Von Neumann Treatment - Polarization State Measurement of a Photon

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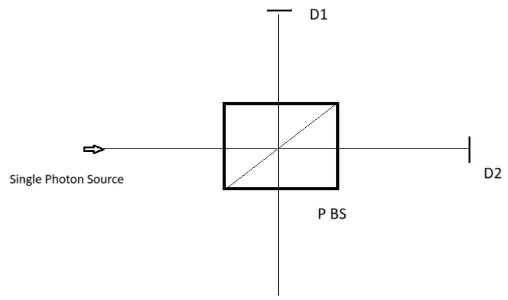

Figure 1: The Setup Diagram

1 Introduction

This is an amateur attempt to explain the detection of polarization qubits and their polarization states using detectors or beam splitters, drawing inspiration from the von Neumann method of orthogonal measurement. I have recently read a portion of Chapter 3 from Preskill's textbook, which discusses the Stern-Gerlach experiment as an example of measuring spin states, such as $+1/2$ or $-1/2$, by applying an external magnetic field. Building on this, I aim to extend the concept to the case of polarization qubits, where instead of spin states $+1/2$ and $-1/2$, we use horizontal ($|H\rangle$) and vertical ($|V\rangle$) polarization states, respectively. My goal is to explore the pointer variable and understand what happens when a detector measures an $|H\rangle$ or $|V\rangle$ state. This exploration is an initial step toward applying these principles to a system of interest.

2 Terms and their meaning

In this setup, we consider a **single photon source** emitting a photon in a polarization state $|\psi\rangle = \alpha |H\rangle + \beta |V\rangle$, where $|H\rangle$ and $|V\rangle$ represent horizontal and vertical polarization states, respectively. The photon passes through a **beam splitter (BS)**, which splits it into two paths: the **direct path** (transmitted) and the **reflected path**. The detectors D_1 and D_2 measure the intensities I_1 and I_2, respectively, which serve as the **pointer variable** outcomes. Additionally, we introduce a controllable parameter λ, which can be turned **ON** or **OFF** to enable or disable the interaction between the photon and the measurement apparatus. Here is a detailed explanation of the key terms:

- λ: A controllable parameter that determines whether the measurement interaction is active ($\lambda = 1$) or inactive ($\lambda = 0$).
- D_1 **and** D_2: Photon detectors placed at the output ports of the beam splitter. D_1 detects the photon in the **direct path**, and D_2 detects the photon in the **reflected path**.
- I_1 **and** I_2: Measurable jhs corresponding to the photon being detected at D_1 ($I_1 = \frac{1}{2}$) or D_2 ($I_2 = -\frac{1}{2}$). These values serve as the pointer variable outcomes.
- P: The momentum conjugate to the pointer variable I. It represents the dynamical variable of the measurement apparatus that couples to the photon's polarization state.
- α **and** β: Complex coefficients in the photon's polarization state $|\psi\rangle = \alpha |H\rangle + \beta |V\rangle$. They determine the probability amplitudes of the photon being in the $|H\rangle$ or $|V\rangle$ state, with $|\alpha|^2$ and $|\beta|^2$ representing the probabilities of measuring $|H\rangle$ or $|V\rangle$, respectively.
- **Direct Path**: The path where the photon continues straight through the beam splitter. For a **polarizing beam splitter (PBS)**, this path corresponds to detecting **horizontal polarization** ($|H\rangle$).
- **Reflected Path**: The path where the photon is reflected at a 90-degree angle by the beam splitter. For a **polarizing beam splitter (PBS)**, this path corresponds to detecting **vertical polarization** ($|V\rangle$).

3 Formalism

Consider a single photon in the polarization state:

$$|\psi\rangle = \alpha |H\rangle + \beta |V\rangle,$$

where $|H\rangle$ and $|V\rangle$ represent horizontal and vertical polarization states, respectively, and α and β are complex coefficients satisfying $|\alpha|^2 + |\beta|^2 = 1$.

