Homework Assignment 2
Kalman filtering

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Contents

1 System

The system that is considered in this work is of the following form:

$$x_{k+1} = Ax_k + w_k$$

$$y_k = Cx_k + v_k$$

Here x_k is a 2×1 vector representing the position (first component, r_k) and the velocity (second component, ν_k) of a moving object. For every k the object has moved during a fixed time step $T = 0.1$, which gives the matrix A the following expression:

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

The system noise w_k is Gaussian i.i.d. white noise with covariance matrix

$$Q = \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix} q$$

Here q has the value 0.2. Measurements of the state x are made by two measuring sensors, each producing scalar measurement results $y_{1,k} = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_{1,k}$ and $y_{2,k} = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_{2,k}$ respectively. These are then combined by a fusion centre to form the measurement vector y_k. The noise terms here are also Gaussian i.i.d. white noise. They have the variances σ_1^2 and σ_2^2. It should be pointed out that both $y_{1,k}$ and $y_{2,k}$ are measurements of the position r_k only. There is no measurement of the velocity.

2 Methods

The fusion centre can combine the measurements $y_{1,k}$ and $y_{2,k}$ in two ways. One way is combining them into a vector

$$y_k = \begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix}.$$

This gives a new look for the measurement part of the original state space model. The C-matrix and the noise vector v_k can be found from the expression

$$y_k = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix}.$$

The covariance matrix R_v of the noise vector v_k becomes a diagonal matrix with the variances of the two noise components on the diagonal. This is because $v_{1,k}$ and $v_{2,k}$ are independent of each other.

$$R_v = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

At this point everything about the system is known, since the matrix C obviously is

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

The way of combining measurement data that has been discussed here so far will later be referred to as "method 1".

Method 2 on the other hand is when the fusion centre makes a linear combination of the two sensors' measurements and produces

$$y_k = y_{1,k} + y_{2,k}.$$

Inserting the expressions for the individual measurements yields

$$y_k = 2r_k + v_{1,k} + v_{2,k}.$$

By mathematical rules for independent stochastical variables one finds that

$$R_v = \sigma_1^2 + \sigma_2^2$$

for this method and since there were only one position term and one velocity term in x_k the C-matrix becomes

$$C = \begin{bmatrix} 2 & 0 \end{bmatrix}.$$

A modified version of both methods is also introduced. These are the cases when the fusion centre can't receive the individual sensors' measurerments in a perfect way, resulting in an additional noise term for each component of y_k. More precisely method 1 becomes

$$y_k = \begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} + \begin{bmatrix} n_{1,k} \\ n_{2,k} \end{bmatrix}$$

and method 2 becomes

$$y_k = y_{1,k} + y_{2,k} + n_k.$$

These will be referred to as method 1* and method 2*. The three new noise terms are all i.i.d. Gaussian white noise with the same variance $\sigma_n^2 = 0.25$. All components of the original state space model are the same for the modified methods as for their corresponding original methods, except for the measurement noise vectors v_k. This vector becomes

$$v_k = \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix} + \begin{bmatrix} n_{1,k} \\ n_{2,k} \end{bmatrix}$$

in method 1*. The corresponding covariance matrix becomes

$$R_v = \begin{bmatrix} \sigma_1^2 + \sigma_n^2 & 0 \\ 0 & \sigma_2^2 + \sigma_n^2 \end{bmatrix}.$$

In method 2* the measurement noise vector becomes

$$v_k = v_{1,k} + v_{2,k} + n_k$$

which gives a scalar covariance term

$$R_v = \sigma_1^2 + \sigma_2^2 + \sigma_n^2,$$

i.e. simply the sum of the variances.

