# Next-Next-Gen Notes

Object-Oriented Maths

JP Guzman

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Format: $characteristic((subjects),(dependencies)) \iff (conditions(dependencies)) \wedge (conditions(subjects))$
Note: All weaker objects automatically induces notions inherited from stronger objects.
TODO define || abs cross-product and other missing refs
TODO distinguish new condition vs implied proposition
TODO link thms?

## 1 Mathematical Analysis

#### 1.0.1 Formal Logic

$$statement(s,(RegEx)) \iff well\text{-}formedString(s,()) \tag{1}$$

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$$\begin{aligned} proposition((p,t),()) \iff \Big(statement(p,())\Big)\wedge \\ (t = eval(p))\wedge \\ (t = true \veebar t = false) \end{aligned} \tag{2}$$

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$$operator\Big(o,\big((p)_{n\in\mathbb{N}}\big)\Big) \iff proposition\Big(o\big((p)_{n\in\mathbb{N}}\big),()\Big) \tag{3}$$

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$$\begin{aligned} operator(\neg,(p_1)) \iff \Big(proposition((p_1,true),()) \implies ((\neg p_1,false),())\Big)\wedge \\ \Big(proposition((p_1,false),()) \implies ((\neg p_1,true),())\Big) \\ \text{\# an operator takes in propositions and returns a proposition} \end{aligned} \tag{4}$$

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$$\begin{aligned} operator(\neg) \iff \textbf{NOT} \;;\; operator(\vee) \iff \textbf{OR} \;;\; operator(\wedge) \iff \textbf{AND} \;;\; operator(\veebar) \iff \textbf{XOR} \\ operator(\implies) \iff \textbf{IF} \;;\; operator(\impliedby) \iff \textbf{OIF} \;;\; operator(\iff) \iff \textbf{IFF} \end{aligned} \tag{5}$$

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$$\begin{aligned} proposition((false \implies true),true,()) \wedge proposition((false \implies false),true,()) \\ \text{\# truths based on a false premise is not false; ex falso quodlibet principle} \end{aligned} \tag{6}$$

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$$\text{(THM)} : (a \implies b \implies c) \iff (a \implies (b \implies c)) \iff ((a \wedge b) \implies c) \tag{7}$$

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$$predicate(P,(V)) \iff \forall_{v\in V}\Big(proposition\Big((P(v),t),()\Big)\Big) \tag{8}$$

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$$\begin{aligned} 0thOrderLogic(P,()) \iff proposition((P,t),()) \\ \text{\# individual proposition} \end{aligned} \tag{9}$$

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$$1stOrderLogic(P,(V)) \iff \Big(\forall_{v\in V}\Big(0thOrderLogic(v,())\Big)\Big)\wedge$$

$$\left(\forall_{v\in V}\left(proposition\Big(\big(P(v),t\big),()\Big)\right)\right)$$

$$\text{\# propositions defined over a set of the lower order logical statements} \tag{10}$$

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$$quantifier\big(q,(p,V)\big) \iff \Big(predicate\big(p,(V)\big)\Big)\wedge$$
$$\Big(proposition\Big(\big(q(p),t\big),()\Big)\Big)$$
$$\text{\# a quantifier takes in a predicate and returns a proposition} \tag{11}$$

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$$quantifier\big(\forall,(p,V)\big) \iff proposition\Big(\Big(\wedge_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\text{\# universal quantifier} \tag{12}$$

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$$quantifier\big(\exists,(p,V)\big) \iff proposition\Big(\Big(\vee_{v\in V}\big(p(v)\big),t\Big),()\Big)$$
$$\text{\# existential quantifier} \tag{13}$$

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$$quantifier\big(\exists!,(p,V)\big) \iff \exists_{x\in V}\Big(P(x)\wedge\neg\Big(\exists_{y\in V\setminus\{x\}}\big(P(y)\big)\Big)\Big)$$
$$\text{\# uniqueness quantifier} \tag{14}$$

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$$\text{(THM)} : \forall_x p(x) \iff \neg\exists_x\neg p(x)$$
$$\text{\# De Morgan's law} \tag{15}$$

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$$\text{(THM)} : \forall_x\exists_y p(x,y)=\forall_x\neg\forall_y\neg p(x,y)\neq\exists_y\forall_x p(x,y)=\neg\forall_y\neg\big(\forall_x p(x,y)\big)=\neg\forall_y\exists_x\neg p(x,y)$$
$$\text{\# different quantifiers are not interchangeable} \tag{16}$$

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$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{17}$$

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$$\textbf{proof=truths derived from a finite number of axioms and deductions} \tag{18}$$

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$$\textbf{elementary arithmetics=system with substitutions, and some notion of}$$
$$\textbf{addition, multiplication, and prime nuumbers for encoding metamathematics} \tag{19}$$

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$$\textbf{Gödel theorem} \implies \textbf{axiomatic systems equivalent in power to elementary mathematics}$$
$$\textbf{either has unprovable statements or has contradictions} \tag{20}$$

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$$sequenceSet\big((A)_{\mathbb{N}},(A)\big) \iff (Amapinputn)\big((A)_{\mathbb{N}}=\{A(1),A(2),A(3),\ldots\}\big) \tag{21}$$

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$$\textbf{TODO: define union, intersection, complement, etc.} \tag{22}$$

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$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{23}$$

## 1.1 Axiomatic Set Theory

$$\text{============} \; \textbf{N O T = U P D A T E D} \; \text{============} \tag{24}$$

$$\textbf{ZFC set theory} = \textbf{standard form of axiomatic set theory} \tag{25}$$

$$A \subseteq B = \forall_x x \in A \implies x \in B \tag{26}$$

$$(A = B) = A \subseteq B \wedge B \subseteq A \tag{27}$$

$$\in \textbf{ basis} \implies \{x,y\} = \{y,x\} \wedge \{x\} = \{x,x\} \tag{28}$$

$$\in \textbf{ and sets works following the 9 ZFC axioms:} \tag{29}$$

$$\forall_x \forall_y \big(x \in y \veebar \neg(x \in y)\big) \text{ \# E: } \in \text{ is only a proposition on sets} \tag{30}$$

$$\exists_\emptyset \forall_y \neg y \in \emptyset \text{ \# E: existence of empty set} \tag{31}$$

$$\forall_x \forall_y \exists_m \forall_u u \in m \iff u = x \vee u = y \text{ \# C: pair set construction} \tag{32}$$

$$\forall_s \exists_u \forall_x \forall_y (x \in s \wedge y \in x \implies y \in u) \text{ \# C: union set construction} \tag{33}$$

$$x = \{\{a\},\{b\}\} \text{ \# from the pair set axiom} \tag{34}$$

$$u = \cup x = \cup \{\{a\},\{b\}\} = \{a,b\} \tag{35}$$

$$\forall_x \exists!_y R(x,y) \text{ \# functional relation } R \tag{36}$$

$$\begin{gathered} \exists_i \forall_x \exists!_y R(x,y) \implies y \in i \text{ \# C: image } i \text{ of set } m \text{ under a relation } R \text{ is assumed to be a set} \\ \implies \{y \in m \,|\, P(y)\} \text{ \# Restricted Comprehension} \not\Longrightarrow \{y \,|\, P(y)\} \text{ \# Universal Comprehension} \end{gathered} \tag{37}$$

$$\forall_{x \in m} P(x) = \forall_x \big(x \in m \implies P(x)\big) \text{ \# ignores out of scope} \neq \forall_x \big(x \in m \wedge P(x)\big) \text{ \# restricts entirety} \tag{38}$$

$$\forall_m \forall_n \exists_{\mathcal{P}(m)} \big(n \subseteq m \implies n \subseteq \mathcal{P}(m)\big) \text{ \# C: existence of power set} \tag{39}$$

$$\exists_I \Big(\emptyset \in I \wedge \forall_{x \in I} \big(\{x\} \in I\big)\Big) \text{ \# I: axiom of infinity ; } I = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}; I \cong \mathbb{N} \implies \mathbb{N} \textbf{ is a set} \tag{40}$$

$$\forall_x \Big(\big(\emptyset \notin x \wedge x \cap x' = \emptyset\big) \implies \exists_y (\textbf{set of each e} \in x)\Big) \text{ \# C: axiom of choice} \tag{41}$$

$$\forall_x x \neq \emptyset \implies x \notin x \text{ \# F: axiom of foundation covers further paradoxes} \tag{42}$$

$$\text{============} \; \textbf{N O T = U P D A T E D} \; \text{============} \tag{43}$$

## 1.2 Classification of sets

$$space\big((set, structure), ()\big) \iff structure(set)$$
$$\text{\# a space a set equipped with some structure}$$
$$\text{\# various spaces can be studied through structure preserving maps between those spaces} \tag{44}$$

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$$map\big(\phi, (A, B)\big) \iff \Big(\forall_{a \in A} \exists!_{b \in B} \big(\phi(a, b)\big)\Big) \vee$$
$$\Big(\forall_{a \in A} \exists!_{b \in B} \big(b = \phi(a)\big)\Big)$$
$$\text{\# maps elements of a set to elements of another set} \tag{45}$$

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$$domain\big(A, (\phi, A, B)\big) \iff \Big(map\big(\phi, (A, B)\big)\Big) \tag{46}$$

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$$codomain\big(B, (\phi, A, B)\big) \iff \Big(map\big(\phi, (A, B)\big)\Big) \tag{47}$$

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$$image\big(B, (A, q, M, N)\big) \iff \Big(map\big(q, (M, N)\big) \wedge A \subseteq M\Big) \wedge$$
$$\Big(B = \{n \in N \,|\, \exists_{a \in A} \big(q(a) = n\big)\}\Big) \tag{48}$$

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$$preimage\big(A, (B, q, M, N)\big) \iff \Big(map\big(q, (M, N)\big) \wedge B \subseteq N\Big) \wedge$$
$$\Big(A = \{m \in M \,|\, \exists_{b \in B} \big(b = q(m)\big)\}\Big) \tag{49}$$

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$$injection\big(q, (M, N)\big) \iff \Big(map\big(q, (M, N)\big)\Big) \wedge$$
$$\forall_{u, v \in M} \big(q(u) = q(v) \implies u = v\big)$$
$$\text{\# every } m \text{ has at most 1 image} \tag{50}$$

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$$surjection\big(q, (M, N)\big) \iff \Big(map\big(q, (M, N)\big)\Big) \wedge$$
$$\forall_{n \in N} \exists_{m \in M} \big(n = q(m)\big)$$
$$\text{\# every } n \text{ has at least 1 preimage} \tag{51}$$

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$$bijection\big(q, (M, N)\big) \iff \Big(injection\big(q, (M, N)\big)\Big) \wedge$$
$$\Big(surjection\big(q, (M, N)\big)\Big)$$
$$\text{\# every unique } m \text{ corresponds to a unique } n \tag{52}$$

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$$isomorphicSets\big((A, B), ()\big) \iff \exists_{\phi} \Big(bijection\big(\phi, (A, B)\big)\Big) \tag{53}$$

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$$infiniteSet\big(S, ()\big) \iff \exists_{T \subset S} \Big(isomorphicSets\big((T, S), ()\big)\Big) \tag{54}$$

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$$finiteSet\big(S, ()\big) \iff \Big(\neg infiniteSet\big(S, ()\big)\Big) \vee \big(|S| \in \mathbb{N}\big) \tag{55}$$

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$$countablyInfinite\big(S, ()\big) \iff \Big(infiniteSet\big(S, ()\big)\Big) \wedge \Big(isomorphicSets\big((S, \mathbb{N}), ()\big)\Big) \tag{56}$$

$$uncountablyInfinite\big(S,()\big) \iff \Big(infiniteSet\big(S,()\big)\Big) \land \Big(\neg isomorphicSets\big((S,\mathbb{N}),()\big)\Big) \quad (57)$$

$$inverseMap\Big(q^{-1},(q,M,N)\Big) \iff \Big(bijection\big(q,(M,N)\big)\Big)\land \Big(map\Big(q^{-1},(N,M)\Big)\Big)\land \Big(\forall_{n\in N}\exists!_{m\in M}\Big(q(m)=n \implies q^{-1}(n)=m\Big)\Big) \quad (58)$$

$$mapComposition\big(\phi\circ\psi,(\phi,\psi,A,B,C)\big) \iff map\big(\psi,(A,B)\big)\land map\big(\phi,(B,C)\big)\land \forall_{a\in A}\Big(\phi\circ\psi(a)=\phi\big(\psi(a)\big)\Big) \quad (59)$$

$$equivalenceRelation\Big(\sim(\$1,\$2),(M)\Big) \iff \big(\forall_{m\in M}(m\sim m)\big)\land \big(\forall_{m,n\in M}(m\sim n \implies n\sim m)\big)\land \big(\forall_{m,n,p\in M}(m\sim n\land n\sim p \implies m\sim p)\big)$$

# behaves as equivalences should (60)

$$equivalenceClass\big([m]_\sim,(m,M,\sim)\big) \iff [m]_\sim=\{n\in M\,|\,n\sim m\}$$

# set of elements satisfying the equivalence relation with $m$ (61)

$$(\text{THM}) : a\in[m]_\sim \implies [a]_\sim=[m]_\sim \ ; \ [m]_\sim=[n]_\sim \veebar [m]_\sim\cap[n]_\sim=\emptyset$$

# equivalence class properties (62)

$$quotientSet\big(M/\sim,(M,\sim)\big) \iff M/\sim=\{equivalenceClass\big([m]_\sim,(m,M,\sim)\big)\in\mathcal{P}(M)\,|\,m\in M\}$$

# set of all equivalence classes (63)

$$(\text{THM}) : \Big((M,\sim,+) \implies \Big(quotientSet\big(M/\sim,(M,\sim)\big),+_\sim\Big)\Big) \iff \forall_{[r],[s]\in M/\sim}\forall_{a\in[r]}\forall_{b\in[s]}\big([r]+_\sim[s]=[a+b]\big)$$