System and Pointer Variable

The photon interacts with a measurement apparatus, which has a pointer variable I (intensity) and its conjugate momentum P. The pointer variable I can take two values:

$$I_1 = \frac{1}{2} \quad \text{(for horizontal polarization, } |H\rangle\text{)},$$

$$I_2 = -\frac{1}{2} \quad \text{(for vertical polarization, } |V\rangle\text{)}.$$

Im using this convention to be consistent with the stern-Gerlach model

Interaction Hamiltonian

The interaction between the photon's polarization state and the measurement apparatus is described by the Hamiltonian:

$$H = \lambda P I,$$

where:

- P is the momentum conjugate to the pointer variable I. It represents the dynamical variable of the measurement apparatus that couples to the photon's polarization state.
- I is the pointer variable, which takes the values $I_1 = \frac{1}{2}$ for horizontal polarization ($|H\rangle$) and $I_2 = -\frac{1}{2}$ for vertical polarization ($|V\rangle$).
- λ is a controllable parameter that determines the strength of the interaction. When $\lambda = 1$, the interaction is active, and the photon's polarization state becomes entangled with the pointer variable. When $\lambda = 0$, the interaction is turned off, and no measurement occurs.

This Hamiltonian governs the coupling between the system (photon polarization) and the measurement apparatus, leading to the entanglement of the photon's state with the pointer variable I.

Initially, the pointer variable is in an uncoupled state $\psi(0)$, which is not entangled with the photon's polarization state. We describe the initial state of the pointer variable as $\psi(x)$, representing its position in space. Since the pointer variable is an abstract object propagating freely, its initial state $\psi(x)$ can be thought of as a wavefunction describing its spatial distribution.

Time Evolution and Pointer Variable Coupling

The time evolution of the pointer variable is governed by the time evolution operator $U(t)$, derived from the interaction Hamiltonian $H = \lambda P I$. Here, P is the momentum conjugate to the pointer variable's position, λ is a controllable parameter, and I is the pointer variable associated with the measurement outcomes. The time evolution operator is given by:

$$U(t) = e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{\hbar}\lambda P I t}.$$

To analyze the action of $U(t)$, we expand it in the basis where I is diagonal. Let I have eigenstates $|a\rangle$ with eigenvalues I_a, such that $I|a\rangle = I_a|a\rangle$. In this basis, $U(t)$ can be expressed as:

$$U(t) = \sum_a |a\rangle \langle a| \, e^{-\frac{i}{\hbar}\lambda P I_a t}.$$

Here, I_a are the eigenvalues of I, and $|a\rangle$ are the corresponding eigenstates.

We now recall that the momentum operator P generates translations in the position representation. In the position basis, P acts as:

$$P = -i\hbar \frac{d}{dx},$$

and the translation operator $e^{-ix_0 P/\hbar}$ shifts a wavepacket by x_0:

$$e^{-ix_0 P/\hbar}\psi(x) = \psi(x - x_0).$$

This can be seen by Taylor expanding the exponential:

$$e^{-ix_0 P/\hbar} = \exp\left(-x_0 \frac{d}{dx}\right),$$

which acts on a wavepacket $\psi(x)$ to produce:

$$e^{-ix_0 P/\hbar}\psi(x) = \psi(x - x_0).$$

Now, consider the initial state of the system and pointer. Suppose the quantum system starts in a superposition of eigenstates of I, and the pointer variable is initially in a position-space wavepacket $|\phi(x)\rangle$, unentangled with the system. The initial state is:

$$|\Psi(0)\rangle = \left(\sum_a \alpha_a |a\rangle\right) \otimes |\phi(x)\rangle .$$

Applying the time evolution operator $U(t)$ to this state, we get:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = \sum_a \alpha_a |a\rangle \otimes e^{-\frac{i}{\hbar}\lambda P I_a t} |\phi(x)\rangle .$$

Using the fact that $e^{-\frac{i}{\hbar}\lambda P I_a t}$ generates a translation by $x_0 = \lambda I_a t$, the time-evolved state becomes:

$$|\Psi(t)\rangle = \sum_a \alpha_a |a\rangle \otimes |\phi(x - \lambda I_a t)\rangle .$$

This shows that the pointer variable's wavepacket $|\phi(x)\rangle$ is translated by an amount $\lambda I_a t$ depending on the eigenvalue I_a of the observable I. Thus, the final position of the pointer variable encodes the measurement outcome, allowing us to infer the state of the quantum system.

