"The JP - Physics"

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November 5, 2017

1 Mathematics

1.1 Derivative definition and identities

$$\frac{df(x)}{dx} = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1}$$

$$\frac{d(c(f(x))}{dx} = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} = c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = c\frac{df}{dx} \tag{2}$$

$$\begin{aligned} \frac{d(f+g)}{dx} = \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) + g(x+\Delta x)) - (f(x) + g(x))}{\Delta x} = \\ \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = \frac{df}{dx} + \frac{dg}{dx} \end{aligned} \tag{3}$$

$$\frac{df(g(x))}{dx} = \left(\frac{dg(x)}{dx}\right)\left(\frac{df(g(x))}{dg(x)}\right) \tag{4}$$

1.2 Differential equation identities

For linear differential equations:

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0 \tag{5}$$

$$A(x)\frac{d^2y_1}{dx^2} + B(x)\frac{dy_1}{dx} + C(x)y_1 = A(x)\frac{d^2y_2}{dx^2} + B(x)\frac{dy_2}{dx} + C(x)y_2 = 0 \tag{6}$$

$$c_1(A(x)\frac{d^2y_1}{dx^2} + B(x)\frac{dy_1}{dx} + C(x)y_1) + c_2(A(x)\frac{d^2y_2}{dx^2} + B(x)\frac{dy_2}{dx} + C(x)y_2) = 0 \tag{7}$$

$$A(x)(\frac{d^2(c_1y_1)}{dx^2} + \frac{d^2(c_2y_2)}{dx^2})) + B(x)(\frac{d(c_1y_1)}{dx} + \frac{d(c_2y_2)}{dx})) + C(x)(c_1y_1 + c_2y_2) = 0 \tag{8}$$

$$(A(x)(\frac{d^2(c_1y_1 + c_2y_2)}{dx^2} + B(x)(\frac{d(c_1y_1 + c_2y_2)}{dx} + C(x)(c_1y_1 + c_2y_2) = 0) \implies (y = c_1y_1 + c_2y_2) \tag{9}$$

1.3 Integral and fundamental theorem of Calculus

$$\int_0^{x_2} F(x)dx = G(x_2) \tag{10}$$

$$\int_0^{x_2+\Delta x} F(x)dx = G(x_2 + \Delta x) \tag{11}$$

$$G(x_2 + \Delta x) - G(x_2) = F(x_2)\Delta x \tag{12}$$

$$F(x_2) = \frac{G(x_2 + \Delta x) - G(x_2)}{\Delta x} \tag{13}$$

$$F(x) = \frac{dG}{dx} \tag{14}$$

$$\sum_{i=1}^{n}(\Delta x_i) = \sum_{i=1}^{n}(x_i - x_{i-1}) = x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \ldots = -x_0 + x_n = x_n - x_0 \tag{15}$$

1.4 Multivariable calculus

$$\frac{\partial f(x,y)}{\partial x} = \frac{f_y(x)}{dx} \tag{16}$$

$$f(x+\Delta x, y+\Delta y) - f(x,y) = \left.\frac{\partial f}{\partial x}\right|_{x,y} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{x+\Delta x, y} \Delta y \tag{17}$$

$$\Delta f = \left.\frac{\partial f}{\partial x}\right|_{x,y} \Delta x + (\left.\frac{\partial f}{\partial y}\right|_{x,y} + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x,y} \Delta x)\Delta y \tag{18}$$

$$\Delta f = \left.\frac{\partial f}{\partial x}\right|_{x,y} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{x,y} \Delta y + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x,y} \Delta x \Delta y \tag{19}$$

1.5 Pythagoras theorem

$$(a+b)^2 = 4\left(\frac{ab}{2}\right) + c^2 = a^2 + b^2 + 2ab = c^2 + 2ab \tag{20}$$

$$c = \sqrt{a^2 + b^2} \tag{21}$$

1.6 Radians, angular and tangential velocity

$$\Delta C = 2\pi r \frac{\Delta\theta^\circ}{360^\circ} = \left(\pi \frac{\Delta\theta^\circ}{180^\circ}\right) r = \Delta\theta r \tag{22}$$

$$V_r = \frac{d\theta}{dt} = \omega \tag{23}$$

$$V_C = \frac{dC}{dt} = r\omega \tag{24}$$

1.7 Vector products, sum and product identities

$$|r| = \sqrt{r_x(\theta)^2 + r_y(\theta)^2} \tag{25}$$

$$\cos(\theta) = \frac{r_x(\theta)}{|r|} \; ; \; \sin(\theta) = \frac{r_y(\theta)}{|r|} \; ; \; \tan(\theta) = \frac{r_y(\theta)}{r_x(\theta)} \tag{26}$$

Proof for sum and difference identities: idns-proof.pdf (27)

Sum and product identities: FormulaSheet.pdf (28)

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = |A|\cos(\theta_A)|B|\cos(\theta_B) + |A|\sin(\theta_A)|B|\sin(\theta_B) = |A||B|(\cos(\theta_A)\cos(\theta_B) + \sin(\theta_A)\sin(\theta_B)) = |A||B|\cos(\theta) \text{ \# dot product or magnitude product} \tag{29}$$

$$\vec{A} \times \vec{B} = |A||B|\sin(\theta)\hat{\perp}(\theta) \text{ \# cross product or vector product} \tag{30}$$

$$\vec{A} \times \vec{B} = \vec{A} \cdot \vec{B}\tan(\theta)\hat{\perp}(\theta) \tag{31}$$

1.8 Taylor series and exponential function expansion

$$f(x) \approx f(0) \approx f(0) + x\frac{df}{dx}(0) \approx f(0) + x\frac{df}{dx}(0) + \left(\frac{x^2}{2!}\right)\frac{d^2 f}{dx^2}(0) \text{ \# factorials annihilate discrepancies} \tag{32}$$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)\left(\frac{d^n f}{dx^n}(0)\right) \text{ \# derivatives at } x = 0 \text{ extrapolates } f(x) \text{ as it deviates from } x = 0 \tag{33}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots \tag{34}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{35}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \tag{36}$$

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \dots = \cos(x) + i\sin(x) \tag{37}$$

$$e^{ix} + e^{-ix} = (\cos(x) + i\sin(x)) - (\cos(x) - i\sin(x)) = 2\cos x \tag{38}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \; ; \; \sin(x) = \frac{e^{ix} + e^{-ix}}{2i} \tag{39}$$

1.9 Complex numbers

$$z = x + iy \ ; \ z^\star = x - iy \tag{40}$$

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{41}$$

$$(z_1)(z_2) = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \tag{42}$$

$$Re(z) = \frac{z + z^\star}{2} = x \tag{43}$$

$$Im(z) = \frac{z - z^\star}{2i} = y \tag{44}$$

$$zz^\star = (x^2 - (-y^2)) + i(x(-y) + yx) = x^2 + y^2 = |z|^2 \ \# \text{ magnitude squared of z} \tag{45}$$

$$\frac{z_1}{z_2} = \left(\frac{x_2 + iy_2}{x_1 + iy_1}\right)\left(\frac{x_1 - iy_1}{x_1 - iy_1}\right) = \frac{(x_2 + iy_2)(x_1 - iy_1)}{{x_1}^2 + {y_1}^2} \tag{46}$$

$$x = r\cos(\theta) \ ; \ y = r\sin(\theta) \tag{47}$$

$$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = |z|e^{i\theta} \tag{48}$$

2 Classical Mechanics

2.1 Force, work, kinetic energy, Work-Energy theorem

$$!E|| \ F = m\frac{d^2x}{dt^2} = \frac{dp}{dt} = m\frac{dv}{dt} = ma \tag{49}$$

$$SI(F) = (kg)m/s^2 = N = \textbf{Netwons} \tag{50}$$

$$\Delta W = F\Delta x \ \# \text{ force inducing a displacement} \tag{51}$$

$$SI(W) = (kg)m^2/s^2 = Nm = J = \textbf{Joules} \tag{52}$$

$$P = \frac{\Delta W}{\Delta t} = F\frac{\Delta x}{\Delta t} = Fv \tag{53}$$

$$SI(P) = (kg)m^2/(s^2t) = W/t = \textbf{Watt} \tag{54}$$

$$m\frac{dv}{dt}v = m\frac{d\frac{v^2}{2}}{dt} = mav = ma\frac{dx}{dt} \tag{55}$$

$$\frac{m}{2}{v_2}^2 - \frac{m}{2}{v_1}^2 = ma(x_2 - x_1) \tag{56}$$

$$K = \frac{1}{2}mv^2 \text{ \# energy associated with motion} \tag{57}$$

$$SI(K) = SI(W) \tag{58}$$

$$K_2 - K_1 = F(x_2 - x_1) = W_2 - W_1 \tag{59}$$

$$\Delta K = F\Delta x = \Delta W \tag{60}$$

$$v^2 = {v_x}^2 + {v_y}^2 \tag{61}$$

$$\frac{dK}{dt} = \frac{d\left(\frac{mv^2}{2}\right)}{dt} = \frac{md({v_x}^2 + {v_y}^2)}{2dt} = \frac{m(2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt})}{2} = m\frac{dv_x}{dt}v_x + \frac{dv_y}{dt}mv_y = F_xv_X + F_yv_y = \vec{F}\cdot\vec{v} \tag{62}$$

$$dW = dt\frac{dK}{dt} = dt(F_xv_x + F_yv_y) = dt(F_x\frac{dr_x}{dt} + F_y\frac{dr_y}{dt}) = F_xdr_x + F_ydr_y = \vec{F}\cdot d\vec{r} \tag{63}$$

2.2 Potential energy, energy conservation

$$K_2 - K_1 = \int_{x_1}^{x^2} F(x)dx = G_2 - G_1 = U_1 - U_2 \tag{64}$$

$$U = -\int Fdx \text{ \# energy associated with position} \tag{65}$$

$$K_2 + U_2 = K_1 + U_1 = E \text{ \# energy conservation} \tag{66}$$

$$SI(U) = SI(Fx) = SI(K) = SI(E) \tag{67}$$

$$\frac{\partial^2 U}{\partial y\partial x} = \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} = \frac{\partial^2 U}{\partial x\partial y} \text{ \# multivariable conservative force?} \tag{68}$$

2.3 Gravity and inclined plane

$$\vec{F_g} = -\frac{gM_1M_2}{{r_{1,2}}^2}\left(\frac{\vec{r}}{|r|}\right) = -\frac{gMm}{r^2}\hat{r} \tag{69}$$

$$(ma_x\hat{x} + ma_y\hat{y}) = (mg\sin(\theta)\hat{x} + (N - mg\cos(\theta))\hat{y}) \tag{70}$$

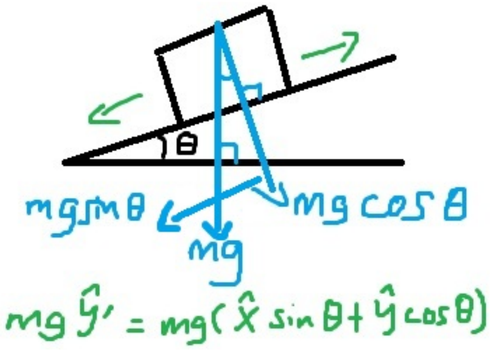

Figure 1: This mgtheta was uploaded via the project menu.

