# LQT, eigenvectors, and drawing graphs

CS 292F

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Lecture 2

$G = (V, E)$ undirected, simple graph

$V = \{1, 2, \dots, n\}$

$L = L_G$ Laplacian matrix

Recall

$LQF: \mathbb{R}^V \to \mathbb{R}$ $\quad x \mapsto x^T L x$

$$x^T L x = \sum_{(i,j) \in E} (x(i) - x(j))^2 \geq 0$$

Connections between LQF & eigen$^{\text{vec}}_{\text{val}}$

If $Lw = \lambda w$ then

$$\frac{w^T L w}{w^T w} = \frac{\lambda w^T w}{w^T w} = \lambda$$

$Lw_i = \lambda_i w_i$ and $w_i$ are <u>orthonormal</u>

$$LW = W\Lambda \quad \text{and} \quad W^T W = I$$

columns $[w_1\ w_2 \cdots w_n]$

$\Lambda = \operatorname{diag}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

($W$ is an orthogonal matrix, ie $W^{-1} = W^T$)

$$WW^T = I$$

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

$L\mathbb{1} = 0$

strict inequality $\Leftrightarrow$ $G$ is connected.

## L as a sum of rank-1 matrices

$$LW = W\Lambda \Rightarrow L = W\Lambda W^T$$

$$W\Lambda W^T = \sum_{i=1}^{n} \underbrace{\lambda_i w_i w_i^T}_{\text{rank } 1} = \sum_{i=2}^{n} \lambda_i w_i w_i^T$$

## Eigenvector Basis

Any vector $x = \sum_{i=1}^{n} c_i w_i$ Actually $c_i = w_i^T x$

Proof: $\sum_i (w_i^T x) w_i = \sum_i w_i w_i^T x = \left(\sum_i w_i w_i^T\right) x$

$= WW^T x = x$

This looks just like $x = \sum_i x(i) \mathbb{1}_i$

# $L$ acts on $x$ elementwise in evec basis

$$Lx = \underline{W}\,\underline{\Lambda}\,\underline{W}^T\Big(\sum_i c_i w_i\Big)$$

$$= \sum_i c_i \underbrace{\Big(\underline{W}\,\underbrace{\underline{\Lambda}\,\underbrace{\underline{W}^T w_i}_{\mathbb{1}_i}}_{\lambda_i \mathbb{1}_i}\Big)}_{\lambda_i w_i}$$

so $$Lx = \sum_i \lambda_i c_i w_i$$

ALL TRUE FOR ANY REAL SYMM MATRIX (except for $\lambda_i = 0$)

<u>Exercise</u>: show $L^k x = \sum_i \lambda_i^k c_i w_i$

(two different ways)

**Thm 1**: Let $A$ be a symm matrix and let $x$ maximize $\dfrac{x^TAx}{x^Tx}$.

Then $x$ is an eigenvector of $A$ with eval $\lambda = x^TAx/x^Tx$.

**Proof**: WLOG take $\|x\| = 1$

(Now we know there is a max, because unit sphere is closed + bounded)

At the max, gradient is 0.

$$\nabla x^Tx = 2x \qquad \nabla x^TAx = 2Ax$$

$$\nabla \frac{x^TAx}{x^Tx} = \frac{(x^Tx)(2Ax) - (x^TAx)(2x)}{(x^Tx)^2}$$

This is 0 $\iff$ $x^Tx(2Ax) = (x^TAx)2x$

$(x^Tx)(Ax) = (x^TAx)x$

$$\iff Ax = \underbrace{\frac{x^TAx}{x^Tx}}_{\text{eval}} \, x \leftarrow \text{evec.}$$

Thm 2 (Rayleigh quotient theorem):
Let $A$ be any symm matrix. Then

$\exists \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $w_1, w_2 \ldots w_n$ orthonormal with

$$AW = W\Lambda \quad \text{and} \quad W^TW = I.$$

Also,

$$w_k = \underset{x \perp w_1, w_2, \ldots w_{k-1}}{\operatorname{argmin}} \frac{x^TAx}{x^Tx} = \underset{x \perp w_{k+1}, \ldots, w_n}{\operatorname{argmax}} \frac{x^TAx}{x^Tx}$$

Proof: Get $w_n$ from thm 1.

Let $A' = A - \lambda_n w_n w_n^T$ and proceed by induction $k = n$ down to 1.
see Spielman sec 2.2 for details

Corollary: Fiedler value $\lambda_2$ (for Laplacian)

$$\left[ \lambda_2 = \min_{1^Tx=0} \frac{x^TLx}{x^Tx} \right]$$