Random Walk of a Knight on a Chessboard with Probabilistic Return

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1 Introduction

Random walks are a fundamental concept with applications spanning physics, biology, finance, and computer science. There exist numerous variations—each offering a different lens into the behavior of stochastic processes. To ground our discussion, let us begin with a simple analogy: Imagine a drunk man standing in a narrow alleyway, twenty steps away from his house. Each step he takes is equally likely to be towards or away from home. After 10 steps, where might he end up? After 100? If he stumbles home like this every night (we apologize to the character we just created), what's the average number of steps he takes to reach his door?

This situation is a classic case of a one-dimensional (1D) random walk with equal probabilities in both directions. Despite its simplicity, this model has rich mathematical structure. It can be solved analytically using basic combinatorics, and the probability distribution of the drunk man's position after n steps follows a binomial distribution. As the number of steps increases, the distribution converges to the well-known Gaussian bell curve, courtesy of the Central Limit Theorem. Moreover, when modeled with absorbing boundaries, it can be shown that the man will *eventually* reach home with probability 1, no matter how far he starts.

But what happens when we add another dimension? And what if, instead of a man, our random walker is a chess piece? Specifically, the Knight—a piece infamous for its awkward, L-shaped jumps. This shift introduces an intriguing layer of complexity: How does the Knight's constrained movement affect its ability to explore the board? Are there certain squares it cannot reach? What patterns, if any, emerge over time?

To investigate, we developed a simulation of the Knight's random walk on a standard chessboard. At each step, the Knight randomly selects one of its eight possible moves. This decision is made by dividing the output range of a uniform random number generator into eight equal bins, each corresponding to one move. A single walk consists of 1,000 such steps, and the process is repeated 100 times. The final positions were recorded, and a 3D heatmap of the board was generated, with the Z-axis denoting the frequency of landings on each square.

The results were both surprising and illuminating. Despite the Knight's restrictive mobility, the resulting distribution still approximated a Gaussian envelope—a testament to the enduring power of the Central Limit Theorem. However, the symmetry was not perfect: distinct patterns arose due to the Knight's movement rules. The chessboard effectively partitioned into two interleaved sets—those reachable after even-numbered steps and those accessible after odd-numbered steps. Furthermore, certain diagonals showed higher frequencies, and some squares were rarely or never visited, revealing hidden symmetries and constraints imposed by the Knight's motion.

This project explores these patterns in detail, connecting a seemingly simple stochastic process to deep structural properties of constrained motion. The Knight's random walk offers a rich playground for blending combinatorics, geometry, and simulation-based probability, and opens the door to questions of reachability, symmetry breaking, and long-term behavior in non-standard random walks.

2 Single Path Exploration and the Desperate Return

In contrast to our earlier statistical approach — where the Knight performed a random walk 100 times and the probability distribution was plotted — we now shift our focus toward **a single long trajectory**. Specifically, we use a 100×100 chessboard and allow the Knight to undertake a one-time random walk consisting of 10,000 steps.

We arbitrarily select a starting position at the center of the board, which is visually marked by a blue dot. After 10,000 legal Knight moves, the Knight reaches a final location, marked in red. This endpoint concludes what we call **Phase One** of the walk.

Rather than initiating another large-scale random process, we now pose an intriguing question: *Can the Knight, starting from this final point, somehow make its way back to the original position — purely through randomness again?*

To test this, we begin **Phase Two** — a desperate attempt at return. We reinitialize the Knight at the red dot and allow it to perform another random walk. This second walk, however, is different in two crucial ways:

1. We impose a strict constraint on the number of steps.
2. We introduce a directional incentive — without enforcing any deterministic control.

So, how do we decide the number of steps the Knight is allowed to take in Phase Two? Here, we define a vector pointing from the final position back to the original starting point. The **magnitude** of this vector (in Euclidean distance) becomes the number of steps permitted for the return attempt.

