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Further applications of clutter domination parameters to projective dimension

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ABSTRACT

We study the relationship between the projective dimension of a squarefree monomial ideal and the domination parameters of the associated graph or clutter. In particular, we show that the projective dimensions of graphs with perfect dominating sets can be calculated combinatorially. We also generalize the well-known graph domination parameter τ to clutters, obtaining bounds on the projective dimension analogous to those for graphs. Through Hochster's Formula, our bounds on projective dimension also give rise to bounds on the homologies of the associated Stanley–Reisner complexes.

1. Introduction

In this paper we continue the theme of [4] and [5] in relating the projective dimension of a squarefree monomial ideal to domination parameters of the associated clutter. This point of view has proved fruitful in recovering and improving both various bounds on projective dimensions of squarefree monomial ideals (see [3], [6]), as well as bounds on the homology of the associated Stanley–Reisner complexes.

The first half of the paper deals with graphs that have perfect dominating sets. We prove that, in this case, the projective dimension of the associated ideal $I(G)$ is exactly

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the size of the complement of any perfect dominating set of G (Theorem 3.6). This provides a class of graphs whose ideals' projective dimensions can be recovered immediately via combinatorial properties of the associated graphs.

In the remainder of the paper, we generalize a domination parameter for graphs studied in [4] to clutters. This invariant is denoted $\tau(\mathcal{C})$. Our main result (Theorem 5.2) asserts that for any clutter $\mathcal{C}$, we have $\mathrm{pd}(\mathcal{C}) \leq |V(\mathcal{C})| - \tau(\mathcal{C})$. The proof, while using the same ideas, is a bit more intricate than the proof for the graph case given in [4].

As usual, bounds on projective dimension translate into bounds on the simplicial homology of the associated Stanley–Reisner complex. For instance, we recover the homological analogue of a result by Barmak on simplicial complex homology (Corollary 5.4).

We also construct an associated graph $K(\mathcal{C})$ for a clutter $\mathcal{C}$, which allows us to bound the homology of the associated Stanley–Reisner complex of $\mathcal{C}$ (Corollary 5.10).

Our paper is organized as follows. In Section 2, we review the background on graphs and their associated ideals, as well as domination parameters. In Section 3, we discuss graphs with perfect dominating sets and properties of the associated ideals. Section 4 is concerned with the background on clutters (or hypergraphs). Finally, in Section 5, we generalize the parameter τ to clutters and prove a result analogous to that proved in [4], and derive some corollaries from this new bound on projective dimension.

2. Background and terminology

Let G be a finite simple graph with vertex set $V(G)$. We often identify $V(G)$ with the set $[n] := \{1, 2, \ldots, n\}$. For $A \subseteq V(G)$, we write $G[A]$ for the corresponding *induced subgraph* of G, which is the subgraph on vertex set A consisting of all edges (v, w) of G where $v, w \in A$. The *independence complex* of G, for which we write $\mathrm{ind}(G)$, is the simplicial complex with vertex set $V(G)$ where $A \subseteq V(G)$ is a face of $\mathrm{ind}(G)$ whenever $G[A]$ contains no edges (that is, no two elements of A are neighbors).

The Stanley–Reisner ideal of $\mathrm{ind}(G)$, for which we write I_G, is the ideal in S given by

$$I_G = \{x_i x_j : (i, j) \text{ is an edge of } G\}.$$

Let k be a fixed field. We set $S = k[x_1, x_2, \ldots, x_n]$. The *projective dimension* of I_G, for which we write $\mathrm{pd}(S/I_G)$ or $\mathrm{pd}(G)$, is the shortest length of a projective (or equivalently, free) resolution of S/I_G. The following alternate characterization of projective dimension can be derived from Hochster's Formula (see, for instance, [7]). Here and in what follows, we write "$\tilde{H}_i(\Sigma) = 0$" to mean that the reduced homology group $\tilde{H}_i(\Sigma)$ is trivial.

The following is an easy corollary of Hochster's Formula.

