# Partitioning and Isoperimetry

CS 292F

April 8, 2021

Lecture 3

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

$$\lambda_1 = 0 \qquad w_1 = \frac{1}{\sqrt{n}} \mathbb{1}$$

$$\lambda_2 = \min_{\mathbb{1}^T x = 0} \frac{x^T L x}{x^T x} \qquad w_2 = \operatorname{argmin}(\text{same})$$

$$\lambda_3 = \min_{\substack{\mathbb{1}^T x = 0 \\ w_2^T x = 0}} \frac{x^T L x}{x^T x} \qquad w_3 = \operatorname{argmin}$$

$$\vdots \qquad\qquad \vdots$$

<u>Note</u>: $\forall x$ with $\mathbb{1}^T x = 0$,

$$\lambda_2 \le \frac{x^T L x}{x^T x}$$

# Graph Drawing

Draw in 1 D.

Want to choose $x(a)$ for $a \in V$ such that neighbors are close.

- $\sum_{(a,b) \in E} (x(a) - x(b))^2$ small.

IE make $x^T L x$ as small as possible.

But $x = 0$ is boring. So require $\|x\| = 1$

But $x = \frac{1}{\sqrt{n}} \mathbf{1}$ is boring

So require $\sum_a x(a) = 0 = \mathbf{1}^T x$

so $x$ is $\underset{\substack{\|x\|=1 \\ \mathbf{1}^T x = 0}}{\operatorname{argmin}} \; x^T L x$ Just $w_2$!

In 2D, put a at $(x(a), y(a))$

$$\min \sum_{a,b \in E} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2 = x^T L x + y^T L y$$

Subject to

$\|x\| = 1$

$\|y\| = 1$

$\mathbf{1}^T x = 0$

$\mathbf{1}^T y = 0$

$y^T x = 0$

gives $x = w_2$

$y = w_3$

See Spielman section 3.2 for details.

# GRAPH PARTITIONING

SPLIT VERTICES EQUALLY, WITH FEW EDGES BETWEEN PARTS (2 parts)

$\tau_{min}$ is NP-hard

Define a "cut vector" $x \in \mathbb{R}^V$

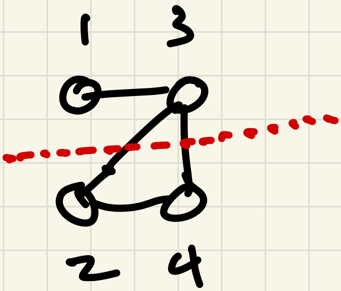
as $x(a) = \pm 1$

$G = A \cup B$

$x = 1_A - 1_B$

$$x^T L x = \sum_{(a,b)\in E} (x(a) - x(b))^2$$

$$= 4 \times (\#\text{ edges crossing cut})$$



$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$x^T \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}}_{L} x = 8$$

$$\min_x (x^T L x)$$

$x(a) \in \pm 1$

$\sum_a x(a) = 0$, i.e. $1^T x = 0$

NP-complete

$\vec{x}$ is a corner of the hypercube in n-D.

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$$\lambda_2 = \min_{\substack{x \\ \|x\|=1 \\ 1^T x = 0}} (x^T L x) \text{ and } \omega_2 = \operatorname{argmin}$$

$x$ is a point on the sphere in n-D.

EASY

FIEDLER VECTOR

# ISOPERIMETRIC RATIOS

Graph $G=(V,E)$

set $S \subseteq V$

DEF Boundary $\partial(S) = \left\{(a,b)\in E : \begin{matrix} a\in S \\ b\notin S \end{matrix}\right\}$

DEF Isoperimetric Ratio

$$\Theta(S) = \frac{|\partial(S)|}{|S|} \quad \begin{matrix}\text{\#edges} \\ \text{\#vtxs}\end{matrix}$$

DEF

$$\Theta_G = \min_{|S|\le \frac{n}{2}} \Theta(S)$$

**THM** For all $S \subseteq V$,

$$\Theta(S) \geq \lambda_2(1-\sigma), \text{ where}$$

$$\sigma = |S|/n$$

THUS $\Theta_G \geq \lambda_2/2$ "$\lambda_2$ big $\Rightarrow$ Graph well connected"

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**PROOF**: $\lambda_2 = \min_{x^T 1 = 0} x^T L x / x^T x$.

