Next-Next-Gen Notes

Object-Oriented Maths

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Format: $characteristic((subjects), (dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$

1 Mathematical Analysis

1.0.1 Formal Logic

$$statement(s, ()) \iff well\text{-}formedString(s, ()) \tag{1}$$

$$\begin{aligned} proposition((p, t), ()) \iff & \Big(statement(p, ())\Big) \wedge \\ & \big(t = eval(p)\big) \wedge \\ & (t = true \veebar t = false) \end{aligned} \tag{2}$$

$$operator\Big(o, \big((p)_{n \in \mathbb{N}}\big)\Big) \iff proposition\Big(o\big((p)_{n \in \mathbb{N}}\big), ()\Big) \tag{3}$$

$$\begin{aligned} operator(\neg, (p_1)) \iff & \Big(proposition((p_1, true), ()) \implies ((\neg p_1, false), ())\Big) \wedge \\ & \Big(proposition((p_1, false), ()) \implies ((\neg p_1, true), ())\Big) \\ & \text{\# an operator takes in propositions and returns a proposition} \end{aligned} \tag{4}$$

$$\begin{aligned} & operator(\neg) \iff \textbf{NOT} \ ; \ operator(\vee) \iff \textbf{OR} \ ; \ operator(\wedge) \iff \textbf{AND} \ ; \ operator(\veebar) \iff \textbf{XOR} \\ & operator(\implies) \iff \textbf{IF} \ ; \ operator(\impliedby) \iff \textbf{OIF} \ ; \ operator(\iff) \iff \textbf{IFF} \end{aligned} \tag{5}$$

$$\begin{aligned} & proposition((false \implies true), true, ()) \wedge proposition((false \implies false), true, ()) \\ & \text{\# truths based on a false premise is not false; ex falso quodlibet principle} \end{aligned} \tag{6}$$

$$\text{(THM)} : (a \implies b \implies c) \iff \big(a \implies (b \implies c)\big) \iff \big((a \wedge b) \implies c\big) \tag{7}$$

$$predicate(P, (V)) \iff \forall_{v \in V} \Big(proposition\Big(\big(P(v), t\big), ()\Big)\Big) \tag{8}$$

$$\begin{aligned} & 0thOrderLogic(P, ()) \iff proposition((P, t), ()) \\ & \text{\# individual proposition} \end{aligned} \tag{9}$$

$$\begin{aligned} 1stOrderLogic(P, (V)) \iff & \Big(\forall_{v \in V} \Big(0thOrderLogic(v, ())\Big)\Big) \wedge \\ & \Big(\forall_{v \in V} \Big(proposition\Big(\big(P(v), t\big), ()\Big)\Big)\Big) \end{aligned}$$

$$\# \text{ propositions defined over a set of the lower order logical statements} \tag{10}$$

$$\mathit{quantifier}(q,(p,V)) \iff \Big(\mathit{predicate}(p,(V))\Big) \land \Big(\mathit{proposition}\Big((q(p),t),()\Big)\Big)$$
$$\# \text{ a quantifier takes in a predicate and returns a proposition} \tag{11}$$

$$\mathit{quantifier}(\forall,(p,V)) \iff \mathit{proposition}\Big(\Big(\land_{v\in V}(p(v)),t\Big),()\Big)$$
$$\# \text{ universal quantifier} \tag{12}$$

$$\mathit{quantifier}(\exists,(p,V)) \iff \mathit{proposition}\Big(\Big(\lor_{v\in V}(p(v)),t\Big),()\Big)$$
$$\# \text{ existential quantifier} \tag{13}$$

$$\mathit{quantifier}(\exists!,(p,V)) \iff \exists_{x\in V}\Big(P(x)\land\neg\Big(\exists_{y\in V\setminus\{x\}}(P(y))\Big)\Big)$$
$$\# \text{ uniqueness quantifier} \tag{14}$$

$$(\text{THM}) : \forall_x p(x) \iff \neg\exists_x\neg p(x)$$
$$\# \text{ De Morgan's law} \tag{15}$$

$$(\text{THM}) : \forall_x\exists_y p(x,y) = \forall_x\neg\forall_y\neg p(x,y) \neq \exists_y\forall_x p(x,y) = \neg\forall_y\neg\big(\forall_x p(x,y)\big) = \neg\forall_y\exists_x\neg p(x,y)$$
$$\# \text{ different quantifiers are not interchangeable} \tag{16}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{17}$$

$$\textbf{proof=truths derived from a finite number of axioms and deductions} \tag{18}$$

$$\textbf{elementary arithmetics=system with substitutions, and some notion of addition, multiplication, and prime nuumbers for encoding metamathematics} \tag{19}$$

$$\textbf{Gödel theorem} \implies \textbf{axiomatic systems equivalent in power to elementary mathematics either has unprovable statements or has contradictions} \tag{20}$$

$$\mathit{sequenceSet}((A)_{\mathbb{N}},(A)) \iff (Amapinputn)((A)_{\mathbb{N}} = \{A(1),A(2),A(3),\ldots\}) \tag{21}$$

$$\textbf{TODO: define union, intersection, complement, etc.} \tag{22}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{23}$$

1.1 Axiomatic Set Theory

$$============ \textbf{N O T = U P D A T E D} ============ \tag{24}$$

$$\textbf{ZFC set theory} = \textbf{standard form of axiomatic set theory} \tag{25}$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \tag{26}$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \tag{27}$$

$$\in \textbf{basis} \implies \{x,y\} = \{y,x\} \wedge \{x\} = \{x,x\} \tag{28}$$

$$\in \textbf{and sets works following the 9 ZFC axioms:} \tag{29}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \text{ \# E: } \in \text{ is only a proposition on sets} \tag{30}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \text{ \# E: existence of empty set} \tag{31}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \text{ \# C: pair set construction} \tag{32}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \text{ \# C: union set construction} \tag{33}$$

$$x = \{\{a\},\{b\}\} \text{ \# from the pair set axiom} \tag{34}$$

$$u = \cup x = \cup\{\{a\},\{b\}\} = \{a,b\} \tag{35}$$

$$\forall_x \exists!_y R(x,y) \text{ \# functional relation } R \tag{36}$$

$$\begin{gathered}\exists_i \forall_x \exists!_y R(x,y) \implies y \in i \text{ \# C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \,|\, P(y)\} \text{ \# Restricted Comprehension } \not\Longrightarrow \{y \,|\, P(y)\} \text{ \# Universal Comprehension}\end{gathered} \tag{37}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \implies P(x)\big) \text{ \# ignores out of scope} \neq \forall_x \big(x \in m \wedge P(x)\big) \text{ \# restricts entirety} \tag{38}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \implies n \subseteq \mathcal{P}(m)\big) \text{ \# C: existence of power set} \tag{39}$$

$$\exists_I \Big(\emptyset \in I \wedge \forall_{x \in I} \big(\{x\} \in I\big)\Big) \text{ \# I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \implies \mathbb{N} \textbf{ is a set} \tag{40}$$

$$\forall_x \Big(\big(\emptyset \notin x \wedge x \cap x' = \emptyset\big) \implies \exists_y (\textbf{set of each e} \in x)\Big) \text{ \# C: axiom of choice} \tag{41}$$

$$\forall_x x \neq \emptyset \implies x \notin x \text{ \# F: axiom of foundation covers further paradoxes} \tag{42}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{43}$$

