# Planar Separators.

If time: Dense eigenvalue algorithms.

CS 292F

Lecture 7

April 26, 2021

CHEEGER: $\nu_2/2 \le \phi_G \le \sqrt{2\nu_2}$

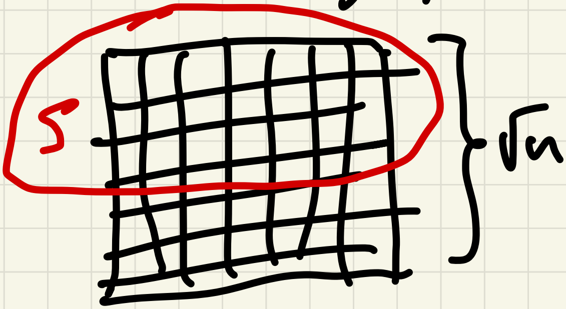
$$\phi_G = \min_S \phi(S) = \frac{|\partial S|}{\min(d(S), d(V-S))}$$

ISOPERIM: $\lambda_2/2 \le \theta_G$

$$\theta_G = \min_{|S| \le \frac{n}{2}} \theta(S) = \frac{|\partial S|}{|S|}$$

$\phi_G$ and $\theta_G$ are "surface/volume" ratios $\longrightarrow$ small $\phi_G \Rightarrow$ "good cuts" (clustering, partitioning)

EG Grid graph:




We showed $\lambda_2 = \Theta\left(\frac{1}{n}\right)$

Also $\nu_2 = \Theta\left(\frac{1}{n}\right)$.

(both tight)

Thus $\exists S \ \phi(S) \leq \frac{c}{\sqrt{n}}$.

Sure! $S$ = top half

$$\phi(S) = \frac{|\partial S|}{\min(d(S), d(V-S))} = \frac{\sqrt{n}}{\sim 2n} = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

The $n$-vertex grid graph has a <u>$\sqrt{n}$-separator</u> of edges.

APPLICATION: solving sparse linear systems of equations.

LIPTON + TARJAN 1977

# PLANAR SEPARATOR THEOREM

Let $G$ be an $n$-vertex planar graph of bounded vertex degree.

Then $\exists$ set of $O(\sqrt{n})$ edges whose removal leaves no connected component with more than $\lceil \frac{n}{2} \rceil$ vertices.

(Note: if degree $\leq \Delta$, size is $O(\sqrt{\Delta n})$)

**PROOF**: Very combinatorial.

Application: all 2D finite element systems can be solved in $O(n^{3/2})$ time.

## SPECTRAL PARTITIONING HEURISTIC (1970s; 1990s in scientific computing)

Fiedler vector $L w_2 = \lambda_2 w_2$

Embed the graph in the real line

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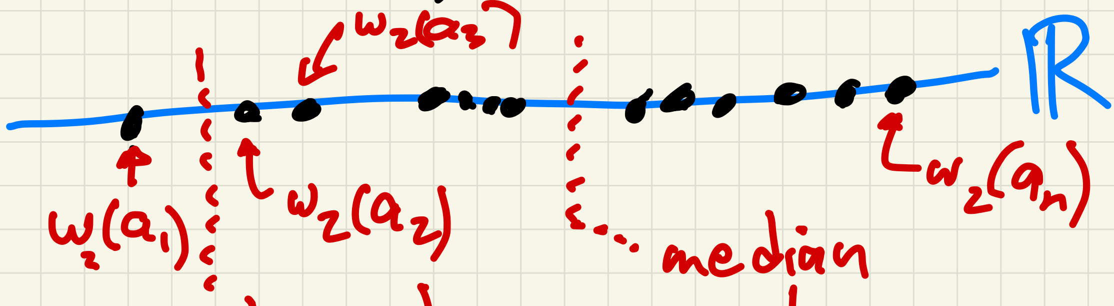


ratio cut: $\Theta^{(S)} = \text{small } \frac{|\partial S|}{|S|}$

median cut

In 1997, Spielman + Teng used spectral graph theory to prove the planar sep thm.

**BIG THEOREM:**

If $G$ is planar with maximum degree $\Delta$, then $\lambda_2(G) \leq \frac{8\Delta}{n}$.

