Thur.

Basic Questions about a laser (= negative conductance oscillator)

1. What are 3 essential elements to construct a laser?
   - ① frequency selective element : LC circuit, delayed feedback loop, Fabry-Perot, ring, distributed feedback cavities
   - ② amplifying element (negative conductance device) : tunnel diode, IMPATT diode, Gunn diode, Josephson junction, pn-junction
   - ③ nonlinear element (amplitude limiting device) : gain saturation

2. What is the electrical circuit element that represents a gain?

   negative differential conductance

3. What is the mechanism for spectral purification in a laser?

   below threshold : frequency selective amplification/deamplification

   above threshold : phase coherent stimulated emission of photons

4. What is the mechanism for amplitude stabilization in a laser?

   gain saturation (relaxation oscillation)

5. What is the energy source (seed) for laser oscillation?

   noise

6. What is the single most important parameter that determines the dynamics of a laser?

   spontaneous emission coupling efficiency = saturation parameter

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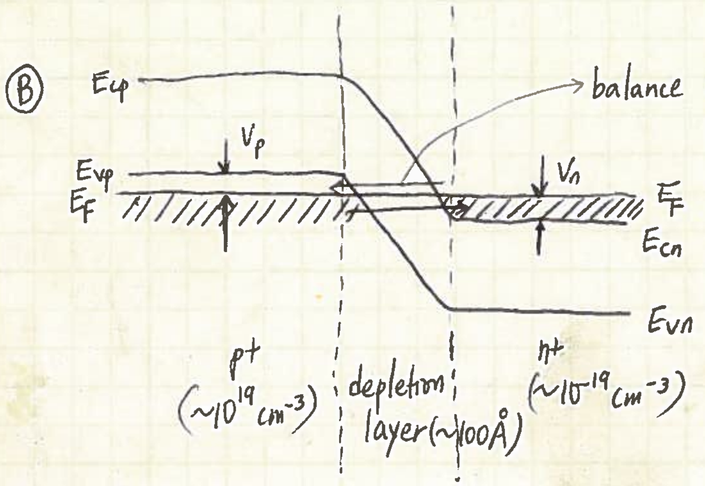
pn Tunnel Junction (Esaki diode)

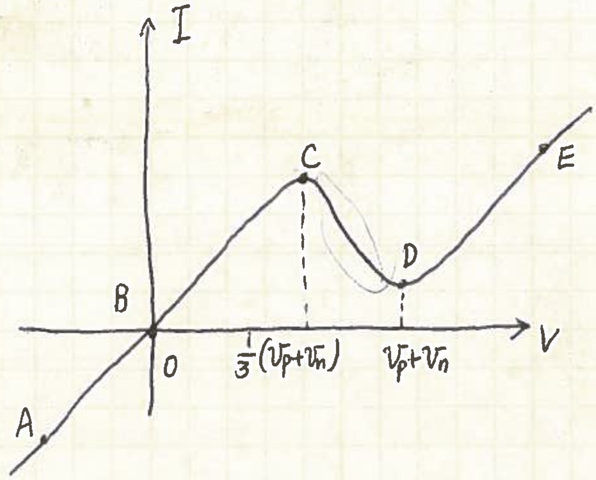


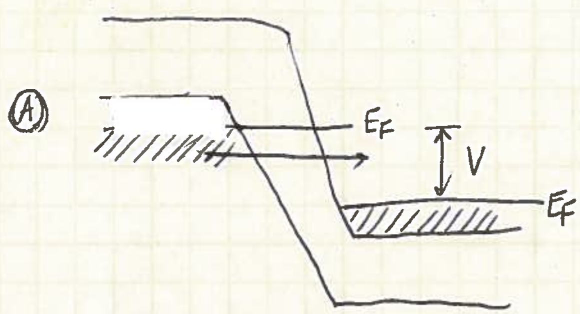
current noise

$$S_I(\omega) = 2qI \coth\left(\frac{qV}{2k_B\Theta}\right)$$

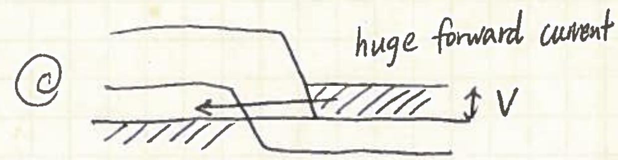
$$V \approx 0 \longrightarrow S_I(\omega) = \frac{4k_B\Theta}{q}$$

$$V > V_T \longrightarrow S_I(\omega) = 2qI$$

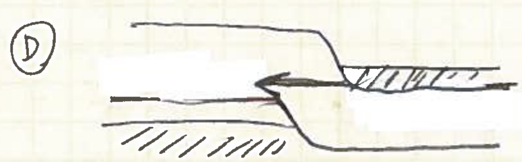




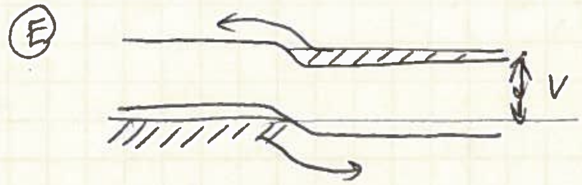
tunnel breakdown



peak tunnel current



no tunnel current



thermal diffusion current

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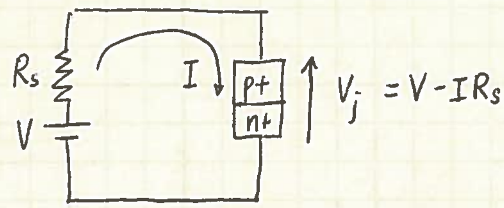
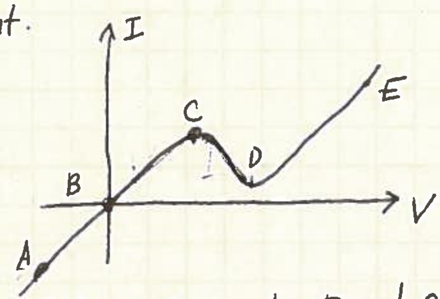
Amplification by a Negative Conductance Element.





