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Minimal resolutions of dominant and semidominant ideals

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A B S T R A C T

We introduce new classes of monomial ideals: dominant, p-semidominant, and GNP ideals. The families of dominant and 1-semidominant ideals extend those of complete and almost complete intersections, while the family of GNP ideals extends that of generic ideals. We show that dominant ideals give a complete characterization of when the Taylor resolution is minimal, 1-semidominant ideals are Scarf, and the minimal resolutions of 2-semidominant ideals can be obtained from their Taylor resolutions by eliminating faces and facets of equal multidegree, in arbitrary order. We also generalize a theorem of Bayer, Peeva and Sturmfels by proving that GNP ideals are Scarf.

1. Introduction

For over half a century mathematicians have tried to obtain minimal resolutions of families of ideals in closed form with little success. A common mark in the construction of these classes of ideals and their corresponding resolutions has been the use of a monomial ordering or, at least, an ordering of the variables. Groebner bases, mapping cones, Borel ideals and the (usually nonminimal) Lyubeznik resolution [7,8,6] are some examples of this phenomenon.

In this paper we introduce dominant, p-semidominant and GNP ideals, which are classes of monomial ideals that distinguish from the objects mentioned above in that they do not require an ordering of the variables; instead, they are characterized by the exponents with which the variables appear in the factorization of the monomial generators. Being easy to describe, these ideals are a useful tool to build interesting examples and counterexamples, such as ideals with similar resolutions, classes of Scarf ideals, methods to force projective dimension to go down, and so on.

The concepts of dominant and 1-semidominant ideal extend those of complete and almost complete intersection in a natural way, and the transition from dominant to 1-semidominant ideal is smooth. The latter definition is obtained from the former by relaxing the defining conditions. However, the combinatorial properties of dominant and 1-semidominant ideals can be radically different. For instance, in section 6 we

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give a condition under which a dominant ideal and a 1-semidominant ideal (that look almost identical) have the largest and smallest possible projective dimensions, respectively.

The concept of dominance resembles the definition of generic ideal [2,3] as we explain in section 4. This resemblance, however, goes beyond the similarity of their definitions. More precisely, in section 5 we introduce the family of GNP ideals using the concepts of dominance and genericity. We show that this family contains that of generic ideals. Then we prove that GNP ideals are Scarf (which extends the fact that generic ideals are Scarf [2,3]). Finally, we show that dominant and 1-semidominant ideals are GNP, which implies that every property of GNP ideals is inherited by dominant and 1-semidominant ideals.

We show that the minimal free resolutions of dominant ideals have some interesting properties. In particular, the Taylor resolution of a monomial ideal is minimal if and only if the ideal is dominant. In other words, dominant ideals give a full and explicit characterization of when the Taylor resolution is minimal.

The minimal resolutions of 1-semidominant ideals are also remarkably simple; they are given by the Scarf complex. Although not as easy to decode as in the first two cases, the minimal resolutions of 2-semidominant ideals can also be expressed in simple terms: informally speaking, they can be obtained from their Taylor resolutions eliminating pairs of face and facet of equal multidegree in arbitrary order, until exhausting all possibilities.

The structure of this paper is as follows. In section 2, we adopt a few conventions, fix the notation that will be used frequently, and describe some background material. Section 3 is technical. There we develop the machinary that will be instrumental in the proofs of the results announced above.

Sections 4, 6, and 7 deal with dominant, 1-semidominant, and 2-semidominant ideals, respectively. In particular, at the end of section 7 we describe all 2-semidominant ideals that are Scarf. In section 5, we study GNP ideals. Finally, in section 8 we describe the minimal resolutions of 3-semidominant ideals and explain how p-semidominant ideals increase in complexity as the value of p grows.

2. Background and notation

Throughout this paper S represents a polynomial ring over an arbitrary field, in a finite number variables. The letter M always denotes a monomial ideal in S. The constructions and definitions that we give below can be found in [6,8].

Construction 2.1. Let $M = (m_1, \ldots, m_q)$ be a monomial ideal. For every subset $\{m_{i_1}, \ldots, m_{i_s}\}$ of $\{m_1, \ldots, m_q\}$, with $1 \leq i_1 < \ldots < i_s \leq q$, we create a formal symbol $[m_{i_1}, \ldots, m_{i_s}]$, called a **Taylor symbol**. The Taylor symbol associated to $\{\}$ will be denoted by $[\varnothing]$. For each $s = 0, \ldots, q$, set F_s equal to the free S-module with basis $\{[m_{i_1}, \ldots, m_{i_s}] : 1 \leq i_1 < \ldots < i_s \leq q\}$ given by the $\binom{q}{s}$ Taylor symbols corresponding to subsets of size s. That is, $F_s = \bigoplus\limits_{i_1<\ldots<i_s} S[m_{i_1}, \ldots, m_{i_s}]$ (note that $F_0 = S[\varnothing]$). Define

$$f_0 : F_0 \to S/M$$
$$s[\varnothing] \mapsto f_0(s[\varnothing]) = s$$

For $s = 1, \ldots, q$, let $f_s : F_s \to F_{s-1}$ be given by

$$f_s\left([m_{i_1}, \ldots, m_{i_s}]\right) = \sum_{j=1}^{s} \frac{(-1)^{j+1}\operatorname{lcm}(m_{i_1}, \ldots, m_{i_s})}{\operatorname{lcm}(m_{i_1}, \ldots, \widehat{m_{i_j}}, \ldots, m_{i_k})}[m_{i_1}, \ldots, \widehat{m_{i_j}}, \ldots, m_{i_k}]$$

and extended by linearity. The **Taylor resolution** $\mathbb{T}_M$ of S/M is the exact sequence

$$\mathbb{T}_M : 0 \to F_q \xrightarrow{f_q} F_{q-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0.$$

Following [6], we define the **multidegree** of a Taylor symbol $[m_{i_1},\ldots,m_{i_s}]$, denoted $\mathrm{mdeg}[m_{i_1},\ldots,m_{i_s}]$, as follows: $\mathrm{mdeg}[m_{i_1},\ldots,m_{i_s}] = \mathrm{lcm}(m_{i_1},\ldots,m_{i_s})$. Sometimes the Taylor symbols $[m_{i_1},\ldots,m_{i_s}]$ will be called **faces**. Abusing the language, we will also say that the monomials $m_{i_1},\ldots,m_{i_s}$ are contained in $[m_{i_1},\ldots,m_{i_s}]$. Finally, a Taylor symbol of the form $[m_{i_1},\ldots,\widehat{m_{i_j}},\ldots,m_{i_s}]$ will be referred to as being **facet** of the face $[m_{i_1},\ldots,m_{i_s}]$.

Construction 2.2. Let $M = (m_1,\ldots,m_q)$ be a monomial ideal. Let $\mathbb{T}_M$ be the Taylor resolution of S/M, and let A be the set of Taylor symbols whose multidegrees are not common to other Taylor symbols; that is, a Taylor symbol $[\sigma]$ is in A if and only if $\mathrm{mdeg}[\sigma] \neq \mathrm{mdeg}[\sigma']$, for every Taylor symbol $[\sigma'] \neq [\sigma]$. For each $s = 0,\ldots,q$, set G_s equal to the free S-module with basis $\{[m_{i_1},\ldots,m_{i_s}] \in A : 1 \leq i_1 < \ldots < i_s \leq q\}$. For each $s = 0,\ldots,q$, let $g_s = f_s \restriction_{G_s}$. It can be proven that the g_s are well defined (more precisely, that $g_s(G_s) \subseteq G_{s-1}$) and that

$$0 \to G_q \xrightarrow{g_q} G_{q-1} \to \cdots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} S/M \to 0$$

is a subcomplex of $\mathbb{T}_M$, which is called the **Scarf complex** of S/M.

Definition 2.3. Let M be a monomial ideal, and let

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0$$

be a free resolution of S/M. We say that a basis element $[\sigma]$ of $\mathbb{F}$ has **homological degree i**, denoted $\mathrm{hdeg}[\sigma] = i$, if $[\sigma] \in F_i$. $\mathbb{F}$ is said to be a **minimal resolution** if for every i, the differential matrix (f_i) of $\mathbb{F}$ has no invertible entries.

Definition 2.4. Let M be a monomial ideal, and let

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0$$

be a minimal free resolution of S/M.

- For every i, the ith **Betti number** $\mathrm{b_i}(S/M)$ of S/M is $\mathrm{b_i}(S/M) = \mathrm{rank}(F_i)$.
- For every $i, j \geq 0$, the **graded Betti number** $\mathrm{b_{ij}}(S/M)$ of S/M, in homological degree i and internal degree j, is

$$\mathrm{b_{ij}}(S/M) = \#\{\text{basis elements } [\sigma] \text{ of } F_i : \deg[\sigma] = j\}.$$

- The **regularity** $\mathrm{reg}(S/M)$ of S/M is

$$\mathrm{reg}(S/M) = \max\{r : \mathrm{b_{i,i+r}}(S/M) \neq 0, \text{ for some } i \geq 0\}.$$

- The **projective dimension** $\mathrm{pd}(S/M)$ of S/M is

$$\mathrm{pd}(S/M) = \max\{i : \mathrm{b_i}(S/M) \neq 0\}.$$

3. Foundational results

The results in this section are foundational in character because they deal with the basic concepts of change of basis and consecutive cancellation, which are a natural avenue leading to the minimal free resolution of a monomial ideal.