Proposition 2.1. *The projective dimension of S/I_G is the smallest integer i such that for all $X \subseteq V(G)$ we have*

$$\tilde{H}_k(\mathrm{ind}(G[X])) = 0$$

for all $k < |X| - i - 1$.

Plugging in $X = V(G)$ into the above proposition yields the following corollary.

Corollary 2.2. *The homology of* $\operatorname{ind}(G)$ *satisfies*

$$\tilde{H}_k(\operatorname{ind}(G)) = 0$$

for $k < |V(G)| - \operatorname{pd}(G) - 1$.

Recall that a vertex of G is *isolated* if it is contained in no edge. We write $\overline{G}$ to denote G with its isolated vertices removed. As isolated vertices of G do not appear in any generator of I_G, it follows that $I_G = I_{\overline{G}}$. More specifically, we have the following.

Observation 2.3. For any graph G, we have $\operatorname{pd}(G) = \operatorname{pd}(\overline{G})$.

Thus, in many of our applications, we can assume that G has no isolated vertices.

Finally, for $x \in V(G)$, we write $N(x)$ to denote the set of neighbors of x in G, and if $X \subseteq V(G)$ we set $N(X) = \cup_{x \in X} N(x)$.

3. Graphs with perfect dominating sets

In [4], we relate the projective dimension of a graph G to the so-called *domination parameters* of G. We briefly recall some of the definitions and results from [4].

Definition 3.1. If G is a graph, an independent set $A \subseteq V(G)$ is an *independent dominating set* of G if $V(G) = A \cup N(A)$. We let $i(G)$ be the smallest size of an independent dominating set of G.

Definition 3.2. Let G be a graph. For a set $A \subseteq V(\overline{G})$, define $\gamma_0(A, G)$ by

$$\gamma_0(A, G) = \min\{|X| : X \subseteq V(G) \text{ and } A \subseteq N(X)\},$$

and define the domination parameter $\tau(G)$ by

$$\tau(G) = \max\{\gamma_0(A, G) : A \subseteq V(\overline{G}) \text{ is independent}\}.$$

The following is well-known, but we state it here for completeness.

Observation 3.3. For any graph G, $|V(G)| - i(G)$ equals $\operatorname{BigHeight}(I_G)$, the BigHeight of I_G. Thus $|V(G)| - i(G) \leq \operatorname{pd}(G)$.

One of the main results in [4] relating the projective dimension of a graph to its domination parameters is the following.

Theorem 3.4. *(See [4].) For any graph G, we have*

$$\operatorname{pd}(G) \le |V(G)| - \tau(G).$$

In order to apply Theorem 3.4 and Observation 3.3, we need the notion of a perfect dominating set. We use the standard definition of graph distance where for $x, y \in V(G)$, $\operatorname{dist}(x, y)$ is the least number of edges in a path from x to y (so that $\operatorname{dist}(x, y) = 1$ iff (x, y) is an edge of G and $\operatorname{dist}(x, y) = 0$ iff $x = y$). We also set $\operatorname{dist}(x, y) = \infty$ if x and y are in different connected components of G.

Definition 3.5. Let G be a graph. A set $A \subseteq V(G)$ is a *perfect dominating set* of G if A is dominating (i.e., $V(G) = A \cup N(A)$) and $\operatorname{dist}(x, y) \ge 3$ for any two vertices $x, y \in A$.

Perfect dominating sets have been studied in depth by those working in graph theory and optimization-type problems (see, for instance, [2]).

Note that, unlike the other forms of graph domination discussed, many graphs have no perfect dominating sets (the simplest example of such a graph is the 5-cycle). In the case when G has a perfect dominating set, however, we can give an exact formula for $\operatorname{pd}(G)$, as shown by the following.

Theorem 3.6. *Suppose G has a perfect dominating set A. Then*

$$\operatorname{pd}(G) = |V(G)| - i(G) = |V(G)| - |A| = \operatorname{BigHeight}(I_G).$$

Proof. First, suppose G has a nonempty set Z of isolated vertices. Then $i(\overline{G}) = i(G) - |Z|$, and so $|V(G)| - i(G) = |V(\overline{G})| + |Z| - (i(\overline{G}) + |Z|) = |V(\overline{G})| - i(\overline{G})$. By Observation 2.3, $\operatorname{pd}(G) = \operatorname{pd}(\overline{G})$, and so we may assume that G has no isolated vertices.

