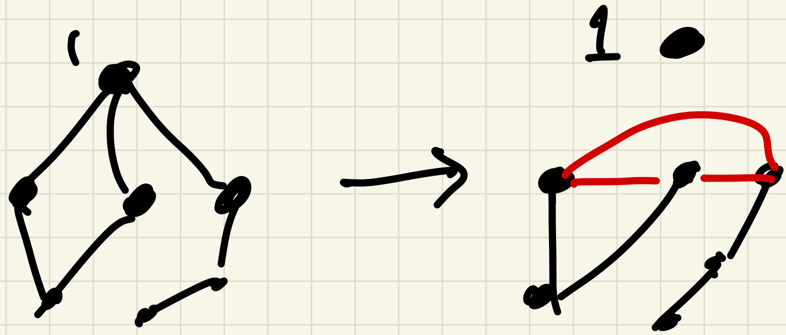
Cholesky Factorization & Intro to Conjugate Gradient

CS 292F

May 10, 2021

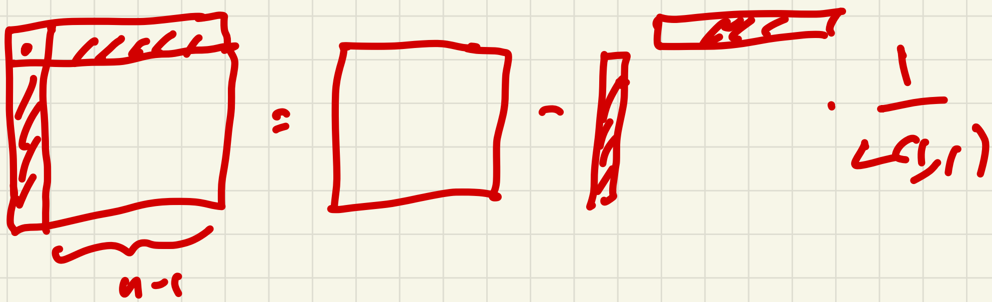
Lecture 11

L:



: L_B

$$L_B = L(2:n,2:n) - \frac{1}{L(1,1)} L(2:n,1)\cdot L(1,2:n)$$



L_B is Laplacian

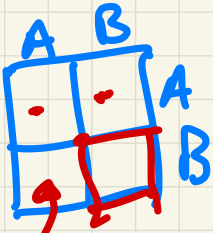
If $i(1)=0$ then $Lv=i \Rightarrow L_B v_B = i_B$

where $v_B = v(2:n)$

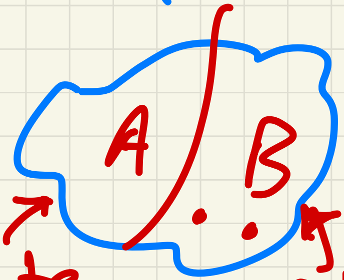
$i_B = i(2:n)$

L_B is an equivalent network to L on vertices $2:n$.

Let $B \subseteq V$ be any set of "boundary" vertices, and $A = V - B$. "interior"

$L =$ 

$L(B,A)$

$G =$ 

eliminate vertices in A (in any order)

fill in some new stuff.

Get:

$$L_B = L(B,B) - L(B,A)\,L(A,A)^{-1}L(A,B)$$

L_B is equivalent to L on the vertices of B.

Note: If G is connected and $|A| \le n-1$, then $L(A,A)$ is nonsingular.

<u>DEF</u>:

L_B is called the <u>Schur complement</u> of A in L. (or "on B")

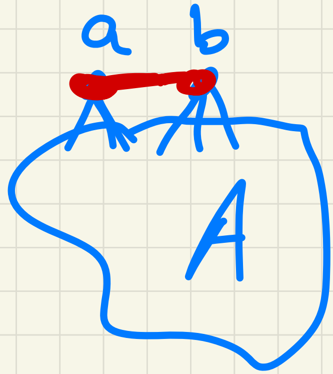
Theorem: If $Lv = i$

and $i(A) = 0$ then $L_B v(B) = i(B)$.

