Effective Resistance, Schur Complements, Gaussian Elimination

CS292F

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Lecture 10

$Lv = i_{EXT}$ maps voltages at vertices to externally applied currents at vertices.

$\underline{1}^T i = 0$ and can take $1^T v = 0$

Every weighted Laplacian mtx can be interpreted this way: weight $c(e)$ on an edge $\Rightarrow$ resistance $1/c(e)$.

(Every graph with positive edge weights)

Can solve for v given i by $v = L^+ i$ (in principle)

TODAY:

EFFECTIVE RESISTANCE

$v = L^{+} i \qquad Lv = i$

Inject 1 unit of current at vertex a and take it out at vertex b.

<u>DEF</u> : $R_{eff}(a,b)$ is the resistance that would account for the observed voltage drop from a to b.

$V = IR$ so if $I = 1$ then $V = R$

So $R_{eff}(a,b) = v(a) - v(b)$

Take $i = \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} a \\ \\ b \\ \\ \end{array} = \mathbb{1}_a - \mathbb{1}_b$

$v = L^{+}(\mathbb{1}_a - \mathbb{1}_b) \qquad v(a) - v(b)$

In fact

$$(\mathbb{1}_a - \mathbb{1}_b)^T L^{+} (\mathbb{1}_a - \mathbb{1}_b) = R_{eff}(a,b).$$

$$\Downarrow$$

$v(a) - v(b).$

DEF $R_{eff}(a,b) \stackrel{\text{def}}{=} (1_a - 1_b)^T L^{+} (1_a - 1_b)$

THM (monotonicity)

Reducing edge weight can only increase effective resistance.

Proof: Let G & H be Laplacians of one graph with different weights

$$c_G(a,b) \geq c_H(a,b)$$

then $L_G \succeq L_H$ by LQF

$$x^T L_G x = \sum_{ab \in E} c(a,b)(x(a) - x(b))^2$$

$\Rightarrow L_G^{+} \preceq L_H^{+}$ (homework 2)

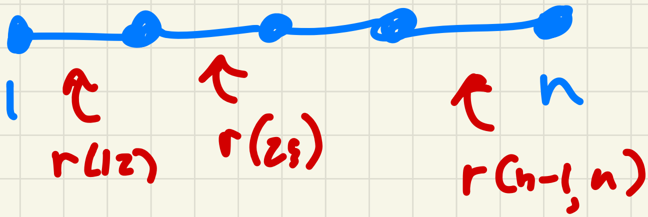
$\Rightarrow \forall x \quad x^T L_G^{+} x \leq x^T L_H^{+} x$

$\Rightarrow R_{eff}(p,q)_G \leq R_{eff}(p,q)_H$

$(x = 1_p - 1_q)$.

Example: series resistors.

Path graph:



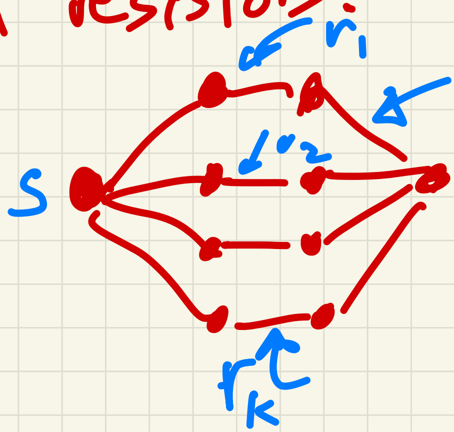
Expect $R_{eff}(1,n) = r(12) + r(23) + \cdots + r(n-1\, n)$

Proof: set $v(a) = r(a\, a{+}1) + r(a{+}1\, a{+}2) + \cdots + r(n)$

$\Rightarrow$ current on edge $(a, a+1)$ is $\dfrac{v(a+1) - v(a)}{r(a, a+1)} = 1.$

So resistances add.