# a quotient set can inherit the operations on the original set if it is well-defined (64)

$$(\text{THM}) : \textbf{axiom of choice} \implies \forall_{[m]_\sim\in M/\sim}\exists_r\big(r\in[m]_\sim\big)$$

# well-defined maps may be defined in terms of chosen representative elements $r$ (65)

## 1.3 Construction of number sets

============ **N O T = U P D A T E D** ============ (66)

$$\textbf{axiom of infinity} \implies \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\ldots\}\cong\mathbb{N} \quad (67)$$

$$\mathbb{N}^\star=\mathbb{N}\backslash\{0\} \quad (68)$$

**addition = successor map:** $\mathbb{N}\to\mathbb{N}=S(n)=\{n\}$ # adds a layer of brackets (69)

**subtraction = predecessor map:** $\mathbb{N}^\star\to\mathbb{N}=P(n)=m\,|\,m\in n$ # removes a layer of brackets (70)

$$S^0 = id \; ; \; n \in \mathbb{N}^\star \implies S^n = S \circ S^{P(n)} \tag{71}$$

$$\textbf{addition} = + : \mathbb{N} \times \mathbb{N} \to \mathbb{N} = +(m,n) = m+n = S^n(m) \tag{72}$$

$$S^x = id = S^0 \implies x = \textbf{additive identity} = 0 \tag{73}$$

$$S^n(x) = 0 \implies x = \textbf{additive inverse} \notin \mathbb{N} \text{ \# git gud smh -\_-} \tag{74}$$

$$\begin{gathered}\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim \textbf{, s.t.:} \\ (m,n) \sim (p,q) \iff m+q = p+n \text{ \# span } \mathbb{Z} \text{ using differences then group equal differences}\end{gathered} \tag{75}$$

$$\mathbb{N} \hookrightarrow \mathbb{Z} : \forall_{n \in \mathbb{N}} n \to [(n,0)] \text{ \# } \mathbb{N} \text{ embedded in } \mathbb{Z} \tag{76}$$

$$+_\mathbb{Z} = [(m +_\mathbb{N} p, n +_\mathbb{N} q)] \text{ \# well-defined and consistent} \tag{77}$$

$$\textbf{multiplication} \ldots M^x = id \implies x = \textbf{multiplicative identity} = 1 \ldots \textbf{multiplicative inverse} \notin \mathbb{N} \tag{78}$$

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^\star) / \sim \textbf{, s.t.: } (x,y) \sim (u,v) \iff x \cdot v = u \cdot y \tag{79}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \forall_{q \in \mathbb{Q}} q \to [(q,1)] \; ; \; \ldots \{x \,|\, x^2 = 2\} \notin \mathbb{Q} \tag{80}$$

$$\mathbb{R} = \textbf{almost homomorphisms on } \mathbb{Z} / \sim \text{ \# http://blog.sigfpe.com/2006/05/defining-reals.html} \tag{81}$$

$$============ \textbf{ N O T = U P D A T E D } ============ \tag{82}$$

## 1.4 Topology

$$\begin{aligned} \textit{topology}\big(\mathcal{O},(M)\big) \iff \big(\mathcal{O} \subseteq \mathcal{P}(M)\big) \land \\ \big(\emptyset, M \in \mathcal{O}\big) \land \\ \Big(\big(F \in \mathcal{O} \land |F| < |\mathbb{N}|\big) \implies \cap F \in \mathcal{O}\Big) \land \\ (C \subseteq \mathcal{O} \implies \cup C \in \mathcal{O}) \\ \text{\# topology is defined by a set of open sets which provide the characteristics needed to define continuity, etc.} \\ \text{\# arbitrary unions of open sets always result in an open set} \\ \text{\# open sets do not contain their boundaries and infinite intersections of open sets may approach and} \\ \text{\# induce boundaries resulting in a closed set} \end{aligned} \tag{83}$$

$$\textit{topologicalSpace}\big((M,\mathcal{O}),()\big) \iff \textit{topology}\big(\mathcal{O},(M)\big) \tag{84}$$

$$\begin{aligned} \textit{open}\big(S,(M,\mathcal{O})\big) \iff \Big(\textit{topologicalSpace}\big((M,\mathcal{O}),()\big)\Big) \land \\ (S \subseteq M) \land (S \in \mathcal{O}) \\ \text{\# an open set do not contains its own boundaries} \end{aligned} \tag{85}$$

$$closed\big(S,(M,\mathcal{O})\big) \iff \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big)\wedge (S\subseteq M)\wedge\big(S\in\mathcal{P}(M)\setminus\mathcal{O}\big)$$
$$\text{\# a closed set contains the boundaries an open set} \quad (86)$$

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$$clopen\big(S,(M,\mathcal{O})\big) \iff \Big(closed\big(S,(M,\mathcal{O})\big)\Big)\wedge\Big(open\big(S,(M,\mathcal{O})\big)\Big) \quad (87)$$

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$$neighborhood\big(U,(a,\mathcal{O})\big) \iff (a\in U\in\mathcal{O})$$
$$\text{\# another name for open set containing } a \quad (88)$$

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$$M=\{a,b,c,d\}\wedge\mathcal{O}=\{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},M\}\implies$$
$$\Big(open\big(X,(M,\mathcal{O})\big)\iff X=\{\emptyset,\{c\},\{a,b\},\{c,d\},\{a,b,c\},M\}\Big)\wedge$$
$$\Big(closed\big(Y,(M,\mathcal{O})\big)\iff Y=\{\emptyset,\{a,b,d\},\{c,d\},\{a,b\},\{d\},M\}\Big)\wedge$$
$$\Big(clopen\big(Z,(M,\mathcal{O})\big)\iff Z=\{\emptyset,\{a,b\},\{c,d\},M\}\Big) \quad (89)$$

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$$chaoticTopology(M)=\{0,M\}\ ;\ discreteTopology=\mathcal{P}(M) \quad (90)$$

## 1.5 Induced topology

$$metric\Big(d(\$1,\$2),(M)\Big)\iff\bigg(map\Big(d,\big(M\times M,\mathbb{R}_0^+\big)\Big)\bigg)$$
$$\Big(\forall_{x,y\in M}\big(d(x,y)=d(y,x)\big)\Big)\wedge$$
$$\Big(\forall_{x,y\in M}\big(d(x,y)=0\iff x=y\big)\Big)\wedge$$
$$\bigg(\forall_{x,y,z}\Big(\big(d(x,z)\le d(x,y)+d(y,z)\big)\Big)\bigg)$$
$$\text{\# behaves as distances should} \quad (91)$$

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$$metricSpace\big((M,d),()\big)\iff metric\big(d,(M)\big) \quad (92)$$

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$$openBall\big(B,(r,p,M,d)\big)\iff\Big(metricSpace\big((M,d),()\big)\Big)\wedge\big(r\in\mathbb{R}^+,p\in M\big)\wedge\big(B=\{q\in M\,|\,d(p,q)<r\}\big) \quad (93)$$

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$$metricTopology\big(\mathcal{O},(M,d)\big)\iff\Big(metricSpace\big((M,d),()\big)\Big)\wedge$$
$$\bigg(\mathcal{O}=\{U\in\mathcal{P}(M)\,|\,\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big)\wedge B\subseteq U\Big)\}\bigg)$$
$$\text{\# every point in the neighborhood has some open ball that is fully enclosed in the neighborhood} \quad (94)$$

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$$metricTopologicalSpace\big((M,\mathcal{O},d),()\big)\iff metricTopology\big(\mathcal{O},(M,d)\big) \quad (95)$$

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$$limitPoint\big(p,(S,M,d)\big)\iff(S\subseteq M)\wedge\forall_{r\in\mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big)\cap S\neq\emptyset\Big)$$
$$\text{\# every open ball centered at } p \text{ contains some intersection with } S \quad (96)$$

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$$interiorPoint\big(p,(S,M,d)\big)\iff(S\subseteq M)\wedge\bigg(\exists_{r\in\mathbb{R}^+}\Big(openBall\big(B,(r,p,M,d)\big)\subseteq S\Big)\bigg)$$

$$\text{\# there is an open ball centered at } p \text{ that is fully enclosed in } S \tag{97}$$

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$$\mathit{closure}\big(\bar{S},(S,M,d)\big) \iff \bar{S}=S\cup\{\mathit{limitPoint}\big(p,(S,M,d)\big)\,|\,p\in M\} \tag{98}$$

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$$\mathit{dense}\big(S,(M,d)\big) \iff (S\subseteq M)\wedge\Bigg(\forall_{p\in M}\Big(p\in\mathit{closure}\big(\bar{S},(S,M,d)\big)\Big)\Bigg)$$
$$\text{\# every of point in } M \text{ is a point or a limit point of } S \tag{99}$$

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$$\mathit{eucD}\big(d,(n)\big) \iff \big(\forall_{i\in\mathbb{N}\wedge i\leq n}(x_i\in\mathbb{R})\big)\wedge\left(d=\sqrt[2]{\sum_{i=1}^{n}{x_i}^2}\right) \tag{100}$$

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$$\mathit{metricTopology}\Bigg(\mathit{standardTopology},\Big(\mathbb{R}^n,\mathit{eucD}\big(d,(n)\big)\Big)\Bigg)$$
$$============\ \textbf{N O T = U P D A T E D}\ ============$$
$$\textbf{L1:}\ \forall_{p\in U=\emptyset}(\ldots)\implies\forall_p\Big((p\in\emptyset)\implies\ldots\Big)\implies\forall_p\big((\textbf{False})\implies\ldots\big)\implies\emptyset\in\mathcal{O}_{standard}$$
$$\textbf{L2:}\ \forall_{p\in\mathbb{R}^n}B(r,p,\mathbb{R}^n,d)\subseteq\mathbb{R}^n\implies M\in\mathcal{O}_{standard}$$
$$\textbf{L4:}\ C\subseteq\mathcal{O}_{standard}\implies\forall_{U\in C}\forall_{p\in U}\exists_{r\in\mathbb{R}^+}\big(B_r(p)\subseteq U\subseteq\cup C\big)\implies\cup C\in\mathcal{O}_{standard}$$
$$\textbf{L3:}\ U,V\in\mathcal{O}_{standard}\implies p\in U\cap V\implies p\in U\wedge p\in V\implies$$
$$\exists_{r\in\mathbb{R}^+}B(r,p,\mathbb{R}^n,d)\wedge\exists_{s\in\mathbb{R}^+}B(s,p,\mathbb{R}^n,d)\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\subseteq U\wedge B\big(min(r,s),q,\mathbb{R}^n,d\big)\subseteq V\implies$$
$$B\big(min(r,s),p,\mathbb{R}^n,eucD\big)\in U\cap V\implies U\cap V\in\mathcal{O}_{standard}$$
$$\text{\# natural topology for } \mathbb{R}^d$$
$$\text{\# could fail on infinite sets since } min \text{ could approach } 0$$
$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{101}$$

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$$\mathit{subsetTopology}\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\iff\mathit{topology}\big(\mathcal{O},(M)\big)\wedge(N\subseteq M)\wedge\big(\mathcal{O}|_N=\{U\cap N\,|\,U\in\mathcal{O}\}\big)$$
$$\text{\# crops open sets outside } N \tag{102}$$

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$$(\text{THM}):\ \mathit{subsetTopology}\big(\mathcal{O}|_N,(M,\mathcal{O},N)\big)\wedge\mathit{topology}\big(\mathcal{O}|_N,(N)\big)\impliedby$$
$$============\ \textbf{N O T = U P D A T E D}\ ============$$
$$\textbf{L1:}\ \emptyset\in\mathcal{O}\implies U=\emptyset\implies\emptyset\cap N=\emptyset\implies\emptyset\in\mathcal{O}|_N$$
$$\textbf{L2:}\ M\in\mathcal{O}\implies U=M\implies M\cap N=N\implies N\in\mathcal{O}|_N$$
$$\textbf{L3:}\ S,T\in\mathcal{O}|_N\implies\exists_{U\in\mathcal{O}}(S=U\cap N)\wedge\exists_{V\in\mathcal{O}}(T=V\cap N)\implies S\cap T=(U\cap N)\cap(V\cap N)$$
$$=(U\cap V)\cap N\wedge U\cap V\in\mathcal{O}\implies S\cap T\in\mathcal{O}|_N$$
$$\textbf{L4:}\ TODO:EXERCISE$$
$$============\ \textbf{N O T = U P D A T E D}\ ============ \tag{103}$$

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$$\mathit{productTopology}\Big(\mathcal{O}_{A\times B},\big((A,\mathcal{O}_A),(B,\mathcal{O}_B)\big)\Big)\iff\Big(\mathit{topology}\big(\mathcal{O}_A,(A)\big)\Big)\wedge\Big(\mathit{topology}\big(\mathcal{O}_B,(B)\big)\Big)\wedge$$
$$\big(\mathcal{O}_{A\times B}=\{(a,b)\in A\times B\,|\,\exists_S(a\in S\in\mathcal{O}_A)\exists_T(b\in T\in\mathcal{O}_B)\}\big)$$
$$\text{\# open in cross iff open in each} \tag{104}$$

### 1.6 Convergence

$$sequence(q,(M)) \iff map(q,(\mathbb{N},M)) \quad (105)$$

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$$sequenceConvergesTo((q,a),(M,\mathcal{O})) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(sequence(q,(M))\Big) \wedge (a \in M) \wedge \Big(\forall_{U \in \mathcal{O} | a \in U} \exists_{N \in \mathbb{N}} \forall_{n > N}(q(n) \in U)\Big)$$