2.4 Circular motion

$$T = \frac{2\pi}{\left(\frac{\Delta\theta}{\Delta t}\right)} = \frac{2\pi}{\omega} \tag{71}$$

$$V_C = \frac{2\pi r}{T} = \omega r \tag{72}$$

$$\vec{x}(t) = |r|(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)) \tag{73}$$

$$\vec{v}(t) = \omega|r|(-\hat{x}\sin(\omega t) + \hat{y}\cos(\omega t)) \tag{74}$$

$$\vec{a}(t) = -\omega^2|r|(+\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)) \tag{75}$$

2.5 Multiple body system

$$\vec{F_{a,b}} = -\vec{F_{b,a}} \tag{76}$$

$$\frac{\vec{F_{a,b}}}{\Delta t} = \frac{-\vec{F_{b,a}}}{\Delta t} = \vec{p_{a,b}} = -\vec{p_{b,a}} \tag{77}$$

$$\frac{d\vec{p_{a,b}}}{dt} = \frac{d(-\vec{p_{b,a}})}{dt} \tag{78}$$

$$\frac{d\vec{p_{a,b}}}{dt} + \frac{d(-\vec{p_{b,a}})}{dt} = \frac{d(\vec{p_{a,b}} + \vec{p_{b,a}})}{dt} = 0 \text{ \# momentum conservation} \tag{79}$$

$$M = m_1 + m_2 \tag{80}$$

$$X = \frac{m_1x_1 + m_2x_2}{M} \text{ \# Center of mass} \tag{81}$$

$$m_1\frac{d^2\vec{x_1}}{dt^2} + m_2\frac{d^2\vec{x_2}}{dt^2} = F_{1,2} + F_{1,e} + F_{2,1} + F_{2,e} = F_{1,e} + F_{2,e} =$$
$$F_e = M\frac{d^2\vec{X}}{dt^2} \text{ \# Center of mass motion depends solely on external forces} \tag{82}$$

$$X = \frac{\int_0^L Mx\frac{dx}{L}}{M} = \frac{1}{L}\int_0^L xdx = \frac{L}{2} \tag{83}$$

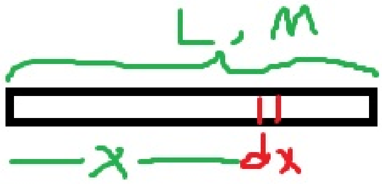

Figure 2: This continuousrod was uploaded via the project menu.

2.6 Rigid body system

$$I = \sum_i^n m_i {r_i}^2 \# \text{ moment of intertia or rotational inertia} \tag{84}$$

$$SI(I) = (kg)m^2 \tag{85}$$

$$L = I\omega \# \text{ angular momentum} \tag{86}$$

$$SI(L) = SI(I\omega) = (kg)m^2/t \tag{87}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \# \text{ angular acceleration} \tag{88}$$

$$SI(\alpha) = SI(L/t) = (kg)m^2/t^2 \tag{89}$$

$$\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0) \tag{90}$$

$$K = \frac{1}{2}\sum_i^n m_i {v_i}^2 = \frac{1}{2}\sum_i^n m_i {r_i}^2\omega^2 = \frac{1}{2}I\omega^2 \tag{91}$$

$$\Delta W = F\Delta x = F(r\Delta\theta) \tag{92}$$

$$\Delta K = \frac{1}{2}I(\omega^2 - {\omega_0}^2) = \frac{1}{2}I({\omega_0}^2 + 2\alpha(\theta_0 - \theta) - {\omega_0}^2) = I\alpha\Delta\theta \tag{93}$$

$$\frac{\Delta W}{\Delta\theta} = \frac{\Delta K}{\Delta\theta} = Fr = I\alpha = \frac{dL}{dt} = \tau \# \text{ torque} \tag{94}$$

$$SI(\tau) = SI(Fx) \tag{95}$$

$$\vec{L} = \vec{r} \times \vec{p} \# \text{ multivariable axis and speed of rotation} \tag{96}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} = \vec{r} \times \vec{F} \tag{97}$$

2.7 Multiple rigid body system

$$\tau = Fr = \sum_i (F_i - N_i) r_i = \sum_i (F_i(\cos(\theta_i) + \sin(\theta_i)) - F_i \cos(\theta_i)) r_i = \sum_i F_i \sin(\theta_i) r_i \tag{98}$$

$$\sum_i \frac{m_i \vec{r_{CM_i}}}{M} = 0 \text{ \# origin is at the center of mass} \tag{99}$$

$$\sum_i \frac{m_i \vec{r_{CM_i}}}{M \Delta t} = \sum_i \frac{m_i \vec{v_{CM_i}}}{M} = \sum_i \vec{p_{CM_i}} = 0 \tag{100}$$

$$I_{CM} = \sum_i m_i \vec{r_{CM_i}}^2 \tag{101}$$

$$\begin{aligned} I = \sum_i m_i \vec{r_i}^2 = \sum_i m_i (\vec{d} + \vec{r_{CM_i}})^2 = \sum_i m_i \vec{d}^2 + 2\vec{d} \sum_i m_i \vec{r_{CM_i}} + \sum_i m_i \vec{r_{CM_i}}^2 = \\ Md^2 + 2\vec{d} \sum_i m_i \vec{r_{CM_i}} + I_{CM} = Md^2 + 2M\vec{d} \sum_i \frac{m_i \vec{r_{CM_i}}}{M} + I_{CM} = Md^2 + I_{CM} \end{aligned} \tag{102}$$

$$\begin{aligned} K = \frac{1}{2} \sum_i^n m_i v_i = \frac{1}{2} \sum_i^n m_i (v_{CM} + v_{i-CM})^2 = \\ \frac{1}{2} \sum_i^n (m_i {v_{CM}}^2 + m_i {v_{i-CM}}^2 + 2 m_i v_{CM} v_{i-CM}) = \\ \frac{1}{2} M v_{CM} + \frac{1}{2} I_{CM} \omega^2 + v_{CM} \sum_i^n m_i v_{CM_i} = \frac{1}{2} M v_{CM} + \frac{1}{2} I_{CM} \omega^2 = K_{CM_T} + K_{CM_R} \end{aligned} \tag{103}$$

2.8 Stationary ladder

$$w - f = 0 \text{ ; } N - Mg = 0 \text{ \# equilibrium constraints} \tag{104}$$

$$wL \sin(\theta) = Mg \frac{L}{2} \sin(\frac{\pi}{2} - \theta) = Mg \frac{L}{2} \cos(\theta) \tag{105}$$

$$w = \frac{Mg}{2} \cot(\theta) = f \leq \mu_S N \leq \mu_S Mg \tag{106}$$

$$\cot(\theta) \leq 2\mu_S \implies \tan(\theta) \geq \frac{1}{2\mu_S} \tag{107}$$

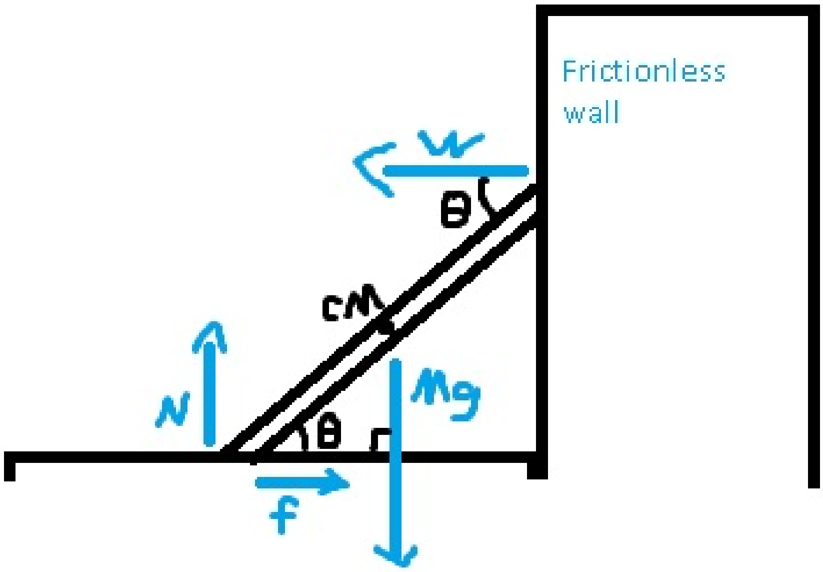

Figure 3: This ftauequilibrium was uploaded via the project menu.

