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Finite atomic lattices and resolutions of monomial ideals

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ABSTRACT

In this paper we primarily study monomial ideals and their minimal free resolutions by studying their associated lcm-lattices. In particular, we formally define the notion of coordinatizing a finite atomic lattice P to produce a monomial ideal whose lcm-lattice is P, and we give a characterization of all such coordinatizations. We prove that all relations in the lattice $\mathcal{L}(n)$ of all finite atomic lattices with n ordered atoms can be realized as deformations of exponents of monomial ideals. We also give structural results for $\mathcal{L}(n)$. Moreover, we prove that the cellular structure of a minimal free resolution of a monomial ideal M can be extended to minimal resolutions of certain monomial ideals whose lcm-lattices are greater than that of M in $\mathcal{L}(n)$.

1. Introduction

Let M be a monomial ideal in a polynomial ring R. We are interested in studying the minimal free resolution of R/M, and specifically understanding the maps in this resolution. Our approach is heavily dependent on using the combinatorial structure of the lattice of least common multiples, or lcm-lattice, associated to M, as well as the set of all such lattices for monomial ideals with a fixed number of generators.

The lcm-lattice was introduced by Gasharov, Peeva, and Welker for the purpose of studying free resolutions of monomial ideals [7]. Specifically the *lcm-lattice* of M, or LCM(M), is the lattice of least common multiples of the minimal generators of M partially ordered by divisibility. They show that two monomial ideals with isomorphic lcm-lattices have isomorphic minimal resolutions. This motivated Phan to ask, and answer in the affirmative, the question of whether or not every finite atomic lattice can be realized as the lcm-lattice of some monomial ideal [12]. Phan's work leads to the point of view that finite atomic lattices are "abstract monomial ideals".

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This paper concerns itself with two main types of results: to give a characterization of how abstract monomial ideals are coordinatized; and to use the set of all finite atomic lattices with a fixed number of atoms to understand families of resolutions of monomial ideals. Theorem 3.2 gives a characterization of how to *coordinatize* a finite atomic lattice by associating a monomial ideal M_P to a finite atomic lattice P such that the lcm-lattice of M_P is isomorphic to P. Moreover, Proposition 3.6 shows that all monomial ideals can be realized as coordinatizations of their lcm-lattices.

Theorem 4.1 is a restatement of Phan's result which shows that the set $\mathcal{L}(n)$ of all finite atomic lattices with n ordered atoms is itself a finite atomic lattice [12], and this idea is central to the rest of our work. Our interest in studying $\mathcal{L}(n)$ is due to the fact that in this setting, total Betti numbers are weakly increasing as one travels up chains in $\mathcal{L}(n)$ (see Theorem 3.3 in [7]). This is related to lower semicontinuity of Betti numbers under deforming the exponents of the monomial ideal. A *deformation of the exponents* of a monomial ideal is a process by which one replaces a given monomial ideal M with a "deformed" monomial ideal M_ϵ, where one or more ties between exponents on minimal generators is broken.

Bayer, Peeva and Sturmfels show that the minimal resolution of M_ϵ provides a (not necessarily minimal) resolution of M (see Theorem 4.3 in [1]). The main result of Section 5 is Theorem 5.1, which states that for any relation $P > Q$ in $\mathcal{L}(n)$ there exists a coordinatization of Q producing a monomial ideal M_Q and a deformation of exponents of M_Q such that the lcm-lattice of the deformed ideal is P.

Ultimately we are interested in understanding questions such as: how can we control deformations of the exponents so that Betti numbers do not increase; and what can we say about the cellular structure of minimal resolutions depending on the location in $\mathcal{L}(n)$? In the remainder of this paper we focus on answering the latter problem.

Both Theorem 5.3 in [11] and Proposition 5.15 in [12] describe simplicial complexes which support a minimal free resolution of a monomial ideal by virtue of being their Scarf complex. Theorem 6.5 generalizes this result by indicating a class of regular CW-complexes which supports the minimal free resolution of an ideal. Additionally it is known that if $P > Q$ in $\mathcal{L}(n)$ then any minimal resolution of P is a resolution of Q (see Theorem 3.3 in [7]). If this resolution of Q is minimal then it inherits any cellular structure which might have existed for the resolution of P. Propositions 6.4 and 6.6 show that to a certain extent the converse is true. Namely that under certain hypotheses any cellular structure for a minimal resolution of Q can be lifted to a minimal resolution of P.

Using this approach, of studying minimal resolutions of monomial ideals by examining the resolutions of "nearby" ideals in $\mathcal{L}(n)$, may tempt one to make a comparison with the upper-semicontinuity of Betti numbers along deformations of projective varieties in the Hilbert scheme. However, a more apropos analogy may be with the MacPhersonian, which is a combinatorial version of the Grassmannian [3], where vector spaces play the role of monomial ideals and oriented matroids replace finite atomic lattices. To develop a theory of combinatorial moduli spaces modeled on both of these examples is a subject for future study.

Further results in this paper are Theorem 4.3, which states that $\mathcal{L}(n)$ is a graded lattice of rank n, and other structural results in Section 4 describing covering relations and meet-irreducibles in this lattice. Additionally, Theorem 6.2 gives a lattice theoretic description of strongly generic monomial ideals.

2. Preliminaries

A *lattice* is a set $(P, <)$ with an order relation $<$ which is transitive and antisymmetric satisfying the following properties:

1. P has a maximum element denoted by $\hat{1}$.
2. P has a minimum element denoted by $\hat{0}$.
3. Every pair of elements a and b in P has a join $a \vee b$ which is the least upper bound of the two elements.
4. Every pair of elements a and b in P has a meet $a \wedge b$ which is the greatest lower bound of the two elements.

If P only satisfies conditions 2 and 4 then it is a *meet-semilattice*. We will use the following (Proposition 3.3.1 in [13]) several times.

Proposition 2.1. *Any meet-semilattice with a unique maximal element is a lattice.*

We define an *atom* of a lattice P to be an element $x \in P$ such that x covers $\hat{0}$ (i.e. $x > \hat{0}$ and there is no element a such that $x > a > \hat{0}$). We will denote the set of atoms as $\text{atoms}(P)$.

Definition 2.2. If P is a lattice and every element in $P - \{\hat{0}\}$ is the join of atoms, then P is an *atomic lattice*. Furthermore, if P is finite, then it is a *finite atomic lattice*.

If P is a lattice, then we define elements $x \in P$ to be *meet-irreducible* if $x \neq a \wedge b$ for any $a > x$, $b > x$. We denote the set of meet-irreducible elements in P by $\text{mi}(P)$. Given an element $x \in P$, the *order ideal* of x is defined to be the set $\lfloor x \rfloor = \{a \in P \mid a \leqslant x\}$. Similarly, we define the *filter* of x to be $\lceil x \rceil = \{a \in P \mid x \leqslant a\}$. The following lemma, which in some sense says that each element in P is uniquely determined by the meet-irreducibles in its filter, will be useful to us later.

Lemma 2.3. *Let P be a finite atomic lattice. Every element $p \in P$ is the meet of all the meet-irreducible elements l such that $l \geqslant p$.*

Proof. Clearly

$$p = \bigwedge_{b \in \lceil p \rceil} b$$

since $p \leqslant b \in \lceil p \rceil$. Note that for some non-meet-irreducible $a \in \lceil p \rceil$ we have that $a = c \wedge d$ for $c, d \in \lceil p \rceil$ then

$$\bigwedge_{b \in \lceil p \rceil,\, b \neq a} b = \bigwedge_{b \in \lceil p \rceil} b$$

since the element a is redundant. Thus we can remove all of the non-meet-irreducibles from $\lceil p \rceil$ and the meet of that set of elements is still p. Thus every element $p \in P$ is the meet of the meet-irreducible elements greater than or equal to p. □

For the purposes of this paper it will often be convenient to consider finite atomic lattices as sets of sets in the following way. Let $\mathcal{S}$ be a set of subsets of $\{1, \ldots, n\}$ with no duplicates, closed under intersections, and containing the entire set, the empty set, and the sets $\{i\}$ for all $1 \leqslant i \leqslant n$. Then it is easy to see $\mathcal{S}$ is a finite atomic lattice by ordering the sets in $\mathcal{S}$ by inclusion. This set obviously has a minimal element, a maximal element, and n atoms, so by Proposition 2.1 we need to show that it is a meet-semilattice. Here the meet of two elements would be defined to be their intersection and since $\mathcal{S}$ is closed under intersections this is a meet-semilattice. Conversely, it is clear that any finite atomic lattice P can be expressed in this way, simply by letting

$$\mathcal{S}_P = \left\{\sigma \;\middle|\; \sigma = \text{supp}(p),\ p \in P\right\},$$

where $\text{supp}(p) = \{a_i \mid a_i \leqslant p,\ a_i \in \text{atoms}(P)\}$.