# each neighborhood of $a$ has a tail-end sequence that does not map to outside points (106)

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(THM) : **convergence generalizes to: the sequence** $q : \mathbb{N} \to \mathbb{R}^d$ **converges against** $a \in \mathbb{R}^d$ **in** $\mathcal{O}_S$ **if:**

$$\forall_{r>0} \exists_{N \in \mathbb{N}} \forall_{n>N}\big(||q(n) - a|| < \epsilon\big) \ \text{\# distance based convergence} \quad (107)$$

### 1.7 Continuity

$$continuous(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)) \iff \Big(topologicalSpace((M,\mathcal{O}_M),())\Big) \wedge$$
$$\Big(topologicalSpace((N,\mathcal{O}_N),())\Big) \wedge \Big(\forall_{V \in \mathcal{O}_N}\Big(preimage(A,(V,\phi,M,N)) \in \mathcal{O}_M\Big)\Big)$$

# preimage of open sets are open (108)

---

$$homeomorphism(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N)) \iff \Big(inverseMap\Big(\phi^{-1},(\phi,M,N)\Big)\Big)$$
$$\Big(continuous(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N))\Big) \wedge \Big(continuous\Big(\phi^{-1},(N,\mathcal{O}_N,M,\mathcal{O}_M)\Big)\Big)$$

# structure preserving maps in topology, ability to share topological properties (109)

---

$$isomorphicTopologicalSpace\Big(((M,\mathcal{O}_M),(N,\mathcal{O}_N)),()\Big) \iff$$
$$\exists_\phi\Big(homeomorphism(\phi,(M,\mathcal{O}_M,N,\mathcal{O}_N))\Big) \quad (110)$$

### 1.8 Separation

$$T0Separate((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U \in \mathcal{O}}\Big((x \in U \wedge y \notin U) \vee (y \in U \wedge x \notin U)\Big)\Big)$$

# each pair of points has a neighborhood s.t. one is inside and the other is outside (111)

---

$$T1Separate((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V}\Big((x \in U \wedge y \notin U) \wedge (y \in V \wedge x \notin V)\Big)\Big)$$

# every point has a neighborhood that does not contain another point (112)

---

$$T2Separate((M,\mathcal{O}),()) \iff \Big(topologicalSpace((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\forall_{x,y \in M \wedge x \neq y} \exists_{U,V \in \mathcal{O} \wedge U \neq V}(U \cap V = \emptyset)\Big)$$

# every point has a neighborhood that does not intersect with a nhbhd of another point - Hausdorff space (113)

---

$$(\text{THM}) : \mathit{T2Separate} \Longrightarrow \mathit{T1Separate} \Longrightarrow \mathit{T0Separate} \quad (114)$$

## 1.9 Compactness

$$\mathit{openCover}(C,(M,\mathcal{O})) \Longleftrightarrow \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big) \wedge (C \subseteq \mathcal{O}) \wedge (\cup C = M)$$

\# collection of open sets whose elements cover the entire space (115)

---

$$\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \Longleftrightarrow \Big(\widetilde{C} \subseteq C\Big) \wedge \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big) \wedge$$
$$\Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big) \wedge \Big(\mathit{finiteSet}\Big(\widetilde{C},()\Big)\Big)$$

\# finite subset of a cover that is also a cover (116)

---

$$\mathit{compact}((M,\mathcal{O}),()) \Longleftrightarrow \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big) \wedge$$
$$\bigg(\forall_{C \subseteq \mathcal{O}} \bigg(\mathit{openCover}(C,(M,\mathcal{O})) \Longrightarrow \exists_{\widetilde{C} \subseteq C} \Big(\mathit{finiteSubcover}\Big(\widetilde{C},(C,M,\mathcal{O})\Big)\Big)\bigg)\bigg)$$

\# every covering of the space is represented by a finite number of nhbhds (117)

---

$$\mathit{compactSubset}(N,(M,\mathcal{O})) \Longleftrightarrow \Big(\mathit{compact}((M,\mathcal{O}),())\Big) \wedge$$
$$\Big(\mathit{subsetTopology}(\mathcal{O}|_N,(M,\mathcal{O},N))\Big) \wedge \Big(\mathit{compact}\Big((N,\mathcal{O}|_N),()\Big)\Big) \quad (118)$$

---

$$\mathit{bounded}(N,(M,d)) \Longleftrightarrow \Big(\mathit{metricSpace}((M,d),())\Big) \wedge (N \subseteq M) \wedge$$
$$\Big(\exists_{r \in \mathbb{R}^+} \forall_{p,q \in n} \big(d(p,q) < r\big)\Big) \quad (119)$$

---

$$(\text{THM}) \text{ Heine-Borel thm.: } \mathit{metricTopologicalSpace}((M,\mathcal{O}_d,d),()) \Longrightarrow$$
$$\forall_{S \subseteq M} \bigg(\Big(\mathit{closed}(S,(M,\mathcal{O}_d)) \wedge \mathit{bounded}(S,(M,\mathcal{O}_d))\Big) \Longleftrightarrow \mathit{compactSubset}(S,(M,\mathcal{O}_d))\bigg)$$

\# when metric topologies are involved, compactness is equivalent to being closed and bounded (120)

## 1.10 Paracompactness

$$\mathit{openRefinement}\Big(\widetilde{C},(C,M,\mathcal{O})\Big) \Longleftrightarrow \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big) \wedge \Big(\mathit{openCover}\Big(\widetilde{C},(M,\mathcal{O})\Big)\Big) \wedge$$
$$\Big(\forall_{\widetilde{U} \in \widetilde{C}} \exists_{U \in C} \Big(\widetilde{U} \subseteq U\Big)\Big)$$

\# a refined cover can be constructed by removing the excess nhbhds and points that lie outside the space (121)

---

$$(\text{THM}) : \mathit{finiteSubcover} \Longrightarrow \mathit{openRefinement} \quad (122)$$

---

$$\mathit{locallyFinite}(C,(M,\mathcal{O})) \Longleftrightarrow \Big(\mathit{openCover}(C,(M,\mathcal{O}))\Big) \wedge$$
$$\forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U} \Big(\mathit{finiteSet}(\{U_c \in C | U \cap U_c \neq \emptyset\},())\Big)$$

\# each point has a neighborhood that intersects with only finitely many sets in the cover (123)

$$paracompact\big((M,\mathcal{O}),()\big) \iff \forall_C \Bigg(openCover\big(C,(M,\mathcal{O})\big) \implies \exists_{\widetilde{C}} \Bigg(locallyFinite\Bigg(openRefinement\Big(\widetilde{C},(C,M,\mathcal{O})\Big),(M,\mathcal{O})\Bigg)\Bigg)\Bigg) \tag{124}$$

# every open cover has a locally finite open refinement

$$\text{(THM)} : metricTopologicalSpace \implies paracompact \tag{125}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{126}$$

$$\begin{aligned} &partitionOfUnitySubjCover\big(\mathcal{F},(C,M,\mathcal{O})\big) \iff \Big(locallyFinite\big(C,(M,\mathcal{O})\big)\Big) \wedge (f \in \mathcal{F}) \wedge \\ &\Bigg(continuous\Bigg(f,\Bigg(M,\mathcal{O},[0,1],subsetTopology\Big(\mathcal{O}|_{[0,1]},\big([0,1],\mathbb{R},standardTopology\big)\Big)\Bigg)\Bigg)\Bigg) \wedge \\ &\Big(\exists_{U_f \in C} \forall_{p \in M} \big(f(p) \neq 0 \implies p \in U_f\big)\Big) \wedge \\ &\Big(\forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U} \big((f_U)_n = \{f \in \mathcal{F} | p \in M \wedge f(p) \neq 0\}\big)\Big) \wedge \\ &\Big(locallyFinite(C,M,\mathcal{O}) \implies finiteSet\big((f_U)_n,()\big)\Big) \wedge \\ &\Bigg(\forall_{p \in M} \exists_{U \in \mathcal{O} | p \in U} \Bigg(\sum_{i=1}^{|(f_U)_n|} (f_U)_i(p) = 1\Bigg)\Bigg) \end{aligned} \tag{127}$$

# useful for defining integrals between overlapping neighborhoods

$$T2Separate\big((M,\mathcal{O}),()\big) \implies \Big(paracompact\big((M,\mathcal{O}),()\big)\Big) \iff \forall_C \Big(openCover\big(C,(M,\mathcal{O})\big) \implies partitionOfUnitySOTCover\big(\mathcal{F},(C,M,\mathcal{O})\big)\Big) \tag{128}$$

$$============ \textbf{N O T = U P D A T E D} ============ \tag{129}$$

### 1.11 Connectedness and path-connectedness

$$connected\big((M,\mathcal{O}),()\big) \iff \Big(topologicalSpace\big((M,\mathcal{O}),()\big)\Big) \wedge \Big(\neg\exists_{A,B \in \mathcal{O}\setminus\emptyset}\big(A \cap B \neq \emptyset \wedge A \cup B = M\big)\Big) \tag{130}$$

# if there is some covering of the space that does not intersect

$$\begin{aligned} &\text{(THM)} : \neg connected\Bigg(\Bigg(\mathbb{R}\setminus\{0\}, subsetTopology\Big(\mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}},\big(\mathbb{R},standardTopology,\mathbb{R}\setminus\{0\}\big)\Big)\Bigg),()\Bigg) \\ &\Longleftarrow \Big(A = (-\infty,0) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge \Big(B = (0,\infty) \in \mathcal{O}_{standard}|_{\mathbb{R}\setminus\{0\}}\Big) \wedge \\ &\big(A \cap B = \emptyset\big) \wedge \big(A \cup B = \mathbb{R}\setminus\{0\}\big) \end{aligned} \tag{131}$$

$$\text{(THM)} : connected\big((M,\mathcal{O}),()\big) \iff \forall_{S \in \mathcal{O}} \Big(clopen\Big(S,(M,\mathcal{O}) \implies \big(S = \emptyset \vee S = M\big)\Big)\Big) \tag{132}$$

$$pathConnected\big((M,\mathcal{O}),()\big) \iff \Big(subsetTopology\Big(\mathcal{O}_{standard}|_{[0,1]},\big(\mathbb{R},standardTopology,[0,1]\big)\Big)\Big) \wedge$$

$$\left(\forall_{p,q\in M}\exists_\gamma\left(continuous\left(\gamma,\left([0,1],\mathcal{O}_{standard}|_{[0,1]},M,\mathcal{O}\right)\right)\wedge\gamma(0)=p\wedge\gamma(1)=q\right)\right) \tag{133}$$

$$\text{(THM)}: pathConnected \implies connected \tag{134}$$

## 1.12 Homotopic curve and the fundamental group

$$\text{============ N O T = U P D A T E D ============} \tag{135}$$

$$\begin{gathered} homotopic(\sim,(\gamma,\delta,M,\mathcal{O}))\iff(map(\gamma,([0,1],M))\wedge map(\delta,([0,1],M)))\wedge \\ (\gamma(0)=\delta(0)\wedge\gamma(1)=\delta(1))\wedge \\ (\exists_H\forall_{\lambda\in[0,1]}(continuous(H,(([0,1]\times[0,1],\mathcal{O}_{standard^2}|_{[0,1]\times[0,1]}),(M,\mathcal{O}))\wedge H(0,\lambda)=\gamma(\lambda)\wedge H(1,\lambda)=\delta(\lambda)))) \\ \#\ H \text{ is a continuous deformation of one curve into another} \end{gathered} \tag{136}$$

$$homotopic(\sim)\implies equivalenceRelation(\sim) \tag{137}$$

$$loopSpace(\mathcal{L}_p,(p,M,\mathcal{O}))\iff\mathcal{L}_p=\{map(\gamma,([0,1],M))|continuous(\gamma)\wedge\gamma(0)=\gamma(1)\}) \tag{138}$$

$$\begin{gathered} concatination(\star,(p,\gamma,\delta))\iff(\gamma,\delta\in loopSpace(\mathcal{L}_p))\wedge \\ (\forall_{\lambda\in[0,1]}((\gamma\star\delta)(\lambda)=\begin{cases}\gamma(2\lambda) & 0\le\lambda<0.5\\ \delta(2\lambda-1) & 0.5\le\lambda\le1\end{cases})) \end{gathered} \tag{139}$$

$$\begin{gathered} group((G,\bullet),())\iff(map(\bullet,(G\times G,G))\wedge \\ (\forall_{a,b\in G}(a\bullet b\in G)) \\ (\forall_{a,b,c\in G}((a\bullet b)\bullet C=a\bullet(b\bullet c))) \\ (\exists_{\boldsymbol{e}}\forall_{a\in G}(\boldsymbol{e}\bullet a=a=a\bullet\boldsymbol{e}))\wedge \\ (\forall_{a\in G}\exists_{a^{-1}}(a\bullet a^{-1}=\boldsymbol{e}=a^{-1}\bullet a)) \\ \#\ \text{characterizes symmetry of a set structure} \end{gathered} \tag{140}$$

$$isomorphic(\cong,(X,\odot),(Y,\ominus)))\iff\exists_f\forall_{a,b\in X}(bijection(f,(X,Y))\wedge f(a\odot b)=f(a)\ominus f(b)) \tag{141}$$

$$\begin{gathered} fundamentalGroup((\pi_{1,p},\bullet),(p,M,\mathcal{O}))\iff(\pi_{1,p}=\mathcal{L}_p/\sim)\wedge \\ (map(\bullet,(\pi_{1,p}\times\pi_{1,p},\pi_{1,p})))\wedge \\ (\forall_{A,B\in\pi_{1,p}}([A]\bullet[B]=[A\star B]))\wedge \\ (group((\pi_{1,p},\bullet),())) \\ \#\ \text{an equivalence class of all loops induced from the homotopic equivalence relation} \end{gathered} \tag{142}$$

$$fundamentalGroup_1\ncong fundamentalGroup_2\implies topologicalSpace_1\ncong topologicalSpace_2 \tag{143}$$

$$\textbf{there exists no known list of topological properties that can imply homeomorphisms} \tag{144}$$

$$\textbf{CONTINUE @ Lecture 6: manifolds} \tag{145}$$

$$\text{============ N O T = U P D A T E D ============} \tag{146}$$

## 1.13 Measure theory

$$\mathit{sigmaAlgebra}(\sigma,(M)) \iff (M\neq\emptyset)\wedge(\sigma\subseteq\mathcal{P}(M))\wedge$$
$$(M\in\sigma)\wedge\Big(\forall_{A\in\sigma}(M\setminus A\in\sigma)\Big)\wedge$$
$$\bigg(\Big(A\subseteq\sigma\wedge\neg\mathit{uncountablyInfinite}(A,())\Big)\implies\cup A\in\sigma\bigg)$$
$$\text{\# behaves as measurable sets should; provides the sufficient structure for defining a measure } \mu \tag{147}$$