Now let A be a perfect dominating set of G, and let $X \subseteq V(G)$ be a set of minimal cardinality such that $A \subseteq N(X)$ (so that $|X| = \gamma_0(A, G)$). First note that $X \cap A = \emptyset$, since no element of A is a neighbor of another. For any $x \in X$, we must have $|N(x) \cap A| = 1$, since if there were $a, a' \in N(x) \cap A$ with $a \ne a'$, we would have $\operatorname{dist}(a, a') \le \operatorname{dist}(a, x) + \operatorname{dist}(x, a') = 2$, contradicting the assumption that A is a perfect dominating set. Thus, for each $x \in X$ there is a unique $a \in A$ with $a \in N(x)$, and so $|X| \ge |A|$. If $|X|$ were greater than $|A|$, then X would not be minimal, and so we must have $|X| = |A|$.

As $|X| = \gamma_0(A, G)$ and A is independent, $\tau(G) \ge |X|$. Similarly, since A is independent and dominating, we have $|A| \ge i(G)$. Observation 3.3 and Theorem 3.4 now give us:

$$\begin{aligned} |V(G)| - |A| \le |V(G)| - i(G) \le \operatorname{pd}(G) &\le |V(G)| - \tau(G) \\ &\le |V(G)| - |X| = |V(G)| - |A|, \end{aligned}$$

meaning

$$\operatorname{pd}(G) = |V(G)| - |A| = |V(G)| - i(G). \qquad \square$$

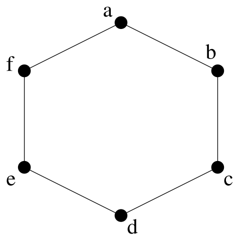


Fig. 1. The 6-cycle C_6.

The following is already known, though we recover it here.

Corollary 3.7. *If G has a perfect dominating set, then any two perfect dominating sets of G have the same cardinality.*

Proof. Theorem 3.6 shows that any perfect dominating set must have cardinality $i(G)$. □

The above corollary can be shown without reference to projective dimension. Indeed, writing $\gamma(G)$ for the smallest cardinality of a set Y such that $V(G) = Y \cup N(Y)$, we have $i(G) \geq \gamma(G)$ (as Y need not be independent). In [4, Theorem 4.1] we show that $\tau(G) \leq \gamma(G)$. Now let G, X, and A be as in the proof of Theorem 3.6. Then $|A| = |X| \leq \tau(G) \leq \gamma(G) \leq i(G) \leq |A|$, and so any perfect dominating set A satisfies $|A| = i(G) = \gamma(G) = \tau(G)$.

Example 3.8. Let C_6 denote the 6-cycle as shown in Fig. 1. Then any pair of antipodal vertices, such as $\{a, d\}$ or $\{b, e\}$ is a perfect dominating set of C_6. Thus $i(C_6) = \tau(C_6) = 2$, and we have $\operatorname{pd}(C_6) = 6 - 2 = 4$.

Many classes of graphs are known to have perfect dominating sets (such as cycles on $3n$ elements and paths). However, a complete classification of such graphs does not seem to be known, even for trees.

For $k \geq 0$, define a *k-star* to be a graph with vertices $v, x_1, x_2, \ldots, x_k$ and edges (v, x_i) for all i (note that a 0-star is just an isolated vertex). We call the vertices x_i *leaves*, and the vertex v the *center*.

Definition 3.9. Build a graph as follows: Begin with a disjoint union of k-stars, where k is allowed to vary. Then add any edges between leaves of these stars (where edges between leaves of the same star are allowed). We call such a graph *star-constructible*.

The following is most likely known, though we were unable to find it in the literature.

Theorem 3.10. *A graph G has a perfect dominating set if and only if it is star-constructible.*

Proof. If G is star-constructible, the centers of the stars is easily seen to form a perfect dominating set of G.

