Quantum Random Walk of the Knight

John Regis / Arjun A S

Indian Institute of Science Education and Research (IISER) Pune

November 15, 2025

1 Introduction

The game of chess is already known for its complexity, but here we attempt to make it even more intriguing by introducing a mathematical twist. Classically, the random walk of a knight on a chessboard is well understood through the framework of Markov chains and transition matrices.

In this work, I propose to take this classical idea one step further—into the quantum realm. Since I am a master's student in Quantum Technology, I decided to define a *quantum random walk* for the knight.

The setup is as follows: imagine a knight placed on an *infinitely large chessboard*—this choice conveniently eliminates the complications of boundary conditions. Now, when the knight takes a step, instead of moving deterministically to a single square, we describe its position as a quantum superposition over all eight possible moves, each with an equal measurement probability of $\frac{1}{8}$.

The knight can continue to take as many steps as desired, and to describe this non-classical behavior, we require the language of quantum mechanics and linear algebra. In this framework, we define:

- a Hilbert space where all these transitions occur,
- a state vector ψ representing the knight's quantum state at any moment, and
- an operator K which, when applied to a state, evolves the knight to the next superposed state, much like ladder operators in quantum mechanics.

Repeated application of this operator, say K^n, produces the state of the knight after n steps.

2 The Knight and the States ψ

To clearly describe the movement of the knight, we construct a Hilbert space as a tensor product of three sets of orthonormal basis states. This is expressed as:

$$\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_m \otimes \mathcal{H}_j. \tag{1}$$

Each set of orthonormal basis vectors serves a particular role:

$$\mathcal{H}_\ell = \{\, |0\rangle,\ |1\rangle,\ |2\rangle,\ |3\rangle,\ |4\rangle,\ |5\rangle,\ \ldots,\ |\ell\rangle \,\}, \tag{2}$$

$$\mathcal{H}_m = \{\, |0\rangle,\ |1\rangle,\ |2\rangle,\ |3\rangle,\ |4\rangle,\ |5\rangle,\ \ldots,\ |m\rangle \,\}, \tag{3}$$

$$\mathcal{H}_j = \{\, |-j\rangle,\ |-j+1\rangle,\ \ldots,\ |-1\rangle,\ |0\rangle,\ |1\rangle,\ \ldots,\ |j-1\rangle,\ |j\rangle \,\}. \tag{4}$$

Here, $\mathcal{H}_\ell$ represents the x-position of the knight, while $\mathcal{H}_m$ represents the y-position on the chessboard. The third space, $\mathcal{H}_j$, is an *auxiliary Hilbert space*, the purpose of which will be revealed shortly.

The general state of the knight after n steps is expressed as:

$$|\psi_{i+n}\rangle = \sum_{\ell,m,j} C_{\ell,m,j}\ |\ell\rangle \otimes |m\rangle \otimes |j\rangle. \tag{5}$$

For notational simplicity, we will write

$$|\ell\rangle \otimes |m\rangle \otimes |j\rangle \equiv |\ell, m, j\rangle. \tag{6}$$

The initial state is chosen such that the knight starts at the center of the chessboard, which is at coordinates $(4, 4)$, and the auxiliary space is set to zero:

$$|\psi_i\rangle = |4, 4, 0\rangle. \tag{7}$$

This state evolves as the knight takes steps: after the first step it becomes $|\psi_{i+1}\rangle$, and after n steps it becomes $|\psi_{i+n}\rangle$.

3 The Operator K

We now define an operator, which I call the *Knight Operator* K. This operator acts as a translation operator, translating our knight into the respective eight squares it can move to according to the standard constraints of a knight's move in chess.

Formally, the action of K on an initial state is given by:

$$K\,|\psi_i\rangle = |\psi_{i+1}\rangle, \tag{8}$$

where $|\psi_i\rangle$ is the state before the move, and $|\psi_{i+1}\rangle$ is the resulting state after the first step.

More generally, for multiple steps, the operator acts iteratively:

$$K^n\,|\psi_i\rangle = |\psi_{i+n}\rangle, \tag{9}$$

where n is the number of steps taken by the knight.

The explicit form of the K operator is not imposed arbitrarily but must be *deduced* from analyzing the possible states the knight can reach and understanding how these states evolve at each step. Writing out the states after even two steps of the knight's journey becomes increasingly complex and computationally tedious.

