Expanders

CS292F

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Lecture 14

Complete graph K_n : $\binom{n}{2} = \frac{n(n-1)}{2}$ edges

$$\text{Laplacian} = \begin{pmatrix} n-1 & & -1 \\ & n-1 & \\ -1 & & \ddots \\ & & & n-1 \end{pmatrix} = nI - \mathbb{1}\mathbb{1}^T$$

Eigenvalues: $\lambda_1 = 0$

If $\mathbb{1}^T x = 0$ then

$$K_n x = nx - \mathbb{1}\mathbb{1}^T x = nx - \mathbb{1} \cdot 0 = nx$$

so $\lambda_2 = \lambda_3 = \cdots = \lambda_n = n$.

Expanders

Intuition: Sparse approximation of K_n

Definitions:

TRADITIONAL: sets of vtxs have large neighbourhoods (large conductance)

SPIELMAN: Eigenvalues of adjacency mtx.
(small)

US: Laplacian eigenvalues.

DEF: An (ε, d)-expander is a graph with n vertices with:

→ every vertex has degree d.

→ $(1-\varepsilon)d \le \lambda_i \le (1+\varepsilon)d$
for $i = 2, 3, \ldots, n$

$$\#\text{edges} = \frac{dn}{2} = \frac{d}{2}n.$$

Theorem: If G is an (ε, d)-expander and $H = \frac{d}{n} K_n$, then

$$(1-\varepsilon) H \preceq G \preceq (1+\varepsilon) H$$

"G is an ε-approximation of H"

Proof: will compare $x^T G x$ and $x^T H x$ for $1^T x = 0$.

If $1^T x = 0$,

$$(1-\varepsilon) d \le \lambda_2 \le \frac{x^T G x}{x^T x} \le \lambda_n \le (1+\varepsilon) d$$

$$\Rightarrow (1-\varepsilon) d\, x^T x \le x^T G x \le (1+\varepsilon) d\, x^T x.$$

$$x^T H x = x^T \frac{d}{n} K_n x = x^T \frac{d}{n} n x = d\, x^T x.$$

$$\Rightarrow (1-\varepsilon) x^T H x \le x^T G x \le (1+\varepsilon) x^T H x.$$

$$\Rightarrow (1-\varepsilon) H \preceq G \preceq (1+\varepsilon) H.$$

G is an ε-"spectral approximation" of H.

Recall $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

$= \max_i |\lambda_i|$ (if A symmetric)

$(1-\varepsilon) H \preceq G \preceq (1+\varepsilon) H$

$\Rightarrow -\varepsilon H \preceq G - H \preceq \varepsilon H$

since evals of H are all 0 or d,

~~evals~~ $-\varepsilon d \leq$ evals$(G-H) \leq \varepsilon d$

$\Rightarrow \|G - H\| \leq \varepsilon d$.

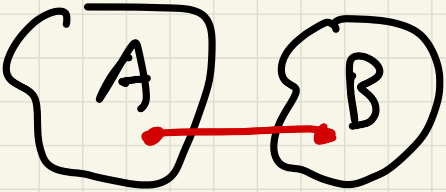
"Random" d-regular graphs are expanders. (WHP)

Explicit constructions of expanders date to ~1980

Spielman does one in ch. 30

Expanders act a lot like random grfs.

Take 2 disjoint sets of vertices



$|A| = \alpha n$ $\quad$ $|B| = \beta n$

$E(A,B)$ = #edges with one end in A + one in B

$$= -1_A^T G 1_B$$ (exercise)

Complete graph: $E_K(A,B) = \alpha\beta n^2$

$$H = \frac{d}{n} K_n \ : \ \underline{-1_A^T H 1_B} = \alpha\beta d n$$

Pick $A+B$ at random in any d-regular grf:

$$E_{expect}(A,B) = (\#\text{edges}) \cdot (\alpha\beta + \beta\alpha)$$

$$= \frac{dn}{2} \cdot 2\alpha\beta = \alpha\beta d n.$$

THM (Spielman 27.3.1):
In an expander, $E(A,B)$ is close to $\alpha\beta dn$ for all sets A & B.

Precisely,

$$\left[\, \left| E(A,B) - \alpha\beta dn \right| \le \varepsilon dn \sqrt{(\alpha-\alpha^2)(\beta-\beta^2)} \,\right],$$

Why?

$$E(A,B) = -\mathbb{1}_A^T G \mathbb{1}_B$$

and $G \approx H$ means

$$-\mathbb{1}_A^T G \mathbb{1}_B \approx -\mathbb{1}_A^T H \mathbb{1}_B = \alpha\beta dn$$

$N(A) = \{$vertices adj to vtxs in $A\}$

<u>Theorem</u> (Tanner): Let G be a d-regular graph that ε-approximates $H = \frac{d}{n} K_n$. Then for all $A \subset V$: $(|A| = \alpha n)$

$$|N(A)| \geq \frac{|A|}{\varepsilon^2(1-\alpha) + \alpha}$$ "expander"

<u>Proof</u>: Let $R = N(A)$, $B = V - R$.

$E(A, B) = 0$. Say $|B| = \beta n$.

Then

$$\alpha \beta \underline{dn} \leq \varepsilon \underline{dn} \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}$$

Say $|R| = |N(A)| = \gamma n$.

$$\alpha^2 \beta^2 \leq \varepsilon^2 (\alpha - \alpha^2)(\beta - \beta^2)$$

$$\alpha \beta \leq \varepsilon^2 (1-\alpha)(1-\beta)$$

$$\alpha\beta \leq \varepsilon^2(1-\alpha)(1-\beta)$$

$$\frac{\beta}{1-\beta} \leq \varepsilon^2 \frac{1-\alpha}{\alpha}$$

$\beta = V - N(A)$
$\beta n \quad n \quad \gamma n$
so $\beta = 1-\gamma$

$$\frac{1-\gamma}{\gamma} \leq \varepsilon^2\left(\frac{1-\alpha}{\alpha}\right)$$

$$\frac{1}{\gamma} - 1 \leq \varepsilon^2\left(\frac{1-\alpha}{\alpha}\right)$$

$$\frac{1}{\gamma} \leq \frac{\varepsilon^2(1-\alpha)+\alpha}{\alpha}$$

$$\gamma \geq \frac{\alpha}{\varepsilon^2(1-\alpha)+\alpha}$$

(times n)

$$|N(A)| \geq \frac{|A|}{\varepsilon^2(1-\alpha)+\alpha}$$

QED