Problem 1.

Prove the following identity: $(\vec{\sigma}\cdot\vec{a})\cdot\left(\vec{\sigma}\cdot\vec{b}\right)=\vec{a}\cdot\vec{b}+\mathrm{i}\vec{\sigma}\cdot\left(\vec{a}\cdot\vec{b}\right)$

$$(\vec{\sigma}\cdot\vec{\alpha})\cdot\left(\vec{\sigma}\cdot\vec{b}\right)=\left(\sigma_x a_x+\sigma_y a_y+\sigma_z a_z\right)\left(\sigma_x b_x+\sigma_y b_y+\sigma_z b_z\right)$$

$$\begin{aligned}=\;& a_x b_x+a_y b_y+a_z b_z+\\ & \sigma_x\sigma_y a_x b_y+\sigma_x\sigma_z a_x b_z+\\ & \sigma_y\sigma_x a_y b_x+\sigma_y\sigma_z a_y b_z+\\ & \sigma_z\sigma_x a_z b_x+\sigma_z\sigma_y a_z b_y\end{aligned}$$

$$\begin{aligned}=\;& \vec{a}\cdot\vec{b}+\\ & \sigma_x\sigma_y a_x b_y+\sigma_x\sigma_z a_x b_z+\\ & \sigma_y\sigma_z a_y b_z+\sigma_y\sigma_x a_y b_x+\\ & \sigma_z\sigma_x a_z b_x+\sigma_z\sigma_y a_z b_y\end{aligned}$$

$$\begin{aligned}=\;& \vec{a}\cdot\vec{b}+\left(\sigma_x\sigma_y a_x b_y+\sigma_y\sigma_x a_x b_y-\sigma_y\sigma_x a_x b_y\right)+\left(\sigma_x\sigma_z a_x b_z+\sigma_z\sigma_x a_x b_z-\sigma_z\sigma_x a_x b_z\right)+\\ & \left(\sigma_y\sigma_z a_y b_z+\sigma_z\sigma_y a_y b_z-\sigma_z\sigma_y a_y b_z\right)+\\ & \sigma_y\sigma_x a_y b_x+\sigma_z\sigma_x a_z b_x+\sigma_z\sigma_y a_z b_y\end{aligned}$$

$$\begin{aligned}=\;& \vec{a}\cdot\vec{b}+\left(\sigma_x\sigma_y a_x b_y+\sigma_y\sigma_x a_x b_y-\sigma_y\sigma_x a_x b_y\right)+\left(\sigma_x\sigma_z a_x b_z+\sigma_z\sigma_x a_x b_z-\sigma_z\sigma_x a_x b_z\right)+\\ & \left(\sigma_y\sigma_z a_y b_z+\sigma_z\sigma_y a_y b_z-\sigma_z\sigma_y a_y b_z\right)+\\ & \sigma_y\sigma_x a_y b_x+\sigma_z\sigma_x a_z b_x+\sigma_z\sigma_y a_z b_y\end{aligned}$$

$$\begin{aligned}=\;& \vec{a}\cdot\vec{b}+\left\{\sigma_x,\sigma_y\right\}a_x b_y-\sigma_y\sigma_x a_x b_y+\\ & \left\{\sigma_x,\sigma_z\right\}a_x b_z-\sigma_z\sigma_x a_x b_z+\\ & \left\{\sigma_y,\sigma_z\right\}a_y b_z-\sigma_z\sigma_y a_y b_z+\\ & \sigma_y\sigma_x a_y b_x+\sigma_z\sigma_x a_z b_x+\sigma_z\sigma_y a_z b_y\end{aligned}$$

$$= \vec{a}\cdot\vec{b} - \sigma_y\,\sigma_x\,a_x\,b_y - \sigma_z\,\sigma_x\,a_x\,b_z - \sigma_z\,\sigma_y\,a_y\,b_z + \sigma_y\,\sigma_x\,a_y\,b_x + \sigma_z\,\sigma_x\,a_z\,b_x + \sigma_z\,\sigma_y\,a_z\,b_y$$

$$= \vec{a}\cdot\vec{b} - \sigma_y\,\sigma_x\left(a_x\,b_y - a_y\,b_x\right) - \sigma_z\,\sigma_x(a_x\,b_z - a_z\,b_x) - \sigma_z\,\sigma_y\left(a_y\,b_z - a_z\,b_y\right)$$

