STAT 221 Homework 2

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2.9

a)

```r
library(astsa)

model <- lm(soi ~ time(soi))

(sum <- summary(model))
```

```
##
## Call:
## lm(formula = soi ~ time(soi))
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.04140 -0.24183  0.01935  0.27727  0.83866
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 13.70367    3.18873   4.298 2.12e-05 ***
## time(soi)   -0.00692    0.00162  -4.272 2.36e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3756 on 451 degrees of freedom
## Multiple R-squared:  0.0389, Adjusted R-squared:  0.03677
## F-statistic: 18.25 on 1 and 451 DF,  p-value: 2.359e-05
```

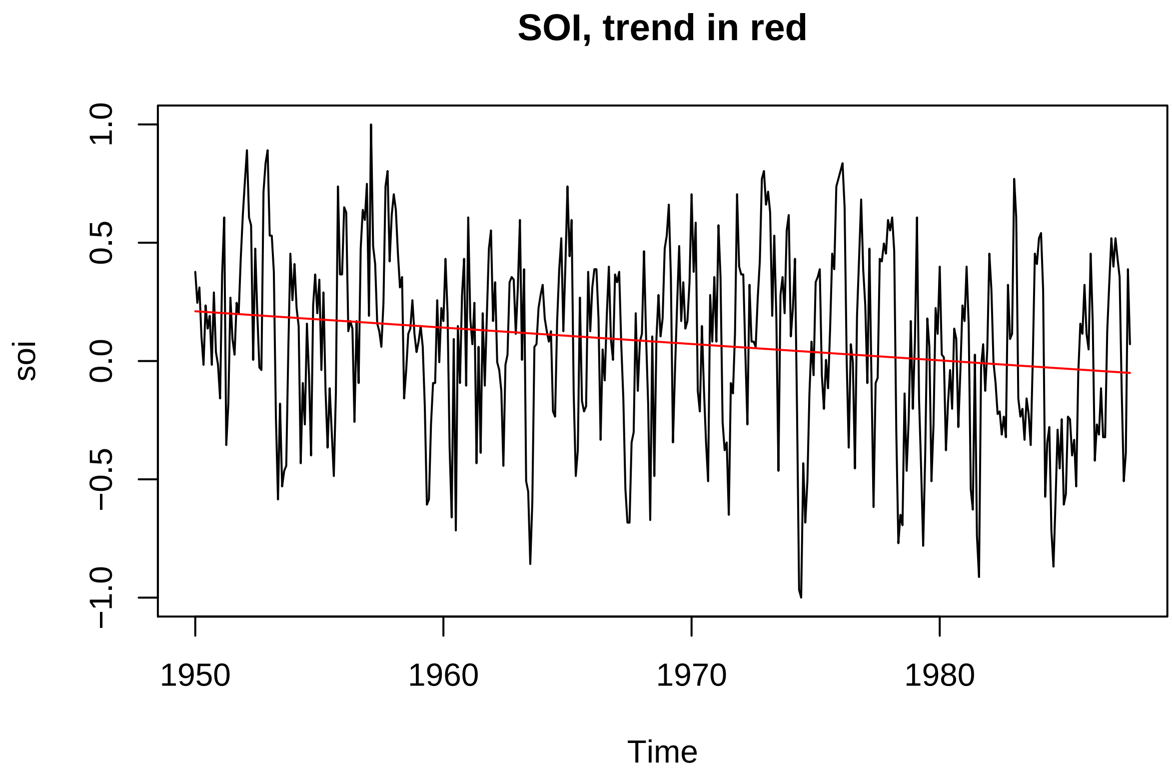
```r
sum$coefficients[2]
```

```
## [1] -0.006919645
```

