# Next-Next-Gen Notes

Object-Oriented Maths

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Note: Operators (op)s preserve type; Relations (rel)s return truths; todo rename default funcs

## 1 Logic and Set Theory

### 1.1 D: Logical Truths and Operators

**undefined terms**: $:=, =, (_), , , , ., $

$$truth[t][] := \left( t = \begin{cases} T \\ F \end{cases} \right) \tag{1}$$

$$operatorOR[\vee][x,y] :=_{1} \left(truth[x][]\right),_{1} \left(truth[y][]\right),_{1} \left( truth[x \vee y][] = \begin{cases} F & x=F, y=F \\ T & x=F, y=T \\ T & x=T, y=F \\ T & x=T, y=T \end{cases} \right) ._{1} \tag{2}$$

$$operatorAND[\wedge][x,y] :=_{1} \left(truth[x][]\right),_{1} \left(truth[y][]\right),_{1} \left( truth[x \wedge y][] = \begin{cases} F & x=F, y=F \\ F & x=F, y=T \\ F & x=T, y=F \\ T & x=T, y=T \end{cases} \right) ._{1} \tag{3}$$

$$operatorNOT[\neg][x] :=_{1} \left(truth[x][]\right),_{1} \left( truth[\neg x][] = \begin{cases} T & x=F \\ F & x=T \end{cases} \right) ._{1} \tag{4}$$

$$operatorXOR[\veebar][x,y] :=_{1} \left(truth[x][]\right),_{1} \left(truth[y][]\right),_{1} \left( truth[x \veebar y][] = \begin{cases} F & x=F, y=F \\ T & x=F, y=T \\ T & x=T, y=F \\ F & x=T, y=T \end{cases} \right) ._{1} \tag{5}$$

$$operatorIF[\implies][x,y] :=_{1} \left(truth[x][]\right),_{1} \left(truth[y][]\right),_{1} \left( truth[x \implies y][] = (\neg x) \vee y = \begin{cases} T & x=F, y=F \\ T & x=F, y=T \\ F & x=T, y=F \\ T & x=T, y=T \end{cases} \right) ._{1}$$

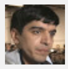
# a counterexample cannot follow from a false precedence, thus the conditional cannot be false (6)

$$operatorOIF[\Longleftarrow][x,y]:=_1\left(truth[x][]\right),_1\left(truth[y][]\right),_1\left(truth[x\Longleftarrow y][]=(\neg y)\vee x=\begin{cases}T & x=F,y=F\\ F & x=F,y=T\\ T & x=T,y=F\\ T & x=T,y=T\end{cases}\right).{}_1 \tag{7}$$

$$operatorIIF[\Longleftrightarrow][x,y]:=_1\left(truth[x][]\right),_1\left(truth[y][]\right),_1 \left(truth[x\Longleftrightarrow y][]=(x\Longrightarrow y)\wedge(y\Longrightarrow x)=\begin{cases}T & x=F,y=F\\ F & x=F,y=T\\ F & x=T,y=F\\ T & x=T,y=T\end{cases}\right).{}_1 \tag{8}$$

## 1.2 P: Boolean Algebra

$$\begin{aligned} booleanAlgebra[(\top,\bot,\otimes,\oplus,\ominus)][]:=_1 {}^{POS-LCom}\left((x\otimes y=y\otimes x),_1(x\oplus y=y\oplus x)\right) &\ \#\ \text{Commutative},_1\\ {}^{POS-LDis}\Big(\left(x\otimes(y\oplus z)=(x\otimes y)\oplus(x\otimes z)\right),_1\left(x\oplus(y\otimes z)=(x\oplus y)\otimes(x\oplus z)\right)\Big) &\ \#\ \text{Distributive},_1\\ {}^{POS-LIdn}\left((x\otimes\top=x),_1(x\oplus\bot=x)\right) &\ \#\ \text{Identity},_1\\ {}^{POS-LCmp}\Big(\left(x\otimes(\ominus x)=\bot\right),_1\left(x\oplus(\ominus x)=\top\right)\Big) &\ \#\ \text{Complement}.{}_1\end{aligned}$$

