λ_2 Bounds, Conductance, Normalized Laplacian

CS 292F

Lecture 5

4-19-2021

Weighted Laplacian: graph with positive edge weights $w(e) = w(i,j)$

$L_G(i,j) = -w(i,j)$ for edge (i,j)

Diagonal of L_G: sum of weights on incident edges.

Positive semidefinite ordering:

Partial order on symmetric matrices

$A \succeq B$ means
- $A - B$ is positive semidefinite
- $x^T A x \geq x^T B x \quad \forall x$
- every $\lambda_i(A-B) \geq 0$

$G \succeq H$ means $L_G \succeq L_H$

implies

$$\lambda_i(G) \geq \lambda_i(H)$$

for all i.

<u>DEF</u> : H c-approximates G

means $cH \succeq G \succeq \frac{1}{c} H$

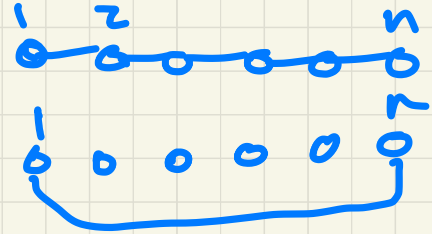
Path graph P_n : estimate λ_2

Last time $\lambda_2(P) \leq \dfrac{12}{n(n+1)}$

LEMMA:

$$(n-1)P_n \succcurlyeq G_{1,n}$$

P_n :
G_{1n} :



PROOF:

Will show $(n-1)x^T P_n x \geq x^T G_{1n} x \quad \forall x.$

$$(n-1)x^T P_n x = \sum_{i=1}^{n-1}(x(i+1)-x(i))^2$$

let $\Delta(i) = x(i+1) - x(i)$

$$= (n-1)\sum_{i=1}^{n-1}(\Delta(i))^2$$

$$= \left(\sum_{i=1}^{n-1} 1^2\right)\left(\sum_{i=1}^{n-1}(\Delta(i))^2\right)$$

$$\geq \left(\sum_{i=1}^{n-1} 1\cdot\Delta(i)\right)^2$$

$$= (x(n)-x(1))^2$$

$$= x^T G x$$

QED

Cauchy-Schwarz:

$$|x^T y| \leq \|x\|\cdot\|y\|$$

or

$$\left(\sum_i x(i)y(i)\right)^2 \leq \left(\sum_i x(i)^2\right)\left(\sum_i y(i)^2\right)$$

Lower bound on $\lambda_2(P_n)$:

compare P_n to K_n $\left(\lambda_2(K_n) = n \right)$

$$K_n = \sum_{1 \le i < j \le n} G_{ij}$$

Path of length $j-i+1$ P_{j-i+1}

$$\le \sum_{i<j} (j-i) \overbrace{\sum_{k=i}^{j-i} G_{k,k+1}}$$

$$\le \sum_{i<j} (j-i) \sum_{k=1}^{n-1} G_{k,k+1}$$

supergraph.

$$= \sum_{1 \le i < j \le n} (j-i) P_n$$

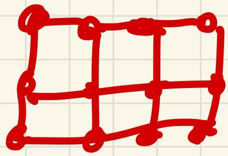
$= c P_n$ for $c = \Theta(n^3)$

actually $c = \dfrac{n(n+1)(n-1)}{6}$

Hence $\lambda_2(P_n) \ge 6 / (n+1)(n-1)$

$$\frac{6}{(n+1)(n-1)} \le \lambda_2(P_n) \le \frac{12}{n(n+1)}$$

Product of graphs

 $= P_3 \times P_4$

$G \times H$ has $n \cdot p$ vertices

n vertices (of G), p vertices (of H)

<u>THM</u> EVALS of $G \times H$ are the $n \cdot p$ <u>sums</u> of the evals of G & of H

EVECS of $G \times H$ have entries that are products of entries in G's & H's evecs.

<u>Proof</u>: Kronecker products.

Recall we showed $\lambda_2(\text{grid } k \times k) = \Theta\left(\frac{1}{n}\right)$

$$G_n = P_k \times P_k \quad (k = \sqrt{n})$$

$\lambda_2(G_n)$ = second smallest sum of λ's of P_k

$$\lambda_2(P_k) = \Theta\left(\frac{1}{k^2}\right) = \Theta\left(\frac{1}{n}\right) = \lambda_2(G_n)$$

<u>Recall</u>: isoperimetric ratio

$$\Theta(S) = \frac{|\partial S|}{|S|}$$

$|\partial S|$ ← # edges in ∂S (between S and $V-S$)

$|S|$ ← # vtxs in $S \subseteq V$

$\Theta_G = \min \Theta(S)$: S has at most $\frac{n}{2}$ vtxs.

Related λ_2 to Θ_G. $\lambda_2 \leq f(\Theta_G)$

<u>DEF</u>: for $S \subseteq V$, $d(S) = \sum_{a \in S} d(a)$ ← degree of vtx a.

<u>DEF</u>: <u>Conductance</u> of set $S \subseteq V$

is $$\phi(S) = \frac{|\partial S|}{\min(d(S), d(V-S))}$$

<u>Conductance</u> of G

is $$\phi_G = \min_{S \subset V} \phi(S)$$

Isoperimetric ratio :: $\dfrac{x^T L x}{x^T x}$

Conductance :: $\dfrac{y^T L y}{y^T D y}$

where $D = \begin{pmatrix} d(1) & & & 0 \\ & d(2) & & \\ & & \ddots & \\ 0 & & & d(n) \end{pmatrix} = \mathrm{diag}(d)$

$$y^T D y = \sum_{a=1}^{n} d(a)\big(y(a)\big)^2$$

Take $x = D^{1/2} y$ $\quad y = D^{-1/2} x$

Then $\dfrac{y^T L y}{y^T D y} = \dfrac{x^T \boxed{D^{-1/2} L D^{-1/2}} x}{x^T x}$

$D^{-1/2} D D^{-1/2} = I$

<u>DEF</u> $N = D^{-1/2} L D^{-1/2}$

is the "normalized Laplacian"

$$N = D^{-1/2} L D^{-1/2} = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

scales row i of L by $d(i)^{-1/2}$

scales col i of L by $d(i)^{-1/2}$

$$N(i,j) = \frac{L(i,j)}{\sqrt{d(i)\,d(j)}}$$

$$N(\text{edge } ij) = -1 \Big/ \sqrt{d(i)\,d(j)}$$

$$N(i,i) = 1$$

$$N(i,j) = 0 \quad \text{non-edge}$$

$$\underbrace{\begin{pmatrix} 3 & & -1 & \\ -1 & 7 & & \\ & & 1000000 & -1 \\ -1 & & 2 & \\ & & & 1 \end{pmatrix}}_{L} \longrightarrow \underbrace{\begin{pmatrix} 1 & & -1/4 & \\ & 1 & & \\ & & 1 & -\frac{1}{10^6} \\ & & & 1 \\ & & & & 1 \end{pmatrix}}_{N}$$