Exploring Random Walks: The Knight's Journey on a Chess Board

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1 Introduction

Imagine yourself standing in the streets, the star-studded night sky makes you wonder if earth is indeed the only planet with life and if there might be alien life on other planets. That's when you see your drunk neighbour standing in the road, twenty steps away from his house. The street is narrow and has tall walls on the sides. Due to obvious reasons, he takes each step at random, either towards his house or away from his house. To make our discussion more concrete, let us say that each step he takes has a 50% chance (Probability 0.5) of being towards home and a 50% chance (Probability 0.5) of being away from home. Where do you think the man would end up after 10 steps? What about 100 steps? If he follows the same routine daily (we apologise to the character we created), what is the average number of steps he takes to get home? The above-mentioned situation is the simplest case of a random walk. The random walk has applications in various fields such as physics, computer science, and biology.

2 Knight on a Chess Board

The knight's move on a chessboard is unique, with its characteristic L-shaped movement. What would happen if a knight moved at random? Are there squares in which it's more likely or less likely to be found after a given number of steps? To answer this, we wrote a Python programme to simulate a randomly moving knight on a chessboard. In a nutshell, the programme generates a random integer from 1 to 8 and each integer is mapped to one of the possible 8 moves of the knight. If the move results in the knight going outside the board, it's discarded; otherwise, the knight's position is updated. We make 1000 moves with the knight and note its final position. And repeat this 10000 times. Then, from the data we obtain, we calculate the probability of the knight ending on a given square and plotted.

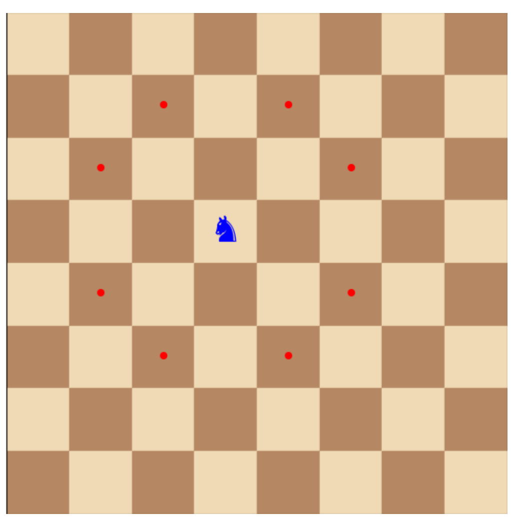
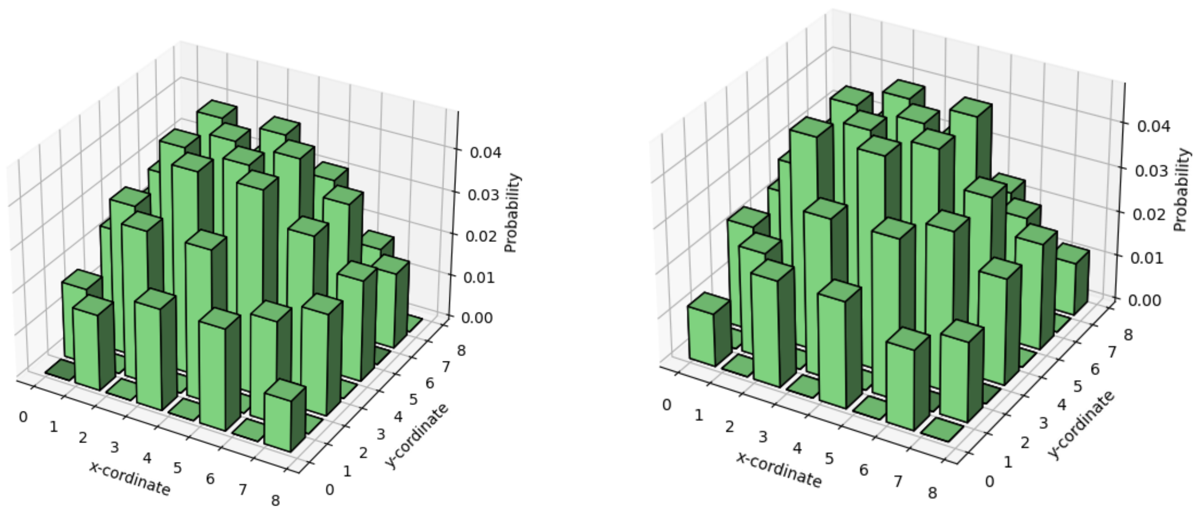


Figure 1: Valid knight moves from square d5 (center).



