

SVD


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CS111

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# MATRIX FACTORIZATIONS

## Gaussian Elimination

$$PA = LU \quad (P \text{ permutation})$$

$${}^n \boxed{P} \boxed{A} = {}^n \triangleleft_L \triangleleft_U^n$$

## Orthogonal Factorization

$$A = QR \quad Q^T Q = I \text{ (orthog)}$$

$${}_m \boxed{A} = {}_m \boxed{Q} \begin{matrix} {}^n \\ \boxed{R} \\ {}_m \end{matrix}$$

# Singular Value Decomposition (SVD)

$$\begin{matrix} n \\ m \end{matrix} \begin{bmatrix} A \end{bmatrix} = \begin{matrix} m \\ m \end{matrix} \begin{bmatrix} U \end{bmatrix} \begin{matrix} m \\ n \end{matrix} \begin{bmatrix} S \end{bmatrix} \begin{matrix} n \\ n \end{matrix} \begin{bmatrix} V^T \end{bmatrix}$$

$$U^T U = I, \text{ orthonormal}$$

$$V^T V = I, \text{ orthonormal}$$

$$\underline{\underline{A = U S V^T}}$$

$$S = \begin{pmatrix} \sigma_0 & & & \\ & \sigma_1 & & \\ & & \sigma_2 & \\ & & & \ddots \\ & & & & \sigma_{k-1} \end{pmatrix}$$

$$k = \min(m, n)$$

singular values of A

$$\sigma_0 \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{k-1} \geq 0$$

note:  $A V = U S$

$$A \begin{bmatrix} \vdots \\ v \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} u \\ \vdots \end{bmatrix}$$

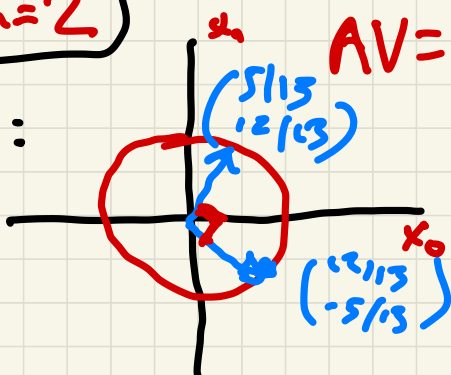
$$[A]^n_m = [ ]^n_m$$

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

A maps the unit sphere in  $\mathbb{R}^n$  to an ellipsoid in  $\mathbb{R}^m$

$$n=m=2$$

$\mathbb{R}^n$ :



$$AV = US$$

$$U = \begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$$

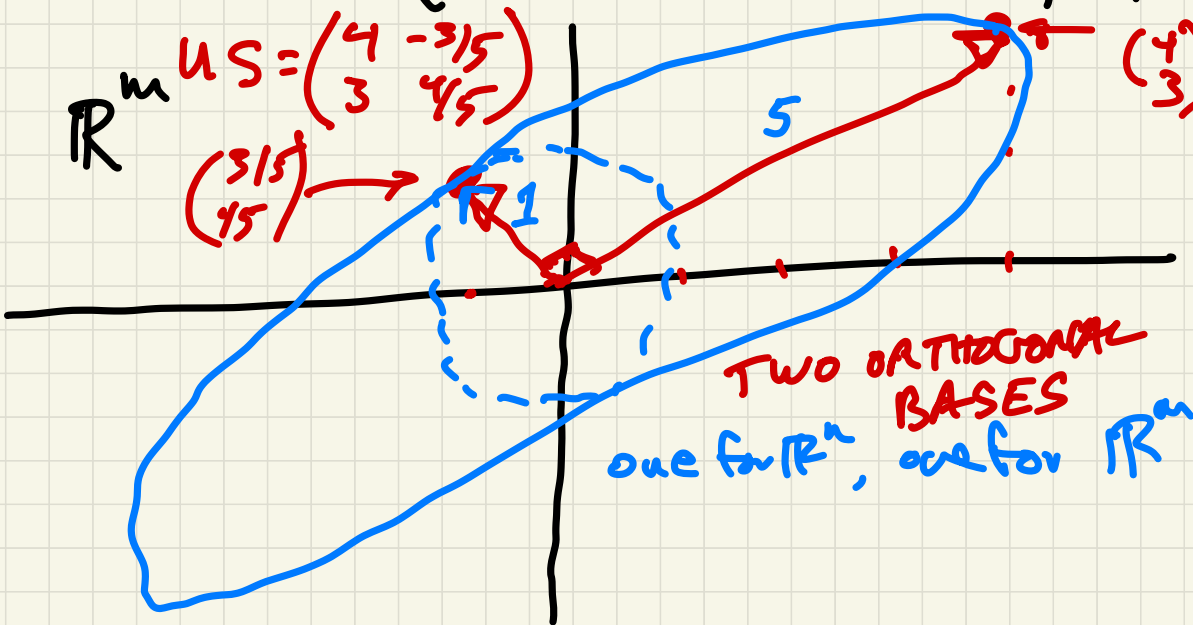
$$V = \begin{pmatrix} 12/13 & 5/13 \\ -5/13 & 12/13 \end{pmatrix}$$

$$A = USV^T = \begin{pmatrix} 0 & 0 \\ 0 & \infty \end{pmatrix}$$

$$S = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_0 = 5, \quad \sigma_1 = 1$$

$$\mathbb{R}^m \quad US = \begin{pmatrix} 4 & -3/5 \\ 3 & 4/5 \end{pmatrix}$$

$$\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$



# SVD THEOREMS

Thm 1:  $\text{rank}(A) = \text{number of nonzero singular values.}$

Thm 2:  $\|A\|_2 = \sigma_0$

$$\|A\|_2 = \max_{\text{nonzero } x} \frac{\|Ax\|_2}{\|x\|_2}$$

matrix norm                      vector norm

The max  $x$  is

$$x = v[:, 0]$$

$$\|x\|_2 = 1$$

$$Ax = \sigma_0 \cdot u[:, 0] \quad \|Ax\| = \sigma_0$$