SVD and Low-Rank Approximation

CS 111

Nov. 17, 2020

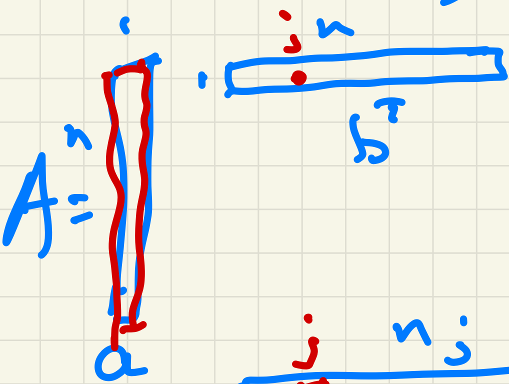
rank of matrix is

→ dimension of the space spanned by the columns

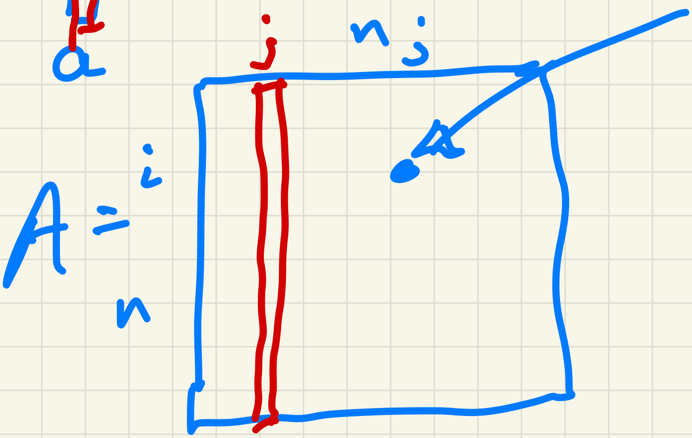
→ max # of linearly independent cols

rank-1 matrices.

Let a and b be n-vectors.



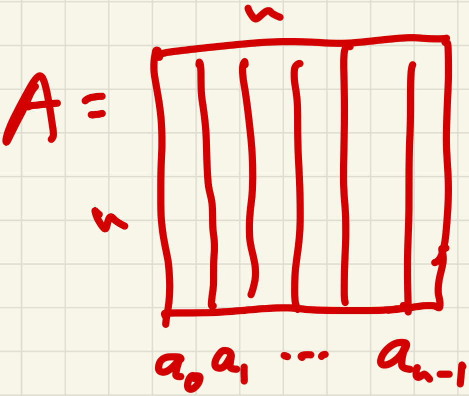
$A = ab^T$

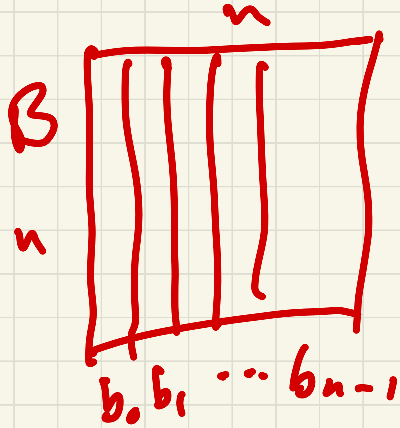


$a_i b_j$

A is the matrix of products of elements of a & b.

A has **rank one** (unless a or b is 0).





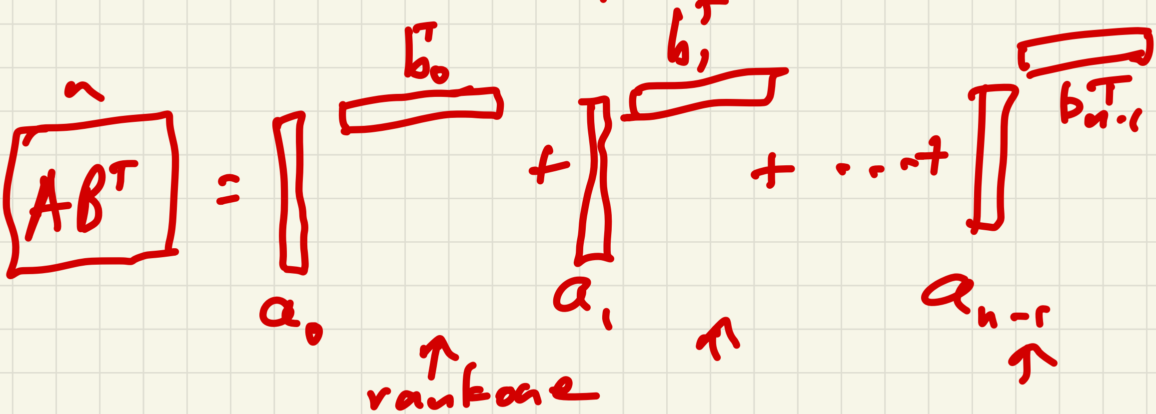
Look at AB^T

$$AB^T[i,j] = \sum_{k=0}^{n-1} A[i,k]\cdot \cancel{B^T[k,j]}$$

$$B[j,k]$$

$$AB^T[i,j] = \sum_{k=0}^{n-1} a_k[i]\, b_k[j]$$

$$AB^T = a_0 b_0^T + a_1 b_1^T + \cdots + a_{n-1} b_{n-1}^T$$



The product of 2 n×n matrices is the sum of n rank-1 matrices.

SVD

$$\underset{m \times n}{A} = U \, S \, V^T$$

$U = \begin{bmatrix} u_0 & u_1 & \cdots \end{bmatrix}$, $V^T = \begin{bmatrix} v_0^T \\ v_1^T \\ \vdots \end{bmatrix}$

$$S = \begin{pmatrix} \sigma_0 & & & 0 \\ & \sigma_1 & & \\ & & \ddots & \\ 0 & & & \sigma_{n-1} \end{pmatrix}$$

$$A = u_0 \cdot \sigma_0 \, v_0^T + u_1 \, \sigma_1 \, v_1^T + \dots + u_{n-1} \, \sigma_{n-1} \, v_{n-1}^T$$

$$A = \sum_{k=0}^{\operatorname{rank}A - 1} \sigma_k u_k v_k^T$$