PERMUTATIONS
and
PERMUTATION MATRICES

Short subject

Oct. 13, 2020

DEF An n-**permutation** is a list of the numbers 0 through $n-1$ in some order

EXAMPLE: $(n=5)$

$$p = [3, 0, 4, 1, 2]$$

DEF An n-by-n **permutation matrix** is a matrix of 0s and 1s with exactly one 1 in each row and column.

EXAMPLE $(n=5)$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

EVERY PERMUTATION CORRESPONDS TO A PERMUTATION MATRIX.

Take $v = \begin{bmatrix} 3.1 \\ 4.1 \\ 5.9 \\ 2.6 \\ 5.3 \end{bmatrix}$ $\qquad v[p] = \begin{bmatrix} 2.6 \\ 3.1 \\ 5.3 \\ 4.1 \\ 5.9 \end{bmatrix}$

$p = [3, 0, 4, 1, 2]$

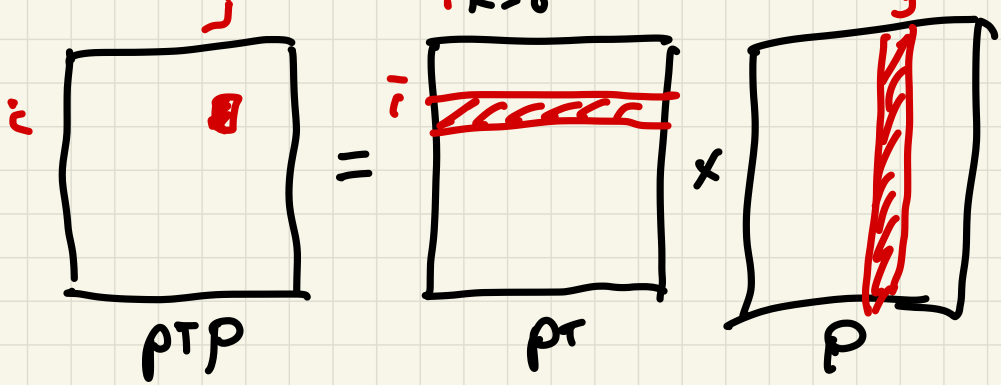
$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3.1 \\ 4.1 \\ 5.9 \\ 2.6 \\ 5.3 \end{bmatrix}}_{v} = \begin{bmatrix} 2.6 \\ 4.1 \\ 5.3 \\ 4.1 \\ 5.4 \end{bmatrix}$$

$$Pv = v[p] \quad (!)$$

THEOREM: THE TRANSPOSE OF A PERMUTATION MATRIX IS ITS INVERSE.

PROOF:

$$(P^T P)[i,j] = \sum_{k=0}^{n-1} P^T[i,k] \cdot P[k,j]$$



$P^T P[i,j]$ is the dot product of

row i of P^T = col i of P

and col j of P

$i \neq j$: DOT PROD is 0

$i = j$: DOT PROD is 1

$$P^T P[i,j] = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i \neq j \end{cases} = I \ (!)$$