

SHEDDING VERTICES AND ASS-DECOMPOSABLE MONOMIAL IDEALS

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ABSTRACT. The shedding vertices of simplicial complexes are studied from an algebraic point of view. Based on this perspective, we introduce the class of ass-decomposable monomial ideals which is a generalization of the class of Stanley-Reisner ideals of vertex decomposable simplicial complexes. The recursive structure of ass-decomposable monomial ideals allows us to find a simple formula for the depth, and in squarefree case, an upper bound for the regularity of such ideals.

INTRODUCTION

A *simplicial complex* Δ on the vertex set V is a collection of subsets of V , such that $\cup_{F \in \Delta} F = V$ and Δ is closed under the operation of taking subsets. The elements of Δ are called *faces*. The maximal faces of Δ , under inclusion, are called the *facets* of Δ and a simplicial complex with facets F_1, \dots, F_m is often denoted by $\langle F_1, \dots, F_m \rangle$. A simplicial complex with only one facet is called a *simplex*. For any face $F \in \Delta$ the *link* and the *deletion* of F in Δ are defined as

$$\begin{aligned}\text{link}_\Delta(F) &= \{G \in \Delta : G \cap F = \emptyset \text{ and } G \cup F \in \Delta\}, \\ \text{del}_\Delta(F) &= \{G \in \Delta : G \cap F = \emptyset\}.\end{aligned}$$

A vertex $v \in V$ is called a *shedding vertex* of Δ if any facet of $\text{del}_\Delta(v)$ is a facet of Δ and Δ is called *vertex decomposable*, if either it is a simplex or else there exists a shedding vertex $v \in V$ such that both $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are vertex decomposable. Vertex decompositions were introduced in the pure case by Provan and Billera [19] and extended to non-pure complexes by Björner and Wachs [3, Section 11]. Vertex decomposable simplicial complexes form a well-behaved class of simplicial complexes and have been studied in several literatures ([1, 2, 6, 9, 16, 17, 18, 20, 24, 26, 27]).

The shedding vertices of simplicial complexes play a crucial role in the combinatorics of simplicial complexes. In [25, Lemma 6] it is shown that if Δ has a shedding vertex such that both $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are shellable,

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then Δ is shellable. It turns out that the vertex decomposable simplicial complexes are shellable and hence the Stanley-Reisner ring of such simplicial complexes are sequentially Cohen-Macaulay. Recall that Δ is called *shellable* if there is an ordering F_1, \dots, F_m of the facets of Δ such that the intersection $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is generated by some maximal proper subsets of F_i , for $i = 1, \dots, m$. Based on the above arguments we have the following well-known implications which both of them are known to be strict.

vertex decomposable \implies shellable \implies sequentially Cohen-Macaulay.

In this paper, we are interested in algebraic counterpart of shedding vertices. Let I_Δ denote the Stanley-Reisner ideal of Δ . Then v is a shedding vertex of Δ precisely when the minimal prime ideals of $I_\Delta + (x_v)$ are those minimal prime ideals of I_Δ that contain x_v , see Proposition 1.1. In view of this characterization of shedding vertices, one of the main results of Section 1 asserts that Δ is sequentially Cohen-Macaulay, if Δ has a shedding vertex such that both $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are sequentially Cohen-Macaulay (cf. Theorem 1.3). This gives a direct proof for the implication

vertex decomposable \implies sequentially Cohen-Macaulay.

Motivated by Proposition 1.1, in Section 2, we introduce a new class of monomial ideals, called ass-decomposable monomial ideals. More precisely, a monomial ideal I is called *ass-decomposable*, if either it is a primary ideal or else there exist a variable $x \notin \sqrt{I}$ and a positive integer k such that $\text{ass}(I, x^k) = \{\mathfrak{p} \in \text{ass}(I) ; x \in \mathfrak{p}\}$, and the monomial ideals (I, x^k) and $I : x^k$ are ass-decomposable. The monomial x^k is called a *decomposing monomial* of I . One can see that squarefree ass-decomposable monomial ideals generated in degree greater than 1, are nothing but the Stanley-Reisner ideals of vertex decomposable simplicial complexes (cf. Corollary 2.2). Though that ass-decomposable monomial ideals are not sequentially Cohen-Macaulay in general, but using the recursive structure of ass-decomposability, we show in Theorem 2.6 that the depth formula of such ideals is as the same as the depth formula for sequentially Cohen-Macaulay modules. More precisely, we have

$$\text{depth}(R/I) = \min\{\dim(R/\mathfrak{p}) ; \mathfrak{p} \in \text{ass}(I)\},$$

in which R denotes the polynomial ring where the ass-decomposable monomial ideal I lives in.

The ass-decomposability, somehow, comes from comparing the associated prime ideals. In this regard, the unmixed property is very effective for I . Indeed ass-decomposability is preserved under taking radical for unmixed ideals (cf. Lemma 2.5). As an immediate consequence of the above depth formula, we derive that unmixed ass-decomposable monomial ideals are Cohen-Macaulay (cf. Corollary 2.8).