For the reverse implication, we induct on the stronger hypothesis that every graph G with a perfect dominating set A is star-constructible, where the vertices in A are the centers of the stars. Indeed, suppose G has a perfect dominating set $A \subseteq V(G)$. Note that, by definition, $N(A) = V(G) \setminus A$. If no two elements in $N(A)$ are neighbors, then G is a disjoint union of the stars with vertex sets $\{a\} \cup N(a)$ for $a \in A$. Otherwise, there is some edge e connecting two elements in $N(A)$. Let G' denote G with this edge deleted. By induction on the number of edges, G' is a star-constructible graph, and A consists of the centers of these stars. Since e connects two leaves of these stars, it follows that G is star-constructible. $\square$

Proposition 3.11. *Let G be a disjoint union of k-stars, and let G' be a graph obtained from G by adding some edges between leaves of these stars. Then* $\mathrm{pd}(G) = \mathrm{pd}(G')$.

Thus, the projective dimension of a star-constructible graph depends only on the number of vertices and the number of stars used in building it.

Proof. Both G and G' have the same perfect dominating set A (which is the centers of the k-stars) and the same number of vertices, so Theorem 3.6 gives that $\mathrm{pd}(G) = |V(G)| - |A| = |V(G')| - |A| = \mathrm{pd}(G')$. $\square$

Here we note an application of Theorem 3.6 to a construction by Francisco and Hà.

Definition 3.12. Let G be a graph. For each vertex $v \in V(G)$, introduce a new vertex v' and a new edge (v, v'). The resultant graph, $w(G)$, is called the *whiskering* of G.

In [8], the author shows that $I_{w(G)}$ is sequentially Cohen–Macaulay for any graph G.

Proposition 3.13. *For any graph G, we have* $\mathrm{pd}(w(G)) = |V(G)|$.

Proof. Let W be the set of vertices added to G to produce $w(G)$. Then $|W| = |V(G)|$. It is easily seen that W is a perfect dominating set of G, so Theorem 3.6 gives us $\mathrm{pd}(w(G)) = |V(w(G))| - |W| = 2|V(G)| - |V(G)| = |V(G)|$. $\square$

4. Clutter background

Definition 4.1. Recall that a *clutter* (or *hypergraph*) $\mathcal{C}$ consists of a finite set of vertices $V(\mathcal{C})$ and a set of edges $E(\mathcal{C}) \subseteq 2^{V(\mathcal{C})}$, such that no edge properly contains another.

In the case when every edge of $\mathcal{C}$ has cardinality 2, we can identify $\mathcal{C}$ with a simple graph. Note that $\mathcal{C}$ may have *isolated* vertices, which are vertices appearing in no edge of $\mathcal{C}$. Note also that these are different from vertices $v \in V(\mathcal{C})$ for which $\{v\} \in E(\mathcal{C})$. We write $\overline{\mathcal{C}}$ to denote the clutter $\mathcal{C}$ with its isolated vertices removed.

Now fix a characteristic zero field k, and let $S = k[x_1, x_2, \dots, x_n]$. For a squarefree monomial m in S, we write $\mathrm{supp}(m)$ for the set $\{i \in [n] : x_i | m\}$. Similarly, if $e \subseteq [n]$, we set

$$x^e = \prod_{i \in e} x_i.$$

Definition 4.2. Let $I \subseteq S$ be a squarefree monomial ideal, and let G be its minimal generating set of monomials. We define a clutter $\mathcal{C}(I)$ by $V(\mathcal{C}(I)) = [n]$ and $E(\mathcal{C}(I)) = \{\mathrm{supp}(m) : m \in G\}$. Similarly, if $\mathcal{C}$ is a clutter with $V(\mathcal{C}) = \{x_1, x_2, \dots, x_n\}$, we define a squarefree monomial ideal $I(\mathcal{C}) \subseteq S$ by $I(\mathcal{C}) = (x^e : e \in E(\mathcal{C}))$.

Using the above definition, one can study properties of squarefree monomial ideals by studying the combinatorics of the corresponding clutters. To this end, we define the following two operations.