$$= \vec{a}\cdot\vec{b} + \mathrm{i}\sigma_z\left(a_x\,b_y - a_y\,b_x\right) - i\,\sigma_y(a_x\,b_z - a_z\,b_x) + \mathrm{i}\sigma_x\left(a_y\,b_z - a_z\,b_y\right)$$

$$= \vec{a}\cdot\vec{b} + i\begin{vmatrix} \sigma_x & \sigma_y & \sigma_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \vec{a}\cdot\vec{b} + \mathrm{i}\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right)$$

In[11]:=

Problem 2.

You are given a density matrix of the form $\rho = \frac{1}{2}\,(\mathbb{1} + \vec{\sigma}\cdot\vec{a})$. Solve parts a) and b) described below. Given the excessively arithmetic nature of this problem (multiplying matrices by themselves and summing their diagonal elements) feel free to use computer algebra systems to alleviate the amount of time spent doing trivial computational tasks.

a) Show that the eigenvalues of ρ are $\frac{1}{2}\,(1 \pm |\,\vec{a}\,|)$

b) The purity of the state has many definitions. One such definition is that $P = \mathrm{tr}\left(\rho^2\right)$, where P is the state's purity and ρ is the density matrix representation of the state. Show that this definition of the purity results in $P = \frac{1}{2}\left(1 + |\,\vec{a}\,|^2\right)$, where the density matrix to use is the one defined in the Problem 2 global problem statement. $\vec{a}$ is a three-dimensional vector that describes the position of the state within the confines of the Bloch sphere ($|\,\vec{a}\,| \leq 1$).

1a) For this problem, it would be required, typically, to required to find the determinant of the matrix formed by ρ-$\lambda\mathbb{1}$ and set that equal to zero. The eigenvalues would be given by the values of λ for which the previous statement is satisfied, i.e. det$(\rho - \lambda\mathbb{1}) = 0$. However, because this author has solved n of these types of problems before (where n is very large), the author will defer to using computer algebra systems to solve the remaining parts of this problem.

In[23]:=

```
p = .5 {{1 + a_z, -I * a_y + a_x}, {I * a_y + a_x, 1 - a_z}};

TraditionalForm[FullSimplify[Eigenvalues[p]]]
```

Out[24]//TraditionalForm=

$$\left\{0.5 - 0.5\sqrt{1.\,a_x^2 + 1.\,a_y^2 + 1.\,a_z^2},\ 0.5\left(\sqrt{1.\,a_x^2 + 1.\,a_y^2 + 1.\,a_z^2} + 1.\right)\right\}$$

In[25]:=

This expression can be rewritten as $\lambda = \frac{1}{2}\left(1 \pm \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}\right)$. Thus, recognizing $\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$ as $|\vec{a}|$ (the magnitude of a vector in $\mathbb{R}^3$), the problem has been solved.

1b) For this problem, it will be required, first, to square the matrix. This, ordinarily, require the author to matrix multiply the matrix by itself. To determine the trace of the matrix it would be required to sum all the elements on the main diagonal of the resulting multiplication. This will not be done by hand (as, again, the author has much experience multiplying matrices and summing scalars). Thus, a computer algebra system will be used to reduce the amount of time spent on computation.

In[26]:= `p.p`

Out[26]=

$$\{\{0.25\,(a_x - i\,a_y)\,(a_x + i\,a_y) + 0.25\,(1 + a_z)^2,$$
$$0.25\,(a_x - i\,a_y)\,(1 - a_z) + 0.25\,(a_x - i\,a_y)\,(1 + a_z)\},$$
$$\{0.25\,(a_x + i\,a_y)\,(1 - a_z) + 0.25\,(a_x + i\,a_y)\,(1 + a_z),$$
$$0.25\,(a_x - i\,a_y)\,(a_x + i\,a_y) + 0.25\,(1 - a_z)^2\}\}$$

In[27]:= `FullSimplify[Tr[p.p]]`

Out[27]= $0.5 + 0.5\,a_x^2 + 0.5\,a_y^2 + 0.5\,a_z^2$

The result can be expressed in the following form : $\frac{1}{2}\left(1 + a_x{}^2 + a_y{}^2 + a_z{}^2\right) = \frac{1}{2}\left(1 + |\vec{a}|^2\right)$. Thus, the problem has been solved.

