Solving $Ax=b$ by LU factorization

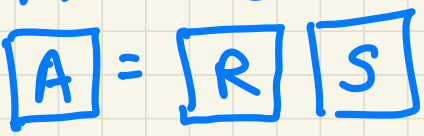
CS111 Lecture

Oct 15, 2020

BIG IDEA: MATRIX FACTORIZATION

To solve $Ax=b$,

if you know $A=RS$



... and if you know how to solve systems with R and S ...

$(RS)x=b$

$R(Sx)=b$

Define $y=Sx$ (even though you don't know x)

① Solve $Ry=b$ for y

② Solve $Sx=y$ for x

... and you're done!

Gaussian Elimination (Row View)

$$
\begin{aligned}
2x_0 + 7x_1 + x_2 + 8x_3 &= 18 \\
x_0 + 5.5x_1 + 8.5x_2 + 5x_3 &= 20 \\
x_1 + 12x_2 + 2.5x_3 &= 15.5 \\
-x_0 - 4.5x_1 - 4.5x_2 + 3.5x_3 &= -6.5
\end{aligned}
$$

$$
\begin{aligned}
2x_0 + 7x_1 + x_2 + 8x_3 &= 18 \\
2x_1 + 8x_2 + x_3 &= 11 \\
x_1 + 12x_2 + 2.5x_3 &= 15.5 \\
-x_1 - 4x_2 + 7.5x_3 &= 2.5
\end{aligned}
$$

$$
\begin{aligned}
2x_0 + 7x_1 + x_2 + 8x_3 &= 18 \\
2x_1 + 8x_2 + x_3 &= 11 \\
8x_2 + 2x_3 &= 10 \\
8x_3 &= 8
\end{aligned}
$$

and so on....

MATRIX VIEW: Gaussian elimination is a matrix factorization!

$$A = \begin{bmatrix} 2 & 7 & 1 & 8 \\ 1 & 5.5 & 8.5 & 5 \\ 0 & 1 & 12 & 2.5 \\ -1 & -4.5 & -4.5 & 3.5 \end{bmatrix}$$

$\Downarrow$

multipliers

transformed matrix

$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 1 & 12 & 2.5 \\ 0 & -1 & -4 & 7.5 \end{bmatrix}$$

$\frac{1}{2}$

0

$-\frac{1}{2}$

multipliers

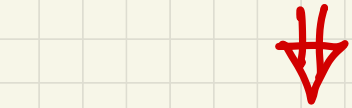
transformed matrix

$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 1 & 12 & 2.5 \\ 0 & -1 & -4 & 7.5 \end{bmatrix}$$

$\frac{1}{2}$

0

$-\frac{1}{2}$



$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$\frac{1}{2}$

$0 \quad \frac{1}{2}$

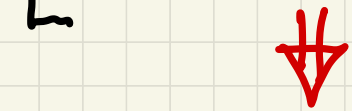
$-\frac{1}{2} \quad -\frac{1}{2}$

multipliers

transformed matrix

$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{2} & & \\ 0 & \frac{1}{2} & \\ -\frac{1}{2} & -\frac{1}{2} & \end{matrix}$$



$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{2} & & \\ 0 & \frac{1}{2} & \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{matrix}$$

NOW THE MULTIPLIERS GO IN L

$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

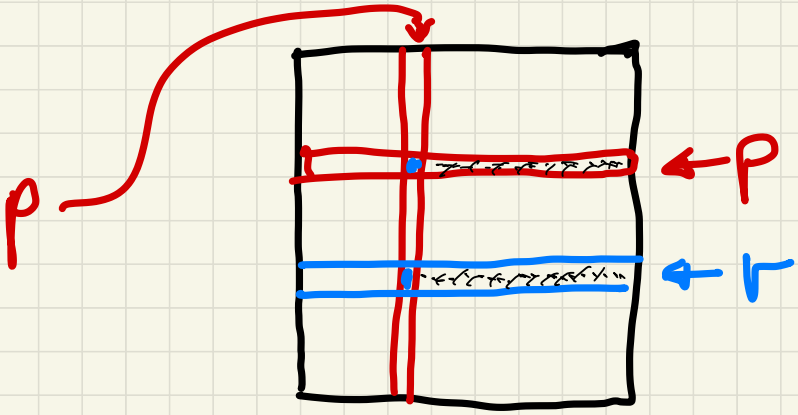
$$\begin{bmatrix} 1/2 & & \\ 0 & 1/2 & \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ -1/2 & -1/2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 & 1 & 8 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

L

U




```
for p = 0 to n-1:
    for r = p+1 to n-1:
        multiplier: A[r,p] / A[p,p]
        subtract multiplier x (end of row p)
            from (end of row r)
```

TRANSFORMS A into U

MULTIPLIERS go in L

WHAT CAN GO WRONG?

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Downarrow$$

Divide by zero pivot!

SOLUTION:
Reorder equations

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$A[p,:] = LU$$

$$PA = LU$$