Take $x = 1_S - \sigma 1$. $\quad x^T 1 = 1_S^T 1 - \sigma 1^T 1 = |S| - \sigma n = 0$

$x^T x = n(\sigma - \sigma^2) = |S|(1-\sigma)$

$$x^T L x = \sum_{(a,b)\in E} (x(a) - x(b))^2 = |\delta S|$$

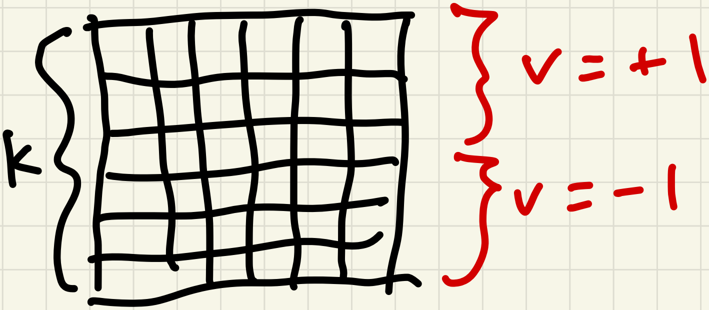
$$\lambda_2 \leq \frac{x^T L x}{x^T x} = \frac{|\delta(S)|}{|S|(1-\sigma)} = \Theta(S)\frac{1}{(1-\sigma)}$$

Thus $\lambda_2(1-\sigma) = \Theta(S)$

TEST VECTOR : $v$ with $\mathbb{1}^T v = 0$

$$\lambda_2 \le \frac{v^T L v}{v^T v}$$

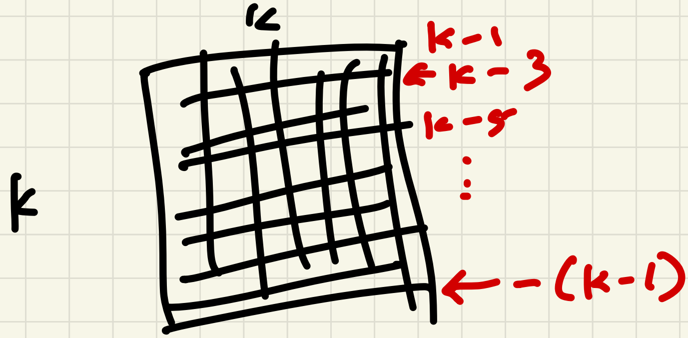
Grid graph: $n = k^2$ vertices




$v^T L v = 4k = 4\sqrt{n}$

$v^T v = n$

$$\lambda_2 \le \frac{v^T L v}{v^T v} = \frac{4}{\sqrt{n}} = O\left(\frac{1}{\sqrt{n}}\right)$$




$n = k^2$

$-(k-1) \le v(a) \le k-1$ for all $a \in V$

$1^T v = 0$

$$v^T v = \left((k-1)^2 + (k-3)^2 + \cdots + (-(k-1))^2\right) \cdot k$$
$$= \Theta(k^4) = \Theta(n^2)$$

$$v^T L v = \sum_{a,b \in E} (v(a) - v(b))^2$$
$$= 2^2 \cdot \Theta(n) = \Theta(n)$$

$$\lambda_2 \le \frac{v^T L v}{v^T v} = \frac{\Theta(n)}{\Theta(n^2)} = \Theta\left(\frac{1}{n}\right)$$

$\lambda_2(\text{planar}) = \Theta\left(\frac{d}{n}\right)$ degree $d$

TIGHT.

BIG THEOREM: $\lambda_2 = O\left(\frac{1}{n}\right)$ for every planar graph of bounded degree.