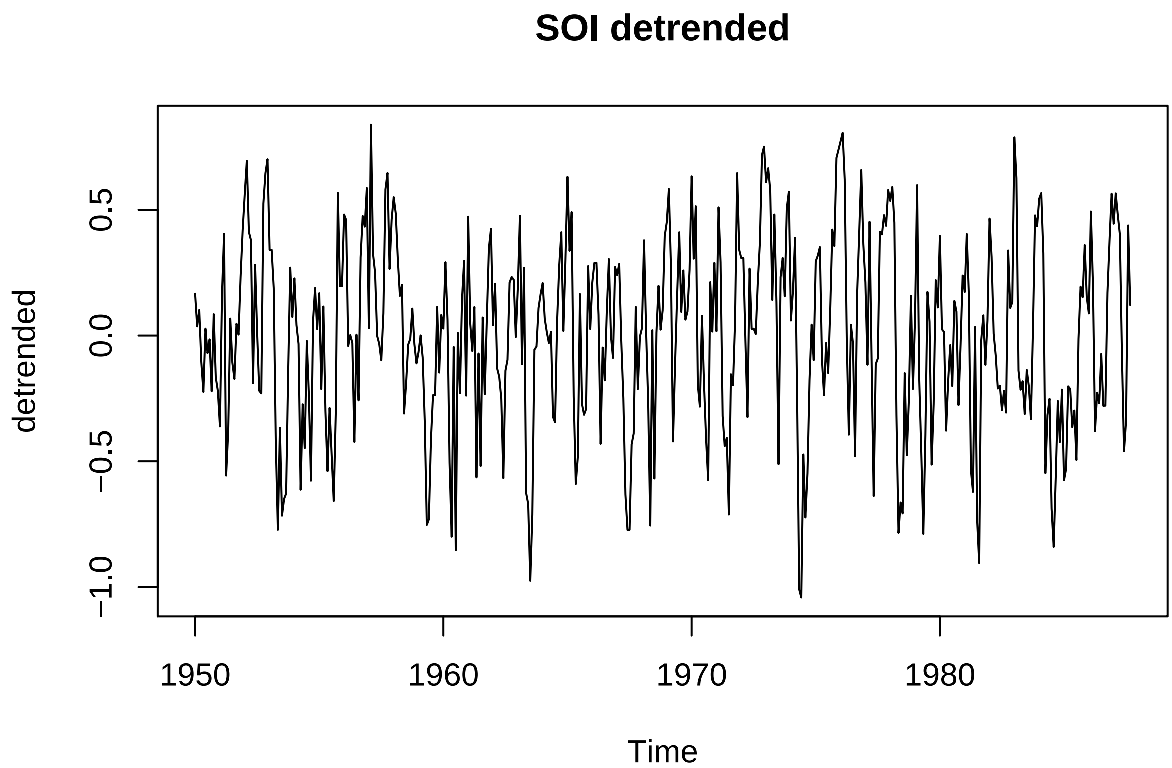
There is a significant trend in SOI in relation to time, where the SOI decreases ~0.007 every year.

b)

```r
plot(soi, main="SOI, trend in red")
lines(soi*0 + predict(model, time(soi)), type="l", col="red")
```



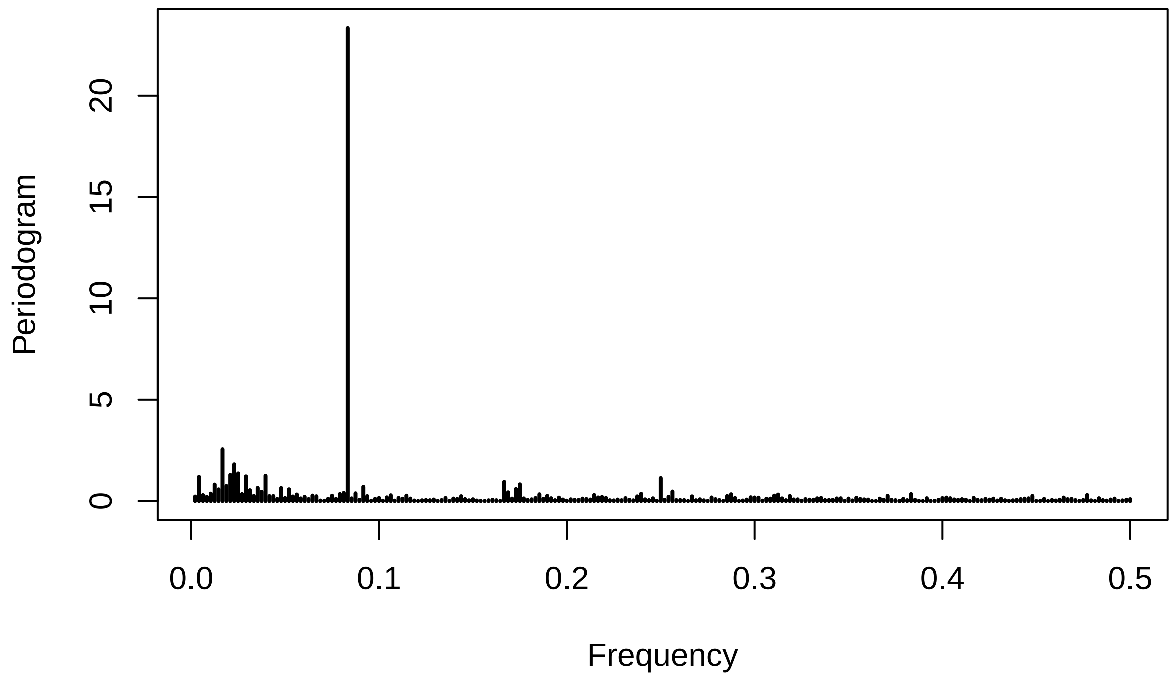

```r
detrended <- soi - predict(model, time(soi))

plot(detrended, main="SOI detrended")
```




```r
library(TSA)
```

```
##
## Attaching package: 'TSA'

## The following objects are masked from 'package:stats':
##
##     acf, arima

## The following object is masked from 'package:utils':
##
##     tar
```

```r
pg <- periodogram(detrended)
```



Two peaks, first at period:

```
max_index <- which(pg$spec == max(pg$spec))
freq <- pg$freq[max_index]

1/freq
```

```
## [1] 12
```

Or every year. Second at period:

```
freq <- pg$freq[which(pg$spec[1:(max_index - 1)] == max(pg$spec[1:(max_index - 1)]))]

1/freq
```

```
## [1] 60
```

Or every five years, which is the probable El Nino cycle.