1.2 Classification of sets

$$\begin{gathered} \mathit{space}\big((\mathit{set}, \mathit{structure}), ()\big) \iff \mathit{structure}(\mathit{set}) \\ \text{\# a space a set equipped with some structure} \end{gathered}$$

$$\text{\# various spaces can be studied through structure preserving maps between those spaces} \tag{44}$$

$$\begin{aligned} map(\phi,(A,B)) \iff &\Big(\forall_{a\in A}\exists!_{b\in B}\big(\phi(a,b)\big)\Big)\vee \\ &\Big(\forall_{a\in A}\exists!_{b\in B}\big(b=\phi(a)\big)\Big) \\ &\text{\# maps elements of a set to elements of another set} \end{aligned} \tag{45}$$

$$domain\big(A,(\phi,A,B)\big) \iff \Big(map\big(\phi,(A,B)\big)\Big) \tag{46}$$

$$codomain\big(B,(\phi,A,B)\big) \iff \Big(map\big(\phi,(A,B)\big)\Big) \tag{47}$$

$$\begin{aligned} image\big(B,(A,q,M,N)\big) \iff &\Big(map\big(q,(M,N)\big)\wedge A\subseteq M\Big)\wedge \\ &\Big(B=\{n\in N\,|\,\exists_{a\in A}\big(q(a)=n\big)\}\Big) \end{aligned} \tag{48}$$

$$\begin{aligned} preimage\big(A,(B,q,M,N)\big) \iff &\Big(map\big(q,(M,N)\big)\wedge B\subseteq N\Big)\wedge \\ &\Big(A=\{m\in M\,|\,\exists_{b\in B}\big(b=q(m)\big)\}\Big) \end{aligned} \tag{49}$$

$$\begin{aligned} injection\big(q,(M,N)\big) \iff &\Big(map\big(q,(M,N)\big)\Big)\wedge \\ &\forall_{u,v\in M}\big(q(u)=q(v)\implies u=v\big) \\ &\text{\# every } m \text{ has at most 1 image} \end{aligned} \tag{50}$$

$$\begin{aligned} surjection\big(q,(M,N)\big) \iff &\Big(map\big(q,(M,N)\big)\Big)\wedge \\ &\forall_{n\in N}\exists_{m\in M}\big(n=q(m)\big) \\ &\text{\# every } n \text{ has at least 1 preimage} \end{aligned} \tag{51}$$

$$\begin{aligned} bijection\big(q,(M,N)\big) \iff &\Big(injection\big(q,(M,N)\big)\Big)\wedge \\ &\Big(surjection\big(q,(M,N)\big)\Big) \\ &\text{\# every unique } m \text{ corresponds to a unique } n \end{aligned} \tag{52}$$

$$isomorphicSets\big((A,B),()\big) \iff \exists_{\phi}\Big(bijection\big(\phi,(A,B)\big)\Big) \tag{53}$$

$$infiniteSet\big(S,()\big) \iff \exists_{T\subset S}\Big(isomorphicSets\big((T,S),()\big)\Big) \tag{54}$$

$$finiteSet\big(S,()\big) \iff \Big(\neg infiniteSet\big(S,()\big)\Big)\vee\big(|S|\in\mathbb{N}\big) \tag{55}$$

$$countablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big)\wedge\Big(isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \tag{56}$$

$$uncountablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big)\wedge\Big(\neg isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \tag{57}$$

$$inverseMap\Big(q^{-1},(q,M,N)\Big) \iff \Big(bijection(q,(M,N))\Big)\wedge \Big(map\Big(q^{-1},(N,M)\Big)\Big)\wedge \Big(\forall_{n\in N}\exists!_{m\in M}\Big(q(m)=n \implies q^{-1}(n)=m\Big)\Big) \tag{58}$$

$$mapComposition(\phi\circ\psi,(\phi,\psi,A,B,C)) \iff map(\psi,(A,B))\wedge map(\phi,(B,C))\wedge \forall_{a\in A}\Big(\phi\circ\psi(a)=\phi\big(\psi(a)\big)\Big) \tag{59}$$

$$equivalenceRelation\big(\sim,(M)\big) \iff (\forall_{m\in M}(m\sim m))\wedge (\forall_{m,n\in M}(m\sim n\implies n\sim m))\wedge (\forall_{m,n,p\in M}(m\sim n\wedge n\sim p\implies m\sim p)) \text{ \# behaves as equivalences should} \tag{60}$$

$$equivalenceClass\big([m],(m,M,\sim)\big) \iff [m]=\{n\in M\,|\,n\sim m\} \text{ \# set of elements satisfying the equivalence relation with } m \tag{61}$$

$$\text{(THM)}: a\in[m]\implies[a]=[m]\ ;\ [m]=[n]\veebar[m]\cap[n]=\emptyset \text{ \# equivalence class properties} \tag{62}$$

$$quotientSet\big(M/\sim,(M,\sim)\big) \iff M/\sim=\{[m]\in\mathcal{P}(M)\,|\,m\in M\} \text{ \# set of all equivalence classes} \tag{63}$$

$$\text{(THM)}: \textbf{axiom of choice}\implies\forall_{[m]\in M/\sim}\exists_r\big(r\in[m]\big) \text{ \# well-defined maps may be defined in terms of chosen representative elements } r \tag{64}$$

1.3 Construction of number sets

$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{65}$$

$$\textbf{axiom of infinity}\implies\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\ldots\}\cong\mathbb{N} \tag{66}$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \tag{67}$$

$$\textbf{addition}=\textbf{successor map: }\mathbb{N}\to\mathbb{N}=S(n)=\{n\} \text{ \# adds a layer of brackets} \tag{68}$$

$$\textbf{subtraction}=\textbf{predecessor map: }\mathbb{N}^\star\to\mathbb{N}=P(n)=m\,|\,m\in n \text{ \# removes a layer of brackets} \tag{69}$$

$$S^0=id\ ;\ n\in\mathbb{N}^\star\implies S^n=S\circ S^{P(n)} \tag{70}$$

$$\textbf{addition}=+:\mathbb{N}\times\mathbb{N}\to\mathbb{N}=+(m,n)=m+n=S^n(m) \tag{71}$$

$$S^x=id=S^0\implies x=\textbf{additive identity}=0 \tag{72}$$

$$S^n(x)=0 \implies x=\textbf{additive inverse} \notin \mathbb{N} \text{ \# git gud smh -\_-} \tag{73}$$

$$\begin{gathered}\mathbb{Z}=\mathbb{N}\times\mathbb{N}/\sim\textbf{, s.t.:}\\ (m,n)\sim(p,q) \iff m+q=p+n \text{ \# span } \mathbb{Z} \text{ using differences then group equal differences}\end{gathered} \tag{74}$$

$$\mathbb{N}\hookrightarrow\mathbb{Z}:\forall_{n\in\mathbb{N}} n\to[(n,0)] \text{ \# } \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{75}$$

$$+_{\mathbb{Z}}=[(m+_{\mathbb{N}}p, n+_{\mathbb{N}}q)] \text{ \# well-defined and consistent} \tag{76}$$

$$\textbf{multiplication}\ldots M^x=id \implies x=\textbf{multiplicative identity}=1\ldots\textbf{multiplicative inverse}\notin\mathbb{N} \tag{77}$$

$$\mathbb{Q}=(\mathbb{Z}\times\mathbb{Z}^\star)/\sim\textbf{, s.t.: } (x,y)\sim(u,v) \iff x\cdot v=u\cdot y \tag{78}$$

$$\mathbb{Z}\hookrightarrow\mathbb{Q}\forall_{q\in\mathbb{Q}} q\to[(q,1)]\ ;\ \ldots\{x\,|\,x^2=2\}\notin\mathbb{Q} \tag{79}$$

$$\mathbb{R}=\textbf{almost homomorphisms on } \mathbb{Z}/\sim \text{ \# http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{80}$$

$$\text{============ \textbf{N O T = U P D A T E D} ============} \tag{81}$$

1.4 Topology

$$\begin{gathered}topology(\mathcal{O},(M)) \iff (\mathcal{O}\subseteq\mathcal{P}(M))\land\\ (\emptyset, M\in\mathcal{O})\land\\ \Big((F\in\mathcal{O}\land|F|<|\mathbb{N}|) \implies \cap F\in\mathcal{O}\Big)\land\\ (C\subseteq\mathcal{O} \implies \cup C\in\mathcal{O})\\ \text{\# topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.}\\ \text{\# arbitrary unions of open sets always result in an open set}\\ \text{\# open sets do not contain their boundaries and infinite intersections of open sets may approach and}\\ \text{\# induce boundaries resulting in a closed set}\end{gathered} \tag{82}$$

