Eigenvalues and Eigenvectors

CS 111

Nov 19, 2020

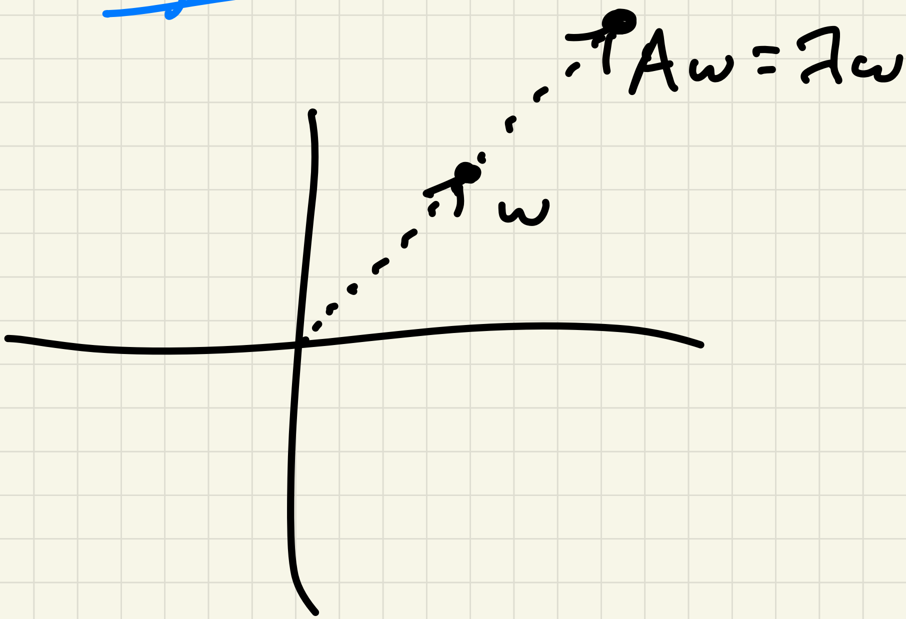
Let A be a square matrix $(n \times n)$.

If $\boxed{Aw = \lambda w}$ for

a nonzero vector w and a number λ,

w is an <u>eigenvector</u> of A

λ is <u>eigenvalue</u> of A.



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A w = \underset{\uparrow \atop 1.}{} w \quad \text{for all } w.$$

$\Rightarrow \lambda = 1$ is the only eigenvalue

every nonzero vector is an eigenvector.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\underline{\lambda = 1}}$$

$$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \lambda = 2$$

$$A \begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \quad \lambda = 3$$

If w is an eigenvector for <u>eval</u> λ then so is any nonzero multiple of w.

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ evec

$\lambda = 1$ eval

$$A \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$\begin{pmatrix} a \\ b \end{pmatrix}$ independent of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow a \neq 0$

$$A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a+b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad a = \lambda a \Rightarrow \lambda = 1$$

$$a+b = \lambda b \Rightarrow a+b = b$$

$$\Rightarrow a = 0$$

oops.

This A has only one linearly independent eigenvector. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>Theorem</u>: Every matrix has a nonzero evec (and an eval).

An $n \times n$ matrix has at most n linearly independent evecs.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \\ a \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \lambda = 1 \qquad A\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \lambda = -1$$

$$A\underbrace{\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}}_{w} = \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix} = iw \quad \text{so } \lambda = i$$

and $\lambda = -i$ works too.

Eigenvalues and eigenvectors can be complex.

$$i = \sqrt{-1}$$

SYMMETRIC MATRICES

$A = A^T$.

THEOREM: If A is n-by-n and symmetric, then:

1. All evals of A are real.
2. A has n linearly independent evecs.
3. The eigenvectors can be chosen to be orthogonal to each other.

Say $\underline{Aw_0 = \lambda_0 w_0}$, $\underline{Aw_1 = \lambda_1 w_1}$ $\underline{Aw_2 = \lambda_2 w_2}$

$\ldots \quad Aw_{k-1} = \lambda_{k-1} w_{k-1}$

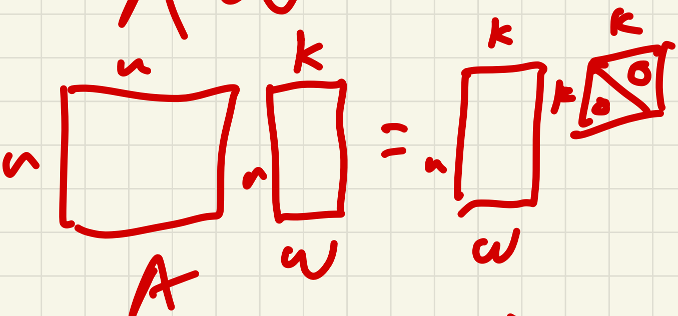
are all the evals & evecs of A.

Write $W = \left[\, w_0 \,\middle|\, w_1 \,\middle|\, \cdots \,\middle|\, w_{k-1} \,\right]$

and $S = \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{k-1} \end{pmatrix}$.

Then

$$AW = WS$$



(whether A is symmetric or not)

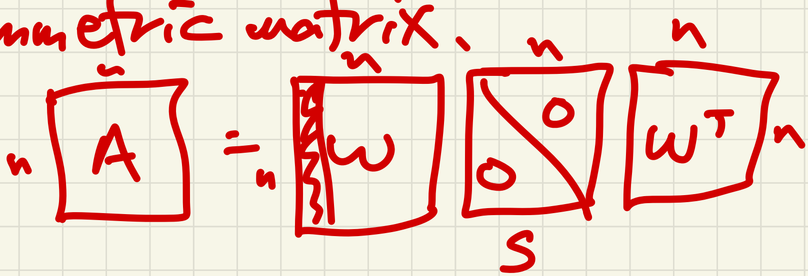
If A is symmetric,

→ $k = n$ (n lin indep evecs)

→ can choose them to be orthogonal + unit length.

$$\underline{AW = WS} \Rightarrow A = WSW^T$$

Eigenvalue factorization of a symmetric matrix.



$$A = \sum_{i=0}^{n-1} \lambda_i w_i w_i^T$$

$A: \lambda_0, \lambda_1$

$$Aw = \lambda w$$

Look at

$$B = A - \alpha I$$

$$Bw = (A - \alpha I)w$$

$$= Aw - \alpha w$$

$$= \lambda w - \alpha w$$

$$= (\lambda - \alpha) w$$

So w is evec of B

but with eval $\lambda - \alpha$