Definition 3.1. Let M be a monomial ideal and let

$$0 \to F_q \xrightarrow{f_q} \cdots \to F_{j+2} \xrightarrow{f_{j+2}} F_{j+1} \xrightarrow{f_{j+1}} F_j \xrightarrow{f_j} F_{j-1} \to \cdots \to F_0 \to S/M \to 0$$

be a free resolution of S/M.

Let $U = \{[u_1], \cdots, [u_h]\}$ be a basis of F_{j+1} and let $V = \{[v_1], \cdots, [v_g]\}$ be a basis of F_j. Suppose a_{rs} is an invertible entry of the differential matrix

$$(f_{j+1})_{U,V} = \begin{pmatrix} a_{11} & \cdots & a_{1s} & \cdots & a_{1h} \\ \vdots & & \vdots & & \vdots \\ a_{r1} & \cdots & a_{rs} & \cdots & a_{rh} \\ \vdots & & \vdots & & \vdots \\ a_{g1} & \cdots & a_{gs} & \cdots & a_{gh} \end{pmatrix}$$

The change of basis $U' = \{[u_1]', \cdots, [u_h]'\}$, where $[u_s]' = [u_s]$ and $[u_i]' = [u_i] - \frac{a_{ri}}{a_{rs}}[u_s]$ for all $i \neq s$; and $V' = \{[v_1]', \ldots, [v_g]'\}$, where $[v_r]' = \sum_{i=1}^{g} a_{is}[v_i]$ and $[v_i]' = [v_i]$, for all $i \neq r$ will be called **the standard change of basis** (around a_{rs}).

Since the results in the following lemma are known in some form to experts, we give statements without proof (for a detailed proof, see [1]).

Lemma 3.2. *With the notation used in Definition 3.1, if we make a standard change of basis around a_{rs}, the following properties hold:*

(i) $\operatorname{mdeg}[u_i]' = \operatorname{mdeg}[u_i]$, *for all* $i = 1, \ldots, h$; $\operatorname{mdeg}[v_i]' = \operatorname{mdeg}[v_i]$, *for all* $i = 1, \ldots, g$.
(ii) The differential matrix $(f_{j+1})_{U',V'}$ is of the form

$$(f_{j+1})_{U',V'} = \begin{pmatrix} b_{1,1} & \ldots & b_{1,s-1} & 0 & b_{1,s+1} & \ldots & b_{1,h} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{r-1,1} & \ldots & b_{r-1,s-1} & 0 & b_{r-1,s+1} & \ldots & b_{r-1,h} \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ b_{r+1,1} & \ldots & b_{r+1,s-1} & 0 & b_{r+1,s+1} & \ldots & b_{r+1,h} \\ \vdots & & \vdots & \vdots & & & \vdots \\ b_{g,1} & \ldots & b_{g,s-1} & 0 & b_{g,s+1} & \ldots & b_{g,h} \end{pmatrix}$$

(iii) Let $1 \leq c \leq g$ and $1 \leq d \leq h$. If $c \neq r$ and $d \neq s$, then $b_{cd} = a_{cd} - \frac{a_{rd}a_{cs}}{a_{rs}}$.
(iv) The differential matrix $(f_{j+2})_{T,U'}$ is obtained from $(f_{j+2})_{T,U}$ by turning the sth row into a row of zeros, and the differential matrix $(f_j)_{V',W}$ is obtained from $(f_j)_{V,W}$ by turning the rth column into a column of zeros. (Here we assume that T and W are bases of F_{j+2} and F_{j-1}, respectively.)

Lemma 3.2 has several important implications that we discuss next. We continue to use the notation introduced in that lemma.

Remark 3.3. It is obvious that when we make a standard change of basis, some of the basis elements $[u_i]$ and $[v_i]$ change. However, since the free modules $S[u_i]$ and $S[u_i]'$ (respectively $S[v_i]$ and $S[v_i]'$) are isomorphic, and given that by Lemma 3.2 (i), $[u_i]$ and $[u_i]'$ (respectively $[v_i]$ and $[v_i]'$) are abstract objects with the same multidegree, we can assume that the basis elements $[u_i]$ and $[v_i]$ do not change. Therefore, after making a standard change of basis, we can interpret that we have two different representations

$$\cdots \to \bigoplus S[u_i] \xrightarrow{(f)} \bigoplus S[v_i] \to \cdots$$

and

$$\cdots \to \bigoplus S[u_i]' \xrightarrow{(f)'} \bigoplus S[v_i]' \to \cdots$$

of the same free resolution of S/M, or we can interpret that we have two representations

$$\cdots \to \bigoplus S[u_i] \xrightarrow{(f)} \bigoplus S[v_i] \to \cdots$$

and

$$\cdots \to \bigoplus S[u_i] \xrightarrow{(f)'} \bigoplus S[v_i] \to \cdots$$

of two different free resolutions of S/M. We will choose the second interpretation. This way, if we identify the basis of $\mathbb{T}_M$ with a simplicial complex, when we make a standard change of basis or a consecutive cancellation, the basis of the new resolution can be identified with a subset of the simplicial complex and we can still speak in terms of faces and facets.

Remark 3.4. In the same fashion that we identified the differential map f_{j+1} with the differential matrix $(f_{j+1})_{U,V} = (a_{rs})$ (here a_{rs} represents an arbitrary entry, not necessarily invertible), we can identify the sth basis element $[u_s]$ of F_{j+1} with the column vector (δ_{is}), where $\delta_{is} = 0$ if $i \neq s$, and $\delta_{ss} = 1$. Similarly, the image $f_{j+1}([u_s]) = \sum_{i=1}^{g} a_{is}[v_i]$ of $[u_s]$ can be identified with the sth column vector $(f_{j+1})_{U,V} \cdot (\delta_{is}) = (a_{is})$ of (a_{rs}). Thus each entry a_{rs} is the coefficient of $[v_r]$ when $f_{j+1}([u_s])$ is expressed in terms of the basis $V = \{[v_1], \ldots, [v_g]\}$. Notice that there is a bijective correspondence between the entries a_{rs} of $(f_{j+1})_{U,V}$ and the ordered pairs $([u_s], [v_r])$ of basis elements $[u_s]$ and $[v_r]$ in homological degrees $j+1$ and j, respectively. This means that the entry a_{rs} of $(f_{j+1})_{U,V}$ can be written $a_{\tau\sigma}$, where $[\sigma]$ is the sth basis element of U and $[\tau]$ is the rth basis element of V. That is, instead of using subscripts that denote the number of row and column where the entry is placed, we can use subscripts that identify the basis elements that generate this entry. Most of the time we will choose the notation $a_{\tau\sigma}$ over a_{rs} and will say that $a_{\tau\sigma}$ is determined by $[\sigma]$ and $[\tau]$.

Remark 3.5. Since f_{j+1} is graded of degree 0, if $a_{rs} \neq 0$ we must have

$$\operatorname{mdeg}[u_s] = \operatorname{mdeg} f_{j+1}([u_s]) = \operatorname{mdeg}\left(\sum_{i=1}^{g} a_{is}[v_i]\right) = \operatorname{mdeg}(a_{rs}[v_r]) = \operatorname{mdeg} a_{rs} \operatorname{mdeg}[v_r]$$

Hence, $a_{rs} = 0$ or $\operatorname{mdeg} a_{rs} = \dfrac{\operatorname{mdeg}[u_s]}{\operatorname{mdeg}[v_r]}$.

With the notation introduced in Remark 3.4: $a_{\tau\sigma} = 0$ or $\operatorname{mdeg} a_{\tau\sigma} = \dfrac{\operatorname{mdeg}[\sigma]}{\operatorname{mdeg}[\tau]}$. In particular, if $a_{\tau\sigma}$ is invertible then $\operatorname{mdeg}[\sigma] = \operatorname{mdeg}[\tau]$.

Now let $b_{\tau\sigma}$ be the entry determined by $[\sigma]$ and $[\tau]$ in $(f_{j+1})_{U',V'}$. Reasoning as before, we get $b_{\tau\sigma} = 0$ or $\operatorname{mdeg} b_{\tau\sigma} = \dfrac{\operatorname{mdeg}[\sigma]}{\operatorname{mdeg}[\tau]}$.

(Informally speaking, the multidegrees of the entries do not change under standard changes of bases.) In particular, if $a_{\tau\sigma}$ is invertible then, $b_{\tau\sigma} = 0$ or $b_{\tau\sigma}$ is also invertible.

Remark 3.6. It follows from Lemma 3.2 (ii) and (iv) that after making a standard change of basis around a_{rs}, it is possible to make the consecutive cancellation $0 \to S[u_s]' \to S[v_r]' \to 0$. With the interpretation we adopted in Remark 3.3 and the notation we introduced in Remark 3.4, the preceding observation can be restated as follows: after making a standard change of basis around $a_{\tau\sigma}$, the resulting resolution admits the consecutive cancellation $0 \to S[\sigma] \to S[\tau] \to 0$.

We close this section introducing the following terminology. After making a standard change of basis around an invertible entry $a_{\tau\sigma}$ of a resolution $\mathbb{F}$, we obtain a new resolution $\mathbb{F}'$ such that $\mathbb{F} = \mathbb{F}' \oplus (0 \to S[\sigma] \to S[\tau] \to 0)$. From now on, the consecutive cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ will be called **standard cancellation**, and we will say that $\mathbb{F}'$ is obtained from $\mathbb{F}$ by means of a standard cancellation.

4. Dominant ideals

We are ready to address the study of our first family of monomial ideals, the dominant ideals. This study includes the construction of their minimal free resolutions as well as an analysis of their combinatorial properties.

