Eigenvalues of Specific Graphs

Mon Apr 12

CS 292F

Lecture 4

K_n = complete graph.

$$L = \begin{pmatrix} n-1 & -1 & & -1 \\ -1 & n-1 & -1 & \\ -1 & -1 & \ddots & -1 \\ -1 & -1 & & n-1 \end{pmatrix}$$

<u>EVALS</u> : $\lambda_1 = 0 \quad w_1 = \mathbb{1}$

<u>THM</u> : $\lambda_2 = \lambda_3 = \cdots = \lambda_n = n$

<u>PROOF</u>: Let $\mathbb{1}^T x = 0$ and compute Lx:

$$(Lx)(a) = (n-1)\,x(a) - \sum_{j \neq a} x(j)$$

$$= n\,x(a) - \underbrace{\sum_{j=1}^{n} x(j)}_{\mathbb{1}^T x}$$

$$= n\,x(a)$$

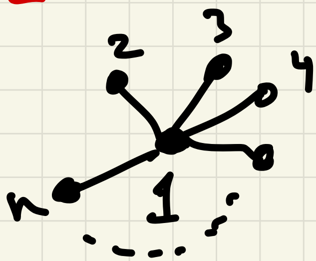
$Lx = nx$ for all $x \perp \mathbb{1}$

(all $n-1$ dimensions of such vectors)

NOTE:

$$(Lx)(a) = \sum_{(a,j)\in E} \big(x(a) - x(j)\big)$$

STAR GRAPH



$$L = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & & 0 & \\ -1 & & 1 & & \\ -1 & 0 & & 1 & \\ & & & & 1 \end{pmatrix}$$

$\lambda_1 = 0 \quad w_1 = \mathbb{1}$

$\Rightarrow x = \mathbb{1}_a - \mathbb{1}_b$ is an evec.

$(Lx)(a) = x(a) - x(c) = x(a) = 1$ with eval 1.

$(Lx)(b) = x(b) - x(c) = x(b) = -1$

$(Lx)(i) = 0 \quad i \neq a, b$

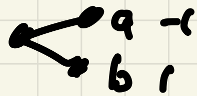
$\mathbb{1}_2 - \mathbb{1}_3,\ \mathbb{1}_3 - \mathbb{1}_4,\ \mathbb{1}_4 - \mathbb{1}_5, \ldots,\ \mathbb{1}_{n-1} - \mathbb{1}_n$

are all evecs with eval $\underline{1}$

(n-2 of them).

$0 = \lambda_1,\ \lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 1 \quad \lambda_n = ?$

$\lambda_0 = 0$ and $(n-2)$ $\lambda_i = 1$



THM For any graph, $\sum_{i=1}^{n} \lambda_i = 2(\#\text{edges})$

Pf: For any matrix, $\sum \lambda_i = \text{Trace}$ (= sum of diagonals)

For the star,

$\#\text{edges} = n-1$ so $\sum \lambda_i = 2n-2$

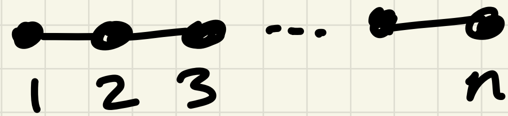
we have $n-2$, so $\underline{\lambda_n = n}$.

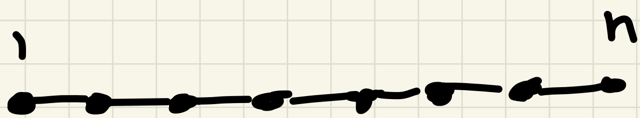
EVEC w_n:

$w_n(a)$ is constant for $2 \le a \le n$

$w_n(1)$ make $\mathbf{1}^T w_n = 0$

so $w_n = \begin{pmatrix} -(n-1) \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Path graph: $P_n =$ 

$f(x)$ 

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \sim \frac{f(a+1) - f(a)}{1}$$

$$\frac{d^2 f}{dx^2} = \lim_{h \to 0} \frac{-f(x-h) + 2f(x) - f(x+h)}{h^2}$$

$$\sim -f(a-1) + 2f(a) - f(a+1)$$

$$f = \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{pmatrix}$$

$$(Lf)(a) =$$

$$(Lf) \cong \frac{d^2 f}{dx^2}$$

Bounds on $\lambda_2(P_n)$ (n even)

upper bound: test vector $1^T x = 0$

$$x = \begin{pmatrix} n-1 \\ n-3 \\ \vdots \\ -(n-1) \end{pmatrix} \quad \text{note } 1^T x = 0$$

$$\lambda_2(P_n) \le \frac{x^T L x}{x^T x}$$

$$= \frac{\sum_{i=1}^{n-1} (x(i) - x(i+1))^2}{\sum_{i=1}^{n} (x(i))^2} = \frac{\sum_{i=1}^{n-1} 2^2}{\sum \quad {}^2}$$

$$= \frac{4(n-1)}{\Theta(n^3)} \quad \left(\text{actually } \frac{4(n-1)}{(n+1)n(n-1)/3}\right)$$

$$= \frac{12}{n(n+1)}$$

LOWER BOUND NEXT – AFTER WE GET SOME MACHINERY.

Let $G=(V,E,w)$ be an undirected graph with $V=\{1,\dots,n\}$ and <u>weights</u> $w(e)>0$ on edge e.

<u>DEF</u> WEIGHTED LAPLACIAN L_G:

$$L_G(i,j) = \begin{cases} -w(i,j) = -w(j,i) & \text{if } (i,j)\in E \\ \sum_{k\neq i} w(i,k) & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

<u>Note</u> If $D = \begin{array}{cc} \begin{pmatrix} w_{ij} & & 0 \\ & \ddots & \\ 0 & & w \end{pmatrix} & \text{edges} \\ \text{edges} & \end{array}$

and U = incidence matrix of G

$$U = \begin{array}{cc} & \text{edges} \\ \text{vtx} & \boxed{} \end{array}$$

Then $L_G = UDU^T$.

Löwner (?) ordering of graphs.

(partial ordering $\succeq$)

DEF

For matrix, $A \succeq O$ means A is symmetric + positive semidefinite

$A \succ O$: A has $\lambda > 0$.

$$A \succeq B \text{ means } A - B \succeq O$$

For weighted graphs, G, H

$$G \succeq H \text{ means } L_G - L_H \succeq O$$

$$A \succeq O \iff \forall x \quad x^T A x \geq 0$$

$$G \succeq H \iff \underbrace{\forall x \quad x^T L_G x \geq x^T L_H x}.$$

If G is weighted graph and $c \geq 0$

then cG means the graf with weights $cw(\cdot)$

If $c \geq 1$ then

$$cG \succcurlyeq G$$

<u>THM</u>: If $G \precsim H$ then

$$\forall k \ \lambda_k(G) \leq \lambda_k(H).$$

<u>Proof</u>: uses a slightly stronger version of RQT called Courant-Fischer.

THM If H is a subgraph of G (with all n vertices) then $H \preceq G$

Proof:
$$x^T G x = \sum_{e \in E_G} (x(i) - x(j))^2$$
$$\geq$$
$$x^T H x = \sum_{ij \in E_H} (x(i) - x(j))^2$$

DEF For a constant $c \geq 1$, graph H is a <u>c-approximation</u> of G

means
$$cH \succeq G \succeq \frac{1}{c} H$$