(a) After 1000 (even) moves.

(b) After 1001 (odd) moves.

Figure 2: Probability distributions after an even and odd number of moves.

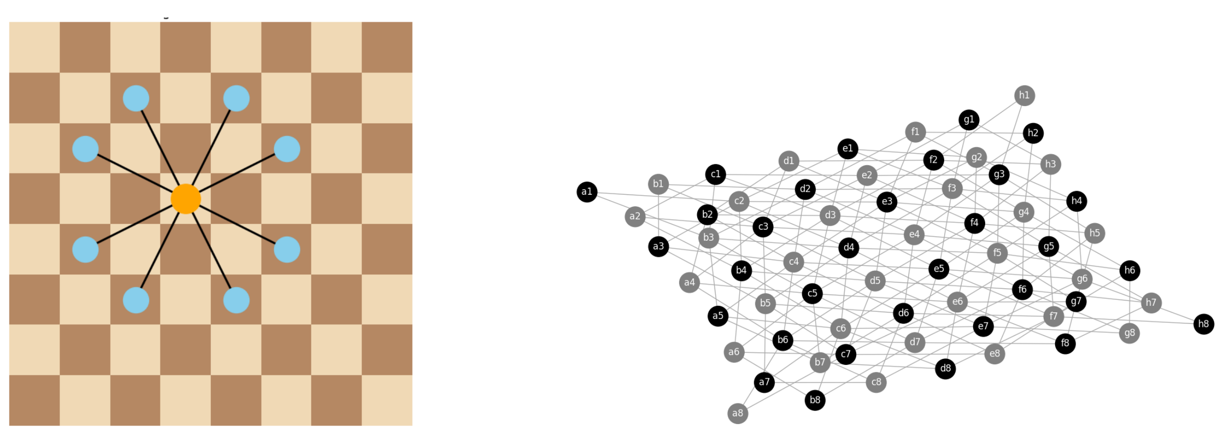
We see that the knight is not found in half of the squares after an odd number of moves, and the other half of the squares after an even number of moves. (Intuitively, this is because the knight always moves to a square of a different colour than what it is present in)

It can also be seen that there is a greater probability of the knight being found in the central squares, followed by the edges, and least likely in the corners.

3 Graphs and Markov Chains

We then tried to explain this behaviour of the knight's random walk with the little mathematics we knew.

First, we converted the chessboard into a graph (or network) such that each square represents a node and two such nodes are linked if and only if a knight can move from one node to the other in a single move. The resulting graph we get is a bipartite graph. (which means that nodes representing white squares are connected only and only to nodes representing black squares, as can be seen in Fig. 3)



(a) Links corresponding to a few nodes

(b) Complete graph (network) of the chess board

Figure 3: Probability distributions after an even and odd number of moves.

The problem of the knight moving on a chessboard is exactly the same as a walker moving from one node to the next at random along the links of the above network.

4 A Simpler Example

Let's work out some ideas in a simple cyclic graph to get an intuition before we go ahead with the knight's graph.

Time for some abstraction!! The most efficient way to represent a network is by an adjacency matrix. This is a matrix such that the element (i,j) = 1 if there is a link between nodes i and j, element (i,j) = 0 if there is no link between nodes (i,j).