The effective resistance in B is the same:

$$R_L^{eff}(a,b) = R_{L_B}^{eff}(a,b) \text{ for all } a,b \in B.$$

Notice that if $B = \{a, b\}$



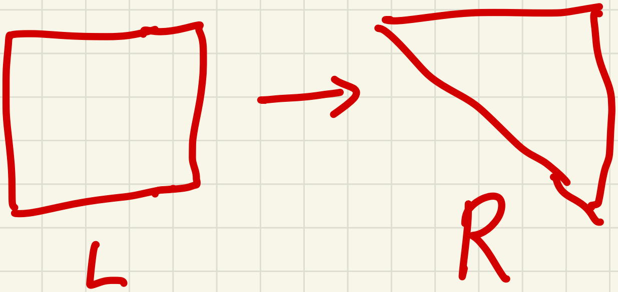
then L_B is 2x2

$$\begin{array}{c|cc} & a & b \\ \hline a & & \bullet \\ b & & \end{array}$$

and $L_B(1,2)$ is

$$\frac{-1}{R_{eff}(a,b)}.$$

Cholesky factorization to solve $Lv = i$ for v.



$$Lv = i \longrightarrow Rv = \hat{i}$$

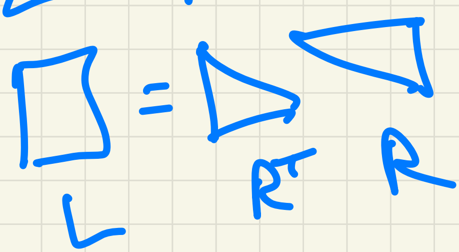
apply the GE steps to $\bar{i}$.

Then can find v by backsubstitution.
(find $v(n)$, then $v(n-1)$, ..., $v(1)$)

This gives a way to solve symmetric positive semidefinite linear systems.

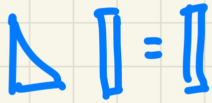
Can also use GE to <u>factor</u>

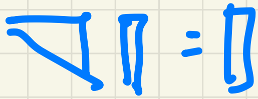
$$L = R^T R \quad \text{for upper triangular } R.$$



"Cholesky factorization"

If $L = R^T R$, to solve $Lv = i$

1. solve $R^T y = i$ 

2. solve $Rv = y$ 

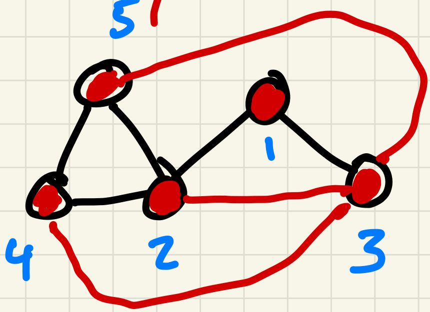
then $R^T R v = R^T y = i$.

What's the nonzero structure of R?

(the graph of $R^T + R$)

?

Cholesky graph game

$G =$ 

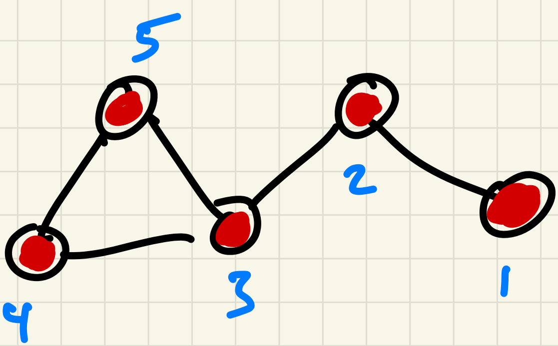
matrix: L_G

For each vertex in turn:

① mark the vertex "eliminated"

② add edges between all its unmarked neighbours

Added edges are called "fill" and the result is the graph of $R+R^T$.

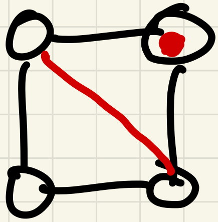


Play the game in a different order $\Rightarrow$ different fill.

matrix: $P^T L_G P$ for a permutation matrix P.

Problem: Given a graph, what order gives the least fill?

(nonzero in general)



This is NP-hard.

But it's very relevant to solving big sparse linear systems.

① (Parter): If G is a tree, there is an ordering with no added fill.

Proof: there must be a vertex of degree 1. eliminate it first.

② The $\sqrt{n}$-by-$\sqrt{n}$ grid graph with n vertices has an ordering with $O(n \log n)$ fill (George, "Nested dissection").