Parallel resistors:



$$R_{eff}(s,t) = \frac{1}{1/r_1 + 1/r_2 + \cdots + 1/r_k}.$$

Graph G (weights)

"Interesting" vertices boundary B

"Uninteresting" interior $I = V - B$.

Fix $i(a)$ for $a \in B$

Let $i(a) = 0$ for $a \in I$. $i = Lv$

$i = Lv$ $L(a,b) = \begin{cases} -c(a,b) & a \neq b \\ d(a) = \sum_{(a,p) \in E} c(a,p) & \text{if } a = b. \end{cases}$

$d(a) = c(a,a)$

Let $B = \{2, 3, \ldots, n\}$

$I = \{1\}$.

1 is the only uninteresting vertex $\boxed{L v = i}$

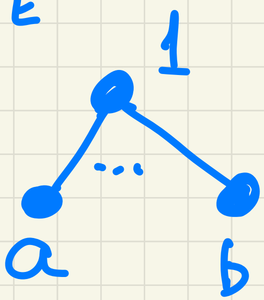
Want: new matrix, $(n-1)$ by $(n-1)$ L_B

L_B maps $v_B = v(2:n)$ to $i_B = i(2:n)$

given $i(1) = 0$.

$i(1) = 0 \Rightarrow v$ is harmonic at 1

$$\Rightarrow 0 = i(1) = L(1,1)v(1) + \sum_{(a,1)\in E} L(1,a)v(a)$$



$\Rightarrow$

$$v(1) = -\sum_{a1\in E} \frac{L(1,a)}{L(1,1)} v(a)$$

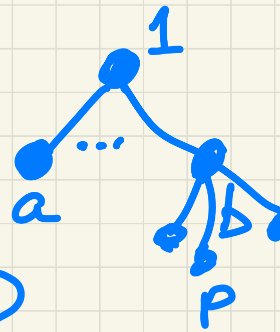
Want to substitute this in

for $v(1)$ in equations $2, 3, \dots, n$:

$$v(1) = -\sum_{a1\in E} \frac{L(1,a)}{L(1,1)}\, v(a)$$

$i = Lv$

$v(1)$ only appears in equations for $i(b)$ where $(1,b)\in E$.



For such b,

$$i(b) = L(b,b)v(b) + \sum_{(p,b)\in E} L(b,p)v(p)$$

becomes:

$$i(b) = L(b,b)v(b) + \sum_{\substack{pb\in E\\ p\neq 1}} L(b,p)v(p) - \sum_{a1\in E} \frac{L(b,1)L(1,a)}{L(1,1)}\, v(a)$$

pull out the term $a=b$.

$$i(b) = \left[L(b,b) - \frac{L(b,1)^2}{L(1,1)}\right] v(b)$$

$$+ \sum_{\substack{pb\in E\\ p\neq 1}} L(b,p)v(p) - \sum_{\substack{a1\in E\\ a\neq b}} \frac{L(b,1)L(1,a)}{L(1,1)}\, V(a)$$

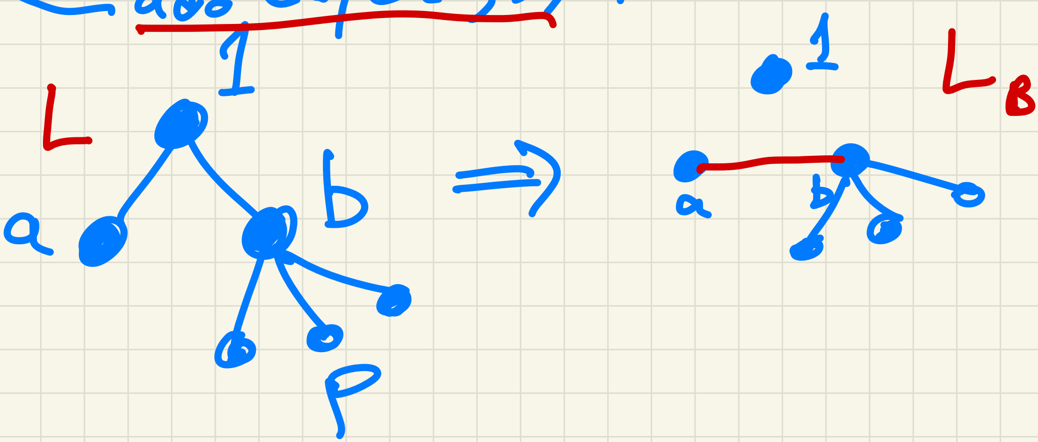
$$i(b) = \left[L(b,b) - \frac{L(b,1)^2}{L(1,1)} \right] v(b) + \sum_{\substack{pb \in E \\ p \neq 1}} L(b,p)\, v(p) - \sum_{\substack{a1 \in E \\ a \neq b}} \frac{L(b,1) L(1,a)}{L(1,1)} v(a)$$

