Index of Notation and Definitions

CS 292F: Graph Laplacians and Spectra

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There is a lot of variation in terminology and notation in the field of Laplacian matrix computation and spectral graph theory. Indeed, even "Laplacian matrix" is defined differently by different authors!

This list gives the versions of notation, terminology, and definitions that we will use in CS 292F. I mostly follow the conventions of Dan Spielman's notes, though I prefer not to use greek letters for vectors. I will keep adding to this list during the quarter.

1. Unless otherwise stated, a *graph* $G = (V, E)$ is always an undirected graph whose n vertices are the integers 1 through n, with no multiple edges or loops.

2. The *degree* of a vertex is the number of edges incident on it, or equivalently (because we don't allow multiple edges or loops) the number of its neighboring vertices.

3. A graph is said to be *regular* if every vertex has the same degree.

4. A graph is said to be *connected* if, for every choice of two vertices i and j, there is a *path* of edges from i to j. The *connected components* of a graph are its maximal connected subgraphs.

5. K_n is the *complete graph*, which has n vertices and all $n(n-1)/2$ possible edges.

6. P_n is the *path graph*, which has n vertices and $n-1$ edges in a single path.

7. S_n is the *star graph*, which has n vertices, one with degree $n-1$ and $n-1$ with degree 1.

8. H_k is the *hypercube graph*, which has $n = 2^k$ vertices, all of degree k. Vertices i and j have an edge between them if i and j differ by a power of 2. Equivalently, we can identify each vertex with a subset of $\{1, \ldots, k\}$, with edges to just those subsets formed by adding or deleting one element.

9. G_e or $G_{(i,j)}$ is the graph with n vertices and only one edge $e = (i, j)$.

10. We will write a *vector* as a lower-case latin letter, possibly with a subscript, like x or w_2. We often think of an n-vector as a set of labels for the n vertices of a graph; in that case element i of vector x is written as $x(i)$, and we may write $x \in \mathbb{R}^V$ instead of $x \in \mathbb{R}^n$. In linear algebraic expressions, vectors are column vectors.

11. Two special vectors are $\mathbf{0}$, the vector of all zeros, and $\mathbf{1}$, the vector of all ones.

12. If i is a vertex then $\mathbf{1}_i$ is the *characteristic vector* of i, which is zero except for $\mathbf{1}_i(i) = 1$. Similarly if S is a set of vertices, then $\mathbf{1}_S$ is the vector that is equal to one on the elements of S and zero elsewhere.

13. If d is an n-vector, $\mathrm{diag}(d)$ is the n-by-n diagonal matrix with the elements of d on the diagonal. If A is any n-by-n matrix, $\mathrm{diag}(A)$ is the n-vector of the diagonal elements of A.

14. The *Laplacian* of graph G is the n-by-n matrix L whose diagonal element $L(i,i)$ is the degree of vertex i, and whose off-diagonal element $L(i,j)$ is -1 if $(i,j) \in E$ and 0 if $(i,j) \notin E$. This matrix, which we (and Spielman) just call the Laplacian, is sometimes called the *combinatorial Laplacian* to distinguish it from the normalized Laplacian (to be defined later). Note that $L\mathbf{1} = \mathbf{0}$.

15. L_e or $L_{(i,j)}$ is the n-by-n Laplacian matrix of the graph with n vertices and only one edge $e = (i,j)$. This matrix has only four nonzero elements, two 1's on the diagonal and two -1's in positions (i,j) and (j,i); thus

$$L_{(i,j)} = (\mathbf{1}_i - \mathbf{1}_j)(\mathbf{1}_i - \mathbf{1}_j)^T.$$

    The Laplacian of any graph $G = (V,E)$ is the sum of the Laplacians of its edges,

$$L_G = \sum_{e \in E} L_e.$$

16. The *Laplacian quadratic form* (or just LQF) is $x^T L x$, where L is a particular graph's Laplacian and x is a variable n-vector. Its value for a particular vector x is

$$x^T L x = \sum_{(i,j) \in E} (x(i) - x(j))^2.$$