Definition 4.1. Given a set G of monomials in S, we say that

- An element $m \in G$ has a **dominant variable** x (with respect to G) if for all $m' \in G \setminus \{m\}$, the exponent with which x appears in the factorization of m is larger than the exponent with which x appears in the factorization of m'; that is, there exists a positive k such that $x^k \mid m$ and $x^k \nmid m'$, for all $m' \neq m$.
- An element $m \in G$ is a **dominant monomial** (with respect to G) if it has a dominant variable.
- The set G is a **dominant set** if every $m \in G$ is dominant.
- A monomial ideal M is a **dominant ideal** if its minimal generating set is dominant.

Example 4.2. The ideals $M_1 = (x^3y, xy^2z, xz^2)$ and $M_2 = (wx, y^3, z^2)$ are dominant, while $M_3 = (x^2, y^2, xy)$ is not.

Some comments are in order. First, notice that the concept of dominant monomial always depends on a reference set. For example, the ideal M_3 introduced above is not dominant because xy is not dominant in the minimal generating set $\{x^2, y^2, xy\}$; however, xy is dominant in the proper subset $\{x^2, xy\}$.

Second, the definitions of dominant ideal and generic ideal are based on properties of the exponents of the monomial generators. (Recall that an ideal is generic if no variable appears with the same nonzero exponent in more than one monomial generator.) Despite this similarity, dominant and generic ideals are generally different. In Example 4.2, for instance, M_1 is dominant but not generic, while M_3 is generic but not dominant.

Finally, observe that if a monomial ideal is a complete intersection (recall that M is a complete intersection if each variable divides at most one minimal generator), its monomial generators are dominant because they do not have variables in common (such is the case with M_2). It follows that the ideal itself is dominant. Thus, monomial complete intersections are a subset of the family of dominant ideals.

Let us now study some properties derived from the concept of dominance. The following lemma will be quoted often throughout this work.

Lemma 4.3. *Let M be a monomial ideal with minimal generating set G. If $[\sigma_1]$ and $[\sigma_2]$ are two basis elements of $\mathbb{T}_M$ with $\operatorname{mdeg}[\sigma_1] = \operatorname{mdeg}[\sigma_2]$, then $[\sigma_1]$ and $[\sigma_2]$ contain the same dominant monomials of G.*

Proof. Let L_1 and L_2 be the sets of monomials contained in $[\sigma_1]$ and $[\sigma_2]$, respectively. Then $\operatorname{lcm}(L_1) = \operatorname{lcm}(L_2)$. If neither L_1 nor L_2 contains dominant elements of G, there is nothing to prove.

Suppose now that one of these sets, call it L_i, contains a dominant monomial m of G. We will show that the other set, call it L_j, contains m as well. Since m has a dominant variable x, there is a positive k such that $x^k \mid m$ and $x^k \nmid m'$, for all m' in $G \setminus \{m\}$. In particular, $x^k \nmid m'$ for all m' in $L_j \setminus \{m\}$. That is, $x^k \nmid \mathrm{lcm}(L_j \setminus \{m\})$. On the other hand, $x^k \mid \mathrm{lcm}(L_i) = \mathrm{lcm}(L_j)$.

Hence, $L_j \neq L_j \setminus \{m\}$, which means that m is in L_j. We have proven that each dominant element m of G which is in one of $[\sigma_1]$ and $[\sigma_2]$ is also contained in the other. □

In the following theorem we construct the minimal resolutions of dominant ideals. This theorem yields, in addition, an explicit characterization of when the Taylor resolution is minimal.

Theorem 4.4. *Let M be a monomial ideal. Then $\mathbb{T}_M$ is minimal if and only if M is dominant.*

Proof. ($\Rightarrow$) Suppose that M is not dominant. Then its minimal generating set G contains a nondominant monomial n. Let $\sigma = G$ and $\tau_m = G \setminus \{m\}$. This means that $n \mid \ lcm\,(\tau_n)$ and thus, $\mathrm{mdeg}[\sigma] = \mathrm{mdeg}\,[\tau_n]$. So, the top differential map sends $[\sigma] \mapsto \sum\limits_{m \neq n} a_m\,[\tau_m] \pm 1\,[\tau_n]$. Since the coefficient ± 1 of $[\tau_n]$ is invertible, $\mathbb{T}_M$ is not minimal, a contradiction.
($\Leftarrow$) If $[\sigma] = [m_1, \ldots, m_j]$ and $[\tau_i] = [m_1, \ldots, \widehat{m_i}, \ldots, m_j]$ for all i, then

$$f_j\left([\sigma]\right) = \sum_{i=1}^{j} a_{\tau_i \sigma}[\tau_i],$$

where $a_{\tau_i\sigma} = (-1)^{i+1}\dfrac{\mathrm{mdeg}[\sigma]}{\mathrm{mdeg}[\tau_i]}$. Since m_i is dominant, it follows from Lemma 4.3 that $a_{\tau_i\sigma}$ is not invertible. This means that the differential matrices of $\mathbb{T}_M$ do not have invertible entries and hence, $\mathbb{T}_M$ is minimal. □

The fact that $\mathbb{T}_M$ is minimal when M is dominant also follows from Lemma 3.1 in [2].

Corollary 4.5. *Dominant ideals are Scarf.*

Proof. If two basis elements $[\sigma_1]$, $[\sigma_2]$ of $\mathbb{T}_M$ have the same multidegree, according to Lemma 4.3, they contain the same dominant monomials. Since all monomials of the minimal generating set are dominant, $[\sigma_1] = [\sigma_2]$. □

It follows from Lemma 4.3 that if M is dominant, no facet $[\tau_i]$ of $[\sigma]$ has the same multidegree as $[\sigma]$. However, Corollary 4.5 shows that an even stronger statement is true: if M is dominant, all basis elements of $\mathbb{T}_M$ have different multidegrees.

Example 4.6. Let $M = (x^2, xy, y^3)$. The Taylor resolution $\mathbb{T}_M$ of S/M is

$$0 \to S[x^2, xy, y^3] \xrightarrow{\begin{pmatrix} x \\ -1 \\ y^2 \end{pmatrix}} \begin{matrix} S[xy, y^3] \\ \oplus \\ S[x^2, y^3] \\ \oplus \\ S[x^2, xy] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -y^3 & -y \\ -y^2 & 0 & x \\ x & x^2 & 0 \end{pmatrix}} \begin{matrix} S[x^2] \\ \oplus \\ S[xy] \\ \oplus \\ S[y^3] \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} S[\varnothing] \to S/M \to 0$$

Notice that M is not a dominant ideal since xy is nondominant. It follows from Theorem 4.4 that $\mathbb{T}_M$ is not minimal, which is consistent with the fact that one of the differential matrices contains an invertible entry -1.

In contrast to the previous example, the next one contains a Taylor Resolution which is minimal.

Example 4.7. Let $M = (x^2, xz, y^3)$. The Taylor resolution $\mathbb{T}_M$ of S/M is

$$0 \to S[x^2,xz,y^3] \xrightarrow{\begin{pmatrix} x \\ -z \\ y^3 \end{pmatrix}} \begin{matrix} S[xz,y^3] \\ \oplus \\ S[x^2,y^3] \\ \oplus \\ S[x^2,xz] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -y^3 & -z \\ -y^3 & 0 & x \\ xz & x^2 & 0 \end{pmatrix}} \begin{matrix} S[x^2] \\ \oplus \\ S[xz] \\ \oplus \\ S[y^3] \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xz & y^3 \end{pmatrix}} S[\varnothing] \to S/M \to 0$$

In this example, M is dominant. According to Theorem 4.4, the Taylor Resolution $\mathbb{T}_M$ is minimal, which is consistent with the fact that none of the differential matrices contains invertible entries.

Having obtained the minimal free resolutions of the dominant ideals, we can now study the combinatorial properties of the family. We will adopt the following notation: $\operatorname{reg}(S/M)$, $\operatorname{pd}(S/M)$, and $\mathrm{b_i}(S/M)$ will represent the regularity, projective dimension, and ith Betti number of S/M, respectively.

Theorem 4.8 *(Regularity of dominant ideals). Let M be a dominant ideal with minimal generating set $G = \{m_1, \ldots, m_q\}$.*

Let $h = \deg(\mathrm{mdeg}[m_1, \ldots, m_q])$. Then $\operatorname{reg}(S/M) = h - q$.

Proof. Since $[m_1, \ldots, m_q]$ is a basis element in homological degree q, it follows that $b_{qh} \neq 0$. Thus, $\operatorname{reg}(S/M) \geq h - q$. We will prove that if $b_{ij} \neq 0$, then $h - q \geq j - i$, which will complete the proof.

Let $[\sigma] = [m_{r_1}, \ldots, m_{r_i}]$ be a basis element of $\mathbb{T}_M$ with $\deg(\mathrm{mdeg}[\sigma]) = j$. Let $m \in G \setminus \{m_{r_1}, \ldots, m_{r_i}\}$. Since different monomial generators have different dominant variables, it follows that

$$\deg(\mathrm{mdeg}[m_{r_1}, \ldots, m_{r_i}, m]) \geq \deg(\mathrm{mdeg}[m_{r_1}, \ldots, m_{r_i}]) + 1$$

Then, after applying the preceding reasoning $q - i$ times, we get

$$\begin{aligned} h &= \deg(\mathrm{mdeg}[m_1, \ldots, m_q]) \\ &= \deg\left(\mathrm{mdeg}[m_{r_1}, \ldots, m_{r_i}, m_{s_1}, \ldots, m_{s_{q-i}}]\right) \\ &\geq \deg(\mathrm{mdeg}[m_{r_1}, \ldots, m_{r_i}]) + (q - i) \\ &= j + q - i \end{aligned}$$

This implies that $h - q \geq j - i$. □

Next, we give a characterization of when the Taylor resolution is minimal. For a different characterization, see [4].