3.2

a)

By expanding “backwards” in time, we have

$$\begin{aligned}
x_t &= \phi x_{t-1} + w_t \\
&= \phi(\phi x_{t-2} + w_{t-1}) + w_t \\
&= \phi^2 x_{t-2} + \phi w_{t-1} + w_t \\
&\vdots \\
&= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}
\end{aligned}$$

When $k = t$ we get

$$\begin{aligned}
x_t &= \phi^t x_0 + \sum_{j=0}^{t-1} \phi^j w_{t-j} \\
&= \phi^t w_0 + \sum_{j=0}^{t-1} \phi^j w_{t-j} \\
&= \sum_{j=0}^{t} \phi^j w_{t-j}
\end{aligned} \tag{1}$$

b)

We have

$$\begin{aligned}
E(x_t) &= E(\sum_{j=0}^{t} \phi^j w_{t-j}) \\
&= \sum_{j=0}^{t} \phi^j E(w_{t-j}) = 0
\end{aligned} \tag{2}$$

as $E(w_i) = 0 \, \forall i$

c)

As $Var(w_i) = \sigma_w^2 \, \forall i$, we have

$$
\begin{aligned}
Var(x_t) &= Var(\sum_{j=0}^{t} \phi^j w_{t-j}) \\
&= \sum_{j=0}^{t} (\phi^j)^2 Var(w_{t-j}) \\
&= \sigma_w^2 \sum_{j=0}^{t} \phi^{2j} \\
&= \sigma_w^2 \frac{1-\phi^{2(t+1)}}{1-\phi^2}
\end{aligned} \tag{3}
$$

by geometric series formula, as $|\phi| < 1 \implies |\phi^2| < 1$.

d)

$$
\begin{aligned}
cov(x_{t+h}, x_t) &= E[(\sum_{j=0}^{t+h} \phi^j w_{t+h-j})(\sum_{j=0}^{t} \phi^j w_{t-j})] \\
&= E[(w_{t+h} + \cdots + \phi^h w_t + \phi^{h+1} w_{t-1} + \cdots + \phi^{h+t} w_0)(w_t + \phi w_{t-1} + \cdots + \phi^t w_0]) \\
&= \sigma_w^2 \sum_{j=0}^{t} \phi^{h+2j} \\
&= \sigma_w^2 \phi^h \sum_{j=0}^{t} \phi^{2j} \\
&= \phi^h \sigma_w^2 \frac{1-\phi^{2(t+1)}}{1-\phi^2} \\
&= \phi^h Var(x_t)
\end{aligned} \tag{4}
$$

e) We have $E(x_t) = 0$ constant, however $\gamma(h) = \phi^h Var(x_t) = \phi^h \sigma_w^2 \frac{1-\phi^{2(t+1)}}{1-\phi^2}$ depends on time. Hence, the series is not stationary.

f) We have $t \to \infty \implies \gamma(h) \to \phi^h \sigma_w^2 \frac{1}{1-\phi^2}$ depends only on h and not on t, hence it is "asymptotically stationary."

g) As we have just proved that this process estimates an AR(1) process, a AR(1) process could be simulated by simulating n i.i.d $N(0,1)$ noise values as $w_t, t = 1..n$ and then calculating x_t by

$$
\begin{cases}
x_t = \phi x_{t-1} + w_t, t = 1..n \ldots \\
x_0 = w_0
\end{cases} \tag{5}
$$

h) We have, with $k = t$,

$$
\begin{aligned}
x_t &= \phi^t x_0 + \sum_{j=0}^{t-1} \phi^j w_{t-j} \\
&= \phi^t w_0/\sqrt{1-\phi^2} + \sum_{j=0}^{t-1} \phi^j w_{t-j} \implies \\
Var(x_t) &= \frac{(\phi^t)^2}{1-\phi^2}\sigma_w^2 + \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} \\
&= \frac{\sigma_w^2}{1-\phi^2}(\phi^{2t} + 1 - \phi^{2t}) \\
&= \frac{\sigma_w^2}{1-\phi^2}
\end{aligned}
\tag{6}
$$

So $Var(x_t)$ is constant, and $E(x_t)$ is still constant, hence the series is stationary.

3.6

We have

$$
\begin{aligned}
x_t &= -.9x_{t-2} + w_t \implies \\
x_t + .9x_{t-2} &= w_t \implies \\
(1+0.9B^2)x_t &= w_t
\end{aligned}
\tag{7}
$$

So $\phi(z) = 1 + 0.9z^2$, which has the roots z_i:

```
(z <- polyroot(c(1,0,0.9)))
```

