Homework 1 - Implementations of Quantum Computing (QIC 750)

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1. Prove the following identities

a) [A,B] = - [B,A]
b) [A+B,C] = [A,C] + [B,C]
c) [A,BC] = [A,B]C + B[A,C]
d) [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0

1a) [A,B] = AB-BA (by definition of the bracket)
= -(BA-AB) (by distribution of scalars over multiplication)
= -[B,A] (by definition of the bracket)

1b) [A+B,C] = (A+B)C - C(A+B)
= AC + BC - CA - CB (by distribution of vector multiplication)
= (AC - CA) + (BC - CB) (rearranging)
= [A,C] + [B,C] (by definition of the bracket)

1c) [A,BC] = ABC-BCA
= ABC (- BAC +BAC) - BCA (by using the additive zero of the vector space)
= (ABC - BAC) + (BAC - BCA) (by associativity of addition)
= [A,B]C + B[A,C] (by definition of the bracket)

1d) (this problem walks the reader through a proof of the oft-cited Jacobi Identity useful in $\mathbb{R}^3$ vector analysis (cross-product)

$$\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\
&= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC \\
&= (ABC - ABC) + (ACB - ACB) + (CBA - CBA) + (BAC - BAC) + (CAB - CAB) \\
&= 0
\end{aligned}$$

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2. Coherent States

A coherent state of a 1D simple harmonic oscillator is defined to be an eigenstate of the annihilation operator $\hat{a}$: $\hat{a}\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle$ where λ is, in general, complex.

a) What is the average number of excitations in a state $\left|\lambda\right\rangle$ (where, according to the above convention $\left|\lambda\right\rangle$ is an eigenstate of the annihilation operator)?

b) Using the Heisenberg representation of this operator, $\hat{x}(t)$, calculate the expectation value of this operator for state $\left|\lambda\right\rangle$.

c) Show that the state $\left|\lambda\right\rangle = e^{\frac{-|\lambda|^2}{2}} e^{\lambda a^\dagger}\left|0\right\rangle$ is a normalized coherent state.

d) Express $\left|\lambda\right\rangle$ as $\sum_{n=0}^{\infty} f(n)\left|n\right\rangle$. Show that the probablitiy distribution $|f(n)|^2$ is Poisson-distributed in n.

2a) $\left\langle\lambda\right|\hat{N}\left|\lambda\right\rangle = \left|\lambda\right\rangle a^\dagger a\left|\lambda\right\rangle = \left\langle\lambda\right|\lambda^* \lambda\left|\lambda\right\rangle = |\lambda|^2$

2b) The actual problem statement directs me to my notes for the expression of $\hat{x}(t)$. This makes it rather unfortunate for me to have notes. However, I am going to choose to see this as an opportunity to reinforce elementary concepts.

The Heisenberg representation of an operator is chosen in such a way as to transfer the time-dependence from the state ket to the operator. However, in doing so we must be careful to make sure that all expectation values are the same at the end (so as to keep the predictions the same). In order to keep the expectation values the same, I am led down the path of using unitary matrices to transform the operator. Note that if I transform a state ket by a unitary U that the inner product is maintained :
$\langle\beta|\beta\rangle = \langle\psi|\, U^\dagger\, U\,|\psi\rangle = \langle\psi|\psi\rangle$ since $U^\dagger\, U = \mathbb{1}$.

Thus, consider a transformation that acts on the time-dependent state $|\psi(\mathrm{t})\rangle$ such that $U\,|\psi(t)\rangle = |\psi(t_0)\rangle$ where t_0 is a particular time during the evolution of $|\psi\rangle$. Then, the effect of some operator $\hat{A}$ on the state $|\psi\,(\mathrm{t})\rangle$ can be expressed as $\hat{A}\,|\psi\,(t)\rangle = \hat{A}\,U\,|\psi(t_0)\rangle$ and the expectation of $\hat{A}$ is given by $\langle\psi(t_0)\,|\,U^\dagger\,\mathrm{AU}\,|\psi(t_0)\,\rangle$. Now, since all the time dependence is stored in between the kets, we can call $U^\dagger\,\mathrm{AU} = A(t)$. This is the Heiseberg representation of the operator A.