If a p^+-n^+ junction diode is biased at $V>0$ between the points B and C,

the differential conductance $\frac{dI}{dV}$ is "positive"

increased current $I \nearrow \rightarrow V_j \searrow \rightarrow I \searrow$ decreased current

decreased current $I \searrow \rightarrow V_j \nearrow \rightarrow I \nearrow$ increased current

This negative feedback mechanism attenuates a current fluctuation

$\therefore$ sys is stable, small fluctuation

If a p^+-n^+ junction diode is biased at $V>0$ btw C & D,

the differential conductance $\frac{dI}{dV}$ is "negative".

increased current $I \nearrow \rightarrow V_j \downarrow \rightarrow I \nearrow\nearrow \rightarrow V_j \downarrow\downarrow \rightarrow I \uparrow\uparrow\uparrow \rightarrow$

decreased current $I \downarrow \rightarrow V_j \nearrow \rightarrow I \downarrow\downarrow \rightarrow V_j \nearrow\nearrow \rightarrow I \downarrow\downarrow\downarrow \rightarrow$

This positive feedback mechanism amplifies a current fluctuation

$\Downarrow$

A negative conductance element such as a tunnel diode features a gain and is used as an amplifier and an oscillator.

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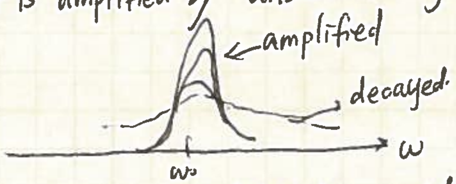
Negative Conductance (van der Pol) Oscillators

Physical process of "self-oscillation"

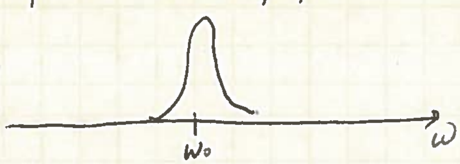
- An amplifying element generates a broad band noise.

when gain is max, noise power is max.

- A frequency selective circuit stores a specific frequency component that is amplified by unsaturated gain

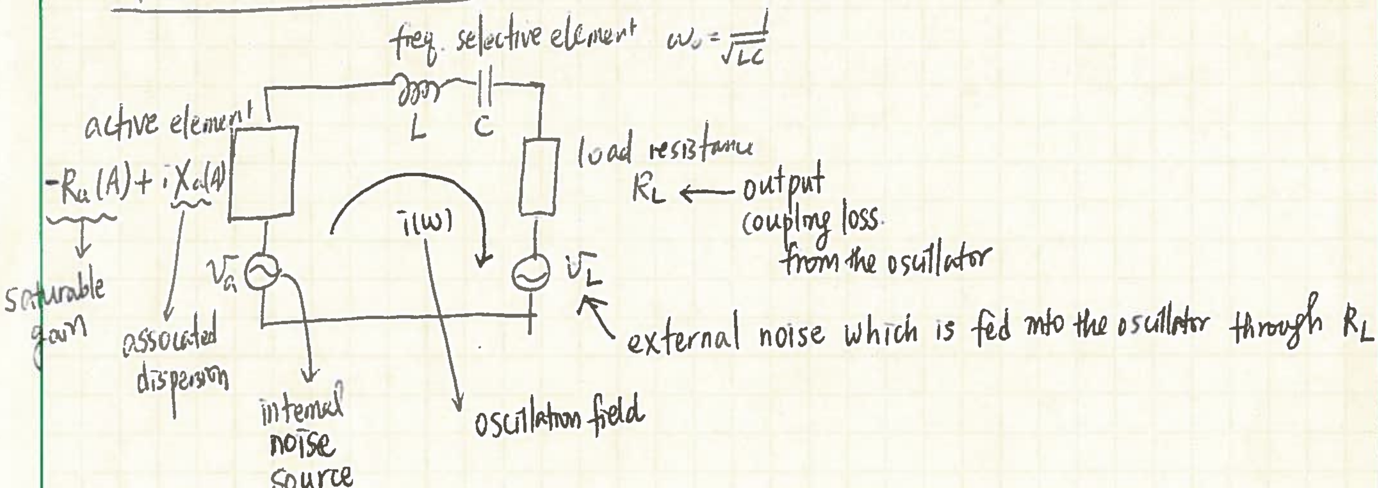


- A nonlinear element stops this amplification process "~~specific~~ regenerative amplification" and stabilization with amplitude via "self-feedback" regulation



→ stabilized amplitude, narrow band signal.

Electrical circuit model



Van der Pol oscillator

Van der Pol → the first person to study the noise properties of a negative conductance oscillator

Phil. Mag. Series 7, 3.65 (1927)

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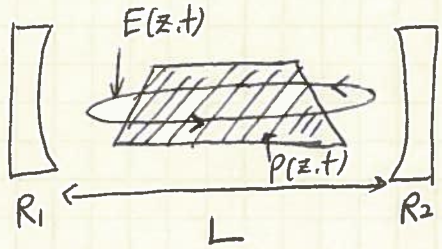

← typical laser oscillar with a Fabry-Perot cavity

R_2 ← mirror reflectivities

photon decay rate $\frac{\omega}{Q} = \frac{c}{2L} \ln \frac{1}{R_1 R_2}$

Q = cavity-Q- value
L = cavity length.

an oscillation field : electric field

$$E(z.t) = \frac{1}{2} \sum_n \underbrace{E_n(t)}_{\text{amplitude}} e^{i(\omega_n t + \underset{\text{phase}}{\phi_n})} u_n(z) + C.C$$

an inverted medium : atomic polarization ensemble

$$P(z.t) = \frac{1}{2} \sum_n P_n(t) e^{i(\omega_n t + \phi_n)} u_n(z) + C.C$$

$u_n(z)$ → a spatial mode ftn

n = a longitudinal mode

Assume a single transverse mode operation.