Problem 3.

Determine the magnetizion of a single spin in an NMR system. Writing the magnetization as the expectation of the σ_+ operator $\sigma_+ \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$:

$\langle M_1(t)\rangle = \mathrm{Tr}\left(\rho\left(\sigma_+{}^{(1)} \otimes \left(\sigma_\uparrow{}^{(2)} + \sigma_\downarrow{}^{(2)}\right) = \mathrm{Tr}\left(\rho\left(\sigma_+{}^{(1)} \otimes \sigma_\uparrow{}^{(2)}\right)\right) + \mathrm{Tr}\left(\rho\left(\sigma_+{}^{(1)} \otimes \sigma_\downarrow{}^{(2)}\right)\right)\right.\right.$.

Note that $\sigma_\uparrow{}^{(2)} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\sigma_\downarrow{}^{(2)} \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ so that $\sigma_\uparrow{}^{(2)} + \sigma_\downarrow{}^{(2)} = \mathbb{1}$. Note that the trace is linear and the tensor product is distributive. These two important

properties will be used to solve this problem.

To begin solving this problem, the

$$\langle M_1(t)\rangle = \mathrm{tr}\left(\hat\rho(t)\left(\hat\sigma_1{}^+(t)\oplus\mathbb{1}\right)\right) = \mathrm{tr}\left(\hat\rho(t)\left(\hat\sigma_1{}^+\oplus\left(\sigma_\uparrow{}^{(2)}+\sigma_\downarrow{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(e^{-\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\,\rho(0)\,e^{\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\left(\hat\sigma_1{}^+\oplus\left(\sigma_\uparrow{}^{(2)}+\sigma_\downarrow{}^{(2)}\right)\right)\right),\ \text{by definition of } \rho(t)$$

$$= \mathrm{tr}\left(p(0)\,e^{\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\left(\hat\sigma_1{}^+\oplus\left(\sigma_\uparrow{}^{(2)}+\sigma_\downarrow{}^{(2)}\right)\right)e^{-\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\right),\ \text{by cyclic trace property}$$

$$= \mathrm{tr}\left(p(0)\,e^{\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\left(\hat\sigma_1{}^+\oplus\sigma_\uparrow{}^{(2)}\right)e^{-\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\right)+\mathrm{tr}\left(p(0)\,e^{\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\left(\hat\sigma_1{}^+\oplus\sigma_\downarrow{}^{(2)}\right)e^{-\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}}\right),$$

by distributive tensor product and linear trace properties

Before I segue into simplifying the previous expression, a few operator identities will be exposited.

$$[\sigma_z, \sigma^+] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}-\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 4\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 2\,\sigma^+$$

$$\sigma_z\,\sigma^+ = \sigma^+ = -\sigma^+\,\sigma_z$$

$$[\sigma_z, \sigma_\uparrow] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 0$$

$$\sigma_z\,\sigma_\uparrow = \sigma_\uparrow = \sigma_\uparrow\,\sigma_z$$

$$[\sigma_z, \sigma_\downarrow] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 0$$

$$\sigma_z\,\sigma_\downarrow = -\sigma_\downarrow = \sigma_\downarrow\,\sigma_z$$

$$\mathrm{tr}\left(p(0)\,e^{\mathrm{i}\mathrm{k}_j\frac{t}{\hbar}}\left(\hat\sigma_1{}^+\oplus\sigma_\uparrow{}^{(2)}\right)e^{-\mathrm{i}\mathrm{k}_j\frac{t}{\hbar}}\right)$$