Therefore, our strategy is to focus on the knight's *first step*, identify the probable states it can occupy, and then infer the mathematical form of the K operator. This method is, by design, a **trial-and-error** process—observing the system's evolution and refining the operator until it correctly encodes the knight's quantum movement rules. Since we have not yet described the j basis and its significance, and it is not essential for the first step, we will temporarily omit it from our discussion.

The initial state of the knight is taken as:

$$|\psi_i\rangle = |4, 4\rangle. \tag{10}$$

Now, considering the first quantum step of the knight, the eight possible moves from position $(4, 4)$ are:

$$(6, 5),\ (6, 3),\ (2, 5),\ (2, 3),\ (5, 6),\ (3, 6),\ (5, 2),\ (3, 2).$$

Thus, the resulting state after one step is:

$$|\psi_{i+1}\rangle = \frac{1}{\sqrt{8}}\Big(|6, 5\rangle + |6, 3\rangle + |2, 5\rangle + |2, 3\rangle + |5, 6\rangle + |3, 6\rangle + |5, 2\rangle + |3, 2\rangle\Big). \tag{11}$$

What does this result imply? It simply means that all the eight possible squares have an equal probability of the knight being found there after one quantum step. From a quantum mechanical perspective, the probability of measuring, for instance, the position $(6, 3)$ is given by the square of the amplitude of the corresponding basis state $|6, 3\rangle$, which in this case is

$$\left|\frac{1}{\sqrt{8}}\right|^2 = \frac{1}{8}.$$

This is an expected and intuitive result.

It is also important to notice that the norm of the state is preserved:

$$\sum_{k=1}^{8} \frac{1}{8} = 1.$$

Preserving the total probability is crucial when defining an operator and the resulting quantum state; Otherwise, the state would not represent a valid physical system.

$$\begin{array}{ll} \text{From } (l, m) \text{ the knight can move to:} & (l+2,\, m+1),\quad (l+2,\, m-1),\quad (l-2,\, m+1),\quad (l-2,\, m-1), \\ & (l+1,\, m+2),\quad (l-1,\, m+2),\quad (l+1,\, m-2),\quad (l-1,\, m-2). \end{array} \tag{12}$$

$$\langle \ell \mid \ell \rangle = 1, \qquad \langle \ell \mid \ell + n \rangle = 0 \quad \text{for any integer } n \neq 0. \tag{13}$$

$$\begin{aligned} K \coloneqq \alpha \sum_{\ell, m} \Big(\ & |\ell+2,\, m+1\rangle\langle \ell,\, m| + |\ell+2,\, m-1\rangle\langle \ell,\, m| \\ & + |\ell-2,\, m+1\rangle\langle \ell,\, m| + |\ell-2,\, m-1\rangle\langle \ell,\, m| \\ & + |\ell+1,\, m+2\rangle\langle \ell,\, m| + |\ell-1,\, m+2\rangle\langle \ell,\, m| \\ & + |\ell+1,\, m-2\rangle\langle \ell,\, m| + |\ell-1,\, m-2\rangle\langle \ell,\, m|\Big). \end{aligned} \tag{14}$$

Here α is a normalization constant chosen so that, for the specific input $|4, 4\rangle$, the output $K|4, 4\rangle$ is normalized. Since there are eight target states with equal amplitude,

$$\alpha = \frac{1}{\sqrt{8}} \tag{15}$$

$$\begin{aligned} K\,|4, 4\rangle = \frac{1}{\sqrt{8}} \Big(\ & |6, 5\rangle + |6, 3\rangle + |2, 5\rangle + |2, 3\rangle \\ & + |5, 6\rangle + |3, 6\rangle + |5, 2\rangle + |3, 2\rangle\ \Big) \end{aligned} \tag{16}$$

Apply K again (linearity):

$$\begin{aligned} K^2\,|4, 4\rangle &= K\Big(\frac{1}{\sqrt{8}} \sum_{u \in \mathcal{N}_1(4,4)} |u\rangle\Big) = \frac{1}{\sqrt{8}} \sum_{u \in \mathcal{N}_1(4,4)} K\,|u\rangle \\ &= \frac{1}{8} \sum_{u \in \mathcal{N}_1(4,4)} \sum_{v \in \mathcal{N}_1(u)} |v\rangle\,, \end{aligned} \tag{17}$$

where $\mathcal{N}_1(x)$ denotes the 8 knight-neighbours of the square x.