I just looked at the lecture notes after realizing I had no idea what $x(t)$ is. It turns out x(t) is a well-defined operator : $x(\mathrm{t}) = \mathrm{k}\left(a(0)\,e^{-\mathrm{i}\omega\mathrm{t}} + a^\dagger(0)\,e^{\mathrm{i}\omega\mathrm{t}}\right)$, where k is a normalization constant with appropriate units. I just need to calculate the expectation value of this operator.

$$\langle\lambda|\,x(t)\,|\lambda\rangle = \langle\lambda|\,k\left(a(0)\,e^{-\mathrm{i}\omega\mathrm{t}} + a^\dagger(0)\,e^{\mathrm{i}\omega\mathrm{t}}\right)|\lambda\rangle$$
$$= \langle\lambda|\,k * a(0)\,e^{-\mathrm{i}\omega\mathrm{t}}\,|\lambda\rangle + \langle\lambda|\,k * a^\dagger(0)\,e^{\mathrm{i}\omega\mathrm{t}}\,|\lambda\rangle$$
$$= \mathrm{k}e^{-\mathrm{i}\omega\mathrm{t}}\,\lambda + \mathrm{k}e^{\mathrm{i}\omega\mathrm{t}}\,\lambda^*$$
$$= k\,|\,\lambda\,|\left(e^{-\mathrm{i}\omega\mathrm{t}}\,e^{\mathrm{i}\phi} + e^{\mathrm{i}\omega\mathrm{t}}\,e^{-\mathrm{i}\phi}\right)$$
$$= 2\,k\,|\,\lambda\,|\cos(\omega\mathrm{t} - \phi)$$

Now, k is just a constant of proportionality : $\mathrm{k} = \sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}}$. The final result can be expressed as $\langle\lambda|\,x(t)\,|\lambda\rangle = \sqrt{\frac{2\,\hbar}{\mathrm{m}\omega}}\,|\,\lambda\,|\cos(\omega\mathrm{t} - \phi)$.

2 c) To show that the state is normalized all we have to do is take the inner product with itself and show that that number is 1. So :

$\langle\lambda|\lambda\rangle = \langle 0| e^{\frac{-|\lambda|^2}{2}} e^{\lambda^* a} e^{\frac{-|\lambda|^2}{2}} e^{\lambda a^\dagger} |0\rangle$
$= e^{-|\lambda|^2} \langle 0| e^{\lambda^* a} e^{\lambda a^\dagger} |0\rangle$

Now, $[\lambda^* a, \lambda a^\dagger] = |\lambda|^2$
So, $[\lambda^* a, [\lambda^* a, \lambda a^\dagger]] = 0$ and $[\lambda a^\dagger, [\lambda^* a, \lambda a^\dagger]] = 0$

According to the Baker-Campbell-Hausdorff Formula:

$e^{A+B} = e^{\frac{-[A,B]}{2}} e^A e^B = e^{\frac{[A,B]}{2}} e^B e^A$ whenever [A,[A,B]]=[B,[A,B]]=0
Thus, $e^{-|\lambda|^2} e^{\lambda^* a} e^{\lambda a^\dagger} = e^{\lambda a^\dagger} e^{\lambda^* a}$
So, $\langle\lambda|\lambda\rangle = e^{\frac{-|\lambda|^2}{2}} \langle 0| e^{\lambda a^\dagger} e^{\lambda^* a} |0\rangle$
$= \langle 0|(1 + \lambda a^\dagger + \frac{(\lambda a^\dagger)^2}{2} + \frac{(\lambda a^\dagger)^3}{3!} + \ldots + \frac{(\lambda a^\dagger)^n}{n!})\ (1 + \lambda^* a + \frac{(\lambda^* a)^2}{2} + \frac{(\lambda^* a)^3}{3!} + \ldots + \frac{(\lambda^* a)^n}{n!})|0\rangle$

The effect of successively higher terms of a is to annihilate the $|0\rangle$ state ket;
similarly, the effect of successively higher terms of $a^\dagger$ is
to annihilate the $\langle 0|$ state bra (i.e. $a|0\rangle = 0|0\rangle = 0 = \langle 0| 0 = \langle 0| a^\dagger$). Thus,
the only terms that survive are the constant terms (the "1's" in the expansion).