$$topologicalSpace((M,\mathcal{O}),()) \iff topology(\mathcal{O},(M)) \tag{83}$$

$$\begin{gathered}open(S,(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big)\land\\ (S\subseteq M)\land(S\in\mathcal{O})\\ \text{\# an open set do not contains its own boundaries}\end{gathered} \tag{84}$$

$$\begin{gathered}closed(S,(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big)\land\\ (S\subseteq M)\land\big(S\in\mathcal{P}(M)\setminus\mathcal{O}\big)\\ \text{\# a closed set contains the boundaries an open set}\end{gathered} \tag{85}$$

$$clopen(S,(M,\mathcal{O})) \iff \Big(closed(S,(M,\mathcal{O}))\Big)\land\Big(open(S,(M,\mathcal{O}))\Big) \tag{86}$$

$$neighborhood(U,(a,\mathcal{O})) \iff (a\in U\in\mathcal{O})$$

$$\text{\# another name for open set containing } a \quad (87)$$

$$M=\{a,b,c,d\}\wedge\mathcal{O}=\{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},M\}\implies \Big(\mathit{open}(X,(M,\mathcal{O}))\iff X=\{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},M\}\Big)\wedge \Big(\mathit{closed}(Y,(M,\mathcal{O}))\iff Y=\{\emptyset,\{a,b,d\},\{c,d\},\{a,b\},\{d\},M\}\Big)\wedge \Big(\mathit{clopen}(Z,(M,\mathcal{O}))\iff Z=\{\emptyset,\{a,b\},\{c,d\},M\}\Big) \quad (88)$$

$$\mathit{chaoticTopology}(M)=\{0,M\}\ ;\ \mathit{discreteTopology}=\mathcal{P}(M) \quad (89)$$

1.5 Induced topology

$$\mathit{distance}(d,(M))\iff\Big(\forall_{x,y\in M}\Big(d(x,y)=d(y,x)\in\mathbb{R}_0^+\Big)\Big)\wedge \Big(\forall_{x,y\in M}\big(d(x,y)=0\iff x=y\big)\Big)\wedge \Big(\forall_{x,y,z}\Big(\big(d(x,z)\le d(x,y)+d(y,z)\big)\Big)\Big) \quad \text{\# behaves as distances should} \quad (90)$$

$$\mathit{metricSpace}((M,d),())\iff\mathit{distance}(d,(M)) \quad (91)$$

$$\mathit{openBall}(B,(r,p,M,d))\iff\Big(\mathit{metricSpace}((M,d),())\Big)\wedge (r\in\mathbb{R}^+,p\in M)\wedge (B=\{q\in M\,|\,d(p,q)<r\}) \quad (92)$$

$$\mathit{metricTopology}(\mathcal{O},(M,d))\iff\Big(\mathit{metricSpace}((M,d),())\Big)\wedge \Big(\mathcal{O}=\{U\in\mathcal{P}(M)\,|\,\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\Big(\mathit{openBall}(B,(r,p,M,d))\wedge B\subseteq U\Big)\}\Big)$$
$$\text{\# every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (93)$$

$$\mathit{metricTopologicalSpace}((M,\mathcal{O},d),())\iff\mathit{metricTopology}(\mathcal{O},(M,d)) \quad (94)$$

$$\mathit{limitPoint}(p,(S,M,\mathcal{O},d))\iff\Big(\mathit{metricTopologicalSpace}((M,\mathcal{O},d),())\Big)\wedge(S\subseteq M)\wedge \forall_{r\in\mathbb{R}^+}\Big(\mathit{openBall}(B,(r,p,M,d))\wedge B\cap S\neq\emptyset\Big)$$
$$\text{\# every open ball centered at } p \text{ contains some intersection with } S \quad (95)$$

$$\mathit{interiorPoint}(p,(S,M,\mathcal{O},d))\iff\Big(\mathit{metricTopologicalSpace}((M,\mathcal{O},d),())\Big)\wedge(S\subseteq M)\wedge \Big(\exists_{r\in\mathbb{R}^+}\Big(\mathit{openBall}(B,(r,p,M,d))\wedge B\subseteq S\Big)\Big)$$
$$\text{\# there is an open ball centered at } p \text{ that is fully enclosed in } S \quad (96)$$

$$\mathit{closure}(\bar{S},(S,M,\mathcal{O},d))\iff\bar{S}=S\cup\{p\in M\,|\,\mathit{limitPoint}(p,(S,M,\mathcal{O},d))\} \quad (97)$$

$$\textit{dense}\big(S,(M,\mathcal{O},d)\big) \iff (S\subseteq M)\wedge\Big(\forall_{p\in M}\Big(p\in \textit{closure}\big(\bar{S},(S,M,\mathcal{O},d)\big)\Big)\Big)$$
$$\text{\# every of point in } M \text{ is a point or a limit point of } S \tag{98}$$

$$\textit{eucD}\big(d,(n)\big) \iff \big(\forall_{i\in\mathbb{N}\wedge i\leq n}(x_i\in\mathbb{R})\big)\wedge\left(d=\sqrt[2]{\sum_{i=1}^{n}{x_i}^2}\right) \tag{99}$$

$$\textit{metricTopology}\Big(\textit{standardTopology},\Big(\mathbb{R}^n,\textit{eucD}\big(d,(n)\big)\Big)\Big)$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============$$
$$\mathbf{L1:}\ \forall_{p\in U=\emptyset}(\ldots)\implies\forall_p\Big(\big(p\in\emptyset\big)\implies\ldots\Big)\implies\forall_p\big((\mathbf{False})\implies\ldots\big)\implies\emptyset\in\mathcal{O}_{standard}$$
$$\mathbf{L2:}\ \forall_{p\in\mathbb{R}^n}B(r,p,\mathbb{R}^n,d)\subseteq\mathbb{R}^n\implies M\in\mathcal{O}_{standard}$$
$$\mathbf{L4:}\ C\subseteq\mathcal{O}_{standard}\implies\forall_{U\in C}\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\big(B_r(p)\subseteq U\subseteq\cup C\big)\implies\cup C\in\mathcal{O}_{standard}$$
$$\mathbf{L3:}\ U,V\in\mathcal{O}_{standard}\implies p\in U\cap V\implies p\in U\wedge p\in V\implies$$
$$\exists_{r\in\mathbb{R}^+}B(r,p,\mathbb{R}^n,d)\wedge\exists_{s\in\mathbb{R}^+}B(s,p,\mathbb{R}^n,d)\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\subseteq U\wedge B\big(min(r,s),q,\mathbb{R}^n,d\big)\subseteq V\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\in U\cap V\implies U\cap V\in\mathcal{O}_{standard}$$
$$\text{\# natural topology for } \mathbb{R}^d$$
$$\text{\# could fail on infinite sets since } min \text{ could approach } 0$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{100}$$

$$\textit{subsetTopology}\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\iff\textit{topology}\big(\mathcal{O},(M)\big)\wedge(N\subseteq M)\wedge\big(\mathcal{O}|_N=\{U\cap N\,|\,U\in\mathcal{O}\}\big)$$
$$\text{\# crops open sets outside } N \tag{101}$$

$$\text{(THM)}:\textit{subsetTopology}\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\wedge\textit{topology}\big(\mathcal{O}|_N,(N)\big)\impliedby$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============$$
$$\mathbf{L1:}\ \emptyset\in\mathcal{O}\implies U=\emptyset\implies\emptyset\cap N=\emptyset\implies\emptyset\in\mathcal{O}|_N$$
$$\mathbf{L2:}\ M\in\mathcal{O}\implies U=M\implies M\cap N=N\implies N\in\mathcal{O}|_N$$
$$\mathbf{L3:}\ S,T\in\mathcal{O}|_N\implies\exists_{U\in\mathcal{O}}(S=U\cap N)\wedge\exists_{V\in\mathcal{O}}(T=V\cap N)\implies S\cap T=(U\cap N)\cap(V\cap N)$$
$$=(U\cap V)\cap N\wedge U\cap V\in\mathcal{O}\implies S\cap T\in\mathcal{O}|_N$$
$$\mathbf{L4:}\ TODO:EXERCISE$$
$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{102}$$

$$\textit{productTopology}\Big(\mathcal{O}_{A\times B},\big((A,\mathcal{O}_A),(B,\mathcal{O}_B)\big)\Big)\iff\Big(\textit{topology}\big(\mathcal{O}_A,(A)\big)\Big)\wedge\Big(\textit{topology}\big(\mathcal{O}_B,(B)\big)\Big)\wedge$$
$$\big(\mathcal{O}_{A\times B}=\{(a,b)\in A\times B\,|\,\exists_S(a\in S\in\mathcal{O}_A)\exists_T(b\in T\in\mathcal{O}_B)\}\big)$$
$$\text{\# open in cross iff open in each} \tag{103}$$

