CS 292F.300 Graph Laplacians

Introduction

Mon March 29

Lecture 1

Graph $G = (V, E)$ undirected simple

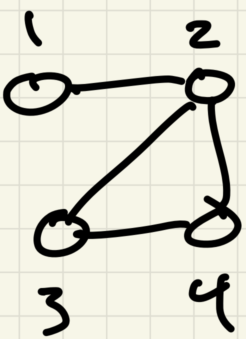
$V = \{1, \dots, n\}$

DEF The Laplacian matrix L_G is an $n \times n$ symmetric matrix

$L(i,i)$ = degree of vertex i

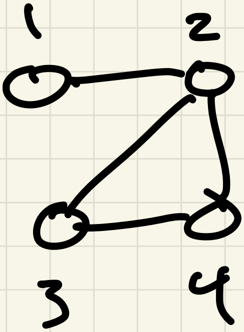
$L(i,j) = -1$ if $(i,j) \in E$

$L(i,j) = 0$ otherwise.



$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

G $\qquad\qquad$ L_G



$$L_G = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

G $\qquad$ L_G

We will often think of a vector as a set of labels for vertices.

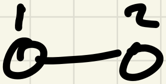
$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(n) \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbb{1}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow i \qquad \mathbb{1}_i(i) = 1$$

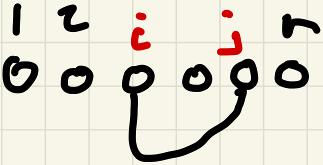
$$L\mathbb{1} = \mathbf{0}$$

Every row sums to 0.

L is singular!



$$L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



$e = (i,j)$

$$L_e = \begin{array}{c} \\ \begin{pmatrix} 0 & & & & 0 \\ & 1 & -1 & & \\ & -1 & 1 & & 0 \\ 0 & & & & \end{pmatrix} \\ i \;\; j \end{array} \begin{array}{l} n \\ \leftarrow i \\ \leftarrow j \\ \end{array}$$

$L_e = L_{(i,j)}$ is the Laplacian of a 1-edge graph.

$$L_{(i,j)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

$$= (\mathbb{1}_i - \mathbb{1}_j)(\mathbb{1}_i - \mathbb{1}_j)^T$$

$$L_G = \sum_{e \in E} L_e = \sum_{(i,j) \in E} (\mathbb{1}_i - \mathbb{1}_j)(\mathbb{1}_i - \mathbb{1}_j)^T$$

$\cancel{(i,j) \in E}$

DEF Laplacian Quadratic Form (LQF)

$$x^T L x = x^T \left(\sum_{(i,j)\in E} (\mathbb{1}_i - \mathbb{1}_j)(\mathbb{1}_i - \mathbb{1}_j)^T \right) x$$

specific graph Laplacian

variable vector

$$= \sum_E \left(x^T (\mathbb{1}_i - \mathbb{1}_j) \right) \left((\mathbb{1}_i - \mathbb{1}_j)^T x \right)$$

$(x_i - x_j)$ $(x_i - x_j)$

$$= \sum_{(i,j)\in E} (x_i - x_j)^2$$

Eigenvalues + eigenvectors:

$$Aw = \lambda w$$

matrix, vec, number

w : eigenvector

λ : eigenvalue

L_G is real & symmetric $\Rightarrow$ it has n orthogonal eigenvectors $w_1, w_2, \ldots, w_n$ (unit length)

and n real eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$

with $L_G w_i = \lambda_i w_i$

$$W = [w_1 \; w_2 \cdots w_n] \qquad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

so $L_G W = W \Lambda$ eval/evec

and $W^T W = I$

$w_i^T w_i = 1$

$w_i^T w_j = 0 \quad i \ne j$

For any vector x,

$$x^T L x = \sum_{ij \in E} (x(i) - x(j))^2 \geq 0$$

$\Rightarrow$ L is positive semidefinite

$\Rightarrow$ all evals ≥ 0

$L\mathbb{1} = 0$

So we write

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \leq \lambda_n$$

How about λ_2 ?

$$0 = \lambda_1 \le \lambda_2 \le \lambda_3 \cdots \le \lambda_n$$

Theorem: G is connected $\Longleftrightarrow \lambda_2 \neq 0$

$\Rightarrow$: If $w_2(a) \neq w_2(b)$, $\qquad Lw_2 = \lambda_2 w_2$

$\exists$ a path $a - \overset{i}{-} \overset{j}{-} - b$ and some edge (i,j) with $x(i) \neq x(j)$

$w_2(i) \neq w_2(j) \Rightarrow (w_2(i) - w_2(j))^2 > 0$

$$\Rightarrow \sum_{ij \in E} (x(i) - x(j))^2 > 0$$

$\Rightarrow \quad w_2^T L w_2 > 0. \Rightarrow Lw_2 \neq 0$

$\Rightarrow \lambda_2 \neq 0$

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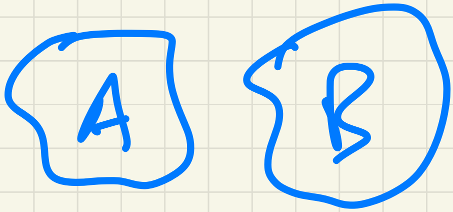
Suppose G is disconnected.

$G =$ 

$\mathbb{1}_A$ $\qquad$ $\mathbb{1}_B$

$L_G \mathbb{1}_A = 0$ ✓

$L_G \mathbb{1}_B = 0$ ✓

$$L_G = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \underbrace{\begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbb{1}_A} = 0$$

$\mathbb{1} = \mathbb{1}_A + \mathbb{1}_B$

$\Rightarrow \lambda_2 = 0$ (at least 2 zero evals).

<u>THM</u> For any graph, the number of connected components is the multiplicity of 0 as an eval, that is, $\max_i(\lambda_i = 0)$