Note: $e^{\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}} = 1 + i\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right) - \left(\frac{k_j t}{\hbar}\right)^2\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)^2}{2!} - i\left(\frac{k_j t}{\hbar}\right)^3\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)^3}{3!} + \left(\frac{k_j t}{\hbar}\right)^4\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)^4}{4!} + \ldots + \left(\frac{k_j t}{\hbar}\right)^n\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)^n}{n!}$

$$= 1 + i\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right) - \left(\frac{k_j t}{\hbar}\right)^2\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)}{2!} - i\left(\frac{k_j t}{\hbar}\right)^3\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)}{3!} +$$

$$\left(\frac{k_j t}{\hbar}\right)^4\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)}{4!} + \ldots + \left(i\,\frac{k_j t}{\hbar}\right)^n\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)^n}{n!}$$

$$1 + i\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1\otimes\sigma_z{}^2\right) - \left(\frac{k_j t}{\hbar}\right)^2\frac{1}{2!} - i\left(\frac{k_j t}{\hbar}\right)^3\frac{\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)}{3!} + \left(\frac{k_j t}{\hbar}\right)^4\frac{1}{4!} + \ldots$$

$$= \mathbb{1}\cos\left(\frac{k_j t}{\hbar}\right) + i\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\sin\left(\frac{k_j t}{\hbar}\right)$$

Thus, succinctly, for reference:

$$e^{-\mathrm{i}\mathrm{H}_j\frac{t}{\hbar}} = \mathbb{1}\cos\left(\frac{k_j t}{\hbar}\right) - i\left(\sigma_z{}^1\otimes\sigma_z{}^2\right)\sin\left(\frac{k_j t}{\hbar}\right)$$

Now, considering the first half of $\langle M_1(t)\rangle$

$$\mathrm{tr}\left(p(0)\, e^{\mathrm{i}\mathrm{H}_j \frac{t}{\hbar}}\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right) e^{-\mathrm{i}\mathrm{H}_j \frac{t}{\hbar}}\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\mathbb{1}\cos\left(\frac{k_j t}{\hbar}\right) + i\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\left(\mathbb{1}\cos\left(\frac{k_j t}{\hbar}\right) - i\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos^2\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right) - \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right) + \right.\right.\right.$$

$$\left.\left.\left.\mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right) + \sin^2\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos^2\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right) - \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(-\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right) + \right.\right.\right.$$

$$\left.\left.\left.\mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right) + \sin^2\left(\frac{k_j t}{\hbar}\right)\left(-\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos^2\left(\frac{k_j t}{\hbar}\right) - \sin^2\left(\frac{k_j t}{\hbar}\right) + \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right) + \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos\left(2\,\frac{k_j t}{\hbar}\right) + 2\,\mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos\left(2\,\frac{k_j t}{\hbar}\right) + \mathrm{i}\sin\left(2\,\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\, e^{i\,\frac{2 k_j t}{\hbar}}\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)$$

$$= \mathrm{tr}\left(p(0)\, e^{i\,\frac{\mathrm{Jt}}{2}}\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)$$

Considering the second half of
$\langle M_1(t)\rangle$:

$$\mathrm{tr}\left(p(0)\, e^{i\,\frac{k_j t}{\hbar}}\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right) e^{-i\,\frac{k_j t}{\hbar}}\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\mathbb{1}\cos\left(\frac{k_j t}{\hbar}\right) + i\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right)\left(\mathbb{1}\cos\left(\frac{k_j t}{\hbar}\right) - i\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos^2\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right) - \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right) + \right.\right.\right.$$

$$\left.\left.\left.\mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right) + \sin^2\left(\frac{k_j t}{\hbar}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right)\left(\sigma_z{}^1 \otimes \sigma_z{}^2\right)\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos^2\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\downarrow}{}^{(2)}\right) - \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(-\hat{\sigma}_1{}^{+} \oplus -\sigma_{\downarrow}{}^{(2)}\right) + \right.\right.\right.$$

$$\left.\left.\left.\mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\left(\hat{\sigma}_1{}^{+} \oplus -\sigma_{\downarrow}{}^{(2)}\right) + \sin^2\left(\frac{k_j t}{\hbar}\right)\left(-\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos^2\left(\frac{k_j t}{\hbar}\right) - \sin^2\left(\frac{k_j t}{\hbar}\right) - \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right) - \mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos\left(2\,\frac{k_j t}{\hbar}\right) - 2\,\mathrm{i}\cos\left(\frac{k_j t}{\hbar}\right)\sin\left(\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\left(\left(\cos\left(2\,\frac{k_j t}{\hbar}\right) - \mathrm{i}\sin\left(2\,\frac{k_j t}{\hbar}\right)\right)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)\right)$$

$$= \mathrm{tr}\left(p(0)\, e^{-I\,\frac{2 k_j t}{\hbar}}\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)$$

$$= \mathrm{tr}\left(p(0)\, e^{-i\,\frac{\mathrm{Jt}}{2}}\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)$$