1.6 Convergence

$$\textit{sequence}\big(q,(M)\big)\iff\textit{map}\big(q,(\mathbb{N},M)\big) \tag{104}$$

$$\textit{sequenceConvergesTo}\big((q,a),(M,\mathcal{O})\big)\iff\Big(\textit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\Big(\textit{sequence}\big(q,(M)\big)\Big)\wedge(a\in M)\wedge\Big(\forall_{U\in\mathcal{O}|a\in U}\exists_{N\in\mathbb{N}}\forall_{n>N}\big(q(n)\in U\big)\Big)$$
$$\text{\# each neighborhood of } a \text{ has a tail-end sequence that does not map to outside points} \tag{105}$$

(THM) : **convergence generalizes to: the sequence** $q:\mathbb{N}\to\mathbb{R}^d$ **converges against** $a\in\mathbb{R}^d$ **in** $\mathcal{O}_S$ **if:**

$$\forall_{r>0}\exists_{N\in\mathbb{N}}\forall_{n>N}\big(||q(n)-a||<\epsilon\big) \text{ \# distance based convergence} \tag{106}$$

1.7 Continuity

$$\mathit{continuous}(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}_M),())\Big)\wedge \Big(\mathit{topologicalSpace}((N,\mathcal{O}_N),())\Big)\wedge\Big(\forall_{V\in\mathcal{O}_N}\Big(\mathit{preimage}(A,(V,\phi,M,N))\in\mathcal{O}_M\Big)\Big) \text{ \# preimage of open sets are open} \tag{107}$$

$$\mathit{homeomorphism}(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)) \iff \Big(\mathit{inverseMap}\big(\phi^{-1},(\phi,M,N)\big)\Big) \Big(\mathit{continuous}(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N))\Big)\wedge\Big(\mathit{continuous}\big(\phi^{-1},(N,\mathcal{O}_N,M,\mathcal{O}_M)\big)\Big) \text{ \# structure preserving maps in topology, ability to share topological properties} \tag{108}$$

$$\mathit{isomorphicTopologicalSpace}\Big(((M,\mathcal{O}_M),(N,\mathcal{O}_N)),()\Big) \iff \exists_\phi\Big(\mathit{homeomorphism}(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N))\Big) \tag{109}$$

1.8 Separation

$$\mathit{T0Separate}((M,\mathcal{O}),()) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big)\wedge \Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U\in\mathcal{O}}\Big(\big(x\in U\wedge y\notin U\big)\vee\big(y\in U\wedge x\notin U\big)\Big)\Big) \text{ \# each pair of points has a neighborhood s.t. one is inside and the other is outside} \tag{110}$$

$$\mathit{T1Separate}((M,\mathcal{O}),()) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big)\wedge \Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U,V\in\mathcal{O}\wedge U\neq V}\Big(\big(x\in U\wedge y\notin U\big)\wedge\big(y\in V\wedge x\notin V\big)\Big)\Big) \text{ \# every point has a neighborhood that does not contain another point} \tag{111}$$

$$\mathit{T2Separate}((M,\mathcal{O}),()) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big)\wedge \Big(\forall_{x,y\in M\wedge x\neq y}\exists_{U,V\in\mathcal{O}\wedge U\neq V}\big(U\cap V=\emptyset\big)\Big) \text{ \# every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space} \tag{112}$$

$$\text{(THM)} : \mathit{T2Separate}\implies\mathit{T1Separate}\implies\mathit{T0Separate} \tag{113}$$

1.9 Compactness

$$\mathit{openCover}(C,(M,\mathcal{O})) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big)\wedge(C\subseteq\mathcal{O})\wedge(\cup C=M) \text{ \# collection of open sets whose elements cover the entire space} \tag{114}$$

$$\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \iff \Big(\widetilde{C}\subseteq C\Big)\wedge\Big(\mathit{openCover}\big(C,(M,\mathcal{O})\big)\Big)\wedge$$
$$\Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big)\wedge\Big(\mathit{finiteSet}\Big(\widetilde{C},()\Big)\Big)$$
$$\text{\# finite subset of a cover that is also a cover} \tag{115}$$

$$\mathit{compact}\big((M,\mathcal{O}),()\big) \iff \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big)\wedge$$
$$\Bigg(\forall_{C\subseteq\mathcal{O}}\Bigg(\mathit{openCover}\big(C,(M,\mathcal{O})\big)\implies\exists_{\widetilde{C}\subseteq C}\Big(\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big)\Big)\Bigg)\Bigg)$$
$$\text{\# every covering of the space is represented by a finite number of nhbhds} \tag{116}$$

$$\mathit{compactSubset}\big(N,(M,\mathcal{O}_d,d)\big)\iff\Big(\mathit{compact}\big((M,\mathcal{O}),()\big)\Big)\wedge\Big(\mathit{subsetTopology}\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\Big) \tag{117}$$

$$\mathit{bounded}\big(N,(M,d)\big)\iff\Big(\mathit{metricSpace}\big((M,d),()\big)\Big)\wedge(N\subseteq M)\wedge$$
$$\Big(\exists_{r\in\mathbb{R}^+}\forall_{p,q\in n}\big(d(p,q)<r\big)\Big) \tag{118}$$

$$\text{(THM) HeineBorel: } \mathit{metricTopologicalSpace}\big((M,\mathcal{O}_d,d),()\big)\implies$$
$$\forall_{S\in\mathcal{P}(M)}\Big(\Big(\mathit{closed}\big(S,(M,\mathcal{O}_d)\big)\wedge\mathit{bounded}\big(S,(M,\mathcal{O}_d)\big)\Big)\iff\mathit{compactSubset}\big(S,(M,\mathcal{O}_d)\big)\Big)$$
$$\text{\# when metric topologies are involved, compactness is equivalent to being closed and bounded} \tag{119}$$

1.10 Paracompactness

$$\mathit{openRefinement}\Big(\widetilde{C},(C,M,\mathcal{O})\Big)\iff\Big(\mathit{openCover}\big(C,(M,\mathcal{O})\big)\Big)\wedge\Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big)\wedge$$
$$\Big(\forall_{\widetilde{U}\in\widetilde{C}}\exists_{U\in C}\Big(\widetilde{U}\subseteq U\Big)\Big)$$
$$\text{\# a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space} \tag{120}$$

$$\text{(THM) : } \mathit{finiteSubcover}\implies\mathit{openRefinement} \tag{121}$$

$$\mathit{locallyFinite}\big(C,(M,\mathcal{O})\big)\iff\Big(\mathit{openCover}\big(C,(M,\mathcal{O})\big)\Big)\wedge$$
$$\forall_{p\in M}\exists_{U\in\mathcal{O}|p\in U}\Big(\mathit{finiteSet}\big(\{U_c\in C|U\cap U_c\neq\emptyset\},()\big)\Big)$$
$$\text{\# each point has a neighborhood that intersects with only finitely many sets in the cover} \tag{122}$$

$$\mathit{paracompact}\big((M,\mathcal{O}),()\big)\iff$$
$$\forall_C\Bigg(\mathit{openCover}\big(C,(M,\mathcal{O})\big)\implies\exists_{\widetilde{C}}\Bigg(\mathit{locallyFinite}\Big(\mathit{openRefinement}\Big(\widetilde{C},(C,M,\mathcal{O})\Big),(M,\mathcal{O})\Big)\Bigg)\Bigg)$$
$$\text{\# every open cover has a locally finite open refinement} \tag{123}$$

