# BOUNDING THE SOCLES OF POWERS OF SQUAREFREE MONOMIAL IDEALS

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Abstract. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in $n$ variables over a field $K$ and $I \subset S$ a squarefree monomial ideal. In the present paper we are interested in the monomials $u \in S$ belonging to the socle $\mathrm{Soc}(S/I^k)$ of $S/I^k$, i.e., $u \notin I^k$ and $ux_i \in I^k$ for $1 \leq i \leq n$. We prove that if a monomial $x_1^{a_1} \cdots x_n^{a_n}$ belongs to $\mathrm{Soc}(S/I^k)$, then $a_i \leq k-1$ for all $1 \leq i \leq n$. We then discuss squarefree monomial ideals $I \subset S$ for which $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$, where $x_{[n]} = x_1x_2\cdots x_n$. Furthermore, we give a combinatorial characterization of finite graphs $G$ on $[n] = \{1, \dots, n\}$ for which $\operatorname{depth} S/(I_G)^2 = 0$, where $I_G$ is the edge ideal of $G$.

## Introduction

The depth of powers of an ideal (especially, a monomial ideal) of the polynomial ring has been studied by many authors. In the present paper, we are interested in the socle of powers of a squarefree monomial ideal.

Let $K$ be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in $n$ variables over $K$, and $I \subset S$ a graded ideal. We denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal of $S$. An element $f + I \in S/I$ is called a *socle element* of $S/I$ if $x_if \in I$ for $i = 1, \dots, n$. Thus $f + I$ is a non-zero socle element of $S/I$ if $f \in I : \mathfrak{m} \setminus I$. The set of socle elements $\mathrm{Soc}(S/I)$ of $S/I$ is called the *socle* of $S/I$. Notice that $\mathrm{Soc}(S/I)$ is a $K$-vector space isomorphic to $(I : \mathfrak{m})/I$. One has $\operatorname{depth} S/I = 0$ if and only if $\mathrm{Soc}(S/I) \neq \{0\}$.

In the case that $I$ is a monomial ideal, a case which we mainly consider here, $\mathrm{Soc}(S/I)$ is generated by the residue classes of monomials. If $u$ and $v$ are monomials not belonging to $I$, then $u+I = v+I$, if and only if $u = v$. Thus, if $u$ is a monomial, it is convenient to write $u \in \mathrm{Soc}(S/I)$ and to call $u$ a socle element of $S/I$ if $u+I \in \mathrm{Soc}(S/I)$ and $u+I \neq 0$. In other words, $u \in \mathrm{Soc}(S/I)$ if and only if $u \notin I$ and $ux_i \in I$ for all $1 \leq i \leq n$.

The present paper is organized as follows. First, in Section 1, we show that, for a squarefree monomial ideal $I \subset S$, if a monomial $x_1^{a_1} \cdots x_n^{a_n}$ is a socle element of $S/I^k$, then $a_i \leq k-1$ for all $1 \leq i \leq n$ (Corollary 1.2). Second, in Section 2, the edge ideal $I_G$ arising from a finite graph $G$ is discussed. We give a combinatorial characterization of $G$ for which $\operatorname{depth} S/(I_G)^2 = 0$ (Theorem 2.1).

Let $I \subset S$ be a squarefree monomial ideal. Then Corollary 1.2 says that one has $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$ if and only if $x_{[n]}^{k-1} \notin I^k$, where $x_{[n]} = x_1x_2\cdots x_n$. In Section 3, we study squarefree monomial ideals $I \subset S$ with $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$. It is proved that, for a squarefree monomial ideal $I \subset S$ with $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$, one has $k < n$ and $\operatorname{depth} S/I^j > 0$ for $j < k$ (Corollary 3.2). Furthermore, for a squarefree monomial ideal $I \subset S$ generated in degree $d$ with $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$, we show that if $d > ((k-1)n+1)/k$, then $\operatorname{depth} S/I^k > 0$ and that

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if $d = ((k-1)n+1)/k$ and $\operatorname{depth} S/I^k = 0$, then $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$ and $\operatorname{depth} S/I^\ell = 0$ for all $\ell \geq k$ (Corollary 3.4).