Recall that a finitely generated \mathbb{Z}^n -graded module M is said to satisfy the *Stanley's inequality*, if $\text{sdepth}(M) \geq \text{depth}(M)$, where $\text{sdepth}(M)$ denote the Stanley depth of M . In fact, Stanley [23] conjectured that all \mathbb{Z}^n -graded modules over n -dimensional polynomial rings satisfy the Stanley's inequality. This conjecture has been disproved in [11]. However, it is still interesting to find some classes of modules which satisfy Stanley's inequality. For a survey on this topic, we refer the reader to [13]. The unmixed ass-decomposable monomial ideals satisfy the Stanley's inequality (cf. Corollary 2.7).

The last part of Section 2 is devoted to find an upper bound for the Castelnuovo-Mumford regularity of squarefree ass-decomposable monomial ideals. We show that the regularity of a squarefree ass-decomposable monomial ideal is at most the index of irreducibility of the ideal and we characterize when this upper bound holds.

1. SHEDDING VERTICES OF SIMPLICIAL COMPLEXES

A *simplicial complex* Δ on the vertex set $V = V(\Delta)$ is a collection of subsets of V , called *faces* of Δ , such that (i) $\{v\} \in \Delta$, for all $v \in V$, (ii) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. The set of maximal faces of Δ with respect to inclusion, which are called *facets*, is denoted by $\mathcal{F}(\Delta)$. If $\mathcal{F}(\Delta) = \{F_1, \dots, F_m\}$, we often write $\Delta = \langle F_1, \dots, F_m \rangle$ and Δ is called a *simplex*, if $\mathcal{F}(\Delta)$ is a singleton. Also the i th pure skeleton of Δ is the simplicial complex $\Delta^{[i]} = \langle F \in \Delta : |F| = i \rangle$.

All over this section, Δ is a simplicial complex on $V = [n]$ and $R = \mathbb{K}[x_1, \dots, x_n]$ denotes the polynomial ring in n variables over a field \mathbb{K} . For each subset $F \subseteq [n]$ we set

$$\mathbf{x}_F = \prod_{i \in F} x_i.$$

The *Stanley-Reisner ideal* of Δ is the ideal I_Δ of R , which is generated by those squarefree monomials \mathbf{x}_F with $F \notin \Delta$. A simplicial complex Δ is called *Cohen-Macaulay* if the quotient ring $\mathbb{K}[\Delta] = R/I_\Delta$ is Cohen-Macaulay and Δ is called *sequentially Cohen-Macaulay*, if all pure i -skeletons of Δ are Cohen-Macaulay. It turns out that Δ is sequentially Cohen-Macaulay if and only if the quotient ring R/I_Δ is sequentially Cohen-Macaulay [10, Theorem 3.3].

For a subset $F \subseteq [n]$, let $\bar{F} = [n] \setminus F$ and $P_{\bar{F}}$ be the prime ideal $(x_i ; i \in [n] \setminus F)$. It follows from [5, Theorem 5.1.4] that

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\bar{F}},$$

is a minimal prime decomposition of I_Δ .

The *link* of a face F in Δ is defined as

$$\text{link}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\},$$

and the *deletion* of F is the simplicial complex

$$\text{del}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset\}.$$

A vertex $v \in V(\Delta)$ is called a *shedding* vertex of Δ if any facet of $\text{del}_\Delta(v)$ is a facet of Δ . It turns out that v is a shedding vertex of Δ if and only if no face of $\text{link}_\Delta(v)$ is a facet of $\text{del}_\Delta(v)$. A simplicial complex Δ is called *vertex decomposable*, if either it is a simplex or else there is a shedding vertex $v \in V(\Delta)$ such that both $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are vertex decomposable.

The following proposition yields an algebraic counterpart of shedding vertices and will be useful for introducing a class of monomial ideals as a generalization of Stanley-Reisner ideals of vertex decomposable simplicial complexes for non-squarefree cases.

Proposition 1.1. *The following statements are equivalent for a vertex v of a simplicial complex Δ .*

- (i) v is a shedding vertex of Δ ,
- (ii) $\text{ass}(I_\Delta, x_v) = \{\mathfrak{p} \in \text{ass}(I_\Delta) : x_v \in \mathfrak{p}\}$.

Proof. Let $I_\Delta = \cap_{i=1}^r \mathfrak{p}_i$ be a minimal prime decomposition of I_Δ . We may assume that $x_v \in \cap_{i=1}^s \mathfrak{p}_i$ and $x_v \notin \cup_{i=s+1}^r \mathfrak{p}_i$, for some $1 \leq s \leq r$. For all $1 \leq i \leq r$, let $\mathfrak{p}_i = P_{\bar{F}_i}$ where $F_i \in \Delta$.