Lastly, there are two different simplicial complexes that one can associate to a poset P. The *order complex* $\Delta(P)$ is the simplicial complex where the vertices are the elements of P and the facets correspond to maximal chains of P (and k-faces are the k-chains of P). The *crosscut complex* $\Gamma(P, C)$ is defined in the following way. A subset $C \subset P$ is called a *crosscut* if the following three conditions

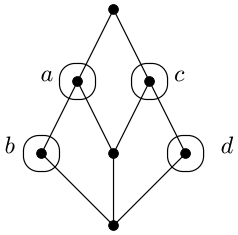


Fig. 1. A finite atomic lattice P with meet-irreducibles identified and labeled with variables.

hold: (1) C is an antichain, (2) for every finite chain σ in P there exists some element in C comparable to each element in σ, and (3) if $A \subset C$ is bounded then the join $\bigvee A$ or the meet $\bigwedge A$ exists in P. Then $\Gamma(P, C)$ is the simplicial complex with vertices in C and faces corresponding to the bounded subsets of C. For the purposes of this paper, we will only use a specific form of the crosscut complex. Note that if P is a finite atomic lattice then the atoms of P form a crosscut. Moreover the atoms of P are still a crosscut of the poset $P - \{\hat{0}\}$ and as these are the posets of interest in this paper we will use the notation $\Gamma(P)$ to mean $\Gamma(P - \{\hat{0}\}, \text{atoms}(P))$. Moreover, we will often make use of the following fact that $\Delta(P)$ is homotopy equivalent to $\Gamma(P, C)$ [4].

3. Coordinatizations

One of the main results (Theorem 5.1) of [12] is to show that every finite atomic lattice is in fact the lcm-lattice of a monomial ideal. This result gives a very specific construction which produces a monomial ideal whose lcm-lattice has a given lattice structure. In this section we generalize this construction and demonstrate that all monomial ideals can be obtained via this generalized construction.

Example 3.1. Fig. 1 shows an example of Phan's original construction (see [12]). Here the meet-irreducibles in the open interval

$$(\hat{0}, \hat{1}) = \{p \in P \mid \hat{0} < p < \hat{1}\}$$

are circled in the figure and are assigned a variable. Phan's construction then defines a monomial ideal with 3 generators by assigning to each atom a_i the product of all the variables assigned to elements which are not above a_i. In this case we get the ideal (cd, bd, ab) in $k[a, b, c, d]$, and in fact the lcm-lattice of this monomial ideal has the same Hasse diagram as the lattice P in the figure.

We aim to generalize this construction by allowing non-meet-irreducibles to be labeled, and to allow lattice elements to be labeled with monomials rather than just variables. Define a *labeling* of a finite atomic lattice P to be any assignment of non-trivial monomials $\mathcal{M} = \{m_{p_1}, \ldots, m_{p_t}\}$ to some set of elements $p_i \in P$. It will be convenient to think of unlabeled elements as having the label 1. Define a monomial ideal $M_{P,\mathcal{M}}$ to be the ideal generated by monomials

$$x(a) = \prod_{p \in \lceil a \rceil^c} m_p$$

for each $a \in \text{atoms}(P)$ where $\lceil a \rceil^c$ means take the complement of $\lceil a \rceil$ in P. We say that the labeling $\mathcal{M}$ is a *coordinatization* if $M_{P,\mathcal{M}}$ has lcm-lattice isomorphic to P.

The following theorem and proof are a generalization of Theorem 5.1 in [12].

Theorem 3.2. *Any labeling $\mathcal{M}$ of elements in a finite atomic lattice P by monomials satisfying the following two conditions will yield a coordinatization of the lattice P.*

(C1) *If* $p \in \mathrm{mi}(P)$ *then* $m_p \neq 1$ (*i.e. all meet-irreducibles are labeled*).
(C2) *If* $\gcd(m_p, m_q) \neq 1$ *for some* $p, q \in P$ *then* p *and* q *must be comparable* (*i.e. each variable only appears in monomials along one chain in* P).

Before we give the proof, we will prove the following lemma which will simplify parts of the argument.

Lemma 3.3. *If* $p \in \lceil q \rceil^c$ *for some* $p, q \in P$ *where* P *is a finite atomic lattice, then* $\lfloor p \rfloor \subset \lceil q \rceil^c$.

Proof. If $b \in \lfloor p \rfloor$ then $b \leqslant p$. Moreover if $p \in \lceil q \rceil^c$ then p is not greater than q (note it need not be less than q). So suppose b were not in $\lceil q \rceil^c$ then $b \geqslant q$ which would imply that $p \geqslant q$ which is a contradiction. □

Now for the proof of Theorem 3.2.

Proof of Theorem 3.2. Let P' be the lcm-lattice of $M_{P,\mathcal{M}}$. We just need to show that P' is isomorphic to P. For $b \in P$ define $f : P \to P'$ to be the map such that

$$f(b) = \prod_{p \in \lceil b \rceil^c} m_p.$$

We will show that this map is well-defined and that it is an isomorphism of lattices. In particular we must show that it is a lattice homomorphism (preserves joins and meets) as well as a bijection on sets.

First we need to show that f is a bijection on atoms, in particular that the set of minimal generators for $M_{P,\mathcal{M}}$ has the same cardinality as atoms(P). In other words we must show $f(a_i) \nmid f(a_j)$ for any pair i, j. By Lemma 2.3, we know that the sets $\mathrm{mi}(P) \cap \lceil a_i \rceil$ are distinct for all i. Since the a_i are atoms and thus non-comparable, none of these sets are a subset of any of the others. Thus these properties (sets are distinct and not subsets of each other) also hold for the sets $\mathrm{mi}(P) \cap \lceil a_i \rceil^c$. Now we are guaranteed that there exists an element q such that $q \in \mathrm{mi}(P) \cap \lceil a_i \rceil^c$ and $q \notin \mathrm{mi}(P) \cap \lceil a_j \rceil^c$, and similarly an element $q' \in \mathrm{mi}(P) \cap \lceil a_j \rceil^c$ and not contained in $\mathrm{mi}(P) \cap \lceil a_i \rceil^c$. If $q < q'$ then $q \in \lfloor q' \rfloor$ and Lemma 3.3 leads to a contradiction with the fact that $q \notin \lceil a_j \rceil^c$. Similarly q' cannot be less than q thus the two elements are not comparable in P.

Now, since q is meet-irreducible, condition (C1) implies that it must be non-trivially labeled. Let x_q be a variable dividing the monomial m_q which labels q. Suppose $p \in \lceil a_j \rceil^c$ is such that m_p is divisible by x_q. Since $q \notin \lceil a_j \rceil^c$ we have $p \neq q$ and condition (C2) implies that either $p < q$ or $q < p$. If $q < p$ then $q \in \lfloor p \rfloor$ so by Lemma 3.3 we get that $q \in \lceil a_j \rceil^c$ which is a contradiction. So we must have that $p < q$ for any $p \in \lceil a_j \rceil^c$ where x_q divides m_p. In other words, if c_q is the chain in P consisting of elements whose monomial labels are divisible by x_q then we have that $c_q \cap \lceil a_j \rceil^c \subset \lceil a_i \rceil^c$. Moreover we will emphasize that this is a strict inclusion since $q \notin \lceil a_j \rceil^c$. Thus if x_q^t is the highest power of x_q dividing $f(a_i)$ then x_q^t does not divide $f(a_j)$ and as such $f(a_i)$ does not divide $f(a_j)$. Similarly, $f(a_j)$ does not divide $f(a_i)$. Thus, P and P' have the same number of atoms and the map f is a bijection on these atoms.

Now, note that

$$\lceil b \rceil^c = \bigcup_{a_i \in \mathrm{supp}(b)} \lceil a_i \rceil^c. \tag{3.1}$$

In order to show that f is well-defined (i.e. that $f(b)$ as defined above is in fact in P') we will show that

$$f(b) = \mathrm{lcm}\big\{ f(a_i) \bigm| a_i \in \mathrm{supp}(b) \big\}. \tag{3.2}$$

This will also imply that f is join-preserving and a surjection. By Eqs. (3.1) and (3.2), we know that

$$f(b) = \prod m_p$$

where $p \in \lceil a_i \rceil^c$ for at least one $a_i \in \operatorname{supp}(b)$. It is clear that $f(a_j)$ divides $f(b)$ for all $a_j \in \operatorname{supp}(b)$. We then need to show that if $x_i^{m_i}$ is the highest power of x_i dividing $f(b)$ then there is some $a_j \in \operatorname{supp}(b)$ such that $x_i^{m_i}$ divides $f(a_j)$. This follows from the fact that x_i only divides monomials that label elements in one chain of P. Indeed let q be the largest element of this chain which is in $\lceil b \rceil^c$. Then there exists $a_j \in \operatorname{supp}(b)$ such that $q \in \lceil a_j \rceil^c$. Since any $p \leqslant q$ is in both $\lceil b \rceil^c$ and $\lceil a_j \rceil^c$, the power of x_i in $f(b)$ and $f(a_j)$ will be equal.