\# Note: I sometimes get too lazy to refer to $POS-LCom$.  (9)

$${}^{INS-LBAl}\left(booleanAlgebra[T,F,\wedge,\vee,\neg][]\right)$$

\# Proven by way of cases or truth tables (10)

$$\begin{aligned} {}^{THM-Dual-1}\Bigg(booleanAlgebra[(T,F,\wedge,\vee,\neg)][] \Longleftrightarrow_1 &\\ \left((x\wedge y=y\wedge x),_1(x\vee y=y\vee x)\right) &\ \#\ \text{Commutative},_1\\ \Big(\left(x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)\right),_1\left(x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)\right)\Big) &\ \#\ \text{Distributive},_1\\ \left((x\wedge T=x),_1(x\vee F=x)\right) &\ \#\ \text{Identity},_1\\ \left((x\wedge\neg x=F),_1(x\vee\neg x=T)\right) &\ \#\ \text{Complement}.{}_1 \Longleftrightarrow_2\\ \left((x\vee y=y\vee x),_2(x\wedge y=y\wedge x)\right) &\ \#\ \text{Reordered Commutative},_2\\ \Big(\left(x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)\right),_2\left(x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)\right)\Big) &\ \#\ \text{Reordered Distributive},_2\\ \left((x\vee F=x),_2(x\wedge T=x)\right) &\ \#\ \text{Reordered Identity},_2\\ \left((x\vee\neg x=T),_2(x\wedge\neg x=F)\right) &\ \#\ \text{Reordered Complement}.{}_2 \Longleftrightarrow\\ booleanAlgebra[(F,T,\vee,\wedge,\neg)][]\Bigg) &\\ {}^{THM-Dual-2}_{THM-Dual-1}\left(booleanAlgebra[(T,F,\wedge,\vee,\neg)][] \Longleftrightarrow booleanAlgebra[(F,T,\vee,\wedge,\neg)][]\right) &\\ {}^{\substack{THM-Dual\\ THM-Dual-2\\ INS-LBAl}} booleanAlgebra[(F,T,\vee,\wedge,\neg)][]() &\end{aligned}$$

\# Boolean Algebra Duality follows from the swap symmetry of $(\wedge,T)$ and $(\vee,F)$ within the axioms (11)

$$\begin{aligned} {}^{THM-LUNt-1}\left((x\vee y=T=x\vee z)\wedge(x\wedge y=F=x\wedge z)\right)\Longrightarrow_1 &\\ {}^{\substack{THM-LUNt-2\\ INS-LBAl\\ POS-LIdn}}(y=y\wedge T),_1 &\\ {}^{\substack{THM-LUNt-3\\ THM-LUNt-1}}\left(y\wedge T=y\wedge(x\vee z)\right),_1 &\end{aligned}$$

$$\underset{POS-LDis}{THM-LUNt-4}\big(y\wedge(x\vee z)=(y\wedge x)\vee(y\wedge z)\big),_1$$

$$\underset{\substack{POS-LCom\\THM-LUNt-4}}{THM-LUNt-5}\big((y\wedge x)\vee(y\wedge z)=(x\wedge z)\vee(y\wedge z)\big),_1$$

$$\underset{\substack{POS-LCom\\POS-LDis}}{THM-LUNt-6}\big((x\wedge z)\vee(y\wedge z)=z\wedge(x\vee y)\big),_1$$

$$\underset{THM-LUNt-1}{THM-LUNt-7}\big(z\wedge(x\vee y)=z\wedge T\big),_1$$

$$\underset{POS-LIdn}{THM-LUNt-8}\big(z\wedge T=z\big)._1$$

$$\underset{\substack{THM-LUNt-1\\THM-LUNt-2\\THM-LUNt-3\\THM-LUNt-4\\THM-LUNt-5\\THM-LUNt-6\\THM-LUNt-7\\THM-LUNt-8}}{THM-LUNt}\Big(\big((x\vee y=T=x\vee z)\wedge(x\wedge y=F=x\wedge z)\big)\implies(y=z)\Big)$$

# Uniqueness of Complements (12)

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$$\underset{\substack{INS-LBAl\\POS-LIdn}}{THM-LDom-1}\big(x\vee T=(x\vee T)\wedge T\big)$$

$$\underset{POS-LCmp}{THM-LDom-2}\big((x\vee T)\wedge T=(x\vee T)\wedge(x\vee\neg x)\big)$$

$$\underset{POS-LDis}{THM-LDom-3}\big((x\vee T)\wedge(x\vee\neg x)=x\vee(T\wedge\neg x)\big)$$

$$\underset{POS-LIdn}{THM-LDom-4}\big(x\vee(T\wedge\neg x)=x\vee\neg x\big)$$

$$\underset{POS-LCmp}{THM-LDom-5}\big(x\vee\neg x=T\big)$$

$$\underset{\substack{THM-LDom-1\\THM-LDom-2\\THM-LDom-3\\THM-LDom-4\\THM-LDom-5}}{THM-LDom-6}\big(x\vee T=T\big)$$

$$\underset{\substack{THM-LDom-6\\THM-Dual}}{THM-LDom}\big((x\vee T=T),(x\wedge F=F)\big)$$