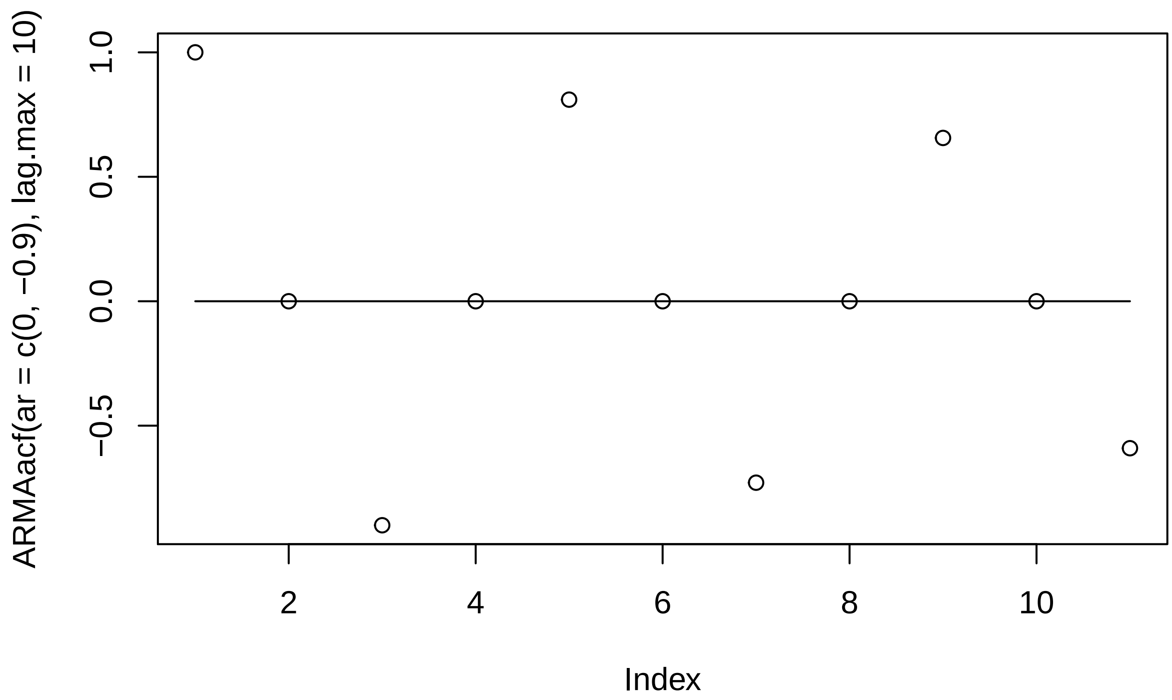
```
## [1] 0+1.054093i 0-1.054093i
```

```
abs(polyroot(c(1,0,0.9)))
```

```
## [1] 1.054093 1.054093
```

Hence $|z_i| > 1 \, \forall i$ and z_i complex conjugate, as such the acf will have periodic behavior, see plot.

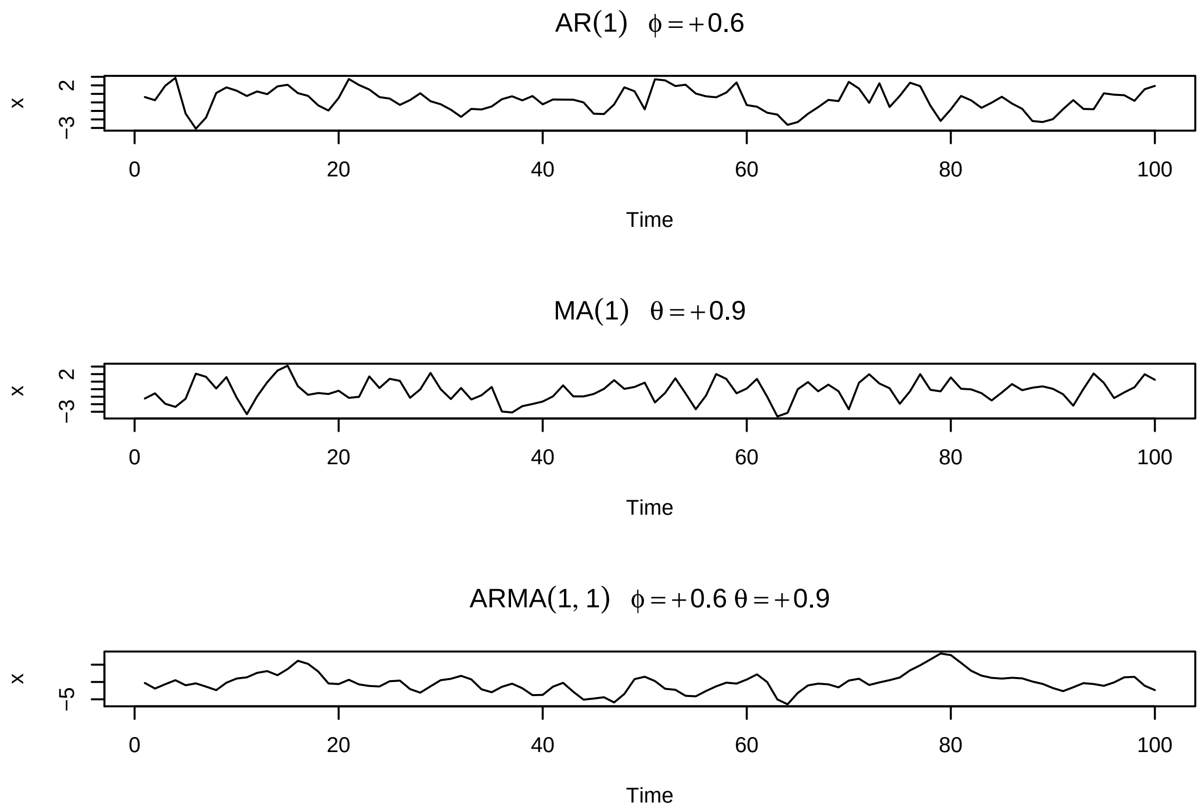
```
plot(ARMAacf(ar = c(0, -0.9), lag.max = 10))
lines(rep(0, 11))
```