Corollary 4.9 *(Characterization of the minimal Taylor resolution). Let M be a monomial ideal minimally generated by q monomials. The following statements are equivalent:*

(i) $\mathbb{T}_M$ *is minimal.*
(ii) M *is dominant.*
(iii) $\mathrm{b_i}(S/M) = \binom{q}{i}$ *for all* i.
(iv) $\operatorname{pd}(S/M) = q$.
(v) $\mathbb{T}_M$ *agrees with the Scarf complex of* S/M.

Proof. The equivalence of (i), (ii), (iii) and (v) is immediate, as is (iii) $\Rightarrow$ (iv). We complete the proof by showing that (iv) $\Rightarrow$ (i).

Assume that the Taylor Resolution is not minimal. Then, by Theorem 4.4, M is not dominant. Thus there exists a nondominant monomial m in the minimal generating set G of M. Let $\sigma = G$ and $\tau = G \setminus \{m\}$. Then $m \mid \mathrm{lcm}(\tau)$ and hence, $\mathrm{mdeg}[\sigma] = \mathrm{mdeg}[\tau]$. Since $[\sigma]$ and $[\tau]$ are face and facet in homological degrees q and $q-1$ respectively, it follows that the qth differential matrix (d_q) of $\mathbb{T}_M$ contains an invertible entry. After making a consecutive cancellation in homological degrees q and $q-1$, we obtain a new resolution $\mathbb{F}$ of S/M. But the rank of the free module in homological degree q of $\mathbb{T}_M$ is 1, which implies that the rank of the free module in homological degree q of $\mathbb{F}$ is 0. Hence, the length of $\mathbb{F}$ is less than q, a contradiction. $\square$

The following two remarks are now trivial but show that dominant ideals are as good as we could expect. First, the Taylor resolution of S/M is usually highly nonminimal, while the Scarf complex is often strictly contained in the minimal free resolution of S/M. However, these two complexes agree when and only when M is dominant. Second, two dominant ideals whose minimal generating sets have the same cardinality must have the same projective dimension and the same total Betti numbers. This is immediate from Corollary 4.9 (iii) and (iv).

5. GNP ideals

The concept of generic ideal was introduced by Bayer, Peeva, and Sturmfels [2], in 1998. In their article they proved that generic ideals are minimally resolved by the Scarf complex. This is a beautiful fact that inspired several other papers and similar constructions. In particular, the concept of generic ideal inspired the definition of GNP ideal, which we give next. In this section we prove that GNP ideals are Scarf.

Definition 5.1. We will say that M is **generic in its nondominant part (GNP)** if no variable appears with the same nonzero exponent in the factorization of two nondominant generators of M. In other words, if $M = (m_1, \ldots, m_q, n_1, \ldots, n_p)$ (where $n_1, \ldots, n_p$ are the nondominant generators), we say that M is generic in its nondominant part if $(n_1, \ldots, n_p)$ is generic.

The following are examples of GNP ideals: $M_1 = (a^3b, b^3c, abc)$, $M_2 = (a^3b, b^3c, abc, a^2b^2)$.

Lemma 5.2. *Let M be a GNP ideal. Suppose that m is a multidegree of $\mathbb{T}_M$, common to two Taylor symbols. Then M has a generator n with the following property:*
If $A = \{[l_1, \ldots, l_r] \in \mathbb{T}_M : \mathrm{mdeg}[l_1, \ldots, l_r] = m, r \in \mathbb{N}, l_i \neq n$ for all $i\}$, and $B = \{[l_1, \ldots, l_r, n] \in \mathbb{T}_M : \mathrm{mdeg}[l_1, \ldots, l_r, n] = m, r \in \mathbb{N}\}$ then the map $f : A \to B$, given by $f([l_1, \ldots, l_r]) = [l_1, \ldots, l_r, n]$ is bijective.

Proof. Among all Taylor symbols of multidegree m, let $[\mu]$ be one with minimal homological degree. Let $[\mu] = [m_1, \ldots, m_q, n_1, \ldots, n_p]$, where the m_i are dominant and the n_i are nondominant. Given that m is common to at least two Taylor symbols, there must be a generator n of M, such that $n \notin \{m_1, \ldots, m_q, n_1, \ldots, n_p\}$ and $n \mid m$. We will prove that n satisfies the thesis of this lemma.

Since f is one-to-one by construction, we only need to prove that f is onto.

Let $[\sigma] = [l_1, \ldots, l_r, n] \in B$, and let $[\tau] = [l_1, \ldots, l_r]$. Suppose that m can be factored in the form $m = x_1^{\alpha_1} \ldots x_j^{\alpha_j}$. Let β_i be the exponent with which a variable x_i appears in the factorization of n. If $\beta_i < \alpha_i$, then $x_i^{\alpha_i} \mid \mathrm{mdeg}[\tau]$. Suppose now that $\beta_i = \alpha_i$. Since $\mathrm{mdeg}[\mu] = m$, the variable x_i must appear with exponent α_i in the factorization of one of $m_1, \ldots, m_q, n_1, \ldots, n_p$. Such a monomial must be dominant, for n is nondominant and M is GNP. It follows that x_i appears with exponent α_i in the factorization of m_s, for some $1 \leq s \leq q$. Then $m_s \mid \mathrm{mdeg}[\mu] = \mathrm{mdeg}[\sigma]$. Since m_s is dominant, we must have that $m_s \in \{l_1, \ldots, l_r, n\}$. More precisely, $m_s \in \{l_1, \ldots, l_r\}$. Thus $m_s \mid \mathrm{mdeg}[\tau]$ and hence, $x_i^{\alpha_i} \mid \mathrm{mdeg}[\tau]$.

Given that i is arbitrary, $m \mid \mathrm{mdeg}[\tau]$. That is, $\mathrm{mdeg}[\tau] = m$, and $[\tau] \in A$. $\square$

With the notation of the lemma, the set of all Taylor symbols of multidegree m is the disjoint union of A and B.

For the rest of this section, the symbols M, m, n, A, B, and f have the meaning with which they were introduced in Lemma 5.2. In addition, the smallest and the largest homological degrees for which there is a Taylor symbol with multidegree m will be denoted v and w, respectively. For each $v \leq t < w$, we define:

$$A_t = \{[\tau] \in A : \mathrm{hdeg}[\tau] = t\}. \text{ (Thus } A = \bigcup_{t=v}^{w-1} A_t.\text{)}$$

Moreover, for practical purposes, the set A_t will be represented in the form

$$A_t = \{[\tau_1^t], \ldots, [\tau_{k_t}^t]\}$$

Finally, we define

$$B_t = \{[\sigma_1^t], \ldots, [\sigma_{k_t}^t]\}, \text{ where } [\sigma_i^t] = f[\tau_i^t], \text{ for all } i = 1, \ldots, k_t.$$

Lemma 5.3. *Let $v \leq t < w$. Suppose that $\mathbb{F}$ is a resolution of S/M, obtained from $\mathbb{T}_M$ by means of standard cancellations of the form*

$$0 \to S[\sigma] \to S[\tau] \to 0,$$

where $[\tau] \in \bigcup_{j=v}^{t-1} A_j$, and $[\sigma] = f([\tau])$. Let $1 \leq r < k_t$. Then, starting with $\mathbb{F}$, it is possible to make the following sequence of cancellations:

$$0 \to S[\sigma_1^t] \to S[\tau_1^t] \to 0, \cdots, 0 \to S[\sigma_r^t] \to S[\tau_r^t] \to 0.$$

If $\mathbb{F}_r$ is the resolution obtained after making these cancellations, then the entries $b_{\epsilon,\sigma_{r+1}^t}$ and $a_{\epsilon,\sigma_{r+1}^t}$ of the $(t+1)$th differentials of $\mathbb{F}_r$ and $\mathbb{T}_M$, respectively, are equal.

Proof. By induction on r. Let $r = 1$. Note that the entries of the $(t+1)$th differentials of $\mathbb{F}$ and $\mathbb{T}_M$ are equal, for $\mathbb{F}$ was obtained from $\mathbb{T}_M$ by means of standard cancellations in homological degrees $\leq t$. Since $[\tau_1^t]$ is a facet of $[\sigma_1^t]$, and $\mathrm{mdeg}[\tau_1^t] = \mathrm{mdeg}[\sigma_1^t] = m$, the entry $a_{\tau_1^t,\sigma_1^t}$ of the $(t+1)$th differential of $\mathbb{T}_M$ is invertible. Hence, it is possible to make the cancellation $0 \to S[\sigma_1^t] \to S[\tau_1^t] \to 0$ in $\mathbb{F}$, which yields a resolution $\mathbb{F}_1$ of S/M. Now the entries b_{ϵ,σ_2^t} of the $(t+1)$th differential of $\mathbb{F}_1$ are

$$b_{\epsilon,\sigma_2^t} = a_{\epsilon,\sigma_2^t} - \frac{a_{\tau_1^t,\sigma_2^t} a_{\epsilon,\sigma_1^t}}{a_{\tau_1^t,\sigma_1^t}} = a_{\epsilon,\sigma_2^t}$$

because $[\tau_1^t]$ is not a facet of $[\sigma_2^t]$.