$$\text{(THM) : } \mathit{metricTopologicalSpace}\implies\mathit{paracompact} \tag{124}$$

$$============\ \mathbf{N\ O\ T = U\ P\ D\ A\ T\ E\ D}\ ============ \tag{125}$$

$$partitionOfUnitySubjCover\big(\mathcal{F},(C,M,\mathcal{O})\big) \iff \Big(locallyFinite\big(C,(M,\mathcal{O})\big)\Big) \wedge (f \in \mathcal{F}) \wedge$$
$$\Bigg(continuous\bigg(f,\Big(M,\mathcal{O},[0,1],subsetTopology\Big(\mathcal{O}|_{[0,1]},\big([0,1],\mathbb{R},standardTopology\big)\Big)\Big)\bigg)\Bigg) \wedge$$
$$\Big(\exists_{U_f \in C}\forall_{p\in M}\big(f(p)\neq 0 \implies p \in U_f\big)\Big) \wedge$$
$$\Big(\forall_{p\in M}\exists_{U\in\mathcal{O}|p\in U}\big((f_U)_n = \{f\in\mathcal{F}|p\in M \wedge f(p)\neq 0\}\big)\Big) \wedge$$
$$\Big(locallyFinite(C,M,\mathcal{O}) \implies finiteSet\big((f_U)_n,()\big)\Big) \wedge$$
$$\left(\forall_{p\in M}\exists_{U\in\mathcal{O}|p\in U}\left(\sum_{i=1}^{|(f_U)_n|}(f_U)_i(p) = 1\right)\right)$$
$$\text{\# useful for defining integrals between overlapping neighborhoods} \tag{126}$$

$$T2Separate\big((M,\mathcal{O}),()\big) \implies \Big(paracompact\big((M,\mathcal{O}),()\big)\Big) \iff$$
$$\forall_C\Big(openCover\big(C,(M,\mathcal{O})\big) \implies partitionOfUnitySOTCover\big(\mathcal{F},(C,M,\mathcal{O})\big)\Big) \tag{127}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{128}$$

1.11 Connectedness and path-connectedness

$$connected\big((M,\mathcal{O}),()\big) \iff \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge \Big(\neg\exists_{A,B\in\mathcal{O}\setminus\emptyset}\big(A\cap B\neq\emptyset \wedge A\cup B = M\big)\Big)$$
$$\text{\# if there is some covering of the space that does not intersect} \tag{129}$$

$$\text{(THM)}: \neg connected\bigg(\Big(\mathbb{R}\setminus\{0\},subsetTopology\Big(\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}},\big(\mathbb{R},standardTopology,\mathbb{R}\setminus\{0\}\big)\Big)\Big),()\bigg)$$
$$\impliedby \Big(A=(-\infty,0)\in\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge \Big(B=(0,\infty)\in\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge$$
$$\big(A\cap B=\emptyset\big) \wedge \big(A\cup B = \mathbb{R}\setminus\{0\}\big) \tag{130}$$

$$\text{(THM)}: connected\big((M,\mathcal{O}),()\big) \iff \forall_{S\in\mathcal{O}}\Big(clopen\Big(S,(M,\mathcal{O}) \implies \big(S=\emptyset \vee S=M\big)\Big)\Big) \tag{131}$$

$$pathConnected\big((M,\mathcal{O}),()\big) \iff \Big(subsetTopology\Big(\mathcal{O}_{standard}|_{[0,1]},\big(\mathbb{R},standardTopology,[0,1]\big)\Big)\Big) \wedge$$
$$\bigg(\forall_{p,q\in M}\exists_\gamma\Big(continuous\Big(\gamma,\big([0,1],\mathcal{O}_{standard}|_{[0,1]},M,\mathcal{O}\big)\Big) \wedge \gamma(0)=p \wedge \gamma(1)=q\Big)\bigg) \tag{132}$$

$$\text{(THM)}: pathConnected \implies connected \tag{133}$$

1.12 Homotopic curve and the fundamental group

$$============ \textbf{N O T = U P D A T E D} ============ \tag{134}$$

$$homotopic(\sim,(\gamma,\delta,M,\mathcal{O})) \iff (map(\gamma,([0,1],M)) \wedge map(\delta,([0,1],M)))\wedge$$
$$(\gamma(0)=\delta(0) \wedge \gamma(1)=\delta(1))\wedge$$
$$(\exists_H \forall_{\lambda\in[0,1]}(continuous(H,(([0,1]\times[0,1],\mathcal{O}_{standard^2}|_{[0,1]\times[0,1]}),(M,\mathcal{O}))\wedge H(0,\lambda)=\gamma(\lambda)\wedge H(1,\lambda)=\delta(\lambda))))$$
$$\text{\# } H \text{ is a continuous deformation of one curve into another} \quad (135)$$

$$homotopic(\sim) \implies equivalenceRelation(\sim) \quad (136)$$

$$loopSpace(\mathcal{L}_p,(p,M,\mathcal{O})) \iff \mathcal{L}_p=\{map(\gamma,([0,1],M))|continuous(\gamma)\wedge\gamma(0)=\gamma(1)\} \quad (137)$$

$$concatination(\star,(p,\gamma,\delta)) \iff (\gamma,\delta\in loopSpace(\mathcal{L}_p))\wedge$$
$$(\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda)=\begin{cases}\gamma(2\lambda) & 0\le\lambda<0.5\\ \delta(2\lambda-1) & 0.5\le\lambda\le 1\end{cases})) \quad (138)$$

$$group((G,\bullet),()) \iff (map(\bullet,(G\times G,G)))\wedge$$
$$(\forall_{a,b\in G}(a\bullet b\in G))$$
$$(\forall_{a,b,c\in G}((a\bullet b)\bullet C=a\bullet(b\bullet c)))$$
$$(\exists_{\boldsymbol{e}}\forall_{a\in G}(\boldsymbol{e}\bullet a=a=a\bullet\boldsymbol{e}))\wedge$$
$$(\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1}=\boldsymbol{e}=a^{-1}\bullet a))$$
$$\text{\# characterizes symmetry of a set structure} \quad (139)$$

$$isomorphic(\cong,(X,\odot),(Y,\ominus))) \iff \exists_f\forall_{a,b\in X}(bijection(f,(X,Y))\wedge f(a\odot b)=f(a)\ominus f(b)) \quad (140)$$

$$fundamentalGroup((\pi_{1,p},\bullet),(p,M,\mathcal{O})) \iff (\pi_{1,p}=\mathcal{L}_p/\sim)\wedge$$
$$(map(\bullet,(\pi_{1,p}\times\pi_{1,p},\pi_{1,p})))\wedge$$
$$(\forall_{A,B\in\pi_{1,p}}([A]\bullet[B]=[A\star B]))\wedge$$
$$(group((\pi_{1,p},\bullet),()))$$
$$\text{\# an equivalence class of all loops induced from the homotopic equivalence relation} \quad (141)$$

$$fundamentalGroup_1\ncong fundamentalGroup_2 \implies topologicalSpace_1\ncong topologicalSpace_2 \quad (142)$$

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there exists no known list of topological properties that can imply homeomorphisms (143)

CONTINUE @ Lecture 6: manifolds (144)

============ **N O T = U P D A T E D** ============ (145)