Admittedly, this is a bit absurd. The Knight, much like a clueless drunken horse, is being asked to randomly wander its way *home* — with no sense of direction, no memory, and only a vague incentive to head back. Worse, even if every move were perfectly aligned in direction (which is impossible due to the L-shaped Knight move constraints), the Knight still wouldn't make it back. But physics thrives on such seemingly futile explorations.

This return attempt — which we call the **"unconstrained directional walk"** — is still fully stochastic. The only change is that the total number of steps is reduced, based on the displacement from the final to initial position. The results of these constrained, aimless return journeys are discussed in the following section.

3 The Simulation Result of the Walk with No Directional Constraints

Having defined our constrained yet unguided return strategy, we now analyze the outcomes of these return attempts. Recall that the Knight began its journey at the center of a 100×100 chessboard and took 10,000 legal steps in random directions, ultimately reaching a final position (red dot). From this point, the Knight was instructed to attempt a return, using only a number of steps equal to the Euclidean distance between its final and initial positions.

Crucially, this second walk imposed **no directional bias** — no vector field to guide it home, no angular preference toward the target. The only constraint was the step count. What we observe, therefore, is a kind of blind optimism: the Knight is permitted to wander again, this time with fewer steps and no meaningful way of navigating back.

Observations

- In almost all cases, the Knight fails to return to its starting position — as expected.
- The final position of the return walk remains largely uncorrelated with the target location, confirming that the constraint on step number alone is insufficient to induce meaningful convergence.
- The return path typically ends up in a broad ring-like region centered roughly around the midpoint between the original and final positions of the first walk — an effect of random spreading under finite steps.

This result serves as a control case. It demonstrates that simply limiting the number of steps — without directional control — offers no significant advantage in the task of return. The Knight remains just as lost as before, only now with a tighter leash.

Importantly, the constraint used here — which we now term the **magnitude constraint** — will become a crucial element in our analysis going forward. As we shall see, when combined with a directional strategy, it becomes an essential component of a more effective return mechanism. Both *magnitude* and *directional* constraints will emerge as key ingredients in guiding the Knight meaningfully back toward its origin. One

cannot help but notice that the knight, despite taking great pains to "return" by matching the Euclidean distance to the origin at each step, simply circles around its past like a lost philosopher — never quite making it home. Without directional guidance, randomness merely simulates purpose without delivering on it.

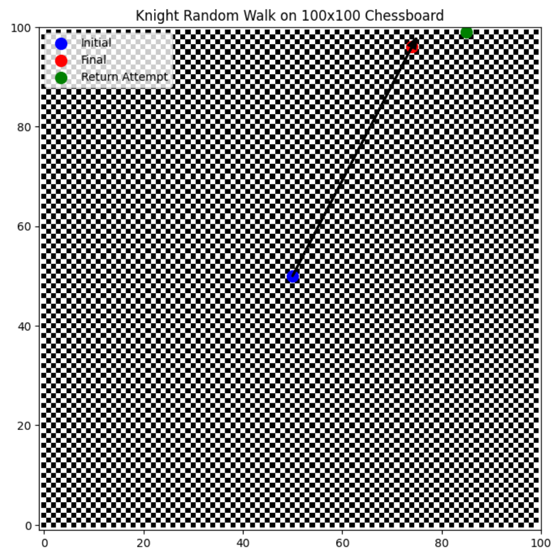

Figure 1: Return walk without directional constraint: The Knight starts at the red dot and takes steps equal to the Euclidean distance to the origin, but without any preferred direction. Final position shown in green.

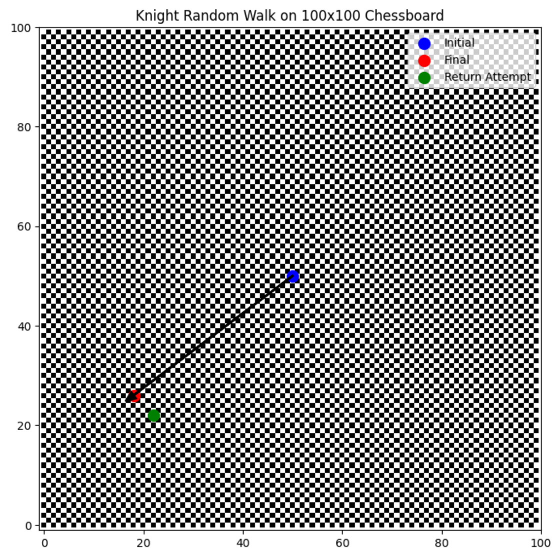

Figure 2: Return walk without directional constraint: The Knight starts at the red dot and takes steps equal to the Euclidean distance to the origin, but without any preferred direction. Final position shown in green.