Let us assume that the lemma holds for $r = l - 1$. That is, we assume that starting with $\mathbb{F}$, it is possible to make the cancellations

$$0 \to S[\sigma_1^t] \to S[\tau_1^t] \to 0, \cdots, 0 \to S[\sigma_{l-1}^t] \to S[\tau_{l-1}^t] \to 0,$$

to obtain a resolution $\mathbb{F}_{l-1}$ of S/M. We also assume that the entries b_{ϵ,σ_l^t} and a_{ϵ,σ_l^t} of the $(t+1)$th differentials of $\mathbb{F}_{l-1}$ and $\mathbb{T}_M$, respectively, are equal.

We will show that the lemma holds for $r = l$. Notice that the entry $a_{\tau_l^t,\sigma_l^t}$ of the $(t+1)$th differential of $\mathbb{T}_M$ is invertible. By induction hypothesis, $b_{\tau_l^t,\sigma_l^t} = a_{\tau_l^t,\sigma_l^t}$, which implies that it is possible to make the cancellation $0 \to S[\sigma_l^t] \to S[\tau_l^t] \to 0$ in $\mathbb{F}_{l-1}$. The entries $b_{\epsilon,\sigma_{l+1}^t}$ of the resulting resolution $\mathbb{F}_l$ satisfy:

$$b_{\epsilon,\sigma_{l+1}^t} = a_{\epsilon,\sigma_{l+1}^t} - \frac{a_{\tau_l^t,\sigma_{l+1}^t} a_{\epsilon,\sigma_l^t}}{a_{\tau_l^t,\sigma_l^t}} = a_{\epsilon,\sigma_{l+1}^t}$$

because $[\tau_l^t]$ is not a facet of $[\sigma_{l+1}^t]$. □

Corollary 5.4. *Let $v \le t < w$. Starting with $\mathbb{T}_M$, it is possible to make the following sequence of standard cancellations*

$$\begin{array}{l} 0 \to S[\sigma_1^v] \to S[\tau_1^v] \to 0, \cdots, 0 \to S[\sigma_{k_v}^v] \to S[\tau_{k_v}^v] \to 0, \\ \vdots \\ 0 \to S[\sigma_1^t] \to S[\tau_1^t] \to 0, \cdots, 0 \to S[\sigma_{k_t}^t] \to S[\tau_{k_t}^t] \to 0. \end{array}$$

Proof. The proof is by induction on t.

If $t = v$ then the hypotheses of Lemma 5.3 are satisfied because $\bigcup_{j=v}^{t-1} A_j = \varnothing$ and $\mathbb{F} = \mathbb{T}_M$. Thus, this corollary holds by Lemma 5.3.

Suppose now that Corollary 5.4 holds for $t = l - 1$. That is, suppose that starting with $\mathbb{T}_M$, it is possible to make the cancellations

$$\begin{array}{l} 0 \to S[\sigma_1^v] \to S[\tau_1^v] \to 0, \cdots, 0 \to S[\sigma_{k_v}^v] \to S[\tau_{k_v}^v] \to 0, \\ \vdots \\ 0 \to S[\sigma_1^{l-1}] \to S[\tau_1^{l-1}] \to 0, \cdots, 0 \to S[\sigma_{k_{l-1}}^{l-1}] \to S[\tau_{k_{l-1}}^{l-1}] \to 0. \end{array}$$

Then by Lemma 5.3, starting with the resulting resolution $\mathbb{F}$, it is possible to make the cancellations

$$0 \to S[\sigma_1^l] \to S[\tau_1^l] \to 0, \cdots, 0 \to S[\sigma_{k_l}^l] \to S[\tau_{k_l}^l] \to 0 \quad \square$$

Corollary 5.5. *The minimal resolution of S/M does not contain any Taylor symbols of multidegree m.*

Proof. Let $t = w - 1$. By Corollary 5.4, starting with $\mathbb{T}_M$, it is possible to make the cancellations

$$\begin{array}{l} 0 \to S[\sigma_1^v] \to S[\tau_1^v] \to 0, \cdots, 0 \to S[\sigma_{k_v}^v] \to S[\tau_{k_v}^v] \to 0, \\ \vdots \\ 0 \to S[\sigma_1^{w-1}] \to S[\tau_1^{w-1}] \to 0, \cdots, 0 \to S[\sigma_{k_{w-1}}^{w-1}] \to S[\tau_{k_{w-1}}^{w-1}] \to 0. \end{array}$$

This means that the resulting resolution $\mathbb{F}$ contains no Taylor symbols of multidegree m and hence, neither does the minimal resolution of S/M. □

The next result extends a celebrated theorem in [2].

Theorem 5.6. *GNP ideals are Scarf.*

Proof. By Corollary 5.5, the minimal resolution of S/M contains no Taylor symbols of multidegree m, where m is an arbitrary multidegree of $\mathbb{T}_M$, common to two or more Taylor symbols. □

6. Semidominant ideals

In this section we introduce the semidominant ideals by slightly modifying the definition of dominance in such a way that the resulting family does not overlap with the family of dominant ideals and yet retains some of its rich properties.

Definition 6.1. Let G be a set of monomials in S. We say that G is **semidominant** if exactly one monomial of G is not dominant. A monomial ideal M is called a **semidominant ideal** if its minimal generating set is semidominant. When a semidominant set G is expressed in the form $G = \{m_1, \ldots, m_q, n\}$ we will assume that $m_1, \ldots, m_q$ are dominant and n is nondominant.

Example 6.2. The ideals $M_1 = (x^2, y^3, xy)$ and $M_2 = (xy, z^2, yz)$ are semidominant, $M_3 = (x^2z, y^3, yz^3)$ is dominant, and $M_4 = (xy, yz, xz)$ is neither dominant nor semidominant.

Recall that a monomial ideal $M = (l_1, ..., l_q)$ is an almost complete intersection if, for some j, $(l_1, ..., \widehat{l_j}, ..., l_q)$ is a complete intersection.

Proposition 6.3. *Monomial almost complete intersections are either dominant or semidominant ideals.*

Proof. Let $M = (l_1, ..., l_q, l)$ be a monomial almost complete intersection, where $l_1, ..., l_q$ form a regular sequence and hence have no variable in common. Note that for all i, $l_i \nmid l$. Then there is a variable x_i that appears with a larger exponent in the factorization of l_i than in that of l. Therefore, x_i is a dominant variable for l_i, which means that l_i is a dominant monomial. □

Theorem 6.4. *Semidominant ideals are Scarf.*

Proof. Since the nondominant part of a semidominant ideal consists of exactly one monomial, semidominant ideals are GNP. By Theorem 5.6, semidominant ideals are Scarf. □

Corollary 6.5. *Monomial almost complete intersections are Scarf.*

Proof. Since dominant and semidominant ideals are GNP, this result follows from Proposition 6.3 and, Theorem 5.6. □

Example 6.6. Let $M = (x^3y, y^2z, xz^2, xyz)$. Note that M is semidominant, xyz being the nondominant generator. By Theorem 6.4, M is Scarf. Now, the multidegrees that are common to more than one basis element of $\mathbb{T}_M$ are x^3y^2z, x^3yz^2, xy^2z^2, and $x^3y^2z^2$ as one can determine by simple inspection. Hence, the basis of the minimal resolution $\mathbb{F}$ of S/M is obtained from the basis of $\mathbb{T}_M$ by eliminating the elements that have one of the multidegrees mentioned above. This leads to the following resolution:

$$\mathbb{F}: \quad 0 \to \begin{array}{c} S[x^3y, xyz] \\ \oplus \\ S[y^2z, xyz] \\ \oplus \\ S[xz^2, xyz] \end{array} \xrightarrow{(f_2)} \begin{array}{c} S[x^3y] \\ \oplus \\ S[y^2z] \\ \oplus \\ S[xz^2] \\ \oplus \\ S[xyz] \end{array} \xrightarrow{(f_1)} S[\varnothing] \xrightarrow{(f_0)} S/M \to 0$$

Corollary 6.7. *Let M be a semidominant ideal with minimal generating set $G = \{m_1, \ldots, m_q, n\}$.*

(i) The projective dimension of S/M is the cardinality of the largest dominant subset of G that contains n.
(ii) Let $B_j = \{[m_{t_1}, \ldots, m_{t_j}] : n \nmid \mathrm{lcm}(m_{t_1}, \ldots, m_{t_j})\}$. Then the Betti numbers of S/M are given by the formula

$$\mathrm{b_i}\,(S/M) = \#B_i + \#B_{i-1}$$

Proof. Let $\mathbb{F}$ be the Scarf complex of S/M and $A = \big\{\big([m_{i_1}, \ldots, m_{i_j}, n], [m_{i_1}, \ldots, m_{i_j}]\big) : n \mid \operatorname{lcm}(m_{i_1}, \ldots, m_{i_j})\big\}$.
(i) Let $r = \max\{\#(D) : D$ is a dominant subset of G that contains $n\}$.

Let $\{m_{t_1}, \ldots, m_{t_{r-1}}, n\}$ be a dominant subset of G. Then $n \nmid \operatorname{lcm}\big(m_{t_1}, \ldots, m_{t_{r-1}}\big)$. Thus $\big([m_{t_1}, \ldots, m_{t_{r-1}}, n], [m_{t_1}, \ldots, m_{t_{r-1}}]\big)$ is not in A and, therefore, $[m_{t_1}, \ldots, m_{t_{r-1}}, n]$ is a basis element of the minimal resolution $\mathbb{F}$. Thus, $\operatorname{pd}(S/M) \geq r$. Now, if $[\sigma]$ is a basis element of $\mathbb{T}$, in homological degree $k > r$, then $[\sigma]$ must be of the form: $[\sigma] = [m_{s_1}, \ldots, m_{s_k}]$ or $[\sigma] = [m_{s_1}, \ldots, m_{s_{k-1}}, n]$.