---

$$\mathit{measurableSpace}((M,\sigma),()) \iff \mathit{sigmaAlgebra}(\sigma,(M)) \tag{148}$$

---

$$\mathit{measurableSet}(A,(M,\sigma)) \iff \Big(\mathit{measurableSpace}((M,\sigma),())\Big)\wedge(A\in\sigma) \tag{149}$$

---

$$\mathit{measure}(\mu,(M,\sigma)) \iff \Big(\mathit{measurableSpace}((M,\sigma),())\Big)\wedge\left(\mathit{map}\left(\mu,\left(\sigma,\left(\overline{\mathbb{R}}\right)_0^+\right)\right)\right)\wedge\Big(\mu(\emptyset)=0\Big)\wedge$$
$$\left(\Big((A)_{\mathbb{N}}\subseteq\sigma\wedge\forall_{i\in\mathbb{N}}\forall_{j\in\mathbb{N}\setminus\{i\}}\left(A_i\cap A_j=\emptyset\right)\Big)\implies\mu\left(\cup_{i\in\mathbb{N}}(A_i)\right)=\sum_{i\in\mathbb{N}}(\mu(A_i))\right)$$
$$\text{\# enforces meaningful concepts of measures such as precise additivity} \tag{150}$$

---

$$(\mathrm{THM}):\ \mathit{measure}(\mu,(M,\sigma))\implies$$
$$\Big(\forall_{A,B\in\sigma}\left(A\subseteq B\implies\mu(A)\leq\mu(B)\right)\Big)\wedge$$
$$\left((A)_{\mathbb{N}}\subseteq\sigma\implies\mu\left(\cup_{i\in\mathbb{N}}(A_i)\right)\leq\sum_{i\in\mathbb{N}}(\mu(A_i))\right)\wedge$$
$$\Big(\left((B)_{\mathbb{N}}\subseteq\sigma\wedge\forall_{i\in\mathbb{N}}(B_i\subseteq B_{i+1})\wedge B=\cup(B)_{\mathbb{N}}\right)\implies\lim_{n\to\infty}\left(\mu(B_n)\right)=\mu(B)\Big)\wedge$$
$$\Big(\left((C)_{\mathbb{N}}\subseteq\sigma\wedge\forall_{i\in\mathbb{N}}(C_{i+1}\subseteq C_i)\wedge C=\cap(C)_{\mathbb{N}}\right)\implies\lim_{n\to\infty}\left(\mu(C_n)\right)=\mu(C)\Big)$$
$$\text{\# immediate implications of the measurable set } A\in\sigma \text{ axioms and the measure } \mu \text{ axioms} \tag{151}$$

---

$$\mathit{measureSpace}((M,\sigma,\mu),()) \iff \mathit{measure}(\mu,(M,\sigma)) \tag{152}$$

---

$$\mathit{finiteMeasure}(\mu,(M,\sigma)) \iff \Big(\mathit{measure}(\mu,(M,\sigma))\Big)\wedge$$
$$\Big(\exists_{(A)_{\mathbb{N}}\subseteq\sigma}\Big(\cup\left((A)_{\mathbb{N}}\right)=M\wedge\forall_{n\in\mathbb{N}}\left(\mu(A_n)<\infty\right)\Big)\Big) \tag{153}$$

---

$$\mathit{generatedSigmaAlgebra}(\sigma(\zeta),(\zeta,M)) \iff \Big(G=\{\sigma\subseteq\mathcal{P}(M)\,|\,\mathit{sigmaAlgebra}(\sigma,(M))\}\Big)\wedge\Big(\sigma(\zeta)=\cap G\Big)$$
$$\text{\# smallest } \sigma\text{-algebra containing the generating set } \zeta \tag{154}$$

---

$$(\mathrm{THM}):\ \exists_{\zeta\subseteq M}\Big(\mathit{generatedSigmaAlgebra}(\sigma(\zeta),(\zeta,M))=\mathit{sigmaAlgebra}(\sigma,(M))\Big) \tag{155}$$

---

$$\mathit{borelSigmaAlgebra}(\sigma(\mathcal{O}),(M,\mathcal{O})) \iff \Big(\mathit{topologicalSpace}((M,\mathcal{O}),())\Big)\wedge$$
$$\Big(\mathit{generatedSigmaAlgebra}(\sigma(\mathcal{O}),(\mathcal{O},M))\Big)$$
$$\text{\# } \sigma\text{-algebra induced by a topology} \tag{156}$$

$$\mathit{standardSigma}(\sigma_s,()) \iff \left(\mathit{borelSigmaAlgebra}\left(\sigma_s,\left(\mathbb{R}^d,\mathit{standardTopology}\right)\right)\right) \tag{157}$$

---

$$\mathit{lebesgueMeasure}(\lambda,()) \iff \left(\mathit{measure}\left(\lambda,\left(\mathbb{R}^d,\mathit{standardSigma}\right)\right)\right)\wedge$$
$$\left(\lambda\left(\times_{i=1}^{d}([a_i,b_i))\right)=\sum_{i=1}^{d}\left(\sqrt[2]{(a_i-b_i)^2}\right)\right)$$
$$\#\ \text{natural measure for } \mathbb{R}^d \tag{158}$$

---

$$\mathit{measurableMap}(f,(M,\sigma_M,N,\sigma_N)) \iff \left(\mathit{measurableSpace}((M,\sigma_M),())\right)\wedge$$
$$\left(\mathit{measurableSpace}((N,\sigma_N),())\right)\wedge\left(\forall_{B\in\sigma_N}\left(\mathit{preimage}(A,(B,f,M,N))\in\sigma_M\right)\right)$$
$$\#\ \text{preimage of measurable sets are measurable} \tag{159}$$

---

$$\mathit{pushForwardMeasure}(f\star\lambda_M,(f,M,\sigma_M,\mu_M,N,\sigma_N)) \iff \left(\mathit{measureSpace}((M,\sigma_M,\mu_M),())\right)\wedge$$
$$\left(\mathit{measurableSpace}((N,\sigma_N),())\right)\wedge\left(\mathit{measurableMap}(f,(M,\sigma_M,N,\sigma_N))\right)\wedge$$
$$\left(\forall_{B\in N}\left(f\star\lambda_M(B)=\mu_M\left(\mathit{preimage}(A,(B,f,M,N))\right)\right)\right)\wedge\left(\mathit{measure}(f\star\lambda_M,(N,\sigma_N))\right)$$
$$\#\ \text{natural construction of a measure based primarily on measurable map} \tag{160}$$

---

$$\mathit{nullSet}(A,(M,\sigma,\mu)) \iff \left(\mathit{measureSpace}((M,\sigma,\mu),())\right)\wedge(A\in\sigma)\wedge(\mu(A)=0) \tag{161}$$

---

$$\mathit{almostEverywhere}(p,(M,\sigma,\mu)) \iff \left(\mathit{measureSpace}((M,\sigma,\mu),())\right)\wedge\left(\mathit{predicate}(p,(M))\right)\wedge$$
$$\left(\exists_{A\in\sigma}\left(\mathit{nullSet}(A,(M,\sigma,\mu))\implies\forall_{n\in M\setminus A}(p(n))\right)\right)$$
$$\#\ \text{the predicate holds true for all points except the points in the null set} \tag{162}$$

## 1.14 Lebesque integration

$$\mathit{simpleTopology}(\mathcal{O}_{simple},()) \iff \mathcal{O}_{simple}=\mathit{subsetTopology}\left(\mathcal{O}|_{\mathbb{R}_0^+},\left(\mathbb{R},\mathit{standardTopology},\mathbb{R}_0^+\right)\right) \tag{163}$$

---

$$\mathit{simpleSigma}(\sigma_{simple},()) \iff \mathit{borelSigmaAlgebra}\left(\sigma_{simple},\left(\mathbb{R}_0^+,\mathit{simpleTopology}\right)\right) \tag{164}$$

---

$$\mathit{simpleFunction}(s,(M,\sigma)) \iff \left(\mathit{measurableMap}\left(s,\left(M,\sigma,\mathbb{R}_0^+,\mathit{simpleSigma}\right)\right)\right)\wedge$$
$$\left(\mathit{finiteSet}\left(\mathit{image}\left(B,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right)$$
$$\#\ \text{if the map takes on finitely many values on } \mathbb{R}_0^+ \tag{165}$$

---

$$characteristicFunction\left(X_A,(A,M)\right) \iff (A \subseteq M) \wedge \left(map\left(X_A,(M,\mathbb{R})\right)\right) \wedge \left(\forall_{m\in M}\left(X_A(m)=\begin{cases}1 & m\in A\\ 0 & m\notin A\end{cases}\right)\right) \tag{166}$$

---

$$\text{(THM)}: simpleFunction\left(s,(M,\sigma_M)\right) \implies \left(finiteSet\left(image\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\right) \wedge \left(characteristicFunction\left(X_A,(A,M)\right)\right) \wedge \left(\forall_{m\in M}\left(s(m)=\sum_{z\in Z}\left(z\cdot X_{preimage\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)}(m)\right)\right)\right) \tag{167}$$

---

$$exStandardSigma\left(\overline{\sigma_s},()\right) \iff \overline{\sigma_s}=\{A\subseteq\overline{\mathbb{R}} \mid A\cap R\in standardSigma\}$$
$$\text{\# ignores } \pm\infty \text{ to preserve the points in the domain of the measurable map} \tag{168}$$

---

$$nonNegIntegrable\left(f,(M,\sigma)\right) \iff \left(measurableMap\left(f,\left(M,\sigma,\overline{\mathbb{R}},exStandardSigma\right)\right)\right) \wedge \left(\forall_{m\in M}\left(f(m)\geq 0\right)\right) \tag{169}$$

---

$$nonNegIntegral\left(\int_M (f d\mu),(f,M,\sigma,\mu)\right) \iff \left(measureSpace\left((M,\sigma,\mu),()\right)\right) \wedge \left(measureSpace\left(\left(\overline{\mathbb{R}},exStandardSigma,lebesgueMeasure\right),()\right)\right) \wedge \left(nonNegIntegrable\left(f,(M,\sigma)\right)\right) \wedge (\int_M (f d\mu)=sup(\{\sum_{z\in Z}\left(z\cdot\mu\left(preimage\left(A,\left(\{z\},s,M,\mathbb{R}_0^+\right)\right)\right)\right) \mid \forall_{m\in M}(s(m)\leq f(m)) \wedge simpleFunction\left(s,(M,\sigma)\right) \wedge finiteSet\left(image\left(Z,\left(M,s,M,\mathbb{R}_0^+\right)\right),()\right)\}))$$
$$\text{\# lebesgue measure on } z \text{ reduces to } z \tag{170}$$

---

$$explicitIntegral \iff \int (f(x)\mu(dx))=\int (f d\mu)$$
$$\text{\# alternative notation for lebesgue integrals} \tag{171}$$

---

$$\text{(THM)}: nonNegIntegral\left(\int (f d\mu),(f,M,\sigma,\mu)\right) \wedge nonNegIntegral\left(\int (g d\mu),(g,M,\sigma,\mu)\right) \implies$$
$$\text{(THM) Markov inequality: } \left(\forall_{z\in\mathbb{R}_0^+}\left(\int (f d\mu)\geq z\cdot\mu\left(preimage\left(A,\left([z,\infty),f,M,\overline{\mathbb{R}}\right)\right)\right)\right)\right) \wedge \left(almostEverywhere\left(f=g,(M,\sigma,\mu)\right) \implies \int (f d\mu)=\int (g d\mu)\right) \left(\int (f d\mu)=0 \implies almostEverywhere\left(f=0,(M,\sigma,\mu)\right)\right) \wedge \left(\int (f d\mu)\leq\infty \implies almostEverywhere\left(f<\infty,(M,\sigma,\mu)\right)\right) \tag{172}$$

---

$$\text{(THM) Mono. conv.: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \mathit{measurableMap}\left(f_n, \left(M,\sigma,\overline{R},\mathit{exStandardSigma}\right)\right) \wedge 0 \leq f_{n-1} \leq f_n\}\right) \wedge \left(\mathit{map}\left(f,\left(M,\overline{\mathbb{R}}\right)\right)\right) \wedge \left(\forall_{m\in M}\left(f(m) = \mathit{sup}(f_n(m) | f_n \in (f)_{\mathbb{N}})\right)\right) \implies \left(\lim_{n\to\infty}\left(\int_M (f_n d\mu)\right) = \int_M (f d\mu)\right)$$

$$\text{\# lengths now depend on } M,\sigma \text{ and limits can be pulled in or out of an integral} \quad (173)$$