(i) \Rightarrow (ii) Fix $s+1 \leq i \leq r$. Since $v \in F_i$ is a shedding vertex of Δ , we conclude that $F_i \setminus \{v\} \in \text{link}_\Delta(v)$ is not a facet of $\text{del}_\Delta(v)$. Hence there exists $G \in \mathcal{F}(\text{del}_\Delta(v))$ such that $F_i \subset G$. Indeed, $G \in \mathcal{F}(\Delta)$ and $v \notin G$. Thus $G = F_t$, for some $1 \leq t \leq s$. Moreover, $\bar{F}_t = \bar{G} \subseteq \bar{F}_i \cup \{v\}$ and consequently $\mathfrak{p}_t = \mathfrak{p}_{\bar{F}_t} \subseteq (\mathfrak{p}_i, x_v)$. This shows that

$$(I_\Delta, x_v) = \left(\bigcap_{j=1}^s \mathfrak{p}_j \right) \cap \left(\bigcap_{i=s+1}^r (\mathfrak{p}_i, x_v) \right) = \bigcap_{j=1}^s \mathfrak{p}_j,$$

is a minimal prime decomposition of $I_\Delta + (x_v)$.

(ii) \Rightarrow (i) Our hypothesis implies that for all $s+1 \leq i \leq r$, there exists $1 \leq j_i \leq s$ such that $\mathfrak{p}_{j_i} \subseteq (\mathfrak{p}_i, x_v)$. Translating this in terms of facets of Δ , we conclude that no facet of $\text{link}_\Delta(v)$ is a facet of $\text{del}_\Delta(v)$. \square

For a vertex v of Δ , let $I_\Delta(v) = (x_v ; x_v x_w \in I_\Delta)$. The following remark enables us to find the minimal primary decomposition of $I_\Delta : x_v$ and $I + (x_v)$ in terms of those of I , when v is a shedding vertex of Δ .

Remark 1. Let v be a shedding vertex of Δ and $I_\Delta = \cap_{i=1}^r \mathfrak{p}_i$ be a minimal prime decomposition of I_Δ . Assume without loss of generality that $x_v \in \cap_{i=1}^s \mathfrak{p}_i$ and $x_v \notin \cup_{i=s+1}^r \mathfrak{p}_i$ for some $1 \leq s \leq r$. Let $\mathfrak{p}_i = P_{\bar{F}_i}$ for $F_i \in \mathcal{F}(\Delta)$ and $i = 1, \dots, r$. Then $v \notin \cup_{i=1}^s F_i$ and $v \in \cap_{i=s+1}^r F_i$. For $i = s+1, \dots, r$, let $Q_i = F_i \setminus \{v\}$. Then

- (1) $\text{del}_\Delta(v) = \langle F_1, \dots, F_s \rangle$.
- (2) $\text{link}_\Delta(v) = \langle Q_{s+1}, \dots, Q_r \rangle$.
- (3) $(I_\Delta, x_v) = \cap_{i=1}^s \mathfrak{p}_i = I_{\text{del}_\Delta(v)} + (x_v)$.
- (4) $I_\Delta : x_v = \cap_{i=s+1}^r \mathfrak{p}_i = I_{\text{link}_\Delta(v)} + I_\Delta(v)$.
- (5) If Δ is vertex decomposable, then $\text{link}_\Delta(w)$ is also vertex decomposable, for any vertex w (cf. [27, Proposition 3.7]).

It is known that if Δ has a shedding vertex such that both $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are shellable, then Δ is shellable ([25, Lemma 6]). Note that the class of shellable simplicial complexes is a subclass of sequentially Cohen-Macaulay complexes, so it is natural to ask if the result holds replacing shellable simplicial complexes with sequentially Cohen-Macaulay complexes. Our algebraic characterization of shedding vertices enables us to obtain the later result for sequentially Cohen-Macaulay complex as well (cf. Theorem 1.3). Recall that a finitely generated (graded) R -module M is called *sequentially Cohen-Macaulay*, if there exists a filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = M \quad (1)$$

of (graded) submodules of M such that each quotient M_i/M_{i-1} is Cohen-Macaulay and

$$\dim M_1/M_0 < \dim M_2/M_1 < \dots < \dim M_r/M_{r-1}.$$

The following observation yields a simple method for computing the depth of a sequentially Cohen-Macaulay module and will be used in Theorem 1.3.

