Laplacian Matrices of Graphs

CS 111

Nov 24, 2020

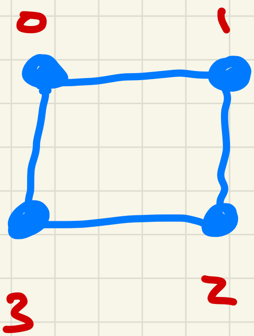
UNDIRECTED GRAPHS

Graph $G = (V, E)$

vertices $0, 1, 2, \ldots, n-1$ → V

Edges → E
Each edge is 2 vertices.

Cycle Graph



$n = 4$ vertices
4 edges

$$V = \{0, 1, 2, 3\}$$

$$E = \{(0\ 1), (1\ 2), (2\ 3), (3\ 0)\}$$

Edge $(1\ 2)$ is the same as $(2\ 1)$

No "loops": $(2\ 2)$ can't be an edge.

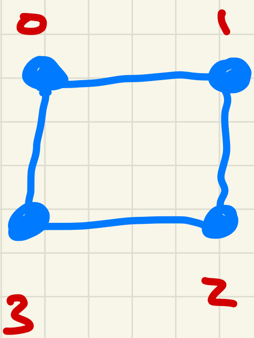
LAPLACIAN MATRIX OF A GRAPH

DEF: Given a graph G with vertices $0, 1, \ldots, n-1$, the Laplacian Matrix is

$L(G)$ is an $n \times n$ symmetric matrix with

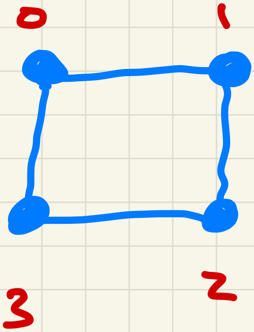
$$L[i,j] = \begin{cases} -1 & \text{if } (i,j) \text{ is an edge} \\ 0 & \text{if } i \neq j \text{ an } (i,j) \text{ not edge} \\ \text{degree of vertex } i & \text{if } i = j. \end{cases}$$

degree = # neighbors

$G =$ 

$$L(G) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

L is always symmetric

$G =$ 

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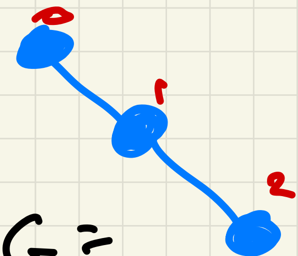
<u>PATH GRAPH</u>

$G =$ 

$$L(G) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

<u>THEOREM</u>

For any graph,

$$L \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

That is, all the row sums are 0.

That is, 0 is an eigenvalue of L with eigenvector $\mathbf{1}$.

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THEOREM

For any graph, L is positive semidefinite. (SPSD)

That is, all the evals of L are ≥ 0.

That is, $x^T L x \geq 0$ for all nonzero n-vectors x.