-LAsc}{\begin{matrix} THM-LAsc-29 \\ THM-Dual \end{matrix}}((x \vee (y \vee z) = (x \vee y) \vee z), (x \wedge (y \wedge z) = (x \wedge y) \wedge z)).$$

\# Associative (19)

$$\overset{THM-LDMr-1}{POS-LDis}((x \vee y) \vee (\neg x \wedge \neg y) = ((x \vee y) \vee \neg x) \wedge ((x \vee y) \vee \neg y)),$$

$$\overset{THM-LDMr-2}{\begin{matrix} POS-LCom \\ THM-LAsc \end{matrix}}(((x \vee y) \vee \neg x) \wedge ((x \vee y) \vee \neg y) = ((x \vee \neg x) \vee y) \wedge ((\neg y \vee y) \vee x)),$$

$$\overset{THM-LDMr-3}{POS-LCmp}(((x \vee \neg x) \vee y) \wedge ((\neg y \vee y) \vee x) = (T \vee y) \wedge (T \vee x)),$$

$$\overset{THM-LDMr-4}{THM-LDom}((T \vee y) \wedge (T \vee x) = T \wedge T),$$

$$\overset{THM-LDMr-5}{THM-LIdm}(T \wedge T = T),$$

$$\begin{matrix} THM-LDMr-6 \\ THM-LDMr-1 \\ THM-LDMr-2 \\ THM-LDMr-3 \\ THM-LDMr-4 \\ THM-LDMr-5 \end{matrix}((x \vee y) \vee (\neg x \wedge \neg y) = T).$$

$$\overset{THM-LDMr-7}{THM-LDis}((x \vee y) \wedge (\neg x \wedge \neg y) = (x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y)),$$

$$\overset{THM-LDMr-8}{\begin{matrix} POS-LCom \\ THM-LAsc \end{matrix}}((x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y) = ((x \wedge \neg x) \wedge \neg y) \vee ((y \wedge \neg y) \wedge \neg x)),$$

$$\overset{THM-LDMr-9}{POS-LCmp}(((x \wedge \neg x) \wedge \neg y) \vee ((y \wedge \neg y) \wedge \neg x) = (F \wedge \neg y) \vee (F \wedge \neg x)),$$

$$\overset{THM-LDMr-10}{THM-LDom}((F \wedge \neg y) \vee (F \wedge \neg x) = F \vee F),$$

$$\overset{THM-LDMr-11}{THM-LIdm}(F \vee F = F),$$

$$\begin{matrix} THM-LDMr-12 \\ THM-LDMr-7 \\ THM-LDMr-8 \\ THM-LDMr-9 \\ THM-LDMr-10 \\ THM-LDMr-11 \end{matrix}((x \vee y) \wedge (\neg x \wedge \neg y) = F).$$

$$\overset{THM-LDMr-13}{\begin{matrix} THM-LDMr-6 \\ THM-LDMr-12 \\ POS-LCmp \end{matrix}}(((x \vee y) \vee (\neg x \wedge \neg y) = T = (x \vee y) \vee \neg(x \vee y)), ((x \vee y) \wedge (\neg x \wedge \neg y) = F = (x \vee y) \wedge \neg(x \vee y))),$$

$$\overset{THM-LDMr-14}{\begin{matrix} THM-LDMr-13 \\ THM-LUNt \end{matrix}}(\neg x \wedge \neg y = \neg(x \vee y)),$$

$$\overset{THM-LDMr}{\begin{matrix} THM-LDMr-14 \\ THM-Dual \end{matrix}}((\neg x \wedge \neg y = \neg(x \vee y)), (\neg x \vee \neg y = \neg(x \wedge y))).$$

\# Boolean De Morgan's Laws (20)

$000TODOIFPROPERTIES$ (21)