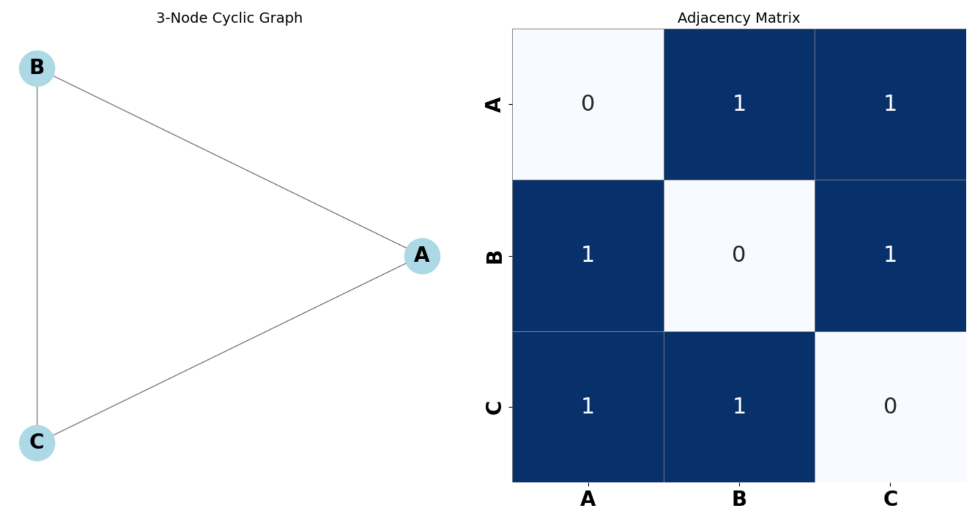


Figure 4: The Adjacency matrix for a cyclic network with three nodes

Now consider what happens when we divide each column of the adjacency matrix by the sum of all elements of the same column. The element (i,j) of the new matrix (Transition matrix) represents the probability of a walker on node i moving to node j in one timestep. (given that the walker is equally likely to move to any adjacent node.)

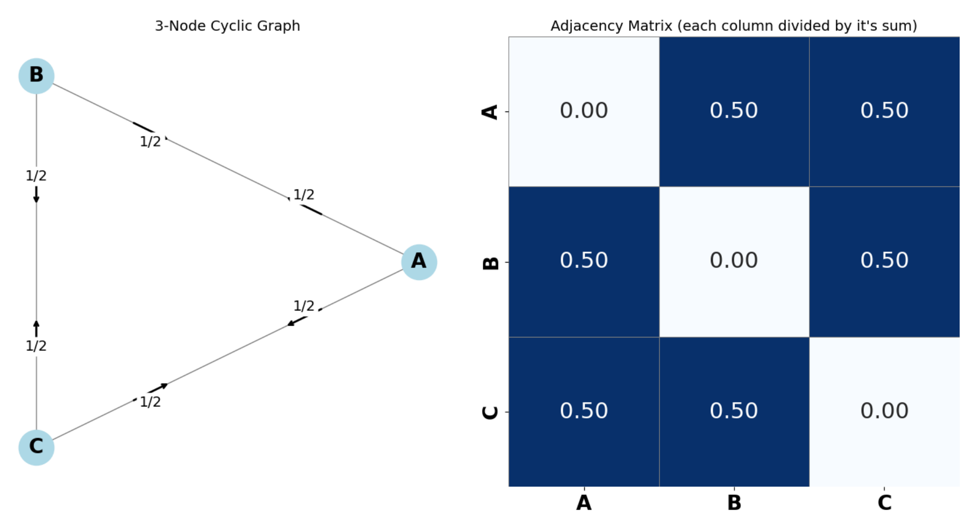


Figure 5: The Transition matrix for a cyclic network with three nodes

5 Powers of the Transition Matrix

Let's dive deeper into the transition matrix T from our three-node cyclic graph. For this graph, the transition matrix is:

$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

What happens if we multiply this matrix by itself to get T^2? The new matrix tells us the probabilities of moving from one node to another in exactly two steps. Let's calculate it:

$$T^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Each entry in T^2, denoted $(T^2)_{ij}$, is the probability of starting at node i and ending at node j after two steps.

For example, look at $(T^2)_{11} = \frac{1}{2}$. This is the probability of starting at node 1 and returning to node 1 after two steps. From node 1, the walker can:

- Move to node 2 (probability $\frac{1}{2}$) and then back to node 1 (probability $\frac{1}{2}$), giving $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
- Move to node 3 (probability $\frac{1}{2}$) and then back to node 1 (probability $\frac{1}{2}$), also giving $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Adding these, we get $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, which matches $(T^2)_{11}$.

Now, consider $(T^2)_{12} = \frac{1}{4}$, the probability of going from node 1 to node 2 in two steps. One path is:

- Node 1 to node 3 (probability $\frac{1}{2}$), then node 3 to node 2 (probability $\frac{1}{2}$), giving $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

There's no other way to reach node 2 in exactly two steps, so the probability is $\frac{1}{4}$.

This shows that T^2 gives us all the two-step probabilities. In general, if we compute T^k, the entry $(T^k)_{ij}$ tells us the probability of going from node i to node j in exactly k steps. This is a powerful tool to understand how our random walker moves over time!