Thus: $\langle M_1(t)\rangle$ is given by : $e^{-i\,\frac{\mathrm{Jt}}{2}}\,\mathrm{tr}\left(p(0)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right) + e^{i\,\frac{\mathrm{Jt}}{2}}\,\mathrm{tr}\left(p(0)\left(\hat{\sigma}_1{}^{+} \oplus \sigma_{\uparrow}{}^{(2)}\right)\right)$

Problem 4.

Consider the following Hamiltonian : $H = \frac{\hbar\omega_1}{2}\sigma_z{}^{(1)} + \frac{\hbar\omega_2}{2}\sigma_z{}^{(2)} + \frac{\hbar J}{2}\sigma_z \otimes \sigma_z$, where tensor products with identity (in the second and third terms) have been left out for succinctness. Note that $\sigma_z{}^{(1)}$ represents $\sigma_z \otimes \mathbb{1}$, $\sigma_z{}^{(2)} = \mathbb{1} \otimes \sigma_z$. The problem is, first, to simplify the following expression : $e^{\frac{-i\phi}{2}\sigma_x{}^{(1)}}\, e^{-i\omega_1 \frac{t}{2}\sigma_z{}^{(1)}}\, e^{\frac{-i\phi}{2}\sigma_x{}^{(1)}}$. It will be shown that this particular sequence reverses the phase of the middle expression when $\phi = \pi$ (i.e. the entire expression reduces to $e^{i\omega_1 \frac{t}{2}\sigma_z{}^{(1)}}$). The last half of the problem is to reduce the following expression : $e^{\frac{-iHt}{\hbar}}\, e^{\frac{-i\pi}{2}\sigma_x{}^{(1)}}\, e^{\frac{-iHt}{\hbar}}\, e^{\frac{-i\pi}{2}\sigma_x{}^{(1)}}$. It will be shown that this sequence of operators reduces to $e^{-i\omega_2 t\sigma_z{}^{(2)}}$. This has the effect of removing the coupling from the evolution of the system. This result demonstrates that the particular pulse sequence has the consequence of removing the spin-coupling from the evolution of the system. Each spin evolves freely. To assist with these problems, a few preliminary results/identities will be stated.

Note that a Pauli operator expressed as a complex exponential can be expressed as a sum of a sine and a cosine as follows:

$$D_x{}^{(1)}(\phi) = e^{-i\sigma_x{}^{(1)}\frac{\phi}{2}} = \cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)\sigma_x{}^{(1)}$$

$$e^{-i\omega_1 \frac{t}{2}\sigma_z{}^{(1)}} = \cos\left(\omega_1 \frac{t}{2}\right)\mathbb{1} - i\sin\left(\omega_1 \frac{t}{2}\right)\sigma_z{}^{(1)}$$

Now, to reduce the first expression $e^{\frac{-i\phi}{2}\sigma_x{}^{(1)}}\, e^{-i\omega_1 \frac{t}{2}\sigma_z{}^{(1)}}\, e^{\frac{-i\phi}{2}\sigma_x{}^{(1)}}$.

Using these two relationships

$$D_x{}^{(1)}(\phi)\, e^{-i\omega_1 \frac{t}{2}\sigma_z{}^{(1)}}\, D_x{}^{(1)}(\phi) = \left(\cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)\sigma_x{}^{(1)}\right)\left(\cos\left(\omega_1 \frac{t}{2}\right)\mathbb{1} - i\sin\left(\omega_1 \frac{t}{2}\right)\sigma_z{}^{(1)}\right)\left(\cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)\sigma_x{}^{(1)}\right)$$