3 Relativity

3.1 Lorentz transformations

$$S = (x,t)\ ;\ S' = (x',t')\ ;\ x = x' = t = t' = 0 \ \#\ \text{light pulse event from origin} \tag{108}$$

$$!\mathrm{E}||\ x' = (x-ut)\gamma\ ;\ x = (x'+ut')\gamma \ \#\ \text{inertial coordinate transform postulate} \tag{109}$$

$$!\mathrm{E}||\ x = ct\ ;\ x' = ct' \ \#\ \text{EM relativity postulate} \tag{110}$$

$$c = \frac{t}{x} = \frac{t'}{x'} \tag{111}$$

$$x'x = (xx' + xut' - x'ut - u^2tt')\gamma^2 \tag{112}$$

$$\gamma^2 = \frac{1}{(1 + u\frac{t'}{x'} - u\frac{t}{x} - \frac{u^2tt'}{xx'})} = \frac{1}{1 - u^2\frac{t}{x}\frac{t'}{x'}} = \frac{1}{1-\frac{u^2}{c^2}} \implies \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{113}$$

$$t' = \frac{1}{u}\left(\frac{x}{\gamma} - x'\right) = \frac{1}{u}\left(\frac{x}{\gamma} - \gamma(x-ut)\right) = \frac{\gamma}{u}\left(\frac{x}{\gamma^2} - x + ut\right) = \frac{\gamma}{u}(x(1-\frac{u^2}{c^2}) - x + ut) = \tag{114}$$

$$\frac{\gamma}{u}(ut - x\frac{u^2}{c^2}) = \gamma(t - x\frac{u}{c^2}) = \frac{t - x\frac{u}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}} \ \#\ \text{x', t' are linear combinations of x, t} \tag{115}$$

$$\Delta x' = x'_2 - x'_1 = (\Delta x - u\Delta t)\gamma \tag{116}$$

$$\Delta t' = (\Delta t - u\frac{\Delta x}{c^2})\gamma \tag{117}$$

$$v' = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x - u\Delta t}{\Delta t - u\frac{\Delta x}{c^2}} = \frac{v - u}{1 - u\frac{v}{c^2}} \quad (118)$$

3.2 Dependent events

$$(sgn(\Delta t') = -sgn(\Delta t)) \implies (u\frac{\Delta x}{c^2} > \Delta t) \implies (1 > \frac{u}{c} > \frac{c\Delta t}{\Delta x}) \implies (\Delta x > c\Delta t) \quad (119)$$

$$(sgn(\Delta t') = sgn(\Delta t)) \implies (\Delta x < c\Delta t) \text{ \# dependent events are in range of light signal propagation} \quad (120)$$

3.3 Distance in space-time

$$\beta = \frac{u}{c} \quad (121)$$

$$x_0{}' = ct' = \frac{ct - c\frac{x}{c}\frac{u}{c}}{\sqrt{1-\beta^2}} = \frac{x_0 - \beta x_1}{\sqrt{1-\beta^2}} \text{ \# time-like component} \quad (122)$$

$$x_1{}' = x' = \frac{x_1 - \beta x_0}{\sqrt{1-\beta^2}} \text{ \# space-like component} \quad (123)$$

$$x_0{}'^2 - x_1{}'^2 = \frac{x_0{}^2 + \beta^2 x_1{}^2 - 2x_0\beta x_1 - x_1{}^2 - \beta^2 x_0{}^2 + 2x_1\beta x_0}{1-\beta^2} = \quad (124)$$

$$\frac{x_0{}^2(1-\beta^2) - x_1{}^2(1-\beta^2)}{1-\beta^2} = x_0{}^2 - x_1{}^2 = S^2 \text{ \# space-time interval} \quad (125)$$

$$X = (x_0, \vec{r}) = (x_0, x_1, x_2, x_3) = (ct, \vec{r}) \text{ \# space-time four-vector} \quad (126)$$

$$L_a \cdot L_b = l_{a_0}l_{b_0} - l_{a_1}l_{b_1} - l_{a_2}l_{b_2} - l_{a_3}l_{b_3} \text{ \# four-vector dot product} \quad (127)$$

$$X^2 = X \cdot X = x_0{}^2 - |\vec{r}|^2 = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 = S^2 \quad (128)$$

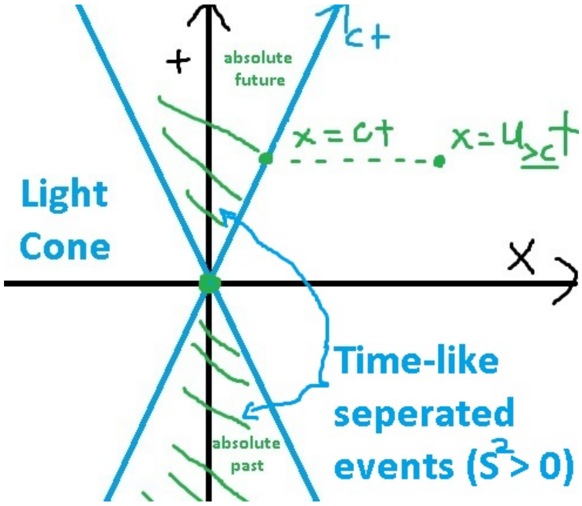

Figure 4: This lightcone was uploaded via the project menu.

3.4 Vectors in space-time

$$(\Delta S')^2 = (\Delta S)^2 = (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t)^2(1 - \frac{v^2}{c^2}) \tag{129}$$

$$\Delta S = \sqrt{(c\Delta t)^2 - (\Delta x)^2} = c\Delta t\sqrt{1 - \frac{\Delta x^2}{\Delta t^2 c^2}} = c\Delta t\sqrt{1 - \frac{v^2}{c^2}} \tag{130}$$

$$\tau \implies \Delta x = 0\ ;\ v = 0\ \#\ \text{scalar proper time in the reference frame of the object} \tag{131}$$

$$\Delta S = c\Delta t\sqrt{1 - \frac{v^2}{c^2}} = c\Delta\tau\sqrt{1 - \frac{0}{c^2}} = c\Delta\tau \tag{132}$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \tag{133}$$

$$V = \frac{dX}{d\tau} = \frac{dX}{dt}\frac{dt}{d\tau} = \frac{dX}{dt}\gamma = \left(\frac{d(ct)}{dt}, \frac{d\vec{r}}{dt}\right)\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(c, \frac{d\vec{r}}{dt}\right)\ \#\ \text{vector via scaling with an invariant} \tag{134}$$

$$P = m_0 V = m_0\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(c, \frac{d\vec{r}}{dt}\right) = \left(\frac{m_0 c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{m_o\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = (P_0, P_1, P_2, P_3) \tag{135}$$

$$cP_0 = \frac{cm_0 c}{\sqrt{1 - \frac{v^2}{c^2}}} = cm_0 c(1 - \frac{v^2}{c^2})^{\frac{-1}{2}} = \tag{136}$$

$$cm_0 c\left(1\left(\frac{1}{0!}\right) + \left(\frac{-1}{2}\right)\left(\frac{-v^2}{c^2}\right)\left(\frac{1}{1!}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2} - 1\right)\left(\frac{-v^2}{c^2}\right)^2\left(\frac{1}{2!}\right) + \ldots\right) = \tag{137}$$

$$\left(m_0 c^2 + \frac{1}{2}m_0 v^2 + \frac{3}{8}m_0\frac{v^4}{c^2} + \ldots\right) = E_0 \tag{138}$$

$$P = \left(\frac{E_0}{c}, \frac{\vec{p}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \text{ \# energy-momentum four-vector} \tag{139}$$

$$P \cdot P = \left(\frac{mc}{1-\frac{v^2}{c^2}}\right)^2 - \left(\frac{mv}{1-\frac{v^2}{c^2}}\right)^2 = \frac{m^2c^2 - m^2v^2}{1-\frac{v^2}{c^2}} = m^2c^2 \tag{140}$$

$$P \cdot P = {P_0}^2 - {P_1}^2 = \left(\frac{E}{c}\right)^2 - {P_1}^2 = m^2c^2 \tag{141}$$

$$E^2 = m^2c^4 + p^2c^2 \text{ \# energy-mass equivalence} \tag{142}$$

$$\frac{dP}{d\tau} = 0 \text{ \# invariant energy-momentum conservation} \tag{143}$$

3.5 Photon energy-momentum conservation

$$\text{!E|| } K = \left(\frac{\omega}{c}, \vec{K}\right) \text{ ; } \omega = |\vec{K}|c \text{ \# photon energy-momentum four-vector} \tag{144}$$

$$K \cdot K = \left(\frac{\omega}{c}\right)^2 - \vec{K} \cdot \vec{K} = |\vec{K}|^2 - |\vec{K}|^2 = 0 = m_K^2c^2 = 0c^2 = 0 \tag{145}$$

$$P_b = P_a + K \tag{146}$$

$$\left(\frac{m_bc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m_bv}{\sqrt{1-\frac{v^2}{c^2}}}\right) = (mc, 0) + \left(\frac{\omega}{c}, \vec{K}\right) \tag{147}$$

$$\frac{m_bc}{\sqrt{1-\frac{v^2}{c^2}}} = mc + \frac{\omega}{c} \text{ ; } \frac{m_bv}{\sqrt{1-\frac{v^2}{c^2}}} = 0 + \vec{K} \tag{148}$$

$${P_b}^2 = m_b^2c^2 = (P_a + K)^2 = P_a^2 + K^2 + 2P_ak = {m_a}^2c^2 + 0 + 2(m_ac, 0) \cdot \left(\frac{\omega}{c}, \vec{K}\right) = \tag{149}$$

$${m_a}^2c^2 + 2m_a\omega = {m_b}^2c^2 \text{ \# bypasses } v \text{ terms} \tag{150}$$

$$m_b = \sqrt{{m_a}^2 + \frac{2m_a\omega}{c^2}} \tag{151}$$

$$v = \vec{K}\frac{\sqrt{1-\frac{v^2}{c^2}}}{m_b} = \vec{K}\frac{\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{{m_a}^2 + \frac{2m_a\omega}{c^2}}} \tag{152}$$