1.13 Measure theory

$$sigmaAlgebra(\sigma,(M)) \iff (M\neq\emptyset)\wedge(\sigma\subseteq\mathcal{P}(M))\wedge$$
$$(M\in\sigma)\wedge\Big(\forall_{A\in\sigma}(M\setminus A\in\sigma)\Big)\wedge$$
$$\Big(\Big(A\subseteq\sigma\wedge\neg uncountablyInfinite(A,())\Big)\implies\cup A\in\sigma\Big)$$
$$\text{\# behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu \quad (146)$$

$$measurableSpace((M,\sigma),()) \iff sigmaAlgebra(\sigma,(M)) \quad (147)$$

$$\mathit{measurableSet}\big(A,(M,\sigma)\big) \Longleftrightarrow \Big(\mathit{measurableSpace}\big((M,\sigma),()\big)\Big) \wedge (A \in \sigma) \tag{148}$$

$$\mathit{measure}\big(\mu,(M,\sigma)\big) \Longleftrightarrow \Big(\mathit{measurableSpace}\big((M,\sigma),()\big)\Big) \wedge \left(\mathit{map}\left(\mu,\left(\sigma,\left(\overline{\mathbb{R}}\right)_0^+\right)\right)\right) \wedge \Big(\mu(\emptyset)=0\Big) \wedge$$
$$\left(\Big((A)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}\forall_{j\in\mathbb{N}\setminus\{i\}}\big(A_i \cap A_j = \emptyset\big)\Big) \Longrightarrow \mu\big(\cup_{i\in\mathbb{N}}(A_i)\big) = \sum_{i\in\mathbb{N}}\big(\mu(A_i)\big)\right)$$
$$\# \text{ enforces meaningful concepts of measures such as precise additivity} \tag{149}$$

$$(\mathrm{THM}) : \mathit{measure}\big(\mu,(M,\sigma)\big) \Longrightarrow$$
$$\Big(\forall_{A,B\in\sigma}\big(A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)\big)\Big) \wedge$$
$$\left((A)_{\mathbb{N}} \subseteq \sigma \Longrightarrow \mu\big(\cup_{i\in\mathbb{N}}(A_i)\big) \leq \sum_{i\in\mathbb{N}}\big(\mu(A_i)\big)\right) \wedge$$
$$\left(\Big((B)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}(B_i \subseteq B_{i+1}) \wedge B = \cup(B)_{\mathbb{N}}\Big) \Longrightarrow \lim_{n\to\infty}\big(\mu(B_n)\big) = \mu(B)\right) \wedge$$
$$\left(\Big((C)_{\mathbb{N}} \subseteq \sigma \wedge \forall_{i\in\mathbb{N}}(C_{i+1} \subseteq C_i) \wedge C = \cap(C)_{\mathbb{N}}\Big) \Longrightarrow \lim_{n\to\infty}\big(\mu(C_n)\big) = \mu(C)\right)$$
$$\# \text{ immediate implications of the measurable set } A \in \sigma \text{ axioms and the measure } \mu \text{ axioms} \tag{150}$$

$$\mathit{measureSpace}\big((M,\sigma,\mu),()\big) \Longleftrightarrow \mathit{measure}\big(\mu,(M,\sigma)\big) \tag{151}$$

$$\mathit{finiteMeasure}\big(\mu,(M,\sigma)\big) \Longleftrightarrow \Big(\mathit{measure}\big(\mu,(M,\sigma)\big)\Big) \wedge$$
$$\left(\exists_{(A)_{\mathbb{N}}\subseteq\sigma}\Big(\cup\big((A)_{\mathbb{N}}\big) = M \wedge \forall_{n\in\mathbb{N}}\big(\mu(A_n) < \infty\big)\Big)\right) \tag{152}$$

$$\mathit{generatedSigmaAlgebra}\big(\sigma(\zeta),(\zeta,M)\big) \Longleftrightarrow \Big(G = \{\sigma \subseteq \mathcal{P}(M) \,|\, \mathit{sigmaAlgebra}\big(\sigma,(M)\big)\}\Big) \wedge \big(\sigma(\zeta) = \cap G\big)$$
$$\# \text{ smallest } \sigma\text{-algebra containing the generating set } \zeta \tag{153}$$

$$(\mathrm{THM}) : \exists_{\zeta\subseteq M}\Big(\mathit{generatedSigmaAlgebra}\big(\sigma(\zeta),(\zeta,M)\big) = \mathit{sigmaAlgebra}\big(\sigma,(M)\big)\Big) \tag{154}$$

$$\mathit{borelSigmaAlgebra}\big(\sigma(\mathcal{O}),(M,\mathcal{O})\big) \Longleftrightarrow \Big(\mathit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big) \wedge$$
$$\Big(\mathit{generatedSigmaAlgebra}\big(\sigma(\mathcal{O}),(\mathcal{O},M)\big)\Big)$$
$$\# \ \sigma\text{-algebra induced by a topology} \tag{155}$$

$$\mathit{standardSigma}(\sigma_s,()) \Longleftrightarrow \left(\mathit{borelSigmaAlgebra}\left(\sigma_s,\left(\mathbb{R}^d,\mathit{standardTopology}\right)\right)\right) \tag{156}$$

$$\mathit{lebesgueMeasure}(\lambda,()) \Longleftrightarrow \left(\mathit{measure}\left(\lambda,\left(\mathbb{R}^d,\mathit{standardSigma}\right)\right)\right) \wedge$$
$$\left(\lambda\Big(\times_{i=1}^d\big([a_i,b_i)\big)\Big) = \sum_{i=1}^d\left(\sqrt[2]{(a_i-b_i)^2}\right)\right)$$

$$\# \text{ natural measure for } \mathbb{R}^d \tag{157}$$

$$\mathit{measurableMap}(f,(M,\sigma_M,N,\sigma_N)) \iff \Big(\mathit{measurableSpace}((M,\sigma_M),())\Big) \wedge \Big(\mathit{measurableSpace}((N,\sigma_N),())\Big) \wedge \Big(\forall_{B\in\sigma_N}\Big(\mathit{preimage}(A,(B,f,M,N)) \in \sigma_M\Big)\Big)$$
$$\# \text{ preimage of measurable sets are measurable} \tag{158}$$

$$\mathit{pushForwardMeasure}(f\star\lambda_M,(f,M,\sigma_M,\mu_M,N,\sigma_N)) \iff \Big(\mathit{measureSpace}((M,\sigma_M,\mu_M),())\Big) \wedge \Big(\mathit{measurableSpace}((N,\sigma_N),())\Big) \wedge \Big(\mathit{measurableMap}(f,(M,\sigma_M,N,\sigma_N))\Big) \wedge \left(\forall_{B\in N}\left(f\star\lambda_M(B)=\mu_M\Big(\mathit{preimage}(A,(B,f,M,N))\Big)\right)\right) \wedge \Big(\mathit{measure}(f\star\lambda_M,(N,\sigma_N))\Big)$$
$$\# \text{ natural construction of a measure based primarily on measurable map} \tag{159}$$

$$\mathit{nullSet}(A,(M,\sigma,\mu)) \iff \Big(\mathit{measureSpace}((M,\sigma,\mu),())\Big) \wedge (A\in\sigma) \wedge \big(\mu(A)=0\big) \tag{160}$$

$$\mathit{almostEverywhere}(p,(M,\sigma,\mu)) \iff \Big(\mathit{measureSpace}((M,\sigma,\mu),())\Big) \wedge \Big(\mathit{predicate}(p,(M))\Big) \wedge \Big(\exists_{A\in\sigma}\Big(\mathit{nullSet}(A,(M,\sigma,\mu)) \implies \forall_{m\in M\setminus A}(p(m))\Big)\Big)$$
$$\# \text{ the predicate holds true for all points except the points in the null set} \tag{161}$$