Which, when completely expanded yields:

$$\cos^2\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\mathbb{1} - i\cos\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\sin\left(\frac{\phi}{2}\right)\sigma_x{}^{(1)} -$$
$$i\cos^2\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_z{}^{(1)} - \cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_z{}^{(1)}\,\sigma_x{}^{(1)}$$
$$-i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\sigma_x{}^{(1)} - \sin^2\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\mathbb{1} -$$
$$\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_x{}^{(1)}\,\sigma_z{}^{(1)} + i\sin^2\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_x{}^{(1)}\,\sigma_z{}^{(1)}\,\sigma_x{}^{(1)}$$

$$\sigma_z\,\sigma_x = i\sigma_y = -\sigma_x\,\sigma_z$$
$$\sigma_x\,\sigma_z\,\sigma_x = i\sigma_x\,\sigma_y = -\sigma_z$$

Using the above relationships the expression above can be expressed as:

$$\cos^2\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\mathbb{1} - i\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\sigma_x{}^{(1)} -$$
$$i\cos^2\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_z{}^{(1)} - i\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_{y^{(1)}}$$
$$-i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\sigma_x{}^{(1)} - \sin^2\left(\frac{\phi}{2}\right)\cos\left(\omega_1\,\frac{t}{2}\right)\mathbb{1} +$$
$$i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_y{}^{(1)} - i\sin^2\left(\frac{\phi}{2}\right)\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_z{}^{(1)}$$

Cancelling/Combining like terms:

$$\cos(\phi)\cos\left(\omega_1\,\frac{t}{2}\right)\mathbb{1} - i\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_z{}^{(1)} - i\sin(\phi)\cos\left(\omega_1\,\frac{t}{2}\right)\sigma_x{}^{(1)}$$

If $\phi = \pi$ the above expression reduces to:

$$-\cos\left(\omega_1\,\frac{t}{2}\right)\mathbb{1} - i\sin\left(\omega_1\,\frac{t}{2}\right)\sigma_z{}^{(1)} = e^{i\pi}\,e^{i\omega_1\,\frac{t}{2}\,\sigma_z{}^{(1)}}$$

Since, at most, the term $e^{i\pi}$ will introduce a global phase then it can be ignored.

Tackling the second part of problem 4. The first attempt the author will make is brute force expansion of the exponentials.

$$e^{-\frac{iHt}{\hbar}} = e^{-ikH} = 1 - ikH - \frac{k^2 H^2}{2!} - i\,\frac{k^3 H^3}{3!} + \ldots =$$

$$1 - ik(a\sigma_1 + b\sigma_2 + c\sigma_3) - \frac{k^2 (a\sigma_1 + b\sigma_2 + c\sigma_3)\,(a\sigma_1 + b\sigma_2 + c\sigma_3)}{2!} - \frac{ik^3 (a\sigma_1 + b\sigma_2 + c\sigma_3)\,a\sigma_1 + b\sigma_2 + c\sigma_3\,a\sigma_1 + b\sigma_2 + c\sigma_3}{3!} + \ldots$$

$$= 1 - ik(a\sigma_1 + b\sigma_2 + c\sigma_3) - \frac{k^2}{2!}\left(a^2 + b^2 + c^2 + ab\sigma_3 + ac\sigma_2 + ba\sigma_3 + bc\sigma_1 + ca\sigma_2 + cb\sigma_1\right) -$$
$$i\,\frac{k^3}{3!}\left(a^2 + b^2 + c^2 + ab\sigma_3 + ac\sigma_2 + ba\sigma_3 + bc\sigma_1 + ca\sigma_2 + cb\sigma_1\right)(a\sigma_1 + b\sigma_2 + c\sigma_3) + \ldots$$

$$= 1 - ik(a\sigma_1 + b\sigma_1 + c\sigma_3) - \frac{k^2}{2!}\left(a^2 + b^2 + c^2 + ab\sigma_3 + ac\sigma_2 + ba\sigma_3 + bc\sigma_1 + ca\sigma_2 + cb\sigma_1\right) -$$
$$i\,\frac{k^3}{3!}\Big(\left(a^2 + b^2 + c^2\right)a\sigma_1 + \left(a^2 + b^2 + c^2\right)b\sigma_2 + \left(a^2 + b^2 + c^2\right)c\sigma_3 + a^2\,b\sigma_2 + ab^2\,\sigma_1 + abc +$$
$$a^2\,c\sigma_3 + abc + ac^2\,\sigma_1 + ba^2\,\sigma_2 + b^2\,a\sigma_1 + abc + abc + b^2\,c\sigma_3 +$$
$$bc^2\,\sigma_2 + a^2\,c\sigma_3 + abc + ac^2\,\sigma_1 + abc + b^2\,c\sigma_3 + c^2\,b\sigma_2\Big) + \ldots$$