# Domination (13)

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$$\underset{\substack{INS-LBAl\\POS-LIdn}}{THM-LIdm-1}\big(x\vee x=(x\vee x)\wedge T\big)$$

$$\underset{POS-LCmp}{THM-LIdm-2}\big((x\vee x)\wedge T=(x\vee x)\wedge(x\vee\neg x)\big)$$

$$\underset{POS-LDis}{THM-LIdm-3}\big((x\vee x)\wedge(x\vee\neg x)=x\wedge(x\vee\neg x)\big)$$

$$\underset{POS-LCmp}{THM-LIdm-4}\big(x\wedge(x\vee\neg x)=x\wedge T\big)$$

$$\underset{POS-LIdn}{THM-LIdm-5}\big(x\wedge T=x\big)$$

$$\underset{\substack{THM-LIdm-1\\THM-LIdm-2\\THM-LIdm-3\\THM-LIdm-4\\THM-LIdm-5}}{THM-LIdm-6}\big(x\vee x=x\big)$$

$$\underset{\substack{THM-LIdm-6\\THM-Dual}}{THM-LIdm}\big((x\vee x=x),(x\wedge x=x)\big)$$

# Idempotent (14)

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$$\underset{\substack{INS-LBAl\\POS-LIdn}}{THM-LInv-1}\big(\neg\neg x=\neg\neg x\vee F\big)$$

$$\underset{POS-LCmp}{THM-LInv-2}\big(\neg\neg x\vee F=\neg\neg x\vee(x\wedge\neg x)\big)$$

$$\underset{POS-LDis}{THM-LInv-3}\big(\neg\neg x\vee(x\wedge\neg x)=(\neg\neg x\vee x)\wedge(\neg\neg x\vee\neg x)\big)$$

$$\underset{POS-LCmp}{THM-LInv-4}\big((\neg\neg x\vee x)\wedge(\neg\neg x\vee\neg x)=(\neg\neg x\vee x)\wedge T\big)$$

$$\underset{POS-LCmp}{THM-LInv-5}\big((\neg\neg x\vee x)\wedge T=(\neg\neg x\vee x)\wedge(x\vee\neg x)\big)$$

$$\underset{POS-LDis}{THM-LInv-6}\big((\neg\neg x\vee x)\wedge(x\vee\neg x)=x\vee(\neg\neg x\wedge\neg x)\big)$$

$$\underset{POS-LCmp}{THM-LInv-7}\big(x\vee(\neg\neg x\wedge\neg x)=x\vee F\big)$$

$$\underset{POS-LIdn}{THM-LInv-8}\big(x\vee F=x\big)$$

$$\begin{array}{l}{}^{THM-LInv}_{THM-LInv-1,\ THM-LInv-2,\ THM-LInv-3,\ THM-LInv-4,\ THM-LInv-5,\ THM-LInv-6,\ THM-LInv-7,\ THM-LInv-8}(\neg\neg x = x)\end{array}$$

\# Involution (15)

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$${}^{THM-LAbs-1}_{INS-LBAl,\ POS-LIdn}\big(x \vee (x \wedge y) = (x \wedge T) \vee (x \wedge y)\big)$$

$${}^{THM-LAbs-2}_{POS-LDis}\big((x \wedge T) \vee (x \wedge y) = x \wedge (T \vee y)\big)$$

$${}^{THM-LAbs-3}_{THM-LDom}\big(x \wedge (T \vee y) = x \wedge T\big)$$

$${}^{THM-LAbs-4}_{POS-LIdn}(x \wedge T = x)$$

$${}^{THM-LAbs-5}_{THM-LAbs-1,\ THM-LAbs-2,\ THM-LAbs-3,\ THM-LAbs-4}\big(x \vee (x \wedge y) = x\big)$$

$${}^{THM-LAbs}_{THM-LAbs-5,\ THM-Dual}\Big(\big(x \vee (x \wedge y) = x\big), \big(x \wedge (x \vee y) = x\big)\Big)$$

\# Absorption (16)