1.14 Lebesque integration

$$\mathit{simpleTopology}(\mathcal{O}_{simple},()) \iff \mathcal{O}_{simple} = \mathit{subsetTopology}\left(\mathcal{O}|_{\mathbb{R}_0^+},\left(\mathbb{R},\mathit{standardTopology},\mathbb{R}_0^+\right)\right) \tag{162}$$

$$\mathit{simpleSigma}(\sigma_{simple},()) \iff \mathit{borelSigmaAlgebra}\left(\sigma_{simple},\left(\mathbb{R}_0^+,\mathit{simpleTopology}\right)\right) \tag{163}$$

$$\mathit{simpleFunction}(s,(M,\sigma)) \iff \left(\mathit{measurableMap}\left(s,\left(M,\sigma,\mathbb{R}_0^+,\mathit{simpleSigma}\right)\right)\right) \wedge \left(\mathit{finiteSet}\left(\mathit{image}\left(B,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right)$$
$$\# \text{ if the map takes on finitely many values on } \mathbb{R}_0^+ \tag{164}$$

$$\mathit{characteristicFunction}(X_A,(A,M)) \iff (A\subseteq M) \wedge \Big(\mathit{map}(X_A,(M,\mathbb{R}))\Big) \wedge \left(\forall_{m\in M}\left(X_A(m)=\begin{cases}1 & m\in A\\ 0 & m\notin A\end{cases}\right)\right) \tag{165}$$

$$(\text{THM}) : \mathit{simpleFunction}(s,(M,\sigma_M)) \implies \left(\mathit{finiteSet}\left(\mathit{image}\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right) \wedge$$

$$\left(characteristicFunction\left(X_A,(A,M)\right)\right)\wedge\left(\forall_{m\in M}\left(s(m)=\sum_{z\in Z}\left(z\cdot X_{preimage\left(A,(\{z\},s,M,\mathbb{R}_0^+)\right)}(m)\right)\right)\right) \tag{166}$$

$$exStandardSigma\left(\overline{\sigma_s},()\right)\iff\overline{\sigma_s}=\{A\subseteq\overline{\mathbb{R}}\,|\,A\cap R\in standardSigma\}$$
$$\text{\# ignores }\pm\infty\text{ to preserve the points in the domain of the measurable map} \tag{167}$$

$$nonNegIntegrable\left(f,(M,\sigma)\right)\iff measurableMap\left(f,\left(M,\sigma,\overline{\mathbb{R}},exStandardSigma\right)\right)\wedge\left(\forall_{m\in M}\left(f(m)\geq 0\right)\right) \tag{168}$$

$$nonNegIntegral\left(\int_M(fd\mu),(f,M,\sigma,\mu)\right)\iff\left(measureSpace((M,\sigma,\mu),())\right)\wedge\left(measureSpace\left(\left(\overline{\mathbb{R}},exStandardSigma,lebesgueMeasure\right),()\right)\right)\wedge\left(measurableMap\left(f,\left(M,\sigma,\overline{\mathbb{R}},\overline{\sigma_s}\right)\right)\right)\wedge(\int_M(fd\mu)=sup(\{\sum_{z\in Z}\left(z\cdot\mu\left(preimage\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)\right)\right)\,|\,\forall_{m\in M}(s(m)\leq f(m))\wedge simpleFunction(s,(M,\sigma))\wedge finiteSet\left(image\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\}))$$
$$\text{\# lebesgue measure on } z \text{ reduces to } z \tag{169}$$

$$explicitIntegral\iff\int(f(x)\mu(dx))=\int(fd\mu)$$
$$\text{\# alternative notation for lebesgue integrals} \tag{170}$$

$$\text{(THM)}: nonNegIntegral\left(\int(fd\mu),(f,M,\sigma,\mu)\right)\wedge nonNegIntegral\left(\int(gd\mu),(g,M,\sigma,\mu)\right)\implies$$
$$\text{(THM) Markov inequality: }\left(\forall_{z\in\mathbb{R}_0^+}\left(\int(fd\mu)\geq z\cdot\mu\left(preimage\left(A,\left(\{z\},f,M,\overline{\mathbb{R}}\right)\right)\right)\right)\right)\wedge\left(almostEverywhere(f=g,(M,\sigma,\mu))\implies\int(fd\mu)=\int(gd\mu)\right)\left(\int(fd\mu)=0\implies almostEverywhere(f=0,(M,\sigma,\mu))\right)\wedge\left(\int(fd\mu)\leq\infty\implies almostEverywhere(f<\infty,(M,\sigma,\mu))\right) \tag{171}$$

$$\text{(THM) Mono. conv.: }\left((f)_{\mathbb{N}}=\{f_n\,|\,\wedge measurableMap\left(f_n,\left(M,\sigma,\overline{R},exStandardSigma\right)\right)\wedge 0\leq f_{n-1}\leq f_n\}\right)\wedge\left(map\left(f,\left(M,\overline{\mathbb{R}}\right)\right)\right)\wedge\left(\forall_{m\in M}\left(f(m)=sup(f_n(m)|f_n\in(f)_{\mathbb{N}})\right)\right)\implies\left(\lim_{n\to\infty}\left(\int_M(f_nd\mu)\right)=\int_M(fd\mu)\right)$$
$$\text{\# lengths now depend on } M,\sigma \text{ and limits can be pulled in or out of an integral} \tag{172}$$

$$\text{(THM)}: nonNegIntegral\left(\int(fd\mu),(f,M,\sigma,\mu)\right)\wedge nonNegIntegral\left(\int(gd\mu),(g,M,\sigma,\mu)\right)\implies\left(\forall_{\alpha\in\mathbb{R}_0^+}\left(\int((f+\alpha g)d\mu)=\int(fd\mu)+\alpha\int(gd\mu)\right)\right) \tag{173}$$

$$\# \text{ integral acts linearly and commutes finite summations} \tag{173}$$

$$\text{(THM)}: \left((f)_{\mathbb{N}} = \{ f_n \mid \wedge measurableMap\left(f_n, \left(M, \sigma, \overline{R}, exStandardSigma\right)\right) \wedge 0 \le f_n \} \right) \implies \left(\int \left(\left(\sum_{n=1}^{\infty} (f_n) \right) d\mu \right) = \sum_{n=1}^{\infty} \left(\int (f_n d\mu) \right) \right)$$
$$\# \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n\to\infty} \sum_{i=1}^{n} f_n \text{ since } f_n \ge 0 \text{ and it commutes with integral from monotone conv.} \tag{174}$$

$$49:05 Section 3: Integrable(without non) \tag{175}$$

2 Statistics

2.1 Overview

$$randomExperiment(X, (\Omega)) \iff \forall_{\omega \in \Omega}(outcome(\omega, (X))) \tag{176}$$

$$sampleSpace(\Omega, (X)) \iff \Omega = \{\omega | outcome(\omega, (X))\} \tag{177}$$

$$event(A, (\Omega)) \implies A \subseteq \Omega \;\#\text{ that is of interest} \tag{178}$$

$$eventOccured(A, (\omega, \Omega)) \iff \omega \in A, \Omega \wedge event(A, (\Omega)) \tag{179}$$

$$\begin{aligned} algebra(\mathcal{F}_0, (\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega)) \wedge \\ (\Omega \in \mathcal{F}_0) \wedge \\ (\forall_{A \in \mathcal{F}_0}(A^C \in \mathcal{F}_0)) \wedge \\ (\forall_{A,B \in \mathcal{F}_0}(A \cup B \in \mathcal{F}_0)) \\ \#\text{ but this is unable to capture some countable events} \end{aligned} \tag{180}$$

$$\begin{aligned} \sigma\text{-}algebra(\mathcal{F}, (\Omega)) \iff (\mathcal{F}_0 \subseteq \mathcal{P}(\Omega)) \wedge \\ (\Omega \in \mathcal{F}) \wedge \\ (\forall_{A \in \mathcal{F}}(A^C \in \mathcal{F})) \wedge \\ (\forall_{F \subseteq \mathcal{F}}(\neg uncountablyInfinite(F, ()) \implies \cup F \in \mathcal{F})) \end{aligned} \tag{181}$$