---

$$\text{(THM) : } \mathit{nonNegIntegral}\left(\int (f d\mu),(f,M,\sigma,\mu)\right) \wedge \mathit{nonNegIntegral}\left(\int (g d\mu),(g,M,\sigma,\mu)\right) \implies \left(\forall_{\alpha\in\mathbb{R}_0^+}\left(\int ((f+\alpha g)d\mu) = \int (f d\mu) + \alpha \int (g d\mu)\right)\right)$$

$$\text{\# integral acts linearly and commutes finite summations} \quad (174)$$

---

$$\text{(THM) : } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \mathit{measurableMap}\left(f_n,\left(M,\sigma,\overline{R},\mathit{exStandardSigma}\right)\right) \wedge 0 \leq f_n\}\right) \implies \left(\int\left(\left(\sum_{n=1}^{\infty}(f_n)\right) d\mu\right) = \sum_{n=1}^{\infty}\left(\int (f_n d\mu)\right)\right)$$

$$\text{\# } \sum_{n=1}^{\infty} f_n \text{ can be treated as } \lim_{n\to\infty}\sum_{i=1}^{n} f_n \text{ since } f_n \geq 0 \text{ and it commutes with integral from monotone conv.} \quad (175)$$

---

$$\mathit{integrable}(f,(M,\sigma)) \iff \left(\mathit{measurableMap}\left(f,\left(M,\sigma,\overline{\mathbb{R}},\mathit{exStandardSigma}\right)\right)\right) \wedge \left(\forall_{m\in M}\left(f(m) = \mathit{max}(f(m),0) - \mathit{max}(0,-f(m))\right)\right) \wedge \left(\mathit{measureSpace}(M,\sigma,\mu) \implies \left(\int\left(\mathit{max}(f(m),0)d\mu\right) < \infty \wedge \int\left(\mathit{max}(0,-f(m))d\mu\right) < \infty\right)\right)$$

$$\text{\# extra condition prevents the occurrence of the indeterminate } \infty - \infty \quad (176)$$

---

$$\mathit{integral}\left(\int (f d\mu),(f,M,\sigma,\mu)\right) \iff \left(\mathit{nonNegIntegral}\left(\int (f^+ d\mu),\left(\mathit{max}(f,0),M,\sigma,\mu\right)\right)\right) \wedge \left(\mathit{nonNegIntegral}\left(\int (f^- d\mu),\left(\mathit{max}(0,-f),M,\sigma,\mu\right)\right)\right) \wedge \left(\mathit{integrable}(f,(M,\sigma))\right) \wedge \left(\int (f d\mu) = \int (f^+ d\mu) - \int (f^- d\mu)\right)$$

$$\text{\# arbitrary integral in terms of nonnegative integrals} \quad (177)$$

---

$$\text{(THM) : } \left(\mathit{map}(f,(M,\mathbb{C}))\right) \implies \left(\int (f d\mu) = \int \left(\mathit{Re}(f) d\mu\right) - \int \left(\mathit{Im}(f) d\mu\right)\right) \quad (178)$$

---

$$\text{(THM) : } \mathit{integral}\left(\int (f d\mu),(f,M,\sigma,\mu)\right) \wedge \mathit{integral}\left(\int (g d\mu),(g,M,\sigma,\mu)\right) \implies \left(\mathit{almostEverywhere}(f \leq g,(M,\sigma,\mu)) \implies \int (f d\mu) \leq \int (g d\mu)\right) \wedge \left(\forall_{m\in M}(f(m),g(m),\alpha \in \mathbb{R}) \implies \int ((f+\alpha g)d\mu) = \int (f d\mu) + \alpha \int (g d\mu)\right) \quad (179)$$

---

$$\text{(THM) Dominant convergence: } \left((f)_{\mathbb{N}} = \{f_n \mid \wedge \mathit{measurableMap}\left(f_n, \left(M, \sigma, \overline{R}, \mathit{exStandardSigma}\right)\right)\}\right) \wedge$$
$$\left(\mathit{map}(f, (M, \overline{\mathbb{R}}))\right) \wedge \left(\mathit{almostEverywhere}\left(f(m) = \lim_{n\to\infty}(f_n(m)), (M, \sigma, \mu)\right)\right) \wedge$$
$$\left(\mathit{nonNegIntegral}\left(\int(gd\mu), (g, M, \sigma, \mu)\right)\right) \wedge \left(\left|\int(gd\mu)\right| < \infty\right) \wedge \left(\mathit{almostEverywhere}(|f_n| \le g, (M, \sigma, \mu))\right)$$
$$\text{\# if all } f_n(m) \text{ are bounded by some integrable } |g(m)| \implies$$
$$\text{\# then all } f_n(m) \text{ including } f \text{ satisfy bounded and integrable properties}$$
$$\left(\forall_{\phi \in \{f\} \cup (f)_{\mathbb{N}}}\left(\mathit{integrable}(\phi, (M, \sigma))\right)\right) \wedge \left(\lim_{n\to\infty}\left(\int(|f_n - f| d\mu) = 0\right)\right) \wedge \left(\lim_{n\to\infty}\left(\int(f_n d\mu)\right) = \int(f d\mu)\right) \quad (180)$$

## 1.15 Vector space and structures

$$\mathit{vectorSpace}((V, +, \cdot), ()) \iff \left(\mathit{map}(+, (V \times V, V))\right) \wedge \left(\mathit{map}(\cdot, (\mathbb{R} \times V, V))\right) \wedge$$
$$(\forall_{v,w \in v}(v + w = w + v)) \wedge$$
$$\left(\forall_{v,w,x \in v}((v + w) + x = v + (w + x))\right) \wedge$$
$$(\exists_{\boldsymbol{0} \in V} \forall_{v \in V}(v + \boldsymbol{0} = v)) \wedge$$
$$\left(\forall_{v \in V} \exists_{-v \in V}(v + (-v) = \boldsymbol{0})\right) \wedge$$
$$\left(\forall_{a,b \in \mathbb{R}} \forall_{v \in V}(a(b \cdot v) = (ab) \cdot v)\right) \wedge$$
$$(\exists_{1 \in \mathbb{R}} \forall_{v \in V}(1 \cdot v = v)) \wedge$$
$$\left(\forall_{a,b \in \mathbb{R}} \forall_{v \in V}((a + b) \cdot v = a \cdot v + b \cdot v)\right) \wedge$$
$$\left(\forall_{a \in \mathbb{R}} \forall_{v,w \in V}(a \cdot (v + w) = a \cdot v + a \cdot w)\right)$$
$$\text{\# behaves similar as vectors should i.e., additive, scalable, linear distributive} \quad (181)$$

---

$$\mathit{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \iff \left(\mathit{vectorSpace}((V, +, \cdot), ())\right) \wedge \left(\mathit{map}(\langle \$1, \$2 \rangle, (V \times V, \mathbb{R}))\right) \wedge$$
$$\left(\forall_{v,w \in V}(\langle v, w \rangle = \langle w, v \rangle)\right) \wedge$$
$$\left(\forall_{v,w,x \in V} \forall_{a,b \in \mathbb{R}}(\langle av + bw, x \rangle = a\langle v, x \rangle + b\langle w, x \rangle)\right) \wedge$$
$$\left(\forall_{v \in V}(\langle v, v \rangle) \ge 0\right) \wedge \left(\forall_{v \in V}(\langle v, v \rangle) = 0 \iff v = \boldsymbol{0}\right)$$
$$\text{\# the sesquilinear or 1.5 linear map inner product provides info. on distance and orthogonality} \quad (182)$$

---

$$\mathit{innerProductSpace}\left((V, +, \cdot, \langle \$1, \$2 \rangle), ()\right) \iff \mathit{innerProduct}(\langle \$1, \$2 \rangle, (V, +, \cdot)) \quad (183)$$

---

$$\mathit{vectorNorm}(||\$1||, (V, +, \cdot)) \iff \left(\mathit{vectorSpace}((V, +, \cdot), ())\right) \wedge \left(\mathit{map}\left(||\$1||, \left(V, \mathbb{R}_0^+\right)\right)\right) \wedge$$
$$\left(\forall_{v \in V}(||v|| = 0 \iff v = \boldsymbol{0})\right) \wedge$$
$$\left(\forall_{v \in V} \forall_{s \in \mathbb{R}}(||sv|| = |s|||v||)\right) \wedge$$
$$\left(\forall_{v,w \in V}(||v + w|| \le ||v|| + ||w||)\right)$$
$$\text{\# magnitude of a point in a vector space} \quad (184)$$

---

$$\textit{normedVectorSpace}\Big((V,+,\cdot,||\$1||),()\Big) \iff \Big(\textit{vectorSpace}((V,+,\cdot),())\Big) \wedge \Big(\textit{vectorNorm}(||\$1||,(V,+,\cdot))\Big) \tag{185}$$

---

$$\begin{aligned} \textit{vectorMetric}\Big(d(\$1,\$2),(V,+,\cdot)\Big) \iff \Big(\textit{vectorSpace}((V,+,\cdot),())\Big)\wedge \\ \left(\textit{metric}\Big(d(\$1,\$2),(V)\Big)\vee\left(\textit{map}\Big(d,\Big(V\times V,\mathbb{R}_0^+\Big)\Big)\right)\right. \\ \Big(\forall_{x,y\in V}\left(d(x,y)=d(y,x)\right)\Big)\wedge \\ \Big(\forall_{x,y\in V}\left(d(x,y)=0\iff x=y\right)\Big)\wedge \\ \left.\left(\forall_{x,y,z\in V}\Big(\left(d(x,z)\le d(x,y)+d(y,z)\right)\Big)\right)\right) \\ \text{\# behaves as distances should} \end{aligned} \tag{186}$$

---

$$\begin{aligned} \textit{metricVectorSpace}\Big(\Big(V,+,\cdot,d(\$1,\$2)\Big),()\Big) \iff \Big(\textit{vectorSpace}((V,+,\cdot),())\Big)\wedge \\ \Big(\textit{vectorMetric}\Big(d(\$1,\$2),(V,+,\cdot)\Big)\Big) \end{aligned} \tag{187}$$

---

$$\begin{aligned} \textit{innerProductNorm}\Big(||\$1||,(V,+,\cdot,\langle\$1,\$2\rangle)\Big) \iff \Big(\textit{innerProductSpace}\Big((V,+,\cdot,\langle\$1,\$2\rangle),()\Big)\Big)\wedge \\ \Big(\forall_{v\in V}\Big(||v||=\sqrt[2]{\langle v,v\rangle}\Big)\implies \textit{vectorNorm}(||\$1||,(V,+,\cdot))\Big) \end{aligned} \tag{188}$$

---

$$\begin{aligned} \textit{normInnerProduct}\Big(\langle\$1,\$2\rangle,(V,+,\cdot,||\$1||)\Big) \iff \Big(\textit{normedVectorSpace}\Big((V,+,\cdot,||\$1||),()\Big)\Big)\wedge \\ \Big(\forall_{u,v\in V}\Big(2||u||^2+2||v||^2=||u+v||^2+||u-v||^2\Big)\Big)\wedge \\ \left(\forall_{v,w\in V}\left(\langle v,w\rangle=\frac{||v+w||^2-||v-w||^2}{4}\right)\implies \textit{innerProduct}(\langle\$1,\$2\rangle,(V,+,\cdot))\right) \end{aligned} \tag{189}$$

---

$$\begin{aligned} \textit{normMetric}\Big(d(\$1,\$2),(V,+,\cdot,||\$1||)\Big) \iff \Big(\textit{normedVectorSpace}\Big((V,+,\cdot,||\$1||),()\Big)\Big)\wedge \\ \Big(\forall_{v,w\in V}\left(d(v,w)=||v-w||\right)\implies \textit{vectorMetric}\Big(d(\$1,\$2),(V,+,\cdot)\Big)\Big) \end{aligned} \tag{190}$$

---

$$\begin{aligned} \textit{metricNorm}\Big(||\$1||,\Big(V,+,\cdot,d(\$1,\$2)\Big)\Big) \iff \Big(\textit{metricVectorSpace}\Big(\Big(V,+,\cdot,d(\$1,\$2)\Big),()\Big)\Big)\wedge \\ \Big(\forall_{u,v,w\in V}\forall_{s\in\mathbb{R}}\Big(d\big(s(u+w),s(v+w)\big)=|s|d(u,v)\Big)\Big)\wedge \\ \Big(\forall_{v\in V}\left(||v||=d(v,\boldsymbol{0})\right)\implies \textit{vectorNorm}(||\$1||,(V,+,\cdot))\Big) \end{aligned} \tag{191}$$

---

$$\begin{aligned} \textit{orthogonal}\Big((v,w),(V,+,\cdot,\langle\$1,\$2\rangle)\Big) \iff \Big(\textit{innerProductSpace}\Big((V,+,\cdot,\langle\$1,\$2\rangle),()\Big)\Big)\wedge \\ (v,w\in V)\wedge\left(\langle v,w\rangle=0\right) \\ \text{\# the inner product also provides info. on orthogonality} \end{aligned} \tag{192}$$

---

$$\textit{normal}\Big(v,(V,+,\cdot,\langle\$1,\$2\rangle)\Big) \iff \Big(\textit{innerProductSpace}\Big((V,+,\cdot,\langle\$1,\$2\rangle),()\Big)\Big)\wedge(v\in V)\wedge\left(\langle v,v\rangle=1\right)$$

$$\text{\# the vector has unit length} \tag{193}$$

$$\text{(THM) Cauchy-Schwarz inequality: } \forall_{v,w\in V}\left(\langle v,w\rangle \leq ||v||||w||\right) \tag{194}$$

$$basis\left((b)_n,(V,+,\cdot,)\right) \iff \left(vectorSpace\left((V,+,\cdot),()\right)\right) \wedge \left(\forall_{v\in V}\exists_{(a)_n\in\mathbb{R}^n}\left(v=\sum_{i=1}^{n}(a_ib_i)\right)\right) \tag{195}$$

$$\begin{aligned} orthonormalBasis\Big((b)_n,(V,+,\cdot,\langle \$1,\$2\rangle)\Big) \iff \Big(innerProductSpace\Big((V,+,\cdot,\langle \$1,\$2\rangle),()\Big)\Big)\wedge \\ \Big(basis\left((b)_n,(V,+,\cdot)\right)\Big)\wedge\Big(\forall_{v\in(b)_n}\Big(normal\Big(v,(V,+,\cdot,\langle \$1,\$2\rangle)\Big)\Big)\Big)\wedge \\ \Big(\forall_{v\in(b)_n}\forall_{w\in(b)_n\setminus\{v\}}\Big(orthogonal\Big((v,w),(V,+,\cdot,\langle \$1,\$2\rangle)\Big)\Big)\Big) \end{aligned} \tag{196}$$