ONE ROW OF L_B

update to diagonal $L(b,b)$ to get $L_B(b,b)$

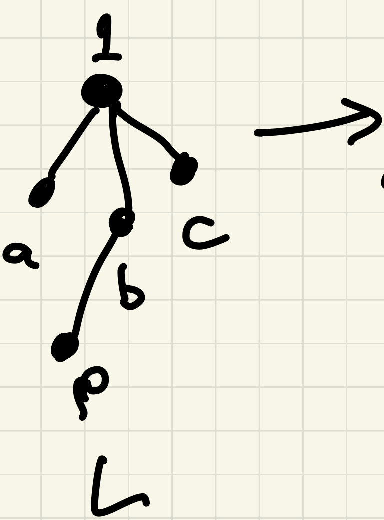
remove edge$(1,b)$

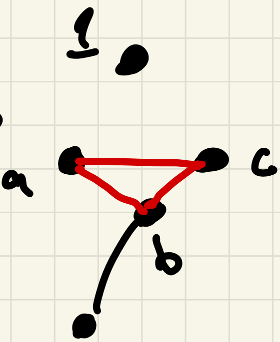
add edge(a,b) for all a adj to 1.



VERTEX ELIMINATION RULE:

To get L_B from L, add edges between all vertices adjacent to vertex 1, then delete vertex 1 and its incident edges.

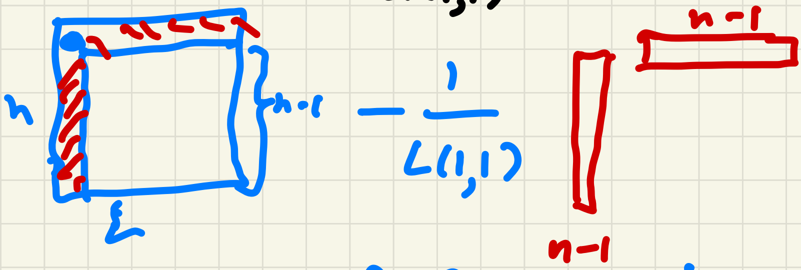




New edges have "filled in" among the adjacencies of the eliminated vertex.

Now

$$L_B = L(2{:}n, 2{:}n) - \frac{1}{L(1,1)} L(2{:}n, 1) L(1, 2{:}n)$$



This is one step of Gaussian elimination!

Theorem: L_B is a weighted Laplacian.

Proof: (1) L_B is symmetric.

(2) added off-diag elts $\frac{-L(b,1)L(1,a)}{L(1,1)}$ are negative.

(3) Sum of changes to row b is:

$$\frac{-L(b,1)^2}{L(1,1)} - \sum_{\substack{(a,1)\in E \\ a\neq b}} \frac{L(b,1)L(1,a)}{L(1,1)} = \frac{L(b,1)}{L(1,1)} \underbrace{\sum_{(a,1)\in E} L(a,1)}_{\text{sum of row 1 of } L = 0}$$

Therefore row b of L_B still sums to 0. QED