$$= 1 - ik(a\sigma_1 + b\sigma_1 + c\sigma_3) - \frac{k^2}{2!}\left(a^2 + b^2 + c^2 + ab\sigma_3 + ac\sigma_2 + ba\sigma_3 + bc\sigma_1 + ca\sigma_2 + cb\sigma_1\right) -$$
$$i\,\frac{k^3}{3!}\Big(\left(a^2 + b^2 + c^2\right)a\sigma_1 + \left(a^2 + b^2 + c^2\right)b\sigma_2 + \left(a^2 + b^2 + c^2\right)c\sigma_3 + 2\,a^2\,b\sigma_2 + 2\,ab^2\,\sigma_1 + 6\,abc +$$
$$2\,a^2\,c\sigma_3 + 2\,ac^2\,\sigma_1 + 2\,cb^2\,\sigma_3 + 2\,bc^2\,\sigma_2\Big) + \ldots$$

$$= 1 - ik(a\sigma_1 + b\sigma_1 + c\sigma_3) - \frac{k^2}{2!}\left(a^2 + b^2 + c^2 + ab\sigma_3 + ac\sigma_2 + ba\sigma_3 + bc\sigma_1 + ca\sigma_2 + cb\sigma_1\right) -$$
$$i\,\frac{k^3}{3!}\left(a^3\,\sigma_1 + b^3\,\sigma_2 + c^3\,\sigma_3 + 3\,ab^2\,\sigma_1 + 3\,ac^2\,\sigma_1 + 3\,a^2\,b\sigma_2 + 3\,c^2\,b\sigma_2 + 3\,a^2\,c\sigma_3 + 3\,b^2\,c\sigma_3\right) + \ldots$$

This looks terrible so I am going to take a step back. By Baker-Campbell-Hausdorff $e^{x+y} = e^{\frac{-1}{2}[x,y]}\,e^x\,e^y$ if and only if $[x, [x, y]] = 0$. So, allow $a\sigma_1 + b\sigma_2 = \sigma_A$ and $c\sigma_3 = \sigma_B$. So, $e^{a\sigma_1 + b\sigma_2 + c\sigma_3} = e^{\sigma_A + \sigma_B}$. Now, consider-

$$[\sigma_A, \sigma_B] = [a\sigma_1 + b\sigma_2, c\sigma_3] = [a\sigma_1, c\sigma_3] + [b\sigma_2, c\sigma_3] = ac(\sigma_1\,\sigma_3 - \sigma_3\,\sigma_1) + bc(\sigma_2\,\sigma_3 - \sigma_3\,\sigma_2) = ac(\sigma_2 - \sigma_2) + bc(\sigma_1 - \sigma_1) =$$
$$0 \left(\text{replacing } \sigma_2\,\sigma_3 \text{ by } \sigma_1 \text{ and } \sigma_1\,\sigma_3 \text{ by } \sigma_2 \text{ is readily verified by considering } \sigma_i\,\sigma_j \text{ for } i, j \text{ that take on the values of } 1,\ 2 \text{ and } 3\right).$$

Thus, $[\sigma_A, [\sigma_a, \sigma_B]] = 0$. Now, $e^{\sigma_A + \sigma_B} = e^{\sigma_A}\,e^{\sigma_B}$. However, since σ_1 and σ_2 also commute, $e^{\sigma_A} = e^{a\sigma_1}\,e^{b\sigma_2}$. Thus, $e^{a\sigma_1 + b\sigma_2 + c\sigma_3} = e^{a\sigma_1}\,e^{b\sigma_2}\,e^{c\sigma_3}$ (note that this is a very special Hamiltonian that can accomplish this. It may be assumed by some students of Quantum Mechanics that this is trivially true in all cases. However, in general it is not the case and must be demonstrated explicitly for a given Hamiltonian/sum of operators).