### 1.16 Subvector space

$$subspace\left((U,\circ),(V,\circ)\right) \iff \Big(space\left((V,\circ),()\right)\Big)\wedge(U\subseteq V)\wedge\Big(space\left((U,\circ),()\right)\Big) \tag{197}$$

$$\begin{aligned} subspaceSum\left(U+W,(U,W,V,+)\right) \iff \Big(subspace\left((U,+),(V,+)\right)\Big)\wedge\Big(subspace\left((W,+),(V,+)\right)\Big)\wedge \\ \left(U+W=\{u+w\,|\,u\in U\wedge w\in W\}\right) \end{aligned} \tag{198}$$

$$subspaceDirectSum\left(U\oplus W,(U,W,V,+)\right) \iff (U\cap W=\emptyset)\wedge\Big(subspaceSum\left(U\oplus W,(U,W,V,+)\right)\Big) \tag{199}$$

$$\begin{aligned} orthogonalComplement\Big(W^\perp,(W,V,+,\cdot,\langle \$1,\$2\rangle)\Big) \iff \\ \Big(subspace\Big((W,+,\cdot,\langle \$1,\$2\rangle),\Big(innerProductSpace\Big((V,+,\cdot,\langle \$1,\$2\rangle),()\Big)\Big)\Big)\Big)\wedge \\ \Big(W^\perp=\Big\{v\in V\,|\,w\in W\wedge orthogonal\Big((v,w),(V,+,\cdot,\langle \$1,\$2\rangle)\Big)\Big\}\Big) \end{aligned} \tag{200}$$

$$\begin{aligned} orthogonalDecomposition\Big(\Big(W,W^\perp\Big),(W,V,+,\cdot,\langle \$1,\$2\rangle)\Big) \iff \\ \Big(orthogonalComplement\Big(W^\perp,(W,V,+,\cdot,\langle \$1,\$2\rangle)\Big)\Big)\wedge\Big(subspaceDirectSum\Big(V,\Big(W,W^\perp,V,+\Big)\Big)\Big) \end{aligned} \tag{201}$$

$$\text{(THM) if } V \text{ is finite dimensional, then every vector has an orthogonal decomposition:} \tag{202}$$

### 1.17 Banach and Hilbert Space

$$\begin{aligned} cauchy\Big((s)_{\mathbb{N}},\Big(V,d(\$1,\$2)\Big)\Big) \iff \Big(metricSpace\Big(\Big(V,d(\$1,\$2)\Big),()\Big)\Big)\wedge\left((s)_{\mathbb{N}}\subseteq V\right) \\ \Big(\forall_{\epsilon>0}\exists_{N\in\mathbb{N}}\forall_{m,n\geq N}\left(d(s_m,s_n)<\epsilon\right)\Big) \end{aligned}$$

$$\text{\# distances between some tail-end point gets arbitrarily small} \tag{203}$$

---

$$complete\Big(\big(V, d(\$1,\$2)\big), ()\Big) \iff \Big(\forall_{(s)_{\mathbb{N}} \subseteq V} \exists_{s \in V} \Big(cauchy\Big((s)_{\mathbb{N}}, \big(V, d(\$1,\$2)\big)\Big) \implies \lim_{n \to \infty} \big(d(s, s_n)\big) = 0\Big)\Big)$$

$$\text{\# or converges within the induced topological space}$$
$$\text{\# in complete spaces, the weaker notion of cauchy is enforced to be equivalent to convergence} \tag{204}$$

---

$$banachSpace\big((V, +, \cdot, ||\$1||), ()\big) \iff \Big(normMetric\big(d(\$1,\$2), (V, ||\$1||)\big)\Big) \wedge \Big(complete\big(V, d(\$1,\$2)\big), ()\Big)$$
$$\text{\# a complete normed vector space} \tag{205}$$

---

$$hilbertSpace\big((V, +, \cdot, \langle\$1,\$2\rangle), ()\big) \iff \Big(innerProductNorm\big(||\$1||, (V, +, \cdot, \langle\$1,\$2\rangle)\big)\Big) \wedge$$
$$\Big(normMetric\big(d(\$1,\$2), (V, ||\$1||)\big)\Big) \wedge \Big(complete\big(V, d(\$1,\$2)\big), ()\Big)$$
$$\text{\# a complete inner product space} \tag{206}$$

---

$$\text{(THM)} : hilbertSpace \implies banachSpace \tag{207}$$

---

$$separable\big((V, d), ()\big) \iff \Big(\exists_{S \subseteq V} \Big(dense\big(S, (V, d)\big) \wedge countablyInfinite\big(S, ()\big)\Big)\Big)$$
$$\text{\# needs only a countable subset to approximate any element in the entire space} \tag{208}$$

---

$$\text{(THM)} : hilbertSpace\Big(\Big(\big(V, +, \cdot, \langle\$1,\$2\rangle\big), ()\Big), ()\Big) \implies$$
$$\Big(\exists_{(b)_{\mathbb{N}} \subseteq V} \Big(orthonormalBasis\big((b)_{\mathbb{N}}, (V, +, \cdot, \langle\$1,\$2\rangle)\big) \wedge countablyInfinite\big((b)_{\mathbb{N}}, ()\big)\Big) \iff$$
$$separable\Big(\Big(V, \sqrt{\langle\$1-\$2, \$1-\$2\rangle}\Big), ()\Big)\Big)$$
$$\text{\# separability in hilbert spaces is equivalent to the existence of a countable orthonormal basis} \tag{209}$$

## 1.18 Matrices, Operators, and Functionals

$$linearOperator\big(L, (V, +_V, \cdot_V, W, +_W, \cdot_W)\big) \iff \Big(map\big(L, (V, W)\big)\Big) \wedge \Big(vectorSpace\big((V, +_V, \cdot_V), ()\big)\Big) \wedge$$
$$\Big(vectorSpace\big((W, +_W, \cdot_W), ()\big)\Big) \wedge \Big(\forall_{v_1, v_2 \in V} \forall_{s_1, s_2 \in \mathbb{R}} \big(L(s_1 \cdot_V v_1 +_V s_2 \cdot_V v_2) = s_1 \cdot_W L(v_1) +_W s_2 \cdot_W L(v_2)\big)\Big) \tag{210}$$

---

$$matrix\big(L, (n, m)\big) \iff \Big(linearOperator\big(L, (\mathbb{R}^m, +_m, \cdot_m, \mathbb{R}^n, +_n, \cdot_n)\big)\Big)$$
$$\text{\# rows=dimensions, cols=vectors} \tag{211}$$

---

$$eigenvector\big(v, (L, V, +, \cdot)\big) \iff \Big(linearOperator\big(L, (V, +, \cdot, V, +, \cdot)\big)\Big) \wedge \Big(\exists_{\lambda \in \mathbb{R}} \big(L(v) = \lambda v\big)\Big) \tag{212}$$

---

$$eigenvalue\big(\lambda, (v, L, V, +, \cdot)\big) \iff \Big(eigenvector\big(v, (L, V, +, \cdot)\big)\Big) \tag{213}$$

---

$$\textit{identityOperator}(I,(A)) \iff \Big(\textit{matrix}(A,(n,n))\Big) \wedge (AI=IA=A) \tag{214}$$

$$\textit{inverseOperator}\Big(A^{-1},(A)\Big) \iff \Big(A^{-1}A=\textit{identityOperator}(I,(A))\Big)$$
$$\text{\# gauss-jordan elimination: } E[A|I]=[I|E]=[I|A^{-1}] \tag{215}$$

$$\text{(THM)}: (AB)^{-1}(AB)=I=B^{-1}A^{-1}AB \tag{216}$$

$$\textit{transposeOperator}\Big(A^T,(A)\Big) \iff \Big(\Big(A^T\Big)_{m,n}=(A)_{n,m}\Big) \vee \textit{adjoint}\Big(A^T,(A)\Big) \tag{217}$$

$$\textit{symmetricOperator}(A,()) \iff \Big(A=\textit{transposeOperator}\Big(A^T,(A)\Big)\Big) \vee \Big(\textit{selfAdjoint}(A,())\Big) \tag{218}$$

$$\text{(THM)}: (AB)^T=B^TA^T \wedge \Big(A^T\Big)^{-1}=\Big(A^{-1}\Big)^T \tag{219}$$

$$\textit{triangularOperator}(A,()) \iff \Big(\textit{matrix}(A,(n,n))\Big) \wedge \Big(\forall_{x<n}\forall_{0<i<x}(A_{i,i}=0)\Big) \tag{220}$$

$$\textit{decomposeLU}(LU(A),(A)) \iff \Big(\textit{matrix}(A,(n,n))\Big) \wedge \Big(\exists_E\Big(EA=\textit{triangularOperator}(U,())\Big)\Big) \wedge$$
$$\Big(LU(A)=E^{-1}U=A\Big)$$
$$\text{\# lower triangle are all 0; useful for solving linear equations} \tag{221}$$

$$\textit{Img}(\textit{Img}(A),(A)) \iff \Big(\textit{matrix}(A,(n,m))\Big) \wedge (\textit{Img}(A)=\{Av\in\mathbb{R}^n \,|\, v\in\mathbb{R}^m\})$$
$$\text{\# the column space; not always a subspace since } A \text{ can map to a set not containing } \boldsymbol{0} \tag{222}$$

$$\textit{Ker}(\textit{Ker}(A),(A)) \iff \Big(\textit{matrix}(A,(n,m))\Big) \wedge (\textit{Ker}(A)=\{v\in\mathbb{R}^m \,|\, Av=\boldsymbol{0}\in\mathbb{R}^n\})$$
$$\text{\# the null or solution space; always a subspace due to linearity } Av+Aw=\boldsymbol{0}=A(v+w) \tag{223}$$

$$\text{(THM) general linear solution: } (Ax_p=b) \wedge (x_n\in \textit{Ker}(A)) \implies \Big(Ax_p+Ax_n=b+0=A(x_p+x_n)=b\Big) \tag{224}$$

$$\textit{independentOperator}(A,()) \iff \Big(\textit{matrix}(A,(n,m))\Big) \wedge \Big(\neg\exists_{v\in\mathbb{R}^m\setminus\boldsymbol{0}_m}(Av=0) \iff \textit{Ker}(A)=\{\boldsymbol{0}_m\}\Big)$$
$$\text{\# also equivalent to invertible operator} \tag{225}$$

$$\textit{dimensonality}(N,(A)) \iff \Big(\textit{matrix}(A,(n,m))\Big) \wedge \Big(N=\textit{inf}\Big(\{|(b)_n|\,|\,\textit{basis}((b)_n,(A))\}\Big)\Big) \tag{226}$$

$$\textit{rank}(r,(A)) \iff \Big(\textit{matrix}(A,(n,m))\Big) \wedge \Big(\textit{dimensonality}(r,(A))\Big) \tag{227}$$

$$\text{(THM)}: \Big(\textit{matrix}(A,(n,m))\Big) \implies \Big(\textit{dimensionality}(\textit{Ker}(A))=n-\textit{rank}(r,(A))\Big)$$
$$\text{\# number of free variables} \tag{228}$$

$$transposeNorm(||x||,()) \iff \left(||x|| = \sqrt{x^T x}\right) \quad (229)$$

---

$$(\text{THM}) : P = P^T = P^2 \quad (230)$$

---

$$orthogonalVectors\big((x,y),()\big) \iff \left(||x||^2 + ||y||^2 = ||x+y||^2\right) \iff$$
$$\left(x^T x + y^T y = (x+y)^T (x+y) = x^T x + y^T y + x^T y = y^T x\right) \iff$$
$$\left(0 = \frac{x^T x + y^T y - \left(x^T x + y^T y\right)}{2} = \frac{x^T y + y^T x}{2} = x^T y\right) \iff \left(0 = \sum_i (x_i y_i) \vee \int (x(u)y(u)du)\right)$$
$$\# \text{ vector and functional orthogonality} \quad (231)$$

---

$$orthogonalOperator\Big(Q,\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big) \iff \left(orthonormalBasis\left(Q^T,\left(V,+,\cdot,\$1^T,\$2\right)\right)\right) \vee \left(Q^T Q = I\right) \quad (232)$$

---

$$(\text{THM}) : orthogonalOperator\Big(Q,\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big) \implies \left(Q^T Q Q^{-1} = I Q^{-1} = Q^T = Q^{-1}\right) \quad (233)$$

---

$$orthogonalProjection\big(P_A b,(A,b)\big) \iff \Big(matrix\big(A,(n,m)\big)\Big) \wedge \Big(matrix\big(b,(m,1)\big)\Big) \wedge$$
$$\left(\exists_{c\in\mathbb{R}^m}\left(A^T(b - P_A b) = 0 = A^T(b - Ac)\right) \iff\right.$$
$$\left. A^T b = A^T A c \iff c = \left(A^T A\right)^{-1} A^T b \iff P_A b = Ac = \left(A\left(A^T A\right)^{-1} A^T\right) b\right)$$
$$\# \; A, A^T \text{ may not necessarily be invertible} \quad (234)$$

---

$$(\text{THM}) : independentOperator(A,()) \implies independentOperator(A^T A,()) \quad (235)$$

---

$$eigenvectors(X,(A,V,+,\cdot,||\$1||)) \iff (normedVectorSpace((V,+,\cdot,||\$1||),())) \wedge$$
$$(X = \{v \in V \mid ||v|| = 1 \wedge eigenvector(v,(A,V,+,\cdot))\}) \quad (236)$$