Definition 4.3. Let $\mathcal{C}$ be a clutter, and let $A \subseteq V(\mathcal{C})$. We define two related clutters, $\mathcal{C} + A$ and $\mathcal{C} : A$, as follows.

- The edges of $\mathcal{C} + A$ are the minimal sets of $E(\mathcal{C}) \cup \{A\}$, and the vertex set of $\mathcal{C} + A$ is $V(\mathcal{C})$.
- The edges of $\mathcal{C} : A$ are the minimal sets of $\{e \smallsetminus A : e \in E(\mathcal{C})\}$, and the vertex set of $\mathcal{C} : A$ is $V(\mathcal{C}) \smallsetminus A$.

Observation 4.4. Let $\mathcal{C}$ be a clutter. If $A \subseteq V(\mathcal{C})$, then $(I(\mathcal{C}), x^A) = I(\mathcal{C} + A)$ and $I(\mathcal{C}) : x^A = I(\mathcal{C} : A)$.

As in the graphical case, for $I \subseteq S$ is an ideal, the *projective dimension* of S/I is the shortest length of a projective (or free) resolution of S/I. If $\mathcal{C}$ is a clutter, we write $\mathrm{pd}(\mathcal{C})$ to mean the projective dimension of $S/I(\mathcal{C})$. The following lemma is well-known, but we state it here for completeness.

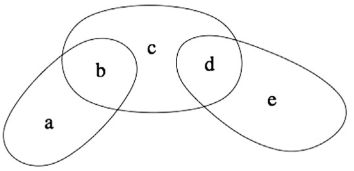
Lemma 4.5. *Let $\mathcal{C}$ be a clutter. Then for any $A \subseteq V(\mathcal{C})$, we have*

$$\mathrm{pd}(\mathcal{C}) \le \max\{\mathrm{pd}(\mathcal{C} + A), \mathrm{pd}(\mathcal{C} : A)\}.$$

Proof. Consider the following natural short exact sequence:

$$0 \to \frac{S}{I(\mathcal{C}) : x^A} \to \frac{S}{I(\mathcal{C})} \to \frac{S}{(I(\mathcal{C}), x^A)} \to 0.$$

This yields $\mathrm{pd}(S/I(\mathcal{C})) \le \max\{\mathrm{pd}(S/(I(\mathcal{C}) : x^A)), \mathrm{pd}(S/(I(\mathcal{C}), x^A)\}$, and the lemma follows from the above observation. $\square$



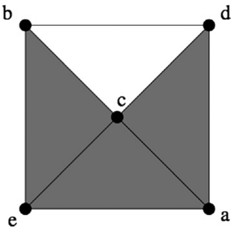


Fig. 2. A clutter $\mathcal{C}$ and its independence complex $\operatorname{ind}(\mathcal{C})$.

As in Section 2, we can define the *independence complex* $\operatorname{ind}(\mathcal{C})$ of a clutter $\mathcal{C}$ to be the simplicial complex whose faces are subsets of $V(\mathcal{C})$ containing no edge, and we say a set $A \subseteq V(\mathcal{C})$ is *independent* if it contains no edges. In Fig. 2, the clutter $\mathcal{C}$ has edges $\{a, b\}$, $\{b, c, d\}$, and $\{d, e\}$. As in Section 2, Hochster's Formula relates the projective dimension of $S/I(\mathcal{C})$ to $\operatorname{ind}(\mathcal{C})$.

Proposition 4.6. *The projective dimension of $S/I(\mathcal{C})$ is the smallest integer i such that for all $X \subseteq V(\operatorname{ind}(\mathcal{C}))$ we have*

$$\tilde{H}_k(\operatorname{ind}(\mathcal{C}[X])) = 0$$

for all $k < |X| - i - 1$ (where the induced clutter $\mathcal{C}[X]$ is defined in the obvious way).

And, as in Section 2, the previous proposition yields bounds on the homology of $\operatorname{ind}(\mathcal{C})$. Namely, plugging in $X = V(\mathcal{C})$ to the previous proposition gives us the following.

