Student: John Rizzo

Course: CS590-A Algorithms

Instructor: Dr. William Hendrix

Due Date: January 31, 2025

Description: Homework 1 Algorithms

Problem 1

Definition 1. *$a \mid b$ ("a divides b") if and only if there exists some integer k such that $b = ak$. Equivalently, $a \mid b$ if and only if b has a remainder of 0 when divided by a (see question 2).*

1. Using the formal definition of divisibility above, prove that there exist positive integers a, b, and c such that $a \mid bc$, but $a \nmid b$ and $a \nmid c$.

   Proof. Let $a = 6$, $b = 2$, and $c = 3$. Then $bc = 6$ and $a \mid bc$ since $6 = 6 \cdot 1$. However, $a \nmid b$ and $a \nmid c$ since $2 = 6 \cdot 0 + 2$ and $3 = 6 \cdot 0 + 3$. ■

   Theorem 1. *The Division Algorithm. For any integers a and b where $b \neq 0$, there exist a unique pair of integers q and r such that $a = qb + r$ and $0 \leq r \leq b$. The integers q and r are known as the quotient and remainder of $a \div b$, respectively.*

2. Using the formal definition of the remainder above, prove that if n and m are positive integers such that n has a remainder of r when divided by m and $r < \sqrt{m}$, n^2 has a remainder of r^2 when divided by m.

   Proof. Let n, and m be positive integers such that $n \div m$ has a remainder r.

   By Theorem 1, $m = qn + r$ and $m = qn^2 + r^2$.

   $\sqrt{m} = \sqrt{qn^2 + r^2}$

   $\sqrt{m} = \sqrt{qn^2} + \sqrt{r^2}$

   $\sqrt{m} = n\sqrt{q} + r$.

3. Use the formal definition of Big-Oh to prove that if $f(n) = n^x + an^y$, where a, x, and y are positive integers such that $x > y$, $f(n) = O(n^x)$.

   Big-Oh is defined as $f(n) = O(g(n))$ if there exists a positive constant c and a positive integer n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.

   Solution. In this case, $f(n) = n^x + an^y$ and $g(n) = n^x$. Let us assume y = x. Then $f(n) = n^x + an^x$ which can be reduced to $f(n) = n^x(1 + a)$. Let $1 + a = c$ and $n_0 = 1$ to establish the upper bound. Then $f(n) = n^x(1 + a) \leq cn^x$ for all $n \geq n_0$. Therefore, $f(n) = O(n^x)$. ■

4. Use the formal definition of Big-Omega to prove that if $f_1(n)$, $f_2(n)$, $g_1(n)$, and $g_2(n)$ are functions such that $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, $f_1(n) + f_2(n) = \Omega(\max(g_1(n), g_2(n)))$.

   Solution. Big Omega is defined as $f(n) = \Omega(g(n))$ if there exists a positive constant c and a positive integer n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$.

   For $f_1(n)$, $n \geq c_1 n$ for say $n_0 = 0$ and $c_1 = 0.5$. The same is true for $f_2(n)$ and $g_2(n)$ All of the functions are separated by a constant factor. Given that $f_1(n)$ and $f_2(n)$ were both larger than $g_1(n)$ and $g_2(n)$ the sum of the two must be larger than the maximum of either $g_1(n)$ or $g_2(n)$.$\blacksquare$