3 Kalman filter

The Kalman filter should be used in the filtering form in this work, i.e. not in the predictor form. This means that the filtering is carried out using the following equations. The expression for the apostriori estimate $\hat{x}_{k|k}$ uses the apriori estimate $\hat{x}_{k|k-1}$ and the measurement y_k and has the following form:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{f,k}(y_k - C\hat{x}_{k|k-1}). \tag{1}$$

Here $K_{f,k}$ is the Kalman gain and it is given by

$$K_{f,k} = P_{k|k-1}C^T(R_v + CP_{k|k-1}C^T)^{-1}. \tag{2}$$

The error covariance matrix's apriori estimate $P_{k|k-1}$ is used to compute its aposteriori estimate with the expression

$$P_{k|k} = P_{k|k-1} - P_{k|k-1}C^T(CP_{k|k-1}C^T + R_v)^{-1}CP_{k|k-1}. \tag{3}$$

The apriori estimates are computed according to

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k}$$

and

$$P_{k+1|k} = AP_{k|k}A^T + Q$$

since there's no input signal u_k present.

4 Questions

The following points are the questions that are supposed to be answered for both method 1 and method 2.

- Specifying what the matrices A and C are, as well as calculating the covariance matrices Q and R_v. The answers to this is already given in the section "Methods" above.
- Implementing and simulating the Kalman filter for the system assuming that the initial condition x_0 is also Gaussian with mean $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and that it has the covariance matrix Q. The implementations are given in a separate MATLAB script file.
- Producing a graph of the position r_k and its estimates $\hat{r}_{k|k-1}$ and $\hat{r}_{k|k}$ with k going from 1 to 100. The results are presented in the leftmost graphs in figure 1 and figure 2 in the "Graphs" section.
- Producing a graph of the velocity ν_k and its estimates $\hat{\nu}_{k|k-1}$ and $\hat{\nu}_{k|k}$ with k going from 1 to 100. The results are presented in the middle graphs in figure 1 and figure 2 in the "Graphs" section.
- Producing a graph of the traces of the error covariance matrices $P_{k|k-1}$ and $P_{k|k}$ with k going from 1 to 100. The results are presented in the rightmost graphs in figure 1 and figure 2 in the "Graphs" section.
- Finding out what happens to $\hat{r}_{k|k-1}$ as k goes to infinity. It appears to be less jumpy for higher values of k and to follow the real r_k curve better. For high k the $\hat{r}_{k|k-1}$ curve rarely intersects the r_k curve (if at all) whereas this happens all the time for low k. Simulations with k going up to 100, 1000 and 10000 were made to check this.
- Finding out what happens to $P_{k|k-1}$ as k approaches infinity. In the figures 1 and 2 it is clearly seen that $P_{k|k-1}$ converges to a value of 0.58 of the units in the graphs in both method 1 and method 2.

So it turns out that the trace of the error covariance matrix $P_{k|k-1}$ looks the same for both method 1 and method 2. The last question is:

- Does this apply for method 1* and method 2* too?

Figures 3 and 4 in the "Graphs" section displays similar graphs for the methods 1* and 2* as figures 1 and 2 do for the methods 1 and 2. When looking at the rightmost graphs in figures 3 and 4 it can clearly be seen that the answer is no, i.e. that the traces of $P_{k|k-1}$ don't look the same for the modified methods.

5 Graphs

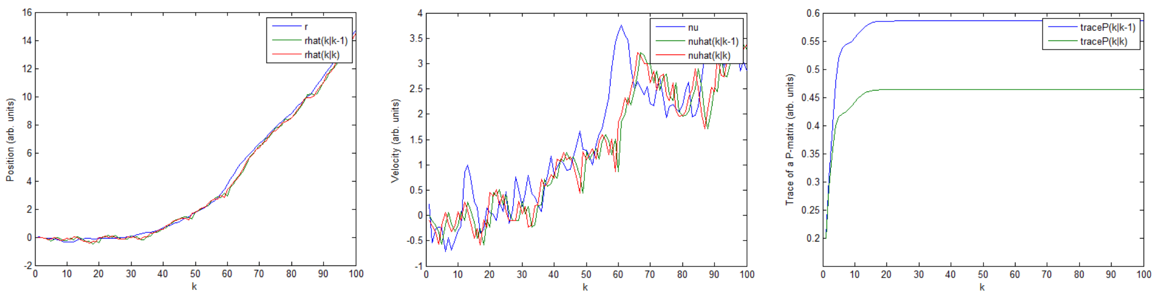

Figure 1: The position and its estimates (left), the velocity and its estimates (center) and the traces of the covariance matrices (right) of a Kalman filter using method 1.