Corollary 4.7. *The homology of $\operatorname{ind}(\mathcal{C})$ satisfies*

$$\tilde{H}_k(\operatorname{ind}(\mathcal{C})) = 0$$

for $k < |V(\mathcal{C})| - \operatorname{pd}(\mathcal{C}) - 1$.

5. The parameter τ for clutters

Definition 5.1. Let $\mathcal{C}$ be a clutter. For a set $A \subseteq V(\mathcal{C})$, let $\gamma_0(A, \mathcal{C})$ denote the least cardinality of a set $X \subseteq V(\mathcal{C}) \setminus A$ such that every $a \in A$ is in an edge of $\mathcal{C}$ with some $x \in X$. We say the set X is a *dominating set* of A in $\mathcal{C}$. Define an invariant $\tau(\mathcal{C})$ by

$$\tau(\mathcal{C}) = \max\{\gamma_0(A, \mathcal{C}) : A \subseteq V(\mathcal{C}) \text{ is independent and has no isolated vertices}\}.$$

The parameter $\tau(\mathcal{C})$ generalizes the well-known graph domination parameter of the same name to clutters. Theorem 5.2 generalizes Theorem 4.4 from [4] to clutters.

Theorem 5.2. *For any clutter $\mathcal{C}$, we have* $\operatorname{pd}(\mathcal{C}) \leq |V(\mathcal{C})| - \tau(\mathcal{C})$.

Proof. Let $A \subseteq V(\mathcal{C})$ be an independent set with $\gamma_0(A, \mathcal{C}) = \tau(\mathcal{C})$, let $X \subseteq V(\mathcal{C}) \setminus A$ be the associated set of vertices, and fix $x \in X$. We examine the two cases of Lemma 4.5. In both cases, we induct on the number of vertices of $\mathcal{C}$. As we may assume $\mathcal{C}$ has no isolated vertices, the base case for this induction is when $\mathcal{C}$ consists of one vertex and one edge. In this case $I(\mathcal{C}) = (x)$ and $\tau(\mathcal{C}) = 0$, so the bound holds: $\mathrm{pd}(\mathcal{C}) \le |V(\mathcal{C})| - \tau(\mathcal{C}) = 1 - 0$.

Case 1. $\mathrm{pd}(\mathcal{C}) \le \mathrm{pd}(\mathcal{C} + x)$. Let $B \subseteq V(\mathcal{C})$ be the set of vertices only appearing in edges with x, and assume that $B \neq \emptyset$. Note that $I(\mathcal{C} + x) = (I(\mathcal{C} - x - B), x)$, and thus $\mathrm{pd}(\mathcal{C} + x) = \mathrm{pd}(\mathcal{C} - x - B) + 1$. Let Q be the set of isolated vertices in $\mathcal{C} - x - B$, and write A as a disjoint union $A = A' \sqcup A_Q \sqcup A_B$ where $A_Q = A \cap Q$, $A_B = A \cap B$, and $A' = A \setminus (A_Q \cup A_B)$. Because every edge of $\mathcal{C} - x - B$ is an edge of $\mathcal{C}$, the set A' is independent in $\mathcal{C} - x - B$, and by construction it contains no isolated vertices. Let Y be a dominating set of A' in $\mathcal{C} - x - B$ with $|Y| = \gamma_0(A', \mathcal{C} - x - B)$. For each vertex $a \in A_Q$, choose a vertex $z \in V(\mathcal{C}) \setminus A$ such that z and a are neighbors (this is possible since A is independent), and call the resulting set Z. Then $|Z| \le |A_Q|$, by construction.

