CS 111: Math Review: Answer key

1.
$$A^T = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \end{pmatrix}, A^2 = \begin{pmatrix} 11 & -4 & 2 \\ 2 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix}, A^T A = \begin{pmatrix} 10 & -3 & 5 \\ -3 & 2 & 0 \\ 5 & 0 & 9 \end{pmatrix}.$$

2. $||(3,1,4,1,5)^T||_2 = \sqrt{52} \approx 7.2111$

3.
$$\begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$$

   In `numpy`, `A = np.array([[2, -3, 1], [0, 2, 3], [1, 0, 1]])` and `b = np.array([1, 7, 4])`. The 1-dimensional `numpy` array b can represent either a column vector or a row vector; Python's matrix-vector multiplication operator `@` will do the right thing.

4.
$$x = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

5. There are many answers to this. Here's one: Let `A = np.array([[1, 2], [2, 4]])` and `b = np.array([3, 3])`. (In math notation we write $b = (3,3)^T$, which is a column vector because of the transpose.) Explanation 1 (column view): Matrix A is singular, so the space spanned by its columns is only one-dimensional, and it consists of multiples of the vector $(1,2)^T$, which do not include b. Explanation 2 (row view): The two lines described by the rows of $Ax = b$ are parallel and hence do not intersect. Explanation 3 (brute force view): No matter what x is, the second entry of Ax will be equal to twice the first entry of Ax, which rules out b.

6. There are many answers to this. Here's one: Take A to be the same matrix as in the previous problem, and $b = (3,6)^T$. Two solutions are $x = (1,1)^T$ and $x = (3,0)^T$.

7. No, it's not possible to have exactly two solutions to $Ax = b$. If x and y are two different solutions, then there are infinitely many solutions: $x + \alpha(y - x)$ is a solution for every α.

8. A has two eigenvalues, 3 and 5. Any multiple of $(1,1)^T$ is an eigenvector corresponding to 3, and any multiple of $(1,-1)^T$ is an eigenvector corresponding to 5.

9. $f'(x) = 21x^2 - 4x + 4$.

10. $\partial z/\partial x = e^{y/2}$, and $\partial z/\partial y = (x/2)e^{y/2}$.

11. $f(x) = x^3/3 - \cos x + c$ for some constant c (any constant will do).

12. The height is maximum when the derivative dh/dt is zero. $dh/dt = 1280 - 32t$, which is zero when $t = 40$, at which time the height is $h = 25600$ feet. The projectile hits the ground when $h = 0$, which means $1280t = 16t^2$, which means $t = 1280/16 = 80$ seconds after firing. (The other solution to $h = 0$ is of course $t = 0$.)

13. The gradient $\nabla f = (\partial f/\partial x, \partial f/\partial y)^T = (2x - y, -x + 6y)^T$ is the direction of steepest ascent. Thus any multiple of $-\nabla f(3,2) = (-4,-9)^T$ points in the direction of steepest descent.