3.9

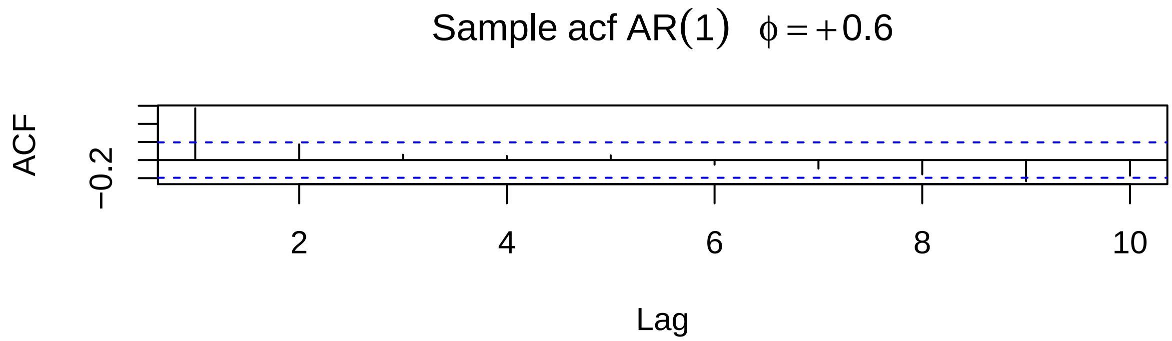
First we'll plot the simulated series

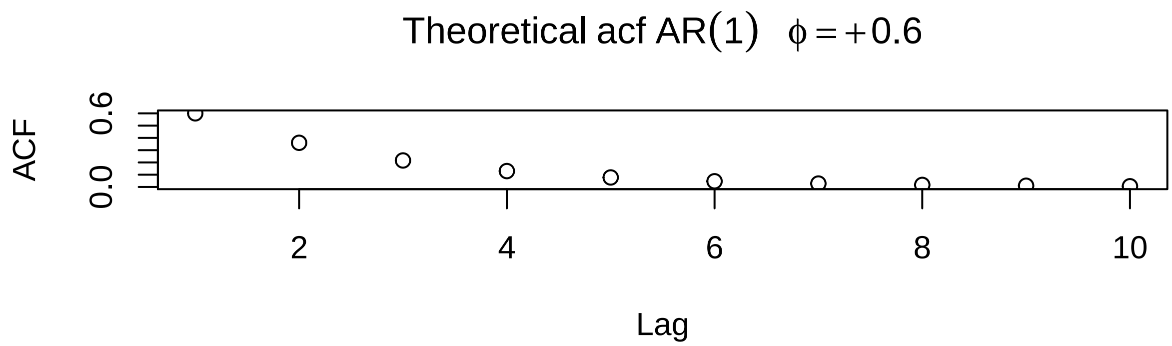
```r
set.seed(2020)
ar <- arima.sim(list(order=c(1,0,0), ar=.6), n=100)
ma <- arima.sim(list(order=c(0,0,1), ma=.9), n=100)
arma <- arima.sim(list(order=c(1,0,1), ar=.6, ma=.9), n=100)

par(mfrow=c(3, 1))
plot(ar, ylab="x", main=(expression(AR(1)~~~phi==+.6)))
plot(ma, ylab="x", main=(expression(MA(1)~~~theta==+.9)))
plot(arma, ylab="x", main=(expression(ARMA(1, 1)~~~phi==+.6~theta==+.9)))
```



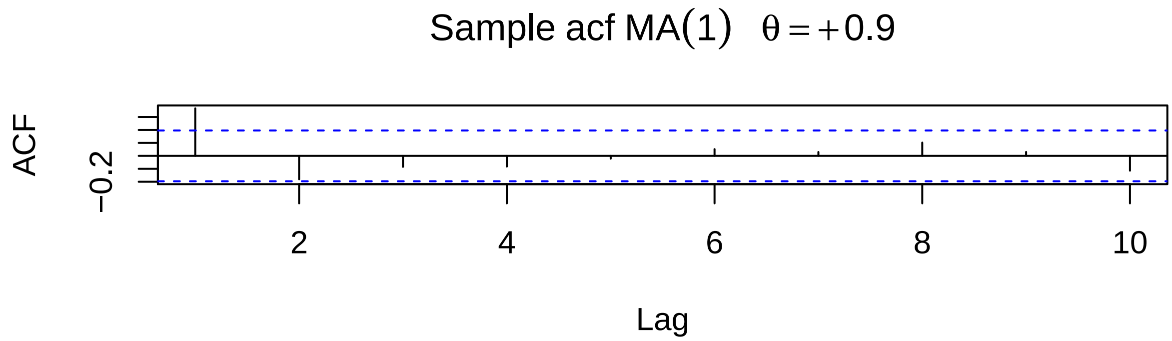
Next, we'll plot and compare the sample acf with the theoretical acf.

