Cheeger Inequalities

CS 292F

Lecture 6

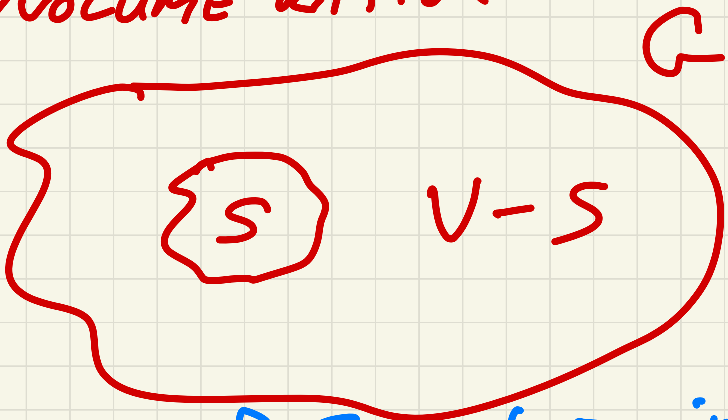
April 21, 2021

Conductance $S \subseteq V = \dfrac{|\partial S|}{\min(d(S), d(V-S))} =: \phi(S)$

where $d(S) = \sum_{a \in S} (\text{degree of vertex } a)$

SURFACE/VOLUME RATIO:

In the plane, $|\partial S|$ scales like $\sqrt{d(S)}$

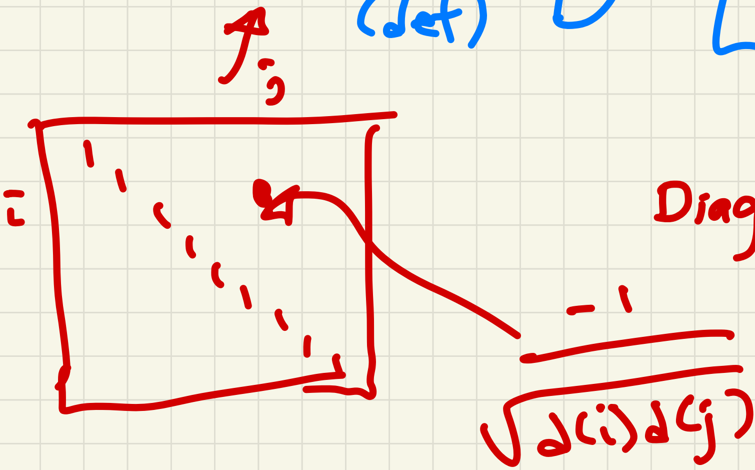


Conductance of G $\quad \phi_G = \min_{S \subseteq V} \phi(S)$

smaller $\phi_G \Rightarrow$ easier to cut off a chunk.
(clustering, partitioning)

Study Φ by "Normalized Laplacian"

$$N = N_G \underset{(def)}{=} D^{-1/2} L_G D^{-1/2}$$



Diag(vertex degrees)

$$\frac{x^T N x}{x^T x} = \frac{y^T L y}{y^T D y} \quad \text{for } D^{1/2} y = x$$

so, the eigenvalues of N $\quad Nx = \nu x$

are related to the generalized eigenproblem $\quad Ly = \nu D y$

$N \succeq 0$ (easy), evals $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$

As with L, 0 is an eval of N with multiplicity $=$ # connected component of G.

$\nu_1 = 0$

Eigenvector for $\nu_1 = 0$ is

$$d^{1/2} = \begin{pmatrix} \sqrt{d(1)} \\ \sqrt{d(2)} \\ \vdots \\ \sqrt{d(n)} \end{pmatrix}.$$

<u>Proof</u>:

$$N d^{1/2} = D^{-1/2} L \underbrace{D^{-1/2} d^{1/2}}_{\mathbb{1}} = D^{-1/2} \underbrace{L \mathbb{1}}_{0} = 0$$

Recall we showed $\lambda_2/2 \le \Theta_G$

THEOREM $\dfrac{\nu_2}{2} \le \Phi_G \le \sqrt{2\nu_2}$

Note: $\nu_2 = \min\limits_{x \perp d^{1/2}} \dfrac{x^T N x}{x^T x} = \min\limits_{y \perp d} \dfrac{y^T L y}{y^T D y}$

Proof of $\nu_2 \le \Phi_G$:

use a test vector y:

For a set $S \subseteq V$, choose y to be constant on S and on V−S.

To get $y \perp d$:

$$y(i) = \begin{cases} d(V-S) & i \in S \\ -d(S) & i \in V-S \end{cases} \quad \text{so } y = d(V-S)\mathbb{1}_S - d(S)\mathbb{1}_{V-S}$$

$y(i) = -d(S)$ for $i \in V-S$, $d(V-S)$ for $i \in S$

$$d^T y = \sum_i d(i) y(i) = \sum_{i \in S} d(i) d(V-S) + \sum_{i \in V-S} d(i) d(V)$$

$$= d(V-S) \sum_{i \in S} d(i) - d(S) \sum_{i \in V-S} d(i)$$

$$= d(V-S) d(S) - d(S) d(V-S)$$

$$= 0$$

so $y \perp d$

$$\frac{y^T L y}{y^T D y}:$$

$$y^T L y = \sum_{ij} (y(i) - y(j))^2 = \sum_{(ij)\in\partial S} (-d(s) - d(v-s))^2$$

$$= \sum_{ij\in\partial S} (d(v))^2 = d(v)^2 |\partial S|$$

$$y^T D y = \sum_i d(i) y(i)^2$$

$$= \sum_{i\in S} d(i)\, d(v-s)^2 + \sum_{i\notin S} d(i)\, d(s)^2$$

$$= d(s)(d(v-s))^2 + d(v-s)(d(s))^2$$

$$= d(s)\, d(v-s)(d(v-s) + d(s))$$

$$= d(s)\, d(v-s)\, d(v)$$

$$\frac{y^T L y}{y^T D y} = \frac{d(v)^2 |\partial S|}{d(s)\, d(v-s)\, d(v)} = \frac{d(v)\, |\partial S|}{d(s)\, d(v-s)}$$

$$\frac{y^T L y}{y^T D y} = \frac{d(V)^2 |\partial S|}{d(S)\, d(V-S)\, d(V)} = \frac{d(V)\, |\partial S|}{d(S)\, d(V-S)}$$

