Orthogonality and QR factorization

CS 111

Oct. 29, 2020

BIG IDEA: ORTHOGONALITY

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \quad y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

dot product
=
inner product

$$x \cdot y = x^T y = \sum_{i=0}^{n-1} x_i y_i$$

THM: $x^T x = \|x\|_2^2$ $\left(\|x\|_2 = \sqrt{\sum_{i=0}^{n-1} x_i^2} \right)$

THM: If $x^T y = 0$ and $x \neq 0, y \neq 0$

then x and y are perpendicular vectors

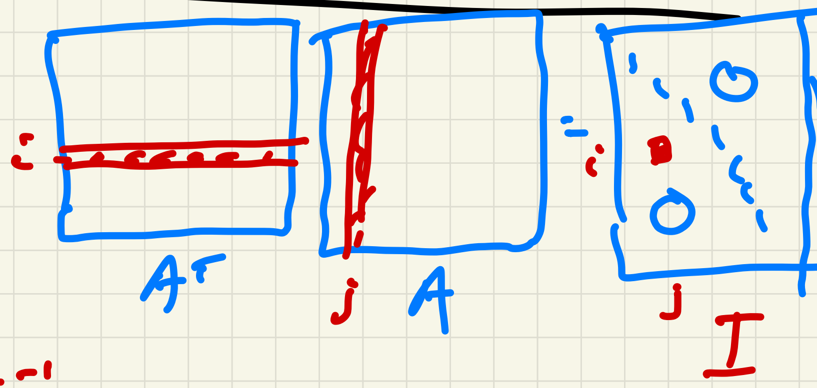
(orthogonal $\equiv$ perpendicular)

ORTHOGONAL MATRICES

SORT OF LIKE NUMBERS WITH ABSOLUTE VALUE 1

DEF: Matrix A is orthogonal if it's square and $A^TA = I$ (identity)

$$A^{-1} = A^T$$



$$(A^TA)_{ij} = \sum_{k=0}^{n-1} A^T[i,k] \cdot A[k,j]$$

$$= \sum_{k=0}^{n-1} a_{ki} a_{kj} = A[:,i] \cdot A[:,j]$$

$$= I_{ij} = \begin{cases} i=j : 1 \\ i \neq j : 0 \end{cases}$$

- columns of A have length 1 (2-norm)
- columns of A are mutually perpendicular

Columns perpendicular + unit length

(A is orthog $\Rightarrow$ A^T is orthog)

Rows are perpendicular + unit length

An orthogonal matrix applies a rotation or reflection or combination to n-space.

THM: Orthogonal matrices don't change the length of a vector, that is,

$$\|Qv\|_2 = \|v\|_2 \text{ for all } v$$
all orthog. Q

$$\|Q\|_2 = \max_{v \neq 0} \frac{\|Qv\|_2}{\|v\|_2} = \max_{v} \frac{\|v\|_2}{\|v\|_2} = 1$$

Orthog mtxs have norm 1.

$$\kappa_2(Q) = \|Q\|_2 \cdot \|Q^{-1}\|_2 = 1 \cdot 1 = 1$$

Orthogonal matrices are perfectly well-conditioned.

Solve $Qx=b$ for x:

$$\underbrace{Q^TQ}_{I}x = Q^Tb \qquad x = Q^Tb.$$

$$\|x\|_2 = \|b\|_2$$

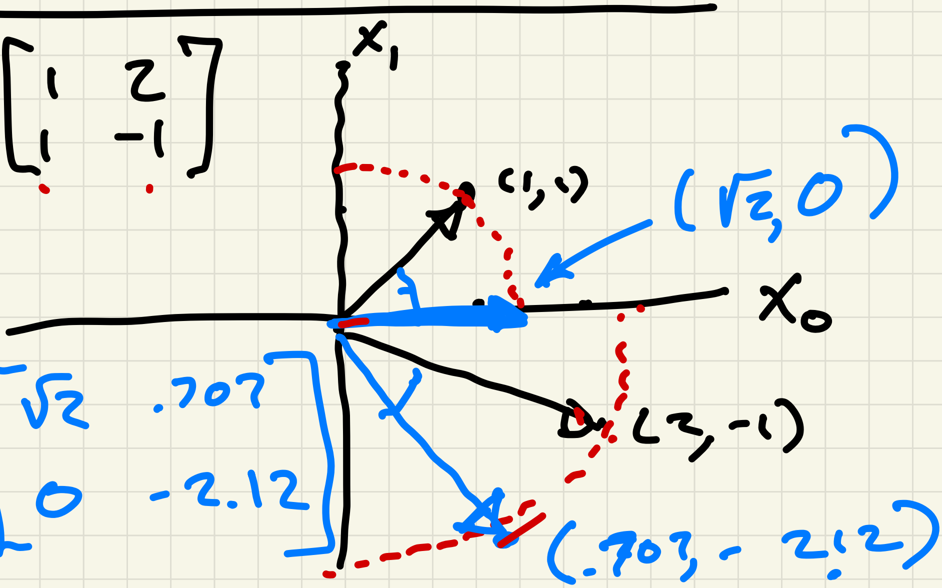
$$\text{error} = x_{\text{exact}} - x$$

$$\text{residual} = b - Qx$$

$$r = Q\cdot e \Rightarrow \|r\|_2 = \|e\|_2$$

QR Factorization

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$



$$R = \begin{bmatrix} \sqrt{2} & .707 \\ 0 & -2.12 \end{bmatrix}$$

$$A = QR$$

THM Every matrix A can be written as A = QR with Q orthogonal + R upper triangular.

$$\boxed{A} = \boxed{Q}\ \boxed{R}$$