---

$$det(det(A),(A,V,+,\cdot,||\$1||)) \iff (eigenvectors(X,(A,V,+,\cdot,||\$1||))) \wedge$$
$$(det(A) = \prod_{x\in X} (eigenvalue(\lambda,(x,A,V,+,\cdot))))$$
$$\# \text{ DEFINE; exterior algebra wedge product area??} \quad (237)$$

---

$$tr(tr(A),(A,V,+,\cdot,||\$1||)) \iff (eigenvectors(X,(A,V,+,\cdot,||\$1||))) \wedge$$
$$(tr(A) = \sum_{x\in X} (eigenvalue(\lambda,(x,A,V,+,\cdot))))$$
$$\# \text{ DEFINE} \quad (238)$$

---

$$(\text{THM}) : independentOperator(A,()) \iff det(A) \neq 0 \quad (239)$$

---

$$(\text{THM}) : A = A^T = A^2 \implies Tr(A) = dimensionality(N,(A)) \; \# \text{ counts dimensions} \quad (240)$$

---

$$(normalOperator(A,())) \iff A^T A = A A^T$$
$$\# \text{ DEFINE} \quad (241)$$

$$diagonalOperator(A,()) \iff (normalOperator(A,())) \wedge (triangularOperator(A,())) \quad (242)$$

$$characteristicEquation((A-\lambda I)x=0,(A)) \iff (Ax=\lambda x \implies Ax-\lambda x=(A-\lambda I)x=0) \wedge$$
$$(x \neq \boldsymbol{0} \implies eigenvalue(0,(x,A-\lambda I) \implies \prod_{\lambda_i \in \Lambda} = 0 = det(A-\lambda I)))$$
$$\text{\# characterizes eigenvalues} \quad (243)$$

$$eigenDecomposition(S\Lambda S^{-1},(A,V,+,\cdot,||\$1||)) \iff (S \subseteq (eigenvectors(X,(A,V,+,\cdot,||\$1||))^T)) \wedge$$
$$(diagonalOperator(\Lambda,())\{1\}^n = (\lambda)_n = \{\lambda \in \mathbb{R} \,|\, s \in S^T \wedge eigenvalue(\lambda,s,A,V)\})$$
$$(independentOperator(S,())) \wedge (\exists_{S^{-1}}(AS=S\Lambda \implies A=S\Lambda S^{-1})) \quad (244)$$

$$\text{(THM)}: eigenDecomposition(S\Lambda S^{-1},(A,V,+,\cdot,||\$1||)) \implies A^2=(A)(A)=S\Lambda S^{-1}S\Lambda S^{-1}=S\Lambda^2 S^{-1} \quad (245)$$

$$\text{(THM)}: spectralDecomposition(Q\Lambda Q^T,(A,V,+,\cdot,||\$1||)) \iff (symmetricOperator(A,())) \implies$$
$$(\exists_Q(eigenDecomposition(Q\Lambda Q^{-1},(A,V,+,\cdot,\$1^T\$1)) \wedge orthogonalOperator(Q,(V,+,\cdot,\$1^T\$2)) \wedge (\lambda)_n \in \mathbb{R}^n))$$
$$\text{\# if symmetric and eigenvalues are real, then there exists orthonormal eigenbasis} \quad (246)$$

$$hermitianAdjoint(A^H,(A)) \iff (A^H=\overline{A}^T) \Longleftarrow (\langle A,A\rangle = \overline{A}^T A \in \mathbb{R})$$
$$\text{\# complex analog to adjoint} \quad (247)$$

$$hermitianOperator(A,()) \iff A=A^H$$
$$\text{\# complex analog to symmetric operator} \quad (248)$$

$$unitaryOperator(Q^HQ,(Q)) \iff Q^HQ=I$$
$$\text{\# complex analog to orthogonal operator} \quad (249)$$

$$positiveDefiniteOperator(A,(V,+,\cdot,||\$1||)) \iff (\forall_{x \in V \setminus \{\boldsymbol{0}\}}(x^TAx>0)) \vee$$
$$(\forall_{x \in eigenvectors(X,(A,V,+,\$1^T\$1))}(eigenvalue(\lambda,(x,A,V,+,\cdot)) \implies \lambda>0))$$
$$\text{\# acts like a positive scalar where any vector only scales and cannot reflect against its perpendicular axis} \quad (250)$$

$$\text{(THM)}: positiveDefiniteOperator(A^TA) \Longleftarrow \forall_{x \in V \setminus \{\boldsymbol{0}\}}(x^TA^TAx=(Ax)^T(Ax)=||Ax||>0) \quad (251)$$

$$semiPositiveDefiniteOperator(A,(V,+,\cdot,||\$1||)) \iff (\forall_{x \in V \setminus \{\boldsymbol{0}\}}(x^TAx \geq 0)) \vee$$
$$(\forall_{x \in eigenvectors(X,(A,V,+,\$1^T\$1))}(eigenvalue(\lambda,(x,A,V,+,\cdot)) \implies \lambda \geq 0))$$
$$\text{\# acts like a nonnegative scalar} \quad (252)$$

$$\text{(THM)}: symmetricOperator(A^TA) \Longleftarrow (A^TA=(A^TA)^T=A^TA^{TT}=A^TA) \quad (253)$$

$$similarOperators((A,B),()) \iff (matrix(A,(n,n))) \wedge (matrix(B,(n,n))) \wedge (\exists_M(B=M^{-1}AM)) \quad (254)$$

$$\text{(THM)}: (similarOperators((A,B),()) \wedge Ax=\lambda x) \implies (\exists_M(M^{-1}Ax=\lambda M^{-1}x=M^{-1}AMM^{-1}x=BM^{-1}x))$$
$$\text{\# similar operators have the same eigenvalues but } M^{-1} \text{ shifted eigenvectors} \quad (255)$$

$$singularValueDecomposition(Q\Sigma R^T,(A,V,+,\cdot,\langle\$1,\$2\rangle)) \iff (orthogonalOperator(R,(V,+,\cdot,\$1^T\$2)))\wedge (orthogonalOperator(Q,(Img(A),+,\cdot,\$1^T\$2)))\wedge(semiPositiveDefiniteOperator(\Sigma,(V,+,\cdot,\$1^T\$1)))\wedge (AR=Q\Sigma)\wedge(A=Q\Sigma R^{-1}=Q\Sigma R^T)\wedge(symmetricOperator(A^TA))\wedge(symmetricOperator(AA^T))\wedge (A^TA=R\Sigma^TQ^TQ\Sigma R^T=R\Sigma^T\Sigma R^T)\wedge(spectralDecomposition(R(\Sigma^T\Sigma)R^T,(A^TA,V,+,\cdot,\$1^T\$1)))\wedge (AA^T=Q\Sigma R^TR\Sigma^TQ^T=Q\Sigma\Sigma^TQ^T)\wedge(spectralDecomposition(Q(\Sigma\Sigma^T)Q^T,(AA^T,V,+,\cdot,\$1^T\$1)))\wedge (diagonalOperator(\Sigma^T\Sigma)\implies normalOperator(\Sigma^T\Sigma)=\Sigma\Sigma^T=\Sigma_{\sigma^2})\wedge(\Sigma=\Sigma_{\sqrt[2]{\sigma^2}}) \tag{256}$$

---

$$leftInverseOperator(A_L^{-1},(A))\iff(matrix(A,(n,m)))\wedge(rank(A)=n<m)\wedge (A_L^{-1}A=I=((A^TA)^{-1}A^T)A) \tag{257}$$

---

$$rightInverseOperator(A_R^{-1},(A))\iff(matrix(A,(n,m)))\wedge(rank(A)=m<n)\wedge (AA_R^{-1}=I=A(A^T(AA^T)^{-1})) \tag{258}$$

---

$$denseMap\Big(L,\big(D,H,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\iff(D\subseteq H)\wedge\Big(linearOperator\big(L,(D,+,\cdot,H,+,\cdot)\big)\Big)\wedge \bigg(innerProductTopology\Big(\mathcal{O},\big(H,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\bigg)\wedge\bigg(dense\Big(D,\big(H,\mathcal{O},d(\$1,\$2)\big)\Big)\bigg) \tag{259}$$

---

$$mapNorm\Big(||L||,\big(L,V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)\iff \Big(linearOperator\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big)\wedge \Big(normedVectorSpace\big((V,+_V,\cdot_V,||\$1||_V),()\big)\Big)\wedge\Big(normedVectorSpace\big((W,+_W,\cdot_W,||\$1||_W),()\big)\Big)\wedge \left(||L||=sup\left(\left\{\frac{||Lf||_W}{||f||_V}\,|\,f\in V\right\}\right)=sup\Big(\big\{||Lf||_W\,|\,f\in V\wedge||f||=1\big\}\Big)\right) \tag{260}$$

---

$$boundedMap\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)\iff \bigg(mapNorm\Big(||L||,\big(L,V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)<\infty\bigg) \tag{261}$$

---

$$\neg boundedMap\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)\Longleftarrow (U\subset V)\wedge\bigg(\infty=mapNorm\Big(||L||_U,\big(L,U,+_U,\cdot_U,||\$1||_U,W,+_W,\cdot_W,||\$1||_W\big)\Big)\leq||L||\bigg) \tag{262}$$

---

$$extensionMap\Big(\widehat{L},(L,V,D,W)\Big)\iff(D\subseteq V)\wedge\Big(linearOperator\big(L,(D,+_D,\cdot_D,W,+_W,\cdot_W)\big)\Big)\wedge \Big(linearOperator\big(\widehat{L},(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big)\wedge\Big(\forall_{d\in D}\big(\widehat{L}(d)=L(d)\big)\Big) \tag{263}$$

---

$$adjoint\Big(L^T,\big(L,V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big)\iff\Big(hilbertSpace\big((V,+_V,\cdot_V,\langle\$1,\$2\rangle_V),()\big)\Big)\wedge \Big(hilbertSpace\big((W,+_W,\cdot_W,\langle\$1,\$2\rangle_W),()\big)\Big)\wedge\Big(linearOperator\big(L,(V,+_V,\cdot_V,W,+_W,\cdot_W)\big)\Big)\wedge \bigg(\forall_{v\in V}\forall_{w\in W}\Big(\big(\langle Lv,w\rangle_W=\langle v,L^Tw\rangle_V\big)\vee\big((Lv)^Tw=v^TL^Tw\big)\Big)\bigg) \tag{264}$$

\# target operator that acts similar to the domain operator

$$selfAdjoint\Big(L,\big(V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big)\iff$$
$$L=adjoint\Big(L^T,\big(L,V,+_V,\cdot_V,\langle\$1,\$2\rangle_V,W,+_W,\cdot_W,\langle\$1,\$2\rangle_W\big)\Big)$$
$$\#\text{ also a generalization of symmetric matrices}\quad(265)$$

---

$$compactMap(L,(V,+_V,\cdot_V,W,+_W,\cdot_W))\iff\Bigg(boundedMap\Big(L,\big(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W\big)\Big)\Bigg)\wedge$$
$$\Bigg(\forall_{v\in V}\Bigg(openBall\Big(B,\big(1.0,v,V,d_V(\$1,\$2)\big)\Big)\implies$$
$$compactSubset\Bigg(closure\Big(\overline{L(B)},image(L(B),(B,L,V,W)),W,d_W(\$1,\$2)\Big),(W,\mathcal{O}_W)\Bigg)\Bigg)\Bigg)\quad(266)$$

---

(THM) Spectral thm.:
$$\Bigg(selfAdjoint\Big(L,\big(V,+,\cdot,\langle\$1,\$2\rangle,V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\Bigg)\wedge\Big(compactMap(L,(V,+,\cdot,V,+,\cdot))\Big)\implies$$
$$\Bigg(\exists_{(e)_{\mathbb{N}}\subseteq V}\Bigg(orthonormalBasis\Big((e)_{\mathbb{N}},\big(V,+,\cdot,\langle\$1,\$2\rangle\big)\Big)\wedge\forall_{e_n\in(e)_{\mathbb{N}}}\Big(eigenvector(e_n,(L,V,+,\cdot))\Big)\Bigg)\Bigg)\implies$$
$$\Bigg(\exists_{(\lambda)_{\mathbb{N}}\subseteq\mathbb{R}^n}\forall_{e_n\in(e)_{\mathbb{N}}}\exists_{\lambda_n\in(\lambda)_{\mathbb{N}}}\Bigg(eigenvalue(\lambda_n,(e_n,L,V,+,\cdot))\wedge\lim_{n\to\infty}(\lambda_n=0)\wedge L=\sum_{n=1}^{\infty}\Big(\lambda_n e_n {e_n}^T\Big)\Bigg)\Bigg)$$
$$\#\text{ TODO intuition}\quad(267)$$

## 1.19 Function spaces

$$curLp(\mathcal{L}^p,(p,M,\sigma,\mu))\iff(p\in\mathbb{R})\wedge(1\leq p<\infty)\wedge$$
$$\Big(\mathcal{L}^p=\{map(f,(M,\mathbb{R}))\,|\,measurableMap(f,(M,\sigma,\mathbb{R},standardSigma))\wedge\int(|f|^p d\mu)<\infty\}\Big)\quad(268)$$

---

$$vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu))\iff\Big(curLp(\mathcal{L}^p,(p,M,\sigma,\mu))\Big)\wedge\Big(\forall_{f,g\in\mathcal{L}^p}\forall_{m\in M}((f+g)(m)=f(m)+g(m))\Big)\wedge$$
$$\Big(\forall_{f\in\mathcal{L}^p}\forall_{s\in\mathbb{R}}\forall_{m\in M}((s\cdot f)(m)=(s)f(m))\Big)\wedge\Big(vectorSpace((\mathcal{L}^p,+,\cdot),())\Big)\quad(269)$$

