Index of Notation and Definitions

CS 292F: Graph Laplacians and Spectra

Version of April 5, 2021

There is a lot of variation in terminology and notation in the field of Laplacian matrix computation and spectral graph theory. Indeed, even "Laplacian matrix" is defined differently by different authors!

This list gives the versions of notation, terminology, and definitions that we will use in CS 292F. I mostly follow the conventions of Dan Spielman's notes, though I prefer not to use greek letters for vectors. I will keep adding to this list during the quarter.

1. Unless otherwise stated, a *graph* $G = (V, E)$ is always an undirected graph whose n vertices are the integers 1 through n, with no multiple edges or loops.

2. The *degree* of a vertex is the number of edges incident on it, or equivalently (because we don't allow multiple edges or loops) the number of its neighboring vertices.

3. A graph is said to be *regular* if every vertex has the same degree.

4. A graph is said to be *connected* if, for every choice of two vertices i and j, there is a *path* of edges from i to j. The *connected components* of a graph are its maximal connected subgraphs.

5. K_n is the *complete graph*, which has n vertices and all $n(n-1)/2$ possible edges.

6. P_n is the *path graph*, which has n vertices and $n-1$ edges in a single path.

7. S_n is the *star graph*, which has n vertices, one with degree $n-1$ and $n-1$ with degree 1.

8. H_k is the *hypercube graph*, which has $n = 2^k$ vertices, all of degree k. Vertices i and j have an edge between them if i and j differ by a power of 2. Equivalently, we can identify each vertex with a subset of $\{1, \ldots, k\}$, with edges to just those subsets formed by adding or deleting one element.

9. G_e or $G_{(i,j)}$ is the graph with n vertices and only one edge $e = (i, j)$.

10. We will write a *vector* as a lower-case latin letter, possibly with a subscript, like x or w_2. We often think of an n-vector as a set of labels for the n vertices of a graph; in that case element i of vector x is written as $x(i)$, and we may write $x \in \mathbb{R}^V$ instead of $x \in \mathbb{R}^n$. In linear algebraic expressions, vectors are column vectors.

11. Two special vectors are $\mathbf{0}$, the vector of all zeros, and $\mathbf{1}$, the vector of all ones.

12. If i is a vertex then $\mathbf{1}_i$ is the *characteristic vector* of i, which is zero except for $\mathbf{1}_i(i) = 1$. Similarly if S is a set of vertices, then $\mathbf{1}_S$ is the vector that is equal to one on the elements of S and zero elsewhere.

13. If d is an n-vector, $\mathrm{Diag}(d)$ is the n-by-n diagonal matrix with the elements of d on the diagonal. If A is any n-by-n matrix, $\mathrm{diag}(A)$ is the n-vector of the diagonal elements of A.

14. The *Laplacian* of graph G is the n-by-n matrix L whose diagonal element $L(i,i)$ is the degree of vertex i, and whose off-diagonal element $L(i,j)$ is -1 if $(i,j) \in E$ and 0 if $(i,j) \notin E$. This matrix, which we (and Spielman) just call the Laplacian, is sometimes called the *combinatorial Laplacian* to distinguish it from the normalized Laplacian (to be defined later). Note that $L\mathbf{1} = \mathbf{0}$.

15. L_e or $L_{(i,j)}$ is the n-by-n Laplacian matrix of the graph with n vertices and only one edge $e = (i,j)$. This matrix has only four nonzero elements, two 1's on the diagonal and two -1's in positions (i,j) and (j,i); thus

$$L_{(i,j)} = (\mathbf{1}_i - \mathbf{1}_j)(\mathbf{1}_i - \mathbf{1}_j)^T.$$

The Laplacian of any graph $G = (V,E)$ is the sum of the Laplacians of its edges,

$$L_G = \sum_{e \in E} L_e.$$

16. The *Laplacian quadratic form* (or just LQF) is $x^T L x$, where L is a particular graph's Laplacian and x is a variable n-vector. Its value for a particular vector x is

$$x^T L x = \sum_{(i,j) \in E} (x(i) - x(j))^2.$$

17. If $Aw = \lambda w$ for any square matrix A, nonzero vector w, and scalar λ, then λ is an *eigenvalue* of A and w is an *eigenvector* associated with λ.

