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Accounts and models for spatial demographic analysis 3: rates and life tables

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Abstract. The paper begins by distinguishing, with the aid of the Lexis diagram that plots age against time, three kinds of demographic rate: age-group rates, period rates, and life-table rates. There are single-region and multiregion versions of those rates. In order to measure multiregional life-table rates, life-table accounts are developed together with an accounts based model that estimates the full accounts matrix from available data. These multiregional rates are then used to construct multiregional life tables akin to those recently proposed by Rogers. It is shown that the calculations involved in measuring the survivorship probabilities of the life table can be succinctly summarized in Stone's fundamental matrix. The detailed connections between life-table accounts and age-group accounts are explored, and the possibility of age-group life tables raised. The conclusion is reached that the age-group accounts are the appropriate ones for generating rates for use in population projection models, and that the life-table accounts are the appropriate ones for generating rates for use in actuarial calculations.

1 Introduction

In earlier papers (Rees and Wilson, 1973, and Wilson and Rees, 1974), we introduced accounting concepts into spatial demographic analysis and showed how, given certain assumptions, models could be built both for historical analysis and for projection. We paid particular attention to the proper definition of the various demographic rates, and our work in this respect has relevance both to single-region and multiregion demographic models. In section 2 of this paper, we review our definitions using Lexis (1875) diagrams and using the methods of Pressat (1972). This shows that there are three kinds of rates: age-group rates, period rates and life-table rates. We use these names for the three kinds of rates although later in the paper it turns out that they are rather inexact descriptions. The terms age-group-cohort rates, noncohort-period rates and exact age-cohort rates are more precise descriptions, but not used because they are cumbersome. The first two types were employed in our previous papers, and we use the third type to connect our earlier papers with more traditional 'life-table' demography, including the multiregion work developed by Rogers (1973a, 1973b). The connections are established in section 3, where we introduce the notion of 'life-table accounts'. In section 4, we develop an accounts based model for multiregional life tables. The connections between life-table and age-group accounts are made in section 5. Sections 6 and 7 contain outlines of how multiregional life tables can be constructed from a set of rates based on accounts. Section 8 raises the possibility of age-group life tables and a number of concluding comments are made in section 9.

In our second paper in this series, we presented our results for a general case, where age groups in the population could have any length (as could the time period for the analysis) provided they were consecutive, mutually exclusive and exhaustive, but noted also the advantages of the simple case where all age groups except the open-ended last age group are of equal length, the length being equal to the time period. In this paper we use the simpler assumptions, partly for ease of presentation and partly because there is little reason in life-table analysis to do anything else.

2 Rates

The basic Lexis diagram for a single region is shown in figure 1. There is one time period, t to $t+T$, labelled θ_t^T and of length T, on the simple assumption that age groups 1, 2, 3, ..., Ω correspond to age intervals $(0, T)$, $(T, 2T)$, ..., with the last group considered open ended. If the time and age scales are equal a lifeline is then a 45° line, examples of which are shown. The start of a line, a birth, is shown by a diamond symbol (◇) and must, of course, occur at age zero. The end of a line, a death, is shown as a black dot (●). There is a lifeline for each individual. The Lexis diagram can easily be extended to cover several time periods and several regions (cf. Rogers, 1973b), as shown in figure 2. Additionally, a *move* can now be shown from one region to another. Definitions of migration have to be constructed very carefully in relation to such moves, and we shall return to this question shortly.

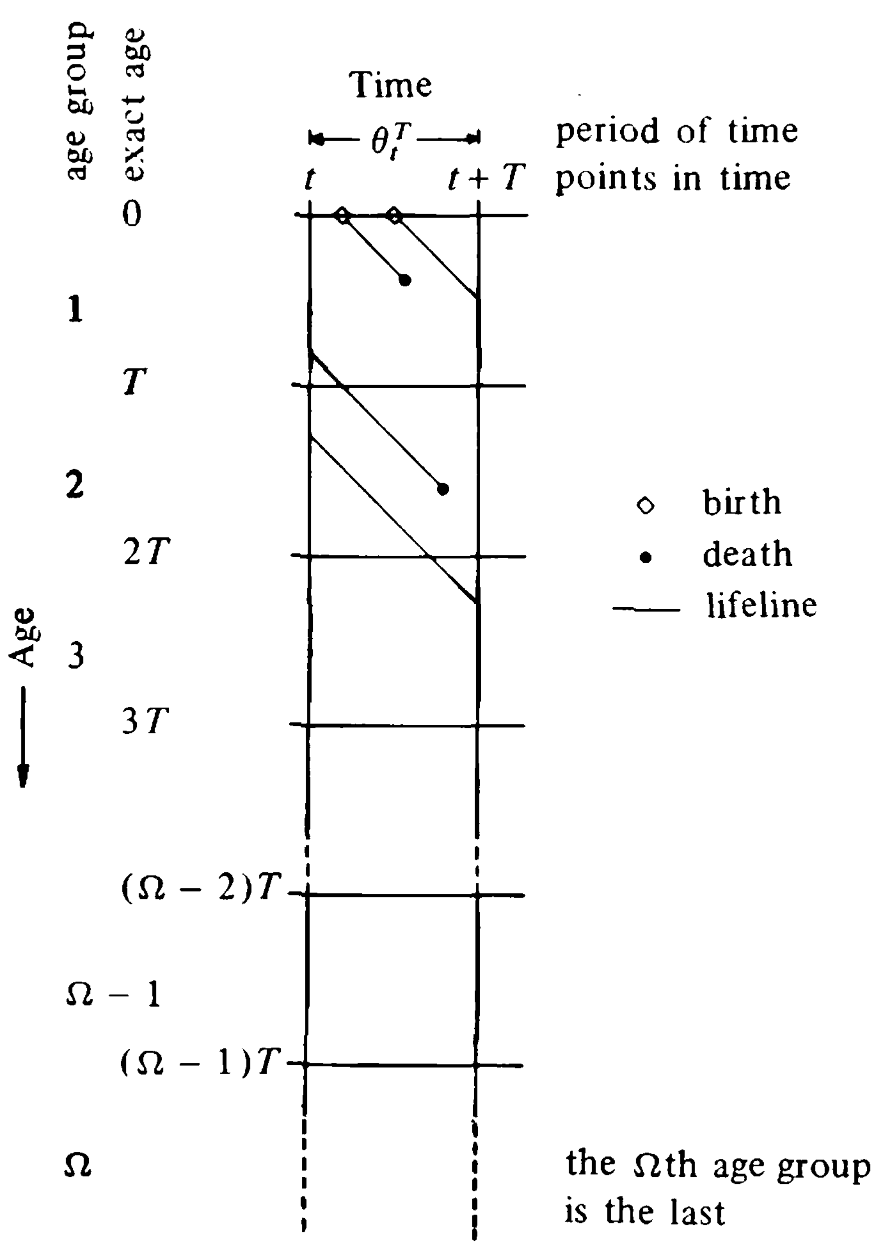

Figure 1. The basic Lexis diagram.

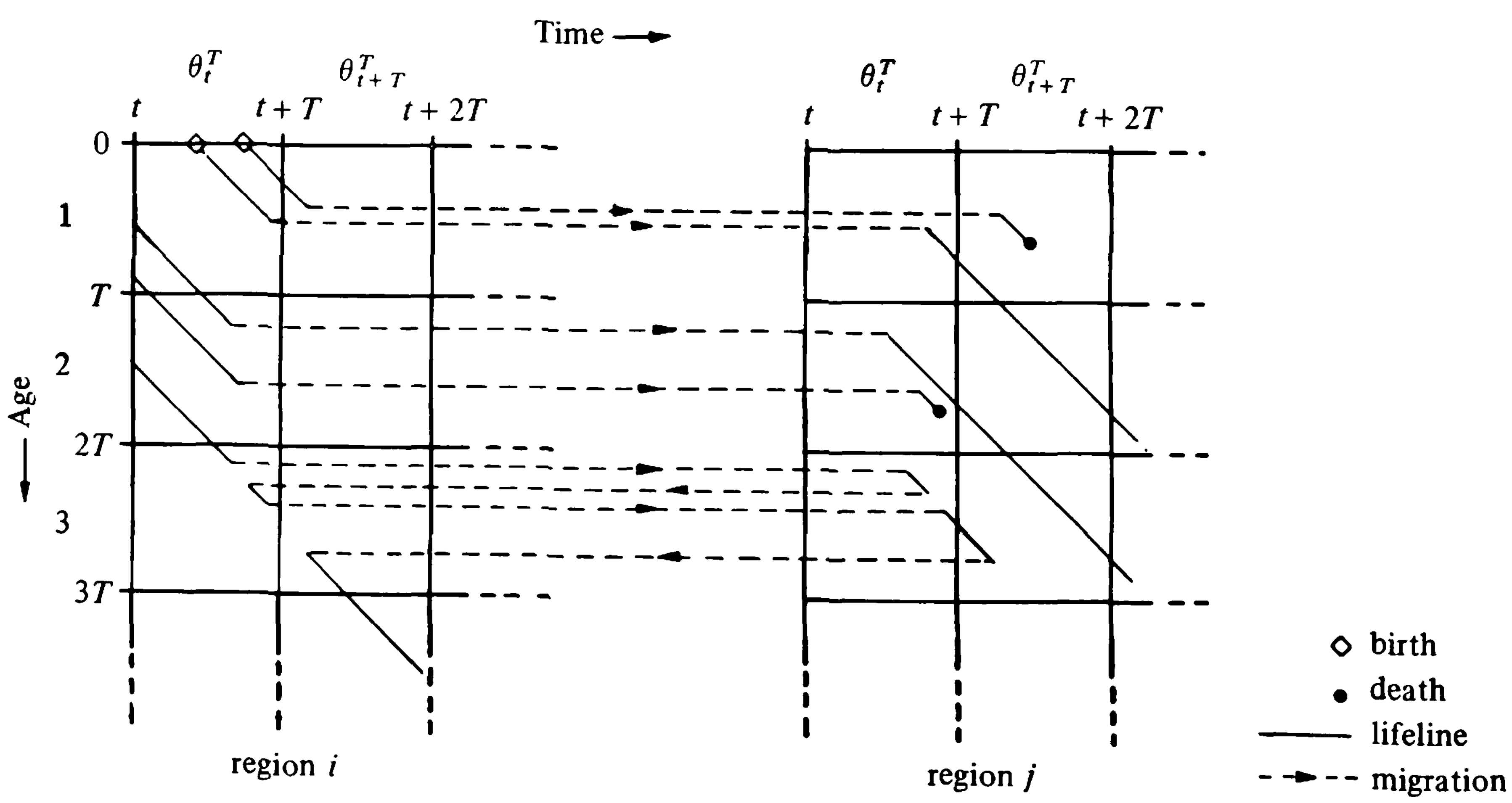

Figure 2. The multiregional Lexis diagram.

It can now be easily seen that, for the two-region example of our first paper, the sixteen different types of demographic event can be represented on such diagrams. This is done in figure 3. The eight flows originating in region i are shown in the top part of the diagram and the eight flows starting in the rest of the world are shown in the bottom part. We recall that K refers to population, $\beta(i)$ to a birth in location i, $\delta(i)$ means death in region i, an unmodified i, R (or j, k etc.) refers to existence in that region at the beginning of the period, if it is on the left of the superscript list, and to survival in the region at the end of the period, if it is on the right of the superscript list. The time labels t and $t+T$, with t referring to the beginning of a period T years in length and $t+T$ to the point in time that ends the period, are understood. Thus,

$K^{\beta(i)i}$ refers to persons born in region i who survive at the end of the period there,
$K^{\beta(i)\delta(i)}$ refers to persons born in region i who die there before the end of the period,
$K^{\beta(i)R}$ refers to persons born in region i who migrate to region R and survive there at the end of the period,
$K^{\beta(i)\delta(R)}$ refers to persons born in region i who migrate to region R and die there,

with similar definitions for $K^{\beta(R)R}$, $K^{\beta(R)\delta(R)}$, $K^{\beta(R)i}$, and $K^{\beta(R)\delta(i)}$. For persons alive at time t we recognize that

K^{ii} denotes persons alive in region i at time t who survive there at time $t+T$,
$K^{i\delta(i)}$ denotes persons alive in region i at time t who die there in the period,
K^{iR} denotes persons alive in region i at time t who migrate to region R and survive there at time $t+T$,
$K^{i\delta(R)}$ denotes persons alive in region i at time t who migrate to region R and die there in the period,

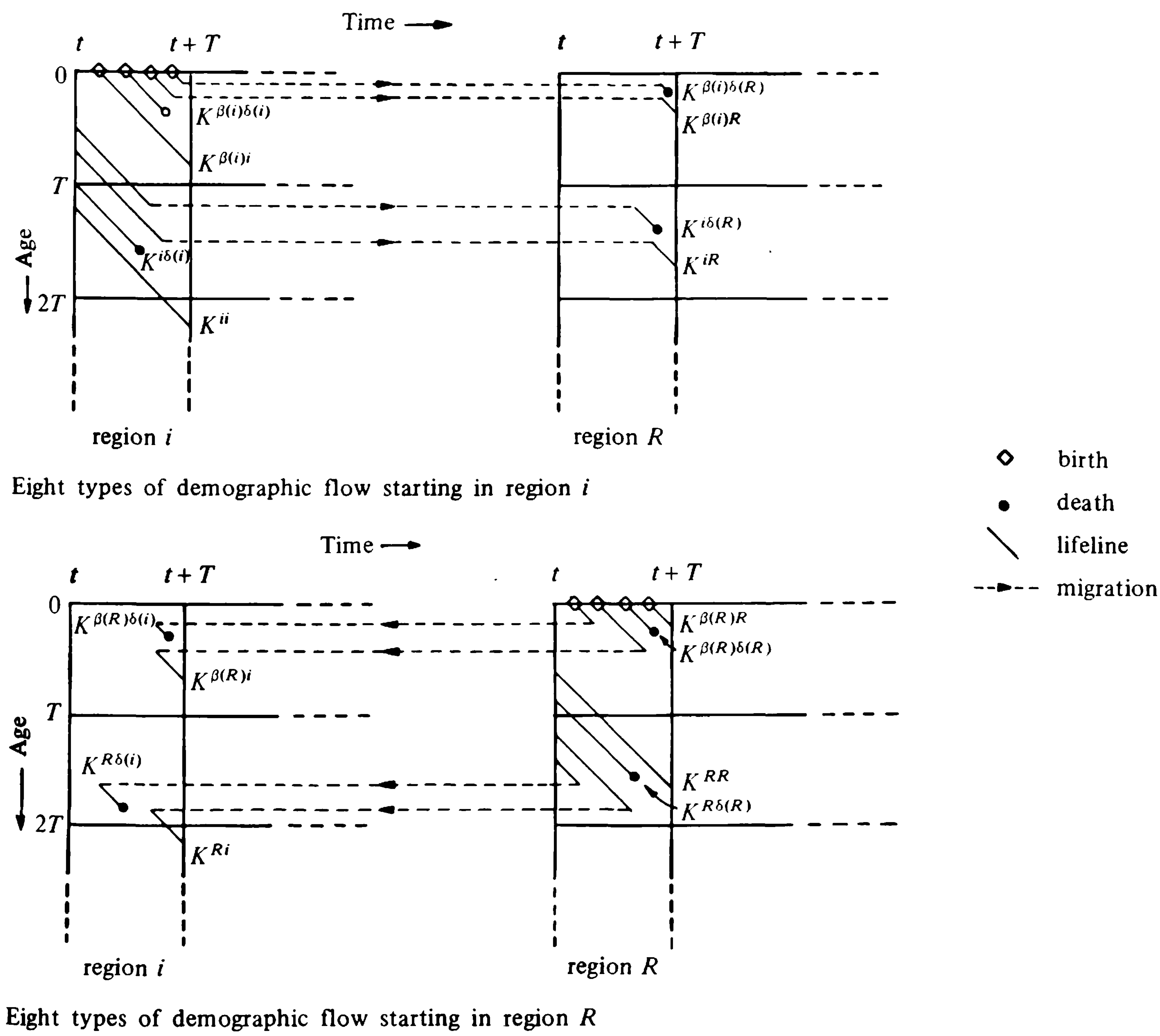

Figure 3. The sixteen demographic flows shown in a multiregional Lexis diagram.

with similar definitions for K^{RR}, $K^{R\delta(R)}$, K^{Ri}, and $K^{R\delta(i)}$. These definitions are stated for two regions, i (of interest) and R (the rest of the world), but can be applied to any set of regions $i, j, k, ..., N$. Once these flows are placed in a Lexis diagram it means we have attached age-group labels to the initial and final state definitions. Age disaggregation was discussed in detail in our second paper (Wilson and Rees, 1974).