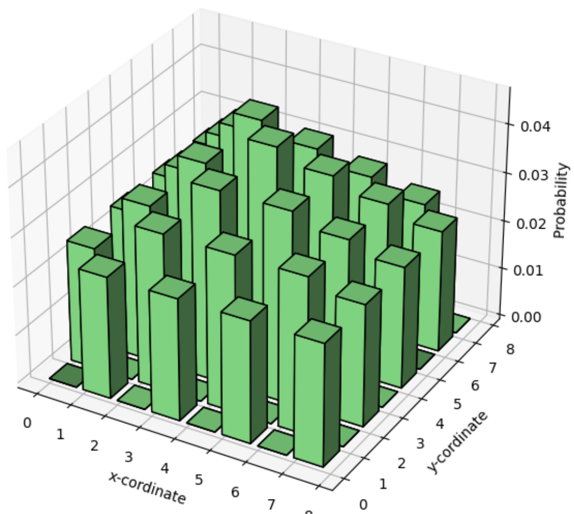
- **Generalising this,** the element (i, j) of the Nth power of the transition matrix represents the probability of the random walker reaching node j from node i in N timesteps.
- **Further,** it can be proved [APPENDIX] that the transition matrices, on acting on any state for a large number of timesteps, on will lead them to a stationary distribution, where the probability of a walker being at a node is proportional to the degree (number of links) of that node.

6 Applying results on the Knight's graph

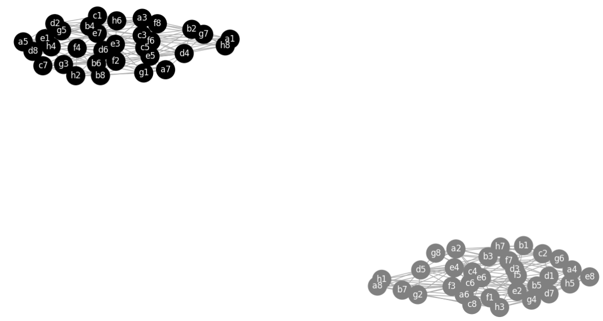
On finding the probability distribution analytically, using the above-mentioned method, we see that the exact same distribution is reproduced.As the distribution oscillates between even and odd timesteps, each of the two can be separately shown to converge, and the higher probability at the centre and lower probability at the corners are explained by the presence ofnodes of higher and lower degrees(degree of a node is the number of connections it has to other nodes).

7 Other chess pieces

We did the same simulation for other chess pieces and found the probability distribution after 1000 moves.

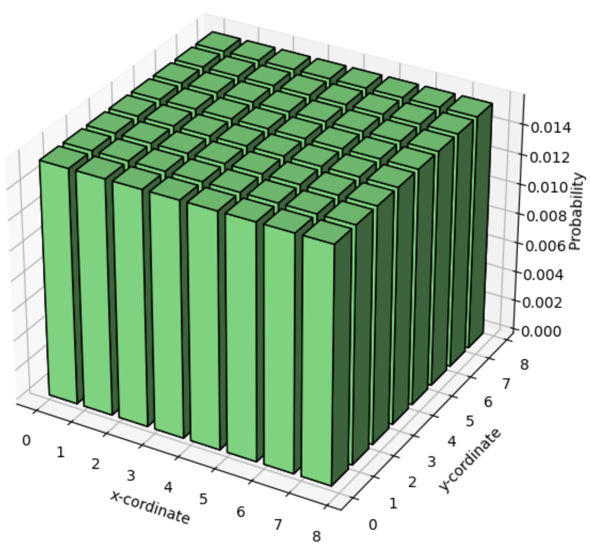


(a) Probability distribution of bishop.

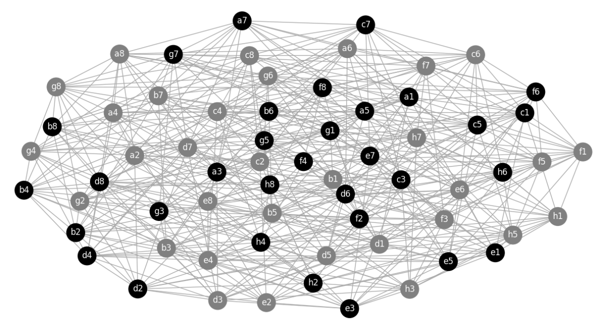


(b) Graph formed by bishop.

Figure 6: Bishop.