## 1. Socles of powers of squarefree monomial ideals

We begin with

**Proposition 1.1.** *Let $I$ be a monomial ideal. For $i = 1, \ldots, n$ set*

$$c_i = \max\{\deg_{x_i}(u) : u \in G(I)\},$$

*and let $x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of $S/I$. Then $a_i \leq c_i - 1$ for $i = 1, \ldots, n$.*

*Proof.* Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of $S/I$. Thus $u \notin I$ and $u \in I : \mathfrak{m}$. Suppose that $a_i \geq c_i$ for some $i$. Since $x_i u \in I$, there exists $v \in G(I)$ which divides $x_i u$.

It follows that $\deg_{x_j}(v) \leq \deg_{x_j}(x_i u) = \deg_{x_j}(u)$ for $j \neq i$, and $\deg_{x_i}(v) \leq c_i \leq \deg_{x_i}(u)$. Therefore, $v$ divides $u$, and hence $u \in I$, a contradiction. $\square$

**Corollary 1.2.** *Let $I$ be a squarefree monomial ideal, and let $x_1^{a_1} \cdots x_n^{a_n}$ be a socle element of $S/I^k$. Then*

$$a_i \leq k-1 \quad \text{for} \quad i = 1, \ldots, n.$$

## 2. Edge ideals whose square has depth zero

We consider the case of edge ideals.

**Theorem 2.1.** *Let $I = I_G \subset S = K[x_1, \ldots, x_n]$ be the edge ideal of graph $G$ on the vertex set $[n]$. The following conditions are equivalent:*

(a) $\operatorname{depth} S/I^2 = 0$*;*

(b) *$G$ is a connected graph containing a cycle $C$ of length $3$, and any vertex of $G$ is a neighbor of $C$.*

*Moreover, $x_{[n]} \in \operatorname{Soc}(S/I^2)$ if and only if $G$ is cycle of length $3$.*

*Proof.* (b) $\Rightarrow$ (a): Suppose that $G$ has a cycle of length 3, say, $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are edges of $G$ and that, for each $4 \leq j \leq n$, one of $\{1,j\}$, $\{2,j\}$ and $\{3,j\}$ is an edge of $G$. It then follows immediately that the monomial $u = x_1x_2x_3$ satisfies $u \notin I^2$ and $u \in I^2 : \mathfrak{m}$. Hence $\operatorname{depth} S/I^2 = 0$, as required. This argument also shows that $x_{[n]} \in \operatorname{Soc}(S/I^2)$ if and only if $G$ is cycle of length 3

(a) $\Rightarrow$ (b): Let $I = I_G$ be the edge ideal of a finite graph $G$ with $\operatorname{depth} S/I^2 = 0$. Then there exists a monomial $u$ with $u \notin I^2$ such that $u \in I^2 : \mathfrak{m}$. Let $H$ denote the induced subgraph of $G$ whose vertices are those $i \in [n]$ such that $x_i$ divides $u$. Since $u \notin I^2$ it follows that $H$ cannot possess two disjoint edges. If $H$ possesses an isolated vertex $i$, then $x_i u \notin I^2$. This contradict $u \in I^2 : \mathfrak{m}$. Hence $H$ is connected without disjoint edges. Thus $H$ must be either a cycle of length 3, or a line of length at most 2.

First, if $H$ is a line of length 1, i.e., $H$ is an edge of $G$, then we may assume that $u = x_1^{a_1} x_2^{a_2}$ with each $a_i \geq 1$. If each $a_i \geq 2$, then $u \in I^2$, a contradiction. Let $a_1 = 1$ and $u = x_1 x_2^{a_2}$. Then $ux_2 \notin I^2$. This contradict $u \in I^2 : \mathfrak{m}$.

