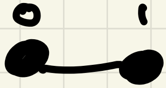
The Laplacian Quadratic Form

CS111

Nov 24, 2020

$G =$ 

$L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$e_{01} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$e_{01} e_{01}^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} =: L_{01}$$

$G_{ij} =$ 

n vertices
one edge (i,j)

$$L_{ij} = \begin{array}{cc} & \begin{matrix} \; & \; & i & \; & j & \; \end{matrix} \\ \begin{matrix} 0 \\ \\ i \\ \\ j \\ n-1 \end{matrix} & \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & -1 & \\ & & & 0 & & \\ & & -1 & & 1 & \\ & & & & & 0 \end{bmatrix} \end{array}$$

$$e_{ij} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \leftarrow j \\ \\ \end{matrix}$$

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eij = np.zeros(n)
eij[i] = 1
eij[j] = -1
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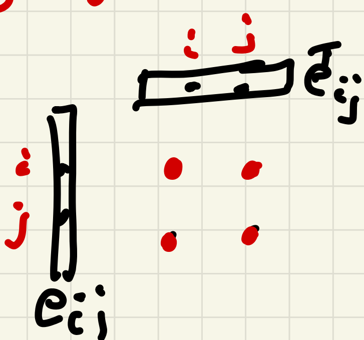
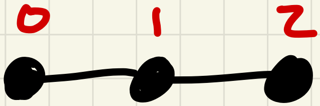
$$e_{ij} e_{ij}^T = L_{ij}$$



$G =$ 

$$e_{01} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad L_{01} = e_{01} e_{01}^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_{12} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad L_{12} = e_{12} e_{12}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L(G) = L_{01} + L_{12} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

For <u>any</u> graph $G = (V, E)$,

$$L(G) = \sum_{\substack{\text{edges} \\ ij \in E}} L_{ij} = \sum_{ij \in E} e_{ij} e_{ij}^T$$

(here $e_{ij} = (0, 0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0, \underset{j}{-1}, 0, \ldots 0)^T$

$$L(G) = \sum_{\substack{\text{edges} \\ ij \in E}} L_{ij} = \sum_{ij \in E} e_{ij} e_{ij}^T$$

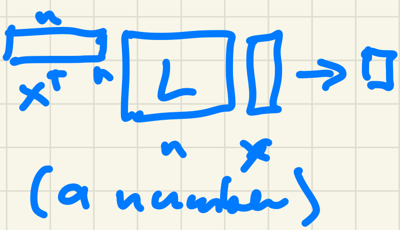
n vertices

Let x be any n-vector.

$$x = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{bmatrix}$$

LAPLACIAN QUADRATIC FORM

$$x^T L x = x^T \Big(\sum_{\substack{\text{edges} \\ ij}} e_{ij} e_{ij}^T \Big) x$$



row vector

col

row

col

$$= \sum_{ij \in E} x^T e_{ij} e_{ij}^T x$$

$$x^T L x = \sum_{ij \in E} (x^T e_{ij})(e_{ij}^T x)$$

$$x^T L x = \sum_{ij \in E} (\underbrace{x^T e_{ij}}_{x(i)-x(j)})(\underbrace{e_{ij}^T x}_{x(i)-x(j)})$$

$$x^T e_{ij} = \boxed{x(0)\ x(1) \cdots x(n-1)} \begin{array}{c} \boxed{\begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{matrix}} \end{array} \begin{array}{l} \\ \\ i \\ \\ j \\ \\ \end{array} = x(i) - x(j)$$

$$e_{ij}^T x = x(i) - x(j)$$

THEREFORE

$$x^T L x = \sum_{ij \in E} (x(i) - x(j))^2 \geq 0$$

(Therefore L is SPSD)

ALSO

$x^T L x = 0$ iff $x(i) = x(j)$ for all edges ij

$\Rightarrow$ if the graph is connected, then x must be constant. $\begin{pmatrix} a \\ a \\ a \\ \vdots \\ a \end{pmatrix}$