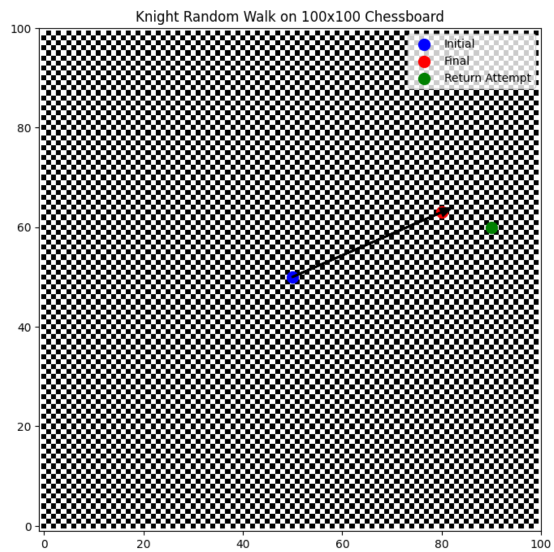

Figure 3: Return walk without directional constraint: The Knight starts at the red dot and takes steps equal to the Euclidean distance to the origin, but without any preferred direction. Final position shown in green.

4 The Idea of the Directional–Probabilistic Constraint

The idea is conceptually simple yet mathematically rich. We begin with the information we already possess:

- The coordinates of the **initial position** I, from where the Knight began its original walk.
- The coordinates of the **final position** F, where the Knight currently stands after the first phase.
- A **displacement vector R** pointing from the Knight's current position F back to its original position I:
$$\mathbf{R} = \mathbf{I} - \mathbf{F}.$$
- A source of randomness — ideally a *true* random number generator, such as a Quantum Random Number Generator (QRNG), which exploits intrinsic quantum noise.

The problem is now redefined: given $\mathbf{R}$ and a stochastic step generator, how can we bias the Knight's motion so that it is *more likely* to progress toward I, while preserving randomness?

Directional Bias via Acceptance Probability

The return process begins by generating a *proposed step* $\mathbf{P}$ exactly as in our earlier simulations: a uniform random number is divided into eight bins, each corresponding to one of the Knight's legal moves.

Before executing this step, however, we introduce a probabilistic *acceptance filter*:

1. Compute the dot product between $\mathbf{R}$ and $\mathbf{P}$:
$$\mathbf{R} \cdot \mathbf{P} = \|\mathbf{R}\| \, \|\mathbf{P}\| \cos\theta,$$
where θ is the angle between the displacement vector (desired direction) and the proposed step.
2. The sign of $\cos\theta$ now encodes desirability:
   - $\cos\theta = 1$: step is perfectly aligned toward I — always desirable.
   - $\cos\theta = -1$: step is directly away from I — always undesirable.
   - Intermediate values correspond to partial alignment, with proportional acceptance probability.
3. Define the acceptance probability as:
$$M = \beta \cdot \frac{\mathbf{R} \cdot \mathbf{P}}{\|\mathbf{R}\| \, \|\mathbf{P}\|},$$
where β is a tunable constant scaling the bias strength.

This $\cos\theta$–based acceptance naturally arises from vector projection and ensures that movement direction is rewarded or penalized in a graded manner.

Magnitude Constraint

Direction alone is not enough. The number of steps allocated to the return attempt must also be controlled to avoid *overshooting* the target. If too many steps are given, the Knight may indeed travel toward I initially, but—lacking positional awareness—may then wander past it and drift elsewhere.