Maxwell equation with a driving term.

$$-\frac{\partial^2 E}{\partial z^2} + \mu_0 \sigma \frac{\partial E}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2}$$

← driving term by inverted atoms

damping due to internal loss

↓

equations of motion for the amplitude and phase of the oscillation field.

$$\dot{E}_n = -\frac{1}{2}\left(\frac{\omega_n}{Q}\right) E_n - \frac{1}{2}\left(\frac{\omega_n}{\varepsilon_0}\right) Im(P_n)$$

← gain/loss: quadrature phase of P_n

$$\dot{\phi}_n = (\Omega_n - \omega_n) - \frac{1}{2}\left(\frac{\omega_n}{\varepsilon_0}\right) \frac{1}{E_n} Re(P_n)$$

← dispersion in-phase of P_n

$\frac{\omega_n}{Q} = \frac{\sigma}{\varepsilon_0}$ photon decay rate

Ω_n = empty cavity resonant freq.

ω_n = actual oscillation freq.

Complex susceptibility
$P_n = \varepsilon_0 \chi_n E_n$
$= \varepsilon_0 (\chi_{nr} + i \chi_{ni}) E_n$

$$\Rightarrow \left.\begin{aligned} \dot{E}_n &= -\frac{1}{2}\frac{\omega_n}{Q} E_n - \frac{1}{2}\omega_n \chi_{ni} E_n \\ \dot{\phi}_n &= (\Omega_n - \omega_n) - \frac{1}{2}\omega_n \chi_{nr} \end{aligned}\right\}$$

$\boxed{\text{laser Master eq.}}$

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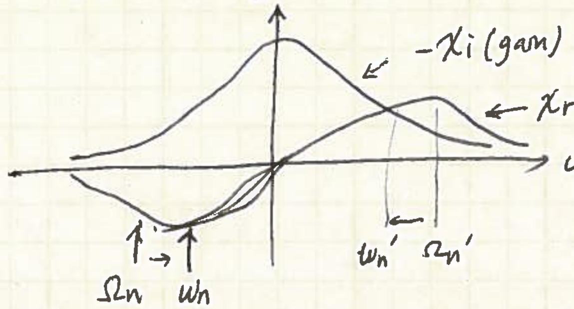
Steady-state solutions of the laser master eq.

$$\frac{\omega_n}{Q} = \omega_n \chi_{ni} \quad \leftarrow \text{gain = loss, threshold condition}$$

$$\omega_n = \Omega_n - \frac{1}{2}\omega_n \chi_{nr} \quad \leftarrow \text{oscillation freq.}$$



← complex electric susceptibility $\chi = \chi_r + i\chi_i$ of an <u>inverted system</u>

ω_n is always pulled toward the gain <u>center</u> from the empty cavity freq. Ω_n due to the dispersion.

If the cavity internal electric field amplitude is normalized to represent a "photon amplitude",

the stored energy is given by

$$E = \hbar\omega A_0^2 = \frac{1}{2} L A^2 \longrightarrow \text{the eq. oscillation current amplitude in the LC circuit}$$

e.g. $I(\omega) = A\cos\omega t$

A_0: photon field amplitude. $A_0^2 = n$, photon #

$$\therefore A_0 = \sqrt{\frac{L}{2\hbar\omega}}\, A$$ relation btw the photon field amplitude A_0 & the eq. current amplitude A

$\therefore$ The output optical power from the laser,

$$P_{out} = \hbar\omega A_0^2 \underbrace{\left(\frac{\omega}{Q_{ex}}\right)}_{\text{photon decay rate} = \frac{R_L}{L}} = \frac{1}{2} R_L A^2$$

The internally generated power

$$P_{in} = \hbar\omega A_0^2 \cdot \underbrace{\omega \chi_i(A_0)}_{\text{photon amplification rate} = \frac{R_a(A)}{L}} = \frac{1}{2} R_a A^2$$

The actual oscillation freq.

$$\omega = \Omega - \frac{1}{2}\underbrace{\omega\chi_r(A_0)}_{\text{dispersion} = \frac{X_a(A)}{L}} = \Omega - \frac{1}{2L} X_a(A)$$

1:1 correspondance table

	electrical circuit	Q. Electro
oscillation field amp	$\sqrt{\frac{L}{2}} A$	$\sqrt{\hbar\omega} A_0$
photon decay rate	$\frac{R_L}{L}$	$\frac{\omega}{Q_{ex}}$
photon amplification rate	$\frac{R_a(A)}{L}$	$\omega\chi_i(A)$
Dispersion	$\frac{X_a(A)}{L}$	$\omega\chi_r(A)$

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$$\omega L - \frac{1}{\omega C} = \frac{L}{\omega}\left(\omega^2 - \frac{1}{LC}\right) = \frac{L}{\omega}(\omega+\omega_0)(\omega-\omega_0) \approx 2L(\omega-\omega_0)$$

if ω is close to ω_0.

$\therefore$ circuit eq. becomes

$$Re\left\{\left[R_L - R_a(A) - \frac{\partial R_a}{\partial A}\Delta A + i2L(\omega-\omega_0) + i\left(X_a(A) + \frac{\partial X_a}{\partial A}\Delta A\right)\right](A+\Delta A)e^{i(\omega t+\Delta\phi)}\right\}$$

$$= v_a(t) + v_L(t)$$

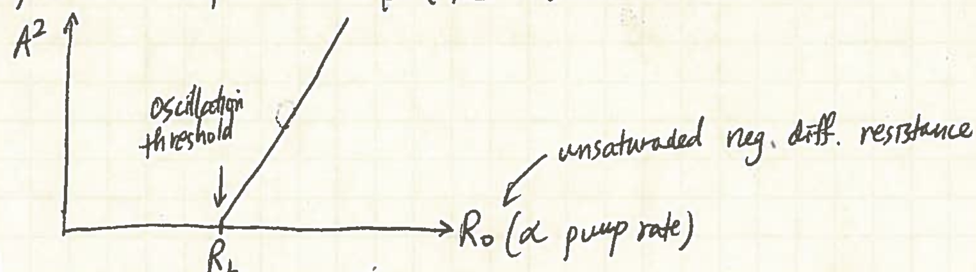
replacing $i\Omega = i(\omega-\omega_0)$ by $\frac{d}{dt}$, & taking the time derivative