That f is meet-preserving follows from the above and the fact that $\operatorname{supp}(p \wedge p') = \operatorname{supp}(p) \cap \operatorname{supp}(p')$, since

$$\begin{aligned} f\big(p \wedge p'\big) &= \operatorname{lcm}\big\{ f(a_i) \mid a_i \in \operatorname{supp}\big(p \wedge p'\big) \big\} \\ &= \operatorname{lcm}\big\{ f(a_i) \mid a_i \in \operatorname{supp}(p) \cap \operatorname{supp}\big(p'\big) \big\} = f(p) \wedge f\big(p'\big). \end{aligned}$$

Finally, we need to show that this map is injective. Suppose we have two elements $a, b \in P$ such that $f(a) = f(b)$. First we will rule out the possibility that $a < b$. If $a < b$ then $\lceil a \rceil^c \subset \lceil b \rceil^c$, and by Lemma 2.3 $\operatorname{mi}(P) \cap \lceil a \rceil^c \subset \operatorname{mi}(P) \cap \lceil b \rceil^c$. Since each meet-irreducible needs to be non-trivially labeled we get that $f(a)$ strictly divides $f(b)$ which is contrary to our assumption that $f(a) = f(b)$. Similarly we rule out that $b < a$. Moreover, if a and b are not comparable in P we get that $\operatorname{mi}(P) \cap \lceil a \rceil^c$ and $\operatorname{mi}(P) \cap \lceil b \rceil^c$ are not subsets of each other. By the same argument as the one used earlier to show that $f(a_i)$ does not divide $f(a_j)$ when a_i and a_j are atoms, we see that $f(a)$ and $f(b)$ do not divide each other. Again this contradicts the assumption that $f(a) = f(b)$. Thus the only possibility left is that $a = b$, and f is injective. □

To complete our discussion of coordinatizations, we will also show that every monomial ideal is in fact a coordinatization of its lcm-lattice. Let M be a monomial ideal with n generators and let P_M be its lcm-lattice. For notational purposes, denote P_M as the set consisting of elements denoted $\overline{p}$ which represent the monomials occurring in P_M. Now define the abstract finite atomic lattice P where the elements in P are formal symbols p satisfying the relations $p < p'$ if and only if $\overline{p} < \overline{p}'$ in P_M. In other words, P is the abstract finite atomic lattice isomorphic to P_M obtained by simply forgetting the data of the monomials in P_M. Define a labeling of P in the following way, let $\mathcal{D}$ be the set consisting of monomials m_p for each $p \in P$ defined as:

$$m_p = \frac{\gcd\{\bar{t} \mid t > p\}}{\overline{p}}, \tag{3.3}$$

where by convention $\gcd\{\bar{t} \mid t > p\}$ for $p = \hat{1}$ is defined to be $\bar{\hat{1}}$. Note that m_p is a monomial since clearly $\overline{p}$ divides $\bar{t}$ for all $t > p$.

Lemma 3.4. *If p is a meet-irreducible then m_p defined in (3.3) satisfies $m_p \neq 1$. In other words, all meet-irreducibles are labeled non-trivially.*

Proof. If p is a meet-irreducible then we know that all $t > p$ are greater than or equal to some $p \vee a_i$ where $p \vee a_i$ is the unique element covering p. In particular we know that a_i is not less than p. So, with this knowledge we can easily see that $\overline{p}, \overline{p \vee a_i}$ and $\overline{a_i}$ all divide $\gcd\{\bar{t} \mid t > p\}$. Thus, since $\overline{a_i}$ does not divide $\overline{p}$ we see that $\overline{p} \neq \gcd\{\bar{t} \mid t > p\}$ and $m_p \neq 1$ as needed. □

The next lemma implies that variables in this labeling must lie along chains.

Lemma 3.5. *If* $\gcd\{m_p, m_q\} \neq 1$ *then* p *and* q *must be comparable* (*where* m_p, m_q *are defined in* (3.3)).

Proof. To show this we will show the contrapositive. Suppose p and q are not comparable. Then using $\bar{p} = \prod x_i^{a_i}$ and $\bar{q} = \prod x_i^{b_i}$ we can define the following sets:

$$A = \{i \mid a_i > b_i\}, \qquad B = \{i \mid b_i > a_i\}, \qquad C = \{i \mid a_i = b_i\}.$$

Clearly since $\bar{p}$ and $\bar{q}$ do not divide each other we can see that sets A and B are non-empty, and by definition their intersections are empty.

Now consider the element $\bar{r} = \operatorname{lcm}(\bar{p}, \bar{q})$ and its corresponding element $r \in P$. By definition the exponents on x_i in $\bar{r}$ are a_i if $i \in A$ or $i \in C$, and b_i if $i \in B$. Moreover since r is greater than both p and q, we know that both $\gcd\{\bar{t} \mid t > p\}$ and $\gcd\{\bar{t} \mid t > q\}$ divide $\bar{r}$.

Now we want to show that the $\gcd\{m_p, m_q\} = 1$. For notational purposes let $\prod x_i^{n_i} = \gcd\{\bar{t} \mid t > p\}$. Then we know that if $i \in A$ or C, $n_i = a_i$, and if $i \in B$ then $b_i \geqslant n_i \geqslant a_i$. So when we divide by $\bar{p}$ we are left with a monomial consisting only of variables indexed by B. A similar argument shows that m_q consists only of variables indexed by A. Since $A \cap B = \emptyset$ we have that $\gcd\{m_p, m_q\} = 1$. □

Using this labeling we can prove our desired result that every monomial ideal can be realized as a coordinatization of its lcm-lattice.

Proposition 3.6. *Given* M *a monomial ideal with lcm-lattice* P_M. *If* P *is the abstract finite atomic lattice where* P *is isomorphic to* P_M *as lattices then the labeling* $\mathcal{D}$ *of* P *as defined by* (3.3) *is a coordinatization and the resulting monomial ideal* $M_{P,\mathcal{D}} = M$.

Proof. Lemmas 3.4 and 3.5 both show that $\mathcal{D}$ is a coordinatization of P. It remains to show that the resulting monomial ideal is equal to M.

To show this we will demonstrate that for each atom

$$x(a_i) = \prod_{p \in \lceil a_i \rceil^c} m_p$$

divides $\overline{a_i}$ and vice versa.

To see that $x(a_i) \mid \overline{a_i}$, consider the variable x. We claim that if $p \in \lceil a_i \rceil^c$ and $x^n \mid m_p$ then $x^n \mid \overline{a_i}$. If $x^n \mid m_p$ then $x^n \bar{p} \mid \bar{t}$ for all $t > p$. Moreover, $a_i \vee p > p$ since p is not greater than a_i and $\overline{a_i \vee p} = \operatorname{lcm}(\overline{a_i}, \bar{p})$ by construction, so $x^n \bar{p} \mid \operatorname{lcm}(\overline{a_i}, \bar{p})$. This implies $x^n \mid \overline{a_i}$.

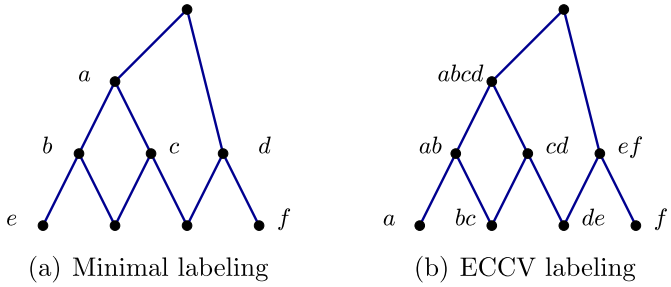
Additionally, if we have a chain in P, $p_1 < \cdots < p_k$ such that $p_k \in \lceil a_i \rceil^c$ and $x^{n_i} \mid m_{p_i}$ then $x^{\sum n_i} \mid \overline{a_i}$. This follows from the fact that $x^{n_i} \mid m_{p_i}$ implies that $x^{n_i} \overline{p_i} \mid \overline{p_{i+1}}$ and further implies that $x^{\sum n_i} \mid \overline{p_k}$. By the previous paragraph we get the desired result, and successfully prove that $x(a_i) \mid \overline{a_i}$.