We shall now use particular sections of Lexis diagrams for a single closed region to show how different kinds of demographic rates can be defined and measured, beginning with death rates. In figure 4(a) part of a Lexis diagram is shown for two age groups, r and $r+1$, and two time periods. The exact age of a person when crossing the line AB is $(r-1)T$, when crossing EF is rT, and when crossing DC is $(r+1)T$. We will call these ages x, $x+T$, and $x+2T$ for convenience. In figure 4(b) we focus on the parallelogram AFCE; that is, on people who were in age-group r at time t. In our previous accounting notation, if the single region is designated as i, there are $K_{r*}^{i*}(t)$ such people at time t. The asterisks refer to summation over possible final states, locations, and age groups. Those who die during the period t to $t+T$ do so either in age group r (ΔAFE) or age group $r+1$ (ΔEFC), and those numbers are $K_{rr}^{i\delta(i)}$ and $K_{rr+1}^{i\delta(i)}$ respectively. Note that the first age-group subscript refers to the beginning of the period and the second to age group at time of death. Corresponding death rates could then be defined as

$$d_{rr}^{ii} = \frac{K_{rr}^{i\delta(i)}}{K_{r*}^{i*}}\,, \tag{1}$$

and

$$d_{rr+1}^{ii} = \frac{K_{rr+1}^{i\delta(i)}}{K_{r*}^{i*}}\,. \tag{2}$$

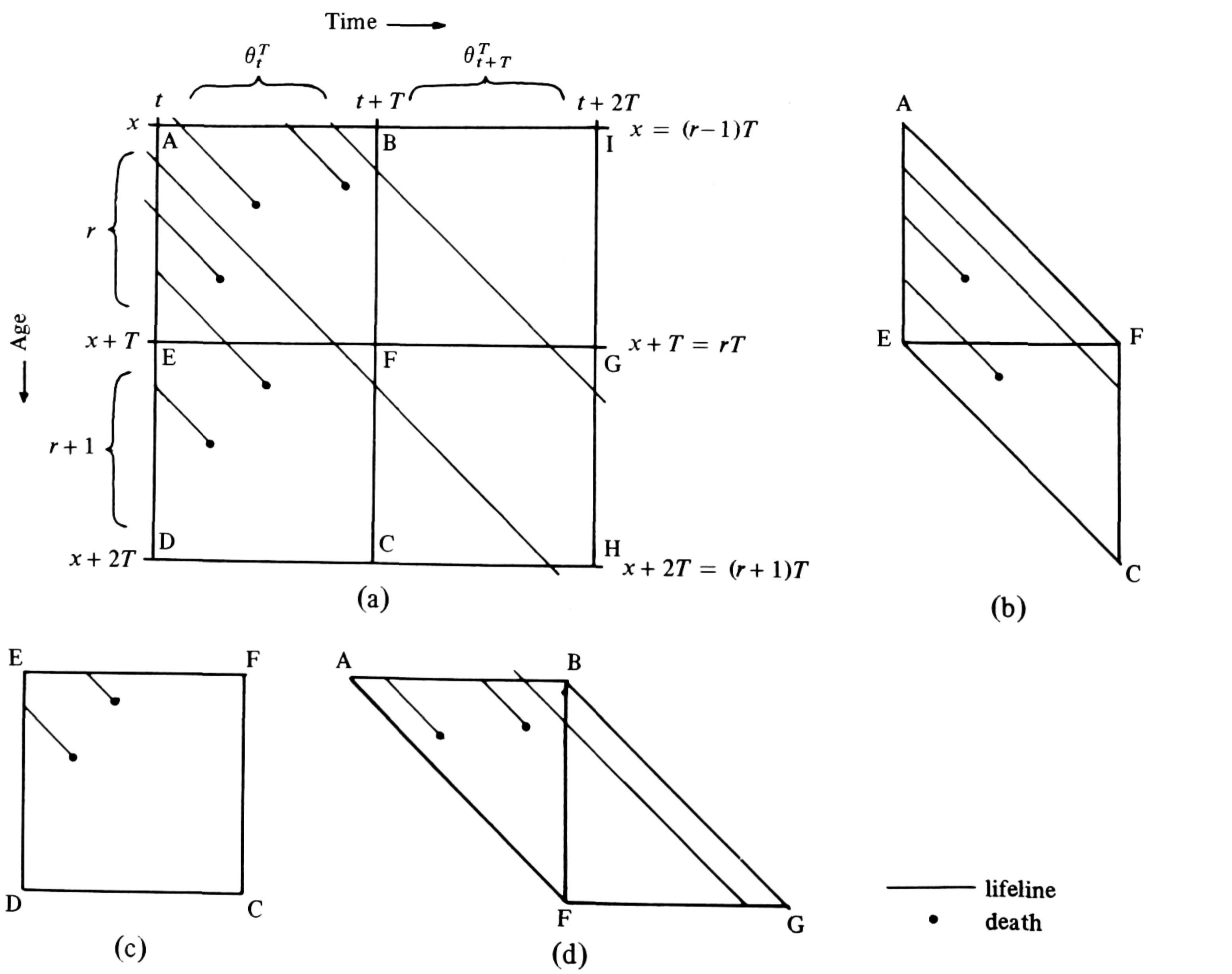

Figure 4. Portions of a Lexis diagram used to illustrate the different rate definitions.

The death rates and death figures refer to period t to $t+T$, and the population in age group r at time t. A death rate for all those in age group r at time t, irrespective of age at death, would be

$$d_{r*}^{ii} = \frac{K_{rr}^{i\delta(i)} + K_{rr+1}^{i\delta(i)}}{K_{r*}^{i*}} \ . \tag{3}$$

These are the rates which turn up in the transition-rate form of the account-based model, but they are not calculated directly because deaths are actually measured by age group at death and not age group at the beginning of the period. Such rates may be called *age-group* rates.

Now consider figure 4(c), which corresponds to the way in which deaths are measured by the Registrars General in Britain. The area EFCD contains all the deaths in age group $r+1$ in the time period t to $t+T$. This total is $K_{*r+1}^{i\delta(i)}$. The population at-risk for this set of deaths is partly made up of those in age group r at the beginning of the period and partly those in age group $r+1$. We previously denoted this by $\hat{K}_{*r+1}^{\mathrm{D}ii}$, and the corresponding death rate would be

$$d_{\square r+1}^{ii} = \frac{K_{*r+1}^{i\delta(i)}}{\hat{K}_{*r+1}^{\mathrm{D}ii}} \ . \text{ (1)} \tag{4}$$

This can be called a *period* death rate. In the account-based model of our previous paper, we used these rates simply because they could be measured directly, and showed how such rates as (1), (2), and (3) could be constructed within the model by using additional assumptions.

Now consider figure 4(d). This identifies the groups of people who attained exact age x sometime during the period t to $t+T$, and the deaths of those who die before they reach exact age $x+T$. Let Q_{x*}^{i*} denote the initial population, and let $Q_{xx+T}^{i\delta(i)}$ be the corresponding number of deaths between exact ages x and $x+T$. Then an appropriate rate is

$$q_{xx+T}^{ii} = \frac{Q_{xx+T}^{i\delta(i)}}{Q_{x*}^{i*}} \ , \tag{5}$$

and it can easily be seen that this is the conventional *life-table rate*. Note that data for two time periods would be needed here.

The three rates are all based on averages. The age-group rate focuses on people who were in an age group at time t, the beginning of the period; the period rate on the population who could have died in age groups during the period; and the life-table rate on the population who attained exact age x during the period t to $t+T$, that is in θ_t^T. These rates are used in different kinds of models. A transition-rate model, such as that of Leslie (1945) or Rogers (1966), demands the use of age-group rates; a life-table model unsurprisingly focuses on 'birth' cohorts, and demands the use of life-table rates. Period rates are used within our own account-based model because they are measurable directly. Age-group rates can also be used. Note, however, that if the account-based model, or some similar procedure, is not used to generate rates such as transition rates, then either more detailed data needs to be made available (as happens in France) or some assumptions have to be made. For example, a transition rate could be calculated [cf. figure 4(b)] if the deaths in ΔAEF

(1) An asterisk replacing a subscript, as we have seen, denotes summation. A $\square$ replacing a subscript on rate variables also implies aggregation over the subscript replaced, but it would not be direct summation. The aggregated rate is then a weighted average of the corresponding disaggregated rates, though usually such aggregated rates are defined directly. Occasionally an asterisk remains appropriate for rates as in equation (3). The notation is a development of that in Wilson and Rees (1974), where an asterisk was used for both purposes.

were taken as half the measured period deaths for age group r (that is, half of ABFE) and those in ΔEFC taken as half the measured period deaths in age group $r+1$ (that is, half of EFCD).

Birth rates in demography are rates at which potential mothers give birth, and an exactly analogous argument to that for deaths can be given in that case. Migration rates generate some harder problems which we shall now consider; we must then return to birth and death rates to see what happens in the multiregion case.

We begin by recalling two features of the definition of migration used in transition-rate or account-based models. First, multiple moves are discounted: migration refers to any change from location at time t to location at time $t+T$. Second, migration—as measured—means 'migration and survival'. On figure 2 actual moves were represented. For the definition of migration now under discussion, therefore, a modified figure 2 can be generated: all multiple moves should be removed and, at any interregion transition, shown as a single horizontal line between 'initial' region and 'final' region. As a convention, this time could be placed at the average time at which a migration took place. This would be the midpoint for 'migration and survival' flows, but earlier for 'migration and death' flows, and later for 'birth and migration' and 'birth, migration and death' flows. Such lifelines are shown in figure 5(a). The migration *rates* which are measured in this way are clearly *age-group rates*, and this fact is very convenient for transition-rate and account-based models.

It would be possible to define *period* migration rates, but it is easy to see that they are rarely measured (except in studies of international migration) and are not used in any current models; so we do not proceed any further with that concept here. It is, however, necessary to explore what a *life-table* migration rate looks like. Again, we discount multiple moves and examine location at the beginning and end of a period; but in this case the period is determined, for an individual, by the time T between one exact age and another, say x (or birth) and $x+T$ (or death if earlier). Thus we

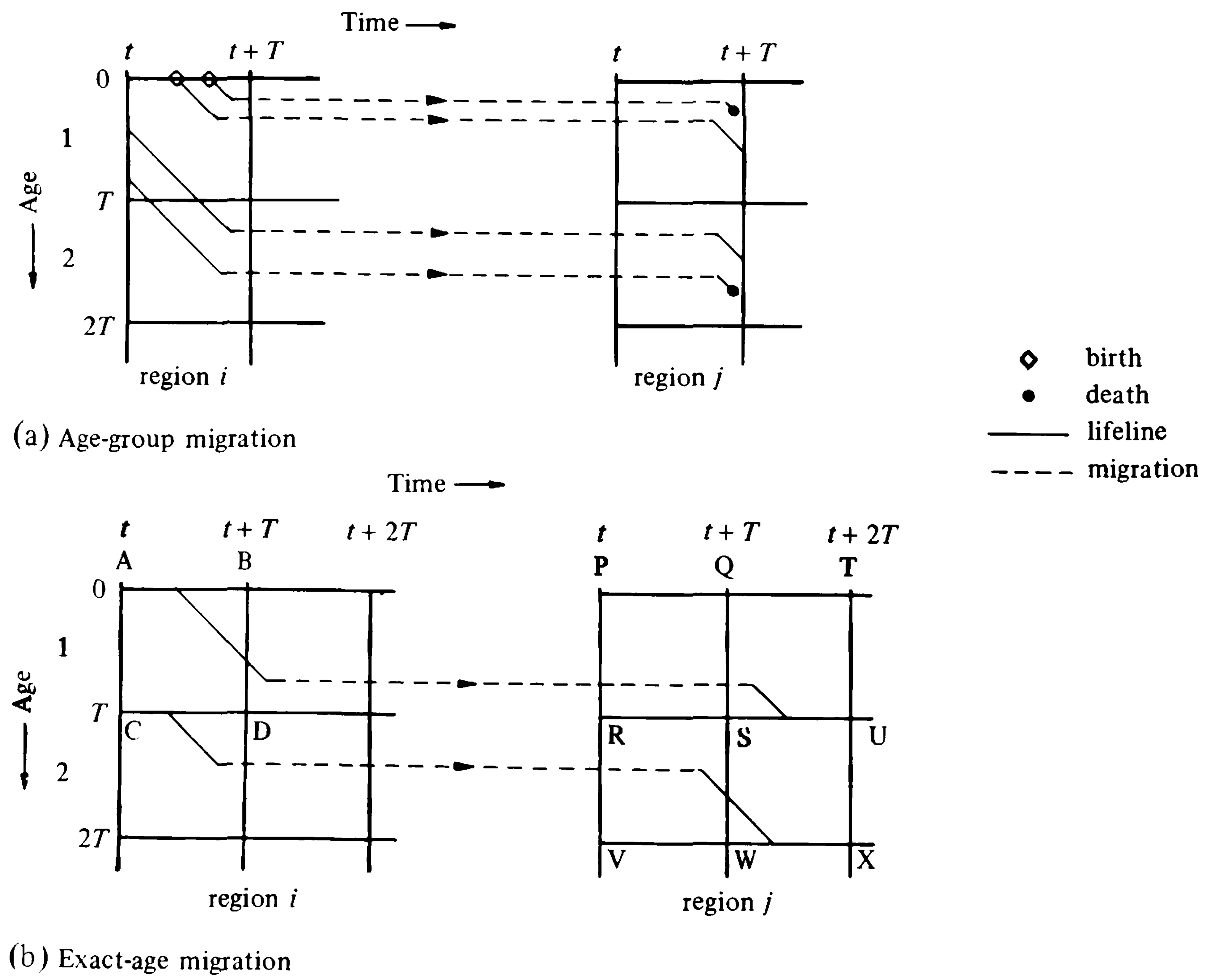

Figure 5. Lexis diagrams showing age-group migration and exact-age migration.

could define 'life-table' migration in this way, discounting multiple moves, and examples of such life lines are shown in figure 5(b). The problem is that no migration data are collected in such a way, and so it will be necessary, for life-table work, to find some way of estimating life-table migration flows from age-group migration data. It can be seen from figure 5(b) that the number of migrating lifelines crossing SU, the desired total would have to be taken as something like the average of those crossing QS and UX, which are measured quantities. But as in much life-table work, data for two time periods would be necessary for this.

The rates discussed in equations (1) to (5) above were examples of the different types of rates for a single region. We can now use the above definition of migration to construct rates for the multiregion case. For age-group and period rates this is a straightforward matter. The equations given still hold for i–i flows, while other rates could be defined by using flows and counts of deaths for migrants. Thus a version of equation (1) for $i \neq j$, giving a death rate for migrants to j, would be

$$d_{rr}^{ij} = \frac{K_{rr}^{i\delta(j)}}{K_{r*}^{i*}} , \qquad i \neq j , \tag{6}$$

and equation (4) would become

$$d_{\square r+1}^{ij} = \frac{K_{*r+1}^{i\delta(j)}}{\hat{K}_{*r+1}^{\mathrm{D}ij}} . \tag{7}$$

In the account-based model, of course, we make the additional assumption that

$$d_{\square r+1}^{ij} = d_{\square r+1}^{\square j} , \tag{8}$$

and use equation (7), within an iterative scheme, to calculate the unknown flow $K_{*r+1}^{i\delta(j)}$. When the accounts have been completed in this way, minor flow transition rates, such as that in equation (6), can be calculated if required.

It is more convenient to tackle the question of multiregion life-table rates more fully after the introduction of life-table accounts, and so we shall postpone this issue until the end of the next section.

3 Life-table accounts

Typical elements of the accounts introduced in our earlier paper were $K_{rs}^{ij}(t, t+T)$, $K_{rs}^{i\delta(j)}(t, t+T)$, $K_{rs}^{\beta(i)j}(t, t+T)$ and $K_{rs}^{\beta(i)\delta(j)}(t, t+T)$. These terms are more disaggregated versions of the K type flows defined in section 2. The r subscript refers to age group of the person at the beginning of the period in K_{rs}^{ij} and $K_{rs}^{i\delta(j)}$, and to age group of the mother at the beginning of the period in $K_{rs}^{\beta(i)j}$ and $K_{rs}^{\beta(i)\delta(j)}$. The s subscript refers to age group attained at the end of the period in K_{rs}^{ij} and $K_{rs}^{\beta(i)j}$, and to age group of death in the $K_{rs}^{i\delta(j)}$ and $K_{rs}^{\beta(i)\delta(j)}$ terms. These terms describe what can happen to the people who were in particular age groups at the beginning of the period. We saw from the discussion of life-table rates that life tables are concerned with groups of people attaining particular exact ages during the time period. We can construct variables, therefore, which 'account' for what can happen to such life-table groups between exact ages. The terms equivalent to the K's above could be defined as

$$Q_{x\,x+T}^{ij}(\theta_t^T, \theta_{t+T}^T) , \qquad Q_{x\,x+T}^{i\delta(j)}(\theta_t^T, \theta_{t+T}^T) , \qquad Q_{0T}^{\beta(i)j}(\theta_t^T, \theta_{t+T}^T) , \qquad Q_{0T}^{\beta(i)\delta(j)}(\theta_t^T, \theta_{t+T}^T) .$$

The first of these, for example, would be the number of people who attained exact age x during period θ_t^T in region i, and who attained exact age $x+T$ in region j during period θ_{t+T}^T.

If $i \neq j$, then the people are migrants in the 'life-table' sense defined in the previous section. The base year population total would be $Q_{x*}^{i*}(\theta_t^T)$ and the final total would be $Q_{*x+T}^{*j}(\theta_{t+T}^T)$.

In defining $Q^{i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T})$, we have to be careful with the second subscript $x+T$; the term, as it stands, means member of the Q^{i*}_{x*} population who died in region j before attaining exact age $x+T$ in period θ^T_{t+T}. It is tempting to try to define $x+T$ as some kind of 'age at death', as with $r+1$ in the $K^{i\delta(j)}_{rr+1}$ term, but this leads to complications and so we resist the temptation for the time being. The birth terms are self explanatory, though note that in this case the first exact-age group is, of course, 0, and the second, T. There is also a problem analogous to that of 'age at death' in defining a suitable 'age of mother' index. We shall return to this shortly.