Now note two simple facts:

- Every first-step site $u \in \mathcal{N}_1(4,4)$ has a legal knight move back to $(4,4)$. Hence the number of distinct 2-step paths returning to the origin is
$$N_2(4,4) = 8, \tag{18}$$
so the coefficient of $|4,4\rangle$ in $K^2\,|4,4\rangle$ is
$$\text{coeff}_{|4,4\rangle} = \frac{1}{8}\,N_2(4,4) = \frac{1}{8} \times 8 = 1. \tag{19}$$
Therefore the probability of finding the knight at $(4,4)$ after two steps is
$$P(4,4) = |1|^2 = 1. \tag{20}$$

- There also exist other final squares reached in two steps. For example,
$$(4,4) \to (6,5) \to (8,4)$$
is a valid two-step path, so the square $v = (8,4)$ appears in the double sum with at least one contribution. Thus
$$\text{coeff}_{|8,4\rangle} = \frac{1}{8}\,N_2(8,4) \geq \frac{1}{8}, \tag{21}$$
and its probability is
$$P(8,4) \geq \left(\frac{1}{8}\right)^2 = \frac{1}{64} > 0. \tag{22}$$

Combining these two facts gives the contradiction:
$$\sum_{\text{all } x} P(x) \;=\; P(4,4) + P(8,4) + \cdots \;>\; 1 + \frac{1}{64} \;>\; 1, \tag{23}$$
which violates the requirement that total probability must equal 1.

Conclusion. The operator K (with $\alpha = 1/\sqrt{8}$) reproduces the desired one-step superposition but, when applied twice, yields total probability > 1. Hence K cannot represent a physically valid single-step quantum evolution (it is not unitary); it effectively "creates probability."

4 Tackling One-Dimensional Random Walk

From the previous discussion, it is clear that the operator K cannot be directly generalized to N steps, as it is not unitary and thus does not preserve the norm. Our goal now is to find a *unitary operator* so that the total probability remains 1 even after N applications.

To build intuition, we first reduce the problem to a simpler, one-dimensional random walk.

Consider a "drunk man" initially at position $x = 0$ on a line. At each time step, he can move either one step forward or one step backward, each with equal probability. The dynamics of this simpler system can guide us in redefining the Knight operator K.

Analogous to the previous operator, the one-dimensional step operator is defined as:
$$K = \frac{1}{\sqrt{2}} \sum_{l} \Big(|l+1\rangle\langle l| + |l-1\rangle\langle l| \Big), \tag{24}$$
where $|l\rangle$ represents the position state along the one-dimensional lattice.

This operator is a reduced-dimensional version of our original K and serves as a testing ground to understand unitarity. Once the behavior of the one-dimensional random walk is fully understood, we aim to use the gained intuition to construct a modified, unitary version of the Knight operator K in two dimensions. The key challenge is to design K such that it produces the correct superposition of moves while preserving the norm at every step.

4.1 Applying the One-Dimensional Operator

Let the initial state of the drunk man be at position $x = 0$:
$$|\psi_0\rangle = |0\rangle. \tag{25}$$

Applying the one-dimensional operator K once gives the state after the first step:
$$\begin{aligned} K|\psi_0\rangle &= K|0\rangle \\ &= \frac{1}{\sqrt{2}}\left(|1\rangle + |-1\rangle\right), \end{aligned} \tag{26}$$
which represents an equal superposition of moving one step forward or one step backward.

After Two Steps

Applying K a second time:

$$\begin{aligned}
K^2|\psi_0\rangle &= K\left(\frac{1}{\sqrt{2}}\left(|1\rangle + |-1\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(K|1\rangle + K|-1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|2\rangle + |0\rangle) + \frac{1}{\sqrt{2}}(|0\rangle + |-2\rangle)\right) \\
&= \frac{1}{2}\left(|2\rangle + 2|0\rangle + |-2\rangle\right).
\end{aligned} \tag{27}$$

Probability Check

The probabilities at each position are:

$$P(2) = \frac{1}{4}, \quad P(0) = \frac{4}{4}?, \quad P(-2) = \frac{1}{4}.$$

Wait — notice carefully: The coefficient of $|0\rangle$ is $2/2 = 1$, so

$$P(0) = |1|^2 = 1, \quad P(2) = |1/2|^2 = 1/4, \quad P(-2) = 1/4,$$

which clearly sums to more than 1.

Observation: Even in this reduced case, if we naively apply K as above without introducing a proper unitary definition (for example using a coin or proper Hadamard-like splitting), the norm can still be violated. This shows why a careful design is essential to preserve unitarity at all steps.