3.6 Particle collision producing additional particles

$$P_E + P_r = \left(\frac{E}{c}, \vec{P}\right) + (mc, 0) = 4(mc, \vec{P_f}) \text{ \# case: when 2 masses + energy = 4 masses} \tag{153}$$

$$\left(\left(\frac{E}{c}, \vec{P}\right) + (mc, 0)\right)^2 = (4(mc, 0))^2 = 16m^2c^2 \tag{154}$$

$$\left(\frac{E}{c} + mc, \vec{P}\right)^2 = \left(\frac{E}{c} + mc\right)^2 - \vec{P}^2 = \frac{E^2}{c^2} + m^2c^2 + 2Em - p^2 = \tag{155}$$

$$2m^2c^2 + p^2 + 2Em - p^2 = m^2c^2 + 2Em = 16m^2c^2 \tag{156}$$

$$E = 7mc^2 \tag{157}$$

4 Harmonic Motion

4.1 Mass and spring system

$$m\frac{d^2x}{dt^2} = -kx \text{ \# harmonic motion with spring constant } k \tag{158}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \text{ \# natural frequency of a mass and spring system ?} \tag{159}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -{\omega_0}^2x \tag{160}$$

$$x = Ae^{\alpha t} \text{ \# guess solution} \tag{161}$$

$$\alpha^2Ae^{\alpha t} + {\omega_0}^2Ae^{\alpha t} = 0 \tag{162}$$

$$(A(\alpha^2 + {\omega_0}^2)e^{\alpha t} = 0) \implies \forall_t((A = 0) \vee (\alpha^2 + {\omega_0}^2 = 0)) \implies (\alpha = \pm i\omega_0) \tag{163}$$

$$x_1 = A_1e^{+i\omega_0 t} \text{ ; } x_2 = A_2e^{-i\omega_0 t} \tag{164}$$

$$x = A_1e^{i\omega t} + A_2e^{-i\omega t} \text{ ; } \omega = \sqrt{\frac{k}{m}} \tag{165}$$

$$x = x^\star = A_1e^{i\omega t} + A_2e^{-i\omega t} = {A_1}^\star e^{-i\omega t} + {A_2}^\star e^{i\omega t} \implies A_1 = {A_2}^\star \text{ \# assume } x \text{ is real} \tag{166}$$

$$x = Ae^{i\omega_0 t} + A^\star e^{-i\omega_0 t} = |A|e^{i\phi}e^{i\omega_0 t} + |A|e^{-i\phi}e^{-i\omega_0 t} = |A|e^{i(\omega_0 t+\phi)} + |A|e^{-i(\omega_0 t+\phi)} = \tag{167}$$

$$2|A|\cos(\omega t + \phi) = C\cos(\omega t + \phi) \tag{168}$$

4.2 Mass and spring system with friction

$$ \tag{169} $$

$$m\frac{d^2x}{dt^2} = -kx - \mu\frac{dx}{dt} \text{ \# harmonic motion with friction } \mu \tag{170}$$

$$m\frac{d^2x}{dt^2} + kx + \mu\frac{dx}{dt} = 0 \tag{171}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x + \left(\frac{\mu}{m}\right)\frac{dx}{dt} = \frac{d^2x}{dt^2} + {\omega_0}^2 x + \gamma\frac{dx}{dt} = 0 \tag{172}$$

$$(x = Ae^{\alpha t}) \implies (A(\alpha^2 + \alpha\gamma + {\omega_0}^2)(e^{\alpha t}) = 0) \tag{173}$$

$$\alpha^2 + \alpha\gamma + \omega_0 = 0 \tag{174}$$

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4{\omega_0}^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - {\omega_0}^2} = \alpha_\pm \tag{175}$$

$$x = Ae^{\alpha_+ t} + Be^{\alpha_- t} = Ae^{-\left(\frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - {\omega_0}^2}\right)t} + Be^{-\left(\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 - {\omega_0}^2}\right)t} \tag{176}$$

$$(\frac{\gamma}{2} > \omega_0) \implies (sgn(\alpha_+) = sgn(\alpha_-) = 1) \text{ \# over-damped relaxation} \tag{177}$$

$$v = \frac{dx}{dt} = \alpha_+ Ae^{\alpha_+ t} + \alpha_- Be^{\alpha_- t} \tag{178}$$

$$x(0) = A + B \; ; \; v(0) = \alpha_+ A + \alpha_- B \tag{179}$$

$$(\frac{\gamma}{2} < \omega_0) \implies (\alpha = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - {\omega_0}^2}) = -\frac{\gamma}{2} \pm i\sqrt{{\omega_0}^2 - \left(\frac{\gamma}{2}\right)^2}) \text{ \# damped oscillation} \tag{180}$$

$$\omega' = (\sqrt{{\omega_0}^2 - \left(\frac{\gamma}{2}\right)^2} \text{ \# damped frequency of a mass and spring system with friction} \tag{181}$$

$$x = Ae^{-\left(\frac{\gamma t}{2} - i\sqrt{{\omega_0}^2 - \left(\frac{\gamma}{2}\right)^2}\right)} + Be^{-\left(\frac{\gamma t}{2} + i\sqrt{{\omega_0}^2 - \left(\frac{\gamma}{2}\right)^2}\right)} = \tag{182}$$

$$(A+B)e^{-\frac{\gamma}{2}t}\cos\left(\left(\sqrt{{\omega_0}^2 - \left(\frac{\gamma}{2}\right)^2}\right)t + \phi\right) = (A+B)e^{-\frac{\gamma}{2}t}\cos\left(\omega' t + \phi\right) \tag{183}$$

4.3 Mass and spring system with friction and driving force

$$m\left(\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + kx\right) = F\cos(\omega t) \text{ \# harmonic motion with a driving force } F \text{ and driving frequency } \omega \tag{184}$$

$$(F\cos(\omega t) = 0) \implies \left(\frac{d^2x_c}{dt^2} + \gamma\frac{dx_c}{dt} + {\omega_0}^2 x_c = 0\right) \implies (x_c = Ae^{i\alpha t}) \text{ \# solution annihilated by operators} \tag{185}$$

$$\left(\frac{\gamma}{2} > \omega_0\right) \implies \left(\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma}{2}^2 - {\omega_0}^2}\right) \tag{186}$$

$$\left(\frac{\gamma}{2} < \omega_0\right) \implies \left(\alpha = -\frac{\gamma}{2} \pm i\sqrt{{\omega_0}^2 - \frac{\gamma}{2}^2} = -\frac{\gamma}{2} \pm i\omega'\right) \tag{187}$$

$$x_c = A_1 e^{-\frac{\gamma t}{2}} e^{i\omega' t} + A_2 e^{-\frac{\gamma t}{2}} e^{-i\omega' t} = |A|e^{i\phi_0}(e^i + e^{-i}) = 2|A|e^{-\frac{\gamma t}{2}}\cos(\omega' t + \phi_0) \tag{188}$$

$$(F\cos(\omega t) \neq 0) \implies \left(\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + {\omega_0}^2 x = \frac{F}{m}\cos(\omega t)\right) \tag{189}$$

$$\left(\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + {\omega_0}^2 z = \frac{F}{m}(\cos(\omega t) + i\sin(\omega t)) = \frac{F}{m}e^{i\omega t}\right) \implies (x = Re(z)) \tag{190}$$

$$z = z_0 e^{i\omega t} \tag{191}$$

$$(-\omega^2 + \omega\gamma + {\omega_0}^2) z_0 e^{i\omega t} = \frac{F}{m}e^{i\omega t} \tag{192}$$

$$z_0 = \frac{\left(\frac{F}{m}\right)}{-\omega^2 + \omega\gamma + {\omega_0}^2} \tag{193}$$

$$z = \frac{\left(\frac{F}{m}e^{i\omega t}\right)}{I(\omega)} = \frac{\left(\frac{F}{m}e^{i\omega t}\right)}{|I(\omega)|e^{i\phi}} = \frac{\left(\frac{F}{m}e^{i\omega t}e^{-i\phi}\right)}{|I(\omega)|} = \frac{\left(\frac{F}{m}e^{i(\omega t - \phi)}\right)}{|I(\omega)|} = \frac{\left(\frac{F}{m}(\cos(\omega t - \phi) + i\sin(\omega t - \phi))\right)}{|I(\omega)|} \tag{194}$$

$$\tan(\phi) = \frac{\omega\gamma}{{\omega_0}^2 - \omega^2} \tag{195}$$

$$x = Re(z) = \frac{\left(\frac{F}{m}\cos(\omega t - \phi)\right)}{|I(\omega)|} = \frac{\left(\frac{F}{m}\cos(\omega t - \phi)\right)}{\sqrt{({\omega_0}^2 - \omega^2)^2 + (\omega\gamma)^2}} = \tag{196}$$

$$\frac{\left(\frac{F}{m}\right)}{\sqrt{({\omega_0}^2 - \omega^2)^2 + (\omega\gamma)^2}}\cos(\omega t - \phi) = x_0\cos(\omega t - \phi) \tag{197}$$

$$\min(({\omega_0}^2 - \omega^2)^2 + (\omega\gamma)^2) \implies \max(x_0) \text{ \# resonance amplifies the amplitude } x_0 \text{ when } \omega_0 \text{ is close to } \omega \tag{198}$$

$$\left(\left(\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + {\omega_0}^2 x\right) = \frac{F}{m}\cos(\omega t) + 0\right) \implies (x_g = x + x_c) \tag{199}$$