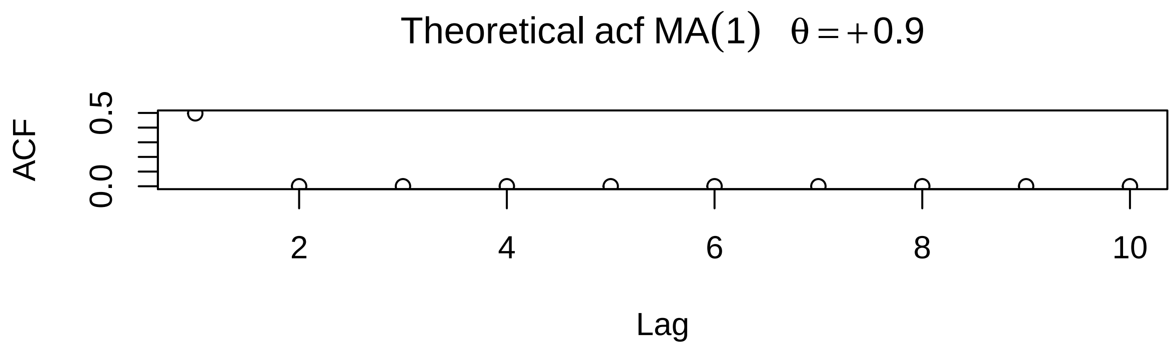
```
par(mfrow=c(2, 1))
plot(acf(ar, lag.max = 10, plot=FALSE), main=(expression(Sample~acf~AR(1)~~~phi==+.6)))
plot(ARMAacf(ar = 0.6, lag.max = 10)[-1], main=(expression(Theoretical~acf~AR(1)~~~phi==+.6)), ylab = "
```



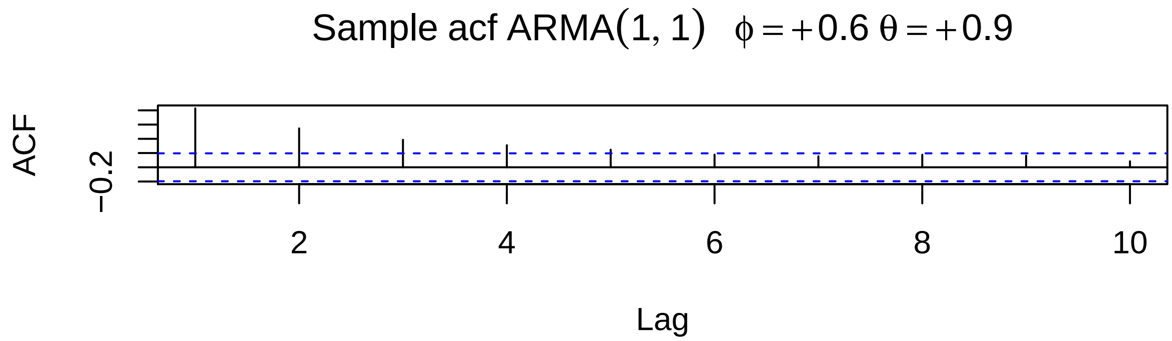


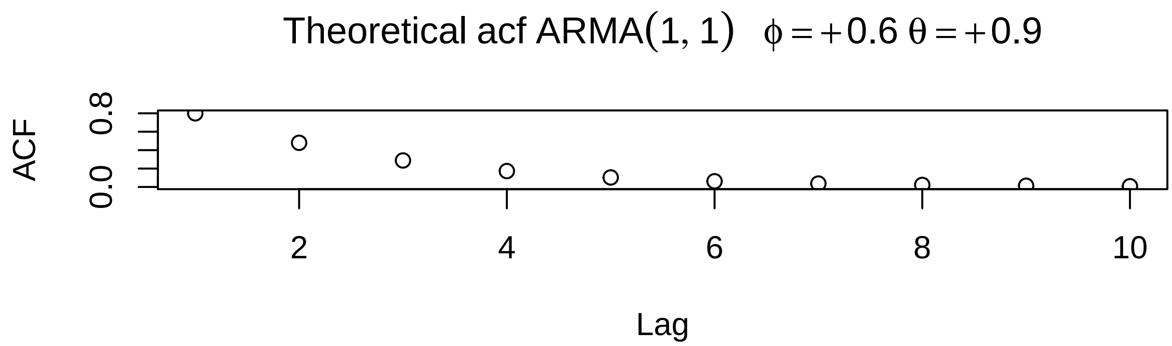

```
par(mfrow=c(2, 1))
plot(acf(ma, lag.max = 10, plot=FALSE), main=(expression(Sample~acf~MA(1)~~~theta==+.9)))
plot(ARMAacf(ma = 0.9, lag.max = 10)[-1], main=(expression(Theoretical~acf~MA(1)~~~theta==+.9)), ylab =
```





