SVD

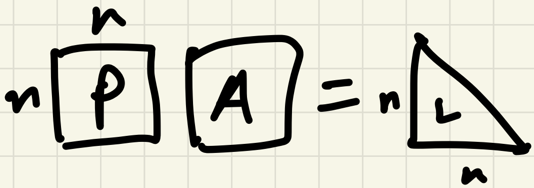
CS111

Nov 12, 2020

MATRIX FACTORIZATIONS

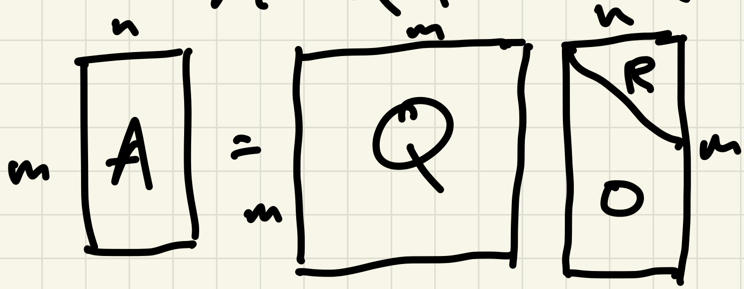
Gaussian Elimination

$PA = LU$ (P permutation)

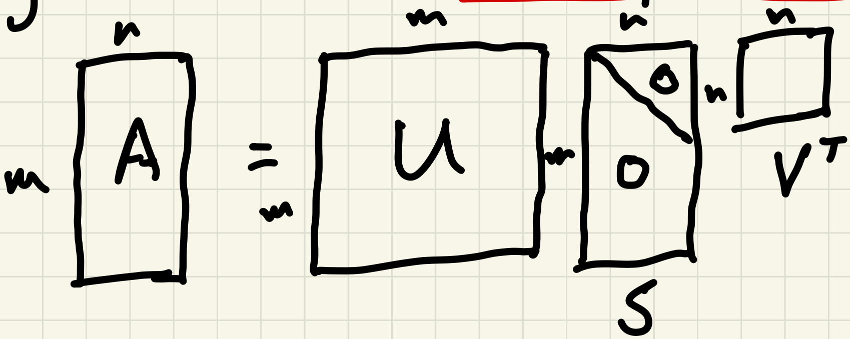


Orthogonal Factorization

$A = QR \qquad Q^TQ = I$ (ortho)



Singular Value Decomposition (SVD)



$U^T U = I$, orthog

$V^T V = I$, orthog.

$A = USV^T$

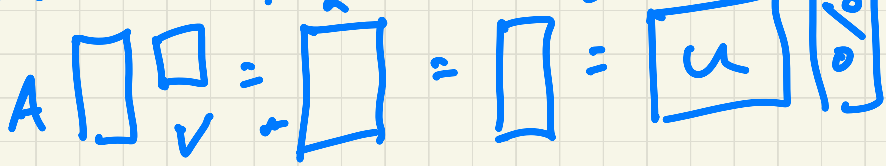
$$S = \begin{pmatrix} \sigma_0 & & & & \\ & \sigma_1 & & & \\ & & \sigma_2 & & \\ & & & \ddots & \\ & & & & \sigma_{k-1} \end{pmatrix}$$

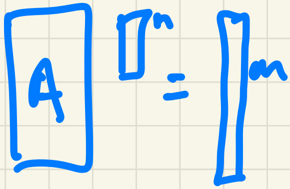
singular values of A

$k = \min(m,n)$

$\sigma_0 \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{k-1} \geq 0$

note: $AV = US$



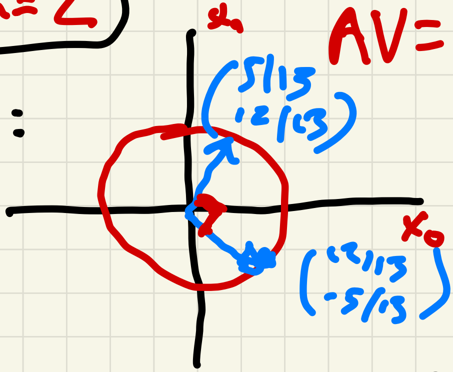


$A : \mathbb{R}^n \to \mathbb{R}^m$

A maps the unit sphere in $\mathbb{R}^n$ to an ellipsoid in $\mathbb{R}^m$

$n = m = 2$

$\mathbb{R}^n$:



$AV = US$

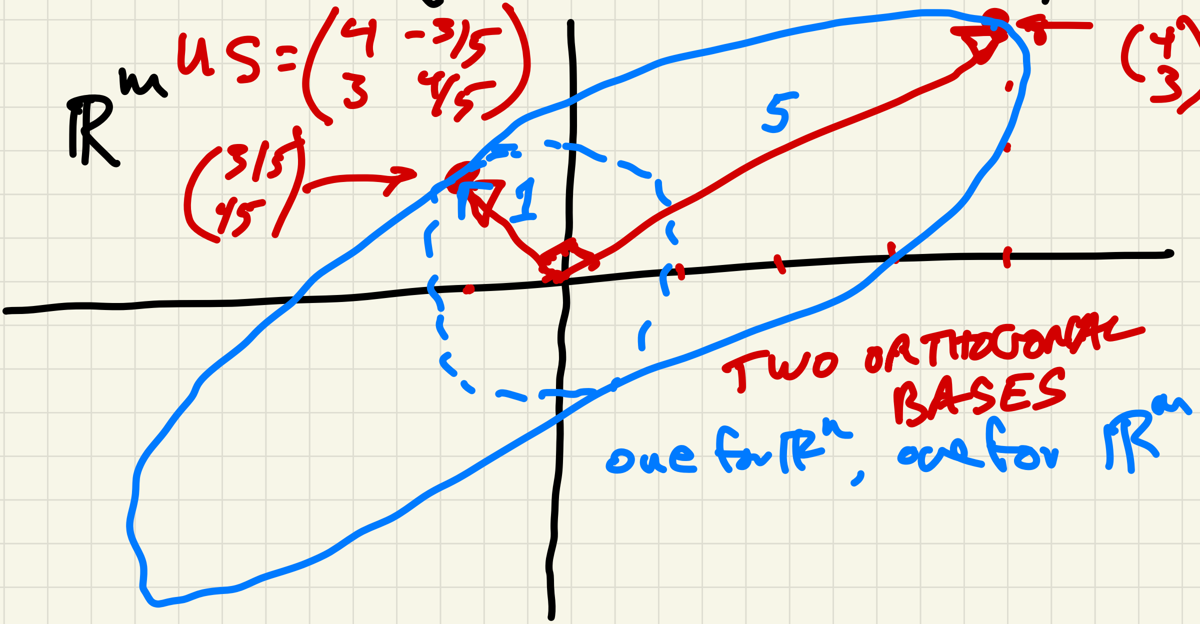
$U = \begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$

$V = \begin{pmatrix} 12/13 & 5/13 \\ -5/13 & 12/13 \end{pmatrix}$

$S = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_0 = 5$, $\sigma_1 = 1$

$A = USV^T = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$

$\mathbb{R}^m$ $US = \begin{pmatrix} 4 & -3/5 \\ 3 & 4/5 \end{pmatrix}$



TWO ORTHOGONAL BASES

one for $\mathbb{R}^n$, one for $\mathbb{R}^m$

SVD THEOREMS

Thm 1: rank(A) = number of nonzero singular values.

Thm 2: $\|A\|_2 = \sigma_0$

$$\underset{\text{matrix norm}}{\|A\|_2} = \max_{\text{nonzero } x} \underset{\text{vector norm}}{\frac{\|Ax\|_2}{\|x\|_2}}$$

The max x is

$x = V[:,0]$ $\qquad \|x\|_2 = 1$

$Ax = \sigma_0 \cdot U[:,0]$ $\qquad \|Ax\| = \sigma_0$