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$${}^{THM-LAsc-1}\Big(\big(A = x \vee (y \vee z)\big), \big(B = (x \vee y) \vee z\big)\Big) \Longrightarrow_1$$

$${}^{THM-LAsc-2}_{THM-LAsc-1}\Big(x \wedge A = x \wedge \big(x \vee (y \vee z)\big)\Big),_1$$

$${}^{THM-LAsc-3}_{THM-LAbs}\Big(x \wedge \big(x \vee (y \vee z)\big) = x\Big),_1$$

$${}^{THM-LAsc-4}_{THM-LAsc-1}\Big(x \wedge B = x \wedge \big((x \vee y) \vee z\big)\Big),_1$$

$${}^{THM-LAsc-5}_{INS-LBAl,\ POS-LDis}\Big(x \wedge \big((x \vee y) \vee z\big) = \big(x \wedge (x \vee y)\big) \vee (x \wedge z)\Big),_1$$

$${}^{THM-LAsc-6}_{THM-LAbs}\Big(\big(x \wedge (x \vee y)\big) \vee (x \wedge z) = x \vee (x \wedge z)\Big),_1$$

$${}^{THM-LAsc-7}_{THM-LAbs}\big(x \vee (x \wedge z) = x\big),_1$$

$${}^{THM-LAsc-8}_{THM-LAbs}\Big(\big(x \wedge (x \vee y)\big) \vee (x \wedge z) = x \vee (x \wedge z)\Big),_1$$

$${}^{THM-LAsc-9}_{THM-LAsc-2,\ THM-LAsc-3,\ THM-LAsc-4,\ THM-LAsc-5,\ THM-LAsc-6,\ THM-LAsc-7,\ THM-LAsc-8}(x \wedge A = x = x \wedge B),_1$$

$${}^{THM-LAsc-10}_{THM-LAsc-1}\Big(\neg x \wedge A = \neg x \wedge \big(x \vee (y \vee z)\big)\Big),_1$$

$${}^{THM-LAsc-11}_{INS-LBAl,\ POS-LDis}\Big(\neg x \wedge \big(x \vee (y \vee z)\big) = (\neg x \wedge x) \vee \big(\neg x \wedge (y + z)\big)\Big),_1$$

$${}^{THM-LAsc-12}_{INS-LBAl,\ POS-LCmp}\Big((\neg x \wedge x) \vee \big(\neg x \wedge (y \vee z)\big) = F \vee \big(\neg x \wedge (y \vee z)\big)\Big),_1$$

$${}^{THM-LAsc-13}_{INS-LBAl,\ POS-LIdn}\Big(F \vee \big(\neg x \wedge (y + z)\big) = \neg x \wedge (y \vee z)\Big),_1$$

$${}^{THM-LAsc-14}_{THM-LAsc-1}\Big(\neg x \wedge B = \neg x \wedge \big((x \vee y) \vee z\big)\Big),_1$$

$${}^{THM-LAsc-15}_{INS-LBAl,\ POS-LDis}\Big(\neg x \wedge \big((x \vee y) \vee z\big) = \big(\neg x \wedge (x \vee y)\big) \vee (\neg x \wedge z)\Big),_1$$

$${}^{THM-LAsc-16}_{INS-LBAl,\ POS-LDis}\Big(\big(\neg x \wedge (x \vee y)\big) \vee (\neg x \wedge z) = \big((\neg x \wedge x) \vee (\neg x \wedge y)\big) \vee (\neg x \wedge z)\Big),_1$$

$${}^{THM-LAsc-17}_{INS-LBAl,\ POS-LCmp}\Big(\big((\neg x \wedge x) \vee (\neg x \wedge y)\big) \vee (\neg x \wedge z) = \big(F \vee (\neg x \wedge y)\big) \vee (\neg x \wedge z)\Big),_1$$

$${}^{THM-LAsc-18}_{INS-LBAl,\ POS-LIdn}\Big(\big(F \vee (\neg x \wedge y)\big) \vee (\neg x \wedge z) = (\neg x \wedge y) \vee (\neg x \wedge z)\Big),_1$$

$${}^{THM-LAsc-19}_{INS-LBAl,\ POS-LDis}\big((\neg x \wedge y) \vee (\neg x \wedge z) = \neg x \wedge (y \vee z)\big),_1$$

$$\substack{THM-LAsc-20\\ THM-LAsc-10\\ THM-LAsc-11\\ THM-LAsc-12\\ THM-LAsc-13\\ THM-LAsc-14\\ THM-LAsc-15\\ THM-LAsc-16\\ THM-LAsc-17\\ THM-LAsc-18\\ THM-LAsc-19}\left(\neg x \wedge A = \neg x \wedge (y \vee z) = \neg x \wedge B\right),_1$$