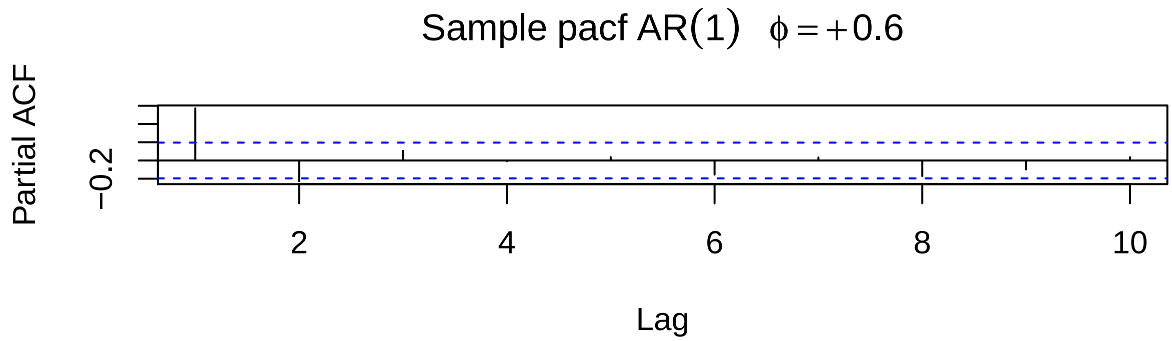

```
par(mfrow=c(2, 1))
plot(acf(arma, lag.max = 10, plot=FALSE), main=(expression(Sample~acf~ARMA(1, 1)~~~phi==+.6~theta==+.9))
plot(ARMAacf(ar = 0.6, ma = 0.9, lag.max = 10)[-1], main=(expression(Theoretical~acf~ARMA(1, 1)~~~phi==
```

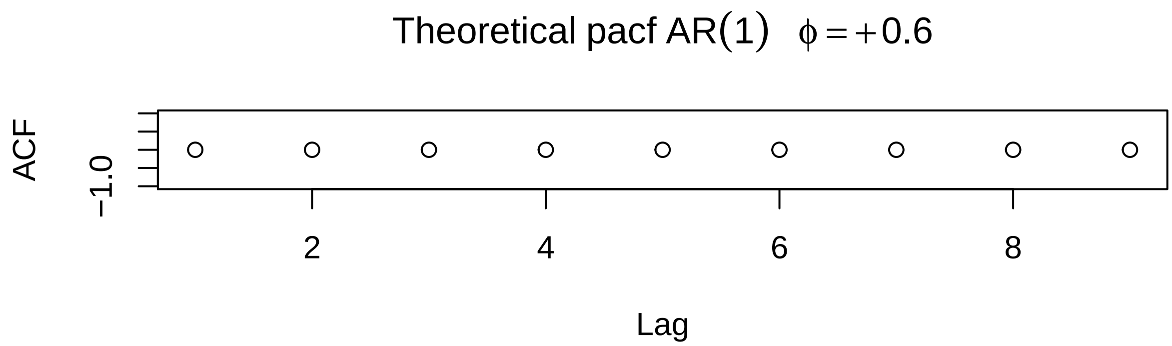




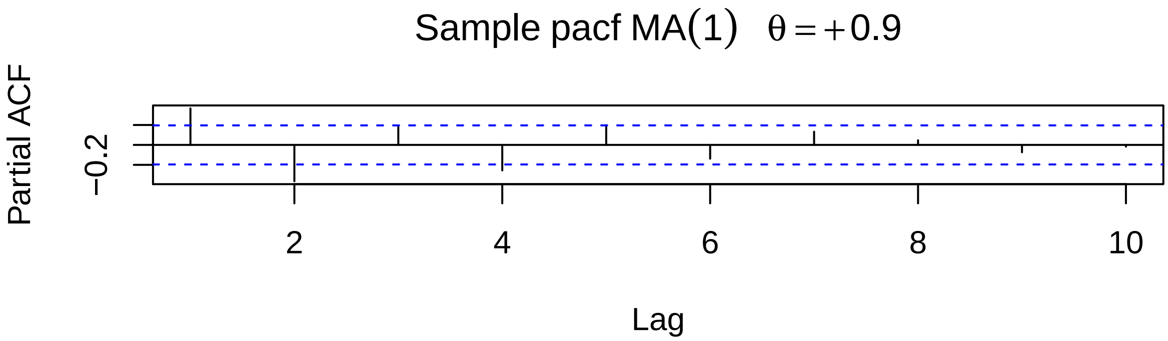
The plots are consistent with the general results of Table 3.1, as the AR(1) as well as ARMA(1,1) process tails off while the MA(1) cutts of after lag 1= q.