To show that $\overline{a_i} \mid x(a_i)$, we first show that if some variable x divides $\overline{a_i}$ there exists some $p \in \lceil a_i \rceil^c$ such that $x \mid m_p$. If we let

$$p = \bigvee_{x \nmid \bar{r}} r,$$

then clearly $x \nmid \bar{p}$ so $p \in \lceil a_i \rceil^c$ but x divides all elements greater than p by construction so $x \mid m_p$. Now to understand what happens when $x^n \mid \overline{a_i}$, we apply this idea successively by defining elements

$$p_i = \bigvee_{x^i \nmid \bar{r}} r.$$



(a) Minimal labeling (b) ECCV labeling

Fig. 2. Two labelings of one lattice.

By inclusion of sets, we see that $p_i \leqslant p_{i+1}$ so we actually get a chain of elements $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ in P. If some elements in the chain are actually equal, for example $p_t = \cdots = p_{t+l}$, then by the above reasoning x^t does not divide $\overline{p_t}$ and $x^{t+l} \mid \bar{r}$ for all $r > p_t$ so we may conclude that m_{p_t} is divisible by x^{l+1}. Since all of the elements in this chain are not greater than a_i by construction, this shows that if $x^n \mid \overline{a_i}$ then $x^n \mid x(a_i)$ as needed.

Therefore for each atom we have that $x(a_i) \mid \overline{a_i}$ and vice versa, thus $M_{P,\mathcal{D}} = M$. □

It is important to note that there exist coordinatizations of finite atomic lattices which are not labelings satisfying the conditions of Theorem 3.2. The previous theorem however implies that any such coordinatization is "equivalent" to a labeling satisfying Theorem 3.2, where by equivalent we mean that both coordinatizations yield the same monomial ideal. It would be interesting to give an explicit formulation for when two coordinatizations are equivalent in this sense or to prove a version of Theorem 3.2 which has weaker hypotheses.

Example 3.7. The labeling of the finite atomic lattice (with minimal element omitted) in Fig. 2(a) obviously satisfies the conditions of 3.2 since only meet-irreducibles are labeled and each variable is used only once. In fact this is the coordinatization which produces a "minimal squarefree monomial ideal" found in [12]. The monomial ideal given by this coordinatization in Fig. 2 is $M = (cdf, def, bef, abce)$.

Example 3.8. The labeling in Fig. 2(b) will be of use to us later. It is an example of the ECCV labeling (Every Chain Covered by a Variable) which is defined as follows. Let $\{c_1, \ldots, c_t\}$ be the set of all maximal chains in $P - \{\hat{0}\}$. Then for variables in the ring $R = k[x_1, \ldots, x_t]$ define the following labeling,

$$\mathcal{M} = \left\{ m_p = \prod_{i : p \in c_i} x_i \;\middle|\; p \in P \right\}.$$

Every meet-irreducible is covered since every element of P is covered and each variable appears only along one chain by definition, so the conditions of 3.2 are satisfied. The example in Fig. 2(b) shows such a coordinatization where the monomial ideal is

$$M = \left(bc^2d^2e^2f^2, ade^2f^2, a^2b^2cf, a^3b^3c^3d^3e\right).$$

The construction of "nearly Scarf" monomial ideals found in [11] and [14] can easily be identified as a specific coordinatization where every element in $P - \{\hat{0}, \hat{1}\}$ is labeled. Additionally, the construction of monomial ideals whose minimal resolutions are supported on trees in [6] can be seen as an instance of coordinatizing a specific lattice defined in terms of the tree (see Section 6.2 in [8]).

4. The set of all atomic lattices with n atoms

It will be useful for us to consider the questions outlined in the introduction about deformations of exponents of monomial ideals in the following setting. Define the set $\mathcal{L}(n)$ to be the set of all finite atomic lattices with n ordered atoms. This set has a partial order where $Q \leqslant P$ if and only if there exists a join-preserving map which is a bijection on atoms from P to Q (note that such a map will also be surjective). In [12], Phan shows the following result.

Theorem 4.1. *With the partial order $\leqslant$, $\mathcal{L}(n)$ is a finite atomic lattice with $2^n - n - 2$ atoms.*

We include a proof here since it currently does not exist in the literature.

Proof. It is clear that $\mathcal{L}(n)$ has both a unique minimal and maximal elements, the lattice where the atoms are also the coatoms, and the Boolean lattice respectively. Thus by Proposition 2.1 in order to show that $\mathcal{L}(n)$ is a lattice it is enough to show that $\mathcal{L}(n)$ is a meet-semilattice.

Here it will be convenient to think of the elements in $\mathcal{L}(n)$ as sets $\mathcal{S}$ as described in Section 2. For any two elements $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathcal{L}(n)$ define their meet to be the intersection of the two sets, denoted $\mathcal{R}$. Clearly $\mathcal{R}$ is the largest set contained in both $\mathcal{S}_1$ and $\mathcal{S}_2$ thus we only need to show that $\mathcal{R}$ is an element of $\mathcal{L}(n)$. Clearly $\mathcal{R}$ contains the sets $\{1, \ldots, n\}$, $\emptyset$, $\{i\}$ for all i since all of these are contained in $\mathcal{S}_1$ and $\mathcal{S}_2$. Lastly, it is closed under intersections, since both $\mathcal{S}_1$ and $\mathcal{S}_2$ are closed under intersections.

To see that $\mathcal{L}(n)$ is a finite atomic lattice, observe that the elements covering the minimal element are of the form

$$\mathcal{S} = \big\{\emptyset, \{1\}, \{2\}, \ldots, \{n\}, \{1, \ldots, n\}, \sigma\big\}$$

where σ is any subset of $\{1, \ldots, n\}$ other than those already in $\mathcal{S}$. As there are $2^n - n - 2$ such subsets, there are $2^n - n - 2$ atoms in $\mathcal{L}(n)$. Moreover, any element $L \in \mathcal{L}(n) - \hat{0}$ is the join of atoms. $\square$

For the rest of this section we will prove some basic results about $\mathcal{L}(n)$. Counting arguments show that $|\mathcal{L}(3)| = 8$ and $|\mathcal{L}(4)| = 545$, and by using a reverse search algorithm on a computer one can see that $|\mathcal{L}(5)| = 702{,}525$ and $|\mathcal{L}(6)| = 66{,}960{,}965{,}307$ (see Appendix A in [8]). Thus the complexity of $\mathcal{L}(n)$ rapidly increases with n. Still there are nice properties that we can show about $\mathcal{L}(n)$ which give it some extra structure.

Most importantly it is necessary to first understand what covering relations look like in $\mathcal{L}(n)$.

Proposition 4.2. *If $P \geqslant Q$ in $\mathcal{L}(n)$ then P covers Q if and only if $|P| = |Q| + 1$.*

Proof. Given that $P \geqslant Q$ we see that $\mathcal{S}_Q \subset \mathcal{S}_P$ (using the notation defined in Section 2). Since every set in $\mathcal{S}$ corresponds to an element in the associated lattice, this implies that $|P| \geqslant |Q|$. It remains to show that they differ by one element when P covers Q.

Suppose they differ by 2 elements, then $\mathcal{S}_P = \mathcal{S}_Q \cup \{\sigma, \beta\}$ where σ and β are subsets of $\{1, \ldots, n\}$ satisfying the conditions that $\sigma \cap \beta$, $\sigma \cap s$, and $\beta \cap s$ are either in $\mathcal{S}_Q$ or $\{\sigma, \beta\}$ for all $s \in \mathcal{S}_Q$. The argument is that in this case P cannot cover Q as there exists a lattice $T \neq P$ satisfying $P > T > Q$. Let $\mathcal{S}_T$ be one of the following:

- $\mathcal{S}_T = \mathcal{S}_Q \cup \{\sigma\}$ if $\sigma \subset \beta$,
- $\mathcal{S}_T = \mathcal{S}_Q \cup \{\beta\}$ if $\beta \subset \sigma$,
- $\mathcal{S}_T = \mathcal{S}_Q \cup \{\sigma\}$ or $\mathcal{S}_T = \mathcal{S}_Q \cup \{\beta\}$ if σ and β are not subsets of each other.

In any of these cases, $T \geqslant Q$ and $|T| = |Q| + 1$. $\square$

The upshot of Proposition 4.2 is the next nice result. It is easy to see that $\mathcal{L}(3) = B_3$ (the Boolean lattice on 3 atoms), whereas $\mathcal{L}(4) \neq B_4$ (and the latter is true for all $n \geqslant 4$ by Theorem 4.4). However, one can ask, what if any are the nice properties of B_n that are retained by $\mathcal{L}(n)$. One answer is the following theorem.