The types of flows corresponding to various account elements are shown in sections of Lexis diagrams in figure 6. All flows relate in the obvious way shown to the 'life-table parallelogram', but in this representation we should note important differences between the events of death and migration on the one hand, and births on the other. This turns out to have a bearing on the 'age of mother' question. Death or a migration can take place anywhere along life lines in the basic parallelogram. Sometimes the event occurs in period θ^T_t, sometimes in θ^T_{t+T}. Births, however, all occur in period θ^T_t as shown in figure 6(e_1), the corresponding event for mothers aged x during θ^T_t, say, *maternity*, can occur, like death or migration, in θ^T_t or θ^T_{t+T}. In figure 6(e_2) we identify the lifelines of the mothers who refer to the *period* births. It can be seen that they are made up of two groups, those who attained age x in θ^T_{t-T} and gave birth in θ^T_t, and those who attained age x in θ^T_t and gave birth in θ^T_t. This creates some problems of defining the appropriate birth rate, to which we shall return

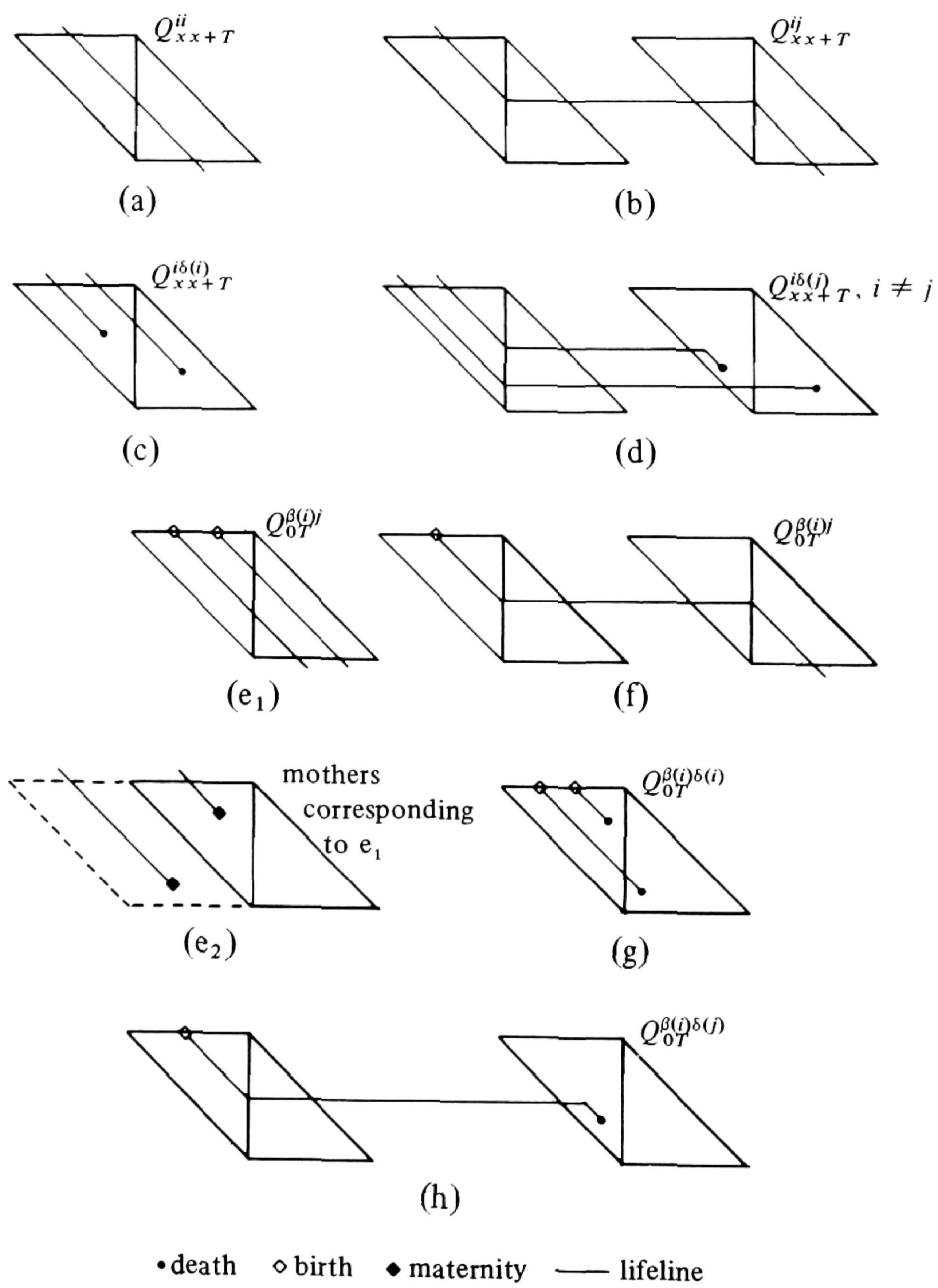

Figure 6. Elements of the accounts shown in a series of Lexis diagrams.

later. Meanwhile we can now disaggregate the terms

$$Q_{0T}^{\beta(i)j}(\theta_t^T, \theta_{t+T}^T) \qquad \text{and} \qquad Q_{0T}^{\beta(i)\delta(j)}(\theta_t^T, \theta_{t+T}^T)$$

to

$$Q_{xT}^{\beta(i)j}(\theta_t^T, \theta_{t+T}^T) \qquad \text{and} \qquad Q_{xT}^{\beta(i)j}(\theta_t^T, \theta_{t+T}^T) \ ,$$

where x now refers to 'age of mother'. It means achievement of exact age x during *either* θ_{t-T}^T *or* θ_t^T, *and* giving birth during θ_t^T.

With the variables now defined a set of life-table accounts can be constructed. These are presented in figure 7. The row and column totals are taken as terms such as $Q_{xx+T}^{i*}(\theta_t^T)$, $Q_{xT}^{\beta(i)*}(\theta_t^T)$, $Q_{xx+T}^{*i}(\theta_{t+T}^T)$, and $Q_{xx+T}^{*\delta(i)}(\theta_t^T, \theta_{t+T}^T)$. We show the summation on the appropriate superscript only, though the subscripts need careful interpretation. In $Q_{xx+T}^{i*}(\theta_t^T)$, for example, note that $x+T$ refers to 'survival *to* $x+T$' or 'death *before* $x+T$'. Note also that the deaths total needs two period labels, while the other totals all refer to a single period.

For each row and column, we can obtain an accounting equation in the usual way. It is sometimes convenient to distinguish the $x = 0$ and $x > 0$ cases, and we also distinguish $i \neq j$ terms.

$$\begin{aligned} Q_{xx+T}^{i*}(\theta_t^T) = {} & Q_{xx+T}^{ii}(\theta_t^T, \theta_{t+T}^T) \\ & + \sum_{j \neq i} Q_{xx+T}^{ij}(\theta_t^T, \theta_{t+T}^T) + Q_{xx+T}^{i\delta(i)}(\theta_t^T, \theta_{t+T}^T) + \sum_{j \neq i} Q_{xx+T}^{i\delta(j)}(\theta_t^T, \theta_{t+T}^T) \ . \end{aligned} \tag{9}$$

$$\begin{aligned} Q_{xT}^{\beta(i)*}(\theta_t^T) = {} & Q_{xT}^{\beta(i)i}(\theta_t^T, \theta_{t+T}^T) \\ & + \sum_{j \neq i} Q_{xT}^{\beta(i)j}(\theta_t^T, \theta_{t+T}^T) + Q_{xT}^{\beta(i)\delta(i)}(\theta_t^T, \theta_{t+T}^T) + \sum_{j \neq i} Q_{xT}^{\beta(i)\delta(j)}(\theta_t^T, \theta_{t+T}^T) \ , \\ & \mu < x < \nu \ , \end{aligned} \tag{10}$$

where μ and ν are appropriately defined exact age limits to the child bearing age groups.

$$Q_{xx+T}^{*\delta(i)}(\theta_t^T, \theta_{t+T}^T) = Q_{xx+T}^{i\delta(i)}(\theta_t^T, \theta_{t+T}^T) + \sum_{j \neq i} Q_{xx+T}^{j\delta(i)}(\theta_t^T, \theta_{t+T}^T) \ , \qquad x > 0 \ . \tag{11}$$

$$Q_{0T}^{*\delta(i)}(\theta_t^T, \theta_{t+T}^T) = \sum_x \left[Q_{xT}^{\beta(i)\delta(i)}(\theta_t^T, \theta_{t+T}^T) + \sum_{j \neq i} Q_{xT}^{\beta(j)\delta(i)}(\theta_t^T, \theta_{t+T}^T) \right] \ . \tag{12}$$

$$Q_{xx+T}^{*i}(\theta_{t+T}^T) = Q_{xx+T}^{ii}(\theta_t^T, \theta_{t+T}^T) + \sum_{j \neq i} Q_{xx+T}^{ji}(\theta_t^T, \theta_{t+T}^T) \ , \qquad x > 0 \ . \tag{13}$$

$$Q_{0T}^{*i}(\theta_{t+T}^T) = \sum_x \left[Q_{xT}^{\beta(i)i}(\theta_t^T, \theta_{t+T}^T) + \sum_{j \neq i} Q_{xT}^{\beta(j)i}(\theta_t^T, \theta_{t+T}^T) \right] \ . \tag{14}$$

These will form the core of a life-table accounts based model in the next section.

The life-table transition rates can now be defined in the usual way, by dividing elements of the accounts by the corresponding row sums. We do this first for the accounts given by equations (9) to (14). This gives a set of multiregion life-table rates. With the appropriate definition of $x+T$ associated with death, equation (14) becomes:

$$q_{xx+T}^{i\delta(i)} = \frac{Q_{xx+T}^{i\delta(i)}}{Q_{x*}^{i*}} \ , \tag{15}$$

and the corresponding minor-flow rate is:

$$q_{xx+T}^{i\delta(j)} = \frac{Q_{xx+T}^{i\delta(j)}}{Q_{x*}^{i*}} \ , \qquad i \neq j \ . \tag{16}$$

We have, of course, dropped the period labels for convenience, q_{xx+T}^{ii} should be written in full as $q_{xx+T}^{ii}(\theta_t^T, \theta_{t+T}^T)$.

				Survival		
				θ^T_{t+T}		
				1	2	N
				T $2T$. $x+T$. ω	T $2T$. $x+T$. ω	T $2T$. $x+T$. ω
Existence in period in region at exact age	θ^T_t	1	T	. Q^{11}_{T2T}	. Q^{12}_{T2T}	. Q^{1N}_{T2T}
			$2T$	.	.	.
			.	.	.	.
			x	Q^{11}_{xx+T}	Q^{12}_{xx+T}	Q^{1N}_{xx+T}
			.	.	.	.
			$\omega-T$	$Q^{11}_{\omega-T\omega}$	$Q^{12}_{\omega-T\omega}$	$Q^{1N}_{\omega-T\omega}$
		2	T	. Q^{21}_{T2T}	. Q^{22}_{T2T}	. Q^{2N}_{T2T}
			$2T$	.	.	.
			.	.	.	.
			x	Q^{21}_{xx+T}	Q^{22}_{xx+T}	Q^{2N}_{xx+T}
			.	.	.	.
			$\omega-T$	$Q^{21}_{\omega-T\omega}$	$Q^{22}_{\omega-T\omega}$	$Q^{2N}_{\omega-T\omega}$
		N	T	. Q^{N1}_{T2T}	. Q^{N2}_{T2T}	. Q^{NN}_{T2T}
			$2T$	.	.	.
			.	.	.	.
			x	Q^{N1}_{xx+T}	Q^{N2}_{xx+T}	Q^{NN}_{xx+T}
			.	.	.	.
			$\omega-T$	$Q^{N1}_{\omega-T\omega}$	$Q^{N2}_{\omega-T\omega}$	$Q^{NN}_{\omega-T\omega}$
Birth in time period in region to mothers aged between exact age, age indicated and that $+T$	θ^T_t	1	T	.	.	.
			α	$Q^{\beta(1)1}_{\alpha T}$	$Q^{\beta(1)2}_{\alpha T}$	$Q^{\beta(1)N}_{\alpha T}$
			x	$Q^{\beta(1)1}_{xT}$	$Q^{\beta(1)2}_{xT}$	$Q^{\beta(1)N}_{xT}$
			$\beta-T$	$Q^{\beta(1)1}_{\beta-TT}$	$Q^{\beta(1)2}_{\beta-TT}$	$Q^{\beta(1)N}_{\beta-TT}$
			$\omega-T$	.	.	.
		2	T	.	.	.
			α	$Q^{\beta(2)1}_{\alpha T}$	$Q^{\beta(2)2}_{\alpha T}$	$Q^{\beta(2)N}_{\alpha T}$
			x	$Q^{\beta(2)1}_{xT}$	$Q^{\beta(2)2}_{xT}$	$Q^{\beta(2)N}_{xT}$
			$\beta-T$	$Q^{\beta(2)1}_{\beta-TT}$	$Q^{\beta(2)2}_{\beta-TT}$	$Q^{\beta(2)N}_{\beta-TT}$
			$\omega-T$	.	.	.
		N	T	.	.	.
			α	$Q^{\beta(N)1}_{\alpha T}$	$Q^{\beta(N)2}_{\alpha T}$	$Q^{\beta(N)N}_{\alpha T}$
			x	$Q^{\beta(N)1}_{xT}$	$Q^{\beta(N)2}_{xT}$	$Q^{\beta(N)N}_{xT}$
			$\beta-T$	$Q^{\beta(N)1}_{\beta-TT}$	$Q^{\beta(N)2}_{\beta-TT}$	$Q^{\beta(N)N}_{\beta-TT}$
			$\omega-T$	.	.	.
Totals				$Q^{\bullet 1}_{\bullet T}$ $Q^{\bullet 1}_{\bullet 2T}$. $Q^{\bullet 1}_{\bullet x+T}$. $Q^{\bullet 1}_{\bullet \omega}$	$Q^{\bullet 2}_{\bullet T}$ $Q^{\bullet 2}_{\bullet 2T}$. $Q^{\bullet 2}_{\bullet x+T}$. $Q^{\bullet 2}_{\bullet \omega}$	$Q^{\bullet N}_{\bullet T}$ $Q^{\bullet N}_{\bullet 2T}$. $Q^{\bullet N}_{\bullet x+T}$. $Q^{\bullet N}_{\bullet \omega}$

Figure 7. Accounts for life tables.

Death between exact ages indicated			
θ^T_{t+T}			
1	2	N	
T $2T$. $x+T$. ω	T $2T$. $x+T$. ω	T $2T$. $x+T$. ω	Totals
. $Q^{1\delta(1)}_{T\,2T}$ . . $Q^{1\delta(1)}_{xx+T}$. $Q^{1\delta(1)}_{\omega-T\omega}$	. $Q^{1\delta(2)}_{T\,2T}$ . . $Q^{1\delta(2)}_{xx+T}$. $Q^{1\delta(1)}_{\omega-T\omega}$	. $Q^{1\delta(N)}_{T\,2T}$ . . $Q^{1\delta(N)}_{xx+T}$. $Q^{1\delta(N)}_{\omega-T\omega}$	$Q^{1\bullet}_{T\bullet}$ . . $Q^{1\bullet}_{x\bullet}$. $Q^{1\bullet}_{\omega-T\bullet}$
. $Q^{2\delta(1)}_{T\,2T}$ . . $Q^{2\delta(1)}_{xx+T}$. $Q^{2\delta(1)}_{\omega-T\omega}$	. $Q^{2\delta(2)}_{T\,2T}$ . . $Q^{2\delta(2)}_{xx+T}$. $Q^{2\delta(2)}_{\omega-T\omega}$	. $Q^{2\delta(N)}_{T\,2T}$ . . $Q^{2\delta(N)}_{xx+T}$. $Q^{2\delta(N)}_{\omega-T\omega}$	$Q^{2\bullet}_{T\bullet}$ . . $Q^{2\bullet}_{x\bullet}$. $Q^{2\bullet}_{\omega-T\bullet}$
. $Q^{N\delta(1)}_{T\,2T}$ . . $Q^{N\delta(1)}_{\omega\omega+T}$. $Q^{N\delta(1)}_{\omega-T\omega}$	. $Q^{N\delta(2)}_{T\,2T}$ . . $Q^{N\delta(2)}_{\omega\omega T}$. $Q^{N\delta(2)}_{\omega-T\omega}$	. $Q^{N\delta(N)}_{T\,2T}$ . . $Q^{N\delta(N)}_{\omega\omega T}$. $Q^{N\delta(N)}_{\omega-T\omega}$	$Q^{N\bullet}_{T\bullet}$ . . $Q^{N\bullet}_{x\bullet}$. $Q^{N\bullet}_{\omega-T\bullet}$
. $Q^{\beta(1)\delta(1)}_{\alpha T}$ $Q^{\beta(1)\delta(1)}_{xT}$ $Q^{\beta(1)\delta(1)}_{\beta-T\,T}$ . .	. $Q^{\beta(1)\delta(2)}_{\alpha T}$ $Q^{\beta(1)\delta(2)}_{xT}$ $Q^{\beta(1)\delta(2)}_{\beta-T\,T}$ . .	. $Q^{\beta(1)\delta(N)}_{\alpha T}$ $Q^{\beta(1)\delta(N)}_{xT}$ $Q^{\beta(1)\delta(N)}_{\beta-T\,T}$ . .	. $Q^{\beta(1)\bullet}_{\alpha\bullet}$ $Q^{\beta(1)\bullet}_{x\bullet}$ $Q^{\beta(1)\bullet}_{\beta-T\bullet}$ . .
. $Q^{\beta(2)\delta(1)}_{\alpha T}$ $Q^{\beta(2)\delta(1)}_{xT}$ $Q^{\beta(2)\delta(1)}_{\beta-T\,T}$ . .	. $Q^{\beta(2)\delta(2)}_{\alpha T}$ $Q^{\beta(2)\delta(2)}_{xT}$ $Q^{\beta(2)\delta(2)}_{\beta-T\,T}$ . .	. $Q^{\beta(2)\delta(N)}_{\alpha T}$ $Q^{\beta(2)\delta(N)}_{xT}$ $Q^{\beta(2)\delta(N)}_{\beta-T\,T}$ . .	. $Q^{\beta(2)\bullet}_{\alpha}$ $Q^{\beta(2)\bullet}_{x\bullet}$ $Q^{\beta(2)\bullet}_{\beta-T\bullet}$ . .
. $Q^{\beta(N)\delta(1)}_{\alpha T}$ $Q^{\beta(N)\delta(1)}_{xT}$ $Q^{\beta(N)\delta(1)}_{\beta-T\,T}$ . .	. $Q^{\beta(N)\delta(2)}_{\alpha T}$ $Q^{\beta(N)\delta(2)}_{xT}$ $Q^{\beta(N)\delta(2)}_{\beta-T\,T}$ . .	. $Q^{\beta(N)\delta(N)}_{\alpha T}$ $Q^{\beta(N)\delta(N)}_{xT}$ $Q^{\beta(N)\delta(N)}_{\beta-T\,T}$ . .	. $Q^{\beta(N)\bullet}_{\alpha\bullet}$ $Q^{\beta(N)\bullet}_{x\bullet}$ $Q^{\beta(N)\bullet}_{\beta-T\bullet}$ . .
$Q^{\bullet\delta(1)}_{0T}$ $Q^{\bullet\delta(1)}_{T\,2T}$. $Q^{\bullet\delta(1)}_{xx+T}$. $Q^{\bullet\delta(1)}_{\omega-T\omega}$	$Q^{\bullet\delta(2)}_{0T}$ $Q^{\bullet\delta(2)}_{T\,2T}$. $Q^{\bullet\delta(2)}_{xx+T}$. $Q^{\bullet\delta(2)}_{\omega-T\omega}$	$Q^{\bullet\delta(N)}_{0T}$ $Q^{\bullet\delta(N)}_{T\,2T}$. $Q^{\bullet\delta(N)}_{xx+T}$. $Q^{\bullet\delta(N)}_{\omega-T\omega}$	

Figure 7 (continued)

The terms now defined can be matched with the usual definitions of life-table analysis. Consider a single 'closed' region, i. Then the main life-table variables are l_x^i, d_x^i, and q_x^i (Pressat, 1972, p.109, with a superscript added), where l_x^i is the total number of survivors in i to exact age x, d_x^i is the number of deaths between exact ages x and $x+T$, and q_x^i is the probability of mortality at age x. The correspondence is

$$l_x^i = Q_{x*}^{i*} \tag{17}$$

$$d_x^i = Q_{xx+T}^{*\delta(i)} \tag{18}$$

$$q_x^i = \frac{d_x^i}{l_x^i} \tag{19}$$

$$= \frac{Q_{xx+T}^{*\delta(i)}}{Q_{x*}^{i*}} . \tag{20}$$

For the single region case, any one set of the three quantities determines the other two, since (Pressat, 1972, p.110)

$$d_x^i = l_x^i - l_{x+T}^i , \tag{21}$$

and

$$d_x^i = l_x^i q_x^i , \tag{22}$$

and equation (19) holds.