18. If A is square and B is nonsingular, then the eigenvalues of BAB^{-1} are the same as those of A, and the eigenvectors of BAB^{-1} are B times the eigenvectors of A.

19. Every Laplacian L is *positive semidefinite*, which (along with symmetry) implies that its n eigenvalues are nonnegative and real. Zero is an eigenvalue of L with multiplicity equal to the number of connected components of the graph G. Therefore, if G is connected, we have $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. In that case the eigenvector w_1 is the constant vector $\mathbf{1}/\sqrt{n}$.

20. The *Fiedler value* of a graph is λ_2, its second-smallest eigenvalue, and the *Fiedler vector* is w_2, the associated eigenvector. The Fiedler value of a graph is also called its *algebraic connectivity.* Note that $\lambda_2 = 0$ iff the graph is not connected.

21. A square matrix Q is *orthogonal* if $Q^T Q = I$, that is, its inverse is its transpose. As vectors, the columns of Q have unit length and are pairwise perpendicular; the same is true of the rows of Q.

22. If the n-by-n matrix A is symmetric, then it possesses n real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ (possibly including duplicates) associated with n mutually orthogonal unit-length eigenvectors $w_1, w_2, \ldots, w_n$. If W is the matrix $[w_1\ w_2\ \ldots\ w_n]$ and Λ is the matrix $\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ then we can summarize this as $AW = W\Lambda$ and $W^T W = I$. We also have $A = W\Lambda W^T$, whence

$$A = \sum_{i=1}^{n} \lambda_i w_i w_i^T.$$

23. If symmetric A and its eigenvalues and eigenvectors are as in (22), any vector x can be written as a linear combination of eigenvectors,

$$x = \sum_{i=1}^{n} \alpha_i w_i,$$

where $\alpha_i = w_i^T x$. Multiplication by A acts termwise on such a sum:

$$A^k x = \sum_{i=1}^{n} \alpha_i \lambda_i^k w_i.$$

24. If symmetric A and its eigenvalues and eigenvectors are as in (22), the *pseudoinverse* of A is

$$A^\dagger = \sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} w_i w_i^T,$$

where the sum is taken over the nonzero eigenvalues of A. If A is nonsingular, $A^\dagger = A^{-1}$. If x is orthogonal to the null space of A (i.e. $x^T w_i = 0$ whenever $\lambda_i = 0$), then

$$A^\dagger A x = A A^\dagger x = x.$$

25. The *positive semidefinite square root* of a positive semidefinite matrix A with eigenvalues and eigenvectors as in (22) is the matrix

$$A^{1/2} = \sum_{i=1}^{n} \lambda_i^{1/2} w_i w_i^T.$$

We write the psd square root of $A^\dagger$ as

$$A^{\dagger/2} = \sum_{\lambda_i \neq 0} \lambda_i^{-1/2} w_i w_i^T.$$

26. The *Rayleigh quotient* of a nonzero vector x and a matrix A is

$$\frac{x^T A x}{x^T x}.$$

If $Ax = \lambda x$, then the Rayleigh quotient of x and A is λ.

27. **Rayleigh quotient theorem**. The eigenvectors of a symmetric matrix A are critical points of its Rayleigh quotient (considered as a real-valued function of an n-vector). Specifically,

$$\lambda_k = \min_{x \perp w_1, \ldots, w_{k-1}} \frac{x^T A x}{x^T x} = \max_{x \perp w_{k+1}, \ldots, w_n} \frac{x^T A x}{x^T x},$$

and the extreme values are attained at $x = w_k$. In particular, therefore, for a Laplacian L the Fiedler value is

$$\lambda_2 = \min_{x \perp \mathbf{1}} \frac{x^T A x}{x^T x},$$

attained at the Fiedler vector w_2.

28. **Courant-Fischer theorem** (a stronger version of the Rayleigh quotient theorem). The eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of a symmetric matrix A are characterized by

$$\lambda_k = \max_{\dim \mathbb{S} = n-k+1} \min_{x \in \mathbb{S}} \frac{x^T A x}{x^T x} = \min_{\dim \mathbb{S} = k} \max_{x \in \mathbb{S}} \frac{x^T A x}{x^T x},$$

where $\mathbb{S}$ ranges over subspaces of $\mathbb{R}^n$. The extreme values are attained at $x = w_k$.