---

$$integralNorm(\wr\wr\$1\wr\wr,(+,\cdot,p,M,\sigma,\mu))\iff\Big(vecLp(\mathcal{L}^p,(+,\cdot,p,M,\sigma,\mu))\Big)\wedge\Bigg(map\Big(\wr\wr\$1\wr\wr,\Big(\mathcal{L}^p,\mathbb{R}_0^+\Big)\Big)\Bigg)\wedge$$
$$\Bigg(\forall_{f\in\mathcal{L}^p}\Bigg(0\leq\wr\wr f\wr\wr=\Big(\int(|f|^p d\mu)\Big)^{1/p}\Bigg)\Bigg)\quad(270)$$

---

$$(\text{THM}):integralNorm(\wr\wr\$1\wr\wr,(+,\cdot,p,M,\sigma,\mu))\implies$$
$$\Big(\forall_{f\in\mathcal{L}^p}\Big(\wr\wr f\wr\wr=0\implies almostEverywhere(f=\boldsymbol{0},(M,\sigma,\mu))\Big)\Big)$$
$$\#\text{ not an expected property from a norm}\quad(271)$$

---

$$Lp\Big(L^p,((+,\cdot,p,M,\sigma,\mu))\Big)\iff\Big(integralNorm(\wr\wr\$1\wr\wr,(+,\cdot,p,M,\sigma,\mu))\Big)\wedge$$

$$\left(L^p = quotientSet\left(\mathcal{L}^p/\sim,\left(\mathcal{L}^p,\left(\wr\wr \$1+(-\$2)\wr\wr = 0\right)\right)\right)\right)$$
$$\text{\# functions in } L^p \text{ that have finite integrals above and below the x-axis} \tag{272}$$

---

$$\text{(THM)} : banachSpace\left(\left(Lp(L^p,(+,\cdot,p,M,\sigma,\mu)),+,\cdot,\wr\wr \$1\wr\wr\right),()\right) \tag{273}$$

---

$$\text{(THM)} : hilbertSpace\left(\left(Lp(L^p,(+,\cdot,2,M,\sigma,\mu)),+,\cdot,\frac{\wr\wr \$1+\$2\wr\wr^2-\wr\wr \$1-\$2\wr\wr^2}{4}\right),()\right) \tag{274}$$

---

$$curL\left(\mathcal{L},(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right) \iff \left(banachSpace\left((W,+_W,\cdot_W,||\$1||_W),()\right)\right)\wedge$$
$$\left(normedVectorSpace\left((V,+_V,\cdot_V,||\$1||_V),()\right)\right)\wedge$$
$$\left(\mathcal{L}=\{f\,|\,boundedMap\left(f,(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right)\}\right) \tag{275}$$

---

$$\text{(THM)} : banachSpace\left(\left(curL\left(\mathcal{L},(V,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right),+,\cdot,mapNorm\right),()\right) \tag{276}$$

---

$$\text{(THM)} : ||L|| \geq \frac{||Lf||}{||f||} \text{ \# from choosing an arbitrary element in the mapNorm sup} \tag{277}$$

---

$$\text{(THM)} : \left(cauchy((f)_{\mathbb{N}},(\mathcal{L},+,\cdot,mapNorm)) \Longrightarrow cauchy\left((f_n v)_{\mathbb{N}},(W,+_W,\cdot_W,||\$1||_W)\right)\right) \Longleftarrow$$
$$\left(\forall_{\epsilon'>0}\forall_{v\in V}\left(||f_n v - f_m v||_W = ||(f_n-f_m)v||_W \leq ||f_n-f_m||\cdot||v||_V\right) < \epsilon\cdot||v||_V = \epsilon'\right)$$
$$\text{\# a cauchy sequence of operators maps to a cauchy sequence of targets} \tag{278}$$

---

$$\text{(THM) BLT thm.:} \left(\left(dense(D,(V,\mathcal{O},d_V)) \wedge boundedMap\left(A,(D,+_V,\cdot_V,||\$1||_V,W,+_W,\cdot_W,||\$1||_W)\right)\right) \Longrightarrow\right.$$
$$\left(\exists!_{\widehat{A}}\left(extensionMap\left(\widehat{A},(A,V,D,W)\right)\right)\wedge||\widehat{A}|| = ||A||\right)\right) \Longleftarrow$$
$$\left(\forall_{v\in V}\exists_{(v)_{\mathbb{N}}\subseteq D}\left(\lim_{n\to\infty}(v_n = v)\right)\right)\wedge\left(\widehat{A}v = \lim_{n\to\infty}(Av_n)\right) \tag{279}$$

## 1.20 Probability Theory

$$0 \tag{280}$$

## 1.21 Underview

$$\tag{281}$$

---

$$curve-fitting/explaining \neq prediction \tag{282}$$

$$ill-definedproblem+solutionspaceconstraints \Longrightarrow well-definedproblem \tag{283}$$

$$x \text{ \# input ; } y \text{ \# output} \tag{284}$$

$$S_n=\{(x_1,y_1),\ldots,(x_n,y_n)\} \text{ \# training set} \tag{285}$$

$$f_S(x)\sim y \text{ \# solution} \tag{286}$$

$$each(x,y)\in p(x,y) \text{ \# training data } x,y \text{ is a sample from an unknown distribution } p \tag{287}$$

$$V(f(x),y)=d(f(x),y) \text{ \# loss function} \tag{288}$$

$$I[f]=\int_{X\times Y} V(f(x),y)p(x,y)dxdy \text{ \# expected error} \tag{289}$$

$$I_n[f]=\frac{1}{n}\sum_{i=1}^{n}V(f(x_i),y_i) \text{ \# empirical error} \tag{290}$$

$$probabilisticConvergence(X,()) \Longleftrightarrow \forall_{\epsilon>0} \lim_{n\to\infty} Pxn-x\leq\epsilon=0 \tag{291}$$

$$I-Ingeneralizationerror \tag{292}$$

$$well-posed:=exists,unique,stable;elseill-posed \tag{293}$$

# 2 Machine Learning

### 2.0.1 Overview

$$X \text{ \# input ; } Y \text{ \# output ; } S(X,Y) \text{ \# dataset} \tag{294}$$

$$\textbf{learned parameters}=\textbf{parameters to be fixed by training with the dataset} \tag{295}$$

$$\textbf{hyperparameters}=\textbf{parameters that depends on a dataset} \tag{296}$$

**validation** = **partitions dataset into training and testing partitions, then evaluates the accuracy of the parameters learned from the training partition in predicting the outputs of the testing partition** # useful for fixing hyperparameters (297)

$$\textbf{cross-validation}=\textbf{average accuracy of validation for different choices of testing partition} \tag{298}$$

$$\textbf{L1}=\textbf{scales linearly ; L2}=\textbf{scales quadratically} \tag{299}$$

$$d=\textbf{distance}=\textbf{quantifies the the similarity between data points} \tag{300}$$

$$d_{L1}(A,B)=\sum_p |A_p-B_p| \text{ # Manhattan distance} \tag{301}$$

$$d_{L2}(A,B)=\sqrt{\sum_p (A_p-B_p)^2} \text{ # Euclidean distance} \tag{302}$$

$$\textbf{kNN classifier}=\textbf{classifier based on } k \textbf{ nearest data points} \tag{303}$$

$$s=\textbf{class score}=\textbf{quantifies bias towards a particular class} \tag{304}$$

$$s_{linear}=f_{c\times 1}(x_{n\times 1},W_{c\times n},b_{c\times 1})=W_{c\times n}x_{n\times 1}+b_{c\times 1} \text{ # linear score function} \tag{305}$$

$$l=\textbf{loss}=\textbf{quantifies the errors by the learned parameters} \tag{306}$$

$$l=\frac{1}{|c_i|}\sum_{c_i} l_i \text{ # average loss for all classes} \tag{307}$$

$$l_{SVM_i}=\sum_{y_i\neq c_i}\max(0,s_{y_i}-s_{c_i}+1) \text{ # SVM hinge class loss function:}$$
$$\text{# ignores incorrect classes with lower scores including a non-zero margin} \tag{308}$$

$$l_{MLR_i}=-\log\left(\frac{e^{s_{c_i}}}{\sum_{y_i} e^{y_i}}\right) \text{ # Softmax class loss function}$$
$$\text{# lower scores correspond to lower exponentiated-normalized probabilities} \tag{309}$$

$$R=\textbf{regularization}=\textbf{optimizes the choice of learned parameters to minimize test error} \tag{310}$$

$$\lambda \text{ # regularization strength hyperparameter} \tag{311}$$

$$R_{L1}(W)=\sum_{W_i}|W_i| \text{ # L1 regularization} \tag{312}$$

$$R_{L2}(W)=\sum_{W_i}{W_i}^2 \text{ # L2 regularization} \tag{313}$$

$$L'=L+\lambda R(W) \text{ # weight regularization} \tag{314}$$

$$\nabla_W L=\overrightarrow{\frac{\partial}{\partial W_i}}L=\textbf{loss gradient w.r.t. weights} \tag{315}$$

$$\frac{\partial L_E}{\partial W_I}=\frac{\partial L_L}{\partial W_I}\frac{\partial L_E}{\partial L_L} \text{ # loss gradient w.r.t. input weight in terms of external and local gradients} \tag{316}$$

$$s=\textbf{forward API} \;;\; \frac{\partial L_L}{\partial W_I}=\textbf{backward API} \tag{317}$$

$$W_{t+1} = W_t - \nabla_{W_t} L \text{ \# weight update loss minimization} \quad (318)$$

$$\textbf{TODO:Research on Activation functions, Weight Initialization, Batch Normalization} \quad (319)$$

$$review5meanvardiscussion/hyperparameteroptimization/babysittinglearning \quad (320)$$

TODO loss L or l ??

# 3 Glossary

chaoticTopology
discreteTopology
topology
topologicalSpace
open
closed
clopen
neighborhood
chaoticTopology
discreteTopology
metric
metricSpace
openBall
metricTopology
metricTopologicalSpace
limitPoint
interiorPoint
closure
dense
eucD
standardTopology
subsetTopology
productTopology
metric
metricSpace
openBall
metricTopology
metricTopologicalSpace
limitPoint
interiorPoint
closure
dense
eucD
standardTopology
subsetTopology
productTopology
sequence
sequenceConvergesTo
sequence
sequenceConvergesTo
continuous
homeomorphism
isomorphicTopologicalSpace
continuous
homeomorphism
isomorphicTopologicalSpace
T0Separate
T1Separate
T2Separate
T0Separate
T1Separate
T2Separate
openCover
finiteSubcover
compact
compactSubset
bounded
openCover
finiteSubcover
compact
compactSubset
bounded
openRefinement
locallyFinite
paracompact
openRefinement
locallyFinite
paracompact
connected
pathConnected
connected
pathConnected
sigmaAlgebra
measurableSpace
measurableSet
measure
measureSpace
finiteMeasure
generatedSigmaAlgebra
borelSigmaAlgebra
standardSigma
lebesgueMeasure
measurableMap
pushForwardMeasure
nullSet
almostEverywhere
sigmaAlgebra
measurableSpace
measurableSet
measure
measureSpace
finiteMeasure
generatedSigmaAlgebra
borelSigmaAlgebra
standardSigma
lebesgueMeasure
measurableMap
pushForwardMeasure
nullSet
almostEverywhere
simpleTopology
simpleSigma
simpleFunction
characteristicFunction
exStandardSigma
nonNegIntegrable
nonNegIntegral
explicitIntegral
integrable
integral
simpleTopology
simpleSigma
simpleFunction
characteristicFunction
exStandardSigma
nonNegIntegrable
nonNegIntegral
explicitIntegral
integrable
integral
vectorSpace
innerProduct
innerProductSpace
vectorNorm
normedVectorSpace
vectorMetric
metricVectorSpace
innerProductNorm
normInnerProduct
normMetric
metricNorm
orthogonal
normal
basis
orthonormalBasis
vectorSpace
innerProduct
innerProductSpace
vectorNorm
normedVectorSpace
vectorMetric
metricVectorSpace
innerProductNorm
normInnerProduct
normMetric
metricNorm
orthogonal
normal
basis
orthonormalBasis
subspace
subspaceSum
subspaceDirectSum
orthogonalComplement
orthogonalDecomposition
subspace
subspaceSum
subspaceDirectSum
orthogonalComplement
orthogonalDecomposition
cauchy
complete
banachSpace
hilbertSpace
separable
cauchy
complete
banachSpace
hilbertSpace
separable
linearOperator
matrix
eigenvector
eigenvalue
identityOperator
inverseOperator
transposeOperator
symmetricOperator
triangularOperator
decomposeLU
Img
Ker
independentOperator
dimensonality
rank
transposeNorm
orthogonalVectors
orthogonalOperator
orthogonalProjection
eigenvectors
det
tr
(
diagonalOperator
characteristicEquation
eigenDecomposition
spectralDecomposition
hermitianAdjoint
hermitianOperator
unitaryOperator
positiveDefiniteOperator
semiPositiveDefiniteOperator
similarOperators
similarOperators

singularValueDecomposition
denseMap
mapNorm
boundedMap
extensionMap
adjoint
selfAdjoint
compactMap
linearOperator
matrix
eigenvector
eigenvalue
identityOperator
inverseOperator
transposeOperator
symmetricOperator
triangularOperator
decomposeLU
Img
Ker
independentOperator
dimensonality
rank
transposeNorm
orthogonalVectors
orthogonalOperator
orthogonalProjection
eigenvectors
det
tr
(
diagonalOperator
characteristicEquation
eigenDecomposition
spectralDecomposition
hermitianAdjoint
hermitianOperator
unitaryOperator
positiveDefiniteOperator
semiPositiveDefiniteOperator
similarOperators
similarOperators
singularValueDecomposition
denseMap
mapNorm
boundedMap
extensionMap
adjoint
selfAdjoint
compactMap
curLp
vecLp
integralNorm
Lp
curL
curLp
vecLp
integralNorm
Lp
curL