In the multiregion case the key relationships are the accounting equations (9)–(14). A whole matrix of rates, $\mathbf{q}$ (equivalent to $\mathbf{H}$ in the age-group accounts), can be defined as

$$\mathbf{q} = \left[\begin{array}{c|c} \{q_{xx+T}^{ij}\} & \{q_{xx+T}^{i\delta(j)}\} \\ \hline \{q_{xT}^{\beta(i)j}\} & \{q_{xT}^{\beta(i)\delta(j)}\} \end{array}\right] \tag{23}$$

$$= \left[\begin{array}{c|c} \{Q_{xx+T}^{ij}/Q_{xx+T}^{i*}\} & \{Q_{xx+T}^{i\delta(j)}/Q_{xx+T}^{i*}\} \\ \hline \{Q_{xT}^{\beta(i)j}/Q_{xT}^{\beta(i)*}\} & \{Q_{xT}^{\beta(i)\delta(j)}/Q_{xT}^{\beta(i)*}\} \end{array}\right] . \tag{24}$$

Note that as with transition rates based on age-group accounts the 'birth rates' are not very interesting ones at this stage, they merely apportion the row totals of births among the possible final states. We discuss the problem of measuring more useful birth rates in the context of the model in the next section.

We can argue, as in the age-groups case, that the accounting basis for life tables gives, through equation (24), the proper definition of the rates in the multiregion case. The problem, which is also a familiar one, is that these rates can only be calculated when all the elements of the accounts are known. Since they are not known from data, it is necessary to build a life-table accounts-based model to accomplish this, and we show how to do this next.

4 A life-table accounts-based model

In the age-group case (Wilson and Rees, 1974) there were seven steps in the model building procedure:

Step 1: assemble known data.

Step 2: manipulate them where necessary.

Step 3: obtain initial values of unknown major flows from the accounting equations, by setting unknown minor flows to zero.

Step 4: calculate at-risk populations.

Step 5: calculate birth and death rates.

Step 6: calculate unknown minor flows.
Step 7: solve the accounting equations for the unknown major flows.

The known data, since it usually refers to time cross-sections and periods, are essentially those which we had before. This implies that we will have to do rather more manipulation in *step 2* in this case. Indeed this turns out to be the main problem; analogous methods can be used for the other steps. We therefore describe the procedures to be used in each step in turn.

Step 1: Assemble known data
Data for two periods θ^T_t and θ^T_{t+T} are clearly essential. We also saw that some of the mothers giving birth 'at age x' come from period θ^T_{t-T}. Thus ideally we need data for three time periods. So, following *step 1* for the age-group model (Wilson and Rees, 1974), we can assume we have K^{i*}_{r*}, $K^{\beta(i)*}_{**}(K^{*m(i)}_{*u})$, $K^{*\delta(i)}_{*s}$, K^{ij}_{*s}, $i \neq j$, $K^{\beta(i)j}_{rs}$, $i \neq j$ for $t-T$, t and $t+T$, where the quantities are cross-sectional, and θ^T_{t-T}, θ^T_t, and θ^T_{t+T} where they refer to periods[2].

Step 2: Manipulation of data
The accounting equations (9)–(14) suggest that we should like to have as 'known' data $Q^{i*}_{xx+T}(\theta^T_t)$, $Q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T})$, $i \neq j$, $Q^{\beta(i)*}_{xT}(\theta^T_t)$, $Q^{\beta(i)j}_{xT}(\theta^T_t, \theta^T_{t+T})$, $i \neq j$, $Q^{*\delta(i)}_{xx+T}(\theta^T_t, \theta^T_{t+T})$, and $Q^{*\delta(i)}_{0T}(\theta^T_t, \theta^T_{t+T})$, and so our task is to estimate these quantities from the data listed in *step 1*. We also need $Q^{i*}_{x*}(\theta^T_{t-T})$ to help get a divisor in calculating birth rates.

(1) $Q^{i*}_{xx+T}(\theta^T_t)$. Even the base-period population is not available directly from data. Figure 8 suggests that a suitable approximation would be

$$Q^{i*}_{xx+T}(\theta^T_t) = 0\cdot5[K^{i*}_{r(x-T)*}(t) + K^{i*}_{r(x)*}(t+T)] \ , \tag{25}$$

where $r(x)$ is the age group 'following' exact age x as indicated on the figure. It remains to be investigated whether this equation could be improved if a full set of age-group accounts were available.

(2) $Q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T})$, $Q^{\beta(i)j}_{xT}(\theta^T_t, \theta^T_{t+T})$. We can treat both kinds of migration and survival flows similarly. Consider figure 9: the available data refers to age-group parallelograms

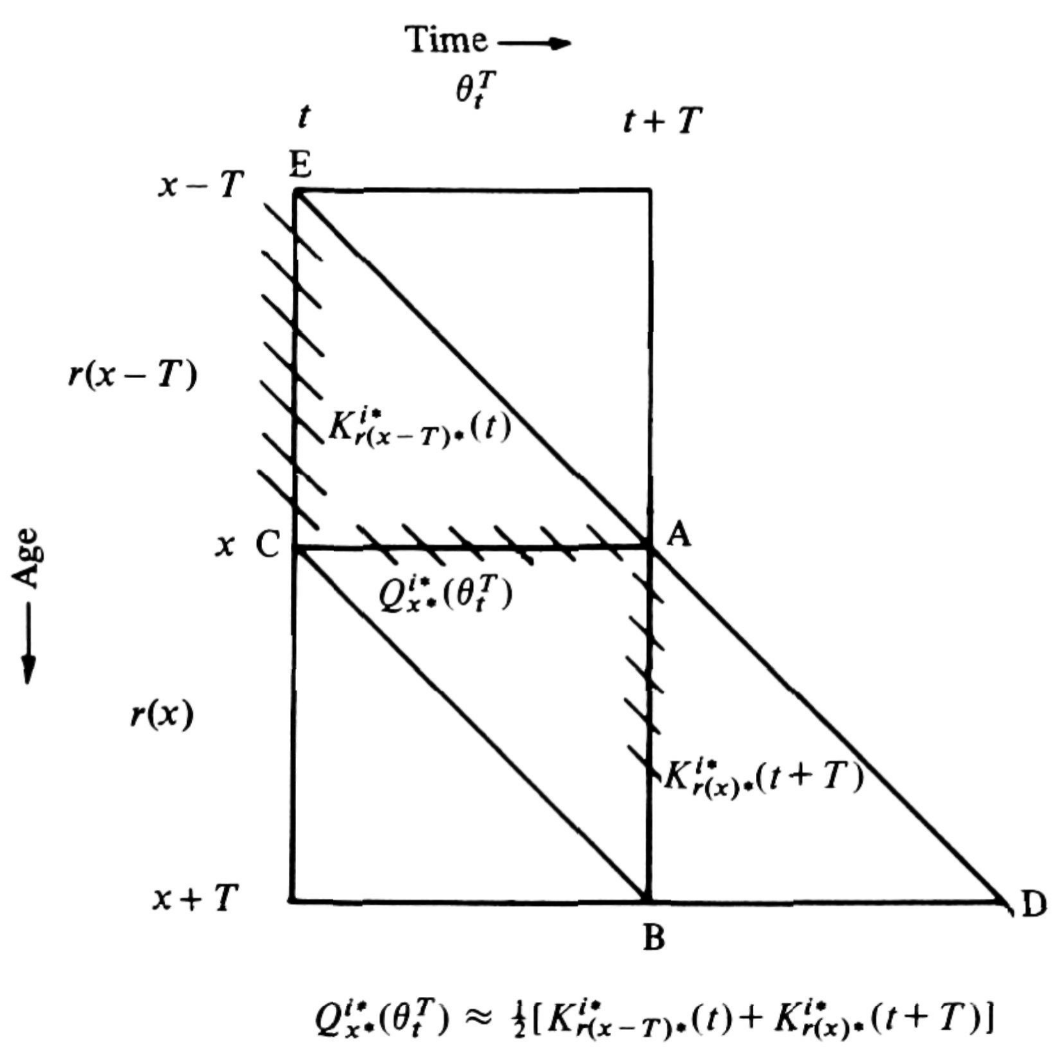

Figure 8. Lexis diagram illustrating one method of estimating numbers in an exact age cohort.

[2] Period θ^T_{t-T} begins at time $t-T$ and ends at time t; period θ^T_t begins at time t and ends at time $t+T$; and period θ^T_{t+T} starts at time $t+T$ and finishes at time $t+2T$.

such as ABCD. A migration taking place in ΔBCD (that is, in period θ_t^T) will be recorded as an ABCD migration, unlike one taking place in CDP, will be part of the CDPQ measured migrations for period θ_{t+T}^T. As an approximation we could take BCD migrations to be half of ABCD migrations, and CDP migrations to be half of CDPQ migrations. Thus:

$$Q^{ij}_{x\,x+T}(\theta_t^T, \theta_{t+T}^T) = 0{\cdot}5[K^{ij}_{r(x-T)r(x)}(t, t+T) + K^{ij}_{r(x)r(x+T)}(t+T, t+2T)] \ . \qquad (26)$$

The corresponding argument for $Q^{\beta(i)j}_{xT}(\theta_t^T, \theta_{t+T}^T)$ is slightly complicated by the age-of-mother subscript x. An appropriate formula would be

$$Q^{\beta(i)j}_{xT}(\theta_t^T, \theta_{t+T}^T) = 0{\cdot}5[K^{\beta(i)j}_{r(x-T)1}(t, t+T) + e_{r(x)}K^{ij}_{12}(t+T, t+2T)] \ . ^{(3)} \qquad (27)$$

(3) $Q^{*\delta(i)}_{x\,x+T}(\theta_t^T, \theta_{t+T}^T)$. Deaths in the life-table parallelogram ABCD are shown in figure 10. In this case we can use the c_{rs} coefficients from the age-group model to find the deaths in Δ's ABD and BCD respectively as:

$$K^{*\delta(i)}_{r(x-T)r(x)}(t, t+T) = c_{r(x-T)r(x)}K^{*\delta(i)}_{*r(x)}(t, t+T) \ , \qquad (28)$$

and

$$K^{*\delta(i)}_{r(x)r(x)}(t+T, t+2T) = c_{r(x)r(x)}K^{*\delta(i)}_{*r(x)}(t+T, t+2T) \ , \qquad (29)$$

so that

$$Q^{*\delta(i)}_{x\,x+T}(\theta_t^T, \theta_{t+T}^T) = K^{*\delta(i)}_{r(x-T)r(x)}(t, t+T) + K^{*\delta(i)}_{r(x)r(x)}(t+T, t+2T) \ . \qquad (30)$$

A similar argument for the first age group gives

$$Q^{*\delta(i)}_{0T}(\theta_t^T, \theta_{t+T}^T) = K^{*\delta(i)}_{01}(t, t+T) + K^{*\delta(i)}_{11}(t+T, t+2T) \ . \qquad (31)$$

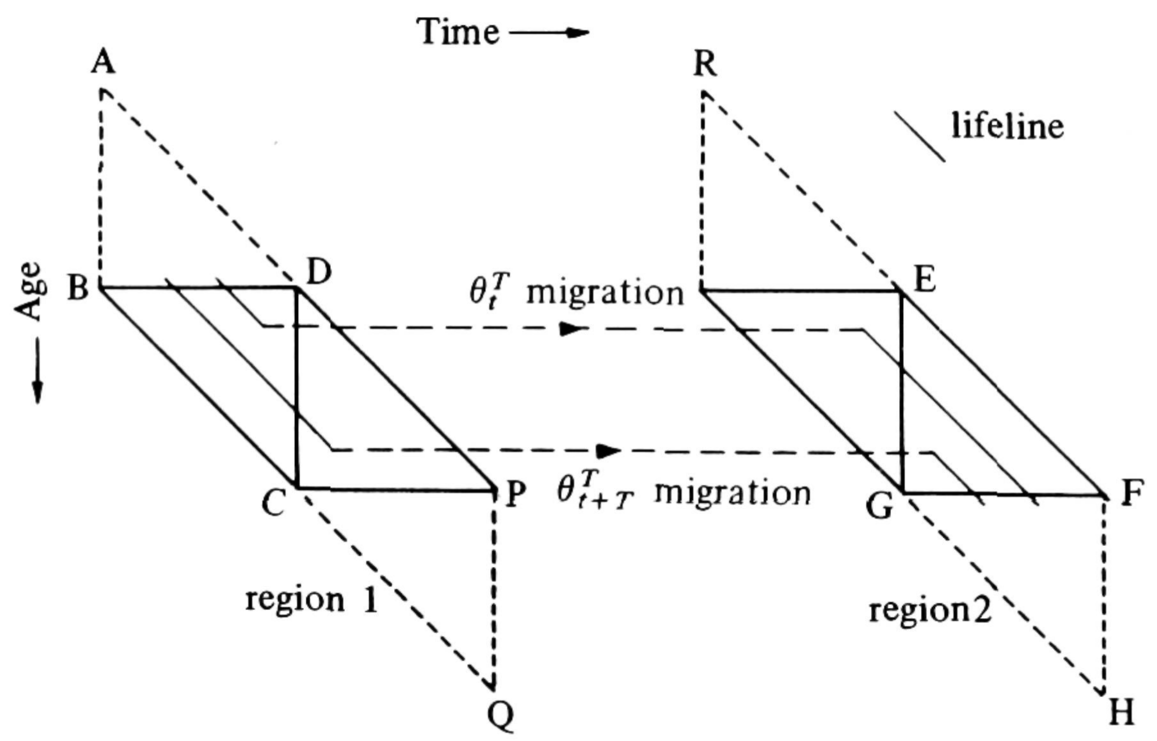

Figure 9. The timing of migrations in a life-table parallelogram.

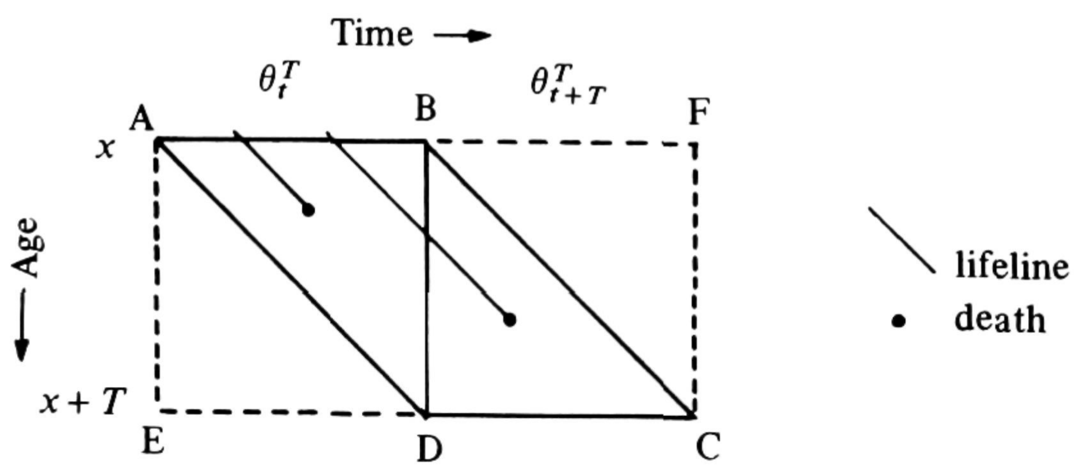

Figure 10. The timing of death in a life-table parallelogram.

(3) Here $e_{r(x)}$ is the proportion of the K^{ij}_{12} flow born to mothers in age group $r(x+T)$ in period t to $t+T$.