$= \langle 0| (1 + 0 + 0 + 0 + \ldots + 0)(1 + 0 + 0 + 0 + \ldots + 0) |0\rangle = 1$

To show that the state is a coherent state it is necessary to consider the action of operator $\hat{a}$ on that state. The given form of $|\lambda\rangle$ can be expressed in an alternative form as an infinite superposition of Fock (number) states.

$|\lambda\rangle = e^{\frac{-|\lambda|^2}{2}} e^{\lambda a^\dagger} |0\rangle = e^{\frac{-|\lambda|^2}{2}} e^{\lambda a^\dagger} |0\rangle = e^{\frac{-|\lambda|^2}{2}} \left(1 + \lambda a^\dagger + \frac{(\lambda a^\dagger)^2}{2!} + \frac{(\lambda a^\dagger)^3}{3!} + \ldots + \frac{(\lambda a^\dagger)^n}{n!}\right) |0\rangle$
$= e^{\frac{-|\lambda|^2}{2}} \left(1 + \lambda a^\dagger + \frac{\lambda^2}{2!} (a^\dagger)^2 + \ldots + \frac{\lambda^n}{n!} (a^\dagger)^n\right) |0\rangle$
$= e^{\frac{-|\lambda|^2}{2}} \left(|0\rangle + \lambda |1\rangle + \frac{\lambda^2}{2!} \sqrt{2!}\ |2\rangle + \frac{\lambda^3}{3!} \sqrt{3!}\ |3\rangle + \ldots + \frac{\lambda^n}{n!} \sqrt{n!}\ |n\rangle\right)$
$= e^{\frac{-|\lambda|^2}{2}} \sum^{\infty}{}_{i=0} \frac{\lambda^i}{\sqrt{i!}} |i\rangle$

The above used the property that $(a^\dagger)^i |0\rangle = \sqrt{i!}\ |i\rangle$. Now, considering the effect of $\hat{a}$ on this state. Note that $a|n\rangle = \sqrt{n}\ |n-1\rangle$

$$a\left|\lambda\right\rangle = e^{\frac{-|\lambda|^2}{2}}\, a \sum^{\infty}{}_{i=0} \frac{\lambda^i}{\sqrt{i!}} \left|i\right\rangle$$

$$= e^{\frac{-|\lambda|^2}{2}} \sum^{\infty}{}_{i=1} \frac{\lambda^i \sqrt{i}}{\sqrt{i!}} \left|i-1\right\rangle$$

$$= e^{\frac{-|\lambda|^2}{2}} \sum^{\infty}{}_{i=0} \frac{\lambda^{i+1}}{\sqrt{i!}} \left|i\right\rangle$$

$$= \lambda\left(e^{\frac{-|\lambda|^2}{2}} \sum^{\infty}{}_{i=0} \frac{\lambda^i}{\sqrt{i!}} \left|i\right\rangle\right)$$

$$= \lambda \left|\lambda\right\rangle$$

Where the 2nd to 3rd line, above, utilized a shift in index and the 3rd to 4th line, above, utilized the identity $\lambda^{j+1} = \lambda\lambda^j$.

Note that the above expression changed in the index (the 1st state was dropped to the 0th state) and the factor of $\sqrt{i}$ hops out since $a\left|n\right\rangle = \sqrt{n}\,\left|n-1\right\rangle$

$$= e^{\frac{-|\lambda|^2}{2}}\left(\sum^{n}{}_{i=0} \frac{\lambda^i}{\sqrt{(i-1)!}} \left|i\right\rangle\right)$$

2d) The goal is to form the state $\left|\lambda\right\rangle = e^{\frac{-|\lambda|^2}{2}} e^{\lambda a^{\dagger}} \left|0\right\rangle$ as a sum of Fock (number) states. I know how the annihilation operator acts on number states. I also know how the annihilation operator acts on number states. Consider the action of the annihilation operator (1.x) on $|\lambda\rangle$ and, separately, on a sum of Fock states (2.x).

1.a $\quad a\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle$

Separately, presupposing the knowledge that $a\left|n\right\rangle = \sqrt{n}\,\left|n-1\right\rangle$

2.a $\quad a\left|\lambda\right\rangle = a\sum^{\infty}{}_{n=0} f(n)\left|n\right\rangle = \sum^{\infty}{}_{n=1} f(n)\sqrt{n}\,\left|n-1\right\rangle$

Next, consider the result of an inner product with a Fock state on the two above states.