Observation 1.2. Let M be a finitely generated d -dimensional sequentially Cohen-Macaulay R -module. By [22, Lemma 5.4], one has

$$H_{\mathfrak{m}}^i(M) = H_{\mathfrak{m}}^i(D_i(M)) = H_{\mathfrak{m}}^i(D_i(M)/D_{i-1}(M)), \quad (2)$$

for all $i = 0, \dots, d$, where $H_{\mathfrak{m}}^i(-)$ denotes the i th local cohomology functor with support in \mathfrak{m} and $D_i(M)$ denotes the largest submodule of M such that $\dim D_i(M) \leq i$. Note that $H_{\mathfrak{m}}^j(D_i(M)/D_{i-1}(M)) = 0$, for all $j < i$, by vanishing theorem of Grothendieck (c.f. [4, Theorem 6.1.2]). Now, by virtue of [14, Proposition 2.5], we obtain:

$$\begin{aligned} \text{depth}(M) &= \min\{i : H_{\mathfrak{m}}^i(M) \neq 0\} \\ &= \min\{\dim R/\mathfrak{p} ; \mathfrak{p} \in \text{ass}(M)\}. \end{aligned} \quad (3)$$

Let I be an ideal in the polynomial ring R and $\text{bight}(I) = \max\{\text{height}(\mathfrak{p}) ; \mathfrak{p} \in \text{ass}(I)\}$ denotes the big height of I . Then the above observation shows that $\text{depth}(R/I) = n - \text{bight}(I)$, if R/I is sequentially Cohen-Macaulay.

The following theorem shows the importance of shedding vertices in inductive process for squarefree monomial ideals.

Theorem 1.3. *Let Δ be a simplicial complex and $I = I_\Delta$ be the Stanley-Reisner ideal of Δ . If Δ has a shedding vertex v such that $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are sequentially Cohen-Macaulay/Cohen-Macaulay/shellable, then*

- (i) $\text{depth}(R/I) = \min \{ \text{depth}(R/(I : x_v)), \text{depth}(R/(I, x_v)) \} = n - \text{bight}(I);$
- (ii) Δ is sequentially Cohen-Macaulay/Cohen-Macaulay/shellable.

Proof. (i) It is well-known that shellable simplicial complexes or Cohen-Macaulay complexes are sequentially Cohen-Macaulay. So it is enough to prove (i) for sequentially Cohen-Macaulay complexes. We note that $(I, x_v) = (x_v) + I_{\text{del}_\Delta(v)}$ is sequentially Cohen-Macaulay complex, by our assumption. Since $\text{link}_\Delta(v)$ is sequentially Cohen-Macaulay, we derive that the ideal $(I : x_v)$ is sequentially Cohen-Macaulay (see Remark 1(4)). Hence by (3), we conclude that

$$\begin{aligned} \text{depth}(R/(I, x_v)) &= \min \{ \dim(R/\mathfrak{p}) : \mathfrak{p} \in \text{ass}((I, x_v)) \}, \\ \text{depth}(R/(I : x_v)) &= \min \{ \dim(R/\mathfrak{p}) : \mathfrak{p} \in \text{ass}((I : x_v)) \}. \end{aligned} \quad (4)$$

It follows from Proposition 1.1 that $\text{ass}(I, x_v) \subset \text{ass}(I)$. Clearly, $\text{ass}(I : x_v) \subset \text{ass}(I)$. So that $\text{ass}(I) = \text{ass}(I, x_v) \cup \text{ass}(I : x_v)$. By [5, Proposition 1.2.13], we have $\text{depth}(R/I) \leq \dim R/\mathfrak{p}$, for all $\mathfrak{p} \in \text{ass}(I)$. Using (4), we conclude that

$$\text{depth}(R/I) \leq \min \{ \text{depth}(R/(I : x_v)), \text{depth}(R/(I, x_v)) \}. \quad (5)$$

On the other hand, from the short exact sequence

$$0 \longrightarrow R/(I : x_v) \longrightarrow R/I \longrightarrow R/(I, x_v) \longrightarrow 0,$$

we obtain

$$\text{depth}(R/I) \geq \min \{ \text{depth}(R/(I : x_v)), \text{depth}(R/(I, x_v)) \}. \quad (6)$$

The inequalities (5) and (6) yield

$$\begin{aligned} \text{depth}(R/I) &= \min \{ \text{depth}(R/(I : x_v)), \text{depth}(R/(I, x_v)) \} \\ &= \min \{ \dim(R/\mathfrak{p}) : \mathfrak{p} \in \text{ass}(I) \} = n - \text{bight}(I). \end{aligned}$$

(ii) First we show that Δ is sequentially Cohen-Macaulay provided that $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are sequentially Cohen-Macaulay. Let $\Delta^{[i]}$ denotes the pure i -skeleton of the simplicial complex Δ and $I_i = I_{\Delta^{[i]}}$. One may easily check that $\text{link}_{\Delta^{[i]}}(v) = (\text{link}_\Delta(v))^{[i-1]}$ and $\text{del}_{\Delta^{[i]}}(v) = (\text{del}_\Delta(v))^{[i]}$. So that $\text{link}_{\Delta^{[i]}}(v)$ and $\text{del}_{\Delta^{[i]}}(v)$ are both Cohen-Macaulay, by our assumption. If $\Delta^{[i]}$ is a simplex, then clearly $\Delta^{[i]}$ is Cohen-Macaulay. So assume that $\Delta^{[i]}$ is not a simplex. Then it is easily seen that v is a shedding vertex in $\Delta^{[i]}$.

