Resistive Networks

CS292F

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Lecture 9

LANCZOS

$$Q_k^T \, A \, Q_k = T_k \qquad (T_k \text{ is } k \times k)$$

n

$$Q_k^T Q_k = I_k$$

evals of T_k.

Then $V_k^T T_k V_k = \Theta$

$$\Theta = \mathrm{Diag}(\theta_1, \theta_2, \theta_3, \ldots, \theta_k)$$

k Ritz values approximate k of the n evals of A.

The extreme Ritz values converge quickly to the extreme evals of A.

As k increases.

Disadvantage: How do you get the middle evals?

Spectral transforms for interior eigenvalues.

Say we want an eigenvalue near σ.

Matrix A, want $Aw = \lambda w$, λ near σ.

Look at $A - \sigma I$. This is symmetric, (Not positive definite)

If $Aw = \lambda w$ then $(A - \sigma I) w = (\lambda - \sigma) w$.

And $A^{-1} w = \frac{1}{\lambda} w$ and $(A - \sigma I)^{-1} w = \frac{1}{\lambda - \sigma} w$

and $\frac{1}{\lambda - \sigma}$ is an "external" eigenvalue of ~~A~~ "extreme"

APPLY LANCZOS TO $(A - \sigma I)^{-1}$.

NEED TO MULTIPLY $v = (A - \sigma I)^{-1} q_k$

$\Rightarrow$ solve $(A - \sigma I) v = q_k$ for v.

STILL SPARSE, BUT NOT POSITIVE DEF.

$Aw = \lambda w$ $\qquad$ $Ax = b$ $\binom{\text{given}}{b}$

SOLVING SYSTEMS OF EQUATIONS

$$Lx = b$$

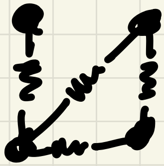
(weighted) Laplacian.
Symmetric
positive semidefinite
not completely general.

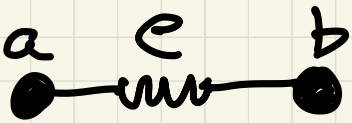
APPLICATIONS:

PDE's; Electrical circuits;
data analysis; graph algorithms
(21st century)

There are surprisingly efficient
algorithms for $Lx=b$, Laplacian.
(ACTIVE RESEARCH)

NETWORKS OF RESISTORS





resistance $r(a,b)$

or $r(e)$

(in Ohms)

current from a to b (in amperes)

$$i(a,b) = -i(b,a)$$

potential at a or at b (voltage)

$v(a)-v(b)$ potential difference.

Ohm's law: $V=IR$ $\quad v(a)-v(b) = i(a,b)\,r(a,b)$

Kirchhoff's current law: $\forall a \in V : \sum_{(a,b)\in E} i(a,b) = 0$

Kirchhoff's voltage law: sum of $v(a)-v(b)$ around any cycle is 0. Automatic.

Resistive network $\equiv$ weighted graph.

DEF

$$U = \begin{array}{c|c} & m \\ n & \begin{bmatrix} & 1 & \leftarrow a & \\ & -1 & \leftarrow b & \end{bmatrix} \\ & \text{edges} \end{array} \text{ vertices}$$

incidence matrix

edge (a,b)

Know that $UU^T = L$ is a Laplacian (unweighted)

(our U is the transpose of Spielman's)

$$R = \begin{array}{c|c} & m \\ m & \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \\ & \text{edges} \end{array} = \underset{e\in E}{\mathrm{Diag}}(r(e)) \qquad v = \begin{pmatrix} v(1) \\ \vdots \\ v(n) \end{pmatrix} \text{ vertex potential } (n)$$

$$i = \begin{pmatrix} i(1) \\ \vdots \\ i(m) \end{pmatrix} = \text{currents of edges } (m)$$

OHM: $Ri = U^T v \Rightarrow i = R^{-1} U^T v$

KIRCHOFF: $\cancel{Ui = 0}$ $\quad Ui = i_{ext}$

Consider an externally applied current at each vertex (or at some vertices): $i_{ext}(v)$

so i_{ext} is an n-vector with $1^T i_{ext} = 0$

$$R i = U^T v \Rightarrow i = R^{-1} U^T v$$

$$U i = i_{ext}$$

$$\underbrace{U R^{-1} U^T}_{\text{weighted Laplacian}} v = i_{ext}$$

$L = U R^{-1} U^T$

$R^{-1} = \mathrm{Diag}_{e \in E}\left(\frac{1}{r(e)}\right)$ Define $c(e) = \frac{1}{r(e)}$

inverse resistance (mhos)

So $L v = i_{ext}$

DEF A vector v of vertex labels is harmonic at vertex a of a weighted graph

$$\equiv \quad v(a) = \frac{\sum_{ab \in E} c(a,b)\, v(b)}{\sum_{ab \in E} c(a,b)}$$

RECALL we defined $d(a) = \sum_{ab \in E} c(a,b)$

If $Lv = i_{ext}$ then v is harmonic at every vertex with no external current.

$Lv = i_{ext}$. we said $\mathbb{1}^T i_{ext} = 0$.

$L(v+\mathbb{1}) = Lv + L\mathbb{1} = Lv$ so WLOG we can take v to have sum 0 as well, $\mathbb{1}^T v = 0$. (replace v by $v - \frac{1}{n}\mathbb{1}^T v$)

$\mathbb{1}^T v = 0 \quad \mathbb{1}^T i_{ext} = 0$

$Lv = i_{ext}$: can think of

$$L : \mathbb{R}^v / \mathbb{1} \to \mathbb{R}^v / \mathbb{1}$$

as an n-1 dimensional linear transformation, which is nonsingular if the graph is connected.

AND IN FACT THE SOLUTION TO $Lv = i_{EXT}$ IS:

$$V = L^{+} i_{ext}$$

pseudoinverse of L.