Theorem 4.3. *$\mathcal{L}(n)$ is a graded lattice of rank $2^n - n - 2$, i.e. this is the length of all maximal chains.*

Proof. The maximal element of $\mathcal{L}(n)$ is the lattice B_n and $|B_n| = 2^n$. The minimal element of $\mathcal{L}(n)$ is the unique lattice on n atoms where the atoms are also the coatoms, it has $n+2$ elements. Then by 4.2 every maximal chain in $\mathcal{L}(n)$ has length $2^n - (n+2)$ and so it is graded of rank $2^n - n - 2$. □

Also for B_3 or equivalently $\mathcal{L}(3)$ we see that all of the meet-irreducible elements are also coatoms. This need not happen in general but with the following theorem we do get a nice description of the meet-irreducibles of $\mathcal{L}(n)$.

Theorem 4.4. *The number of meet-irreducibles in $\mathcal{L}(n)$ is*

$$n\big(2^{n-1} - n\big).$$

Moreover the meet-irreducibles are the lattices of the form

$$L_{i,\sigma} = B_n - \big[\sigma, \{1, \ldots, \hat{i}, \ldots, n\}\big]$$

where $|\sigma| \geqslant 2$, $1 \leqslant i \leqslant n$, $i \notin \sigma$, and $[\sigma, \{1, \ldots, \hat{i}, \ldots, n\}]$ denotes the closed interval $\{p \in B_n \,|\, \sigma \leqslant p \leqslant \{1, \ldots, \hat{i}, \ldots, n\}\}$.

Proof. A lattice $P \in \mathcal{L}(n)$ is meet-irreducible if it is covered by only one element, call it Q. Thinking of P as a collection of subsets $\mathcal{S}_P$ then by Proposition 4.2 that means that there is only one way to add a subset to $\mathcal{S}_P$ so that the resulting set of sets is closed under intersections.

Let X be the set of all lattices $L_{i,\sigma} \in \mathcal{L}(n)$ described above. To see that $L_{i,\sigma} \in \mathcal{L}(n)$ first observe that it has n atoms, a maximal element and a minimal element. It remains to show that it is a meet-semilattice. To show this assume that there are two elements a, b in $L_{i,\sigma}$ which do not have a meet, i.e. their meet was in $[\sigma, \{1, \ldots, \hat{i}, \ldots, n\}]$. If this is the case then both $a \geqslant \sigma$ and $b \geqslant \sigma$ but both are not comparable to $\{1, \ldots, \hat{i}, \ldots, n\}$, so $i \in a$ and $i \in b$. This means that the meet of a and b should contain σ and $\{i\}$ thus their meet cannot be in $[\sigma, \{1, \ldots, \hat{i}, \ldots, n\}]$. So $L_{i,\sigma}$ is a meet-semilattice.

Now we need to show that $X = \mathrm{mi}(\mathcal{L}(n))$. To see that $X \subset \mathrm{mi}(\mathcal{L}(n))$ observe that the only candidates for elements to add to $L_{i,\sigma}$ come from $[\sigma, \{1, \ldots, \hat{i}, \ldots, n\}]$. Choose such a γ such that $\sigma \subset \gamma$. Observe that $\sigma \cup \{i\} \in L_{i,\sigma}$ from the above discussion. So $\gamma \cap (\sigma \cup \{i\}) = \sigma$ which shows that one cannot only add γ to $L_{i,\sigma}$ and still have a set which is closed under intersections. Thus the only element covering $L_{i,\sigma}$ is $L_{i,\sigma} \cup \sigma$.

To see that $\mathrm{mi}(\mathcal{L}(n)) \subset X$, let $L \in \mathrm{mi}(\mathcal{L}(n))$ this means that there is only one element $L' = L \cup \{\sigma\}$ which covers L. From this we can deduce that at minimum L contains a minimal element, a maximal element and all n atoms and does not include σ. We aim to understand the rest of the elements in L. Since L' is the only covering element, we can deduce that for every subset $\gamma \subset \{1, \ldots, n\}$ not in L and not equal to σ there exists a subset $\alpha \subset \{1, \ldots, n\}$ in L such that $\gamma \cap \alpha$ is not in L, because adding γ to L would not produce a covering relation. With this we examine all possible subsets γ which necessarily fit into one of the following three types to see if γ must be in L or not, the three types are:

Type 1: γ is not comparable to σ.
Type 2: $\gamma \subset \sigma$.
Type 3: $\sigma \subset \gamma$.

Considering the subsets γ of type 1 of size 2, it is clear that all such subsets must be elements of L otherwise $L \cup \{\gamma\}$ would also cover L. Once we have all the subsets of size 2, the argument applies for all subsets of type 1 with size 3, and so on until $n-1$. So now L must consist of all of $\hat{0}, \hat{1}$, the atoms, and all subsets of type 1.

Now, with all the subsets of type 1 in L consider the fact that L' needs to cover L or in other words σ intersect any subset of type 1 needs to already be in L. Since we have all subsets that do not include or are not included by σ this means that we need all subsets of σ in L. So, now L must consist of $\hat{0}, \hat{1}$, the atoms, and all subsets of type 1 and 2.

Finally we need to consider subsets of type 3. We start by considering such subsets which differ from σ by only one element (i.e. $\gamma = \sigma \cup \{i\}$ where $i \notin \sigma$). It is easy to see that only one such γ can be included in L since if there were more than one then L would not be closed under intersections as $\sigma \notin L$. Moreover it is necessary for L to include at least one set of the form $\sigma \cup \{i\}$ since otherwise adding either $\sigma \cup \{i\}$ to L or adding $\sigma \cup \{j\}$ to L will produce a lattice which covers L in $\mathcal{L}(n)$. Similar arguments apply for γ' where $\sigma \subset \gamma'$ and $|\gamma'| \geqslant |\sigma| + 2$. Although here it is necessary that if $|\gamma'| = |\sigma| + 2$ then only the γ' which include the $\gamma = \sigma \cup \{i\}$ already included into L be included as well, and so on. So, we see that all of the elements of type 3 that we've included contain i, so it must be true that we've included everything in B_n except for the elements greater than σ and less than $\{1, \ldots, \hat{i}, \ldots, n\}$ since these are precisely the elements of type three not containing i. Thus we have shown that $\mathrm{mi}(\mathcal{L}(n)) \subset X$.

Now, all that remains is to count the number of elements in X. First fix i, there we see that the number of possible $L_{i,\sigma}$ lattices is $2^{n-1} - n$. This is because the number of subsets of size 2 or more in a set of $n-1$ elements is $2^{n-1} - (n-1) - 1$. Finally we see that $|X| = n(2^{n-1} - n)$ by letting i range from 1 to n. □

5. Deformations of exponents

A *deformation of exponents* (Definition 1.4 in [9]) of a monomial ideal $M = (m_1, \ldots, m_t)$ is a choice of vectors $\{\epsilon_1, \ldots, \epsilon_t\}$ where each $\epsilon_i \in \mathbb{R}^n$ (where n is the number of variables), and the following condition is satisfied:

$$m_{is} < m_{js} \quad \text{implies} \quad m_{is} + \epsilon_{is} < m_{js} + \epsilon_{js}, \quad \text{and}$$
$$m_{is} = 0 \quad \text{implies} \quad \epsilon_{is} = 0.$$

Here m_{is} is the exponent of x_s in m_i.

It is easy to see that for any given monomial ideal M and a deformation of exponents M_ϵ, the lcm-lattice of M_ϵ is greater than that of M in $\mathcal{L}(n)$ (see Example 3.4(a) in [7]). The converse, which is the following theorem, indicates that deformations can be as "badly" behaved as possible.

Theorem 5.1. *If $P \geqslant Q$ in $\mathcal{L}(n)$ then there exists a coordinatization of Q which produces a monomial ideal M_Q such that a deformation of exponents of M_Q produces a monomial ideal whose lcm-lattice is isomorphic to P.*

We will need the following lemma in order to simplify the proof of Theorem 5.1.

Lemma 5.2. *Let $f : P \to Q$ be a join-preserving map between lattices. If c is a chain in P then $f(c)$ is a chain in Q.*

Proof. If c is a chain in P then $c = \{p_1, \ldots, p_k\}$, where the elements are listed in increasing order. Since f is join-preserving and thus order-preserving, the image of these elements will be a (possibly shorter) chain in Q. □

Now for the proof of the theorem.

Proof of Theorem 5.1. First, label P with the ECCV labeling from above. Then construct a labeling of Q as follows. Since $P \geqslant Q$ there is a join-preserving map $f : P \to Q$. To each element $q \in Q$ assign the monomial

$$\prod_{j \in I} x_j,$$

where $I = \{j \mid x_j \text{ divides some } m_p \text{ where } p \in f^{-1}(q)\}$. To see that this is in fact a coordinatization we need to check the two conditions in Theorem 3.2. Clearly all meet-irreducibles in Q will be labeled as every $p \in f^{-1}(q)$ is labeled with at least one variable x_j so $I \neq \emptyset$. Moreover because the variables x_j only appeared along chains in P, in this new labeling they will only appear along the image of that chain under f which by Lemma 5.2 will also be a chain in Q. Thus, this labeling is in fact a coordinatization.