Applying part (i) of the statement for the simplicial complex $\Delta^{[i]}$, together with the fact that $R/(I_i, x_v)$ and $R/(I_i : x_v)$ are Cohen-Macaulay, we find

that $\Delta^{[i]}$ is Cohen-Macaulay. Hence Δ is sequentially Cohen-Macaulay. In the case that $\text{link}_\Delta(v)$ and $\text{del}_\Delta(v)$ are Cohen-Macaulay, the same discussion as above shows that Δ is Cohen-Macaulay, while the shellable case follows from [25, Lemma 6]. \square

2. ASS-DECOMPOSABLE MONOMIAL IDEALS

Throughout this section $R = \mathbb{K}[x_1, \dots, x_n]$ denotes the polynomial ring in n variables over a field \mathbb{K} endowed with the standard grading, that is $\deg(x_i) = 1$. Several algebraic interpretations have been considered for the concept of vertex decomposability. A simplicial complex Δ is vertex decomposable if and only if I_{Δ^\vee} is 0-decomposable in the sense of [20], which holds precisely when I_{Δ^\vee} is a vertex splittable ideal introduced in [17]. In the following, we are looking for an algebraic property of I_Δ to characterize the vertex decomposability of Δ . This property is not restricted to squarefree monomial ideals.

Definition 2.1. A monomial ideal I is called *ass-decomposable*, if either it is a primary ideal or else there exist a variable $x_i \notin \sqrt{I}$ and a positive integer k such that

- (i) $\text{ass}(I, x_i^k) = \{\mathfrak{p} \in \text{ass}(I) ; x_i \in \mathfrak{p}\}$.
- (ii) (I, x_i^k) and $I : x_i^k$ are ass-decomposable.

The monomial x_i^k is called a *decomposing monomial* of I .

Remark 2. Let I be a squarefree ass-decomposable monomial ideal with decomposing monomial x_i^k . Let \mathfrak{p} be a minimal prime ideal of I such that $x_i \notin \mathfrak{p}$. Then there exists a minimal prime ideal \mathfrak{q} of I such that $x_i \in \mathfrak{q}$ and $\mathfrak{q} \subseteq \mathfrak{p} + (x_i^k)$. Therefore, $k = 1$.

As a consequence of Proposition 1.1 and Remarks 1 and 2, we derive the following.

Corollary 2.2. I_Δ is ass-decomposable if and only if Δ is vertex decomposable.

Example 2.3. Let $I = \bigcap_{i=1}^4 \mathfrak{q}_i$, where $\mathfrak{q}_1 = (x_1, x_2)$, $\mathfrak{q}_2 = (x_3, x_1)$, $\mathfrak{q}_3 = (x_1, x_2^2, x_3^2)$, $\mathfrak{q}_4 = (x_1, x_3^2, x_4)$, $\mathfrak{q}_5 = (x_3^2, x_4, x_5)$ in $R = \mathbb{K}[x_1, \dots, x_5]$. Then

$$\begin{aligned} I + (x_3^2) &= \bigcap_{i=2}^5 \mathfrak{q}_i \quad ; \quad I : x_3^2 = \mathfrak{q}_1 \\ \left(\bigcap_{i=2}^5 \mathfrak{q}_i \right) + (x_4) &= \mathfrak{q}_4 \cap \mathfrak{q}_5 \quad ; \quad \left(\bigcap_{i=2}^5 \mathfrak{q}_i \right) : x_4 = \mathfrak{q}_2 \cap \mathfrak{q}_3 \\ (\mathfrak{q}_2 \cap \mathfrak{q}_3) + (x_2^2) &= \mathfrak{q}_3 \quad ; \quad (\mathfrak{q}_2 \cap \mathfrak{q}_3) : x_2^2 = \mathfrak{q}_2 \\ (\mathfrak{q}_4 \cap \mathfrak{q}_5) + (x_5) &= \mathfrak{q}_5 \quad ; \quad (\mathfrak{q}_4 \cap \mathfrak{q}_5) : x_5 = \mathfrak{q}_4 \end{aligned}$$

Therefore I is ass-decomposable.

Let $I \subseteq R$ be an ass-decomposable monomial ideal. Since the Stanley-Reisner ideal of a vertex decomposable simplicial complex is sequentially Cohen-Macaulay, it follows from Corollary 2.2 that R/I is sequentially Cohen-Macaulay, provided that I is a squarefree ass-decomposable monomial ideal. The following example shows that this is not the case if I is not squarefree. This example also shows that the radical of an ass-decomposable ideal need not to be an ass-decomposable ideal.