③ Every n-vertex planar graph has an ordering with $O(n \log n)$ fill. (Lipton/Tarjan) ... LONG STORY.

Solving $Ax=b$ (symmetric A
today: positive **definite** $A \succ 0$)

GE is a "direct method"

Iterative methods for $Ax=b$.

Find $x_0, x_1, x_2, \dots$ converge to x.

CONJUGATE GRADIENT.

(closely related to Lanczos)

Krylov subspace: $\mathrm{span}(b, Ab, A^2b, \dots, A^tb)$

$= K_{t+1}(b)$

Is x in $K_{t+1}(b)$ for some t?

An element of $K_{t+1}(b)$ looks like

$$\beta_0 b + \beta_1 Ab + \beta_2 A^2 b + \dots + \beta_t A^t b.$$

(some scalars β_k)

$\beta_0 b + \cdots + \beta_t A^t b = p(A)\, b$ $\quad$ $\boxed{Ax = b}$

where $p(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_t z^t$.

Want $p(A)\, b = x$.

Want $p(A) A x = x$.

Want $(I - p(A)A)\, x = 0$.

So let $q(z) = (1 - z\,p(z))$

Suppose $q(z) = \sum_{k=1}^{t+1} \gamma_k z^k$

Want: $q(0) = 1$ and $q(A)\, x = 0$.

$$q(z) = \sum_{k=1}^{t+1} \gamma_k z^k \qquad \text{want } q(A)x = 0. \qquad Ax = b$$

Suppose $AW = \Lambda W$, $Aw_i = \lambda_i w_i$

$W^T W = I$ for $i = 1, 2, \ldots, n$.

Write x in evec basis as

$$x = \sum_{i=1}^{n} \alpha_i w_i \quad (\text{where } \alpha_i = w_i^T x).$$

$$q(A)x = \sum_{k=1}^{t+1} \gamma_k A^k x = \sum_{k=1}^{t+1} \gamma_k A^k \sum_{i=1}^{n} \alpha_i w_i$$

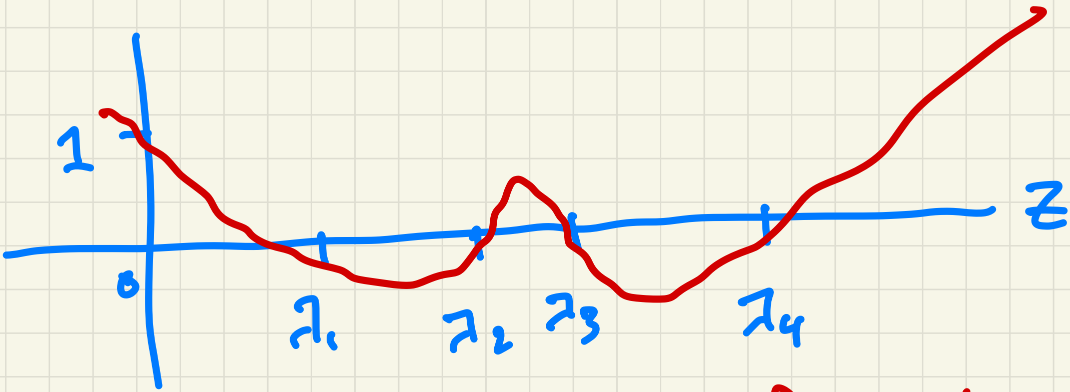
$$= \sum_{k=1}^{t+1} \gamma_k \sum_{i=1}^{n} \alpha_i \lambda_i^k w_i$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{t+1} \gamma_k \lambda_i^k \right) \alpha_i w_i = \sum_{i=1}^{n} q(\lambda_i) \alpha_i w_i$$

so if $q(0) = 1$ and $q(\lambda_i) = 0$ for all i,

then $q(A)x = 0$.

So $q(z)$ needs to go through the $n+1$ points $(0,1)$, $(\lambda_1, 0)$ $(\lambda_2\, 0) \cdots (\lambda_n 0)$ which is possible if degree is n.



The q I want exists if $t \geq n$.

$\Rightarrow$ p exists in K_{t+1}.

$$q(z) = 1 - zp(z).$$

How do we get that q?

Next time...