Preconditioned CG

CS292F

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Lecture 13

CG: minimize $\frac{1}{2}x_t^T A x_t - b^T x_t = f(x_t)$

over all $x_t \in K_t(A, b)$

$p_0 = b$, $x_0 = 0$

<u>for</u> $t = 1, 2, \ldots$

- compute Ap_{t-1}
- A-orthogonalize that against earlier p_i to get p_t.
- compute c_t to minimize $f(c_t p_t)$
- $x_t := x_{t-1} + c_t p_t$
- $r_t = b - Ax_t$ "residual" vector.

$$\min_{x_t \in K_t(b)} \|x_t - x\|_A$$

$\Rightarrow$ gets 0 in exact arithmetic when

$\exists$ polynomial $q_t(z)$ with $q_t(0)=1$ and $q_t(\lambda)=0$ for every eigenvalue λ of A.

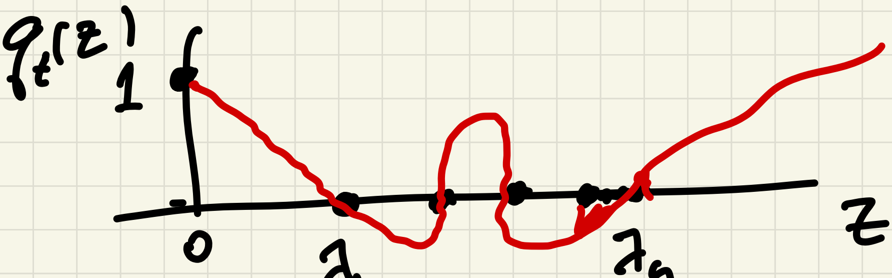
THM

At each step t,

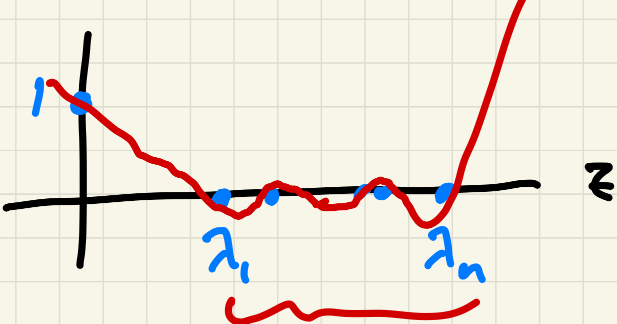
$$\frac{\|x_t - x\|_A}{\|x\|_A} \leq \min_{q_t(z)} \max_{\lambda} |q_t(\lambda)|$$

polynomials of degree t with $q_t(0)=1$

eigenvalues of A



EVALS close together are good.
(Clusters of evals)



how close to 0 can a polynomial be on this whole interval?

$$\kappa(A) \overset{\text{def}}{=} \frac{\lambda_n}{\lambda_1}$$

(if A is symm + pos def)

condition number

Answer: Chebyshev polynomials.

Can use Cheb. polys to get convergence bounds for CG based on $\kappa(A)$, for example:

$$\frac{\|r_t\|}{\|b\|} \le \underbrace{10^{-6}}_{\varepsilon} \quad \text{for } t = O\left(\sqrt{\kappa(A)}\right)$$

proportional to $\log\left(\frac{1}{\varepsilon}\right)$