(4) $Q_{xT}^{\beta(i)*}(\theta_t^T)$. The life-table parallelograms both for infants and mothers are shown in figure 11. Our data refers to age group of mother at maternity, but we can transform it using the method of the age-group model to refer to age groups at the beginning of the period.

$$K_{r*}^{\beta(i)*}(t, t+T) = \sum_u f_{ru} K_{**}^{\beta(i)*}(K_{*u}^{*\mathrm{m}(i)}) \,. \tag{32}$$

f_{ru} is the proportion of mothers in age group u at time of maternity who were in age group r at the beginning of the period. In practice, it can usually be taken to be the same as c_{ru} defined in our previous papers (Wilson and Rees, 1974, p.107). Then the θ_t^T period births we require are those connected to maternities in Δ's BCD and PQR of figure 11. If we take these to be half the births in age-group parallelograms ABCD and PQRS respectively, we have

$$Q_{xT}^{\beta(i)*}(\theta_t^T) = 0{\cdot}5[K_{r(x-T)*}^{\beta(i)*}(t, t+T) + K_{r(x)*}^{\beta(i)*}(t, t+T)] \,. \tag{33}$$

This completes *step 2.*

Step 3: Initial values of unknown major flows

The unknown minor flows will be $Q_{xx+T}^{i\delta(j)}(\theta_t^T, \theta_{t+T}^T)$ and $Q_{xx+T}^{\beta(i)\delta(j)}(\theta_t^T, \theta_{t+T}^T)$. These are set to zero at this stage and initial values of the unknown major flows are calculated from the accounting equations as follows: $Q_{xx+T}^{i\delta(i)}(\theta_t^T, \theta_{t+T}^T)$, $x > 0$ from equation (11); $\sum_x Q_{xT}^{\beta(i)\delta(i)}(\theta_t^T, \theta_{t+T}^T)$ can be calculated from equation (12). Then let e_x be the proportion of these deaths of babies born to 'mothers aged x', so

$$Q_{xT}^{\beta(i)\delta(i)}(\theta_t^T, \theta_{t+T}^T) = e_x \sum_x Q_{xT}^{\beta(i)\delta(i)}(\theta_t^T, \theta_{t+T}^T) \,. \tag{34}$$

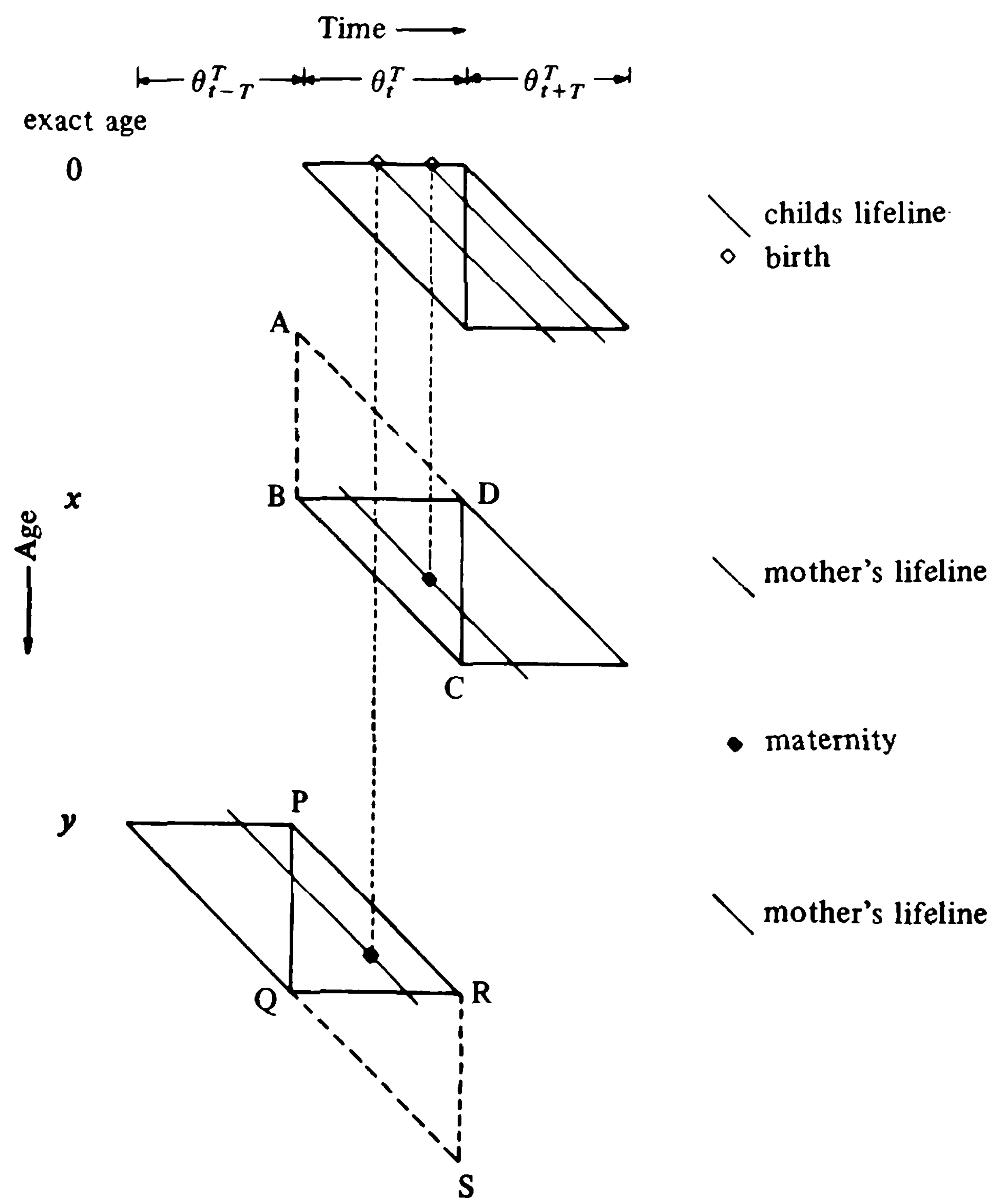

Figure 11. The two types of cohorts of mothers who give birth to children in a time period.

Then $Q^{\beta(i)i}_{xT}(\theta^T_t, \theta^T_{t+T})$ can be obtained from equation (10), and $Q^{ii}_{xx+T}(\theta^T_t, \theta^T_{t+T})$ from equation (9).

Step 4: At-risk population
For persons attaining exact age x during period θ^T_t, the relevant deaths are those outlined in parallelogram ABCD in figure 10. The corresponding at-risk population can be denoted by $\hat{Q}^{\mathrm{D}*i}_{xx+T}(\theta^T_t, \theta^T_{t+T})$. We can define a set of coefficients, ${}^{i}\phi^{\mathrm{D}jk}_{xx+T}$, such that

$$\hat{Q}^{\mathrm{D}*i}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = \sum_{jk} {}^{i}\phi^{\mathrm{D}jk}_{xx+T}\, Q^{jk}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \,, \tag{35}$$

where the summation is over all flows (j, k). ${}^{i}\phi^{\mathrm{D}jk}_{xx+T}$ is the proportion of time that a person in a (j, k) flow is at-risk of dying in, that is resident in, region i, in time T following the attainment of exact age x in θ^T_t. These coefficients would be assumed to be zero unless either j or k equalled i.

We can deal with births similarly. The methods are shown in figure 11 and, since we have two contributions, we define two sets of coefficients ${}^{i}\phi^{\mathrm{B}jk(1)}_{xx+T}$ and ${}^{i}\phi^{\mathrm{B}jk(2)}_{xx+T}$. The population at-risk can be denoted by $\hat{Q}^{\mathrm{B}*i}_{xx+T}(\theta^T_t)$, with only one period label needed in this case. Note that the ϕ's relate to the proportion of time spent in Δ's BCD and PQR—not the full parallelograms—and should be calculated accordingly. Thus;

$$\hat{Q}^{\mathrm{B}*i}_{xx+T}(\theta^T_t) = \sum_{jk} {}^{i}\phi^{\mathrm{B}jk(1)}_{xx+T}\, Q^{jk}_{xx+T}(\theta^T_{t-T}, \theta^T_t) + \sum_{jk} {}^{i}\phi^{\mathrm{B}jk(2)}_{xx+T}\, Q^{jk}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \,. \tag{36}$$

Step 5: Rates
The main birth and death rates for the model can now be calculated as

$$q^{\square\delta(i)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = \frac{Q^{*\delta(i)}_{xx+T}(\theta^T_t, \theta^T_{t+T})}{\hat{Q}^{\mathrm{D}*i}_{xx+T}(\theta^T_t, \theta^T_{t+T})} \,, \tag{37}$$

and

$$q^{\beta(i)\square}_{xx+T}(\theta^T_t) = \frac{Q^{\beta(i)*}_{xT}(\theta^T_t)}{\hat{Q}^{\mathrm{B}*i}_{xx+T}(\theta^T_t)} \,. \tag{38}$$

Step 6: Unknown minor flows
The unknown minor flows are calculated from equations which assume that the overall death rate for a region of *step 5* can be applied to in-migrants.

$$Q^{i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = q^{\square\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T})\, \hat{Q}^{\mathrm{D}i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \,, \tag{39}$$

and

$$Q^{\beta(i)\delta(j)}_{xT}(\theta^T_t, \theta^T_{t+T}) = e_x\, q^{\square\delta(j)}_{0T}(\theta^T_t, \theta^T_{t+T})\, \hat{Q}^{\mathrm{D}*j}_{0T}(\theta^T_t, \theta^T_{t+T}) \,, \tag{40}$$

where we assume that the coefficient e_x, defined for equation (34), can be applied here also. The at-risk populations $\hat{Q}^{\mathrm{D}i\delta(j)}_{xx+T}$ and $\hat{Q}^{\mathrm{D}*\delta(j)}_{0T}$ are subsets of that defined in equation (35). That is

$$\hat{Q}^{\mathrm{D}i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = {}^{i}\phi^{\mathrm{D}ij}_{xx+T}\, Q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T}) + {}^{i}\phi^{\mathrm{D}i\delta(j)}_{xx+T}\, Q^{i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \,, \tag{41}$$

and

$$\hat{Q}^{\mathrm{D}i\delta(j)}_{0T}(\theta^T_t, \theta^T_{t+T}) = {}^{i}\phi^{\mathrm{D}\beta(i)j}_{0T}\, Q^{\beta(i)j}_{0T}(\theta^T_t, \theta^T_{t+T}) + {}^{i}\phi^{\mathrm{D}\beta(i)\delta(j)}_{0T}\, Q^{\beta(i)\delta(j)}_{0T}(\theta^T_t, \theta^T_{t+T}) \,. \tag{42}$$

If these expressions are substituted in equations (39) and (40), we can solve for the flows as follows:

$$Q^{i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = \frac{q^{\square\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T})\, {}^i\phi^{\mathrm{D}ij}_{xx+T}\, Q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T})}{1 - q^{\square\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T})\, {}^i\phi^{\mathrm{D}i\delta(j)}_{xx+T}} \tag{43}$$

$$Q^{\beta(i)\delta(j)}_{xT}(\theta^T_t, \theta^T_{t+T}) = \frac{e_x q^{\square\delta(j)}_{0T}(\theta^T_t, \theta^T_{t+T})\, {}^i\phi^{\mathrm{D}\beta(i)j}_{0T}\, Q^{\beta(i)j}_{0T}(\theta^T_t, \theta^T_{t+T})}{1 - e_x q^{\square\delta(j)}_{0T}(\theta^T_t, \theta^T_{t+T})\, {}^i\phi^{\mathrm{D}\beta(i)\delta(j)}_{0T}} \tag{44}$$

Step 7. Solution of accounting equations
The accounting equations (9)–(14) can then be solved as follows: Equation (11) gives $Q^{i\delta(i)}_{xx+T}(\theta^T_t, \theta^T_{t+T})$, and (12) gives $\sum_x Q^{\beta(i)\delta(i)}_{xT}(\theta^T_t, \theta^T_{t+T})$. Equation (34) can then be used to give $Q^{\beta(i)\delta(i)}_{xT}(\theta^T_t, \theta^T_{t+T})$, and $Q^{\beta(i)i}_{xT}(\theta^T_t, \theta^T_{t+T})$ is obtained from equation (10). Finally, the 'new population' $Q^{*i}_{xx+T}(\theta^T_{t+T})$ and $Q^{*i}_{0T}(\theta^T_{t+T})$ are obtained from equations (13) and (14).

Steps 4 to *7* would then have to be repeated until convergence was achieved.

5 The connections between the life-table accounts and the age-group accounts

Some connections between Q and K type population quantities were made in the specification of the accounts-based model for life-table accounts. We extend these connections in this section.

In general, we can estimate a set of $\{Q^{ij}_{xx+T}\}$ flows by averaging over two sets of $\{K^{ij}_{r(x)r(x+T)}\}$ flows:

$$Q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = 0{\cdot}5[(K^{ij}_{r(x-T)r(x)}(t, t+T) + K^{ij}_{r(x)r(x+T)}(t+T, t+2T)] \tag{45}$$

as in equation (26), and

$$Q^{i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) = 0{\cdot}5[(K^{i\delta(j)}_{r(x-T)r(x)}(t, t+T) + K^{i\delta(j)}_{r(x)r(x+T)}(t+T, t+2T)] \,. \tag{46}$$

This 'minor flow' equation represents an alternative to the account-based-model estimation procedure of section 4. It can be used only when two full sets of age-group accounts are available. The second age-group subscripts in the K terms refer to the age group in which people survive at the end of period, or before attaining which they die. The equation (45) situation is shown in the Lexis diagram of figure 12. Lifeline bundle A represents a Q^{ij} flow which is estimated as a half of bundles B and C.

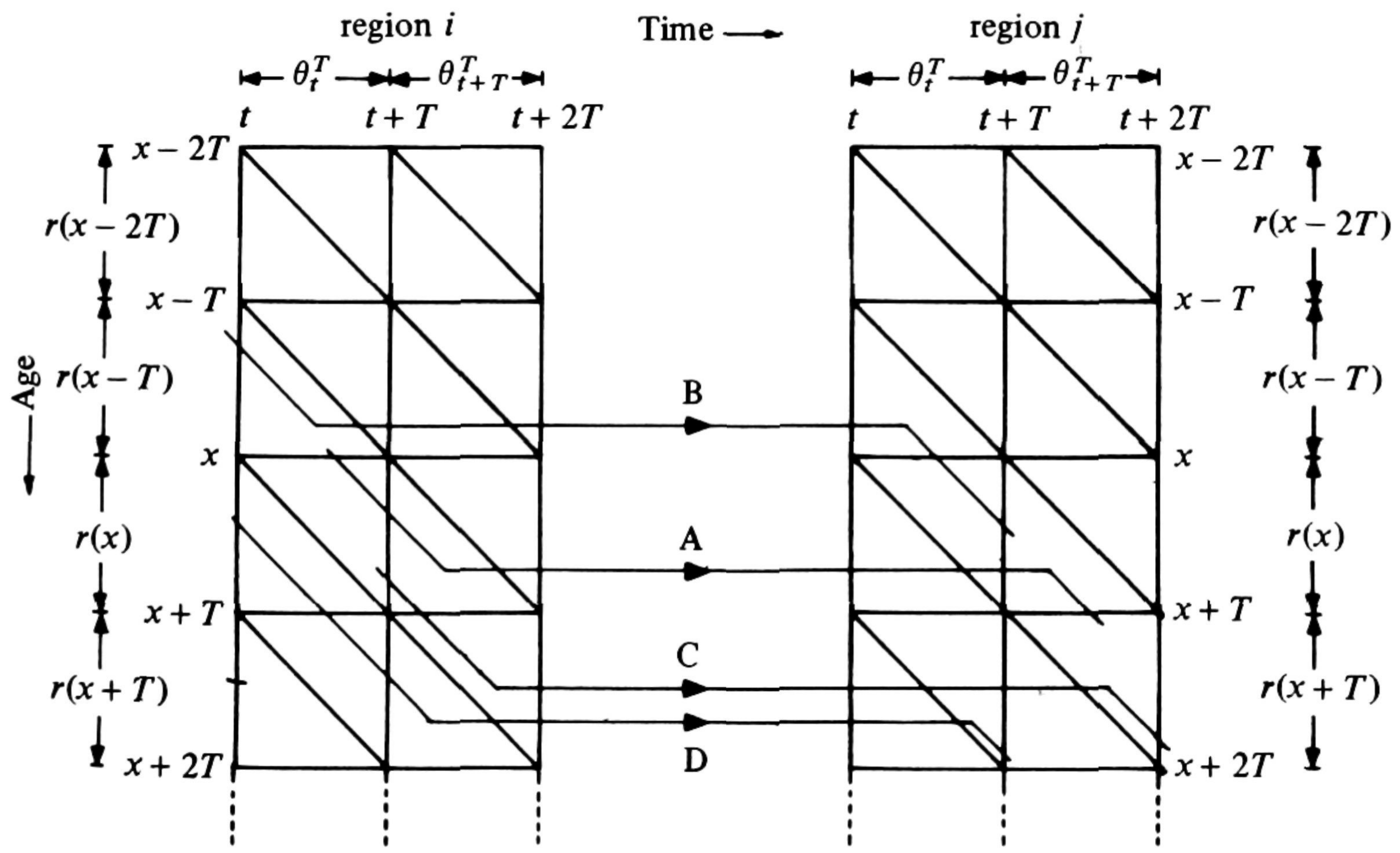

Figure 12. A multiregional Lexis diagram illustrating alternative methods of averaging K type flows to obtain Q type flows.

Table 1. An aggregated version of the age-group accounts for Middle Earth.

Initial state		Final state																Totals
Location of birth or at time t	Age group	Location at time $t+20$								Location of death in period t to $t+20$								
		Region 1: the Shire				Region 2: rest of Middle Earth				Region 1: the Shire				Region 2: rest of Middle Earth				
		Age group at time $t+20$								Death before attainment of age group at $t+20$								
		1 0-19	2 20-39	3 40-59	4 60+	1 0-19	2 20-39	3 40-59	4 60+	1 0-19	2 20-39	3 40-59	4 60+	1 0-19	2 20-39	3 40-59	4 60+	
Region 1: the Shire	0 birth	1977				108				119				7				2211
	1 0-19		808				100				81				11			1000
	2 20-39			526				135				190				49		900
	3 40-59				41				25				419				15	500
	4 60+				22				3				73				2	100
Region 2: rest of Middle Earth	0 birth	214				8131				8				991				9344
	1 0-19		**200**				3269				11				520			4000
	2 20-39			**270**				**2110**				65				**1155**		3600
	3 40-59				50				269				27				1654	2000
	4 60+				5				173				6				216	400
Totals		2191	1008	796	118	8239	3369	2245	470	127	92	255	525	**998**	**531**	1204	1887	24055

Table 2. The h transition rates generated from table 1.