5 Waves

5.1 Wave equation

$$\Psi(x,t) \text{ \# displacement of the medium from a resting position at point } x \text{, time } t \tag{200}$$

$$\textbf{Longitudinal} \iff \vec{\textbf{Medium}} \parallel \vec{\textbf{Signal}} \text{ ; } \textbf{Transverse} \iff \vec{\textbf{Medium}} \perp \vec{\textbf{Signal}} \tag{201}$$

$$\mu = \frac{m}{L} \text{ \# linear density} \tag{202}$$

$$F\sin(\theta + \Delta\theta) - F\sin(\theta) = \mu dx \frac{\partial^2 \Psi}{\partial t^2} \text{ \# tension } F \text{ acting on opposite sides of } \mu dx \tag{203}$$

$$(x \to 0) \implies (\sin(x) \sim x, \cos(x) \sim 1, \tan(x) \sim x) \tag{204}$$

$$F(\theta + \Delta\theta) - F(\theta) = F\Delta\theta = \mu dx \frac{\partial^2 \Psi}{\partial t^2} \tag{205}$$

$$F\frac{\Delta\theta}{dx} = \mu\frac{\partial^2 \Psi}{\partial t^2} = F\frac{\Delta\left(\frac{\Psi}{dx}\right)}{dx} = F\frac{\partial^2 \Psi}{\partial x^2} \tag{206}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 \Psi}{\partial t^2} \tag{207}$$

$$V = \sqrt{\frac{F}{\mu}} \tag{208}$$

$$k = \frac{2\pi}{\lambda} \text{ \# spatial angular frequency ; } \lambda = \frac{2\pi}{k} \text{ \# wavelength} \tag{209}$$

$$\omega = \frac{2\pi}{T} \text{ \# temporal angular frequency ; } T = \frac{2\pi}{\omega} \text{ \# time period} \tag{210}$$

$$f = \frac{1}{T} \text{ \# frequency} \tag{211}$$

$$\Psi(x,t) = A\cos(kx - \omega t) \implies \omega = kV \tag{212}$$

$$V = \sqrt{\frac{F}{\mu}} = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f \text{ \# wave velocity} \tag{213}$$

$$\Psi(x,t) = A\cos(kx - \omega t) = A\cos(kx - \omega t + 2\pi) = A\cos(k(x+\lambda) - \omega t) = A\cos(kx - \omega(t - T)) \tag{214}$$

$$V = \frac{\Delta x}{\Delta t} = \frac{x + \lambda - x}{-t + T + t} = \frac{\lambda}{T} \tag{215}$$

$$E = \frac{1}{2}(\mu dx)(A\omega(1))^2 + 0 = \frac{1}{2}\mu dx A^2\omega^2 \text{ \# Kinetic Energy is at a maximum} \tag{216}$$

$$P = \frac{E}{dx} v = \frac{1}{2} \mu A^2 \omega^2 v \tag{217}$$

$$I = \frac{P}{\textbf{Area}} \text{ \# intensity} \tag{218}$$

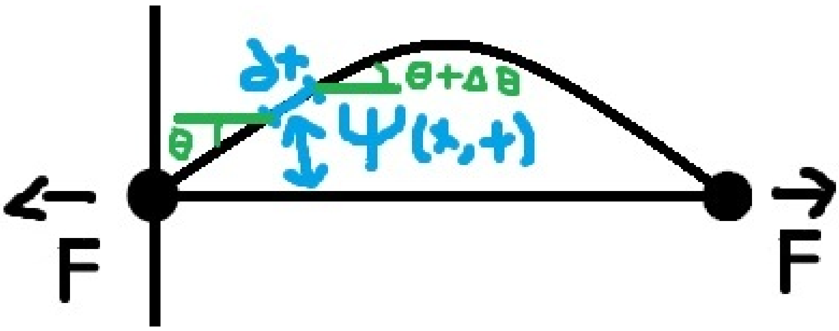

Figure 5: This wave was uploaded via the project menu.

5.2 Doppler effect

$$\lambda' = \lambda - u_s T = \lambda - \frac{u_s}{f} \tag{219}$$

$$f' = \frac{V}{\lambda'} = \frac{V}{\lambda - \frac{u_s}{f}} = f\left(\frac{1}{1 - \frac{u_s}{v}}\right) \text{ \# medium determines } V \tag{220}$$

$$f' = \frac{V'}{\lambda} = \frac{V + u_r}{\lambda} = f(1 + \frac{u_r}{V}) \text{ \# wave determines } \lambda \tag{221}$$

$$f' = \frac{V + u_r}{V + u_s} f \text{ \# general formula ?} \tag{222}$$

5.3 Wave interference and constrained waves

$$\Psi_1 = A\cos(\omega_1 t) \text{ ; } \Psi_2 = A\cos(\omega_2 t) \tag{223}$$

$$\Psi = \Psi_1 + \Psi_2 = A(\cos(\omega_1 t) + \cos(\omega_2 t)) \tag{224}$$

$$\Psi = (A) 2\cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \tag{225}$$

$$\omega_B = 2\frac{\omega_1 - \omega_2}{2} = \omega_1 - \omega_2 \text{ \# } \cos(-x) = \cos(x) \tag{226}$$

$$A = \cos\left(\frac{2\pi L_1}{\lambda}\right) + \cos\left(\frac{2\pi L_2}{\lambda}\right) \text{ \# interference at a point and time} \tag{227}$$

$$L_1 - L_2 = \frac{\lambda}{2} \implies A = \cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) + \cos\left(-\omega t + \frac{2\pi(L_1 - \frac{\lambda}{2})}{\lambda}\right) = \cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) + \cos\left(-\omega t + \frac{2\pi L_1}{\lambda} + \pi\right) = \cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) - \cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) = 0 \tag{228}$$

$$L_1 - L_2 = 0 \implies A = \cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) + \cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) = 2\cos\left(-\omega t + \frac{2\pi L_1}{\lambda}\right) = 2A_1 \tag{229}$$

$$L_1 - L_2 = n\lambda \implies \textbf{constructive interference} \tag{230}$$

$$L_1 - L_2 = \left(n + \frac{1}{2}\right)\lambda \implies \textbf{destructive interference} \tag{231}$$

$$\left.\frac{d\Psi}{dt}\right|_{0 \wedge L} = 0 \implies f = \frac{V}{\lambda} = \frac{V}{\left(\frac{L}{n}\right)} = n\frac{V}{L} \text{ \# frequencies are quantized} \tag{232}$$

6 Fluid Mechanics

6.1 Density, pressure

$$\rho = \frac{m}{V} = \frac{m}{r^3} \text{ \# density} \tag{233}$$

$$SI(\rho) = \frac{kg}{m^3} \tag{234}$$

$$P = \frac{F}{A} = \frac{F}{r^2} \text{ \# pressure} \tag{235}$$

$$SI(P) = \frac{N}{m^2} = \frac{kg}{ms^2} = \textbf{Pascals} \tag{236}$$

$$P_A = 10^3 \textbf{ Pascals} \tag{237}$$

$$P - P_A = P_g \text{ \# gauge pressure} \tag{238}$$

6.2 Fluid equilibrium

$$P_{w_2}A - P_{w_1}A = 0 \tag{239}$$

$$P_{w_2} = P_{w_1} \text{ \# points with the same depths have the same pressure} \tag{240}$$

$$P_{h_2}A - P_{h_1}A - A(h_2 - h_1)g\rho = 0 \tag{241}$$

$$P_{h_2} = P_{h_1} + \rho g(h_2 - h_1) \text{ \# points with the different depths have different pressures} \tag{242}$$

$$P = P_A + \rho g h_A \tag{243}$$

6.3 Hydraulic press

$$P_2 = P_1 = \frac{F_2}{A_2} = \frac{F_1}{A_1} \tag{244}$$

$$F_2 = F_1 \frac{A_2}{A_1} \text{ \# varying areas scales the output force} \tag{245}$$

$$W_2 = F_2 \Delta x_2 = P_2 A_2 \Delta x_2 = P_2 A_1 \Delta x_1 = P_1 A_1 \Delta x_1 = W_1 \tag{246}$$

6.4 Buoyancy

$$F_B = P_2 A - P_1 A = hA\rho g \text{ \# displaced weight} \tag{247}$$

$$\frac{V_s}{V} \rho_w g V - \rho g V = 0 \text{ \# equilibrium on water} \tag{248}$$

$$\frac{V_s}{V} = f_s = \frac{\rho}{\rho_w} \text{ \# fraction submerged in water} \tag{249}$$

6.5 Bernoulli's principle

$$V_{\to 1} = V_{2\to} \implies A_1 v_1 \Delta t = A_2 v_2 \Delta t \text{ \# incompressible fluids conserve internal system volume} \tag{250}$$

$$A_1 v_1 = A_2 v_2 \tag{251}$$

$$E_2 = A_2 \Delta x_2 \rho \frac{{v_2}^2}{2} + A_2 \Delta x_2 \rho g h_2 = A_2 \Delta x_2 \rho \left(\frac{{v_2}^2}{2} + g h_2 \right) \; ; \; E_1 = A_1 \Delta x_1 \rho \left(\frac{{v_1}^2}{2} + g h_1 \right) \tag{252}$$

$$W_2 = P_2 A_2 \Delta x_2 \; ; \; W_1 = P_1 A_1 \Delta x_1 \tag{253}$$

$$A_2 \Delta x_2 \rho \left(\frac{{v_2}^2}{2} + g h_2 \right) - A_1 \Delta x_1 \rho \left(\frac{{v_1}^2}{2} + g h_1 \right) = P_2 A_2 \Delta x_2 - P_1 A_1 \Delta x_1 \text{ \# Work-Energy theorem} \tag{254}$$

$$\frac{1}{dt} \left(A_2 \Delta x_2 \rho \left(\frac{{v_2}^2}{2} + g h_2 \right) - A_1 \Delta x_1 \rho \left(\frac{{v_1}^2}{2} + g h_1 \right) \right) = \frac{1}{dt} \left(P_2 A_2 \Delta x_2 - P_1 A_1 \Delta x_1 \right) \tag{255}$$

$$A_2 v_2 \rho \left(\frac{{v_2}^2}{2} + g h_2 \right) - A_1 v_1 \rho \left(\frac{{v_1}^2}{2} + g h_1 \right) = P_2 A_2 v_2 - P_1 A_1 v_1 \; ; \; A_1 v_1 = A_2 v_2 \tag{256}$$

$$P_2 - P_1 = \rho \left(\frac{{v_2}^2}{2} + g h_2 \right) - \rho \left(\frac{{v_1}^2}{2} + g h_1 \right) \tag{257}$$

$$P_1 + \frac{\rho}{2}{v_1}^2 + \rho g h_1 = P_2 + \frac{\rho}{2}{v_2}^2 + \rho g h_2 \tag{258}$$

$$\Delta v \neq 0 \implies \Delta P \neq 0 \implies \Delta F \neq 0 \text{ \# varying flow rates produces a net force} \tag{259}$$