($\lambda_2 = O(\frac{\Delta}{\sqrt{n}})$ was known)

**Proof:** See Spielman.

# EIGENVALUE + EIGENVECTOR ALGORITHMS

## (Symmetric real matrices) $A = A^T$

$A = A^T \quad Aw_i = \lambda_i w_i \quad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$

$\|w_i\| = 1$

$W^T A W = \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$

$W^T W = I \qquad W = [w_1\ w_2\ \cdots\ w_n]$

matrix factorization $A = W \Lambda W^T$

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All eigenvalue algorithms are iterative.

Because: The roots of any degree-n polynomial are the evals of some $n \times n$ matrix.

BUT that can't be done with $+, -, \times, \div, \sqrt{\ }, \sqrt[3]{\ }$, etc. (ABEL GALOIS)

$\Rightarrow$ need iteration.

If max degree $= \Delta$ then $L_G \preceq \Delta N_G$

so upper bounds on $\lambda_2$ imply upper bounds on $\nu_2$. So S-T implies $\exists$ small conductance cuts.

In fact, you can get

$\boxed{\Theta_G \leq \frac{c}{\sqrt{n}} \text{ for all planar } G \quad (\text{some } c = c(\Delta))}$

Therefore $\exists S$ with

$$\frac{|\partial S|}{|S|} = \Theta(S) \leq \frac{c}{\sqrt{|S|}}$$

and in fact you can get it by a Fiedler vector cut (a ratio cut).

Therefore $\exists S$ with

$$\frac{|\partial S|}{|S|} = \Theta(S) \leq \frac{c}{\sqrt{|S|}}$$

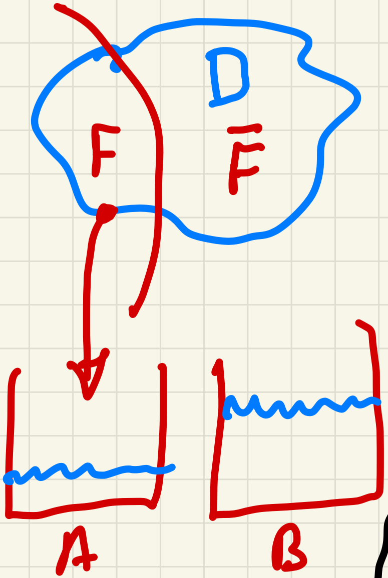
**ALGORITHM** to find a balanced cut:

**Goal**: divide vertices into 2 parts $A, B$ with $\leq \lceil \frac{n}{2} \rceil$ each, and $\Theta(\sqrt{n})$ edges between them.

↑ constant depends on $\Delta$.

**Notation**: Set $A \subseteq V$ mean both the set of vertices and the induced subgraph of $G$ (which is planar).

**Method**:




CODE:

$D = V \quad A = B = \emptyset$

repeat

① Find a small-ratio cut in $D$ into $F, \bar{F}$ with $|F| \le |\bar{F}|$

and $\Theta(F) = \dfrac{|\partial F|}{|F|} \le \dfrac{c}{\sqrt{|D|}}$

② Add $F$ to the smaller of $A$ & $B$

③ $D = \bar{F}$

until $|D| \le 1$

① $A$ & $B \le \lceil \frac{n}{2} \rceil$

$$\min(|A|, |B|) + |F| \le \frac{|A| + |B|}{2} + \frac{|F| + |\bar{F}|}{2} = \frac{n}{2}$$

How many total edges cut?

At each iteration $i$ $\quad \frac{|\partial F|}{|F|} \le \frac{c}{\sqrt{|D|}}$

so we cut at most $c|F| \cdot \frac{1}{\sqrt{|D|}}$ edges.

$$\text{cut}(i) \le \frac{c|F_i|}{\sqrt{|D_i|}} = c\sum_{j=1}^{|F_i|} \frac{1}{\sqrt{|D_i|}}$$

$$= c\sum_{j=|D_i|-|F_i|+1}^{|D_i|} \frac{1}{\sqrt{|D_i|}} \le c\sum_{j=|D_i|-|F_i|+1}^{|D_i|} 1/\sqrt{j}$$

$$\text{totalcut} = \sum_{i=1}^{t} \text{cut}(i) = c\sum_{i=1}^{t} \sum_{j=|D_i|-|F_i|+1}^{|D_i|} 1/\sqrt{j}$$

$$= c\sum_{j=1}^{n} 1/\sqrt{j} \lesssim c\int_1^n \frac{dx}{\sqrt{x}} = 2c\sqrt{x}\Big|_{x=1}^{x=n}$$

$$\le 2c\sqrt{n}.$$

recall $c = c(\Delta)$