If $[\sigma] = [m_{s_1}, \ldots, m_{s_k}]$, then $\{m_{s_1}, \ldots, m_{s_k}, n\}$ cannot be dominant because its cardinality is larger than r. Hence, $n \mid \operatorname{lcm}(m_{s_1}, \ldots, m_{s_k})$, which means that $([m_{s_1}, \ldots, m_{s_k}, n], [\sigma]) \in A$, and thus $[\sigma]$ is not a basis element of $\mathbb{F}$. A similar reasoning shows that if $[\sigma] = [m_{s_1}, \ldots, m_{s_{k-1}}, n]$ then $\big([\sigma], [m_{s_1}, \ldots, m_{s_{k-1}}]\big) \in A$, and thus $[\sigma]$ is not a basis element of $\mathbb{F}$.

Given that every basis element of $\mathbb{T}_M$ in homological degree $k > r$ is excluded from the basis of $\mathbb{F}$, we conclude that $\operatorname{pd}(S/M) = r$.
(ii) The basis elements of $\mathbb{T}_M$ in homological degree i are of the form $[m_{s_1}, \ldots, m_{s_{i-1}}, n]$ or $[m_{t_1}, \ldots, m_{t_i}]$. Since the basis elements of $\mathbb{F}$ are obtained from the basis of $\mathbb{T}_M$ by eliminating those elements which are the first or the second component of a pair $([\sigma], [\tau]) \in A$, it follows that the family of basis elements of $\mathbb{F}$ in homological degree i is: $\{[m_{t_1}, \ldots, m_{t_i}] : n \nmid \operatorname{lcm}(m_{t_1}, \ldots, m_{t_i})\} \cup \big\{[m_{s_1}, \ldots, m_{s_{i-1}}, n] : n \nmid \operatorname{lcm}\big(m_{s_1}, \ldots, m_{s_{i-1}}\big)\big\}$.

The statement of part (ii) is now clear. □

Corollary 6.8. *Let $M = (m_1, \ldots, m_q, n)$ be a semidominant ideal. Then* $\operatorname{pd}(S/M) = 2$ *if and only if for all $i \neq j$, $n \mid \operatorname{lcm}(m_i, m_j)$.*

Proof. ($\Rightarrow$) If $\operatorname{pd}(S/M) = 2$, then the largest dominant subset of $\{m_1, \ldots, m_q, n\}$ that contains n has cardinality 2 Corollary 6.7. Thus every set $\{m_i, m_j, n\}$ is nondominant, which implies that $n \mid \operatorname{lcm}(m_i, m_j)$.
($\Leftarrow$) If $k \geq 2$, then $n \mid \operatorname{lcm}(m_{i_1}, \ldots, m_{i_k})$. Therefore, the set $D = \{m_{i_1}, \ldots, m_{i_k}, n\}$ is not dominant and, according to Corollary 6.7, $\operatorname{pd}(S/M) \leq 2$. Now, $\{m_1, n\}$ is dominant, so $\operatorname{pd}(S/M) = 2$. □

Corollary 6.8 is interesting because it tells us that an ideal M may have maximum projective dimension (i.e., $\operatorname{pd}(S/M) =$ number of generators of M) and another ideal M', obtained by adding one generator to the minimal generating set of M, may have minimum projective dimension (i.e., $\operatorname{pd}(S/M') = 2$). The next example illustrates this phenomenon.

Example 6.9. Let $M = (v^2xyz, vw^2yz, vwx^2z, vwxy^2, wxyz^2)$, and $M' = (v^2xyz, vw^2yz, vwx^2z, vwxy^2, wxyz^2, vwxyz)$. Since M is dominant, $\operatorname{pd}(S/M) = 5$. The semidominant ideal M' obtained from M by adding the generator $vwxyz$ satisfies the condition of Corollary 6.8 and thus $\operatorname{pd}(S/M') = 2$.

Corollary 6.10. *Let M be a semidominant ideal with minimal generating set $G = \{m_1, \ldots, m_q, n\}$. Then*

$$\operatorname{reg}(S/M) = \max\{\deg(\operatorname{mdeg}[\sigma]) - \operatorname{hdeg}[\sigma] : \sigma \subset G,\ n \in \sigma,\ \textit{and } \sigma \textit{ is dominant}\}$$

Proof. Let $\{m_{r_1}, \ldots, m_{r_t}, n\}$ be a dominant set such that

$$\deg(\operatorname{mdeg}[m_{r_1}, \ldots, m_{r_t}, n]) - (t + 1) = c.$$

Then $b_{t+1,t+1+c} \neq 0$. Thus, $\operatorname{reg}(S/M) \geq c$. We will prove that if $b_{ij} \neq 0$, then $c \geq j - i$, which will complete the proof.

By Corollary 6.7 (ii) there are two ways in which we might have $b_{ij} \neq 0$:
(i) the minimal free resolution contains a basis element of the form $[m_{r_1}, \ldots, m_{r_i}]$ such that $\{m_{r_1}, \ldots, m_{r_i}, n\}$ is dominant and $\deg(\operatorname{mdeg}[m_{r_1}, \ldots, m_{r_i}]) = j$;

(ii) the minimal free resolution contains a basis element of the form $[m_{s_1}, \ldots, m_{s_{i-1}}, n]$ such that $\{m_{s_1}, \ldots, m_{s_{i-1}}, n\}$ is dominant and $\deg\big(\text{mdeg}[m_{s_1}, \ldots, m_{s_{i-1}}, n]\big) = j$.

If (i) happens, then $[m_{r_1}, \ldots, m_{r_i}, n]$ is also in the minimal free resolution and

$$\deg\left(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}, n]\right) \geq \deg\left(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}]\right) + 1.$$

It follows from the construction of c that

$$c \geq \deg\left(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}, n]\right) - (i+1) \geq \deg\left(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}]\right) + 1 - (i+1) = j - i.$$

If (ii) happens, then it follows from the construction of c that

$$c \geq \deg\big(\text{mdeg}[m_{s_1}, \ldots, m_{s_{i-1}}, n]\big) - i = j - i. \quad \square$$

Example 6.11. Let $M = (x^3y, y^2z, xz^2, xyz)$ as in Example 6.6. Since we already know the minimal free resolution $\mathbb{F}$ of S/M, we can read off the numbers $\text{pd}\,(S/M)$, $\text{b}_\text{i}\,(S/M)$, and $\text{reg}\,(S/M)$ from $\mathbb{F}$. However, we will calculate these numbers using Corollary 6.7 and Corollary 6.10 which, in some cases, turns out to be a faster alternative.

Observe that the largest dominant sets containing the nondominant generator xyz are $\{x^3y, xyz\}$, $\{y^2z, xyz\}$, and $\{xz^2, xyz\}$. It follows from Corollary 6.7 (i) that $\text{pd}\,(S/M) = 2$.

Besides that, according to Corollary 6.7 (ii), $\text{b}_2\,(S/M)$ is given by the formula:

$$\begin{aligned}\text{b}_2\,(S/M) &= \#\{[m_i, m_j]/n \nmid \text{mdeg}[m_i, m_j]\} \\ &\quad + \#\{[m_i, n]/n \nmid \text{mdeg}[m_i]\} = \#\{\} + \#\{[x^3y, xyz]; [y^2z, xyz]; [xz^2, xyz]\} = 3.\end{aligned}$$

($\text{b}_1\,(S/M)$ and $\text{b}_0\,(S/M)$ are always easily obtained from $\mathbb{T}_M$.) Finally, by Corollary 6.10 we have

$$\begin{aligned}\text{reg}\,(S/M) &= \max\{\deg(\text{mdeg}[x^3y, xyz]) - 2; \deg(\text{mdeg}[y^2z, xyz]) - 2; \deg(\text{mdeg}[xz^2, xyz]) - 2\} \\ &= \max\{5-2;\ 4-2;\ 4-2\} = 3.\end{aligned}$$

All our calculations are consistent with the information encoded in $\mathbb{F}$, as we can easily verify.

7. 2-Semidominant ideals

The concepts of dominance and semidominance lead in a natural way to the more general definition of p-semidominance, which we give next.

Definition 7.1. A set of monomials is called ***p*-semidominant** if it contains exactly p nondominant monomials. A monomial ideal is called ***p*-semidominant** if its minimal generating set is p-semidominant.

With this definition, dominant and semidominant ideals can be thought of as being 0-semidominant and 1-semidominant, respectively. Sometimes, the word semidominant is used to denote 1-semidominant ideals while other times it makes reference to p-semidominant ideals in general (as in the title of this paper). The meaning will be clear from the context.

In this section we will construct the minimal free resolution of 2-semidominant ideals; that is, monomial ideals M with minimal generating set $G = \{m_1, \ldots, m_q, n_1, n_2\}$ where $m_1, \ldots, m_q$ are dominant and n_1 and n_2 are nondominant. $M = (x^2y^2, xz, yz)$, and $M' = (x^3, y^3z^3, x^2y, xyz)$ are examples of 2-semidominant ideals.