Example 2.4. Let $I = \cap_{i=1}^3 \mathfrak{q}_i$, where $\mathfrak{q}_1 = (x_1, x_2)$, $\mathfrak{q}_2 = (x_3, x_4)$ and $\mathfrak{q}_3 = (x_1, x_3^2, x_4)$ in $R = \mathbb{K}[x_1, x_2, x_3, x_4]$. Then

$$\begin{aligned} I + (x_1) &= \mathfrak{q}_1 \cap \mathfrak{q}_3 \quad ; \quad I : x_1 = \mathfrak{q}_2 \\ (\mathfrak{q}_1 \cap \mathfrak{q}_3) + (x_2) &= \mathfrak{q}_1 \quad ; \quad (\mathfrak{q}_1 \cap \mathfrak{q}_3) : x_2 = \mathfrak{q}_3 \end{aligned}$$

Therefore I is ass-decomposable. If R/I is sequentially Cohen-Macaulay, then $R/(\mathfrak{q}_1 \cap \mathfrak{q}_2)$ should be Cohen-Macaulay by [15, Lemma 2.21], which is not the case because $\text{depth } R/(\mathfrak{q}_1 \cap \mathfrak{q}_2) = 1 < 2 = \dim R/(\mathfrak{q}_1 \cap \mathfrak{q}_2)$. Note that $\sqrt{I} = (x_1, x_2) \cap (x_3, x_4)$ is not an ass-decomposable monomial ideal.

In the above example, the quadratic monomial in \mathfrak{q}_3 allows to get ass-decomposability along with producing an associated prime ideal $\mathfrak{p}_3 \supset \mathfrak{p}_2$. More precisely, \sqrt{I} is not ass-decomposable, since \mathfrak{p}_3 is omitted from its associated prime ideals. The following result shows that ass-decomposability of an unmixed monomial ideal implies the ass-decomposability of its radical.

Lemma 2.5. *If I is an unmixed ass-decomposable monomial ideal, then \sqrt{I} is also ass-decomposable.*

Proof. Let $I = \cap_{i=1}^r \mathfrak{q}_i$ be a minimal primary decomposition of I , where \mathfrak{q}_i is \mathfrak{p}_i -primary. As I is unmixed, $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i \neq j$. It follows that $\sqrt{I} = \cap_{i=1}^r \mathfrak{p}_i$ is a minimal prime decomposition of \sqrt{I} .

We use induction on $r = |\text{ass}(I)|$ to show that \sqrt{I} is ass-decomposable. If $r = 1$, we have nothing to prove. So assume that $r > 1$ and the assertion holds for any unmixed ass-decomposable monomial ideal whose number of irreducible components is less than r . Let the variable $x \notin \sqrt{I}$ and the positive number k be as in the Definition 2.1. Without loss of generality, assume that $x \in \cap_{i=1}^t \mathfrak{p}_i \setminus \cup_{j=t+1}^r \mathfrak{p}_j$, for some $1 \leq t < r$. Since $\text{ass}(I, x^k) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$, we conclude that $\text{ass}(\sqrt{I}, x) = \text{ass}(\sqrt{(I, x^k)}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Note that $I + (x^k)$ and $I : x^k = \cap_{i=t+1}^r \mathfrak{q}_i$ satisfy in our induction hypothesis, so that $\sqrt{(I, x^k)} = (\sqrt{I}, x)$ and $\sqrt{(I : x^k)} = \sqrt{I} : x$ are ass-decomposable. Thus \sqrt{I} is ass-decomposable. \square

Remark 3. Let $I = \cap_{i=1}^r \mathfrak{p}_i$ be a minimal prime decomposition of a squarefree monomial ideal I , with $r \geq 2$. It's not difficult to observe that

$$\mathfrak{p}_j^k \not\subseteq (x_s^t) + \mathfrak{p}_i^k \text{ and } (x_s^t) + \mathfrak{p}_i^k \not\subseteq \mathfrak{p}_j^k,$$

for all $1 \leq j \neq i \leq r$, $1 \leq s \leq n$, $k \geq 2$ and $t > 1$. Therefore, $\text{ass}(I, x_s^t)$ has r elements, which follows that the k th symbolic power of I , for $k \geq 2$, is never ass-decomposable. In particular, the converse of Lemma 2.5 does not hold.

Let $I \subseteq R$ be an ass-decomposable monomial ideal. In spite of the fact that R/I is not necessarily sequentially Cohen-Macaulay, but the following result shows the depth of R/I can be computed by the same formula as in (3).

Theorem 2.6. *If I is an ass-decomposable ideal, then*

$$\text{depth}(R/I) = \min\{\text{depth}(R/\mathfrak{p}) ; \mathfrak{p} \in \text{ass}(I)\} = n - \text{bight}(I)$$

Proof. Let $I = \cap_{i=1}^r \mathfrak{q}_i$ be a minimal primary decomposition of the monomial ideal $I \subset R$. We proceed by induction on $r = |\text{ass}(I)|$. If $r = 1$, then R/I is Cohen-Macaulay and there is nothing to prove. Assume that $r > 1$ and the result is true for ass-decomposable monomial ideals with less than r irreducible components. Consider the exact sequence

$$0 \longrightarrow R/(I : x^k) \longrightarrow R/I \longrightarrow R/(I, x^k) \longrightarrow 0.$$