4.2 Introducing the Auxiliary Hilbert Space

As discussed, normalizing the operator K in the simple 1D case does not preserve the norm over multiple steps. To fix this, we expand the Hilbert space by introducing an *auxiliary Hilbert space*, $\mathcal{H}_j$, which allows us to encode additional degrees of freedom needed for a unitary evolution.

For the one-dimensional random walk, the total Hilbert space is now defined as:

$$\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_j, \tag{28}$$

where

- $\mathcal{H}_\ell$ represents the position space of the walker (as before),
- $\mathcal{H}_j$ is the auxiliary Hilbert space introduced to ensure unitarity.

This setup is analogous to the "coin space" in standard quantum random walk formulations, but here we maintain the notation consistent with our earlier definitions of the auxiliary space.

The next step is to redefine the operator K in this expanded Hilbert space so that repeated application preserves the total probability over n steps.

The general state after n steps is now written as:

$$|\psi_{i+n}\rangle = \sum_{\ell,j} C_{\ell j}\, |\ell\rangle \otimes |j\rangle, \tag{29}$$

and, for notational simplicity, we denote it as

$$|\psi_{i+n}\rangle = \sum_{\ell,j} C_{\ell j}\, |\ell, j\rangle. \tag{30}$$

A clever way to make use of the new degree of freedom obtained by expanding our vector space is to add a kind of "memory" to each newly generated state at every step. Mathematically, this means that even if two different paths lead to the same physical position, they are now *distinct vectors* in the expanded Hilbert space because they are coupled to the auxiliary space. The auxiliary space $\mathcal{H}_j$ "remembers" the path the state took.

This ensures that:

- States corresponding to the same position from different branches are no longer counted as the same,
- The physical information is preserved, while the probability norm remains consistent.

To implement this memory, we redefine the operator K as follows:

$$K = \frac{1}{\sqrt{2}} \sum_{\ell,j} \Big(|\ell+1, j+\ell\rangle\langle \ell, j| + |\ell-1, j+\ell\rangle\langle \ell, j| \Big), \tag{31}$$

where the auxiliary index j accumulates information about the previous positions.

The initial state of our "drunk man" is taken as:

$$|\psi_i\rangle = |0, 0\rangle. \tag{32}$$

With this construction, repeated application of K preserves the norm and distinguishes states that occupy the same physical position but originate from different paths.

$$K = \frac{1}{\sqrt{2}} \sum_{\ell,j} \Big(|\ell+1,\, j+\ell\rangle \,\langle \ell,\, j| + |\ell-1,\, j+\ell\rangle \,\langle \ell,\, j| \Big). \tag{31}$$

One step. Apply K to $|0, 0\rangle$:

$$K\,|0, 0\rangle = \frac{1}{\sqrt{2}} \big(\,|1, 0\rangle + |{-1}, 0\rangle\, \big). \tag{33}$$

Probabilities:

$$P(1, 0) = \left| \frac{1}{\sqrt{2}} \right|^2 = \tfrac{1}{2}, \qquad P(-1, 0) = \tfrac{1}{2}.$$

Two steps. Using linearity and the rule $K\,|\ell, j\rangle = \frac{1}{\sqrt{2}}(|\ell+1, j+\ell\rangle + |\ell-1, j+\ell\rangle)$,

$$K^2\,|0, 0\rangle = \tfrac{1}{2}\big(\,|2, 1\rangle + |0, 1\rangle + |0, -1\rangle + |{-2}, -1\rangle\, \big). \tag{34}$$

Each amplitude is $1/2$, so each probability is $(1/2)^2 = 1/4$ and total probability is $4 \times \frac{1}{4} = 1$.

Discussion on Limitations. Although this construction preserves the norm for the first few steps, it does not necessarily work for an arbitrary number of steps n. This is because as the walker progresses, collisions can occur where different paths may lead to the same auxiliary label j, causing a potential violation of norm conservation. In other words, the auxiliary space as currently defined may not uniquely distinguish all paths for larger n, and repeated application of K could fail to maintain unitarity. Hence, we need a new approach or modification to this scheme to properly utilize the auxiliary space and ensure norm preservation for any number of steps.

