The Laplacian and Graph Connectivity

CS 111

Dec 1, 2020

Laplacian Quadratic Form (LQF)

THEOREM FROM LAST TIME

$$x^T L x = \sum_{(i,j)\in E} (x(i) - x(j))^2$$

L: Laplacian matrix of an n-vertex graph

x: n-vector of vertex labels

E: edges of the graph

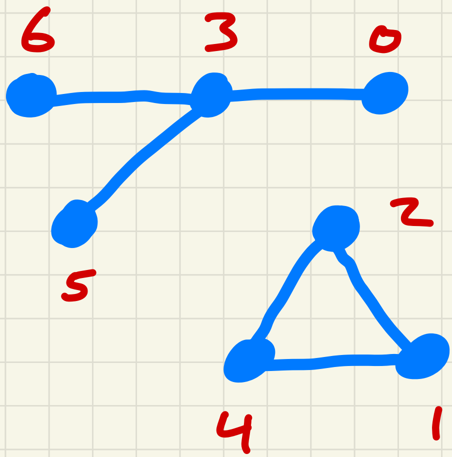
$$\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Theorem: L is symmetric positive semidefinite.

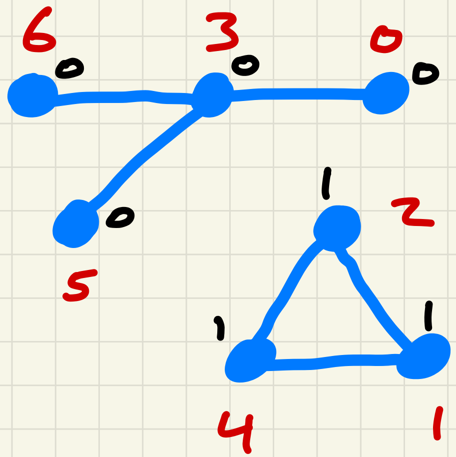
Theorem: If G is connected, then $x^T L x = 0$ if and only if $x = \alpha \cdot \mathbb{1}$ for some α.

What if G is not connected?

A Disconnected Graph

$G =$ 

$$
L = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & & & -1 & & & \\ 1 & & 2 & -1 & & -1 & & \\ 2 & & -1 & 2 & & -1 & & \\ 3 & -1 & & & 3 & & -1 & -1 \\ 4 & & -1 & -1 & & 2 & & \\ 5 & & & & -1 & & 1 & \\ 6 & & & & -1 & & & 1 \end{array}
$$

$G =$ 

$$x = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$L \cdot \mathbb{1} = 0$

$\underline{x^T L x} = 0$ and $Lx = 0$

<u>THEOREM</u>: If G is <u>not</u> connected, and $Lx = 0$, then x is constant on every connected component of the graph.

This G has 2 linearly independent eigenvectors for the eigenvalue 0:

$L \cdot \underline{1} = 0 \cdot \mathbb{1} \qquad L \cdot x = 0 \cdot x$

THEOREM:

Suppose $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ are the eigenvals of $L(G)$.

A) If G is connected,

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1}$$

B) If G has k connected components,

$$0 = \lambda_0 = \lambda_1 = \cdots = \lambda_{k-1} < \lambda_k \le \cdots \le \lambda_{n-1}.$$

THE NUMBER OF CONNECTED COMPONENTS OF A GRAPH IS EQUAL TO THE NUMBER OF LINEARLY INDEPENDENT NULL VECTORS.

DEF The Fiedler value of a graph is λ_1.

NOTE: G is connected $\Leftrightarrow$ its Fiedler value is nonzero

If G is connected, the Fiedler vector is the eigenvector w_1.

$$(Lw_1 = \lambda_1 w_1)$$