Initial state		Final state																Totals
Location of birth or at time t	Age group	Location at time $t+20$								Location of death in period t to $t+20$								
		Region 1: the Shire				Region 2: rest of Middle Earth				Region 1: the Shire				Region 2: rest of Middle Earth				
		Age group at time $t+20$								Death before attainment of age group at $t+20$								
		1 0–19	2 20–39	3 40–59	4 60+	1 0–19	2 20–39	3 40–59	4 60+	1 0–19	2 20–39	3 40–59	4 60+	1 0–19	2 20–39	3 40–59	4 60+	
Region 1: the Shire	0 birth	0·894166				0·048847				0·053822				0·003166				1·000000
	1 0–19		0·808000				0·100000				0·081000				0·011000			1·000000
	2 20–39			0·584444				0·150000				0·211111				0·054444		1·000000
	3 40–59				0·082000				0·050000				0·838000				0·030000	1·000000
	4 60+				0·220000				0·030000				0·730000				0·020000	1·000000
Region 2: rest of Middle Earth	0 birth	0·022902				0·870184				0·000856				0·106057				1·000000
	1 0–19		0·050000				0·817250				0·002750				0·130000			1·000000
	2 20–39			0·075000				0·586111				0·018056				0·320833		1·000000
	3 40–59				0·025000				0·134500				0·013500				0·827000	1·000000
	4 60+				0·012500				0·432500				0·015000				0·540000	1·000000
Totals		—				—				—				—				—

Table 3. The h or age-group transition rates rearranged from table 2.

Initial age group $r(x)$		$h^{11}_{r(x)r(x+20)}$	$h^{12}_{r(x)r(x+20)}$	$h^{1\delta(1)}_{r(x)r(x+20)}$	$h^{1\delta(2)}_{r(x)r(x+20)}$	$h^{21}_{r(x)r(x+20)}$	$h^{22}_{r(x)r(x+20)}$	$h^{2\delta(1)}_{r(x)r(x+20)}$	$h^{2\delta(2)}_{r(x)r(x+20)}$
0	birth	0·894166	0·048847	0·053822	0·003166	0·022902	0·870184	0·000856	0·106057
1	0–19	0·808000	0·100000	0·081000	0·011000	0·050000	0·817250	0·002750	0·130000
2	20–39	0·584444	0·150000	0·211111	0·054444	0·075000	0·586111	0·018056	0·320833
3	40–59	0·082000	0·050000	0·838000	0·030000	0·025000	0·134500	0·013500	0·827000
4	60+	0·220000	0·030000	0·730000	0·020000	0·012500	0·432500	0·015000	0·540000
Note:		Each row sums to 1				Each row sums to 1			

Table 4. The q or life-table rates derived from table 3.

Initial exact age x	q^{11}_{xx+T}	q^{12}_{xx+T}	$q^{1\delta(1)}_{xx+T}$	$q^{1\delta(2)}_{xx+T}$	q^{21}_{xx+T}	q^{22}_{xx+T}	$q^{2\delta(1)}_{xx+T}$	$q^{2\delta(2)}_{xx+T}$
0	0·851083	0·074424	0·067411	0·007083	0·036451	0·843717	0·001803	0·118029
20	0·696222	0·125000	0·146056	0·032722	0·062500	0·701681	0·010403	0·225417
40	0·333222	0·100000	0·524556	0·042222	0·050000	0·360306	0·015778	0·573917
60	0·151000	0·040000	0·784000	0·025000	0·018750	0·283500	0·014250	0·683500
80	0·000000	0·000000	0·969098	0·030902	0·000000	0·000000	0·020423	0·979577
Note:	Each row sums to 1				Each row sums to 1			

Thus, a set of life-table accounts can be estimated by this averaging process. Note that there is no necessary exact relation between the two sets of accounts.

If a second full set of age-group accounts is not available, then more drastic assumptions have to be made. We can assume that the one set of age-group accounts is repeated exactly in the second period, and that

$$K^{ij}_{r(x)r(x+T)}(t+T, t+2T) = K^{ij}_{r(x)r(x+T)}(t, t+T) , \tag{47}$$

and

$$K^{i\delta(j)}_{r(x)r(x+T)}(t+T, t+2T) = K^{i\delta(j)}_{r(x)r(x+T)}(t, t+T) . \tag{48}$$

In terms of figure 12, we assume that lifeline bundle C is the same size as bundle D. Then the connecting equation becomes (with a similar equation for $Q^{i\delta(j)}_{xx+T}$)

$$Q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \approx 0\cdot5[K^{ij}_{r(x-T)r(x)}(t, t+T) + K^{ij}_{r(x)r(x+T)}(t, t+T)] . \tag{49}$$

This assumption can be improved if we express the equations in rate terms, converting the Q flows into q rates, and the K flows into h rates:

$$q^{ij}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \approx 0\cdot5[(h^{ij}_{r(x-T)r(x)}(t, t+T) + h^{ij}_{r(x)r(x+T)}(t, t+T)] , \tag{50}$$

and

$$q^{i\delta(j)}_{xx+T}(\theta^T_t, \theta^T_{t+T}) \approx 0\cdot5[(h^{i\delta(j)}_{r(x-T)r(x)}(t, t+T) + h^{i\delta(j)}_{r(x)r(x+T)}(t, t+T)] . \tag{51}$$

If we had more information about the age group in which persons are when they migrate, we could establish closer connections between life-table and age-group accounts. We could break down more of the age-group Lexis diagram parallelogram figures into Lexis diagram triangle figures, and reassemble the triangles in life-table form. However, because of the possibility of multiple migrations, even this additional information would be insufficient to connect exactly the two sets of accounts. The only exact common denominator between the two sets of accounts is the set of migration life histories of all individuals. From this set, generated from a population register, both types of accounts could be built.

The interpolation method of equations (50) and (51) is useful as a means of generating a set of q transition rates, with which we can build the multiregional life table to be described in the next section. Table 1 sets out an aggregated version of the hypothetical two-region accounts used in Rees and Wilson (1975) to review models of population change. Table 2 shows the set of h transition rates calculated by dividing each element in the interior of table 1 by its row total. These rates are rearranged in table 3 in a form convenient for interpolation using equations (50) and (51), and table 4 shows the resulting q transition rates which we use in generating the example of a multiregional life table in the next section.

6 The multiregional exact age life table

From the matrix of **q** rates generated from the **Q** life-table accounts (or estimated from a set of **h** rates generated from the **K** age-group accounts), we can compute the sets of figures that constitute the multiregional exact age life table. Here, we assume that the reader is reasonably familiar with the methods used to compute a conventional closed system single-region life table[4], and develop the argument for a multiregional exact-age life table. We draw heavily on the work of Rogers (1971, 1973a, 1973b),

[4] If this is not the case, and if the following section gives difficulty, the reader should refer to an account of the life table in a standard demographic text. The best account read by the authors is given in Pressat (1972, chapter 6).

and Rogers and Ledent (1974), though we adopt our own subscript notation that connects to life-table accounts.

The life table is a device for computing the likely life history of a hypothetical cohort called the radix, which is set to some arbitrary size of say, 1000000. The size of the cohort steadily diminishes as it ages until, at some final exact age, it contains no living persons. An alternative view provides a representation wholly in probability terms. Rather than define a cohort of an arbitrary size at birth in a region in a period, we can define a probability of being in that cohort, with a maximum value of 1. In this account of the life table, we substitute a probability of 1·000000 for a radix of 1000000. We also generalise the notion of a cohort so that any collection of persons attaining an exact age in a region in a period can constitute a 'residential exact-age' cohort. A life table can be generated for any residential exact-age cohort.

Let us adopt ${}_{x}^{i}l_{y}^{j}$ as the variable referring to this 'resident' exact-age cohort, where we assign it the following precise meaning: the probability that persons attaining exact age x in region i will attain exact age y in region j. When we are considering a birth cohort of persons attaining exact age 0 in a region in a period, then this variable reads as ${}_{0}^{i}l_{y}^{j}$. Note that when $y = x$, ${}_{x}^{i}l_{x}^{i}$ has a positive value, often of 1 [5], but that ${}_{x}^{i}l_{x}^{j}$, $j \neq i$ has a zero value since one cannot simultaneously be both in region i and region j at exact age x.

These survivorship probabilities can be disaggregated further by considering the probability of a person being in one of the life-table accounts transitions. Let ${}_{x}^{i}l_{yy+T}^{jk}$ be the probability that persons who are in the region i exact age x cohort will attain exact age y in region j, and will migrate to, and survive in, region k to attain exact age $y+T$; and let ${}_{x}^{i}d_{yy+T}^{j\delta(k)}$ be the probability that persons who are in the region i exact-age x cohort will attain exact age y in region j, and will migrate to and then die in region k before attaining exact age $y+T$.

These variables are estimated by multiplying the ${}_{x}^{i}l_{y}^{j}$ survivorship probabilities by the life-table transition rates, q_{yy+T}^{jk} and $q_{yy+T}^{j\delta(k)}$:

$$
{}_{x}^{i}l_{yy+T}^{jk} = {}_{x}^{i}l_{y}^{j}\, q_{yy+T}^{jk} , \tag{52}
$$

$$
{}_{x}^{i}d_{yy+T}^{j\delta(k)} = {}_{x}^{i}l_{y}^{j}\, q_{yy+T}^{j\delta(k)} . \tag{53}
$$

Equations (52) and (53) embody the assumptions that

$$
{}_{x}^{i}q_{yy+T}^{jk} = q_{yy+T}^{jk} , \tag{54}
$$

and that

$$
{}_{x}^{i}q_{yy+T}^{j\delta(k)} = q_{yy+T}^{j\delta(k)} , \tag{55}
$$

where the left-hand variables are the transition rates for each region i exact-age x cohort. Sometimes these assumptions are unreasonable, particularly if region j is the rest of the world, and the transition rates will require modification.

Equations (52) and (53) have, in effect, partitioned the ${}_{x}^{i}l_{y}^{j}$ survivorship probabilities amongst the possible transitions that can occur between exact ages y and $y+T$. We have a 'row' accounting equation for the survivorship probabilities:

$$
{}_{x}^{i}l_{y}^{j} = \sum_{k} {}_{x}^{i}l_{yy+T}^{jk} + \sum_{k} {}_{x}^{i}d_{yy+T}^{j\delta(k)} . \tag{56}
$$

(5) The initial value of a residential cohort, ${}_{x}^{i}l_{x}^{i}$, can be set to one if the single region i is being looked at in a multiregional context. However, if one is looking at a set of regions at the same time, ${}_{x}^{i}l_{x}^{i}$ may be set equal to the proportion of the regional system's population attaining exact age x in a period that falls in region i. In that case the *sum* of the ${}_{x}^{i}l_{x}^{i}$'s for all regions i which are in the regional system would equal unity.

The 'column' accounting equation enables us to compute the next survivorship probability, ${}_x^{i}l_{y+T}^{k}$, as

$$ {}_x^{i}l_{y+T}^{k} = \sum_j {}_x^{ij}l_{yy+T}^{k} \,, \tag{57} $$

which adds up the probabilities of region i exact-age x cohort persons migrating back to, or surviving in, region k to attain exact age $y+T$.

The sequence of calculations that generates the full set of ${}_x^{ij}l_{yy+T}^{k}$'s and ${}_x^{i}d_{yy+T}^{j\delta(k)}$'s from exact age $y = x$ to $y = \omega$, the last exact age beyond which everyone is assumed to die, is as follows.

Step (1). Initial values are assumed for the ${}_x^{i}l_x^{j}$'s for all regions i and all regions j. These are the probabilities of being in the initial exact-age–region cohort. A number of alternatives are possible.

We might adopt 'the single region in a multiregional framework' point of view and set,

$$ {}_x^{i}l_x^{i} = 1 \qquad \text{for all regions } i \,. \tag{58} $$

Or one might set the ${}_x^{i}l_x^{i}$'s equal to the proportions of the system-of-interest's births that region i produces in a particular period, with the constraint that

$$ \sum_i {}_x^{i}l_x^{i} = 1 \,. \tag{59} $$

In either case

$$ {}_x^{i}l_x^{j} = 0 \tag{60} $$

for all regions $j \neq i$, because persons cannot simultaneously attain exact age x in two places. Note that all ${}_x^{i}l_y^{j}$'s in which $j \neq i$, but in which $y > x$, can have positive probabilities.

Step (2). The second step is to compute the ${}_x^{ij}l_{yy+T}^{k}$'s and ${}_x^{i}d_{yy+T}^{j\delta(k)}$'s by using equations (52) and (53) for all sets of regions i, j, and k. Initially, y is set to the value x.

Step (3). We then compute the ${}_x^{i}l_{y+T}^{k}$'s, using equation (57) for all pairs of regions i and k, and summing over regions j.

Step (4). The value of y is updated to that of $y+T$, and the values of the ${}_x^{i}l_y^{j}$'s in equations (52) and (53) are updated to those of the ${}_x^{i}l_{y+T}^{k}$'s, for $k = j$, generated using equation (57).

The calculations then return to *step (2)* to compute the next set of y to $y+T$ transitions. *Steps (3)* and *(4)* follow, with continued cycling through the calculations until all y to $y+T$ transitions have been calculated from $y = x$ to $y = \omega$. The calculations can be repeated for each exact-age–region cohort from exact age 0 to exact age ω. This scheme is summarised in figure 13.

These calculations are illustrated numerically in tables 5 and 6 for one cycle of the outer loop of the computational sequence, for the exact-age 0 cohort for the two regions concerned.

Step (0). The starting value of x is set to 0.

Step (1). The initial values of the ${}_x^{i}l_x^{j}$'s are

$$ {}_0^{1}l_0^{1} = 1{\cdot}000000 \qquad {}_0^{1}l_0^{2} = 0{\cdot}000000 $$

$$ {}_0^{2}l_0^{1} = 0{\cdot}000000 \qquad {}_0^{2}l_0^{2} = 1{\cdot}000000 $$

Step (2). The ${}^{ij}_{x}l^{jk}_{yy+T}$ and ${}^{i}_{x}d^{j\delta(k)}_{yy+T}$ calculations begin:

$${}^{1}_{0}l^{11}_{0T} = {}^{1}_{0}l^{1}_{0}\,q^{11}_{0T} = (1\cdot000000)(0\cdot851083) = 0\cdot851083$$

$${}^{1}_{0}l^{12}_{0T} = {}^{1}_{0}l^{1}_{0}\,q^{12}_{0T} = (1\cdot000000)(0\cdot074424) = 0\cdot074424$$

$${}^{1}_{0}d^{1\delta(k)}_{0T} = {}^{1}_{0}l^{1}_{0}\,q^{1\delta(1)}_{0T} = (1\cdot000000)(0\cdot067411) = 0\cdot067411$$

$${}^{1}_{0}d^{1\delta(2)}_{0T} = {}^{1}_{0}l^{1}_{0}\,q^{1\delta(2)}_{0T} = (1\cdot000000)(0\cdot007083) = 0\cdot007083$$

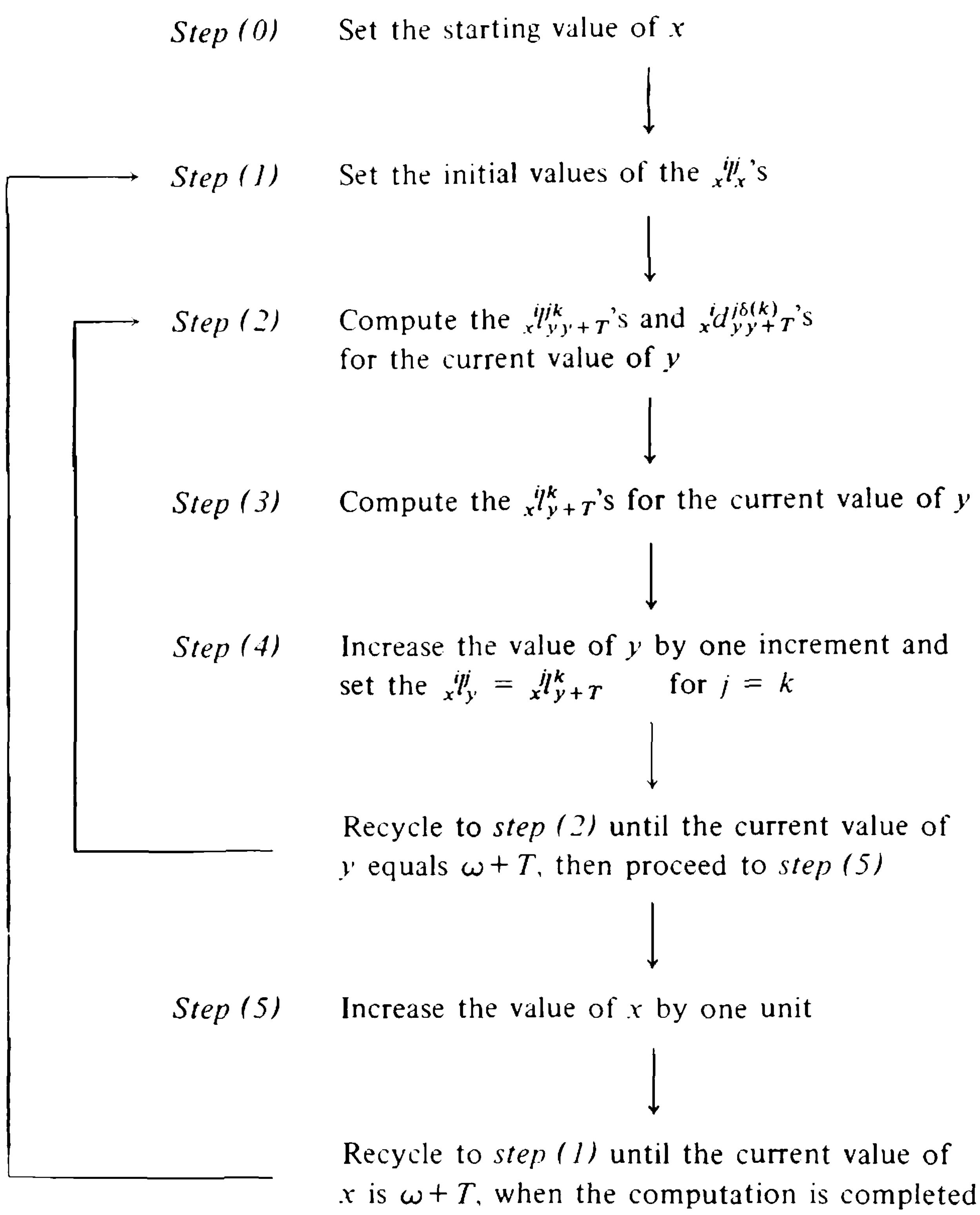

Figure 13. The sequence of life table calculations.