Then $\text{depth}(R/I) \geq \min\{\text{depth}(R/(I : x^k)), \text{depth}(R/(I, x^k))\}$. As $(I : x^k)$ and (I, x^k) satisfy the induction hypothesis, and $\text{ass}(I) = \text{ass}(I : x^k) \cup \text{ass}(I, x^k)$, we derive

$$\text{depth}(R/I) \geq \min\{\text{depth}(R/\mathfrak{p}); \mathfrak{p} \in \text{ass}(I)\},$$

which implies the result together with [5, Proposition 1.2.13]. \square

Corollary 2.7. *If I is an unmixed ass-decomposable ideal then*

$$\text{sdepth}(R/I) \geq \text{depth}(R/I).$$

Proof. Since I is unmixed, $\text{depth}(R/I) = \text{depth}(R/(I : x^k)) = \text{depth}(R/(I, x^k))$, by Theorem 2.6. Now using induction on r and the fact from [21, Lemma 2.2] that

$$\text{sdepth}(R/I) \geq \min\{\text{sdepth}(R/(I : x^k)), \text{sdepth}(R/(I, x^k))\},$$

we get the desired inequality. \square

As Example 2.3 shows, for an ass-decomposable monomial ideal I , the quotient ring R/I is not necessarily sequentially Cohen-Macaulay. However, an unmixed ass-decomposable monomial ideal is Cohen-Macaulay.

Corollary 2.8. *Let I be an unmixed ass-decomposable monomial ideal. Then, R/I is Cohen-Macaulay.*

Proof. Since I is unmixed, $\dim(R/I) = \dim(R/\mathfrak{p})$ for all $\mathfrak{p} \in \text{ass}(I)$. Now, Theorem 2.6 implies the result. \square

In the rest of this section, we study the Castelnuovo-Mumford regularity of squarefree ass-decomposable monomial ideals i.e. the Stanley-Reisner ideals of vertex decomposable simplicial complexes. Let $I \subseteq S$ be a monomial ideal and

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow I \rightarrow 0$$

be graded minimal free resolution of I with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}^{\mathbb{K}}(I)}$, for all i . The numbers $\beta_{i,j}^{\mathbb{K}}(I) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(I, \mathbb{K})_j$ are called the *graded Betti numbers* of I and the *Castelnuovo-Mumford regularity* of $I \neq 0$, $\operatorname{reg}(I)$, is given by

$$\operatorname{reg}(I) = \sup\{j - i : \beta_{i,j}^{\mathbb{K}}(I) \neq 0\}.$$

Note that $\operatorname{reg}(R/I) = \operatorname{reg}(I) - 1$. We say that I has a *d-linear resolution* if I is generated by monomials of degree $d = \operatorname{reg}(I)$.

Proposition 2.9. *Let $I = \bigcap_{i=1}^r \mathfrak{p}_i$ be a minimal prime decomposition of the squarefree monomial ideal I . If I is ass-decomposable then $\operatorname{reg}(R/I) \leq r - 1$.*

Proof. We proceed by induction on r to prove the assertion. If $r = 1$, we have nothing to prove. Assume that $r > 1$ and the assertion holds for all squarefree ass-decomposable monomial ideals with less than r irreducible components. Let x be a decomposing variable of I and consider the short exact sequence

$$0 \longrightarrow R/(I : x)(-1) \longrightarrow R/I \longrightarrow R/(I, x) \longrightarrow 0.$$

Using the above exact sequence along with [12, Corollary 20.19], we get

$$\begin{aligned} \operatorname{reg}(R/I) &\leq \max\{\operatorname{reg}(R/(I : x)) + 1, \operatorname{reg}(R/(I, x))\} \\ &\leq (r - 2) + 1 = r - 1. \end{aligned}$$

The last inequality follows from induction hypothesis. □

If $I = \bigcap_{i=1}^r \mathfrak{p}_i$ is a minimal prime decomposition of a squarefree monomial ideal I , then r is said to be the *index of irreducibility* of I . Let $\mathcal{G}(I)$ denote the minimal set of monomial generators of I and $d(I) = \max\{\deg(u) ; u \in \mathcal{G}(I)\}$. It is immediate consequence of the definition of regularity that $\operatorname{reg}(I) \geq d(I)$.

Proposition 2.10. *Let I be a squarefree ass-decomposable monomial ideal with decomposing variable x and the index of irreducibility r . Then the following statements are equivalent.*

- (i) $\operatorname{reg}(R/I) = r - 1$.
- (ii) $d(I) = r$.

If the above conditions hold, then $I = \mathfrak{p} + xJ$ where \mathfrak{p} is a monomial prime ideal and J is ass-decomposable monomial ideal.