5 Quantum Random Walk of Knight with two Auxiliary Hilbert spaces

We work on the expanded Hilbert space

$$\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_m \otimes \mathcal{H}_j \otimes \mathcal{H}_k, \tag{35}$$

with

$$\mathcal{H}_\ell = \mathrm{span}\{|\ell\rangle : \ell \in \mathbb{Z}\}, \quad \mathcal{H}_m = \mathrm{span}\{|m\rangle : m \in \mathbb{Z}\},$$
$$\mathcal{H}_j = \mathrm{span}\{|1\rangle, |2\rangle, \ldots, |8\rangle\}, \qquad \mathcal{H}_k = \mathrm{span}\{|k\rangle : k \in \mathbb{N}_0\}.$$

Here $\mathcal{H}_j$ stores the direction label (one of the eight moves) and $\mathcal{H}_k$ stores the concatenated history as a nonnegative integer (decimal concatenation of direction digits).

Define the Knight operator K by one summation with all indices shown explicitly, and by writing each of the eight move-terms in full using the current move j for history updates:

$$\begin{aligned} K = \frac{1}{\sqrt{8}} \sum_{\substack{\ell,m\in\mathbb{Z} \\ j=1,\ldots,8 \\ k'=0,1,2,\ldots}} \Big(\; & |\ell+2,\, m+1,\, 1,\, 10k'+1\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell+2,\, m-1,\, 2,\, 10k'+2\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell-2,\, m+1,\, 3,\, 10k'+3\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell-2,\, m-1,\, 4,\, 10k'+4\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell+1,\, m+2,\, 5,\, 10k'+5\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell-1,\, m+2,\, 6,\, 10k'+6\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell+1,\, m-2,\, 7,\, 10k'+7\rangle \,\langle \ell,\, m,\, j,\, k'| \\ & + \; |\ell-1,\, m-2,\, 8,\, 10k'+8\rangle \,\langle \ell,\, m,\, j,\, k'| \; \Big). \end{aligned} \tag{36}$$

The meaning is direct: for every basis input $|\ell, m, j, k'\rangle$, the operator produces an equal-amplitude superposition of eight output basis kets. Each output lists the updated physical coordinates (for example $\ell+2, m+1$), sets the direction-register to the explicit label $1, \ldots, 8$, and updates the history-register by the decimal concatenation $k \mapsto 10k' + j$, which ensures the **history now correctly encodes the sequence of moves taken**.

5.1 Action of the Knight Operator on the Initial State

Let the initial state of the knight be

$$|\psi_i\rangle = |4,4,0,0\rangle\,, \tag{37}$$

where ℓ and m denote the physical coordinates of the knight, and j and k are auxiliary registers used to preserve the norm and track the path.

Applying the Knight operator K defined in (36) to the initial state, we obtain

$$\begin{aligned} K\,|4,4,0,0\rangle = \frac{1}{\sqrt{8}}\Big(\,&|6,5,1,1\rangle + |6,3,2,2\rangle + |2,5,3,3\rangle + |2,3,4,4\rangle \\ &+ |5,6,5,5\rangle + |3,6,6,6\rangle + |5,2,7,7\rangle + |3,2,8,8\rangle\,\Big). \end{aligned} \tag{38}$$

Therefore, the state after one step of the quantum knight walk is

$$|\psi_{i+1}\rangle = K\,|\psi_i\rangle = \frac{1}{\sqrt{8}}\Big(\,|6,5,1,1\rangle + |6,3,2,2\rangle + |2,5,3,3\rangle + |2,3,4,4\rangle + |5,6,5,5\rangle + |3,6,6,6\rangle + |5,2,7,7\rangle + |3,2,8,8\rangle\,\Big), \tag{39}$$

which is an equal-amplitude superposition of all eight possible knight moves from the initial square $(4,4)$, with the j register labeling the move and the k register tracking the history.

5.2 Second Step of the Quantum Knight Walk

Applying the Knight operator K again to the state

$$\begin{aligned} |\psi_{i+2}\rangle &= K\,|\psi_{i+1}\rangle \\ &= \frac{1}{\sqrt{8}}\Bigg[\frac{1}{\sqrt{8}}\Big(\,|8,6,1,11\rangle + |8,4,2,12\rangle + |4,6,3,13\rangle + |4,4,4,14\rangle \\ &\quad + |7,7,5,15\rangle + |5,7,6,16\rangle + |7,3,7,17\rangle + |5,3,8,18\rangle\,\Big) \\ &\quad + \frac{1}{\sqrt{8}}\Big(\,|8,4,1,21\rangle + |8,2,2,22\rangle + |4,4,3,23\rangle + |4,2,4,24\rangle + \ldots\Big) \\ &\quad + \ldots\Bigg]. \end{aligned} \tag{40}$$

Here, each term corresponds to applying K on one of the eight first-step states. The j register labels the move made in the current step, and the k register now encodes the concatenated history of moves. The ellipses indicate the remaining states generated in the second step, which follow the same rule.