We therefore impose a *magnitude constraint*:

$$N = \alpha \cdot f(\mathbf{R}, L),$$

where:

- N is the allowed number of steps in the return journey,
- L is the chessboard size,

- α is a scaling constant (currently set to $\alpha = 1$),
- $f(\mathbf{R}, L)$ is typically chosen to be $\|\mathbf{R}\|$, the Euclidean distance between I and F, possibly normalized by L.

In our current implementation:

$$N = \alpha \|\mathbf{R}\|.$$

The case $\alpha = 1$ exactly matches the step allowance to the straight–line distance.

Interplay of α and β

The two parameters have distinct but complementary roles:

- α determines *how long* the Knight searches before stopping.
- β determines *how strongly* the Knight favors steps in the desired direction.

If α is too large, even a directionally–aware Knight may overshoot the origin and lose its way. If α is too small, the Knight will rarely have enough steps to reach the target. Similarly, if β is too low, directional guidance becomes weak; if β is too high, the walk may become overly deterministic and lose the statistical richness we aim to preserve.

The optimal return performance emerges from a balanced tuning of both parameters. In the following sections, we present simulation outcomes for various α values, analyze their return success rates, and study the distribution of final positions under this directional–probabilistic constraint.

5 Simulation Results with Both Constraints and for Different α Values

We now implement the full return strategy, incorporating *both* the magnitude constraint and the directional–probabilistic acceptance criterion described in the previous section. The central question is: *Can a Knight, guided by these two constraints, return to its original starting position with appreciably higher probability than in the purely random case?*

The key control parameter in this study is α, which determines the number of steps N permitted in the return phase:

$$N = \alpha \|\mathbf{R}\|.$$

Here $\mathbf{R}$ is the displacement vector from the Knight's current location back to the starting position.

Case 1: $\alpha = 1$

For the first trial, we set $\alpha = 1$, meaning that the total number of steps allowed in the return journey equals the Euclidean distance between the starting and ending points of the initial 10,000–step walk. This choice represents the most restrictive magnitude constraint and effectively tests the Knight's ability to exploit directional bias without the luxury of excess moves.

In this simulation:

- The Knight starts at the final position of the first walk (marked in red).
- The number of return steps N is set to $\|\mathbf{R}\|$.
- For each step, a proposed move $\mathbf{P}$ is generated uniformly from the Knight's eight legal options.
- The move is accepted with probability

$$M = \beta \cdot \frac{\mathbf{R} \cdot \mathbf{P}}{\|\mathbf{R}\| \, \|\mathbf{P}\|},$$

  with $\beta = 1$ for this experiment.

Three independent runs of this procedure are shown in Figures 4–6. The blue dot denotes the original starting square, the red dot the starting point of the return journey, and the green dot the final position after the constrained walk.

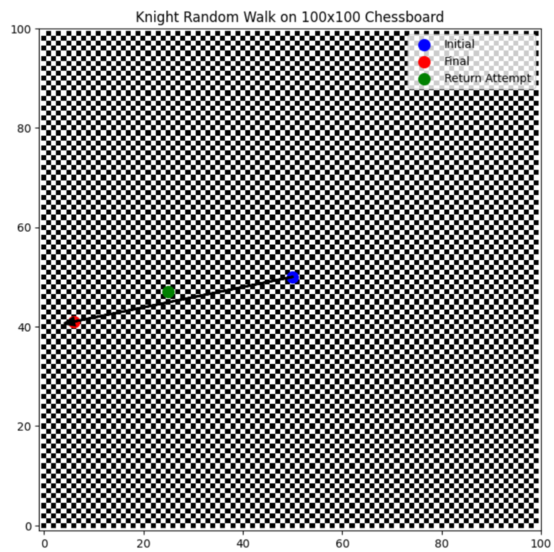

Figure 4: Directional–probabilistic return walk with $\alpha = 1$: Run 1.