It remains to show that there exists ϵ_i for each of the n atoms, such that the monomial ideal obtained for P is a deformation of exponents for the monomial ideal obtained for Q (with these coordinatizations). We do this by considering chains in both P and Q and their relation to each other under the map f.

Let c_j be the chain in P which is labeled by the variable x_j under the ECCV labeling. Note that we can write the monomial associated to an atom a_i as follows

$$\prod_j \prod_{p \in \lceil a_i \rceil_P^c \cap c_j} x_j,$$

where the subscript P indicates that both the order ideal and the complement are in P. Similarly with the coordinatization of Q given above we can think of the monomials for the lattice Q as

$$\prod_j \prod_{q \in \lceil a_i \rceil_Q^c \cap f(c_j)} x_j.$$

Given these descriptions of the monomials generators for each lattice makes it clear that for ϵ_i we want to define

$$\epsilon_{ij} = \left|\lceil a_i \rceil_P^c \cap c_j\right| - \left|\lceil a_i \rceil_Q^c \cap f(c_j)\right|.$$

Observe that if $\lceil a_i \rceil_Q^c \cap f(c_j) = \emptyset$ then every $q \in f(c_j)$ is greater than or equal to a_i (in Q). This implies that a_i is an element in the chain $f(c_j)$ and since f is a bijection on atoms this means that a_i is an element of the chain c_j in P. Thus $\lceil a_i \rceil_P^c \cap c_j = \emptyset$ also. Thus if the exponent on x_j for the monomial m_i is 0 then $\epsilon_{ij} = 0$.

Next we will show that $m_{is} < m_{js}$ implies that $m_{is} + \epsilon_{is} < m_{js} + \epsilon_{js}$. Note that under these coordinatizations these quantities correspond to the following $m_{is} = |\lceil a_i \rceil_Q^c \cap f(c_s)|$ and $m_{is} + \epsilon_{is} = |\lceil a_i \rceil_P^c \cap c_s|$ by the definition of ϵ_{is} (and likewise we get similar expressions with the index js substituting a_j for a_i). First we will examine what the condition $m_{is} < m_{js}$ means in terms of the lattice and specifically how the atoms a_i and a_j interact with the chain $f(c_s)$. To do so we will need the following lemma.

Lemma 5.3. *If c is a maximal chain in a finite atomic lattice P and $p \in \lceil a_i \rceil_P^c \cap c$ then for any $p' < p$ and $p' \in c$, $p' \in \lceil a_i \rceil^c \cap c$.*

Proof. We prove this by contradiction. Assume $p' \notin \lceil a_i \rceil^c \cap c$ then since $\lceil a_i \rceil$ and $\lceil a_i \rceil^c$ partition P into 2 subsets then we must have that $p' \in \lceil a_i \rceil \cap c$. If this is true then $a_i < p'$ but by transitivity this would imply that $a_i < p$ which contradicts the hypothesis that $p \in \lceil a_i \rceil_P^c \cap c$. □

Lemma 5.3 implies that if the chain $f(c_s)$ consists of elements $\hat{0} < a_s < q_1 < \cdots < q_t < \hat{1}$ then $\lceil a_i \rceil_Q^c \cap f(c_s)$ consists of a sub-chain starting at $\hat{0}$ and going continuously to some element call it q_i and likewise $\lceil a_j \rceil_Q^c \cap f(c_s)$ is a sub-chain terminating at an element q_j where $q_i < q_j$. Here the inequality $q_i < q_j$ comes from the fact that $m_{is} < m_{ij}$.

Now to see that $m_{is} + \epsilon_{is} < m_{js} + \epsilon_{js}$ or equivalently that $|\lceil a_i \rceil_P^c \cap c_s| < |\lceil a_j \rceil_P^c \cap c_s|$ we must look at what happens when we lift this chain (and the subsequent sub-chains) back to P. If we consider each of the elements q_α in the chain $f(c_s)$ as the maximal subset of atoms in their support then we would see the following chain of inclusions $\emptyset \subset \{a_s\} \subset \operatorname{supp}_Q q_1 \subset \cdots \subset \operatorname{atoms} Q$. Now we will use this to explain what c_s looks like relative to $f(c_s)$. We can identify atoms Q with atoms P because f is a bijection on atoms, and because f is a join-preserving map any q_α can be understood to also be an element in P where $\operatorname{supp}_Q q_\alpha$ is identified with $\operatorname{supp}_P q_\alpha$ (specifically to get $q_\alpha \in P$ just take the join of the atoms in its support in P rather than in Q). Note also that if q_i is the largest element in $\lceil a_i \rceil_Q^c \cap f(c_s)$ then this means that $a_i \in \operatorname{supp}_Q q_{i+1}$ and therefore $a_i \in \operatorname{supp}_P q_{i+1}$.

Also following from the fact that f is join-preserving we can see that if there are any new elements in the chain c_s then they must be between two of the elements coming from the chain $f(c_s)$ in particular if $p \in c_s$ and $\operatorname{supp}_P p \neq \operatorname{supp}_P q_\alpha$ for any q_α then p is between q_l and q_{l+1} for some l. The conclusion of this is that if $p \in \lceil a_i \rceil_P^c \cap c_s$ then it is necessarily true that $p < q_{i+1} \leqslant q_j$. So we have that $\lceil a_i \rceil_P^c \cap c_s \subset \lceil a_j \rceil_P^c \cap c_s$. Moreover we know that $q_j \notin \lceil a_i \rceil_P^c \cap c_s$ so this is a strict set inclusion thus yielding the inequality $m_{is} + \epsilon_{is} < m_{js} + \epsilon js$.

This collection of ϵ_{ij} gives a deformation of exponents from a monomial ideal whose lcm-lattice is Q to one whose lcm-lattice is P. □

Note that this proof uses the fact that we can represent any deformation of exponents using integer vectors rather than working with real exponents. An obvious corollary to this is the following.

Corollary 5.4. *Given a lattice $Q \in \mathcal{L}(n)$ there exists a coordinatization of Q such that every element in $\lceil Q \rceil$ is the lcm-lattice of a deformation of exponents of that coordinatization.*

Proof. We want to use the method used in the proof of Theorem 5.1. First coordinatize B_n (the maximal element in $\mathcal{L}(n)$) using the ECCV coordinatization, call the resulting monomial ideal M_{B_n}. Now we need to construct coordinatizations for all the elements P' in $\lceil Q \rceil$. By Theorem 5.1 we get that for each $P' \in \lceil Q \rceil$ (including $P' = Q$) we get a coordinatization of P' yielding a monomial ideal $M_{P'}$ such that M_{B_n} is a deformation of exponents.

It remains to show that for each $M_{P'}$ we can find an ϵ_i for $1 \leqslant i \leqslant n$ so that $M_{P'}$ is a deformation of exponents of M_Q. Again for a variable x_j which appears only along one chain c_j we define ϵ_{ij} to be

$$\left|\lceil a_i \rceil_{P'}^c \cap c_j\right| - \left|\lceil a_i \rceil_Q^c \cap f(c_j)\right|. \qquad \square$$

Together these results give evidence that deformations of exponents might be best understood in the context of the set $\mathcal{L}(n)$. In particular given a finite atomic lattice (or "abstract monomial ideal") L, the set of all possible deformations is the set $\lceil L \rceil$. However for a given coordinatization the set $\lceil L \rceil$ may be much larger than the actual set of deformations that occur, nevertheless it gives a good starting point. Alternatively one could view all possible deformations for a given monomial ideal as a fan in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. Specifically if M is in the polynomial ring with n variables and has k monomial generators then it sits in the intersection of hyperplanes in $\mathbb{R}^{nk}$ which are defined by multiple generators having the same exponent on a variable. In this setting, deformations correspond to moving off these intersections, but various partial deformations might yield a combinatorially equivalent monomial ideal. If we compare these two approaches we can quickly see that the set $\lceil L \rceil \in \mathcal{L}(n)$ only lists possible deformations up to combinatorial type thus making it a simpler set to study than the fan.

6. Relationship to cellular resolutions

In this section we prove the main results of this paper which focus on how the geometric structure of minimal resolutions stabilizes as one moves up chains in $\mathcal{L}(n)$. Specifically, Theorem 3.3 in [7] can be interpreted to say that total Betti numbers are weakly increasing along chains in $\mathcal{L}(n)$, thus we want to focus our attention on subsets of $\mathcal{L}(n)$ with fixed total Betti numbers. Given a finite atomic lattice P let $\beta = (\beta_0, \beta_1, \ldots, \beta_t)$ be the vector whose entries are the total Betti numbers of any coordinatization of P. We define $\mathcal{L}(n)_\beta$ to be the "Betti stratum" of $\mathcal{L}(n)$ consisting of all the finite atomic lattices whose total Betti numbers are β. We want to study how the structure of the minimal resolutions stabilizes within these strata. Our approach is to first understand the simpler but well-understood cases where a minimal resolution can be supported on a simplicial complex, in particular the Scarf complex. We then show how theorems of this type can be generalized to non-simplicial complexes. First we must introduce some semantics and notation.