Moreover, it is easy to see that $Y \cup Z \cup x$ is a dominating set of A in $\mathcal{C}$, as any vertex in A' has a neighbor in Y, any vertex in A_Q has a neighbor in Z, and any vertex in A_B is a neighbor of x. Thus, $|Y \cup Z \cup x| \ge \tau(\mathcal{C}) = |X|$. As isolated vertices do not affect the projective dimension of a clutter, we have $\mathrm{pd}(\mathcal{C} - x - B) = \mathrm{pd}(\mathcal{C} - x - B - Q)$. Isolated vertices do not affect the invariant τ either, and so $|Y| \le \tau(\mathcal{C} - x - B - Q)$. By induction on the number of vertices, $\mathrm{pd}(\mathcal{C} - x - B - Q) \le |V(\mathcal{C} - x - B - Q)| - \tau(\mathcal{C} - x - B - Q)$. Thus, we have

$$
\begin{aligned}
\mathrm{pd}(\mathcal{C}) \le \mathrm{pd}(\mathcal{C} + x) &= \mathrm{pd}(\mathcal{C} - x - B) + 1 = \mathrm{pd}(\mathcal{C} - x - B - Q) + 1 \\
&\le |V(\mathcal{C} - x - B - Q)| - \tau(\mathcal{C} - x - B - Q) + 1 \\
&= |V(\mathcal{C})| - 1 - |B| - |Q| - \tau(\mathcal{C} - x - B - Q) + 1 \\
&\le |V(\mathcal{C})| - |B| - |Q| - |Y| \le |V(\mathcal{C})| - 1 - |Z| - |Y| \le |V(\mathcal{C})| - \tau(\mathcal{C}),
\end{aligned}
$$

where in the final line we have used the fact that $|Q| \ge |A_Q| \ge |Z|$ and the assumption that $B \neq \emptyset$.

If instead $B = \emptyset$ then, using the notation above, $Y \cup Z$ is a dominating set of A in $\mathcal{C}$, and so the final line of the above becomes

$$
\mathrm{pd}(\mathcal{C}) \le |V(\mathcal{C})| - |Q| - |Y| \le |V(\mathcal{C})| - |Z| - |Y| \le |V(\mathcal{C})| - \tau(\mathcal{C}).
$$

Case 2. $\mathrm{pd}(\mathcal{C}) \le \mathrm{pd}(\mathcal{C} : x)$. As in the first case, let Q be the set of isolated vertices of $\mathcal{C} : x$ and set $A_Q = A \cap Q$. Let $e_1, e_2, \ldots, e_t$ be all edges of $\mathcal{C}$ contained in $A \cup x$, and set $A' = (A \setminus \left(\bigcup_{i=1}^t e_i\right)) \setminus Q$. By construction, A' is independent in $\mathcal{C} : x$ and has no isolated vertices. Let $Y \subseteq V(\mathcal{C} : x) \setminus A$ be a dominating set of A' with $|Y| = \gamma_0(A', \mathcal{C})$ (and so $|Y| \le \tau(\mathcal{C} : x)$). For every vertex $a \in A_Q$, choose a neighbor z of a in $\mathcal{C}$, and let Z be the collection of all such neighbors (so that $|Z| \le |A_Q| \le |Q|$). Now note that

$Y \cup Z \cup x$ is a dominating set of A in $\mathcal{C}$, and so $|Y \cup Z \cup x| \geq \tau(\mathcal{C})$. As isolated vertices do not affect the projective dimension of a clutter, we have $\mathrm{pd}(\mathcal{C} : x) = \mathrm{pd}((\mathcal{C} : x) - Q)$. By induction on the number of vertices, $\mathrm{pd}((\mathcal{C} : x) - Q) \leq |V((\mathcal{C} : x) - Q)| - \tau((\mathcal{C} : x) - Q)$. Thus,

$$\begin{aligned} \mathrm{pd}(\mathcal{C}) &\leq \mathrm{pd}(\mathcal{C} : x) = \mathrm{pd}((\mathcal{C} : x) - Q) \leq |V((\mathcal{C} : x) - Q)| - \tau((\mathcal{C} : x) - Q) \\ &= |V(\mathcal{C})| - 1 - |Q| - \tau(\mathcal{C} : x) \leq |V(\mathcal{C})| - 1 - |Z| - |Y| \\ &\leq |V(\mathcal{C})| - |Y \cup Z \cup x| \leq |V(\mathcal{C})| - \tau(\mathcal{C}). \qquad \square \end{aligned}$$

Observation 5.3. Combining [5, Remark 3.4] and Theorem 5.2, we can bound the projective dimension of a clutter from above and below: For any clutter $\mathcal{C}$, we have

$$|V(\mathcal{C})| - i(\mathcal{C}) \leq \mathrm{pd}(\mathcal{C}) \leq |V(\mathcal{C})| - \tau(\mathcal{C}).$$

By Corollary 4.7, Theorem 5.2 gives us the following.