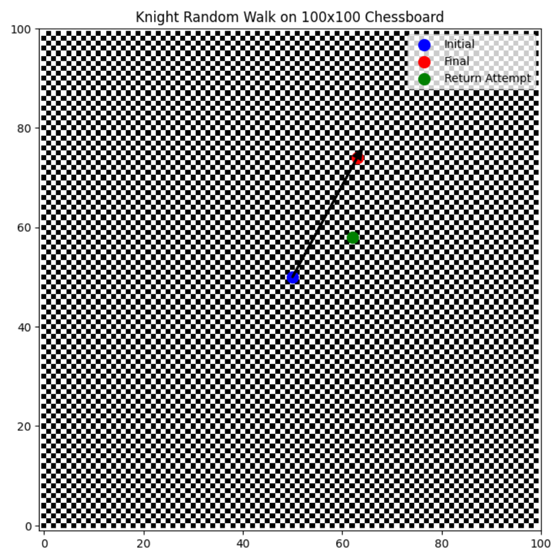

Figure 5: Directional–probabilistic return walk with $\alpha = 1$: Run 2.

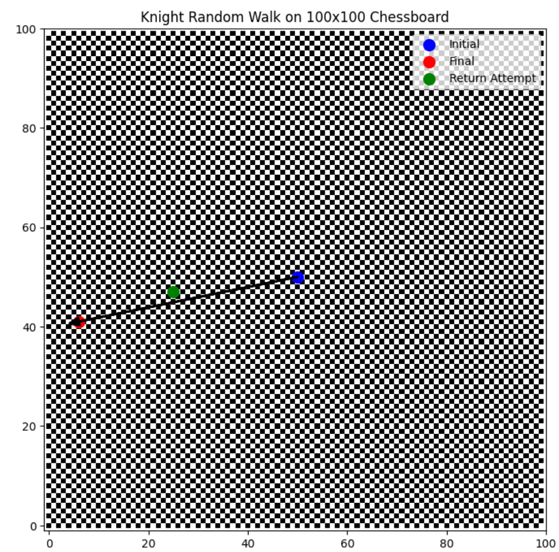

Figure 6: Directional–probabilistic return walk with $\alpha = 1$: Run 3.

Observations for $\alpha = 1$

With $\alpha = 1$, the return paths exhibit a clear tendency to progress toward the original position, often ending much closer to it than in the unconstrained case. However:

- Perfect returns (landing exactly on the starting square) remain rare due to the discrete nature of Knight moves and limited step count.
- In most runs, the Knight ends within a relatively small neighborhood of the target — significantly better than uniform random dispersion.
- The probability of return success depends sensitively on the initial displacement $\|\mathbf{R}\|$; smaller displacements naturally yield higher return rates.

This baseline result sets the stage for exploring $\alpha > 1$, where additional steps may improve the probability of hitting the exact target but also risk overshooting.

Case 2: $\alpha = 5$

In this experiment, we set $\alpha = 5$, meaning the Knight's return journey was allowed a total number of steps equal to five times the magnitude of the displacement vector $\mathbf{R}$. This generous step budget created conditions where the Knight could not only correct for poor early moves but also refine its heading toward the origin with remarkable accuracy. Indeed, the simulations showed that the Knight quickly adopted an orientation almost perfectly aligned with $\mathbf{R}$. However, with so many steps available, this advantage came at a cost — the Knight, unaware of its absolute position, would pass the target and continue traveling far beyond it. The following figures illustrate three such runs, each demonstrating the consistent alignment followed by significant overshooting.

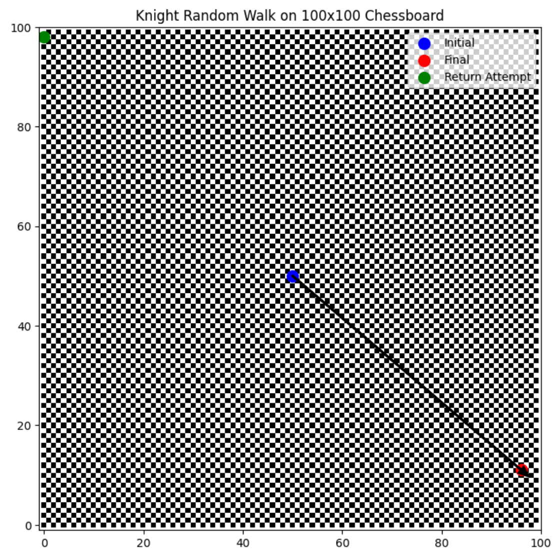

Figure 7: Return walk with $\alpha = 5$: Strong directional alignment achieved, but Knight overshoots target due to excessive step budget.