1.b $\quad \left\langle 0\right| a\left|\lambda\right\rangle = \left\langle 0\right|\lambda\left|\lambda\right\rangle = \lambda\left\langle 0|\lambda\right\rangle = \lambda\left\langle 0\right|\left(\sum^{\infty}{}_{n=0} f(n)\left|n\right\rangle\right) = \lambda f(0)$ (by orthonormality of the Fock states)

2.b $\quad \left\langle 0\right| a\left|\lambda\right\rangle = \left\langle 0\right| a\sum^{\infty}{}_{n=0} f(n)\left|n\right\rangle = \left\langle 0\right|\sum^{\infty}{}_{n=1} f(n)\sqrt{n}\,\left|n-1\right\rangle = f(1)*1*\left\langle 0|0\right\rangle = f(1)$ (by orthonormality of the Fock states)

So $\lambda f(0) = f(1)$

Consider this same procedure using Fock state $\left|1\right\rangle$.

1.c $\quad \left\langle 1\right| a\left|\lambda\right\rangle = \left\langle 1\right|\lambda\left|\lambda\right\rangle = \lambda\left\langle 1|\lambda\right\rangle = \lambda\left\langle 1\right|\left(\sum^{\infty}{}_{n=0} f(n)\left|n\right\rangle\right) = \lambda f(1) = \lambda^2 f(0)$ (by orthonormality of the Fock states)

2.c $\quad \left\langle 1\right| a\left|\lambda\right\rangle = \left\langle 1\right| a\sum^{\infty}{}_{n=0} f(n)\left|n\right\rangle = \left\langle 1\right|\sum^{\infty}{}_{n=1} f(n)\sqrt{n}\,\left|n-1\right\rangle = f(2)*\sqrt{2}*\left\langle 1|1\right\rangle = \sqrt{2}\,f(2)$ (by orthonormality of the Fock states)

So $f(2) = \frac{\lambda^2}{\sqrt{2}} f(0)$

Consider this same procedure using Fock state $|2\rangle$.

1.d $\quad \langle 2| a |\lambda\rangle = \langle 2| \lambda |\lambda\rangle = \lambda \langle 2|\lambda\rangle = \lambda \langle 2|\left(\sum^{\infty}{}_{n=0} f(n) |n\rangle\right) = \lambda \mathrm{f}(2) = \frac{\lambda^3}{\sqrt{2}} f(0)$ (by orthonormality of the Fock states)

2.d $\quad \langle 2| a |\lambda\rangle = \langle 2| a \sum^{\infty}{}_{n=0} f(n) |n\rangle = \langle 2| \sum^{\infty}{}_{n=1} f(n) \sqrt{n}\, |n-1\rangle = f(3) * \sqrt{3} * \langle 2|2\rangle = \sqrt{3}\, f(3)$ (by orthonormality of the Fock states)

So $f(3) = \frac{\lambda^3}{\sqrt{3!}} f(0)$

It seems to be the case that $f(n) = \frac{\lambda^n}{\sqrt{n!}} f(0)$ (see table below)

n	$\langle \mathrm{n}| \mathrm{a} |\lambda\rangle$	$\langle \mathrm{n}| \mathrm{a} \sum^{\infty}{}_{\mathrm{n}=0} \mathrm{f}(\mathrm{n}) |\mathrm{n}\rangle$
0	$\lambda \mathrm{f}(0)$	$\mathrm{f}(1)$
1	$\lambda \mathrm{f}(1)$	$\sqrt{2}\ \mathrm{f}(2)$
2	$\lambda \mathrm{f}(2)$	$\sqrt{3}\ \mathrm{f}(3)$
3	$\lambda \mathrm{f}(3)$	$\sqrt{4}\ \mathrm{f}(4)$
4	$\lambda \mathrm{f}(4)$	$\sqrt{5}\ \mathrm{f}(5)$
...	...	...
n	$\lambda \mathrm{f}(\mathrm{n})$	$\sqrt{\mathrm{n}+1}\ \mathrm{f}(\mathrm{n}+1)$