Lemma 7.2. *Let M be a 2-semidominant ideal. If two basis elements of a resolution of S/M, in consecutive homological degrees, have the same multidegree, then they are face and facet.*

Proof. Let $[\sigma]$ and $[\tau]$ be basis elements in homological degrees $j+1$ and j, respectively. If $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, then $[\sigma]$ and $[\tau]$ must be generated by the same dominant monomials. Given that $[\sigma]$ has one more generator than $[\tau]$, if $[\tau]$ contains no nondominant generator, $[\sigma]$ must contain exactly one. On the other hand, if $[\tau]$ contains one nondominant generator, then $[\sigma]$ must contain both nondominant generators. In every case we see that $[\tau]$ is a facet of $[\sigma]$. $\square$

Our next goal is to prove that the basis of the minimal free resolution of S/M can be obtained from the basis of its Taylor resolution by eliminating pairs of basis elements $[\sigma]$, $[\tau]$ in an arbitrary order, where $[\tau]$ is a facet of $[\sigma]$ and $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, until exhausting all possibilities.

If this idea is going to succeed, we need first to confirm that the following dangerous scenario never occurs. Suppose that $([\sigma_1], [\tau_1])$ and $([\sigma_2], [\tau_2])$ are disjoint pairs of face and facet with $\text{mdeg}[\sigma_i] = \text{mdeg}[\tau_i]$. Let $([\sigma_1], [\tau_1])$ determine the invertible entry a_{rs} of the differential matrix (f_{j+1}) of $\mathbb{T}_M$. Then eliminating $[\sigma_1]$ and $[\tau_1]$ from the basis of $\mathbb{T}_M$ is equivalent to making the standard change of basis around a_{rs}, followed by the standard cancellation $0 \to S[\sigma_1] \to S[\tau_1] \to 0$.

Similarly, $([\sigma_2], [\tau_2])$ define an invertible entry a_{cd} and eliminating $[\sigma_2]$, $[\tau_2]$ from the basis of the Taylor resolution is equivalent to making a standard change of basis around a_{cd}, followed by the standard cancellation $0 \to S[\sigma_2] \to S[\tau_2] \to 0$.

However, when we make the standard change of basis around a_{rs}, the entries of the matrices change. In particular, the entry a_{cd} might become noninvertible, which would prevent us from doing the standard cancellation $0 \to S[\sigma_2] \to S[\tau_2] \to 0$.

In the next lemma we show that this scenario is not possible for 2-semidominant ideals.

Note: We will say that two pairs of basis elements $([\sigma], [\tau])$ and $([\theta], [\pi])$ of $\mathbb{T}_M$ are "disjoint" if $[\sigma] \neq [\theta], [\pi]$ and $[\tau] \neq [\theta], [\pi]$.

Lemma 7.3. *Let M be a 2-semidominant ideal. Let $\mathbb{F}$ be a free resolution of S/M obtained from $\mathbb{T}_M$ by means of standard cancellations. Let $a_{\tau\sigma}$ and $a_{\pi\theta}$ be two invertible entries of $\mathbb{F}$, corresponding to two disjoint pairs of basis elements $([\sigma], [\tau])$ and $([\theta], [\pi])$ of $\mathbb{F}$, respectively. Then after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, it is possible to make the standard cancellation $0 \to S[\theta] \to S[\pi] \to 0$.*

Proof. $[\sigma]$ and $[\tau]$ are basis elements in homological degrees j and $j-1$, respectively, for some j. Thus $a_{\tau\sigma}$ is an entry of the differential matrix (f_j) of $\mathbb{F}$. Similarly, $[\theta]$ and $[\pi]$ are basis elements in some homological degrees k and $k-1$, and $a_{\pi\theta}$ is an entry of the differential matrix (f_k) of $\mathbb{F}$.

In order to prove the lemma, it is enough to show that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, the entry $a'_{\pi\theta}$ of the differential matrix (f'_k) of the new resolution $\mathbb{F}'$ is invertible.

Given that only (f_{j+1}), (f_j) and (f_{j-1}) are affected by the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, if $k \neq j-1, j, j+1$ then $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible. Therefore, we only need to prove that $a'_{\pi\theta}$ is invertible in the following cases: $k = j$; $k = j-1$, $k = j+1$.

Suppose $k = j$. Since $a_{\pi\theta}$ is invertible, $\text{mdeg}[\pi] = \text{mdeg}[\theta]$. Then $a'_{\pi\theta} = 0$ or $a'_{\pi\theta}$ is invertible. Let us assume that $a'_{\pi\theta} = 0$. By Lemma 3.2 (iii), we have that $0 = a'_{\pi\theta} = a_{\pi\theta} - \dfrac{a_{\pi\sigma} a_{\tau\theta}}{a_{\tau\sigma}}$. It follows that $a_{\pi\theta} a_{\tau\sigma} = a_{\pi\sigma} a_{\tau\theta}$ and, since $a_{\pi\theta}$ and $a_{\tau\sigma}$ are invertible, $a_{\pi\sigma}$ and $a_{\tau\theta}$ must be invertible too. In particular, the fact that $a_{\pi\sigma}$ is invertible implies that $\text{mdeg}[\sigma] = \text{mdeg}[\pi]$ which, combined with the hypothesis $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, implies that $\text{mdeg}[\tau] = \text{mdeg}[\pi]$.

In particular, $[\tau]$ and $[\pi]$ contain the same dominant monomials and thus they differ in the nondominant monomials that define them. Since $[\tau]$ and $[\pi]$ appear in the same homological degree, they must contain exactly one nondominant generator each. Then $[\tau]$ and $[\pi]$ are of the form $[\tau] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1]$; $[\pi] =$

$[m_{i_1}, \ldots, m_{i_{j-1}}, n_2]$. Given that $\mathrm{mdeg}[\tau] = \mathrm{mdeg}[\theta]$, and the fact that $[\tau]$ and $[\theta]$ appear in homological degrees $j-1$ and j, respectively, it follows from Lemma 7.2 that $[\tau]$ is a facet of $[\theta]$. Thus θ must be of the form $[\theta] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1, n_2]$. Since $[\tau]$ is also a facet of $[\sigma]$, the same reasoning applies to $[\sigma]$, which means that $[\sigma] = [\theta]$, an absurd. We conclude that $a'_{\pi\theta}$ is invertible.

Now suppose $k = j-1$. In this case $[\tau]$ and $[\theta]$ appear in homological degree $j-1$. Let $[\tau]$ and $[\theta]$ be the rth and sth basis elements, respectively. It follows from Lemma 3.2 iv) that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, the matrix $\left(f'_{j-1}\right)$ of the new resolution $\mathbb{F}'$ is obtained from (f_{j-1}) by eliminating its rth column. Since the entry $a'_{\pi\theta}$ is placed in the sth column of $\left(f'_{j-1}\right)$, we have that $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible.

Finally, suppose $k = j+1$. In this case $[\sigma]$ and $[\pi]$ appear in homological degree j. Let $[\sigma]$ and $[\pi]$ be the uth and vth basis elements, respectively. It follows from Lemma 3.2 iv) that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, the matrix $\left(f'_{j+1}\right)$ of the new resolution $\mathbb{F}'$ is obtained from (f_{j+1}) by eliminating its uth row. Since the entry $a'_{\pi\theta}$ is placed in the vth row of $\left(f'_{j+1}\right)$, we have that $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible. $\square$

Theorem 7.4. *Let M be a 2-semidominant ideal. Let $([\sigma_1], [\tau_1]), \ldots, ([\sigma_k], [\tau_k])$ be k pairs of basis elements of $\mathbb{T}_M$, satisfying the following properties:*

(i) $([\sigma_i], [\tau_i])$ *and* $([\sigma_j], [\tau_j])$ *are disjoint, if $i \neq j$.*
(ii) $[\tau_i]$ *is a facet of* $[\sigma_i]$ *for all $i = 1, \ldots k$.*
(iii) $\mathrm{mdeg}[\sigma_i] = \mathrm{mdeg}[\tau_i]$ *for all $i = 1, \ldots k$.*

Then, starting with $\mathbb{T}_M$, it is possible to make the following sequence of standard cancellations:

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad, 0 \to S[\sigma_k] \to S[\tau_k] \to 0$$

Proof. The proof is by induction on k.

If $k = 2$, the statement holds by Lemma 7.3, with $\mathbb{F} = \mathbb{T}_M$. (The fact that $a_{\tau_1\sigma_1}$ and $a_{\tau_2\sigma_2}$ are invertible follows from the fact that in $\mathbb{T}_M$ faces and facets of equal multidegree always determine an invertible entry.)

Assume that the theorem holds for $k = j-1$. Let $k = j$. Then it is possible to make either of the following two sequences of standard cancellations:

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad, 0 \to S[\sigma_{j-1}] \to S[\tau_{j-1}] \to 0$$

and

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad, 0 \to S[\sigma_{j-2}] \to S[\tau_{j-2}] \to 0, 0 \to S[\sigma_j] \to S[\tau_j] \to 0.$$

This means that after making the first $j-2$ cancellations

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad, 0 \to S[\sigma_{j-2}] \to S[\tau_{j-2}] \to 0$$

either of the following two cancellations can be made:

$$0 \to S[\sigma_{j-1}] \to S[\tau_{j-1}] \to 0$$

and

$$0 \to S[\sigma_j] \to S[\tau_j] \to 0.$$

In other words, after making the first $j-2$ standard cancellations, we obtain a free resolution $\mathbb{F}$, where the entries $a_{\tau_{j-1}\sigma_{j-1}}$ and $a_{\tau_j\sigma_j}$ determined by $([\sigma_{j-1}],[\tau_{j-1}])$ and $([\sigma_j],[\tau_j])$, respectively, are invertible. Therefore, it follows from Lemma 7.3, that after making the cancellation $0 \to S[\sigma_{j-1}] \to S[\tau_{j-1}] \to 0$, the cancellation $0 \to S[\sigma_j] \to S[\tau_j] \to 0$ is still possible. $\square$

The next theorem proves that the basis of the minimal free resolution of S/M can be obtained from the basis of its Taylor resolution by eliminating pairs of faces and facets of equal multidegree in arbitrary order, until exhausting all possibilities.