Corollary 5.4. *Let Δ be a simplicial complex and $\mathcal{C}_\Delta$ its clutter of minimal non-faces. Then $\tilde{H}_i(\Delta) = 0$ for $i < \tau(\mathcal{C}_\Delta) - 1$.*

This allows us to generalize the homological version of the following theorem of Barmak.

Theorem 5.5. *(See [1].) Let G be a graph with vertex set V, and let $A \subseteq V$ be a set of vertices such that the distance between any two members of A is at least 3. If Δ is the independence complex of G, then Δ is $(|A| - 2)$-connected.*

We use the standard definition of clutter distance, namely if x and y are two vertices of a clutter $\mathcal{C}$, let $E_0, E_1, E_2, \dots, E_k$ be a sequence of edges of minimal length such that $x \in E_0$, $y \in E_k$, and $E_i \cap E_{i+1} \neq \emptyset$ for all i. We set $\mathrm{dist}(x, y) = k$.

Definition 5.6. Let A be a subset of the vertices of some clutter $\mathcal{C}$. We say A is a *distance-3 set* if $\mathrm{dist}(a_1, a_2) \geq 3$ for any distinct vertices $a_1, a_2 \in A$.

Corollary 5.7. *Let $\mathcal{C}$ be a clutter, and let $A \subseteq V(\mathcal{C})$ be a distance-3 set. Then the independence complex Δ of $\mathcal{C}$ satisfies $\tilde{H}_i(\Delta) = 0$ for $i \leq |A| - 2$.*

Proof. The set A is clearly independent. Now let $X \subseteq V(\mathcal{C}) \setminus A$ be a dominating set of A in $\mathcal{C}$ of minimal possible cardinality. Note that any $x \in X$ must have exactly one neighbor in A. Indeed, if x had no neighbors in A, then X would not be a minimal cardinality dominating set. If $a_1, a_2 \in A$ were both neighbors of x, we would have $\mathrm{dist}(x, a_1) = \mathrm{dist}(x, a_2) = 1$, meaning $\mathrm{dist}(a_1, a_2) < 3$. Thus $\tau(\mathcal{C}) \geq |X| = |A|$, and the result follows from Corollary 5.4. $\square$

In using the above corollary, one must search for a large distance-3 set A in $V(\mathcal{C})$. However, this is equivalent to the problem of finding a large distance-3 set in an associated graph $K(\mathcal{C})$.

Definition 5.8. For a clutter $\mathcal{C}$, define a graph $K(\mathcal{C})$ on the same vertex set as follows. If $x, y \in V(K(\mathcal{C})) = V(\mathcal{C})$ are distinct, then (x, y) is an edge of $K(\mathcal{C})$ if and only if there exists an edge E of $\mathcal{C}$ with $x, y \in E$.

One can think of $K(\mathcal{C})$ as the graph obtained by replacing each edge of $\mathcal{C}$ with the complete graph on that edge's vertices.

The proof of the following is immediate.

Proposition 5.9. *A set $A \subseteq V(\mathcal{C}) = V(K(\mathcal{C}))$ is a distance-3 set in $\mathcal{C}$ if and only if it is a distance-3 set in $K(\mathcal{C})$.*

Combining Corollary 5.7 and Proposition 5.9, we have the following.

Corollary 5.10. *Let Δ be a simplicial complex, let $\mathcal{C}_\Delta$ be its clutter of minimal non-faces, and let $K(\mathcal{C}_\Delta)$ be the associated graph. If A is a distance-3 set in $K(\mathcal{C}_\Delta)$, then $\tilde{H}_i(\Delta) = 0$ for $i \leq |A| - 2$.*

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