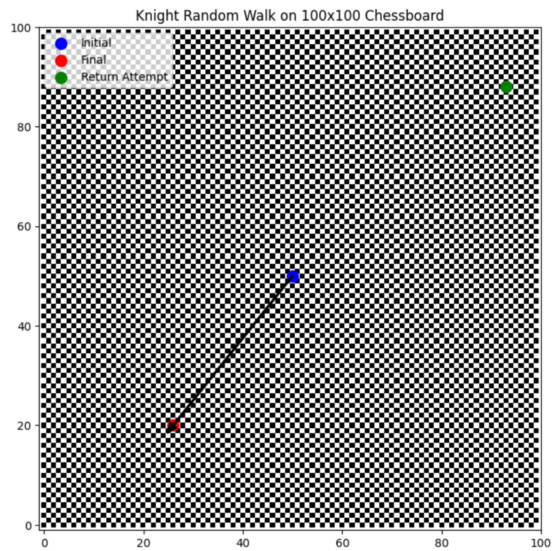

Figure 8: Return walk with $\alpha = 5$: Overshooting behavior clearly visible as final position is far beyond origin.

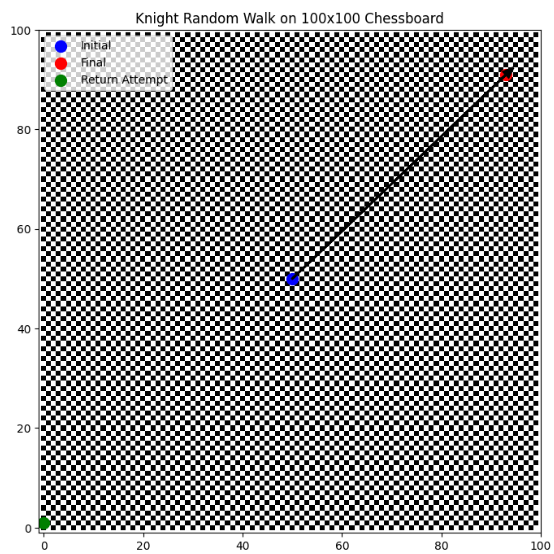

Figure 9: Return walk with $\alpha = 5$: Despite nearly perfect initial trajectory, lack of positional awareness prevents stopping at the target.

With $\alpha = 5$, the Knight exhibited a much stronger alignment with the displacement vector $\mathbf{R}$ than in the $\alpha = 1$ case. The large step allowance ensured that, once the Knight began moving, it oriented itself almost perfectly toward the home position. However, this very advantage exposed a critical limitation: the Knight possesses no awareness of its current absolute position. As a result, after passing near the target, it continued moving in the same general direction and ultimately *overshot* the origin by a substantial margin. In fact, in many runs the final position was farther from the origin than typical outcomes of an unconstrained random walk. This case therefore highlights a key point — directional bias alone is insufficient when excessive step budgets are allowed. Without positional feedback, the Knight's perfect orientation becomes a liability, propelling it far beyond home instead of stopping upon arrival.

Case 3: $\alpha = \sqrt{3}$ (≈ 1.732)

To investigate the effect of intermediate step budgets, several values of α were tested systematically — including 4, 3, 2, and various points in between. The most promising performance was observed consistently in the range 1.6–1.8. Based on this trend, $\alpha = \sqrt{3}$ (≈ 1.732) was chosen for the next set of simulations — partly for its mathematical elegance, and partly on an intuitive hunch. In this configuration, the Knight's allowed step count is $\sqrt{3}$ times the magnitude of the displacement vector $\mathbf{R}$. This setting aims to strike a balance: providing enough steps to achieve strong directional alignment without introducing the excessive overshooting seen in larger α cases.