Table 5. The survivorship probabilities for region 1 (the Shire). Exact-age 0, region 1 cohort.

x	${}^{1}_{0}l^{1}_{x}$	${}^{1}_{0}l^{11}_{xx+T}$	${}^{1}_{0}l^{12}_{xx+T}$	${}^{1}_{0}d^{1\delta(1)}_{xx+T}$	${}^{1}_{0}d^{1\delta(2)}_{xx+T}$
0	1·000000	0·851083	0·074424	0·067411	0·007083
20	0·851083	0·592543	0·106385	0·124306	0·027849
40	0·597195	0·198999	0·059720	0·313262	0·025215
60	0·206929	0·031246	0·008277	0·162232	0·005173
80	0·033437	0·000000	0·000000	0·032404	0·001033
x	${}^{1}_{0}l^{2}_{x}$	${}^{1}_{0}l^{21}_{xx+T}$	${}^{1}_{0}l^{22}_{xx+T}$	${}^{1}_{0}d^{2\delta(1)}_{xx+T}$	${}^{1}_{0}d^{2\delta(2)}_{xx+T}$
0	0·000000	0·000000	0·000000	0·000000	0·000000
20	0·074424	0·004652	0·052222	0·000774	0·016776
40	0·158607	0·007930	0·057147	0·002503	0·091027
60	0·116867	0·002191	0·033132	0·001665	0·079879
80	0·041409	0·000000	0·000000	0·000846	0·040563

Step (3). The ${}_x^{i}l^{jk}_{y+T}$'s begin:

$$ {}^{1}_{0}l^{1}_{T} = {}^{1}_{0}l^{11}_{0T} + {}^{1}_{0}l^{21}_{0T} = 0{\cdot}851083 + 0{\cdot}000000 = 0{\cdot}851083 $$

$$ {}^{1}_{0}l^{2}_{T} = {}^{1}_{0}l^{12}_{0T} + {}^{1}_{0}l^{22}_{0T} = 0{\cdot}074424 + 0{\cdot}000000 = 0{\cdot}074424 $$

and so on.

The meaning of the ${}_x^{i}l^{j}_{y}$ variable is a fairly straightforward extension of the conventional variable. For example, the fact that ${}^{1}_{0}l^{1}_{40} = 0{\cdot}597195$ means that there is a just under 60 per cent chance that persons born in region i will survive there to exact age 40. There is a 0·158607 chance (the value of ${}^{1}_{0}l^{2}_{40}$) that they will survive in region 2 at exact age 40. Note that there is a

$$ (0{\cdot}067411) + (0{\cdot}124306) + (0{\cdot}000000) + (0{\cdot}000774) = (0{\cdot}192491) $$

chance that a person born in region 1 died there before attaining exact age 40, and a 0·051708 chance of dying in region 2 before attaining exact age 40.

This second set of probabilities is the death equivalent of the survivorship probabilities, ${}_x^{i}l^{j}_{x}$, and can be defined as ${}_x^{i}d^{*\delta(k)}_{*z}$, the probability that a person in exact-age x, region i cohort will die in region k before attaining exact age z

$$ {}_x^{i}d^{*\delta(k)}_{*z} = \sum_{j} \sum_{y=x}^{z-T} {}_x^{i}d^{j\delta(k)}_{y\,y+T} \; . \tag{61} $$

The following constraint will always hold

$$ \sum_{j} {}_x^{i}l^{j}_{y} + \sum_{k} {}_x^{i}d^{*\delta(k)}_{*y} = 1 \; , \tag{62} $$

which in our illustration above is

$$ 0{\cdot}597195 + 0{\cdot}158607 + 0{\cdot}192491 + 0{\cdot}051708 = 1 \; . \tag{63} $$

The next stage in generating the multiregional life table is to work out how much time persons in each exact-age region cohort spend in the various regions. What we do is to multiply each ${}_x^{i}l^{jk}_{y\,y+T}$ and ${}_x^{i}d^{j\delta(k)}_{y\,y+T}$ quantity by the proportion of the time interval between exact ages spent in a particular region, and by the length of that time interval in years.

Table 6. The survivorship probabilities for region 2 (the rest of Middle Earth). Exact-age 0, region 2 cohort.

x	${}^{2}_{0}l^{1}_{x}$	${}^{2}_{0}l^{11}_{x\,x+T}$	${}^{2}_{0}l^{12}_{x\,x+T}$	${}^{2}_{0}d^{1\delta(1)}_{x\,x+T}$	${}^{2}_{0}d^{1\delta(2)}_{x\,x+T}$
0	0·000000	0·000000	0·000000	0·000000	0·000000
20	0·036451	0·025378	0·004556	0·005324	0·001193
40	0·078110	0·026028	0·007811	0·040973	0·003298
60	0·055857	0·008434	0·002234	0·043792	0·001396
80	0·012611	0·000000	0·000000	0·012221	0·000390
x	${}^{2}_{0}l^{2}_{x}$	${}^{2}_{0}l^{21}_{x\,x+T}$	${}^{2}_{0}l^{22}_{x\,x+T}$	${}^{2}_{0}d^{2\delta(1)}_{x\,x+T}$	${}^{2}_{0}d^{2\delta(2)}_{x\,x+T}$
0	1·000000	0·036451	0·843717	0·001803	0·118629
20	0·843717	0·052732	0·592020	0·008777	0·190188
40	0·596576	0·029829	0·214950	0·009413	0·342385
60	0·222761	0·004177	0·063153	0·003174	0·152257
80	0·065387	0·000000	0·000000	0·001335	0·064052

Let us define the following variables:

${}^{mi}_{x}L^{jk}_{yy+T}$ is the average number of life years lived in region m between exact ages y and $y+T$ by persons who are in region–exact-age cohort i,x and who attain exact age y in region j and exact age $y+T$ in region k;

${}^{\delta(k)i}_{x}D^{j\delta(k)}_{yy+T}$ is the average number of years between exact ages y and $y+T$ which elapse after persons in region–exact-age cohort i,x have died in region k, having attained exact age y in region j;

${}^{m}\phi^{jk}_{yy+T}$ is the average proportion of the time interval between exact ages y and $y+T$ spent alive in region m by persons who attained exact age y in region j, and exact age $y+T$ in region k;

${}^{\delta(k)}\phi^{j\delta(k)}_{yy+T}$ is the average proportion of the time interval between exact ages y and $y+T$ that elapses after persons, who have attained exact age y in region j, have died in region k before attaining exact age $y+T$ (recalling that T is the time interval in years between exact ages used in the life table). Then we can estimate the life years lived in a region as

$$ {}^{mi}_{x}L^{jk}_{yy+T} = T\,{}^{m}\phi^{jk}_{yy+T}\,{}^{i}_{x}l^{jk}_{yy+T} , \tag{64} $$

and the years elapsed after death as

$$ {}^{\delta(k)i}_{x}D^{j\delta(k)}_{yy+T} = T\,{}^{\delta(k)}\phi^{j\delta(k)}_{yy+T}\,{}^{i}_{x}d^{j\delta(k)}_{yy+T} . \tag{65} $$

In the absence of any empirical averages, we adopt the following values for the ϕ's used:

1 for	${}^{m}\phi^{jk}_{yy+T}$	where $j = m$ and $k = m$;
0·5 for	${}^{m}\phi^{jk}_{yy+T}$	where $j = m$ and $k \neq m$, where $j \neq m$ but $k = m$;
0·5 for	${}^{m}\phi^{j\delta(k)}_{yy+T}$	where $j = m$ and $k = m$;
0·25 for	${}^{m}\phi^{j\delta(k)}_{yy+T}$	where $j = m$ and $k \neq m$, where $j \neq m$ but $k = m$;
0 for	${}^{m}\phi^{jk}_{yy+T}$	where $j \neq m$ and $k \neq m$;
0 for	${}^{m}\phi^{j\delta(k)}_{yy+T}$	where $j \neq m$ and $k \neq m$;
0·5 for	${}^{\delta(m)}\phi^{j\delta(k)}_{yy+T}$	where $k = m$;
0 for	${}^{\delta(m)}\phi^{j\delta(k)}_{yy+T}$	where $k \neq m$.

Table 7 sets out the values of ϕ adopted for our two-region example, and tables 8 and 9 show the results of multiplying the terms in the survivorship probabilities tables (tables 5 and 6) by the appropriate ϕ values. The entries in the table are in

Table 7. Assumed values for the average proportions of the time intervals between exact ages spent in the regions.

Residence in region 1

${}^{1}\phi^{11}_{yy+T}$	${}^{1}\phi^{12}_{yy+T}$	${}^{1}\phi^{1\delta(1)}_{yy+T}$	${}^{1}\phi^{1\delta(2)}_{yy+T}$	${}^{1}\phi^{21}_{yy+T}$	${}^{1}\phi^{22}_{yy+T}$	${}^{1}\phi^{2\delta(1)}_{yy+T}$	${}^{1}\phi^{2\delta(2)}_{yy+T}$
1	0·5	0·5	0·25	0·5	0	0·25	0

Residence in region 2

${}^{2}\phi^{11}_{yy+T}$	${}^{2}\phi^{12}_{yy+T}$	${}^{2}\phi^{1\delta(1)}_{yy+T}$	${}^{2}\phi^{1\delta(2)}_{yy+T}$	${}^{2}\phi^{21}_{yy+T}$	${}^{2}\phi^{22}_{yy+T}$	${}^{2}\phi^{2\delta(1)}_{yy+T}$	${}^{2}\phi^{2\delta(2)}_{yy+T}$
0	0·5	0	0·25	0·5	1	0·25	0·5

Death in region 1

${}^{\delta(1)}\phi^{11}_{yy+T}$	${}^{\delta(1)}\phi^{12}_{yy+T}$	${}^{\delta(1)}\phi^{1\delta(1)}_{yy+T}$	${}^{\delta(1)}\phi^{1\delta(2)}_{yy+T}$	${}^{\delta(1)}\phi^{21}_{yy+T}$	${}^{\delta(1)}\phi^{22}_{yy+T}$	${}^{\delta(1)}\phi^{2\delta(1)}_{yy+T}$	${}^{\delta(1)}\phi^{2\delta(2)}_{yy+T}$
0	0	0·5	0	0	0	0·5	0

Death in region 2

${}^{\delta(2)}\phi^{11}_{yy+T}$	${}^{\delta(2)}\phi^{12}_{yy+T}$	${}^{\delta(2)}\phi^{1\delta(1)}_{yy+T}$	${}^{\delta(2)}\phi^{1\delta(2)}_{yy+T}$	${}^{\delta(2)}\phi^{21}_{yy+T}$	${}^{\delta(2)}\phi^{22}_{yy+T}$	${}^{\delta(1)}\phi^{2\delta(1)}_{yy+T}$	${}^{\delta(2)}\phi^{2\delta(2)}_{yy+T}$
0	0	0	0·5	0	0	0	0·5

proportions of the time interval between exact ages, and can be converted to years by multiplication by T, which, in this case, is 20. The very detailed figures in the interior of the table have been summed in two stages to yield ${}^{mi}_{x}L^{j*}_{yy+T}$ as the average number of life years lived in region m, between exact ages y and $y+T$, by persons in the region-exact-age cohort i,x who attained exact age y in region j; and ${}^{mi}_{x}L^{**}_{yy+T}$ as the average number of years lived in region m, between exact age y and $y+T$, by persons in the region-exact-age cohort i,x.

These variables inform us about the average time spent in various transitions by members of the cohort under consideration. For example, the expected number of years lived in region 2 between exact ages 40 and 60 by persons born in region 2 is

$$ {}^{22}_{0}L^{**}_{40\,60} = (0{\cdot}408142) \times 20 = 8{\cdot}16284 \; . \tag{66} $$

Now the only members of the exact-age x, region i cohort who could have lived in these years are the persons who survive to exact age y. So, to work out the expected number of years lived by persons who survive to exact age y, we divide ${}^{mi}_{x}L^{jk}_{yy+T}$ by ${}^{ij}_{x}l_{y}$ to yield the conditional life expectancy ${}^{mi}_{x}e^{jk}_{yy+T}$,

$$ {}^{mi}_{x}e^{jk}_{yy+T} = \frac{{}^{mi}_{x}L^{jk}_{yy+T}}{{}^{ij}_{x}l_{y}} \; . \tag{67} $$

In the example cited above, the life expectancy in region 2 between exact ages 40 and 60 by persons born in region 2 conditional on survival to exact age 40 is

$$ {}^{22}_{0}e^{**}_{20\,40} = \frac{{}^{22}_{0}L^{**}_{20\,40}}{{}^{2}_{0}l^{*}_{20}} = \frac{8{\cdot}16284}{(0{\cdot}078110+0{\cdot}596576)} = 12{\cdot}09872 \; . \tag{68} $$

Normally we are interested in more aggregated variables than this, namely, the expected number of years lived after a given exact age y, and the life expectancy beyond exact age y given survival to exact age y. We can define the expected number of years in prospect beyond an exact age z as

$$ {}^{mi}_{x}T^{**}_{z*} = \sum_{y=z}^{\omega} [{}^{mi}_{x}L^{**}_{yy+T}] \; , \tag{69} $$

or the sum of the relevant life-years-lived components from exact age z to the final exact-age transition ω to $\omega+T$. Note that although we could define a variable ${}^{mi}_{x}T^{j*}_{z*}$ as

$$ {}^{mi}_{x}T^{j*}_{z*} = \sum_{y=z}^{\omega} [{}^{mi}_{x}L^{j*}_{yy+T}] \; , \tag{70} $$

it is not very meaningful.

Table 10 presents these life-years-lived sums for our two-region example in age-interval proportion terms and in year terms. We then divide each expected life years in prospect by the relevant survivorship probability to yield expectations of life

$$ {}^{mi}_{x}e^{\square *}_{z*} = \frac{{}^{mi}_{x}T^{**}_{z*}}{{}^{i*}_{x}l_{z}} \; , \tag{71} $$

which are listed in table 11 both in age-interval-proportions and in year terms. If we compare the values in table 11 with those in table 10, we see that the length of life expected beyond any age improves as you survive up to that age. The expectation of life anywhere beyond age 80 is, for example, only 0·74848 of a year for persons born in region 1 but for persons born in region 1 who

Table 8. Life years lived and years after death for the cohort of persons born in region 1, expressed in proportions of the time interval T between exact ages.

x	${}^{11}_{0}L^{1*}_{xx+T}$	${}^{11}_{0}L^{11}_{xx+T}$	${}^{11}_{0}L^{12}_{xx+T}$	${}^{11}_{0}L^{1\delta(1)}_{xx+T}$	${}^{11}_{0}L^{1\delta(2)}_{xx+T}$	${}^{11}_{0}L^{2*}_{xx+T}$	${}^{11}_{0}L^{21}_{xx+T}$	${}^{11}_{0}L^{22}_{xx+T}$	${}^{11}_{0}L^{2\delta(1)}_{xx+T}$	${}^{11}_{0}L^{2\delta(2)}_{xx+T}$	${}^{11}_{0}L^{**}_{xx+T}$
0	0·923772	0·851083	0·037212	0·033706	0·001771	0·000000	0·000000	0·000000	0·000000	0·000000	0·923772
20	0·714851	0·592543	0·053193	0·062153	0·006962	0·002520	0·002326	0·000000	0·000194	0·000000	0·717371
40	0·391794	0·198999	0·029860	0·156631	0·006304	0·004591	0·003965	0·000000	0·000626	0·000000	0·396385
60	0·117794	0·031246	0·004139	0·081116	0·001293	0·001512	0·001096	0·000000	0·000416	0·000000	0·119306
80	0·016460	0·000000	0·000000	0·016202	0·000258	0·000212	0·000000	0·000000	0·000212	0·000000	0·016672
	${}^{21}_{0}L^{1*}_{xx+T}$	${}^{21}_{0}L^{11}_{xx+T}$	${}^{21}_{0}L^{12}_{xx+T}$	${}^{21}_{0}L^{1\delta(1)}_{xx+T}$	${}^{21}_{0}L^{1\delta(2)}_{xx+T}$	${}^{21}_{0}L^{2*}_{xx+T}$	${}^{21}_{0}L^{21}_{xx+T}$	${}^{21}_{0}L^{22}_{xx+T}$	${}^{21}_{0}L^{2\delta(1)}_{xx+T}$	${}^{21}_{0}L^{2\delta(2)}_{xx+T}$	${}^{21}_{0}L^{**}_{xx+T}$
0	0·038983	0·000000	0·037212	0·000000	0·001771	0·000000	0·000000	0·000000	0·000000	0·000000	0·038983
20	0·060155	0·000000	0·053193	0·000000	0·006962	0·063130	0·002326	0·052222	0·000194	0·008388	0·123285
40	0·036164	0·000000	0·029860	0·000000	0·006304	0·107252	0·003965	0·057147	0·000626	0·045514	0·143416
60	0·005432	0·000000	0·004139	0·000000	0·001293	0·074584	0·001096	0·033132	0·000416	0·039940	0·080016
80	0·000258	0·000000	0·000000	0·000000	0·000258	0·020494	0·000000	0·000000	0·000212	0·020282	0·020752
	${}^{\delta(1)1}_{0}D^{1*}_{xx+T}$			${}^{\delta(1)1}_{0}D^{1\delta(1)}_{xx+T}$		${}^{\delta(1)1}_{0}D^{2*}_{xx+T}$			${}^{\delta(1)1}_{0}D^{2\delta(1)}_{xx+T}$		${}^{\delta(1)1}_{0}D^{**}_{xx+T}$
0	0·033705			0·033705		0·000000			0·000000		0·033705
20	0·062153			0·062153		0·000386			0·000386		0·062539
40	0·156631	0	0	0·156631	0	0·001252	0	0	0·001252	0	0·157883
60	0·081116			0·081116		0·000832			0·000832		0·081948
80	0·016202			0·016202		0·000423			0·000423		0·016625
	${}^{\delta(2)1}_{0}D^{1*}_{xx+T}$				${}^{\delta(2)1}_{0}D^{1\delta(2)}_{xx+T}$	${}^{\delta(2)1}_{0}D^{2*}_{xx+T}$				${}^{\delta(2)1}_{0}D^{2\delta(2)}_{xx+T}$	${}^{\delta(2)1}_{0}D^{**}_{xx+T}$
0	0·003541				0·003541	0·000000				0·000000	0·003541
20	0·013925				0·013925	0·008388				0·008388	0·022313
40	0·012607	0	0	0	0·012607	0·045514	0	0	0	0·045514	0·058121
60	0·002586				0·002586	0·039940				0·039940	0·042526
80	0·000516				0·000516	0·020282				0·020282	0·020798