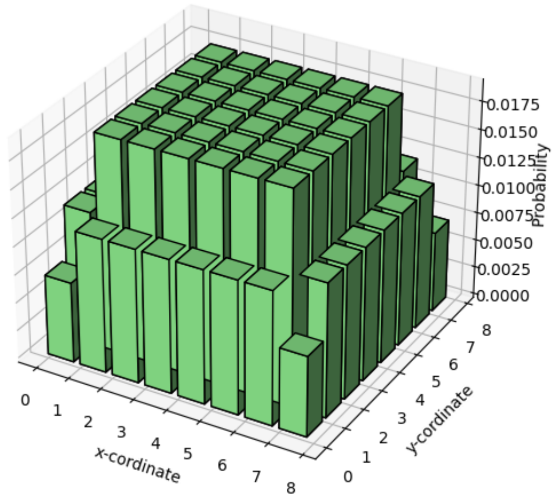


(a) Probability distribution of rook.

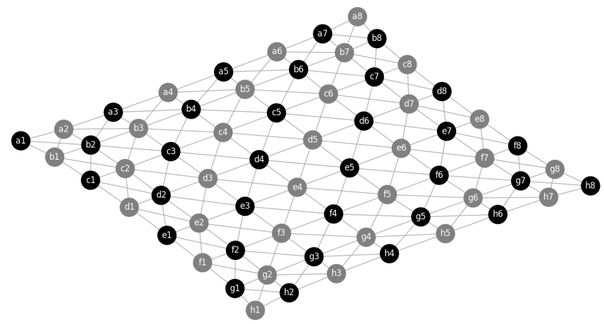


(b) Graph formed by rook.

Figure 7: Rook.

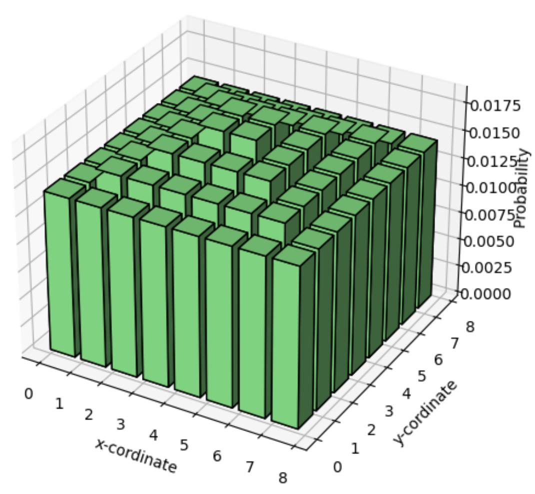


(a) Probability distribution of king.

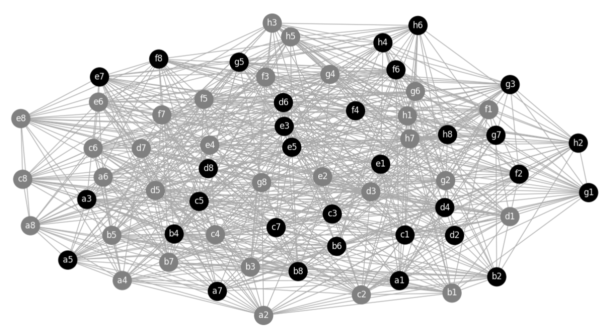


(b) Graph formed by king.

Figure 8: King.



(a) Probability distribution of queen.



(b) Graph formed by queen.

Figure 9: Queen.

8 Appendix: Stationary Distribution of a Random Walk on an Undirected Graph.

Let $G = (V, E)$ be an undirected graph, and let T be the transition matrix of a simple random walk on this graph. The transition probabilities are defined as:

$$T_{ij} = \begin{cases} \frac{1}{\deg(i)} & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

where $\deg(i)$ is the degree of node i, and $i \sim j$ means nodes i and j are connected.

We aim to find a stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ such that:

$$\pi T = \pi \quad \text{and} \quad \sum_i \pi_i = 1$$

Claim

The vector

$$\pi_i = \frac{\deg(i)}{2|E|}$$

is a stationary distribution of the random walk.

Proof

We compute the j-th component of πT:

$$(\pi T)_j = \sum_i \pi_i T_{ij} = \sum_{i\sim j} \pi_i \cdot \frac{1}{\deg(i)} = \sum_{i\sim j} \left(\frac{\deg(i)}{2|E|} \cdot \frac{1}{\deg(i)} \right) = \sum_{i\sim j} \frac{1}{2|E|} = \frac{\deg(j)}{2|E|} = \pi_j$$

Hence, $\pi T = \pi$, satisfying the stationarity condition.

Finally, we check normalization:

$$\sum_i \pi_i = \sum_i \frac{\deg(i)}{2|E|} = \frac{2|E|}{2|E|} = 1$$

Thus, the stationary distribution of a random walk on an undirected graph is proportional to the degree of each node:

$$\pi_i \propto \deg(i)$$