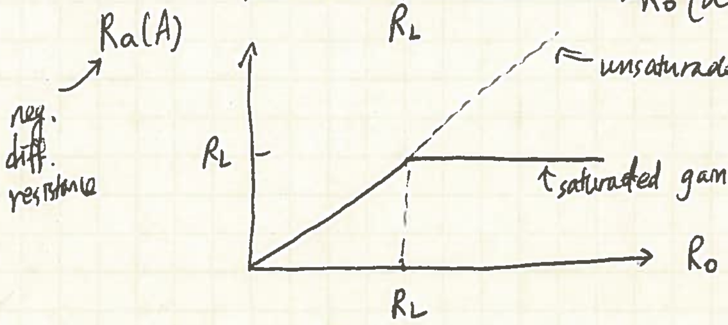
$$Re\left\{\left[R_L - R_a(A) - i2L\left(\omega-\omega_0+\frac{X_a(t)}{2L}\right) + 2L\left(\frac{1}{A}\frac{d\Delta A}{dt} - \frac{1}{2L}\frac{\partial R_a}{\partial A}\Delta A\right) + i2L\left(\frac{d\Delta\phi}{dt} + \frac{1}{2L}\frac{\partial X_a}{\partial A}\Delta A\right)\right]Ae^{i(\omega t+\Delta\phi)}\right\} = v_a(t) + v_L(t)$$

Suppose $\Delta A(t) = \Delta\phi(t) = v_a(t) = v_L(t) = 0$,

steady-state solutions: $R_L = R_a(A) = \frac{R_0}{1+\beta A^2}$, $\omega = \omega_0 - \frac{X_a(A)}{2L}$

steady-state oscill. amplitude $A^2 = \frac{1}{\beta}\left(\frac{R_0}{R_L} - 1\right)$





$\boxed{\text{relaxation oscillation process}}$

① when oscillation field increases above the s.s. value $\left(A^2 = \frac{1}{\beta}\left(\frac{R_0}{R_L}-1\right)\right)$ the saturated gain decreases to below R_L $\Rightarrow$ the circuit has a net loss. oscillation field $\downarrow$

② when oscillation field decrease below the ss. value, the saturaded gain increases above R_L $\Rightarrow$ $\exists$ a net gain $\therefore$ oscillation field amplified.

$\rightarrow$ stabilize both the oscillation field amplitude & the saturaded gain.

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Free-Running Van der Pol Oscillators

regenerative amplification ← { active element pumped by an external energy source
↓
a negative diff. conductance (gain) realized
↓
internal external noise voltages amplified
↓
the fluctuation freq. component of $i(\omega)$ grows near $\omega_0 = \frac{1}{\sqrt{LC}}$ }

Once $R_a = R_L$ the active element negative resistance balances the positive load resistance,

the circuit becomes <u>unstable</u> & spectral shape is purified

the noise grows exponential

∴ circuit oscillates & the steady-state coherent field amp. is established

the steady-state condition is established by the <u>gain</u>-saturation of the active element

then broad band noise v_a, v_L are transformed into

a <u>coherent wave</u> with stabilized amplitude & a well-defined freq.

✓ frequency-selective amplification puri<u>fies</u> <u>spectral profile</u>

✓ gain saturation stabilized the <u>amplide</u>

<u>circuit</u> eq. of $i(\omega)$

$$\left[R_L + i\left(\omega L - \frac{1}{\omega C}\right) - R_a + iX_a\right] i(\omega) = v_a(\omega) + v_L(\omega)$$

<u>Suppose</u> $i(t) \equiv Re\,(i(\omega)) = Re\left[(A+\Delta A)\,e^{i(\omega t + \Delta\phi)}\right]$

$$= (A + \Delta A(t)) \cos(\omega t + \Delta\phi(t))$$

average amplitude

slowly varying amplitud & phase fluctuations

gain saturation of the active element

$$-R_a = \frac{-R_0}{1+\beta A^2}$$

$-R_0$: unsaturated negative differential resistance

$\propto$ Pump rate

β → saturation parameter

if $\left|\frac{\Delta A}{A}\right| \ll 1$, <u>linearize</u> R_a & X_a : $R_a = R_a(A) + \frac{\partial R_a}{\partial A}\Delta A$

$X_a = X_a(A) + \frac{\partial X_a}{\partial A}\Delta A$

Amplitude and Phase Noise of an Internal Field

The small fluctuating parts

$$\left(2L\frac{d}{dt}\Delta A - A\frac{\partial R_a}{\partial A}\Delta A\right)\cos(\omega t+\Delta\phi) - \left(2LA\frac{d}{dt}\Delta\phi + A\frac{\partial X_a}{\partial A}\Delta A\right)\sin(\omega t+\Delta\phi)$$
$$= v_a(t) + v_L(t)$$

Multiplying by $\cos(\omega t+\Delta\phi)$ or $\sin(\omega t+\Delta\phi)$ and integrating over one period of oscillation, $T=\frac{2\pi}{\omega}$, one has

$$2L\frac{d}{dt}\Delta A - A\frac{\partial R_a}{\partial A}\Delta A = \frac{\omega}{\pi}\int_{t-\frac{\pi}{\omega}}^{t+\frac{\pi}{\omega}}(v_a(t^{'})+v_L(t^{'}))\cos(\omega t^{'}+\Delta\phi)dt^{'} = v_{ac}+v_{Lc}$$

$$2LA\frac{d}{dt}\Delta\phi + A\frac{\partial X_a}{\partial A}\Delta A = -\frac{\omega}{\pi}\int_{t-\frac{\pi}{\omega}}^{t+\frac{\pi}{\omega}}(v_a(t^{'})+v_L(t^{'}))\sin(\omega t^{'}+\Delta\phi)dt^{'} = -(v_{as}+v_{Ls}) \quad .$$