Table 9. Life years lived and years after death for the cohort of persons born in region 2, expressed in proportions of the time interval T between exact ages.

x	${}^{12}_{0}L^{1*}_{xx+T}$	${}^{12}_{0}L^{11}_{xx+T}$	${}^{12}_{0}L^{12}_{xx+T}$	${}^{12}_{0}L^{1\delta(1)}_{xx+T}$	${}^{12}_{0}L^{1\delta(2)}_{xx+T}$	${}^{12}_{0}L^{2*}_{xx+T}$	${}^{12}_{0}L^{21}_{xx+T}$	${}^{12}_{0}L^{22}_{xx+T}$	${}^{12}_{0}L^{2\delta(1)}_{xx+T}$	${}^{12}_{0}L^{2\delta(2)}_{xx+T}$	${}^{12}_{0}L^{**}_{xx+T}$
0	0·000000	0·000000	0·000000	0·000000	0·000000	0·018677	0·018226	0·000000	0·000451	0·000000	0·018677
20	0·030616	0·025378	0·002278	0·002662	0·000298	0·028560	0·026366	0·000000	0·002194	0·000000	0·059176
40	0·051246	0·026028	0·003906	0·020487	0·000825	0·017268	0·014915	0·000000	0·002353	0·000000	0·068514
60	0·031796	0·008434	0·001117	0·021896	0·000349	0·002883	0·002089	0·000000	0·000794	0·000000	0·034679
80	0·006209	0·000000	0·000000	0·006111	0·000098	0·000334	0·000000	0·000000	0·000334	0·000000	0·006543
	${}^{22}_{0}L^{1*}_{xx+T}$	${}^{22}_{0}L^{11}_{xx+T}$	${}^{22}_{0}L^{12}_{xx+T}$	${}^{22}_{0}L^{1\delta(1)}_{xx+T}$	${}^{22}_{0}L^{1\delta(2)}_{xx+T}$	${}^{22}_{0}L^{2*}_{xx+T}$	${}^{22}_{0}L^{21}_{xx+T}$	${}^{22}_{0}L^{22}_{xx+T}$	${}^{22}_{0}L^{2\delta(1)}_{xx+T}$	${}^{22}_{0}L^{2\delta(2)}_{xx+T}$	${}^{22}_{0}L^{**}_{xx+T}$
0	0·000000	0·000000	0·000000	0·000000	0·000000	0·921409	0·018226	0·843717	0·000451	0·059015	0·921409
20	0·002576	0·000000	0·002278	0·000000	0·000298	0·715674	0·026366	0·592020	0·002194	0·095094	0·718250
40	0·004731	0·000000	0·003906	0·000000	0·000825	0·403411	0·014915	0·214950	0·002353	0·171193	0·408142
60	0·001466	0·000000	0·001117	0·000000	0·000349	0·142165	0·002089	0·063153	0·000794	0·076129	0·143631
80	0·000098	0·000000	0·000000	0·000000	0·000098	0·032359	0·000000	0·000000	0·000334	0·032025	0·032457
	${}^{\delta(1)2}_{0}D^{1*}_{xx+T}$			${}^{\delta(1)2}_{0}D^{1\delta(1)}_{xx+T}$		${}^{\delta(1)2}_{0}D^{2*}_{xx+T}$			${}^{\delta(1)2}_{0}D^{2\delta(1)}_{xx+T}$		${}^{\delta(1)2}_{0}D^{**}_{xx+T}$
0	0·000000			0·000000		0·000902			0·000902		0·000902
20	0·002662			0·002662		0·004389			0·004389		0·007051
40	0·020487	0	0	0·020487	0	0·004707	0	0	0·004707	0	0·025194
60	0·021896			0·021896		0·001587			0·001587		0·233483
80	0·006111			0·006111		0·000668			0·000668		0·006779
	${}^{\delta(2)2}_{0}D^{1*}_{xx+T}$				${}^{\delta(2)2}_{0}D^{1\delta(2)}_{xx+T}$	${}^{\delta(2)2}_{0}D^{2*}_{xx+T}$				${}^{\delta(2)2}_{0}D^{2\delta(2)}_{xx+T}$	${}^{\delta(2)2}_{0}D^{**}_{xx+T}$
0	0·000000				0·000000	0·059015				0·059015	0·059015
20	0·000597				0·000597	0·095094				0·095094	0·095691
40	0·001649	0	0	0	0·001649	0·171193	0	0	0	0·171193	0·172842
60	0·000698				0·000698	0·076129				0·076129	0·076827
80	0·000195				0·000195	0·032025				0·032025	0·032220

survive to exact age 80 the further expectation of life is 10·00026 years. This figure of 10 years (the 0·00026 is cumulative rounding error) follows from our assumption that persons who die are alive for half the interval between exact ages. Everyone is assumed to die between 80 and 100, and hence are alive for an average of (0·5)20 = 10 years.

The sequence of variables we have computed and the tables containing the numerical values of the variables constitute the multiregional life table. The life table explores the consequences of the operation of a set of **q** rates, constant over time, in terms of the residence, survival and death history of members of particular region, exact-age cohorts (people who attain a particular exact age in the same region in a period).

Table 10. The expected number of life years in prospect beyond exact age z.

	Exact age z	${}^{11}_{0}T^{**}_{z*}$	${}^{21}_{0}T^{**}_{z*}$	${}^{*1}_{0}T^{**}_{z*}$	${}^{12}_{0}T^{**}_{z*}$	${}^{22}_{0}T^{**}_{z*}$	${}^{*2}_{0}T^{**}_{z*}$
Age interval proportions	0	2·173506	0·406452	2·579958	0·187589	2·223889	2·411478
	20	1·249734	0·367469	1·617203	0·168912	1·302480	1·471392
	40	0·532363	0·244184	0·776547	0·109736	0·584230	0·693966
	60	0·135978	0·100768	0·236746	0·041222	0·176088	0·217310
	80	0·016672	0·020752	0·037424	0·006543	0·032457	0·039000
Years	0	43·47012	8·12904	51·59916	3·75178	44·47778	48·22956
	20	24·99468	7·34938	32·34406	3·37824	26·04980	29·42784
	40	10·64726	4·88368	15·53094	2·19472	11·68460	13·87932
	60	2·71956	2·01536	4·73492	0·82444	3·52176	4·34620
	80	0·33344	0·41504	0·74848	0·13086	0·64914	0·78000

Table 11. The expectations of life beyond exact age z.

	Exact age z	${}^{11}_{0}e^{\square *}_{z*}$	${}^{21}_{0}e^{\square *}_{z*}$	${}^{*1}_{0}e^{\square *}_{z*}$	${}^{12}_{0}e^{\square *}_{z*}$	${}^{22}_{0}e^{\square *}_{z*}$	${}^{*2}_{0}e^{\square *}_{z*}$
Age interval proportions	0	2·173506	0·406452	2·579958	0·187589	2·223889	2·411478
	20	1·350324	0·397046	1·747370	0·191909	1·479808	1·671717
	40	0·704368	0·323079	1·027447	0·162648	0·865929	1·028576
	60	0·419950	0·311208	0·731158	0·147952	0·632005	0·779957
	80	0·222751	0·277263	0·500013	0·083887	0·416126	0·500013
Years	0	43·47012	8·12904	51·59916	3·75178	44·47778	48·22956
	20	27·00648	7·94092	34·94740	3·83818	29·59616	33·43434
	40	14·08736	6·46158	20·54894	3·25296	17·31858	20·57152
	60	8·39900	6·22416	14·62316	2·95904	12·64010	15·99914
	80	4·45502	5·54526	10·00026	1·67774	8·32252	10·00026

7 The fundamental matrix and associated model

Stone (1972) has shown that the initial step in generating the life table, namely the generation of survivorship probabilities, can be succinctly represented by the fundamental matrix of Markov chain theory, which is analogous to the matrix multiplier in input–output analysis.

We can define the population-growth model in a life-table context to be

$$\boldsymbol{p}(\theta^T_{t+T}) = \mathbf{C}\boldsymbol{p}(\theta^T_t)+\boldsymbol{b}(\theta^T_{t+T}) \tag{72}$$

where $\boldsymbol{p}$ denotes the column vector of exact-age-cohort populations in period θ^T_{t+T} on the left-hand side, and in period θ^T_t on the right-hand side, where $\mathbf{C}$ is the matrix of exact-age transition coefficients, and $\boldsymbol{b}(\theta_{t+T}^{T})$ is the column vector of births (and immigrants, where relevant) added to $\boldsymbol{p}$ in the period θ_{t+T}^{T}. The $\mathbf{C}$ coefficient matrix in our hypothetical two region example looks as follows:

$$\mathbf{C} = \mathbf{q}' =$$

$$\left[\begin{array}{ccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0{\cdot}851083 & 0 & 0 & 0 & 0 & 0{\cdot}036451 & 0 & 0 & 0 & 0 \\
0 & 0{\cdot}696222 & 0 & 0 & 0 & 0 & 0{\cdot}062500 & 0 & 0 & 0 \\
0 & 0 & 0{\cdot}333222 & 0 & 0 & 0 & 0 & 0{\cdot}050000 & 0 & 0 \\
0 & 0 & 0 & 0{\cdot}151000 & 0 & 0 & 0 & 0 & 0{\cdot}018750 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0{\cdot}074424 & 0 & 0 & 0 & 0 & 0{\cdot}843717 & 0 & 0 & 0 & 0 \\
0 & 0{\cdot}125000 & 0 & 0 & 0 & 0 & 0{\cdot}701681 & 0 & 0 & 0 \\
0 & 0 & 0{\cdot}100000 & 0 & 0 & 0 & 0 & 0{\cdot}360306 & 0 & 0 \\
0 & 0 & 0 & 0{\cdot}040000 & 0 & 0 & 0 & 0 & 0{\cdot}283500 & 0
\end{array}\right] . \tag{73}$$

Transitions in $\mathbf{C}$ are from columns to rows. The reader can check back to table 4 and note the way in which the q_{xx+T}^{11}, q_{xx+T}^{12}, q_{xx+T}^{21}, and q_{xx+T}^{22} rates have been arranged.

Now if the $\boldsymbol{b}$ vector of additions to the population remains constant over time, and $\mathbf{C}$ also stays constant, equation (72) will settle down into an equilibrium in which the $\boldsymbol{p}$ vector of populations is constant. We can, under these conditions, write equation (72) as

$$\boldsymbol{p} = \mathbf{C}\boldsymbol{p}+\boldsymbol{b} \ , \tag{74}$$

because $\boldsymbol{p}(\theta_{t+T}^{T})$ will equal $\boldsymbol{p}(\theta_{t}^{T})$. This equation can be rearranged,

$$\boldsymbol{p}-\mathbf{C}\boldsymbol{p} = \boldsymbol{b} \ , \tag{75}$$

$$(\mathbf{I}-\mathbf{C})\boldsymbol{p} = \boldsymbol{b} \ , \tag{76}$$

where $\mathbf{I}$ is the identity matrix with ones in the principal diagonal, to yield finally

$$\boldsymbol{p} = (\mathbf{I}-\mathbf{C})^{-1}\boldsymbol{b} \ . \tag{77}$$

This inverse matrix $(\mathbf{I}-\mathbf{C})^{-1}$ is what Stone (1972) refers to as the fundamental matrix. It transforms the vector of population inputs into a set of population outputs (cf. economic input-output analysis).

In our simple example the $(\mathbf{I}-\mathbf{C})$ and $(\mathbf{I}-\mathbf{C})^{-1}$ matrices are as follows:

$$(\mathbf{I}-\mathbf{C}) =$$

$$\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0{\cdot}851083 & 1 & 0 & 0 & 0 & -0{\cdot}036451 & 0 & 0 & 0 & 0 \\
0 & -0{\cdot}696222 & 1 & 0 & 0 & 0 & -0{\cdot}062500 & 0 & 0 & 0 \\
0 & 0 & -0{\cdot}333222 & 1 & 0 & 0 & 0 & -0{\cdot}050000 & 0 & 0 \\
0 & 0 & 0 & -0{\cdot}151000 & 1 & 0 & 0 & 0 & -0{\cdot}018750 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0{\cdot}074424 & 0 & 0 & 0 & 0 & -0{\cdot}843717 & 1 & 0 & 0 & 0 \\
0 & -0{\cdot}125000 & 0 & 0 & 0 & 0 & -0{\cdot}701681 & 1 & 0 & 0 \\
0 & 0 & -0{\cdot}100000 & 0 & 0 & 0 & 0 & -0{\cdot}360306 & 1 & 0 \\
0 & 0 & 0 & -0{\cdot}040000 & 0 & 0 & 0 & 0 & -0{\cdot}283500 & 1
\end{array}\right] , \tag{78}$$

and $(\mathbf{I}-\mathbf{C})^{-1} =$ (79)

$$\left[\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0{\cdot}851083 & 1 & 0 & 0 & 0 & 0{\cdot}036451 & 0 & 0 & 0 & 0 \\
0{\cdot}597194 & 0{\cdot}696222 & 1 & 0 & 0 & 0{\cdot}078110 & 0{\cdot}062500 & 0 & 0 & 0 \\
0{\cdot}206929 & 0{\cdot}238246 & 0{\cdot}333222 & 1 & 0 & 0{\cdot}055857 & 0{\cdot}055910 & 0{\cdot}050000 & 0 & 0 \\
0{\cdot}033437 & 0{\cdot}038125 & 0{\cdot}052192 & 0{\cdot}151000 & 1 & 0{\cdot}012611 & 0{\cdot}013300 & 0{\cdot}014306 & 0{\cdot}018750 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0{\cdot}074424 & 0 & 0 & 0 & 0 & 0{\cdot}843717 & 1 & 0 & 0 & 0 \\
0{\cdot}158607 & 0{\cdot}125000 & 0 & 0 & 0 & 0{\cdot}596577 & 0{\cdot}701681 & 1 & 0 & 0 \\
0{\cdot}116867 & 0{\cdot}114660 & 0{\cdot}100000 & 0 & 0 & 0{\cdot}222761 & 0{\cdot}259070 & 0{\cdot}360306 & 1 & 0 \\
0{\cdot}041409 & 0{\cdot}042036 & 0{\cdot}041679 & 0{\cdot}400000 & 0 & 0{\cdot}065387 & 0{\cdot}075683 & 0{\cdot}104147 & 0{\cdot}283500 & 1
\end{array}\right]$$

The elements in each column turn out to be ${}_{x}l^{ij}_{y}$ survivorship probabilities, the ${}_{0}l^{ij}_{y}$ set of which we generated in tables 5 and 6. These 'survivorship from birth' probabilities are set out in the first and sixth columns for regions 1 and 2 respectively; the 'survivorship from exact age 20' probabilities are contained in the second and seventh columns; and so on. The elements of $(\mathbf{I}-\mathbf{C})^{-1}$ indicate the probability of reaching a particular row state given that you started in a particular column state. The matrix model presented in this section can be extended to generate the whole multiregional life table, but we leave that task as an exercise for the reader!

8 Age-group life tables

We could, if we wished, apply the system of equations used to generate the multiregional life table to data from age-group accounts. We would be able to calculate the probability that persons between two exact ages, x and $x+T$, survive T, $2T$, $3T$, ... years beyond a starting point t. Such life tables might well be useful from a population planning point of view. However, they would not be of much use to the actuary or to the individual. If we were constructing a life table from direct data held by an insurance company, we would be averaging over a year or five years many people attaining age x. Given that survival rates change slowly over time, the temporal variance is probably small. There would not be much difference between the survival chances of a person attaining age x at the start of a year and the chances of a person having his xth birthday at the end of a year. However, if we were constructing an age-group life table, we would be averaging over many persons of different exact age at the same point in time. Given that survival rates change radically over age, there would, in the earlier and later age groups, be substantial differences between the survival chances of a person at the beginning of an age group and someone at the end. We leave, therefore, the task of constructing age-group life tables to the interested reader.

9 Concluding comments

We have sought to demonstrate in this paper how the principles of accounting and experience in the construction of age-group accounts may profitably be brought to bear on the problem of constructing life tables that recognise that migration of people between regions occurs. We have drawn on results from the two earlier papers in this "accounts and models" series, and on the discerning work of Rogers and of Stone. However, we should not forget the debt owed to that long deceased German demographer, Lexis, without whose diagram none of the paper would have been possible.

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