Thus $|\lambda\rangle = \sum^{\infty}{}_{n=0} \frac{\lambda^n}{\sqrt{n!}} f(0) |n\rangle = f(0) \sum^{\infty}{}_{n=0} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$

Note that f(0) is $e^{-|\lambda|}$: $\langle\lambda|\lambda\rangle = 1 = f^*(0)\left(\sum^{\infty}{}_{i=0} \frac{(\lambda^*)^i}{\sqrt{i!}} \langle i|\right) f(0) \left(\sum^{\infty}{}_{j=0} \frac{\lambda^j}{\sqrt{j!}} |j\rangle\right)$

$$= |f(0)|^2 \left(\sum^{\infty}{}_{k=0} \frac{(\lambda^*)^k}{\sqrt{k!}} \langle k| \frac{\lambda^k}{\sqrt{k!}} |k\rangle\right)$$

$$= |f(0)|^2 \sum^{\infty}{}_{k=0} \frac{|\lambda|^{2k}}{k!} = |f(0)|^2 e^{|\lambda|^2}$$

So $e^{\frac{-|\lambda|^2}{2}} = |f(0)|^2$. I am free to pick the phase of f(0) since it only contributes a global phase to $|\lambda\rangle$. I will choose f(0) to be real.

$|\lambda\rangle = \sum^{\infty}{}_{n=0} e^{\frac{-|\lambda|^2}{2}} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$(see the last half of 2c)

Thus, $| f(n) |^2 = e^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!}$ which (according to Wikipedia and my notes from an undergraduate course) is the Poisson distribution of the parameter $|\lambda|^2$ in n.

3. Pauli Operators

The Pauli operators can be defined in an orthonormal basis as:
$\sigma_x = |+\rangle\langle-|+|-\rangle\langle+|$, $\sigma_y = -i(|+\rangle\langle-|-|-\rangle\langle+|)$, $\sigma_z = |+\rangle\langle+|-|-\rangle\langle-|$. Using these definitions, show that $[\sigma_i , \sigma_j] = 2\,\mathrm{i}\epsilon_{\mathrm{ijk}}\,\sigma_k$ and that $\{\sigma_i, \sigma_j\} = 2\,\delta_{\mathrm{ij}}$.

3a) Commutator Relationships

Consider, $[\sigma_x, \sigma_y] = \sigma_x\,\sigma_y - \sigma_y\,\sigma_x$
$= (|+\rangle\langle-|+|-\rangle\langle+|)(-i(|+\rangle\langle-|-|-\rangle\langle+|))$
$+i(|+\rangle\langle-|-|-\rangle\langle+|)(|+\rangle\langle-|+|-\rangle\langle+|)$
$= i(|+\rangle\langle+|-|-\rangle\langle-|) + i(|+\rangle\langle+|-|-\rangle\langle-|)$
$= 2\,i(|+\rangle\langle+|-|-\rangle\langle-|) = 2\,\mathrm{i}\sigma_z$

As *a* consequence of the work done in Problem 1 it is clear that $[\sigma_y, \sigma_x] = -2\,\mathrm{i}\sigma_z$

$[\sigma_x, \sigma_z] = \sigma_x\,\sigma_z - \sigma_z\,\sigma_x$
$= (|+\rangle\langle-|+|-\rangle\langle+|)(|+\rangle\langle+|-|-\rangle\langle-|)$
$-(|+\rangle\langle+|-|-\rangle\langle-|)(|+\rangle\langle-|+|-\rangle\langle+|)$
$= (-|+\rangle\langle-|+|-\rangle\langle+|) - (|+\rangle\langle-|-|-\rangle\langle+|)$
$= 2(|-\rangle\langle+|-|+\rangle\langle-|) = -2\,\mathrm{i}\sigma_y$

As *a* consequence of the work done in Problem 1 it is clear that $[\sigma_z, \sigma_x] = 2\,\mathrm{i}\sigma_y$