7 Thermodynamics

7.1 Temperature and ideal gas law

$$T \text{ \# temperature} \tag{260}$$

$$SI(T) = K = \textbf{Kelvin} \tag{261}$$

$$!E|| \; 273.16K = \textbf{water triple point} \; ; \; 0K = \textbf{absolute lowest temperature} \tag{262}$$

$$!E|| \; k = (1.38)10^{-23}\frac{J}{K} \text{ \# Boltzmann constant} \tag{263}$$

$$N = \textbf{number of particles} \tag{264}$$

$$!E|| \; N_0 = (6.02)10^{23} = \textbf{number of particles in } \frac{1kg}{1000} \textbf{ of hydrogen} = \textbf{mole} \text{ \# Avogadro constant} \tag{265}$$

$$n = \frac{N}{N_0} = \textbf{number of moles} \tag{266}$$

$$R = N_0 k = 8.31\frac{J}{K} \text{ \# gas constant} \tag{267}$$

$$!E|| \; PV = NkT = nRT \text{ \# ideal gas law independent of complex intermolecular forces} \tag{268}$$

$$F_{particle} = \frac{\Delta p}{\Delta t} = \frac{mv - (-mv)}{2\frac{\sqrt[3]{V}}{v}} = \frac{2mv}{\frac{2L}{v}} = \frac{mv^2}{L} \tag{269}$$

$$F_{\perp} = \frac{N}{3}F_{particle} = \frac{N}{3}\left(\frac{mv^2}{L}\right) \tag{270}$$

$$P = \frac{F_{\perp}}{A} = \frac{N}{3}\left(\frac{mv^2}{L^3}\right) = \frac{N}{3}\left(\frac{mv^2}{V}\right) \tag{271}$$

$$PV = \frac{N}{3}mv^2 = NkT \tag{272}$$

$$\frac{mv^2}{2} = KE_g = \frac{3}{2}kT \text{ \# temperature as a measure of kinetic energy} \tag{273}$$

$$v \geq 0 \implies T \geq 0 \text{ \# existence of absolute zero temperature} \tag{274}$$

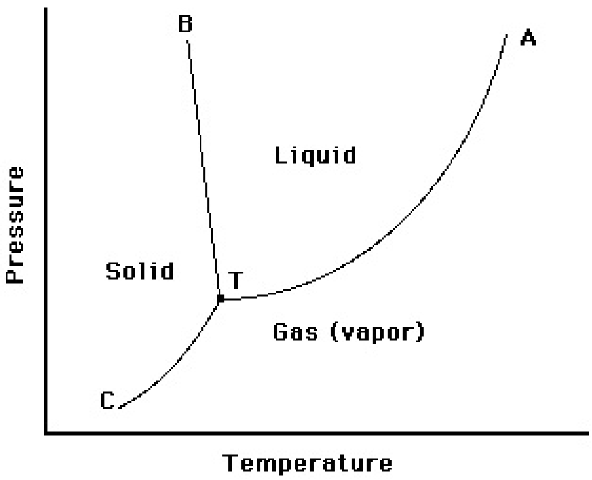

Figure 6: This triplepoint was uploaded via the project menu.

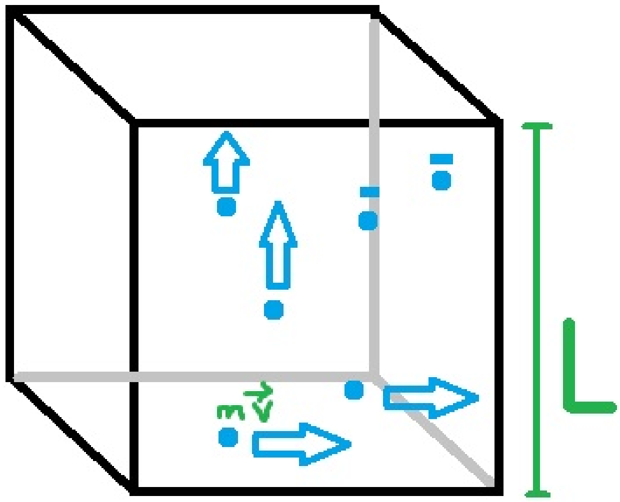

Figure 7: This microPV was uploaded via the project menu.

7.2 Heat and internal gas energy

$$U = KE_g N = \frac{3}{2}NkT = \frac{3}{2}nRT = \frac{3}{2}PV \text{ \# internal energy} \tag{275}$$

$$\Delta U = \Delta Q - \Delta W = \Delta Q - F\Delta x = \Delta Q - PA\Delta x = \Delta Q - P\Delta V \text{ \# change in heat input and work output} \tag{276}$$

7.3 Specific heat

$$nC = \frac{\Delta Q}{\Delta T} = \frac{\Delta U + P\Delta V}{\Delta T} \text{ \# controls the output } \Delta T \text{ per input } \Delta Q \tag{277}$$

$$C_V = \left.\frac{dQ}{ndT}\right|_V = \frac{dU + 0}{ndt} = \frac{3}{2}R \tag{278}$$

$$C_P = \left.\frac{dQ}{ndT}\right|_P = \left.\frac{dU + PdV}{ndT}\right|_P = C_V + \left.\frac{d(PV)}{ndT}\right|_P = C_V + \left.\frac{d(nRT)}{ndT}\right|_P = C_V + R = \frac{5}{2}R \tag{279}$$

$$\gamma = \frac{C_P}{C_V} = \frac{C_V + R}{C_V} = 1 + \frac{R}{C_V} \tag{280}$$

$$W_{cycle} = \oint PdV \neq 0 \text{ \# work of a state is undefined} \tag{281}$$

$$Q_{cycle} = \oint PdV \neq 0 \text{ \# heat of a state is undefined} \tag{282}$$

$$U_{cycle} = \oint PdV = 0 \text{ \# state variable} \tag{283}$$

7.4 Isothermal process

$$\Delta T = 0 \implies \Delta U = 0 \implies \Delta Q = \Delta W \tag{284}$$

$$W = \int_a^b P dV = nRT \int_a^b \frac{dV}{V} = nRT \ln\left(\frac{V_b}{V_a}\right) \tag{285}$$

7.5 Adiabatic process

$$\Delta Q = 0 \implies 0 = \Delta U + P\Delta V = C_V n \Delta T + \frac{nRT}{V}\Delta V \text{ \# adiabatic process} \tag{286}$$

$$\frac{C_V}{R}\frac{\Delta T}{T} + \frac{\Delta V}{V} = 0 \tag{287}$$

$$\frac{C_V}{R}\int_a^b \frac{dT}{T} + \int_a^b \frac{dV}{V} = 0 = \frac{C_V}{R}\ln\left(\frac{T_b}{T_a}\right) + \ln\left(\frac{V_b}{V_a}\right) = \ln\left(\left(\frac{T_b}{T_a}\right)^{\frac{C_V}{R}}\frac{V_b}{V_a}\right) \tag{288}$$

$$\left(\frac{T_b}{T_a}\right)^{\frac{C_V}{R}}\frac{V_b}{V_a} = 1 \tag{289}$$

$$T_b{}^{\frac{C_V}{R}} V_b = T_a{}^{\frac{C_V}{R}} V_a \tag{290}$$

$$T_b V_b{}^{\frac{R}{C_V}} = T_a V_a{}^{\frac{R}{C_V}} \tag{291}$$

$$\frac{P_b V_b}{nR} V_b{}^{\frac{R}{C_V}} = \frac{P_a V_a}{nR} V_a{}^{\frac{R}{C_V}} \tag{292}$$

$$P_b V_b{}^{\frac{1+R}{C_V}} = P_a V_a{}^{\frac{1+R}{C_V}} \tag{293}$$

$$P_b V_b{}^{\gamma} = P_a V_a{}^{\gamma} = c \tag{294}$$

$$W = \int_a^b P dV = c\int_a^b \frac{dV}{V^\gamma} = c\int_a^b V^{-\gamma} dV = c\frac{V_b{}^{1-\gamma} - V_a{}^{1-\gamma}}{1-\gamma} = \frac{P_b V_b^\gamma V_b{}^{1-\gamma} - P_a V_a^\gamma V_a{}^{1-\gamma}}{1-\gamma} = \tag{295}$$

$$\frac{P_b V_b - P_a V_a}{1-\gamma} = \frac{P_a V_a - P_b V_b}{\gamma - 1} = \frac{\Delta(PV)}{1-\gamma} \tag{296}$$

7.6 Heat engine

$$W = Q_a - Q_b \text{ \# some heat is rejected} \tag{297}$$

$$\eta = \frac{W}{Q_a} = \frac{Q_a - Q_b}{Q_a} = 1 - \frac{Q_b}{Q_a} \text{ \# engine efficiency} \tag{298}$$

$$Q_h = nRT_h \ln\left(\frac{V_B}{V_A}\right) \tag{299}$$

$$Q_c = -nRT_c \ln\left(\frac{V_D}{V_C}\right) = nRT_c \ln\left(\frac{V_C}{V_D}\right) \tag{300}$$

$$\eta = 1 - \frac{T_c}{T_h}\frac{\ln(V_C/V_D)}{\ln(V_B/V_A)} \tag{301}$$

$$V_B {T_h}^{\frac{C_V}{R}} = V_C {T_c}^{\frac{C_V}{R}} \tag{302}$$

$$V_A {T_h}^{\frac{C_V}{R}} = V_D {T_c}^{\frac{C_V}{R}} \tag{303}$$

$$\frac{V_B}{V_A} = \frac{V_C}{V_D} \implies \frac{\ln(V_C/V_D)}{\ln(V_B/V_A)} = 1 \tag{304}$$

$$\eta = 1 - \frac{Q_c}{Q_h} = 1 - \frac{T_c}{T_h} \text{ \# heat to work efficiency limit to reversible Carnot engine} \tag{305}$$