Here, $v_{ac}(v_{Lc})$ and $v_{as}(v_{Ls})$ are the cosine and sine components of the internal (external) noise voltages,

$$v_a(t) = v_{ac}\cos(\omega t+\Delta\phi) + v_{as}\sin(\omega t+\Delta\phi)$$

$$v_L(t) = v_{Lc}\cos(\omega t+\Delta\phi) + v_{Ls}\sin(\omega t+\Delta\phi)$$

A resistive saturation parameter s and reactive saturation parameter r are introduced and defined by

$$s \equiv -\frac{A}{R_a(A)}\frac{\partial R_a}{\partial A} \quad ,$$
$$r \equiv \frac{A}{R_a(A)}\frac{\partial X_a}{\partial A} \quad .$$

If one uses $-R_a = \dfrac{-R_0}{1+\beta A^2}$ for saturated gain, the resistive saturation parameter is

$$s = \frac{2\beta A^2}{1+\beta A^2} = \begin{cases} 0 & : \beta A^2 \ll 1 \quad \text{(just above threshold)} \\ 2 & : \beta A^2 \gg 1 \quad \text{(far above threshold)} \end{cases}$$

$$2L\frac{d}{dt}\Delta A - A\frac{\partial R_a}{\partial A}\Delta A = \frac{\omega}{\pi}\int_{t-\frac{\pi}{\omega}}^{t+\frac{\pi}{\omega}}(v_a(t^{'}) + v_L(t^{'}))\cos(\omega t^{'} + \Delta\phi)dt^{'} = v_{ac} + v_{Lc}$$

$$2LA\frac{d}{dt}\Delta\phi + A\frac{\partial X_a}{\partial A}\Delta A = -\frac{\omega}{\pi}\int_{t-\frac{\pi}{\omega}}^{t+\frac{\pi}{\omega}}(v_a(t^{'}) + v_L(t^{'}))\sin(\omega t^{'} + \Delta\phi)dt^{'} = -(v_{as} + v_{Ls}) \quad .$$

$$\Longrightarrow \quad \frac{d}{dt}\Delta A + \frac{sR_L}{2L}\Delta A = \frac{1}{2L}(v_{ac} + v_{Lc})$$

$$\frac{d}{dt}\Delta\phi + \frac{rR_L}{2LA}\Delta A = -\frac{1}{2LA}(v_{as} + v_{Ls})$$

$$v_a(t) = v_{ac}\cos(\omega t + \Delta\phi) + v_{as}\sin(\omega t + \Delta\phi)$$

$$v_L(t) = v_{Lc}\cos(\omega t + \Delta\phi) + v_{Ls}\sin(\omega t + \Delta\phi)$$

The amplitude noise ΔA $\longleftarrow$ cosine components of the internal and external noise voltages

suppressed with the decay rate $\frac{s}{2}\frac{R_L}{L} = \frac{s}{2}\left(\frac{\omega}{Q_e}\right)$

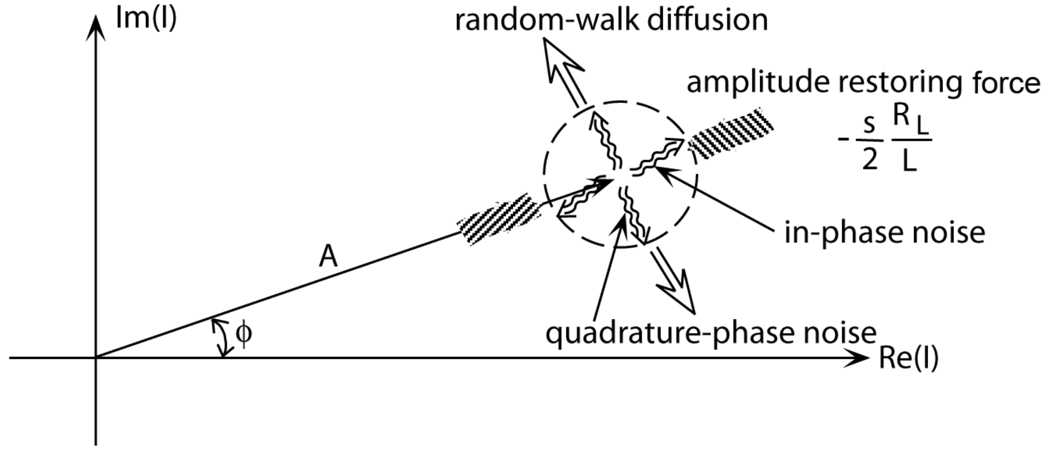
where Q_e is a Q factor due to output coupling loss and $\frac{\omega}{Q_e}$ is the associated photon decay rate. The gain saturation represented by the resistive saturation parameter s operates as a restoring force for the amplitude.

The phase noise $\Delta\phi$ $\longleftarrow$ sine components of the internal and external noise voltages

driven by the amplitude noise via the reactive saturation parameter.

no restoring force for the phase.

the phase of a free-running oscillator diffuses via a random walk, while the amplitude is stabilized to its steady-state value. This effect is shown schematically in Fig. 10.5.



$$\frac{d}{dt}\Delta A + \frac{sR_L}{2L}\Delta A = \frac{1}{2L}(v_{ac} + v_{Lc})$$

$$\frac{d}{dt}\Delta\phi + \frac{rR_L}{2LA}\Delta A = -\frac{1}{2LA}(v_{as} + v_{Ls})$$

$$\xrightarrow{\text{F.T.}} \quad S_{\Delta A}(\Omega) = \frac{1}{s^2 R_L^2} \cdot \frac{S_{ac}(\Omega) + S_{Lc}(\Omega)}{1 + (\Omega/\Omega_c)^2} \quad ,$$

$$S_{\Delta\phi}(\Omega) = \frac{\left(\frac{\omega}{Q_e}\right)^2}{4A^2 R_L^2 \Omega^2}\left[S_{as}(\Omega) + S_{Ls}(\Omega)\right] + \frac{\left(\frac{r}{s}\right)^2 \left(\frac{\omega}{Q_e}\right)^2}{4A^2 R_L^2 \Omega^2} \cdot \frac{S_{ac}(\Omega) + S_{Lc}(\Omega)}{1 + (\Omega/\Omega_c)^2} \quad .$$