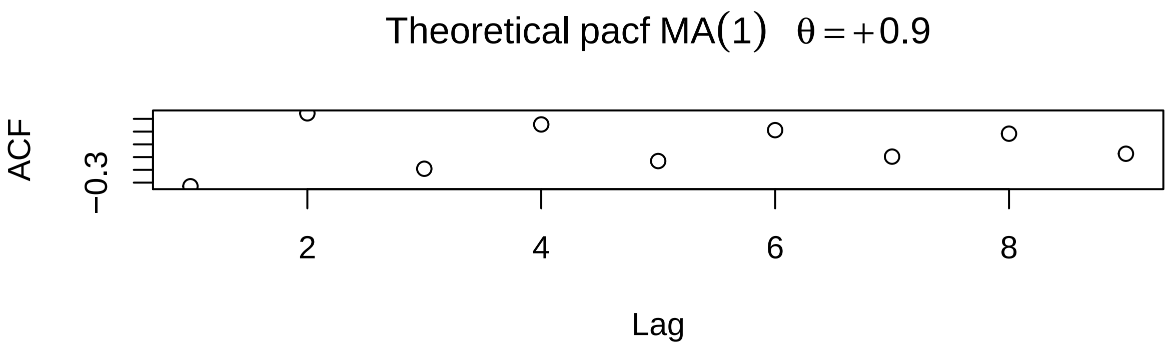
```
par(mfrow=c(2, 1))
plot(pacf(ar, lag.max = 10, plot=FALSE), main=(expression(Sample~pacf~AR(1)~~~phi==+.6)))
plot(ARMAacf(ar = 0.6, lag.max = 10, pacf = TRUE)[-1], main=(expression(Theoretical~pacf~AR(1)~~~phi==+
```



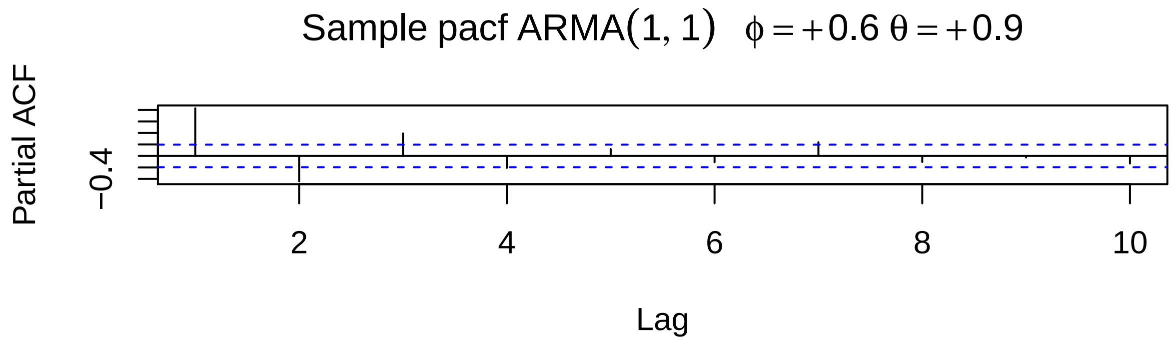


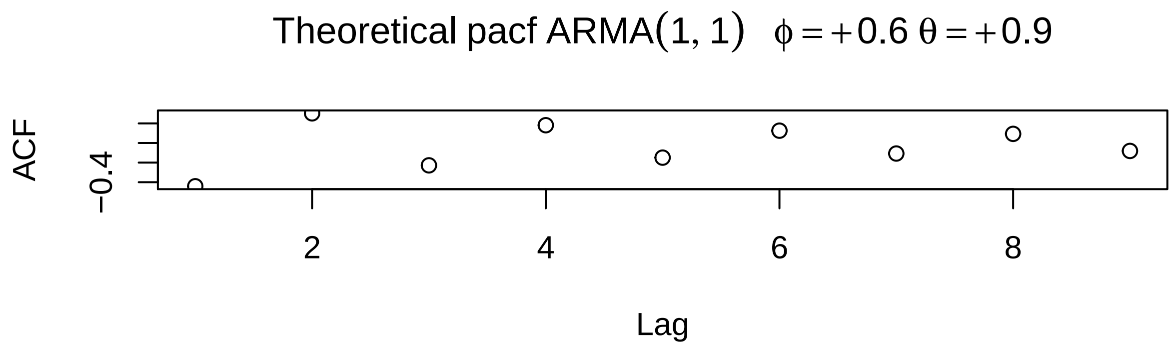

```
par(mfrow=c(2, 1))
plot(pacf(ma, lag.max = 10, plot=FALSE), main=(expression(Sample~pacf~MA(1)~~~theta==+.9)))
plot(ARMAacf(ma = 0.9, lag.max = 10, pacf = TRUE)[-1], main=(expression(Theoretical~pacf~MA(1)~~~theta=
```






```
par(mfrow=c(2, 1))
plot(pacf(arma, lag.max = 10, plot=FALSE), main=(expression(Sample~pacf~ARMA(1, 1)~~~phi==+.6~theta==+.
plot(ARMAacf(ar = 0.6, ma = 0.9, lag.max = 10, pacf = TRUE)[-1], main=(expression(Theoretical~pacf~ARMA
```





The plots are consistent with the general results of Table 3.1, as the MA(1) as well as ARMA(1,1) process tails off while the AR(1) cutts of after lag 1= q.