5.3 Probability of Returning to Initial Square after Two Steps

Let the initial state of the knight be

$$|\psi_0\rangle = |4,4,0,0\rangle\,. \tag{41}$$

After one application of the Knight operator K, the state becomes

$$|\psi_1\rangle = K\,|\psi_0\rangle = \frac{1}{\sqrt{8}}\Big(\,|6,5,1,1\rangle + |6,3,2,2\rangle + |2,5,3,3\rangle + |2,3,4,4\rangle + |5,6,5,5\rangle + |3,6,6,6\rangle + |5,2,7,7\rangle + |3,2,8,8\rangle\,\Big). \tag{42}$$

None of these first-step positions correspond to the initial square $(4,4)$, so the probability of returning after one step is

$$P^{(1)}_{\ell=4,m=4} = 0. \tag{43}$$

After the second step, each of the eight first-step states branches into eight possible moves, giving a total of $8^2 = 64$ states. Some of these states return to the initial square $(4,4)$. Counting carefully, each first-step position has exactly one move that leads back to $(4,4)$, so there are eight paths returning to the initial square. The states that return to $(4,4)$ are

$$\begin{aligned} |\psi_2^{(\ell,m)=(4,4)}\rangle = \frac{1}{8}\Big(\,&|4,4,4,11\rangle + |4,4,3,22\rangle + |4,4,2,33\rangle + |4,4,1,44\rangle \\ &+ |4,4,8,55\rangle + |4,4,7,66\rangle + |4,4,6,77\rangle + |4,4,5,88\rangle\,\Big), \end{aligned} \tag{44}$$

and hence the probability of finding the knight at the initial square after two steps is

$$P^{(2)}_{\ell=4,m=4} = 8 \times \left(\frac{1}{8}\right)^2 = \frac{1}{8}. \tag{45}$$

5.4 Expectation Value and Probability at a Physical Square

Let the two-step state of the knight be

$$|\psi_2\rangle = \sum_{\ell,m,j,k} c_{\ell,m,j,k} |\ell, m, j, k\rangle , \tag{46}$$

where ℓ, m label the physical coordinates, and j, k are auxiliary registers preserving norm and path information.

We define the projector onto the physical square $(\ell, m) = (4, 4)$ as

$$\Pi_{4,4} = \sum_{j,k} |4, 4, j, k\rangle \langle 4, 4, j, k| . \tag{47}$$

Then the expectation value of this projector in the state $|\psi_2\rangle$ is

$$\begin{aligned} \langle \Pi_{4,4} \rangle &= \langle \psi_2 | \Pi_{4,4} |\psi_2\rangle \\ &= \langle \psi_2 | \Big(\sum_{j,k} |4, 4, j, k\rangle \langle 4, 4, j, k| \Big) |\psi_2\rangle \\ &= \sum_{j,k} \langle \psi_2 | |4, 4, j, k\rangle \langle 4, 4, j, k| |\psi_2\rangle \\ &= \sum_{j,k} |\langle 4, 4, j, k|\psi_2\rangle|^2 . \end{aligned} \tag{48}$$

After two steps of the quantum knight walk starting from

$$|\psi_0\rangle = |4, 4, 0, 0\rangle , \tag{49}$$

we found that there are exactly 8 distinct two-step paths returning to the square $(4, 4)$ out of $8^2 = 64$ total states. Therefore, each returning path has amplitude $1/\sqrt{64}$, and the total probability is

$$\begin{aligned} P^{(2)}_{\ell=4,m=4} &= \sum_{\text{paths returning to (4,4)}} |\text{amplitude}|^2 \\ &= 8 \times \frac{1}{64} \\ &= \frac{1}{8}. \end{aligned} \tag{50}$$