,where the noise bandwidth Ω_c is given by

$$\Omega_c = \frac{s}{2}\left(\frac{\omega}{Q_e}\right) = \begin{cases} 0 & : \beta A^2 \ll 1 \quad \text{(just above threshold)} \\ \dfrac{\omega}{Q_e} & : \beta A^2 \gg 1 \quad \text{(far above threshold)} \end{cases}$$

At far above threshold, the amplitude noise spectrum is reduced to

$$S_{\Delta A}(\Omega) = \frac{1}{4R_L^2} \cdot \frac{S_{ac}(\Omega) + S_{Lc}(\Omega)}{1 + \left(\Omega/\frac{\omega}{Q_e}\right)^2}$$

Two limiting cases:

(1) Quantum Limit: $S_{ac}(\Omega) = S_{Lc}(\Omega) = 4\hbar\omega R_L$

$$\longrightarrow \quad S_{\Delta A}(\Omega) = \frac{\frac{2\hbar\omega}{R_L}}{1 + \left(\Omega/\frac{\omega}{Q_e}\right)^2}$$

(2) Thermal Limit: $S_{ac}(\Omega) = S_{Lc}(\Omega) = 8k_B T R_L$

$$\longrightarrow \quad S_{\Delta A}(\Omega) = \frac{\frac{4k_B T}{R_L}}{1 + \left(\Omega/\frac{\omega}{Q_e}\right)^2}$$

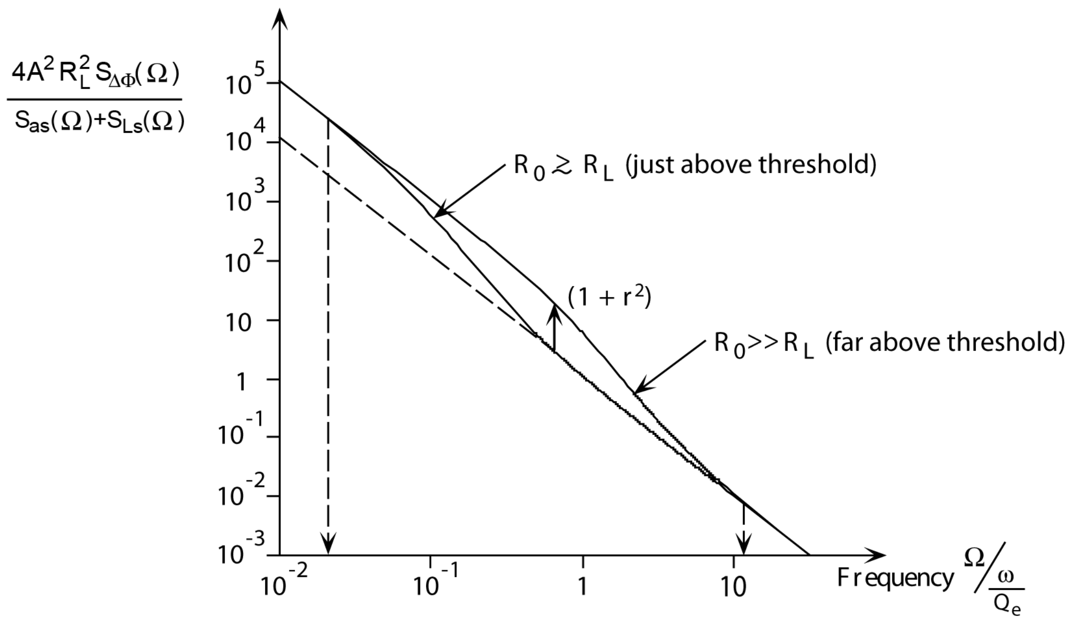
Remember that $\frac{1}{2}LA^2 = \hbar\omega n$,where n = the # of oscillator photons.

$$S_{\Delta n}(\Omega) = \left(\frac{LA}{\hbar\omega}\right)^2 S_{\Delta A}(\Omega) = \begin{cases} 4n \cdot \dfrac{\left(\frac{\omega}{Q_e}\right)^{-1}}{1 + \left(\Omega/\frac{\omega}{Q_e}\right)^2} & : \text{Quantum limit} \\ 4n\left(\dfrac{2k_B T}{\hbar\omega}\right)\dfrac{\left(\frac{\omega}{Q_e}\right)^{-1}}{1 + \left(\Omega/\frac{\omega}{Q_e}\right)^2} & : \text{Thermal limit} \end{cases}$$

Therefore, the variance in the photon number $\langle \Delta n^2 \rangle$ is calculated by the Parseval theorem:

$$\langle \Delta n^2 \rangle \equiv \int_0^\infty S_{\Delta n}(\Omega)\frac{d\Omega}{2\pi} = \begin{cases} n & : \text{Quantum limit} \quad = \text{Poissonian} \\ 2nn_{\text{th}} & : \text{Thermal limit} \quad > \text{Poissonian} \end{cases}$$

The phase noise spectra of a laser oscillator



If we define an instantaneous frequency by

$$\Delta\omega(t) = \frac{d}{dt}\Delta\phi(t) \quad ,$$

the frequency noise spectrum is given by

$$\begin{aligned} S_{\Delta\omega}(\Omega) &= \Omega^2 S_{\Delta\phi}(\Omega) \\ &= \frac{\left(\frac{\omega}{Q_{\rm ex}}\right)^2}{4A^2R_L^2}\left[S_{as}(\Omega) + S_{Ls}(\Omega)\right] + \frac{\left(\frac{r}{s}\right)^2\left(\frac{\omega}{Q_{\rm ex}}\right)^2}{4A^2R_L^2} \cdot \frac{\left[S_{ac}(\Omega) + S_{Lc}(\Omega)\right]}{1 + (\Omega/\Omega_c)^2} \quad . \end{aligned}$$

The frequency noise enhancement factor

$$\begin{aligned} \frac{r}{s} &\equiv \left(\frac{A}{R_a(A)}\frac{\partial X_a}{\partial A}\right) / \left(-\frac{A}{R_a(A)}\frac{\partial R_a}{\partial A}\right) \\ &= -\left(\frac{\partial X_a}{\partial A}\right) / \left(\frac{\partial R_a}{\partial A}\right) \\ &= -\left(\frac{\partial \chi_r}{\partial A_0}\right) /. \left(\frac{\partial \chi_i}{\partial A_0}\right) \quad . \end{aligned}$$