Now, let $H$ be either a cycle of length 3, or a line of length 2. Thus we may assume that $u = x_1^{a_1} x_2^{a_2} x_3^{a_3}$ with each $a_i \geq 1$, where $\{1,2\}$ and $\{1,3\}$ are edges of $G$. Since $u \notin I^2$, it follows that $a_1 = 1$. Thus $u = x_1 x_2^{a_2} x_3^{a_3}$. If $\{2,3\}$ is not an edge of $G$, then $x_2 u \notin I^2$, a contradiction. Hence $\{2,3\}$ is an edge of $G$. Then, since $u \notin I^2$, it follows that $a_2 = a_3 = 1$. Thus $u = x_1x_2x_3$ and $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ are edges of $G$. Let $j \geq 4$. Since $x_j u \in I^2$, it follows that one of $\{1,j\}$, $\{2,j\}$ and $\{3,j\}$ must be an edge of $G$, as desired. $\square$

This result has been shown independently by [5].

## 3. Powers of squarefree monomial ideals with maximal socle

Let $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree monomial ideal. According to Corollary 1.2, $x_{[n]}^{k-1}$ is a socle element of $S/I^k$ if $x_{[n]}^{k-1} \notin I^k$. In that case it is a socle element of $S/I^k$ of maximal degree. The next proposition characterizes those squarefree monomial ideals for which $x_{[n]}^{k-1}$ is indeed a socle element of $S/I^k$.

We consider $I$ as the facet ideal of a simplicial complex $\Delta$. Thus $I = I(\Delta)$ where the set of facets $\mathcal{F}(\Delta)$ of $\Delta$ is given as

$$\mathcal{F}(\Delta) = \{\mathrm{supp}(u) \colon\ u \in G(I)\}.$$

In other words, $G(I(\Delta)) = \{x_F \colon\ F \in \mathcal{F}(\Delta)\}$ where we set $x_F = \prod_{i\in F} x_i$ for $F \subset [n]$.

**Proposition 3.1.** *Let $\Delta$ be a simplicial complex on the vertex set $[n]$, and $I = I(\Delta) \subset S = K[x_1, \ldots, x_n]$ its facet ideal.*

(a) *The following conditions are equivalent:*
   (i) $x_{[n]}^{k-1} \notin I^k$*;*
   (ii) $\bigcap_{i=1}^k F_i \neq \emptyset$ *for all* $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$*.*

(b) *Assuming that $x_{[n]}^{k-1} \notin I^k$, the following conditions are equivalent:*
   (i) $x_j x_{[n]}^{k-1} \in I^k$ *for all* $j$*;*
   (ii) *for each* $j = 1, \ldots, n$*, there exist* $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$ *such that* $\bigcap_{i=1}^k F_i = \{j\}$*.*

*In particular, $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$ if and only if* (a)(ii) *and* (b)(ii) *hold.*

*Proof.* (a) $x_{[n]}^{k-1} \in I^k$ if and only if there exist $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$ such that $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_{[n]}^{k-1}$. This is the case, if and only if no $x_i^k$ divides $x_{F_1} x_{F_2} \cdots x_{F_k}$. This is equivalent to saying that $\bigcap_{i=1}^k F_i = \emptyset$. Thus the desired conclusion follows.

(b) $x_i x_{[n]}^{k-1} \in I^k$ if and only if $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides divides $x_j x_{[n]}^{k-1}$ for some $F_1, \ldots, F_k \in \mathcal{F}(\Delta)$. By (a), $\bigcap_{i=1}^k F_i \neq \emptyset$. Therefore, $x_{F_1} x_{F_2} \cdots x_{F_k}$ divides $x_j x_{[n]}^{k-1}$ if and only if $\bigcap_{i=1}^k F_i = \{j\}$. $\square$

**Corollary 3.2.** *Let $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Let $n > 1$ and suppose that $x_{[n]}^{k-1} \in \mathrm{Soc}(S/I^k)$. Then $k < n$, and* $\operatorname{depth} S/I^j > 0$ *for $j < k$.*

*Proof.* The condition (b)(ii) of Proposition 3.1 guarantees the existence of $F^{(j)} \in \mathcal{F}(\Delta)$ with $j \in F^{(j)}$ and $j+1 \notin F^{(j)}$ for each $1 \leq j < n$ and the existence of $F^{(n)} \in \mathcal{F}(\Delta)$ with $n \in F^{(n)}$ and $1 \notin F^{(n)}$. Then $\bigcap_{j=1}^n F^{(j)} = \emptyset$. Thus if $k \geq n$, then the condition (a)(ii) of Proposition 3.1 is violated, and hence $k < n$.