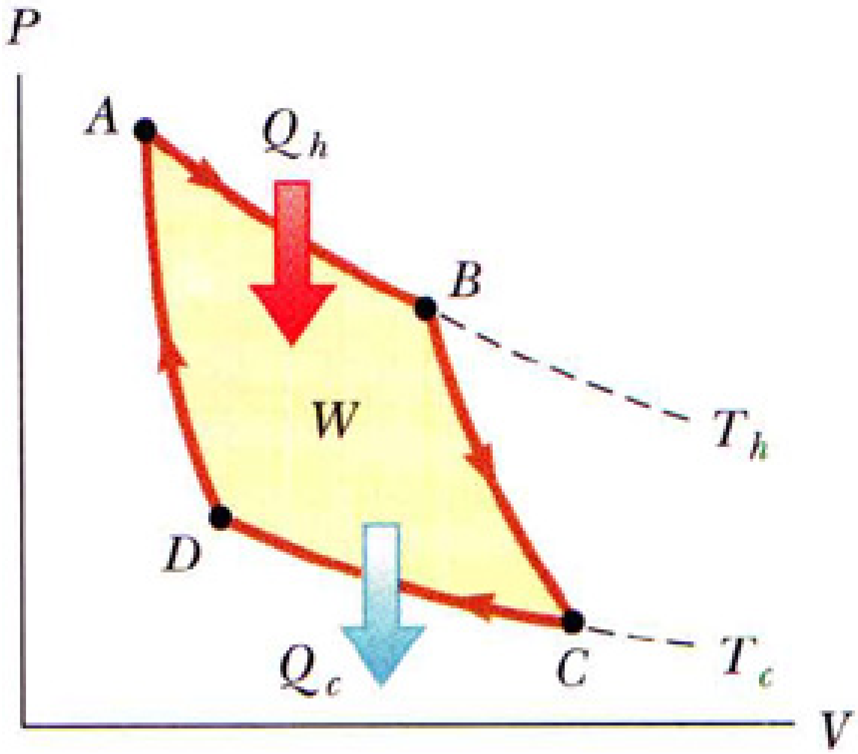

Figure 8: This carnot was uploaded via the project menu.

7.7 Theoretical heat engine efficiency limit

$$!E|| \; Q^{out}_{hot} > Q^{in}_{hot} \text{ \# heat is not allowed to flow into a hotter system} \tag{306}$$

$$Q_{hot}^{out} = Q < \frac{\eta_M}{\eta_L} = Q_{hot}^{in} \text{ \# contradictory} \tag{307}$$

$$\sim \exists_E \eta_E > \eta_L \tag{308}$$

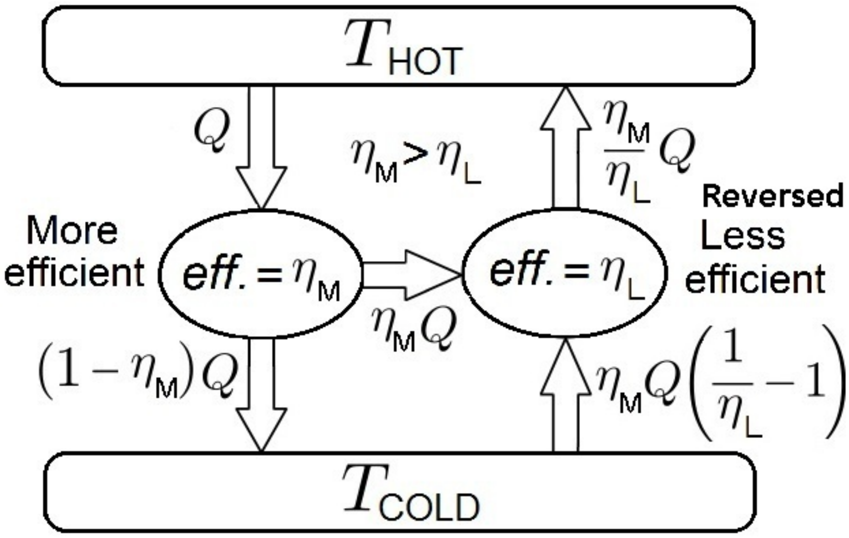

Figure 9: This carnotengine was uploaded via the project menu.

7.8 Entropy change

$$\textbf{carnot cycle} \tag{309}$$

$$\sum_i \Delta Q_i = Q_h + 0 - Q_c - 0 \tag{310}$$

$$\sum_i T_i = T_h + T_h - T_c - T_c \tag{311}$$

$$\sum_i \frac{\Delta Q_i}{T_i} = \frac{Q_h}{T_h} - \frac{Q_c}{T_c} = 0 = \sum_i \Delta S_i \text{ \# manufactured state variable entropy } S \tag{312}$$

$$\text{!E}|| \ \Delta S \geq 0 \text{ \# second law of thermodynamics} \tag{313}$$

$$\textbf{hot to cold heat transfer} \implies \Delta S = \frac{-Q}{T_h} + \frac{Q}{T_c} > 0 \implies \textbf{allowed} \tag{314}$$

$$\textbf{cold to hot heat transfer} \implies \Delta S = \frac{Q}{T_h} + \frac{-Q}{T_c} < 0 \implies \textbf{not allowed} \tag{315}$$

$$\textbf{hot and cold mixing} \implies \Delta S = nC(\int_{T_h}^{\frac{T_h+T_c}{2}} \frac{dT}{T} + \int_{T_c}^{\frac{T_h+T_c}{2}} \frac{dT}{T}) = n \ln \left(\frac{\left(\frac{T_h+T_c}{2}\right)^2}{T_h T_c} \right) \tag{316}$$

$$\left(\frac{T_h + T_c}{2} \right)^2 > T_h T_c \implies {T_h}^2 + 2T_h T_c + {T_c}^2 > 4T_h T_c \implies$$

$$T_h{}^2 - 2T_hT_c + T_c{}^2 = (T_h - T_c)^2 > 0 \implies \Delta S > 0 \implies \textbf{allowed} \tag{317}$$

$$\textbf{hot and cold separating} \implies \Delta S = nC(\int_{\frac{T_h+T_c}{2}}^{T_h} \frac{dT}{T} + \int_{\frac{T_h+T_c}{2}}^{T_c} \frac{dT}{T}) < 0 \implies \textbf{not allowed} \tag{318}$$

7.9 Entropy

$$\Delta T = 0 \implies Isothermal(S_1, S_2) \tag{319}$$

$$S_2 - S_1 = \Delta S = nR\ln\left(\frac{V_2}{V_1}\right) \tag{320}$$

$$V_2 = 2V_1 \implies \Delta S = nR\ln\left(\frac{2V_1}{V_1}\right) = nR\ln(2) = Nk\ln(2) = k\ln(2^N) \tag{321}$$

$$!E|| \; S = k\ln(\Omega) \; ; \; \Omega = \textbf{number of microstates satisfying the macrostate ?} \tag{322}$$

$$S_1 = k\ln(1) \tag{323}$$

$$S_2 - k\ln(2^N) \tag{324}$$

$$\textbf{dispersing gas} \implies S_2 - S_1 = \Delta S = k\ln(2^N) > 0 \implies \textbf{allowed} \tag{325}$$

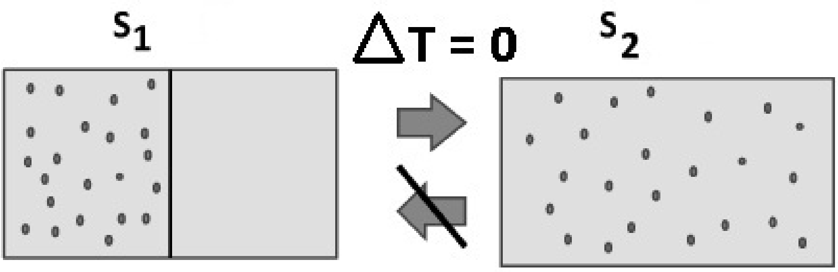

Figure 10: This disorderstate was uploaded via the project menu.

8 Electrodynamics

8.1 Electrical charge and field

$$q \; \# \text{ electrical charge} \tag{326}$$

$$SI(q) = C = \textbf{Coulomb} \tag{327}$$

$$!E|| \; q_0 = (1.60)10^{-19}C \; \# \text{ electron unit charge constant} \tag{328}$$

$$!E|| \; q_n = 0C \; \# \text{ neutron electric charge} \tag{329}$$

$$!E|| \; q_e = -q_0 \; \# \text{ electron electric charge} \tag{330}$$

$$!E|| \; q_p = q_0 \; \# \text{ proton electric charge} \tag{331}$$

$$!E|| \; q = nq_0 \; \# \text{ electrical charge is quantized} \tag{332}$$

$$!E|| \; k_e = \frac{1}{4\pi\epsilon_0} = (8.99)10^9 \frac{Nm^2}{C^2} \; \# \text{ electric force proportionality constant} \tag{333}$$

$$!E|| \; \vec{F_{2,1}} = \frac{q_2 q_1}{|\vec{r_2} - \vec{r_1}|^2} k_e \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|} = \frac{q_2 q_1}{|\vec{r_2} - \vec{r_1}|^2} \frac{1}{4\pi\epsilon_0} \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|} \; \# \text{ Coulomb's law} \tag{334}$$

$$!E|| \; \vec{F_i} = \sum_{j \neq i} \vec{F_{i,j}} \; \# \text{ classical charge superposition} \tag{335}$$

$$F_2 = \frac{q_1}{|\vec{r_2} - \vec{r_1}|^2} k_e \frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|}(q_2) = \vec{E}(\vec{r_2}) q_2 \; \# \text{ electric fields from external charges influences } q_2 \tag{336}$$

$$!E|| \; \frac{\Delta q}{\Delta t} = 0 \; \# \text{ electrical charge conservation} \tag{337}$$