= a linewidth enhancement factor, Henry's α-parameter

Spectral Linewidth

The oscillator circuit impedance = $Z(\omega) = 2L\left[\frac{1}{2}\frac{\omega}{Q} + i(\omega - \omega_0^{'})\right]$

, where $\omega_0^{'} = \omega_0 - \frac{X_a(A)}{2L}$ = actual oscillation frequency

cold cavity resonant frequency

& the effective (or active) Q-value

$$\frac{\omega}{Q} \equiv \frac{\omega}{Q_e} - \frac{\omega}{Q_a} = \frac{R_L}{L} - \frac{R_a(A)}{L}$$

The spectral profile of the oscillating current

$$\overline{|i(\omega)|^2} = \frac{S_{v_a}(\omega_0^{'}) + S_{v_L}(\omega_0^{'})}{4L^2\left[\frac{1}{4}\left(\frac{\omega}{Q}\right)^2 + (\omega - \omega_0^{'})^2\right]}$$

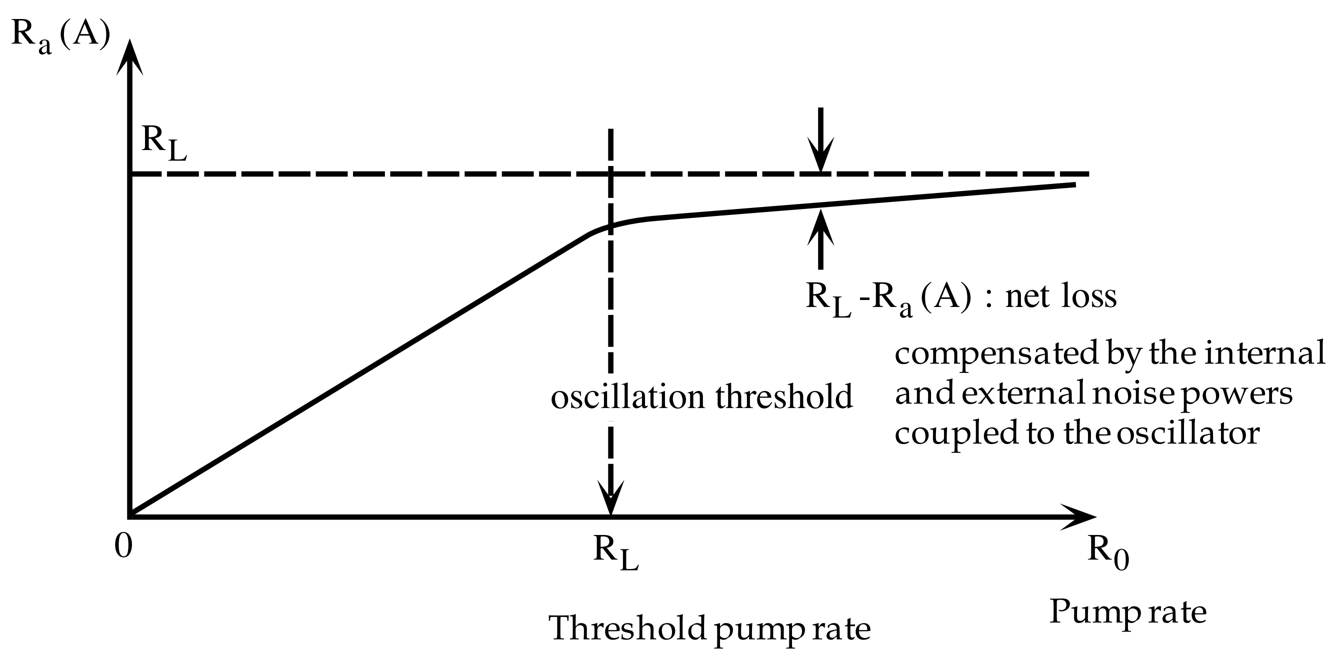
Lorentzian profile with FWHM $\Delta\omega_{1/2} = \frac{\omega}{Q}$

(1) Below the Oscillation Threshold

$$\Delta\omega_{1/2} = \frac{\omega}{Q_e}\left(1 - \frac{R_0}{R_L}\right)$$

gain saturation is negligible, $R_a(A) \simeq R_0$

Decreases linearly with the difference $(R_L - R_0)$



Spectral Linewidth

(2) Above the Oscillation Threshold

Let us first compute the emitted power

$$\begin{aligned} P_e &= R_L \int_0^\infty \overline{\left| i(\omega_0^{'}) \right|^2} \frac{d\omega}{2\pi} \\ &= \frac{R_L}{4L^2 \left(\frac{\omega}{Q}\right)} \left[S_{v_a}(\omega_0^{'}) + S_{v_L}(\omega_0^{'}) \right] \quad . \end{aligned}$$

(2-A) Quantum Limit: $S_{v_a}(\omega_0^{'}) = S_{v_L}(\omega_0^{'}) = 2\hbar\omega_0^{'} R_L$

$$\Delta\omega_{1/2} \equiv \frac{\omega}{Q} = \frac{\hbar\omega_0^{'} \left(\frac{\omega}{Q_e}\right)^2}{P_e} \quad .$$

If one uses $\Delta\nu_{1/2} = \frac{1}{2\pi}\Delta\omega_{1/2}$ and $\Delta\nu_c = \frac{1}{2\pi}\left(\frac{\omega}{Q_e}\right)$, one obtains

$$\Delta\nu_{1/2} = \frac{2\pi h\nu_0^{'} (\Delta\nu_c)^2}{P_e} \quad .$$

This is the Schawlow-Townes linewidth for a laser oscillator.