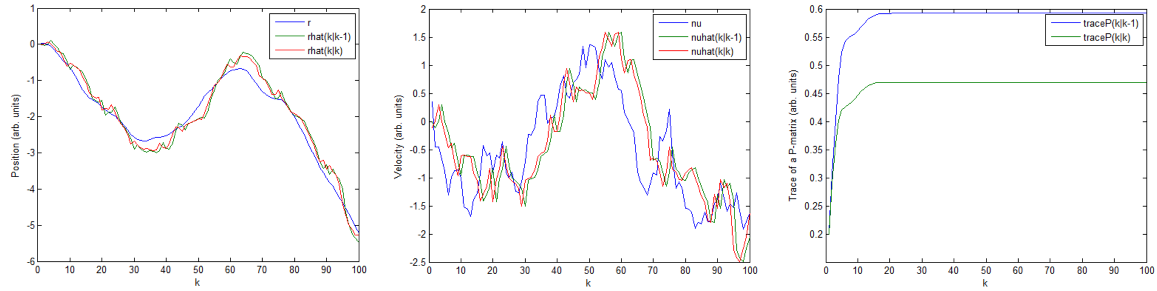

Figure 2: The position and its estimates (left), the velocity and its estimates (center) and the traces of the covariance matrices (right) of a Kalman filter using method 2.

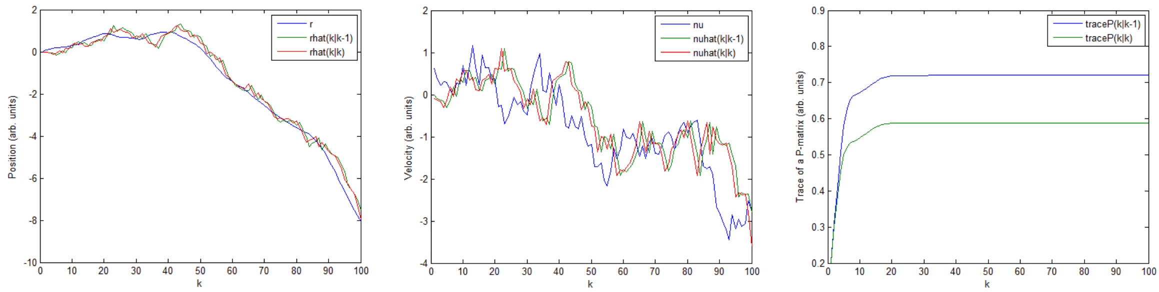

Figure 3: The position and its estimates (left), the velocity and its estimates (center) and the traces of the covariance matrices (right) of a Kalman filter using method 1*.

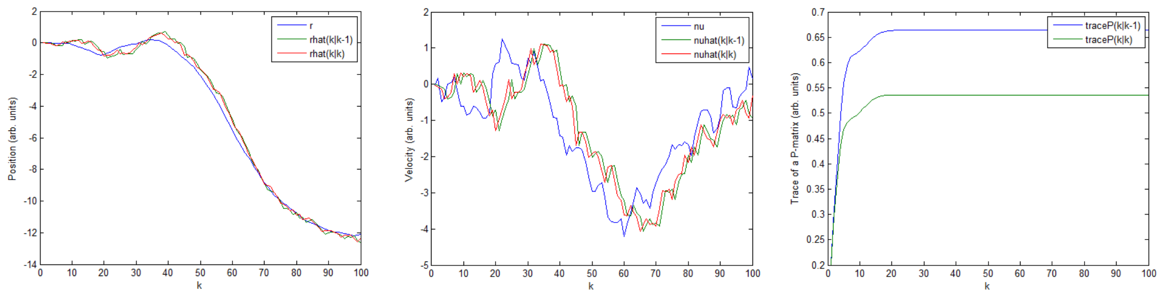

Figure 4: The position and its estimates (left), the velocity and its estimates (center) and the traces of the covariance matrices (right) of a Kalman filter using method 2*.