Theorem 7.5. *Let M be a 2-semidominant ideal. Let $A = \{([\sigma_1],[\tau_1]),\dots,([\sigma_k],[\tau_k])\}$ be a family of pairs of basis elements in $\mathbb{T}_M$, having the following properties:*

(i) $([\sigma_i],[\tau_i])$ *and* $([\sigma_j],[\tau_j])$ *are disjoint, if* $i \neq j$.
(ii) $[\tau_i]$ *is a facet of* $[\sigma_i]$ *for all* $i = 1,\dots k$.
(iii) $\mathrm{mdeg}[\sigma_i] = \mathrm{mdeg}[\tau_i]$ *for all* $i = 1,\dots k$.
(iv) A is maximal with respect to inclusion among the sets satisfying i), ii) and iii).

Then a minimal free resolution $\mathbb{F}$ of S/M can be obtained from $\mathbb{T}_M$ by doing all standard cancellations $0 \to S[\sigma] \to S[\tau] \to 0$, with $([\sigma],[\tau]) \in A$. In symbols,

$$\mathbb{T}_M = \mathbb{F} \oplus \left(\bigoplus_{([\sigma],[\tau])\in A} 0 \to S[\sigma] \to S[\tau] \to 0 \right)$$

Proof. By Theorem 7.4, $\mathbb{F}$ is a resolution of S/M. We claim that $\mathbb{F}$ is minimal. If $\mathbb{F}$ were not minimal, one of its differential matrices would contain an invertible entry. That, in turn, would mean that there exists a pair $([\sigma],[\tau])$ of basis elements of $\mathbb{T}_M$, such that $A \bigcup \{([\sigma],[\tau])\}$ satisfies conditions (i), (ii), and (iii), which contradicts (iv). $\square$

Theorem 7.6 *(Characterization of the scarf 2-semidominant ideals). Let M be a 2-semidominant ideal. Let $B = \{m : m$ is the multidegree of more than one basis element of $\mathbb{T}_M\}$. For each $m \in B$, let $B_m = \{[\sigma] \in \mathbb{T}_M : \mathrm{mdeg}[\sigma] = m\}$. Then M is Scarf if and only if $\#(B_m)$ is even for all $m \in B$.*

Proof. Let $G = \{m_1,\dots,m_q,n_1,n_2\}$ be the minimal generating set of M. Let us denote with $\mathbb{F}$ the minimal resolution of S/M.
($\Rightarrow$) Let $m \in B$. Since M is minimally resolved by the Scarf complex, all elements of B_m are excluded from the basis of $\mathbb{F}$, but if we were to eliminate the elements of B_m making standard cancellations, those elements would be eliminated in pairs. It follows that $\#(B_m)$ is even.
($\Leftarrow$) Let $m \in B$. We need to prove that no element of the basis of $\mathbb{F}$ has multidegree m. Given that basis elements of $\mathbb{T}_M$ with the same multidegree contain the same dominant monomials, what distinguishes these elements is the nondominant monomials that define them. Thus there are at most four basis elements of multidegree m; namely,

$$[\sigma_1] = [m_{i_1},\dots,m_{i_r}]; \quad [\sigma_2] = [m_{i_1},\dots,m_{i_r},n_1];$$
$$[\sigma_3] = [m_{i_1},\dots,m_{i_r},n_2]; \quad [\sigma_4] = [m_{i_1},\dots,m_{i_r},n_1,n_2]$$

The fact that $\#(B_m)$ is even implies that either
(i) $\#(B_m) = 4$ or (ii) $\#(B_m) = 2$.

(i) In this case $([\sigma_2], [\sigma_1])$, $([\sigma_4], [\sigma_3])$ and $\mathbb{T}_M$ satisfy the hypotheses of Lemma 7.3, which means that after making the standard cancellation $0 \to S[\sigma_2] \to S[\sigma_1] \to 0$ in $\mathbb{T}_M$, it is still possible to make the cancellation $0 \to S[\sigma_4] \to S[\sigma_3] \to 0$. Hence, the basis of $\mathbb{F}$ does not contain elements of multidegree m.
(ii) We will show that the two basis elements with multidegree m are face and facet. There are exactly two pairs of basis elements that are not face and facet; these pairs are $[\sigma_2]$, $[\sigma_3]$ and $[\sigma_1]$, $[\sigma_4]$. If we assume that $\text{mdeg}[\sigma_2] = \text{mdeg}[\sigma_3] = m$, then $n_2 \mid \text{mdeg}[\sigma_3] = \text{mdeg}[\sigma_2]$. It follows that $\text{mdeg}[\sigma_4] = \text{mdeg}[\sigma_2]$ and thus $[\sigma_4]$, $[\sigma_2]$ and $[\sigma_3]$ have multidegree m, which is not possible because $\#(B_m) = 2$.

Similarly, if $\text{mdeg}[\sigma_1] = \text{mdeg}[\sigma_4]$, then $n_2 \mid \text{mdeg}[\sigma_4] = \text{mdeg}[\sigma_1]$, which implies that $[\sigma_3]$, $[\sigma_1]$ and $[\sigma_4]$ have multidegree m, which is not possible. Therefore, if $\text{mdeg}[\sigma_i] = \text{mdeg}[\sigma_j] = m$, then $[\sigma_i]$ and $[\sigma_j]$ must be face and facet. Thus they determine an invertible entry of $\mathbb{T}_M$ and it is possible to eliminate $[\sigma_i]$ and $[\sigma_j]$ from the basis of $\mathbb{T}_M$ by means of a standard cancellation. This means that no element of the basis of $\mathbb{F}$ has multidegree m. $\square$

8. Conclusion

A distinctive mark of this paper is that, in order to build the basis of a minimal resolution, we always start with the Taylor resolution and eliminate redundant parts by means of standard cancellations. There is no real advantage in knowing how to make these cancellations, if a class of monomial ideals has a well known resolution as its minimal resolution (this is the case for 1-semidominant ideals). However, when a family of ideals cannot be minimally resolved by one of these known resolutions, the standard cancellations become an alternative. This is especially true if the description of the cancellations is simple (consider the 2-semidominant ideals, for example).

Since the minimal resolutions of 2-semidominant ideals can be obtained from $\mathbb{T}_M$ eliminating faces and facets of equal multidegree, in arbitrary order, it is natural to wonder whether 3-semidominant ideals can be resolved in the same way. Unfortunately, the answer is no, as the next example shows.

Example 8.1. With the assistance of a software system (for instance, Macaulay 2 [5]) it is easy to verify that the 3-semidominant ideal $M = (x^2y^2z^2, xw^2, yw^2, zw)$ is Scarf. Now, there are six basis elements of $\mathbb{T}_M$ with multidegree $m = x^2y^2z^2w^2$ which, therefore, are excluded from the basis of the minimal resolution of S/M. However, if we eliminate pairs of face and facet of equal multidegree as follows:

$$\left([x^2y^2z^2, xw^2, yw^2, zw], [x^2y^2z^2, xw^2, zw]\right), \text{ and then } \left([x^2y^2z^2, xw^2, yw^2], [x^2y^2z^2, yw^2]\right),$$

the remaining basis elements of multidegree m, $[x^2y^2z^2, yw^2, zw]$ and $[x^2y^2z^2, xw^2]$, cannot be eliminated in this way because they are not face and facet. This tells us that the order of the cancellations matters for 3-semidominant ideals.

Something that we infer when considering dominant, 1-, 2-, and 3-semidominant ideals, is that the minimal resolutions of p-semidominant ideals increase in complexity as p grows. Nevertheless, 3-semidominant ideals are a class that we can still handle. We close this article with a description of their minimal resolutions.

Let $M = (l_1, \ldots, l_q)$ be 3-semidominant. The minimal free resolution of S/M can be obtained from $\mathbb{T}_M$ by eliminating pairs of Taylor symbols of equal multidegree, and consecutive homological degrees, in decreasing order of homological degree. More precisely, for $j = 1, \ldots, q-2$, Step j: eliminate pairs $([\sigma], [\tau])$ of Taylor symbols, where $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, $\text{hdeg}[\sigma] - 1 = \text{hdeg}[\tau] = q - j$, in arbitrary order, until exhausting all possibilities.

Notice that in Example 8.1, the pair $([x^2y^2z^2, yw^2, zw], [x^2y^2z^2, xw^2])$ could not be eliminated following the rule of excluding pairs of face and facet. However, these Taylor symbols appear in consecutive homological degrees, and have equal multidegree. Thus, eliminating the pairs $\left([x^2y^2z^2, xw^2, yw^2, zw], [x^2y^2z^2, xw^2, zw]\right)$,

$\big([x^2y^2z^2, xw^2, yw^2], [x^2y^2z^2, yw^2]\big)$, and $([x^2y^2z^2, yw^2, zw], [x^2y^2z^2, xw^2])$ is allowed by the technique just introduced, which is consistent with the fact that the ideal M is Scarf.

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