8.2 Dipole

$$x(q_p) = a \wedge x(q_e) = -a \wedge y(q_p) = y(q_e) = 0 \implies E(x,0) = \hat{x} q k_e \left(\frac{+1}{(x-a)^2} + \frac{-1}{(x+a)^2} \right) = \hat{x} q k_e \left(\frac{4xa}{(x^2 - a^2)^2} \right) \; \# \text{ dipole system} \tag{338}$$

$$p = 2qa \; \# \text{ dipole moment} \tag{339}$$

$$x >> a \implies E(x,0) = \hat{x} q k_e \left(\frac{4xa}{(x^2)^2} \right) = \hat{x} \frac{2qa}{2\pi\epsilon_0} \frac{1}{x^3} = \hat{x} \frac{p}{2\pi\epsilon_0 x^3} \tag{340}$$

$$\vec{\tau} = \vec{F} \times \vec{d} = (Eq)a(\sin(\theta) - \sin(\pi + \theta)) = Eqa2\sin(\theta) = pE\sin(\theta) = \vec{p} \times \vec{E} \; \# \text{ dipole in uniform field} \tag{341}$$

$$U_\tau = -\int pE\sin(\theta) = -pE\cos(\theta) = p \cdot E \tag{342}$$

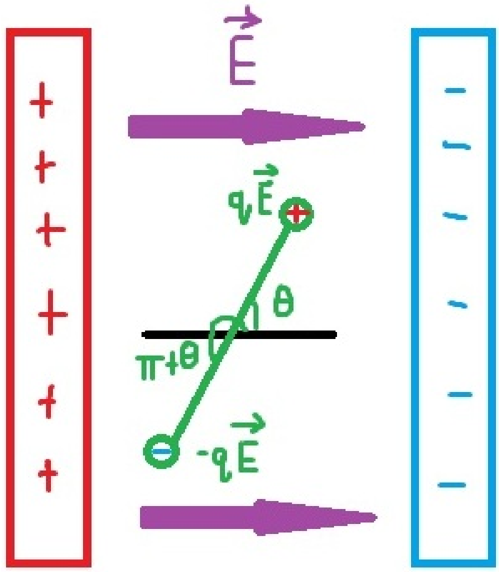

Figure 11: This dipoleuniform was uploaded via the project menu.

8.3 Infinite linear uniform charge

$$\lambda \frac{C}{m} \text{ \# linear charge density} \tag{343}$$

$$dE = dE_y = \frac{\lambda dx k_e}{(\sqrt{x^2+a^2})^2}\cos(\theta) = \frac{\lambda dx k_e}{x^2+a^2}\frac{r_x(\theta)}{|r|} = \frac{\lambda dx k_e}{x^2+a^2}\frac{a}{\sqrt{x^2+a^2}} \tag{344}$$

$$E_y = \int_{-\infty}^{\infty} dE_y = \int_{-\infty}^{\infty} \frac{\lambda dx k_e}{x^2+a^2}\frac{a}{\sqrt{x^2+a^2}} = \lambda a k_e \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^{3/2}} \tag{345}$$

$$x = a\tan(\theta) \implies \frac{dx}{d\theta} = a\sec^2\theta \implies dx = a\sec^2\theta d\theta \tag{346}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^{3/2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2(\theta)d\theta}{((a\tan(\theta))^2+a^2)^{3/2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2(\theta)d\theta}{(a^2\tan^2(\theta)+a^2)^{3/2}} = \\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2(\theta)d\theta}{(a^2(\tan^2(\theta)+1))^{3/2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2(\theta)d\theta}{(a^2\sec^2(\theta))^{3/2}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2(\theta)d\theta}{a^3\sec^3(\theta)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a^2\sec^2(\theta)} = \\ \frac{1}{a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos(\theta) = \frac{1}{a^2}\sin(\theta)|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{a^2}(1-(-1)) = \frac{2}{a^2}\end{aligned} \tag{347}$$

$$E = E_y = \lambda a k_e \frac{2}{a^2} = \frac{\lambda}{2\pi\epsilon_0 a} \textbf{field strength falls linearly } 1/a \tag{348}$$

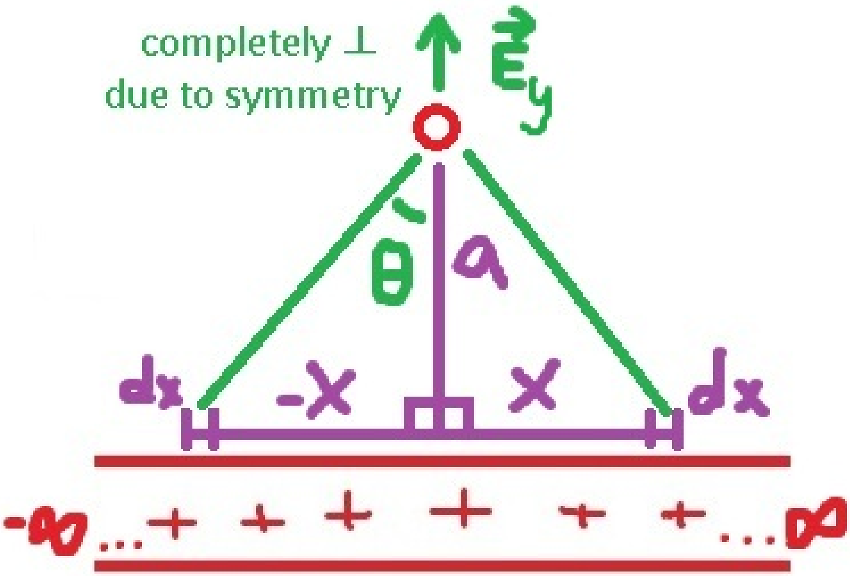

Figure 12: This linearinfinitecharge was uploaded via the project menu.

8.4 Infinite surface uniform charge

$$\sigma \frac{C}{m^2} \text{ \# surface charge density} \tag{349}$$

$$dA = (2\pi r)(dr) \tag{350}$$

$$dE = dE_z = \frac{\sigma dA k_e}{r^2 + a^2} \frac{a}{(r^2 + a^2)^{1/2}} = \frac{\sigma 2\pi r dr k_e}{r^2 + a^2} \frac{a}{(r^2 + a^2)^{1/2}} \tag{351}$$

$$E_z = \int_0^\infty dE_z = \int_0^\infty \frac{(2\pi\sigma a k_e) r dr}{(r^2 + a^2)^{3/2}} = \frac{\sigma a}{4\epsilon_0} \int_0^\infty \frac{2rdr}{(r^2 + a^2)^{3/2}} \tag{352}$$

$$u = r^2 \implies \frac{du}{dr} = 2r \implies du = 2rdr \tag{353}$$

$$\int_0^\infty \frac{du}{(u + a^2)^{3/2}} = (u + a^2)^{-1/2} \left(\frac{-2}{1}\right)\Big|_0^\infty = 0 - \frac{-2}{a^2} = \frac{2}{a^2} \tag{354}$$

$$E = E_z = \frac{\sigma a}{4\epsilon_0} \left(\frac{2}{a^2}\right) = \frac{\sigma}{2\epsilon_0} \tag{355}$$

8.5 Gauss's law

$$S = \textbf{sphere} \implies A = 4\pi r^2 \tag{356}$$

$$d\vec{A} = dA \hat{\perp}(A) \tag{357}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\vec{r}}{r} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \tag{358}$$

$$S \implies \hat{r} \parallel \hat{\perp}(A) \implies \vec{E} \cdot d\vec{A} = EdA\cos(\theta) = d\vec{A} = EdA\cos(0) = EdA \text{ \# alternatively ...} \tag{359}$$

$$\vec{E} \cdot d\vec{A} = \frac{qdA}{4\pi\epsilon_0 r^2} \left(\hat{r} \cdot \hat{\perp}(A)\right) = \frac{qdA}{4\pi\epsilon_0 r^2} \text{ \# amount flow across a small area} \tag{360}$$

$$\phi_{sphere} = \int \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0 r^2} \int dA = \frac{q}{4\pi\epsilon_0 r^2} A = \frac{q}{\epsilon_0} \text{ \# electric flow or flux through spherical surface} \tag{361}$$

$$S = \textbf{closed surface} \tag{362}$$

$$V \text{ \# enclosed volume ; } \partial V \text{ \# volume boundary or surface area} \tag{363}$$

$$\iint_S E_1 \vec{+} E_2 \cdot d\vec{A} = \iint_S \vec{E_1} \cdot d\vec{A} + \vec{E_2} \cdot d\vec{A} = \frac{q_1 + q_2}{\epsilon_0} \text{ \# superposition} \tag{364}$$

$$q_V = \sum_{i \in V} q_i \tag{365}$$

$$\phi = \oiint_{S=\partial V} \vec{E} \cdot d\vec{A} = \frac{q_V}{\epsilon_0} \text{ \# electric flux due to a discrete charge distribution} \tag{366}$$

$$\rho \text{ \# charge density ; } SI(\rho) = \frac{C}{m^3} \tag{367}$$

$$\oiint_{S=\partial V} \vec{E} \cdot d\vec{A} = \iiint_V \rho(x,y,z)dxdydz \text{ \# electric flux due to a continuous charge distribution} \tag{368}$$

8.6 Solid charged sphere

$$S = \textbf{sphere} \tag{369}$$

$$\phi = E4\pi r^2 = \frac{q_V}{\epsilon_0} \tag{370}$$

$$\vec{E}(\vec{r}) = \hat{r} \frac{q_V}{4\pi\epsilon_0} \frac{1}{r^2} \tag{371}$$

$$r_{in} \leq r \tag{372}$$

$$V(r) = \frac{4}{3}\pi r^3 \text{ \# volume of a sphere} \tag{373}$$

$$\vec{E}(\vec{r_{in}}) = \frac{\hat{r_{in}}}{4\pi {r_{in}}^2 \epsilon_0} \frac{q_V}{V(r)} V(r_{in}) = \frac{\hat{r_{in}}}{4\pi {r_{in}}^2 \epsilon_0} \frac{q_V {r_{in}}^3}{r^3} = \hat{r_{in}} \frac{q_V}{4\pi\epsilon_0} \frac{r_{in}}{r^3} \text{ \# increases with } r_{in} \tag{374}$$

8.7 Hollow charged sphere

$$A(r) = \frac{c}{r^2} \text{ \# area of a circular cone base} \tag{375}$$

$$\sigma\frac{A_1}{{r_1}^2} = \sigma c = \sigma\frac{A_2}{{r_2}^2} \text{ \# charges on from opposing sides cancel} \tag{376}$$

Note: RHS is based on the external normal of a closed surface lolol.