Due to the formulas for multigraded Betti numbers in terms of order complexes of intervals in the lcm-lattice (see Theorem 2.1 in [7]), we can refer to the Betti numbers of a finite atomic lattice as opposed to the Betti numbers of a monomial ideal. Specifically for $p \in \mathrm{LCM}(M)$ the formula for computing multigraded Betti numbers in homological degree i is:

$$b_{i,p}(R/M) = \dim \tilde{H}_{i-2}\big(\Delta(\hat{0}, p); k\big),$$

where M is a monomial ideal whose lcm-lattice is P, and $\Delta(\hat{0}, p)$ is the order complex of the open interval from $\hat{0}$ to p. Note also that we can substitute the crosscut complex Γ defined in Section 2 for Δ. Since all monomial ideals with the same lcm-lattice have isomorphic minimal resolutions (see Theorem 3.3 in [7]) this means that if a cell complex supports the minimal resolution of one ideal then it will support the minimal resolution for all possible coordinatizations of the lcm-lattice of that ideal. In this sense, we will discuss the cellular resolution of a finite atomic lattice.

Notice that in a finite atomic lattice, any given element is not necessarily expressed as a unique join of atoms. For example the join of atoms 1 and 2 may be equal to the join of atoms 1, 2, and 3. To account for this ambiguity it will be useful for us to introduce the following set associated to every element in the lattice, define $\mathrm{equiv}_P(p)$ to be the set containing all subsets of the atoms of P whose join in P is equal to p.

It is now easy to define the Scarf complex of a monomial ideal M_P in terms of the lattice P,

$$\mathrm{scarf}(P) = \Gamma\big(\big\{p \in P \;\big|\; \big|\mathrm{equiv}_P(p)\big| = 1\big\}\big) \subset \Gamma(P).$$

For the original definition of the Scarf complex see Section 3 of [1]. Note that because P is a finite atomic lattice $\Gamma(P)$ is the $(n-1)$-simplex and supports the Taylor resolution of any monomial ideal associated to P (for an explanation of the Taylor resolution see Section 2 of [1]).

By results of [1] and [9], if a given monomial ideal is generic or strongly generic then its minimal resolution is the Scarf complex. A strongly generic monomial ideal is one where no variable appears with the same exponent in two or more generators. The weaker condition of being generic implies that if two generators have the same exponent in a variable then there is a third generator which strictly divides their least common multiple. Note however, that there may be monomial ideals whose minimal resolution is the Scarf complex, yet the ideal is not generic. An obvious example of this phenomenon is if one takes a generic monomial ideal and polarizes to obtain a squarefree monomial ideal (see Chapter 21 of [10] for a discussion on polarization). It will have the same Scarf complex which supports the minimal resolution since lcm-lattices are preserved under polarization. It is rare however, for squarefree monomial ideals to be generic since all variables always appear with the same exponent.

However there exist examples which are not simply a polarization of a generic monomial ideal, whose minimal resolution is supported by the Scarf complex, but they are not generic or strongly generic. The following example of a finite atomic lattice illustrates an example where the minimal

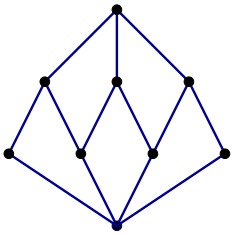


Fig. 3. Lattice resolved by Scarf complex which has no generic coordinatization.

resolution of any coordinatization will be given by the Scarf complex, but where there is no possible coordinatization which produces a generic monomial ideal.

Example 6.1. The lattice P in Fig. 3 is the *augmented face lattice* (face poset, which in this case is a meet-semilattice, plus a maximal element) of a simplicial complex consisting of 4 vertices and 3 edges. Every point in P except for the minimal and maximal elements represents a multidegree that has a non-zero Betti number. This is easy to see since for all of the atoms $\tilde{h}_{-1}(\Gamma(P_{<a_i}), k) = 1$ and for each element p covering an atom $\Gamma(P_{<p})$ consists of two vertices thus $\tilde{h}_0 = 1$. Thus, P will always be resolved by its Scarf complex.

Note however, that every possible coordinatization of P fails to satisfy the conditions to be generic. To see this note that for every coordinatization a variable appears with the same non-zero exponent in either m_1 and m_2, m_2 and m_3, or in m_3 and m_4. Since the meet-irreducibles of P are precisely $p_{12} = a_1 \vee a_2$, $p_{23} = a_2 \vee a_3$, $p_{34} = a_3 \vee a_4$, a_1, and a_4 any coordinatization must cover these. I will focus on just showing that for all coordinatizations there is a variable appearing with the same non-zero exponent for the pair m_1 and m_2. Note that the monomial m_1 is determined by the labelings found on $\{p_{23}, p_{34}, a_2, a_3, a_4\}$ and m_2 is determined by $\{p_{34}, a_1, a_3, a_4\}$. Moreover, p_{34} and a_4 must be non-trivially labeled and if a_3 happens to be non-trivially labeled it also appears in both of these sets. Any variable appearing in the label on p_{34} can at most appear in the label of either a_3, or a_4 and cannot appear anywhere else. Thus this variable must appear with the same exponent in both m_1 and m_2. Moreover, no other generator divides the least common multiple associated to element p_{12} by virtue of the fact that if it did then it would be an atom which is less than the element p_{12} and the only atoms less than p_{12} are a_1 and a_2. So there is no coordinatization of P yielding a generic monomial ideal (i.e. a *generic coordinatization*).

Understanding when generic coordinatizations exist must be determined on a case by case basis. For strongly generic coordinatizations however, the following theorem characterizes precisely when they exist.

Theorem 6.2. *P in $\mathcal{L}(n)$ admits a strongly generic coordinatization if and only if P is a graded finite atomic lattice whose maximal chains have rank n.*

Proof. To show that such graded lattices have strongly generic coordinatizations, coordinatize P using the ECCV coordinatization. Now, we just need to show that the resulting monomial ideal is strongly generic. In other words we need to show that for any variable x_j, if it appears in any two monomials m_{a_i} and m_{a_k} then it has a different exponent. By the definition of the monomials m_{a_i} and m_{a_k} this amounts to showing that the intersections of the complements of the filters $\lceil a_i \rceil$ and $\lceil a_k \rceil$ with the chain corresponding to x_j are different.

Let $c_j = \{\hat{0}, a_{j1}, a_{j1} \vee a_{j2}, \ldots, a_{j1} \vee a_{j2} \vee \cdots \vee a_{jn} = \hat{1}\}$ be the chain which is entirely labeled by the variable x_j. Since P is graded of rank n we know that there are $n+1$ elements in c_j and that the i-th element in the chain is the join of $i-1$ atoms of P. Every set $\lceil a_i \rceil$ intersects c_j at a different spot along the chain, so it is likewise with the complements of these filters. This guarantees that each variable x_j appears with a different exponent in each monomial generator.

Conversely suppose $M = (m_1, \ldots, m_n)$ is a strongly generic monomial ideal. We need to show that its lcm-lattice is graded and that the chains have rank n. Since M is strongly generic then we know that each variable x_i must achieve its maximum degree d_i for exactly one monomial generator m_j. Let $p = \operatorname{lcm}(m_1, \ldots, \widehat{m}_j, \ldots, m_n)$, i.e. omitting generator j. Then we know that the degree of x_i in the monomial p must be less than d_i. Thus $p \neq \hat{1}$ in LCM(M) but $p \vee a_j = \hat{1}$ where a_j is the atom corresponding to the j-th generator, so we can conclude that p is covered by $\hat{1}$ and thus is a coatom.

Now, consider $\lfloor p \rfloor \in \text{LCM}(M)$. This will be LCM($M'$) where $M' = (m_1, \ldots, \widehat{m}_j, \ldots, m_n)$ which is also a strongly generic monomial ideal. Repeating the above argument for perhaps a different variable appearing in the generators of M' we can find a p' such that the maximal element of LCM(M') covers it, i.e. p covers p'. Iterating this process we can find precisely n monomial ideals $M^{(i)}$ so that $M^{(i+1)} \subset M^{(i)}$ and the maximal element in $M^{(i+1)}$ corresponds to a coatom of $M^{(i)}$. This produces a chain of length $n+1$ connecting $\hat{1}$ to $\hat{0}$ in LCM(M) and varying over all choices at each iteration we get all chains of LCM(M). □

The following statement appeared as Proposition 5.14 in [12] and separately as Theorem 5.3 in [11]. An idea of how to prove this can be summarized as taking the face poset of the simplicial complex, making it a lattice by adding a maximal element (if necessary), and then coordinatizing the lattice.