$$\substack{THM-LAsc-21\\ INS-LBAl\\ POS-LDis}\left(A = A \wedge T\right),_1$$

$$\substack{THM-LAsc-22\\ INS-LBAl\\ POS-LCmp}\left(A \wedge T = A \wedge (x \vee \neg x)\right),_1$$

$$\substack{THM-LAsc-23\\ INS-LBAl\\ POS-LDis}\left(A \wedge (x \vee \neg x) = (x \wedge A) \vee (\neg x \wedge A)\right),_1$$

$$\substack{THM-LAsc-24\\ THM-LAsc-9}\left((x \wedge A) \vee (\neg x \wedge A) = (x \wedge B) \vee (\neg x \wedge A)\right),_1$$

$$\substack{THM-LAsc-25\\ THM-LAsc-20}\left((x \wedge B) \vee (\neg x \wedge A) = (x \wedge B) \vee (\neg x \wedge B)\right),_1$$

$$\substack{THM-LAsc-26\\ INS-LBAl\\ POS-LDis}\left((x \wedge B) \vee (\neg x \wedge B) = B \wedge (x \vee \neg x)\right),_1$$

$$\substack{THM-LAsc-27\\ INS-LBAl\\ POS-LCmp}\left(B \wedge (x \vee \neg x) = B \wedge T\right),_1$$

$$\substack{THM-LAsc-27\\ INS-LBAl\\ POS-LIdn}\left(B \wedge T = B\right),_1$$

$$\substack{THM-LAsc-28\\ THM-LAsc-21\\ THM-LAsc-22\\ THM-LAsc-23\\ THM-LAsc-24\\ THM-LAsc-25\\ THM-LAsc-26\\ THM-LAsc-27}\left(A = B\right),_1$$

$$\substack{THM-LAsc-29\\ THM-LAsc-28\\ THM-LAsc-1}\left(x \vee (y \vee z) = (x \vee y) \vee z\right),_1$$

$$\substack{THM-LAsc\\ THM-LAsc-29\\ THM-Dual}\Big(\big(x \vee (y \vee z) = (x \vee y) \vee z\big), \big(x \wedge (y \wedge z) = (x \wedge y) \wedge z\big)\Big)$$

$$\# \text{ Associative} \tag{17}$$

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$$\substack{THM-LDMr-1\\ INS-LBAl\\ POS-LDis}\Big((x \vee y) \vee (\neg x \wedge \neg y) = \big((x \vee y) \vee \neg x\big) \wedge \big((x \vee y) \vee \neg y\big)\Big)$$

$$\substack{THM-LDMr-2\\ POS-LCom\\ THM-LAsc}\Big(\big((x \vee y) \vee \neg x\big) \wedge \big((x \vee y) \vee \neg y\big) = \big((x \vee \neg x) \vee y\big) \wedge \big((\neg y \vee y) \vee x\big)\Big)$$

$$\substack{THM-LDMr-3\\ INS-LBAl\\ POS-LCmp}\Big(\big((x \vee \neg x) \vee y\big) \wedge \big((\neg y \vee y) \vee x\big) = (T \vee y) \wedge (T \vee x)\Big)$$

$$\substack{THM-LDMr-4\\ THM-LDom}\left((T \vee y) \wedge (T \vee x) = T \wedge T\right)$$

$$\substack{THM-LDMr-5\\ THM-LIdm}\left(T \wedge T = T\right)$$

$$\substack{THM-LDMr-6\\ THM-LDMr-1\\ THM-LDMr-2\\ THM-LDMr-3\\ THM-LDMr-4\\ THM-LDMr-5}\left((x \vee y) \vee (\neg x \wedge \neg y) = T\right)$$

$$\substack{THM-LDMr-7\\ INS-LBAl\\ POS-LDis}\left((x \vee y) \wedge (\neg x \wedge \neg y) = (x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y)\right)$$

$$\substack{THM-LDMr-8\\ POS-LCom\\ THM-LAsc}\Big((x \wedge \neg x \wedge \neg y) \vee (y \wedge \neg x \wedge \neg y) = \big((x \wedge \neg x) \wedge \neg y\big) \vee \big((y \wedge \neg y) \wedge \neg x\big)\Big)$$