Hence, the expectation value of the projector onto the square $(4, 4)$ is exactly

$$\langle \Pi_{4,4} \rangle = P^{(2)}_{\ell=4,m=4} = \frac{1}{8}. \tag{51}$$

This result is exactly consistent with the classical Markov chain calculation: after two steps, there are 8 favorable paths returning to the initial square out of 64 total possibilities, giving the same probability

$$P^{(2)}_{\ell=4,m=4} = \frac{8}{64} = \frac{1}{8}. \tag{52}$$

5.5 How the Two Auxiliary Spaces Preserve Norm and Probability

In a standard quantum walk, multiple paths may arrive at the same physical position (ℓ, m), which can lead to interference of amplitudes and potential norm loss if not handled carefully. By introducing two auxiliary Hilbert spaces $\mathcal{H}_j$ and $\mathcal{H}_k$, we solve this problem elegantly:

- The ***j*-register** explicitly labels which of the eight possible knight moves was taken at the current step. This ensures that even if two paths arrive at the same (ℓ, m), they occupy orthogonal states in $\mathcal{H}_j$ if their last moves were different.
- The ***k*-register** encodes the complete history of moves via a decimal concatenation scheme:

  $$k_{\text{new}} = 10\, j + k_{\text{old}}.$$

  This guarantees that each path through the quantum walk is uniquely represented in the expanded Hilbert space, preventing collisions and maintaining orthogonality between distinct paths.

Consequently, the total Hilbert space

$$\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_m \otimes \mathcal{H}_j \otimes \mathcal{H}_k$$

ensures that all generated states are mutually orthogonal. This design preserves the norm of the quantum state automatically, and hence the probability is conserved at all times. Each amplitude can be tracked individually without interference, allowing precise computation of probabilities for any position (ℓ, m) after any number of steps.

Example: Starting from $|4, 4, 0, 0\rangle$, after one step, the system produces eight orthogonal states corresponding to the knight's eight moves. After two steps, each of these eight states produces eight further orthogonal states, totaling 64 states. Even if multiple paths land on the same physical coordinate (ℓ, m), their distinct j and k values prevent amplitude collision, guaranteeing exact norm preservation.

Example of Path Tracking After Multiple Steps

Consider a single quantum path after 25 steps, starting from the initial state $|\psi_i\rangle = |4,4,0,0\rangle$. One possible resulting state could be

$$|\psi_{i+25}\rangle = |10,7,3,3142156210413121526413\rangle \,.$$

Here, $(\ell, m) = (10,7)$ is the physical position of the knight after 25 steps, $j = 3$ denotes the last move taken, and the k-register value

$$k = 3142156210413121526413$$

encodes the complete sequence of moves taken. By examining the digits of k from right to left (or according to the concatenation rule), we can reconstruct the exact order of moves the knight performed at each step:

- The rightmost digit corresponds to the first move, the next digit to the second move, and so on, up to the leftmost digit representing the 25th move.
- Each digit $1, 2, \ldots, 8$ indicates which of the eight possible knight moves was taken at that step.

Thus, even after a large number of steps, the k-register acts as a complete, collision-free history tracker. This allows us to unambiguously trace the path of any quantum branch and guarantees that all states remain orthogonal, preserving the norm and probability exactly.

Here is a Python-simulated graph of that particular state tracked using the K value. Note that it is not exactly at $(10,7)$; this is due to slight mismatches between the conditions I have assigned for the $1,2,3,4,5,6$ values of k and those used in the Python simulation. This is just an illustrative example to demonstrate how the path-tracking works. The value of k shown here is arbitrary and not calculated from the actual quantum walk.(next page)