Let $j < k$ and suppose that $\operatorname{depth} S/I^j = 0$. Then $j \geq 2$, since $I$ is squarefree. Let $u \in \mathrm{Soc}(S/I^j)$; then $ux_i \in I^j$ for all $i$ and hence also $x_{[n]}^{j-1} x_i \in I^j$ for all $i$. Since $n > 1$, the ideal $I$ cannot be a principal ideal, because otherwise $\operatorname{depth} S/I^j > 0$ for all $j$. Hence there exist an integer $i$, say $i = 1$, such that $x_2 x_3 \cdots x_n \in I$. Then

$$x_{[n]}^j = (x_{[n]}^{j-1} x_1)(x_2 x_3 \cdots x_n) \in I^{j+1}.$$

It follows that

$$x_{[n]}^{k-1} = x_{[n]}^j x_{[n]}^{k-j-1} = (x_{[n]}^j x_1^{k-j-1})(x_2 x_3 \cdots x_n)^{k-j-1} \in I^k,$$

a contradiction. $\square$

**Examples 3.3.** (a) The ideal

$$I = (x_1x_2\cdots x_{n-1}, x_1x_n, x_2x_n, \ldots, x_{n-1}x_n)$$

in $S = K[x_1, \ldots, x_n]$ satisfies the conditions (a)(ii) and (b)(ii) of Proposition 3.1 for $k = 2$. Hence $\operatorname{depth}(S/I^2) = 0$.

(b) Let $n = 2d-1$ and $I$ a monomial ideal of $S = K[x_1, \ldots, x_n]$ generated by squarefree monomials of degree $d$. Then the condition (a)(ii) in Proposition 3.1 is satisfied for $k = 2$. Thus if a squarefree monomial $w$ belongs to $\operatorname{Soc}(S/I^2)$, then $w$ must be $x_{[n]}$. Hence $\operatorname{depth} S/I^2 = 0$ if and only if $I$ satisfies for $k = 2$ the condition (b)(ii) in Proposition 3.1.

For example, if $I$ is generated by the following squarefree monomials

$$\begin{aligned}
&x_1x_2\cdots x_d, \quad x_1x_{d+1}x_{d+2}\cdots x_{2d-1},\\
&x_ix_{d+1}x_{d+2}\cdots x_{2d-1} \quad \text{with} \quad 2 \le i \le d,\\
&x_2x_3\cdots x_dx_j \quad \text{with} \quad d+1 \le j \le 2d-1,
\end{aligned}$$

then $\operatorname{depth} S/I^2 = 0$.

Example 3.3(b) shows that for any odd integer $n > 1$ there exists a squarefree monomial ideal $I \subset K[x_1, \ldots, x_n]$ generated in degree $d = (n+1)/2$ such that $\operatorname{depth} S/I^2 = 0$.

On the other hand for a squarefree monomial ideal generated in degree $d > (n+1)/2$ one has $\operatorname{depth} S/I^2 > 0$, as follow from

**Corollary 3.4.** *Let $I \subset K[x_1, \ldots, x_n]$ be a squarefree monomial ideal generated in the single degree $d$.*

(a) *If $d > ((k-1)n+1)/k$, then* $\operatorname{depth} S/I^k > 0$.
(b) *For all positive integer $d, k$ and $n$ such that $d = ((k-1)n+1)/k$, there exists a squarefree monomial ideal $I \subset K[x_1, \ldots, x_n]$ generated in degree $d$ such that* $\operatorname{depth} S/I^k = 0$.
(c) *If $d = ((k-1)n+1)/k$ and* $\operatorname{depth} S/I^k = 0$. *Then $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$ and* $\operatorname{depth} S/I^\ell = 0$ *for all $\ell \ge k$.*

*Proof.* (a) Let $F_1, \ldots, F_k$ subset of $[n]$ of cardinality $d$. We first show by induction on $i$, that $|\bigcap_{j=1}^i F_j| > ((k-i)n+i)/k$. The assertion is trivial for $i = 1$. By using the induction hypothesis, we see that

$$|\bigcap_{j=1}^{i} F_j| \ge |\bigcap_{j=1}^{i-1} F_j| + |F_i| - n > \frac{(k-i+1)n+(i-1)}{k} + \frac{(k-1)n+1}{k} - n = \frac{(k-i)n+i}{k},$$

as desired.