Proposition 6.3. *Every simplicial complex X not equal to the boundary of a simplex is the Scarf complex of some squarefree monomial ideal. If X is acyclic then X supports the minimal resolution of that ideal.*

The intention of the next proposition is to demonstrate that each stratum in $\mathcal{L}(n)$ with fixed total Betti numbers has entire regions of lattices whose minimal resolution is supported by the appropriate Scarf complex.

Proposition 6.4. *Suppose P is a finite atomic lattice where any coordinatization of P is minimally resolved by its Scarf complex. If Q is such that $Q \geqslant P$ in $\mathcal{L}(n)_\beta$ where β is the vector of total Betti numbers of P, then Q is also resolved by its Scarf complex which will equal the Scarf complex of P.*

Proof. First we note two facts, one is when $|\operatorname{equiv}_P(p)| = 1$ then $\Gamma(P_{<p})$ is the boundary of the $i-1$-simplex, where i is the number of atoms below p. In this case, $\beta_{i-2,p} = 1$ and $\beta_{j,p} = 0$ for all $j \neq i-2$. Second we note that an atomic lattice P is resolved by its Scarf complex exactly when $\beta_{i,p} \neq 0$ is equivalent to $|\operatorname{equiv}_P(p)| = 1$ for all $p \in P$. Now to prove the proposition we must show (i) the same property for Q, and (ii) that scarf(Q) equals scarf(P).

Since $Q \geqslant P$ there is a join-preserving map $\psi : Q \to P$ and recall that

$$\operatorname{equiv}_P(p) = \bigcup_{q \in \psi^{-1}(p)} \operatorname{equiv}_Q(q).$$

If $|\operatorname{equiv}_P(p)| = 1$ then there must be only one $q \in \psi^{-1}(p)$ and we must have $\operatorname{equiv}_P(p) = \operatorname{equiv}_Q(q)$. Hence scarf($P$) is contained in scarf($Q$). Also $\beta_{i,q} = \beta_{i,p} = 1$ for the appropriate i. Since Q and P are in the same Betti stratum this accounts for all the Betti numbers of Q.

If $|\operatorname{equiv}_P(p)| \geqslant 2$ then all $q \in \psi^{-1}(p)$ must have $\beta_{i,q} = 0$ since all the Betti numbers of Q are accounted for. For such a q we must then have $|\operatorname{equiv}_Q(q)| \geqslant 2$. This shows part (i). We also get that scarf(Q) is not strictly larger than scarf(P), concluding the proof of part (ii). □

We can also begin to generalize the previous results to account for ideals whose minimal resolutions will not be supported on a simplicial complex. Then the following propositions provide a generalization of Propositions 6.3 and 6.4.

Theorem 6.5. *Let X be a regular cell complex such that X is acyclic, and the augmented face poset P_X of X is a finite atomic lattice on $|X^0| = n$ atoms (i.e. the face poset is a meet-semilattice with n atoms). Then the minimal resolution of any coordinatization of P_X is supported on X.*

Proof. Observe that if P_X is the face lattice of X then labeling X with the monomials in any coordinatization of P_X as prescribed in [2] simply puts the monomial at a point $p \in P_X$ on its corresponding face in X. Moreover, each face in X has a distinct multidegree labeling it. To show that the resolution of any coordinatization is supported on X it is sufficient to show that $X_{\leqslant p}$ is acyclic due to Theorem 3.1a in [5] which reduces checking Proposition 1.2 in [2] for monomials occurring in the lcm-lattice. This is true by construction though since $X_{\leqslant p}$ corresponds to the d-cell that p represents and its boundary.

We will now show the minimality of this resolution. Note that $\Delta(P_{X_{<p}})$ is the barycentric subdivision of $X_{<p}$. In particular, we can easily see that $\tilde{h}_i(\Delta(P_{X_{<p}}), k) = 1$ for $i = d-1$ where p corresponds to a d-cell in X since $X_{<p}$ is the boundary of that d-cell. Thus $\beta_{d+1,p} = 1$ for a d-cell F_p in X.

If P_X is the augmented face lattice of X (i.e. if the $\hat{1}$ element actually needs to be added in) then the above description applies to all $p \in P_X - \hat{1}$. For $p = \hat{1}$, we have by construction that $X_{<p} = X$ and since X was assumed to be acyclic, X still supports the resolution.

Moreover, only the multidegrees corresponding to each $p \in P_X$ can possibly have $\beta_{i,p}$ non-zero (see remark before Theorem 2.1 in [7] and the fact that here P_X is isomorphic to the lcm-lattice). Thus, since we've considered all such p's, X supports the resolution of P. This resolution is minimal since $\beta_{i,p} = 1$ for only one i, i.e. no map has a non-zero constant as an entry of the matrix. □

The following result is a partial generalization of 6.4.

Proposition 6.6. *Let P_X be as in Theorem 6.5. Suppose that Q and P_X are both in $\mathcal{L}(n)_\beta$ where β is the vector of total Betti numbers of P_X and that Q satisfies the following two conditions*:

1. *Q covers P_X.*
2. *$\beta_{i,q} = 1$ for $q = \max(\psi^{-1}(p))$ for any $p \in P_X$ where ψ is the join-preserving map from Q to P_X, and $\beta_{i,q} = 0$ otherwise.*

Then Q has a minimal resolution supported on X.

Proof. Since $Q \in \mathcal{L}(n)_\beta$ we know that the total Betti numbers of the minimal resolution of Q are the same as those of P_X. So, all that needs to be shown is that X supports a resolution of Q, and then since that resolution has the right total Betti numbers it must be minimal. Thus, we just need to show that $X_{\leqslant q}$ is acyclic for all $q \in Q$.

One can observe that

$$\operatorname{equiv}_{P_X}(p) = \bigcup_{q \in \psi^{-1}(p)} \operatorname{equiv}_Q(q).$$

Moreover, $\operatorname{supp}(p)$ is the maximal element in $\operatorname{equiv}_{P_X}(p)$ when it is ordered by inclusion which implies that $\operatorname{supp}(p)$ is the maximal element in $\bigcup_{q \in \psi^{-1}(p)} \operatorname{equiv}_Q(q)$. So we can observe that there are two types of elements $q \in Q$: type (1) being where

$$q = \bigvee_{i \in \operatorname{supp}(p)} a_i$$

for some $p \in P_X$ where the join is in Q; and the type (2) being the $q \in Q$ that are not of type (1). Clearly the elements of type (1) are maximal in the appropriate $\psi^{-1}(p)$. Note also that since Q covers P_X there is precisely one element of type (2) in Q due to Proposition 4.2.

Now we need to label X with the appropriate multidegrees. Notice that each face of X corresponding to $p \in P_X$ will be labeled with the appropriate $q \in Q$ of type (1) which is the maximal element in $\psi^{-1}(p)$. Moreover, no elements in Q of type (2) label any faces in X. Thus when we examine the complexes $X_{\leqslant q}$ for q of type (1) they will all be acyclic since $X_{\leqslant q}$ represents the d-cell corresponding to the element $p \in P_X$ where

$$q = \bigvee_{i \in \operatorname{supp}(p)} a_i.$$

It remains to show that for the one element $q \in Q$ of type (2) we have that $X_{\leqslant q}$ is acyclic. We know that since $\beta_{i,q} = 0$ for this element, $\Delta(Q_{<q})$ is acyclic. Moreover, q is the only element in Q which does not correspond to a face in X, so $Q_{<q}$ is equal to the face lattice of $X_{<q}$. This shows that $X_{<q}$ is acyclic since $\Delta(Q_{<q})$ is homotopy equivalent to $X_{<q}$ by barycentric subdivision. Finally, since q does not label a face of X, it is clear that $X_{<q} = X_{\leqslant q}$ thus concluding the proof. □

Both Proposition 6.3 and Theorem 6.5 are instances of statements saying "if a geometric object X has a certain property then there is a monomial ideal whose resolution is supported on X". We believe that these propositions merely begin to give a description of what types of cell complexes can support resolutions, and that there are more statements of this type that exist with different conditions on the geometric object.

Moreover, both Propositions 6.4 and 6.6 are of the form "if a lattice P has resolutions with some property then all the lattices above it with the same total Betti numbers also have the same property". Statements of this type are pleasantly surprising since it is known that if $Q > P$ then minimal resolutions of Q will be resolutions of P thus certain properties of a minimal resolution of Q descend to properties of some resolution of P. Our statements however say the opposite, that certain properties of a minimal resolution of P can be lifted to a some extent in $\mathcal{L}(n)$ and we believe that these are instances of a much stronger result. This is the subject of ongoing work.

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