$[\sigma_y, \sigma_z] = \sigma_y\,\sigma_z - \sigma_z\,\sigma_y$
$= -i(|+\rangle\langle-|-|-\rangle\langle+|)(|+\rangle\langle+|-|-\rangle\langle-|)$
$+i(|+\rangle\langle+|-|-\rangle\langle-|)(|+\rangle\langle-|-|-\rangle\langle+|)$
$= i(|+\rangle\langle-|+|-\rangle\langle+|) + i(|+\rangle\langle-|+|-\rangle\langle+|)$
$= 2\,i(|+\rangle\langle-|+|-\rangle\langle+|) = 2\,\mathrm{i}\sigma_x$

As *a* consequence of the work done in Problem 1 it is clear that $[\sigma_z, \sigma_y] = -2\,\mathrm{i}\sigma_x$

Thus, by construction, it has been shown that $[\sigma_i , \sigma_j] = 2\,\mathrm{i}\epsilon_{\mathrm{ijk}}\,\sigma_k$

3b) Anticommutator Relationships

Similarly, $\{\sigma_x, \sigma_y\} = \sigma_x\,\sigma_y + \sigma_y\,\sigma_x$
$= -i(|+\rangle\langle-|+|-\rangle\langle+|)(|+\rangle\langle-|-|-\rangle\langle+|)$
$-i(|+\rangle\langle-|-|-\rangle\langle+|)(|+\rangle\langle-|+|-\rangle\langle+|)$
$= -i(-|+\rangle\langle+|+|-\rangle\langle-|) - i(|+\rangle\langle+|-|-\rangle\langle-|)$

$= 0$

$\{\sigma_x, \sigma_z\} = \sigma_x \sigma_z + \sigma_z \sigma_x$
$= (|+\rangle\langle-| + |-\rangle\langle+|)(|+\rangle\langle+| - |-\rangle\langle-|)$
$+(|+\rangle\langle+| - |-\rangle\langle-|)(|+\rangle\langle-| + |-\rangle\langle+|)$
$= (-|+\rangle\langle-| + |-\rangle\langle+|) + (|+\rangle\langle-| - |-\rangle\langle+|)$
$= 0$

$\{\sigma_y, \sigma_z\} = \sigma_y \sigma_z + \sigma_z \sigma_y$
$= -i(|+\rangle\langle-| - |-\rangle\langle+|)(|+\rangle\langle+| - |-\rangle\langle-|)$
$-i(|+\rangle\langle+| - |-\rangle\langle-|)(|+\rangle\langle-| - |-\rangle\langle+|)$
$= i(|+\rangle\langle-| + |-\rangle\langle+|) - i(|+\rangle\langle-| + |-\rangle\langle+|)$
$= 0$

$\{\sigma_x, \sigma_x\} = 2\,\sigma_x{}^2 = 2\,(|+\rangle\langle-| + |-\rangle\langle+|)(|+\rangle\langle-| + |-\rangle\langle+|)$
$= 2\,(|+\rangle\langle+| + |-\rangle\langle-|) = 2\,\mathbb{1}$

$\{\sigma_y, \sigma_y\} = 2\,\sigma_y{}^2 = -2\,(|+\rangle\langle-| - |-\rangle\langle+|)(|+\rangle\langle-| - |-\rangle\langle+|)$
$= -2\,(-|+\rangle\langle+| - |-\rangle\langle-|) = 2\,\mathbb{1}$

$\{\sigma_z, \sigma_z\} = 2\,\sigma_z{}^2 = 2\,(|+\rangle\langle+| - |-\rangle\langle-|)(|+\rangle\langle+| - |-\rangle\langle-|)$
$= 2\,(|+\rangle\langle+| + |-\rangle\langle-|) = 2\,\mathbb{1}$

Thus, it has been shown by construction that $\{\sigma_i\,,\,\sigma_j\} = 2\,\delta_{ij}\,\mathbb{1}$

4 An example of degeneracy

Consider the two observables A nad B represented in a certain basis by the matrices A $\doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$, B $\doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$ with both a and b being real numbers.

a) A exhibits a degenerate spectrum. Does B exhibit a degenerate spectrum?
b) Show that A and B commute.
c) i) Find a set of basis vectors that are eigenstates of both A and B.
ii) Specify the eigenvalues corresponding to these eigenstates.
iii) Are the eigenvalues, alone, enough to characterize the eigenstate (i.e. is there degeneracy in the basis states)?