Now,

$$e^{-i\frac{Ht}{\hbar}} e^{\frac{-i\pi}{2}\sigma_x{}^{(1)}} e^{-i\frac{Ht}{\hbar}} e^{\frac{-i\pi}{2}\sigma_x{}^{(1)}} = \left(e^{\frac{-i\omega_1 t}{2}\sigma_z{}^{(1)}} e^{\frac{-i\omega_2 t}{2}\sigma_z{}^{(2)}} e^{\frac{-iJt}{4}\sigma_z{}^{(1)}\sigma_z{}^{(2)}}\right) e^{\frac{-i\pi}{2}\sigma_x{}^{(1)}} \mathbb{1}\left(e^{\frac{-i\omega_1 t}{2}\sigma_z{}^{(1)}} e^{\frac{-i\omega_2 t}{2}\sigma_z{}^{(2)}} e^{\frac{-iJt}{4}\sigma_z{}^{(1)}\sigma_z{}^{(2)}}\right) e^{\frac{-i\pi}{2}\sigma_x{}^{(1)}}$$

$$= \left(e^{-ia\sigma_1} e^{-ib\sigma_2} e^{-ic\sigma_3}\right)\left(-i\sigma_x{}^{(1)}\right)\left(e^{-ia\sigma_1} e^{-ib\sigma_2} e^{-ic\sigma_3}\right)\left(-i\sigma_x{}^{(1)}\right)$$

$$= -e^{-ib\sigma_2} e^{-ib\sigma_2}\left(e^{-ic\sigma_3} e^{-ia\sigma_1} \sigma_x{}^{(1)} e^{-ia\sigma_1} e^{-ic\sigma_3} \sigma_x{}^{(1)}\right)$$

$$= -e^{-ib\sigma_2} e^{-ib\sigma_2}\left(e^{-ic\sigma_3} e^{-ia\sigma_1} \sigma_x{}^{(1)} e^{-ia\sigma_1} e^{-ic\sigma_3} \sigma_x{}^{(1)}\right)$$

$$= -e^{-2\,ib\sigma_2}\left(e^{-ic\sigma_3}(\cos(a) - i\sin(a)\,\sigma_1)\,\sigma_x{}^{(1)}(\cos(a) - i\sin(a)\,\sigma_1)\,e^{-ic\sigma_3}\right)\sigma_x{}^{(1)}$$

$$= -e^{-2\,ib\sigma_2}\left(e^{-ic\sigma_3}\left(\cos^2(a)\,\sigma_x{}^{(1)} - i\cos(a)\sin(a)\,\sigma_x{}^{(1)}\sigma_1 - i\sin(a)\cos(a)\,\sigma_1\sigma_x{}^{(1)} - \sin^2(a)\,\sigma_1\sigma_x{}^{(1)}\sigma_1\right)e^{-ic\sigma_3}\right)\sigma_x{}^{(1)}$$

$$= -e^{-2\,ib\sigma_2}\left(e^{-ic\sigma_3}\left(\cos^2(a)\,\sigma_x{}^{(1)} + \sin^2(a)\,\sigma_x{}^{(1)}\right)e^{ic\sigma_3}\right)\sigma_x{}^{(1)}$$

$$= -e^{-2\,ib\sigma_2}\left(\sigma_x{}^{(1)}\sigma_x{}^{(1)}\right)$$

$$= -e^{-2\,ib\sigma_2} = -e^{-i\omega_2 t\sigma_2}$$

Where in the second line, the terms involving σ_2 have been extracted since they act in different spaces than any operator with a superscript “(1)” (like $\sigma_x{}^{(1)}$). In the second line the operators have also been re-arranged when commutation relations allow. Lines 2, 3, and 4 utilize the expansion of the “complex-Pauli-exponentials” and trigonometric identities. Line 5 skips a similar expansion of $e^{-ic\sigma_3}$ recognizing the symmetry of the math with the previous 3 lines. The last line utilizes Baker-Campbell-Hausdorff “in reverse” to combine the two complex-Pauli exponentials into a single expression matching that of the desired result. One might retort that the phase is different than expected. For this, I refer the interested reader to the explanation given for a similar dilemma in “part A” of this problem. Thus, it has been shown and thus I am done.