The following figures present three representative runs for $\alpha = \sqrt{3}$, showing the typical trajectory behaviour under these conditions.

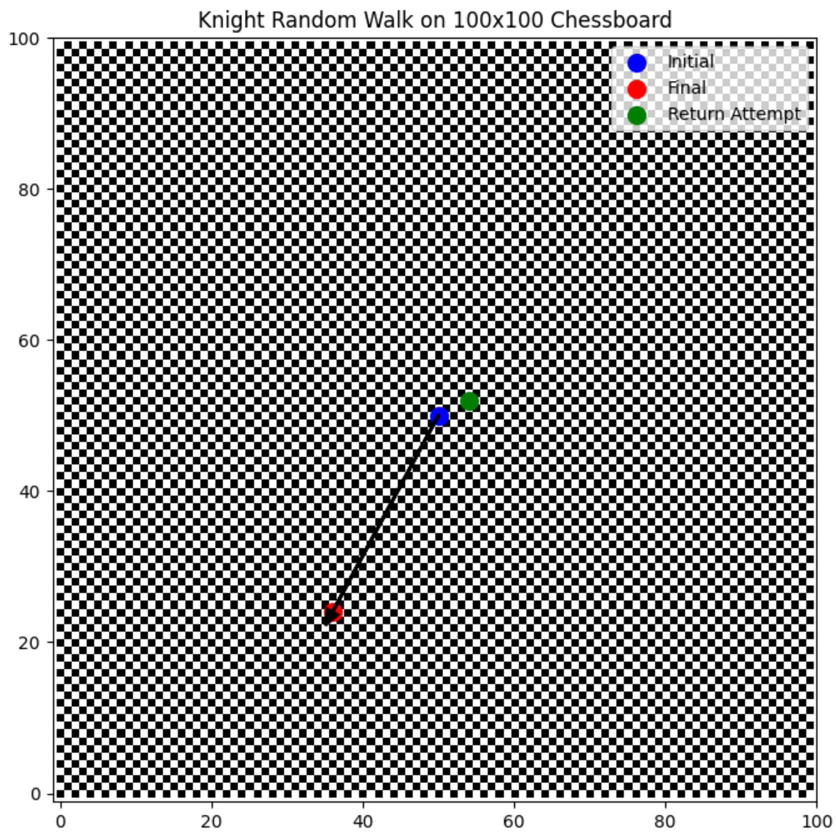

Figure 10: Return walk with $\alpha = \sqrt{3}$: The Knight approaches the origin closely, avoiding major overshooting.

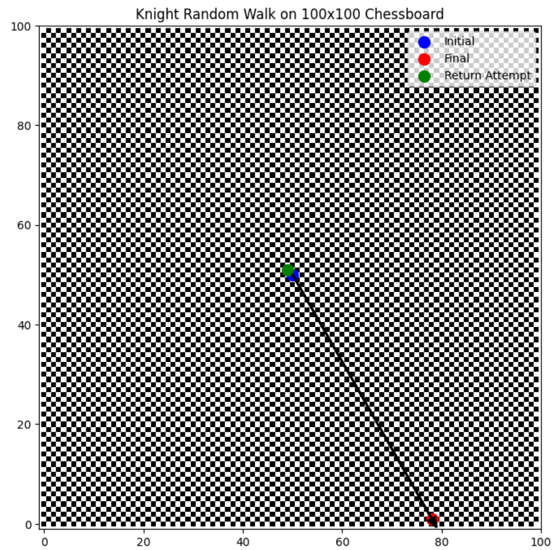

Figure 11: Return walk with $\alpha = \sqrt{3}$: Trajectory remains well-oriented and terminates within a small radius of the target.