$$\substack{THM-LDMr-9\\ INS-LBAl\\ POS-LCmp}\Big(\big((x \wedge \neg x) \wedge \neg y\big) \vee \big((y \wedge \neg y) \wedge \neg x\big) = (F \wedge \neg y) \vee (F \wedge \neg x)\Big)$$

$$\substack{THM-LDMr-10\\ THM-LDom}\left((F \wedge \neg y) \vee (F \wedge \neg x) = F \vee F\right)$$

$$\substack{THM-LDMr-11\\ THM-LIdm}\left(F \vee F = F\right)$$

$$\substack{THM-LDMr-12\\ THM-LDMr-7\\ THM-LDMr-8\\ THM-LDMr-9\\ THM-LDMr-10\\ THM-LDMr-11}\left((x \vee y) \wedge (\neg x \wedge \neg y) = F\right)$$

$$\substack{THM-LDMr-13\\ THM-LDMr-6\\ THM-LDMr-12\\ POS-LCmp}\Big(\big((x \vee y) \vee (\neg x \wedge \neg y) = T = (x \vee y) \vee \neg(x \vee y)\big), \big((x \vee y) \wedge (\neg x \wedge \neg y) = F = (x \vee y) \wedge \neg(x \vee y)\big)\Big)$$

$$\overset{THM-LDMr-14}{\underset{THM-LUNt}{THM-LDMr-13}}(\neg x \wedge \neg y = \neg(x \vee y))$$

$$\overset{THM-LDMr}{\underset{THM-Dual}{THM-LDMr-14}}\Big((\neg x \wedge \neg y = \neg(x \vee y)), (\neg x \vee \neg y = \neg(x \wedge y))\Big)$$

# Boolean De Morgan's Laws (18)

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$$\overset{THM-CtrP-1}{operatorIF}(x \implies y = (\neg x) \vee y)$$

$$\overset{THM-CtrP-2}{\underset{THM-LInv}{POS-LCom}}\Big((\neg x) \vee y = ((\neg\neg y) \vee (\neg x))\Big)$$

$$\overset{THM-CtrP-3}{operatorIF}((\neg\neg y) \vee (\neg x) = (\neg y) \implies (\neg x))$$

$$\overset{THM-CtrP}{THM-CtrP-1\; THM-CtrP-2\; THM-CtrP-3}(x \implies y = (\neg y) \implies (\neg x))$$

# Contrapositive Law (19)

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**MISC IMPLICATION LAWS:**

$$(T \implies x = x)$$

$$(F \implies x = T)$$

$$(x \implies T = T)$$

$$(x \implies F = \neg x)$$

$$\big((x \vee y) \implies z\big) = (x \implies z) \wedge (y \implies z)$$

$$\big(x \implies (y \wedge z) = (x \implies y) \wedge (x \implies z)\big)$$

$$(x \iff y = x \implies y \wedge \neg x \implies \neg y) \qquad (20)$$

## 1.3 Predicates, Sets, Tuples

**undefined terms**: $arg\_(\_), set, \in, \{\_\},$

$$predicate[P][] := truth[P(v_{free})][] \qquad (21)$$

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$$universalQuantifier[\forall][Q,P] :=_1 (predicate[Q][]),_1 (predicate[P][]),_1$$
$$(\forall_{arg_x(Q(x))}(P(x)) = Q(y_{free}) \implies P(y_{free})).{}_1 \qquad (22)$$

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$$existentialQuantifier[\exists][Q,P] := (\exists_{arg_x(Q(x))}(P(x)) = \neg\forall_{arg_x(Q(x))}(\neg P(x))) \qquad (23)$$

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$$uniquenessQuantifier[\exists!][Q,P] := (\exists!_{arg_x(Q(x))}(P(x)) = \exists_{arg_x(Q(x))}(P(x) \wedge \neg\exists_{arg_y(Q(y))}(P(y) \wedge \neg(y = x)))) \qquad (24)$$

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$$relationSetEq[=][X,Y] := (\forall_{arg_z(z \in X \vee z \in Y)}(z \in X \wedge z \in Y)) \qquad (25)$$

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$$operatorIntersection[\bigcap][X] := (z \in \bigcap(X) \iff \forall_{x \in X}(z \in x)) \qquad (26)$$

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$$operatorUnion[\bigcup][X] := (z \in \bigcup(X) \iff \exists_{x \in X}(z \in x)) \qquad (27)$$

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$$orderedPair[<x,y>][] == <x,y> = <a,b> iff x = a and y = b == \{\{x\},\{x,y\}\} \qquad (28)$$