It follows that any intersection of $k$ subsets of $[n]$ of cardinality $d$ admits more than one elements. Therefore $I$ satisfies condition (a)(ii) of Proposition 3.1, but violates condition (b)(ii).

Since condition (a)(ii) is satisfied, it follows from Proposition 3.1 that $x_{[n]}^{k-1} \notin I^k$. Thus, if we assume that $\operatorname{depth} S/I^k = 0$, Corollary 1.2 implies that $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$. However, since condition (b)(ii) is violated, this is not possible.

(b) Suppose that $d = ((k-1)n+1)/k$. Then there exists an integer $r \ge 0$ such that $d = (r+1)k - r$ and $n = (r+1)k + 1$. Consider the monomial ideal $I$ generated by all squarefree monomials of degree $d$ in $K[x_1,\dots,x_n]$. By [2, Corollary 3.4] one has

$$\operatorname{depth} S/I^k = \max\{0, n-k(n-d)-1\}.$$

Since $n-k(n-d)-1 = (r+1)k+1-k(r-1)-1 = 0$, the assertion follows.

(c) Let $u \in \operatorname{Soc}(S/I^k)$, $u = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$. Then, by Corollary 1.2, $a_i \leq k-1$ for all $i$, and hence $\deg u \leq (k-1)n = kd-1$. On the other hand, since $ux_i \in I^k$, it follows that $\deg u + 1 \geq kd$. Thus we conclude that $\deg u = kd-1 = (k-1)n$, which is only possible if $u = x_{[n]}^{k-1}$. Let $\ell > k$ and let $v$ be a generator of $I^{\ell-k}$. Then $uvx_i \in I^{\ell+1}$, but $uv \notin I^\ell$, because

$$\deg uv = (kd-1)+(\ell-k) \leq kd-1+(\ell-k)d = \ell d - 1 < \ell d.$$

This shows that $uv \in \operatorname{Soc}(S/I^\ell)$, and consequently $\operatorname{depth} S/I^\ell = 0$, as required. $\square$

**Examples 3.5.** Let $k \geq 2$, and assume that $d = ((k-1)n+1)/k$. Then $n = (kd-1)/(k-1)$, and this is an integer if and only if $d \equiv 1 \bmod (k-1)$. One solution is $d = k$. Then $n = k+1$. With these data we may choose the ideal $I \subset S = K[x_1,\dots,x_n]$ generated by all squarefree monomials of degree $d = k = n-1$. Then obviously $I$ satisfies the conditions (a)(i) and (b)(i) of Proposition 3.1. Thus $x_{[n]}^{k-1} \in \operatorname{Soc}(S/I^k)$. In particular, $\operatorname{depth} S/I^k = 0$. It is shown in [2] that $\operatorname{depth} S/I^j > 0$ for $j < k$. (This also follows from Corollary 3.2). This example shows that arbitrary high powers of a squarefree monomial ideal may have a maximal socle.

It is known by a result of Brodmann [1] (see also [2]) that the depth function $f(k) = \operatorname{depth} S/I^k$ is eventually constant. In [3] the smallest number $k$ for which $\operatorname{depth} S/I^k = \operatorname{depth} S/I^j$ for all $j \geq k$, is denoted by $\operatorname{dstab}(I)$. In [4] it is conjectured that $\operatorname{dstab}(I) < n$ for all graded ideals in $K[x_1,\dots,x_n]$. Corollary 3.2 together with Corollary 3.4(c) show that this conjecture holds true for a squarefree monomial ideal $I \subset K[x_1,\dots,x_n]$ generated in degree $d = ((k-1)n+1)/k$ for which $\operatorname{depth} S/I^k = 0$.

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