(2-B) Thermal Limit: $S_{v_a}(\omega_0^{'}) = S_{v_L}(\omega_0^{'}) = 4k_B T R_L$

$$\Delta\omega_{1/2} = \frac{2k_B T \left(\frac{\omega}{Q_e}\right)^2}{P_e} \quad ,$$

$$\Delta\nu_{1/2} = \frac{4\pi k_B T (\Delta\nu_c)^2}{P_e} \quad .$$

This is the Shimoda-Takahashi-Townes linewidth for a maser oscillator.

Spontaneous Emission Coupling Efficiency

Recall $P_{\text{out}} = \hbar\omega A_0^2 \left(\frac{\omega}{Q_{\text{ex}}}\right) = \frac{1}{2} R_L A^2$ and $A^2 = \frac{1}{\beta}\left(\frac{R_0}{R_L} - 1\right)$

, where β = saturation parameter

The spectral linewidth $\Delta\omega_{1/2}$ in the quantum limit $\Delta\omega_{1/2} = \left(\frac{\omega}{Q_e}\right) \frac{2\beta\frac{\hbar\omega_0'}{L}}{\left(\frac{R_0}{R_L} - 1\right)}$

$2\beta\hbar\omega_0'/L$ = the inverse of the saturation photon number

$$n_s = \frac{L}{2\hbar\omega_0'\beta}$$

The spontaneous emission coupling efficiency is defined as the fractional rate of spontaneous emission coupled into a single laser mode out of the total spontaneous emission rate.

$$\xi = \frac{1}{n_s} = \frac{2\hbar\omega_0'\beta}{L}$$

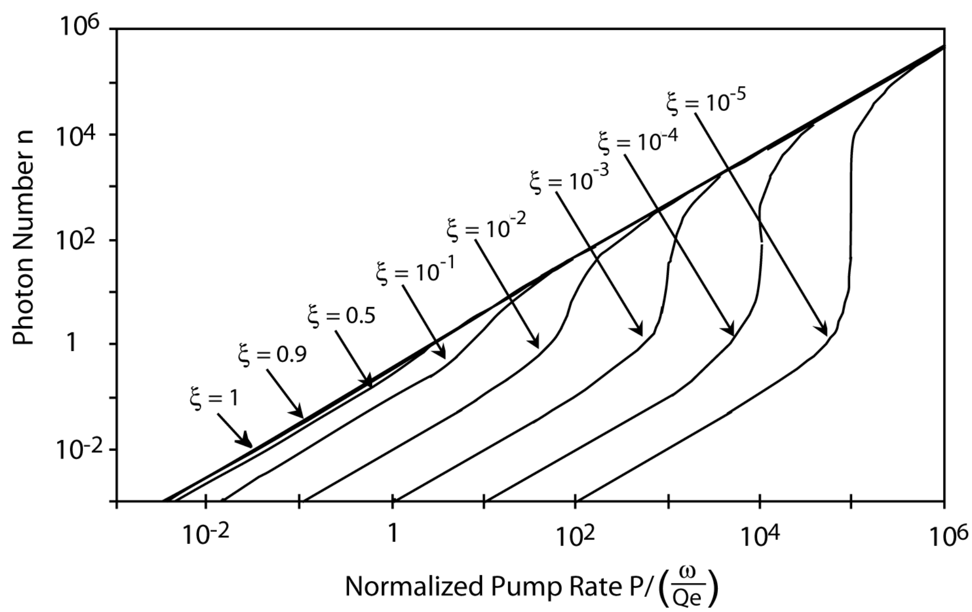
Laser threshold definition: an average photon number of a laser mode = spontaneous emission rate = stimulated emission rate

$$n = \xi P_{\text{th}} \tau_{\text{ph}} = 1 \longrightarrow P_{\text{th}} = \frac{1}{\xi \tau_{\text{ph}}} = \frac{\left(\frac{\omega}{Q_e}\right)}{\xi}$$

Photon number

$$n = \xi P \tau_{\text{ph}} = \xi \frac{P}{\left(\frac{\omega}{Q_e}\right)} \qquad n = n_s\left(P/P_{\text{th}} - 1\right) = \frac{1}{\xi}\left(P/P_{\text{th}} - 1\right).$$

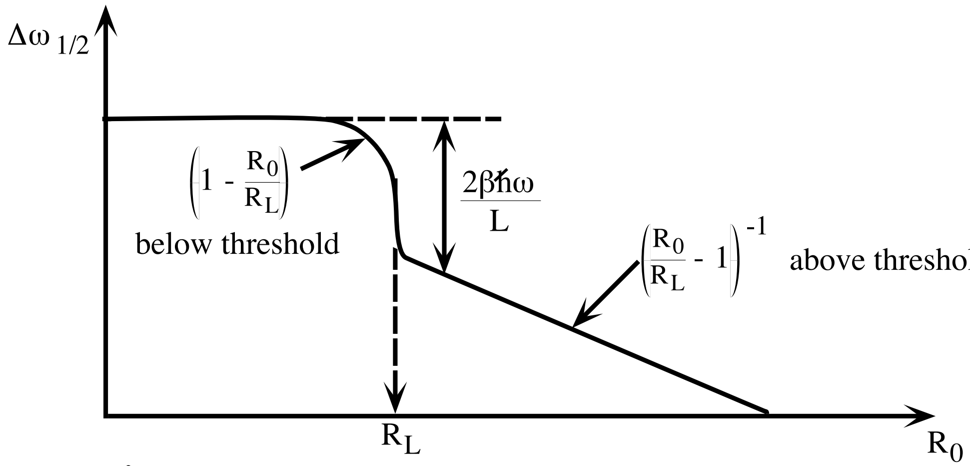
below threshold above threshold



Spectral linewidth

$$\Delta\omega_{1/2} = \frac{\omega}{Q_e}\left(1 - P/P_{\text{th}}\right)$$
below thresholdc

$$\Delta\omega_{1/2} = \frac{\left(\frac{\omega}{Q_e}\right)}{n} = \xi\left(\frac{\omega}{Q_e}\right)\frac{1}{P/P_{\text{th}} - 1}$$
above threshold



Gain saturation

above threshold : the total spontaneous emission rate is pinned at the threshold power (=gain saturation)
Below threshold : the total sponteneous emission rate is linearly increasing with the power