Proof. If (ii) holds, then (i) follows from Proposition 2.9 and the obvious relations $\text{reg}(R/I) = \text{reg}(I) - 1 \geq d(I) - 1$. Now, assume that (i) holds. We use induction on r to obtain (ii). If $r = 1$, there is nothing to prove. Assume that $r > 1$ and the assertion holds for any squarefree ass-decomposable monomial ideal J whose regularity of R/J is less than $r - 1$. Let $I = \cap_{i=1}^r \mathfrak{p}_i$ be a minimal prime decomposition of I such that $x \notin \cup_{i=1}^s \mathfrak{p}_i$ and $x \in \cap_{i=s+1}^r \mathfrak{p}_i$, for some $1 \leq s < r$.

By [7, Lemma 2.10], we have

$$\text{reg}(R/I) \in \{\text{reg}(R/(I : x)) + 1, \text{reg}(R/(I, x))\}.$$

As $\text{reg}(R/(I, x)) \leq r - 2$, by Proposition 2.9, we derive

$$r - 1 = \text{reg}(R/I) = \text{reg}(R/(I : x)) + 1.$$

Thus $\text{reg}(R/(I : x)) = r - 2$. Since $I : x$ is a squarefree ass-decomposable monomial ideal, it follows from induction hypothesis that $d(I : x) = r - 1$. On the other hand, proposition 2.9 implies that $r - 2 \leq s - 1$, and hence $s = r - 1$. Let \mathfrak{p} be the prime ideal generated by $\mathcal{G}(\mathfrak{p}_r) \setminus \{x\}$. Since $\mathfrak{p}_r \subseteq \mathfrak{p}_i + (x)$, for $i = 1, \dots, r - 1$, we get $\mathfrak{p} \subseteq \cap_{i=1}^{r-1} \mathfrak{p}_i$. Therefore, $I : x = \cap_{i=1}^{r-1} \mathfrak{p}_i = \mathfrak{p} + J$ for a monomial ideal J whose support does not contain any $y \in \mathcal{G}(\mathfrak{p}_r)$. Thus

$$I = (\mathfrak{p} + J) \cap (\mathfrak{p}, x) = \mathfrak{p} + xJ. \quad (7)$$

Note that $r - 1 = d(I : x) = d(J)$, hence $d(I) = d(J) + 1 = (r - 1) + 1 = r$, in view of (7). Furthermore, J is ass-decomposable because $\mathfrak{p} + J = \cap_{i=1}^{r-1} \mathfrak{p}_i = I : x$ is ass-decomposable. \square

As a consequence of Propositions 2.9 and 2.10, we have the following result for the edge ideal of chordal graphs. First we recall the required definitions.

Let $G = (V, E)$ be a finite simple graph. The *edge ideal* of G is the ideal $I(G) = (x_v x_w : \{v, w\} \in E)$ in the polynomial ring $R = \mathbb{K}[V]$. A subset $S \subseteq V$ is called *independent* in G if $e \not\subseteq S$, for all $e \in E$. The collection of independent subsets of G forms a simplicial complex Δ_G which is called the *independence complex* of G . It is easy to verify that $I_{\Delta_G} = I(G)$. The graph G is *chordal* if each cycle of length four or more has a chord, an edge joining two vertices that are not adjacent in the cycle. It is known that the independence complex of a chordal graph is vertex decomposable (see [26, Corollary 7] or [9, Theorem 4.1]) hence the edge ideal of chordal graph is squarefree ass-decomposable ideal. Thus applying our results about (square-free) ass-decomposable ideals, we obtain the following corollary. Statement (i) of this corollary can be found in [8, Corollary 5.6] where it is proved by a different method.

Corollary 2.11. *Let $I = I(G) \subseteq R$ be the edge ideal of a chordal graph G and r be the index of irreducibility of I . Then*

- (i) $\text{depth}(R/I) = \min\{\text{depth}(R/\mathfrak{p}) ; \mathfrak{p} \in \text{ass}(I)\} = n - \text{bight}(I).$
- (ii) $\text{reg}(R/I) \leq r - 1.$
- (iii) *The followings are equivalent:*
 - (1) $\text{reg}(R/I) = r - 1;$
 - (2) $r = 2;$
 - (3) G is a star graph.

If one of the above conditions holds, then I has a linear resolution.

Proof. As it is mentioned in the beginning of this corollary, the ideal I is a squarefree ass-decomposable ideal. Hence (i) follows from Theorem 2.6 while (ii) follows from Proposition 2.9. To show that the statements in (iii) are equivalent, first note that (1) and (2) are equivalent by virtue of Proposition 2.10. If (2) holds, then applying Proposition 2.10 once again, we observe that $I = xJ$ where J is a squarefree ass-decomposable ideal. Thus J is a prime monomial ideal and consequently I is the edge ideal of a star graph. Conversely, if G is a star graph with root vertex $\{v\}$, then $I = (x_v x_u ; u \in V(G) \setminus \{v\}) = (x_v) \cap \mathfrak{p}$, where \mathfrak{p} is the prime ideal generated by variables $\{x_u : u \in V(G) \setminus \{v\}\}$. In particular, the index of irreducibility of I is exactly 2. \square

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