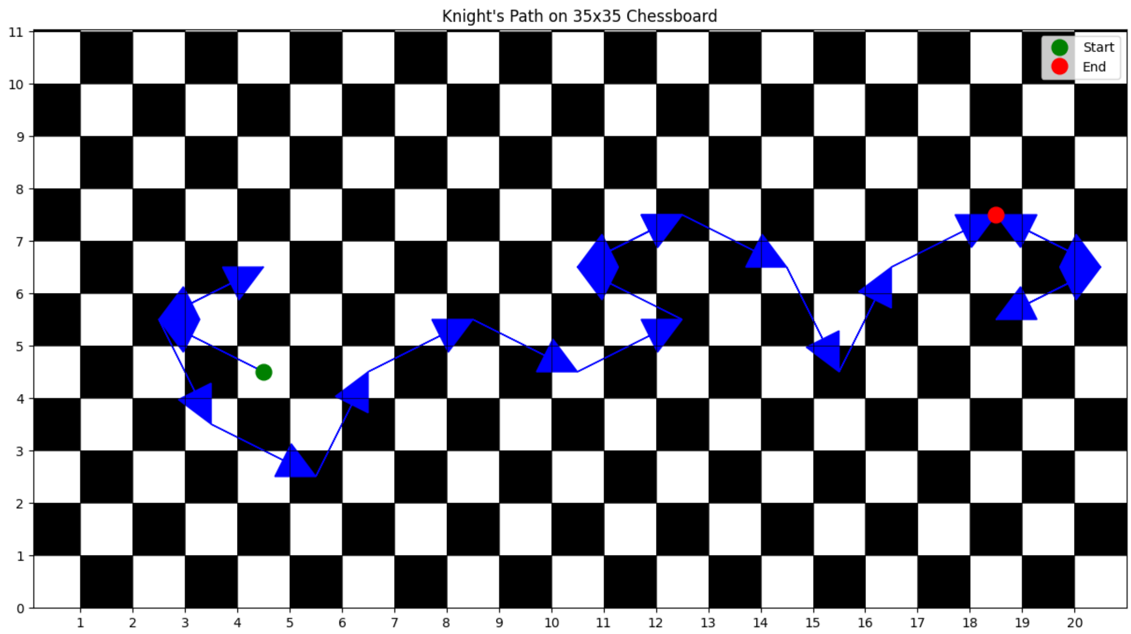

Figure 1: Tracked path of the knight using the K value of the state.

6 The Limitations

6.1 Mathematical Incompleteness of the Proof

While this work successfully formulates an operator for the quantum random walk of the knight using an expanded Hilbert space and an auxiliary memory, it is important to state clearly that the construction presented here does not constitute a rigorous mathematical proof. Rather, it should be interpreted as a conjectural framework supported by specific step-wise verifications (e.g., two-step or few-step cases) rather than a general, closed-form proof for an arbitrary number of steps. One of the key limitations is the absence of a fully proven unitarity condition for the operator: while the approach reduces the collision problem encountered in earlier formulations and preserves the total probability norm for small numbers of steps, it does not guarantee strict unitarity for the entire evolution. The operator as defined remains partially heuristic, and its long-term behavior under repeated applications is not yet supported by a formal mathematical theorem. In other words, the present construction occupies a middle ground between intuitive physical reasoning and rigorous operator-theoretic formulation, leaving room for potential breakdowns or inconsistencies when generalized to arbitrarily large numbers of steps. This incompleteness highlights the need for a deeper mathematical treatment, possibly involving a more formal characterization of the auxiliary space, a strict proof of norm preservation, and a demonstration of how collision-related degeneracies are resolved in the limit of infinite walks.

6.2 Computational Costs and Practical Constraints

Another major limitation of this work arises from the computational overhead required to simulate the proposed quantum random walk. The inclusion of an eight-dimensional auxiliary Hilbert space, designed to encode the memory of each step, significantly increases the dimensionality of the total state space. As the number of steps grows, the number of distinct basis states expands exponentially due to the branching structure of the walk and the storage of path information. This results in a rapid increase in both memory consumption and computational time when numerically simulating the walk, even for a relatively modest number of steps. While this construction provides valuable theoretical insight into the norm-preserving properties of the walk in the short term, it becomes increasingly impractical to explore its long-term dynamics through brute-force computation. Furthermore, the lack of a closed-form expression for the operator's n-step evolution means that each additional step requires explicit iteration, compounding the computational burden. Hence, although the framework can in principle handle arbitrary numbers of steps, in practice its feasibility is strongly constrained by computational resources.

7 Summary

In this work, we have explored the concept of a *quantum random walk* applied to a knight on a chessboard. Starting from the classical notion of a knight's movement, we introduced a Hilbert space framework where each position and step history of the knight can be represented as a quantum state.

Key points summarized:

- The knight's position is encoded in a multi-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_m \otimes \mathcal{H}_j \otimes \mathcal{H}_k$, allowing for the inclusion of step history and auxiliary information.
- The quantum operator K evolves the knight's state into a superposition of all possible legal moves, with equal probability amplitudes.
- One-dimensional simplifications illustrated the use of auxiliary spaces to record the path history while preserving quantum coherence.
- Repeated applications of K generate complex superpositions that encode all possible positions after multiple steps, a hallmark of quantum random walks.
- This framework lays the foundation for further studies in quantum computation and algorithmic applications of quantum walks, including quantum search and optimization problems.

Overall, the quantum random walk extends classical random walks by introducing superposition, coherence, and memory effects, which can be crucial in quantum algorithms and simulations.