

# SVD and Low-Rank Approximation

CS 111

Nov. 17, 2020



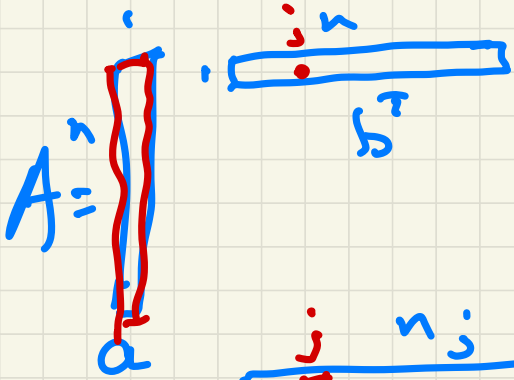
rank of matrix is

→ dimension of the space spanned  
by the columns

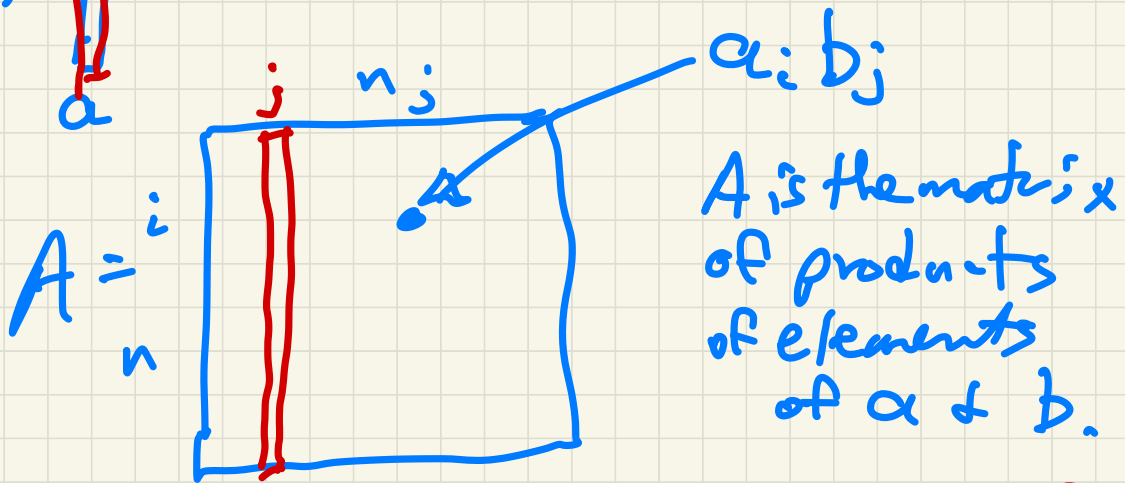
→ max # of linearly independent cols

rank-1 matrices.

Let  $a$  and  $b$  be  $n$ -vectors.



$$A = ab^T$$



A has rank one (unless  $a$  or  $b$  is 0).

$$A = \begin{matrix} & \sim \\ \begin{matrix} | & | & | & | \end{matrix} \\ \sim & \begin{matrix} a_0 & a_1 & \dots & a_{n-1} \end{matrix} \end{matrix}$$

$$B = \begin{matrix} & \sim \\ \begin{matrix} | & | & | & | \end{matrix} \\ \sim & \begin{matrix} b_0 & b_1 & \dots & b_{n-1} \end{matrix} \end{matrix}$$

Look at  $AB^T$

$$AB^T[i,j] = \sum_{k=0}^{n-1} A[i,k] \cdot \cancel{B[k,j]} \quad B[j,k]$$

$$AB^T[i,j] = \sum_{k=0}^{n-1} a_k[i] b_k[j]$$

$$AB^T = a_0 b_0^T + a_1 b_1^T + \dots + a_{n-1} b_{n-1}^T$$

$$\boxed{AB^T} = \begin{matrix} \sim \\ \begin{bmatrix} | \\ | \\ | \end{bmatrix} \end{matrix} + \begin{matrix} \begin{bmatrix} \text{---} \end{bmatrix} \\ b_0^T \end{matrix} + \begin{matrix} \begin{bmatrix} \text{---} \end{bmatrix} \\ b_1^T \end{matrix} + \dots + \begin{matrix} \begin{bmatrix} \text{---} \end{bmatrix} \\ b_{n-1}^T \end{matrix}$$

$\begin{matrix} a_0 & a_1 & & a_{n-1} \\ \uparrow & \uparrow & & \uparrow \\ \text{rank one} & & & \end{matrix}$

The product of 2 nxn matrices is the sum of n rank-1 matrices.

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SVD

$$\begin{matrix} n \\ \boxed{A} \end{matrix} = \begin{matrix} n \times n \\ \boxed{U} \end{matrix} \begin{matrix} n \times n \\ \boxed{S} \end{matrix} \begin{matrix} n \times n \\ \boxed{V^T} \end{matrix}$$

$U = [u_0 \ u_1 \ \dots \ u_{n-1}]$ 
 $S = \begin{pmatrix} \sigma_0 & & & \\ & \sigma_1 & & \\ & & \ddots & \\ & & & \sigma_{n-1} & \\ 0 & & & & 0 \end{pmatrix}$ 
 $V^T = [v_0^T \ v_1^T \ \dots \ v_{n-1}^T]$

$$A = \begin{matrix} \sigma_0 \\ \vdots \\ \sigma_{n-1} \end{matrix} \begin{matrix} \boxed{u_0 u_0^T} \\ \vdots \\ \boxed{u_{n-1} u_{n-1}^T} \end{matrix} + \dots + \begin{matrix} \sigma_1 \\ \vdots \\ \sigma_{n-1} \end{matrix} \begin{matrix} \boxed{u_1 u_1^T} \\ \vdots \\ \boxed{u_{n-1} u_{n-1}^T} \end{matrix}$$

$$A = \sum_{k=0}^{\text{rank } A - 1} \sigma_k u_k u_k^T$$