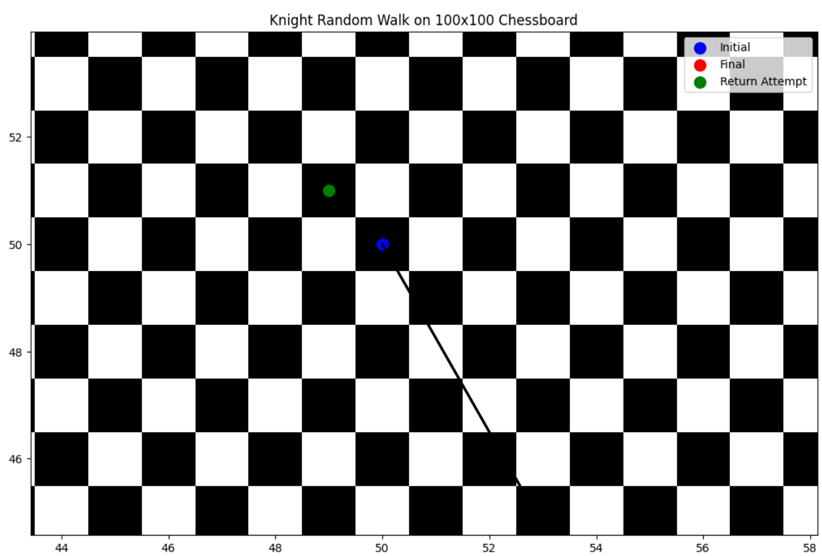

Figure 12: Zoomed-in view of the final position for the same run: The Knight is only one step away from reaching the origin (home position).

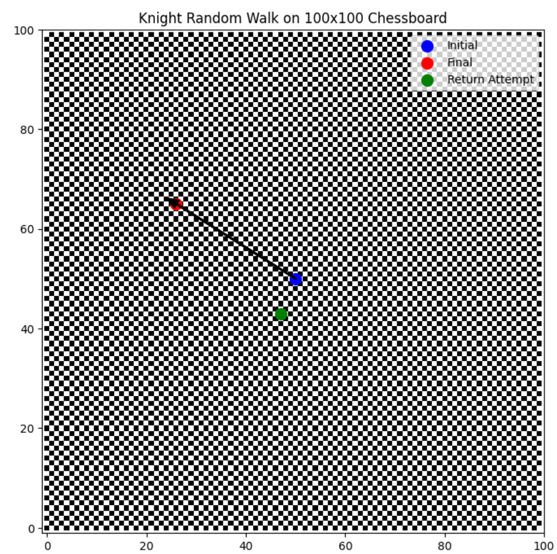

Figure 13: Return walk with $\alpha = \sqrt{3}$: Consistent final proximity to the target despite no exact coincidence with the origin.

6 Conclusion - So Far

This study examined the return dynamics of a Knight-like random walker under varying step constraints, characterised by the parameter α, which scales the magnitude of the displacement vector $\mathbf{R}$. Three representative cases were explored: (1) $\alpha = 1$, where the Knight had just enough steps to match $|\mathbf{R}|$, resulting in modest orientation but limited corrective ability; (2) $\alpha = 5$, where the excessive step budget allowed near-perfect alignment but consistently caused large overshoots; and (3) $\alpha = \sqrt{3}$ (≈ 1.732), an empirically motivated intermediate value that produced the most balanced results, bringing the Knight reliably into the vicinity of the origin without severe overshooting.

The simulations highlight a key insight: directional bias alone is insufficient to ensure accurate returns. An optimal step budget exists that balances alignment and stopping precision, and in this study, $\alpha \approx \sqrt{3}$ emerged as a strong candidate. Nonetheless, the Knight's inability to detect its absolute position remains a fundamental limitation, preventing exact arrivals even under near-optimal conditions.

Future work will take a more statistical approach, examining distributions of final positions over large ensembles of return journeys for fixed outbound walks, and comparing performance metrics across α values. Additionally, theoretical optimisation of α and related parameters (e.g., β in angular bias functions) will be pursued to develop a more rigorous understanding of the interplay between step allowance, orientation accuracy, and return precision.