4a) To determine if B has a degenerate spectrum it is enough to determine the eigenvalues of B. I am interested then, in the case where det(B - $\mathbb{i}\lambda$) =0

$$\begin{vmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -\text{ib} \\ 0 & \text{ib} & -\lambda \end{vmatrix} = 0$$

$$(b-\lambda)\left(\lambda^2 - b^2\right) = 0$$

$$(b-\lambda)(\lambda - b)(\lambda + b) = 0$$

Thus, $\lambda_1 = \lambda_2 = b$ and $\lambda_3 = -b$ so that there is degeneracy in the observable B.
This has been confirmed by using *Mathematica* (below).

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Eigensystem[{{b, 0, 0}, {0, 0, -I*b}, {0, I*b, 0}}]
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{{-b, b, b}, {{0, i, 1}, {0, -i, 1}, {1, 0, 0}}}
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4b) To show that A and B commute it is enough to compute the product AB and subtract the product BA and show that this is zero.

$$[A, B] = \text{AB} - \text{BA}$$

$$= \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\text{ib} \\ 0 & \text{ib} & 0 \end{pmatrix} - \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -\text{ib} \\ 0 & \text{ib} & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

$$= \begin{pmatrix} a\,b & 0 & 0 \\ 0 & 0 & a\,\text{ib} \\ 0 & -a\,\text{ib} & 0 \end{pmatrix} - \begin{pmatrix} a\,b & 0 & 0 \\ 0 & 0 & a\,\text{ib} \\ 0 & -a\,\text{ib} & 0 \end{pmatrix}$$

$$= 0$$

4c) i) To determine the simultaneous set of eigenstates of A and B it is necessary to first find the eigenstates of A and the eigenstates of B.

$$\textbf{Eigensystem}\left[\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\right]$$

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{{-a, -a, a}, {{0, 0, 1}, {0, 1, 0}, {1, 0, 0}}}
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$$\textbf{Eigensystem}\left[\begin{pmatrix} \text{b} & 0 & 0 \\ 0 & 0 & -\text{I}*\text{b} \\ 0 & \text{I}*\text{b} & 0 \end{pmatrix}\right]$$

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{{-b, b, b}, {{0, i, 1}, {0, -i, 1}, {1, 0, 0}}}
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Since the set of eigenvectors for the two operators are not the same then we have to form linear combinations of one of the sets of eigenvectors. We can choose to do this in such a way that the eigenvectors of one operator are transformed into that of the other operator.

So, working with the set of eigenvectors for operator B = $\left\{\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ and labeling the eigenvectors as follows $b_1 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$, $b_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. As long as the eigenvalues of two eigenvectors are the same then I am free to take linear combinations of the corresponding eigenvectors and the result will be an eigenvector of the corresponding operator. Consider $a_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. So,

$i a_i + a_2 = b_1$ and $-i a_1 + a_2 = b_2$. Consider the act of A on $a_1 + i a_2 = b_1$.

$\mathrm{A} b_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ -a \end{pmatrix} + \begin{pmatrix} 0 \\ -i a \\ 0 \end{pmatrix} = -a \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = -a b_1$. Thus, a new eigenstate of A has been generated with eigenvalue $-a$.

Similarly, $a_1 - i a_2 = b_2$ can be shown to be an eigenvalue of A.

$$\mathrm{A} b_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ -a \end{pmatrix} + \begin{pmatrix} 0 \\ i a \\ 0 \end{pmatrix} = -a \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = -a b_2$$

Thus, a mutual set of eigenstates is $\{b_1, b_2, b_3\} = \{a_1 + i a_2, a_1 - i a_2, a_3\}$.

4 c) ii) The corresponding eigenvalues for operator A are $\{-a, -a, a\}$;

for operator B : $\{-b, b, b\}$ given the set of (ordered) eigenvectors $\{b_1, b_2, b_3\} =$

$\{a_1 + i a_2, a_1 - i a_2, a_3\}$ (see 4 c) i) for the definitions of these eigenvectors)

iii) The operators' eigenvectors share

eigenvalues : The b_2 and b_3 eigenvectors both yield b for operator B,

the b_1 and b_2 eigenvectors yield $-a$ for operator A. Thus,

the eigenvalues do not uniquely characterize each state (this is commonly referred to as "degeneracy").