Now $\nu_2 \le \dfrac{y^T L y}{y^T D y} = \dfrac{d(V)\,|\partial S|}{d(S)\, d(V-S)}$

$$\le \frac{2\,|\partial S|}{\min(d(S), d(V-S))} = 2\phi(S)$$

This is true for all S,

so $\nu_2 \le \min\limits_S 2\phi(S) = \Phi_G$.

QED

ϕ : phi (conductance)

Θ : Theta (isoper ratio)

Proof of $\Phi_G \le \sqrt{2\nu_2}$

First part: from set S $\to$ test vector $y \perp d$.

This part: from vector $y \perp d$ $\to$ set S

and show $\phi(S) \le \sqrt{2\nu_2}$

Let $y \perp d$.

Think of $y(i)$ labeling vertex i.

We'll take S to be

$$\underline{S} = \{ a \in V : y(a) < \underline{t} \}$$

for some carefully chosen t.

for all $y \perp d$:
we will show $\exists t$ such that

$$\phi(S) \le \sqrt{2 \frac{y^T L y}{y^T D y}}.$$

Then $\Phi_G = \min \phi(S)$

$$\le \min_{y \perp d} \sqrt{2 \frac{y^T L y}{y^T D y}}$$

$$= \sqrt{2\nu_2}$$

Start with $y \perp d$. Define $\rho = \dfrac{y^T L y}{y^T D y}$.

Will show $\exists S \quad \phi(S) \le \sqrt{2\rho}$

Will show $\exists t : \phi(\{i : y(i) < t\}) \le \sqrt{2\rho}$

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① assume $y(1) \le y(2) \le \cdots \le y(n)$
(by renumbering the vertices)

② Take smallest k such that

$$\sum_{i=1}^{k} d(i) \geq \frac{d(V)}{2}$$

Then shift

$$z = y + \alpha \mathbb{1}$$ to make $z(k) = 0$

so

$$z(1) \le z(2) \le \ldots z(k) = 0 \le \ldots \le z(n)$$

note:

$$z^T L z = (y+\alpha \mathbb{1})^T L (y+\alpha \mathbb{1}) =$$

$$y^T L y + 2\alpha y^T \underbrace{L\mathbb{1}}_{0} + \alpha^2 \mathbb{1}^T \underbrace{L\mathbb{1}}_{0} = y^T L y$$

$$z^T D z = (y+\alpha 1)^T D (y+\alpha 1)$$

$$= y^T D y + \underbrace{2\alpha y^T D 1}_{2\alpha y^T d = 0} + \underbrace{\alpha^2 1^T D 1}_{\alpha^2 d(V) \geq 0}$$

$$\geq y^T D y$$

so $\dfrac{z^T L z}{z^T D z} \leq \dfrac{y^T L y}{y^T D y} = \rho$

3 WLOG, scale z so that

$$z(1)^2 + z(n)^2 = 1$$

note $z(1) < 0$ and $z(n) > 0$.

Want to show $\exists t \in [z(1) \quad z(n)]$

$$z(1) \leq t \leq z(n)$$

such that if $S = \{ i : z(i) < t \}$

then $$\frac{|\partial S|}{\min(d(S), d(V-S))} \leq \sqrt{2\rho}.$$

(and then conclude

$$\phi_G = \min_S \phi(S) \leq \min_y \sqrt{2\rho} = \sqrt{2\nu_2}$$

• CHOOSE A CAREFUL PROBABILITY DISTRIBUTION ON the interval $[z(1) \; z(n)]$

• SHOW

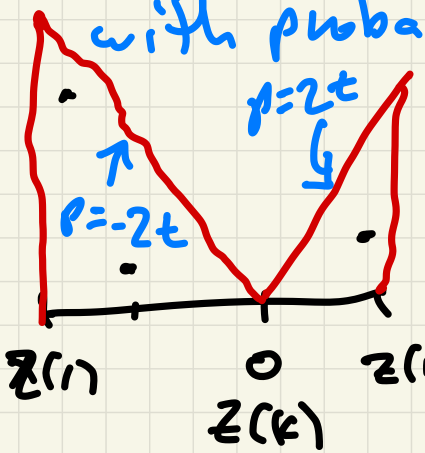
$$\mathbb{E}_t(|\partial S|) \leq \sqrt{2\rho}\, \mathbb{E}_t\left(\min(d(S), d(V-S))\right)$$

• Conclude $\exists t$ for which the inequality holds.

What probability distribution on $z(1) \le t \le z(n)$?

(L. Trevisan 2011)

Choose t between $z(1)$ & $z(n)$ with probability density $2|t|$.



AND THE REST IS JUST ALGEBRA.

$$\Pr[a \le t \le b] = \int_a^b 2|t|\,dt$$

$$= (\operatorname{sign} b)\, b^2 - (\operatorname{sign} a)\, a^2$$

$$\int_{-z(1)}^{z(n)} 2|t|\,dt = z(1)^2 + z(n)^2 = 1$$