3 Statistical Learning Theory

3.1 Overview

$$\tag{182}$$

$$curve - fitting/explaining \neq prediction \tag{183}$$

$$ill - defined problem + solution space constraints \implies well - defined problem \tag{184}$$

$$x \text{ \# input ; } y \text{ \# output} \tag{185}$$

$$S_n = \{(x_1, y_1), \ldots, (x_n, y_n)\} \text{ \# training set} \tag{186}$$

$$f_S(x) \sim y \text{ \# solution} \tag{187}$$

$$each(x,y) \in p(x,y) \text{ \# training data } x,y \text{ is a sample from an unknown distribution } p \tag{188}$$

$$V(f(x),y) = d(f(x),y) \text{ \# loss function} \tag{189}$$

$$I[f] = \int_{X \times Y} V(f(x),y)p(x,y)dxdy \text{ \# expected error} \tag{190}$$

$$I_n[f] = \frac{1}{n} \sum_{i=1}^{n} V(f(x_i), y_i) \text{ \# empirical error} \tag{191}$$

$$probabilisticConvergence(X, ()) \iff \forall_{\epsilon > 0} \lim_{n \to \infty} Pxn - x \leq \epsilon = 0 \tag{192}$$

$$I - Ingeneralizationerror \tag{193}$$

$$well-posed := exists, unique, stable; elseill-posed \tag{194}$$

3.2 Background maths

$$\begin{aligned}
vectorSpace(V, (+, *)) \iff (u, v, w \in V), (c, d \in \mathbb{R} \in F) \wedge \\
(u + v, c * u = c(u) = cu \in V) \wedge \\
(u + v = v + u) \wedge \\
((u + v) + w = u + (v + w)) \wedge \\
(\exists_{\boldsymbol{0}} (u + \boldsymbol{0} = u)) \wedge \\
(\exists_{-u} (u + (-u) = \boldsymbol{0})) \wedge \\
((1)u = u) \\
((cd)u = c(du)) \wedge \\
((c + d)u = cu + du) \wedge \text{ \# linearity} \\
(c(u + v) = cu + cv) \wedge \text{ \# linearity} \\
\text{\# behaves similar to vectors}
\end{aligned} \tag{195}$$

$$\begin{aligned}
innerProduct(\langle \cdot, \cdot \rangle, (V)) \iff (u, v, w \in V), (c \in \mathbb{R} \in F) \wedge \\
(\langle v, w \rangle = \langle w, v \rangle) \wedge \\
(\langle cv, w \rangle = c \langle v, w \rangle) \wedge \\
(\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle) \wedge \text{ \# linearity} \\
(\langle u, u \rangle \geq 0 \in \mathbb{R}_0^+) \wedge \text{ \# metric inducing} \\
(\langle u, u \rangle = 0 \iff u = \boldsymbol{0})
\end{aligned} \tag{196}$$

$$\begin{aligned}
innerProductNorm(|| \cdot ||, (V)) \iff (v, w \in V), (r \in R) \wedge \\
(||v|| = \sqrt{\langle v, v \rangle} \in \mathbb{R}_0^+) \wedge \\
(||v|| = 0 \iff v = \boldsymbol{0}) \wedge
\end{aligned}$$

$$(||rv|| = |r|||v||) \wedge$$
$$(||v+w|| \leq ||v|| + ||w||) \text{ \# triangle inequality} \quad (197)$$

$$normConvergences(v,(V,(v_n)_{n\in\mathbb{N}})) \iff (\{v\} \cup (v_n)_{n\in\mathbb{N}} \subseteq V) \wedge$$
$$(\lim_{n\to\infty} ||v - v_n|| = 0) \quad (198)$$

$$cauchySequence((v_n)_{n\in\mathbb{N}},(V)) \iff$$
$$(\forall_{\epsilon>0} \exists_{n\in\mathbb{N}} \forall_{x,y>n}(||v_x - v_y|| < \epsilon)) \quad (199)$$

$$normConvergences \implies cauchySequence \text{ \# there might be holes in the space} \quad (200)$$

$$completeSpace(V,(innerProductNorm)) \iff (cauchySequence \iff normConvergences) \quad (201)$$

$$completion(R,(Q)) \iff R = QUcauchyUs = Qbar \quad (202)$$

$$hilbertSpace(H,(+,*,\langle\cdot,\cdot\rangle)) \iff (vectorSpace(H,(+,*))) \wedge$$
$$(innerProduct(\langle\cdot,\cdot\rangle,(H))) \wedge$$
$$completeSpace(H,(innerProductNorm)) \quad (203)$$

$$separable(H,()) \iff \exists_{S\subseteq V}(countable(S,()) \wedge Sbar = V) \text{ \# has a countable basis} \quad (204)$$

$$hilbertSpace \wedge seperable \iff \exists countableortho(gonal)normalbasisforspace, allnorm = 1, IP = 0 \quad (205)$$

$$x = \sum < x,v > v \text{ \# countable projection times v} \quad (206)$$

$$0000000000 \quad (207)$$

$$linearOperator(L,(V)) \iff (u,v \in V),(c,d \in \mathbb{R}) \wedge$$
$$(L(cu+dv) = cL(u) + dL(v)) \quad (208)$$

$$adjoint(L^\dagger,(L,V)) \iff (\forall_{u,v\in V} < L(u),v > = < u,L^\dagger(v) >_\dagger) \quad (209)$$

$$selfAdjoint(L,()) \iff L = L^\dagger \quad (210)$$

$$eigenvector(V) \iff Lv = kv \quad (211)$$

$$30mins \quad (212)$$

4 Machine Learning

4.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X,Y) \text{ \# dataset} \quad (213)$$

$$\textbf{learned parameters} = \textbf{parameters to be fixed by training with the dataset} \tag{214}$$

$$\textbf{hyperparameters} = \textbf{parameters that depends on a dataset} \tag{215}$$

$$\textbf{validation} = \textbf{partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition} \text{ \# useful for fixing hyperparameters} \tag{216}$$

$$\textbf{cross-validation} = \textbf{average accuracy of validation for different choices of testing partition} \tag{217}$$

$$\textbf{L1} = \textbf{scales linearly} \; ; \; \textbf{L2} = \textbf{scales quadratically} \tag{218}$$

$$d = \textbf{distance} = \textbf{quantifies the the similarity between data points} \tag{219}$$

$$d_{L1}(A,B) = \sum_p |A_p - B_p| \text{ \# Manhattan distance} \tag{220}$$

$$d_{L2}(A,B) = \sqrt{\sum_p (A_p - B_p)^2} \text{ \# Euclidean distance} \tag{221}$$

$$\textbf{kNN classifier} = \textbf{classifier based on } k \textbf{ nearest data points} \tag{222}$$

$$s = \textbf{class score} = \textbf{quantifies bias towards a particular class} \tag{223}$$

$$s_{linear} = f_{c\times 1}(x_{n\times 1}, W_{c\times n}, b_{c\times 1}) = W_{c\times n} x_{n\times 1} + b_{c\times 1} \text{ \# linear score function} \tag{224}$$

$$l = \textbf{loss} = \textbf{quantifies the errors by the learned parameters} \tag{225}$$

$$l = \frac{1}{|c_i|} \sum_{c_i} l_i \text{ \# average loss for all classes} \tag{226}$$

$$l_{SVM_i} = \sum_{y_i \neq c_i} \max(0, s_{y_i} - s_{c_i} + 1) \text{ \# SVM hinge class loss function:}$$
$$\text{\# ignores incorrect classes with lower scores including a non-zero margin} \tag{227}$$

$$l_{MLR_i} = -\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ \# Softmax class loss function}$$
$$\text{\# lower scores correspond to lower exponentiated-normalized probabilities} \tag{228}$$

$$R = \textbf{regularization} = \textbf{optimizes the choice of learned parameters to minimize test error} \tag{229}$$

$$\lambda \text{ \# regularization strength hyperparameter} \tag{230}$$

$$R_{L1}(W) = \sum_{W_i} |W_i| \text{ \# L1 regularization} \tag{231}$$

$$R_{L2}(W) = \sum_{W_i} {W_i}^2 \text{ \# L2 regularization} \tag{232}$$

$$L' = L + \lambda R(W) \text{ \# weight regularization} \tag{233}$$

$$\nabla_W L = \overrightarrow{\frac{\partial}{\partial W_i}} L = \textbf{loss gradient w.r.t. weights} \tag{234}$$

$$\frac{\partial L_E}{\partial W_I} = \frac{\partial L_L}{\partial W_I} \frac{\partial L_E}{\partial L_L} \text{ \# loss gradient w.r.t. input weight in terms of external and local gradients} \tag{235}$$

$$s = \textbf{forward API} \text{ ; } \frac{\partial L_L}{\partial W_I} = \textbf{backward API} \tag{236}$$

$$W_{t+1} = W_t - \nabla_{W_t} L \text{ \# weight update loss minimization} \tag{237}$$

$$\textbf{TODO:Research on Activation functions, Weight Initialization, Batch Normalization} \tag{238}$$

$$review5meanvardiscussion/hyperparameteroptimization/babysittinglearning \tag{239}$$

TODO loss L or l ??