4 Example Scenario: Photon in Horizontal Polarization State $|H\rangle$

Hamiltonian and Time Evolution Operator

The interaction Hamiltonian is given by:

$$H = \lambda P I,$$

where:

- λ is the coupling strength,
- P is the momentum conjugate to the pointer variable's position x,
- I is the pointer variable, with eigenvalues $I_1 = \frac{1}{2}$ (for $|H\rangle$) and $I_2 = -\frac{1}{2}$ (for $|V\rangle$).

The time evolution operator is:

$$U(t) = e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{\hbar}\lambda P I t}.$$

Expansion in Eigenbasis of I

Since the photon is in the state $|H\rangle$, the pointer variable I takes the value $I_1 = \frac{1}{2}$. Thus, the time evolution operator acting on $|H\rangle$ becomes:

$$U(t)|H\rangle = e^{-\frac{i}{\hbar}\lambda P I_1 t} |H\rangle .$$

Action of the Translation Operator

The momentum operator P generates translations in the position representation. Specifically:

$$e^{-\frac{i}{\hbar}\lambda P I_1 t} |\phi(x)\rangle = |\phi(x - \lambda I_1 t)\rangle .$$

Substituting $I_1 = \frac{1}{2}$, we get:

$$e^{-\frac{i}{\hbar}\lambda P I_1 t} |\phi(x)\rangle = |\phi\left(x - \frac{\lambda t}{2}\right)\rangle .$$

Time-Evolved State

The time-evolved state of the system and pointer is:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = |H\rangle \otimes |\phi\left(x - \frac{\lambda t}{2}\right)\rangle .$$

This shows that the pointer variable's wavepacket $|\phi(x)\rangle$ is translated by $\frac{\lambda t}{2}$ due to the photon being in the $|H\rangle$ state.

Probability of Measuring $|H\rangle$ or $|V\rangle$

To find the probability of measuring the photon in the $|H\rangle$ or $|V\rangle$ state, we project the final state $|\Psi(t)\rangle$ onto $\langle H|$ or $\langle V|$, respectively.

Probability of measuring $|H\rangle$:

$$P(H) = |\langle H| |\Psi(t)\rangle|^2 .$$

Substituting $|\Psi(t)\rangle$:

$$P(H) = \left| \langle H| \left(\alpha |H\rangle \otimes |\phi \left(x - \frac{\lambda t}{2} \right) \rangle + \beta |V\rangle \otimes |\phi \left(x + \frac{\lambda t}{2} \right) \rangle \right) \right|^2 .$$

Expanding the inner product:

$$P(H) = \left| \alpha \langle H| |H\rangle |\phi \left(x - \frac{\lambda t}{2} \right) \rangle + \beta \langle H| |V\rangle |\phi \left(x + \frac{\lambda t}{2} \right) \rangle \right|^2 .$$

Simplifying using orthogonality:

$$P(H) = \left| \alpha |\phi \left(x - \frac{\lambda t}{2} \right) \rangle \right|^2 = |\alpha|^2.$$

Probability of measuring $|V\rangle$, Similarly:

$$P(V) = |\langle V| |\Psi(t)\rangle|^2 = |\beta|^2.$$

Interpretation

- The probability of measuring the photon in the $|H\rangle$ state is $|\alpha|^2$.
- The probability of measuring the photon in the $|V\rangle$ state is $|\beta|^2$.
- These probabilities are consistent with the Born rule and reflect the coefficients of the initial superposition state $|\psi\rangle = \alpha |H\rangle + \beta |V\rangle$.