PSTAT160A Stochastic Processes

Section 2

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Problem 1 - Dobrow Q1.9

Assume that X is uniformly distributed on $\{1, 2, 3, 4\}$. If $X = x$, then Y is uniformly distributed on $\{1, \dots, x\}$. Find

(a) $\mathbb{P}(Y = 2 \mid X = 2)$

(b) $\mathbb{P}(Y = 2)$

(c) $\mathbb{P}(X = 2 \mid Y = 2)$

(d) $\mathbb{P}(X = 2)$

(e) $\mathbb{P}(X = 2, Y = 2)$

Solution

(a) Since $Y \mid X = 2 \sim Unif(1, 2)$, then $\mathbb{P}(Y = 2 \mid X = 2) = \frac{1}{2}$

(b) The marginal probability mass function of Y can be computed as follows

$$\mathbb{P}(Y = 2) = \sum_{x=1}^{4} \mathbb{P}(Y = 2, X = x) = \sum_{x=1}^{4} \mathbb{P}(Y = 2 \mid X = x)\mathbb{P}(X = x)$$

However, notice that $\mathbb{P}(Y = 2 \mid X = 1) = 0$ and that $\mathbb{P}(X = x) = \frac{1}{4}$ for all $x \in \{1, 2, 3, 4\}$. Thus,

$$\begin{aligned}
\mathbb{P}(Y = 2) &= \mathbb{P}(Y = 2 \mid X = 2)\mathbb{P}(X = 2) + \mathbb{P}(Y = 2 \mid X = 3)\mathbb{P}(X = 3) + \mathbb{P}(Y = 2 \mid X = 4)\mathbb{P}(X = 4) \\
&= \frac{1}{4}\Big(\mathbb{P}(Y = 2 \mid X = 2) + \mathbb{P}(Y = 2 \mid X = 3) + \mathbb{P}(Y = 2 \mid X = 4)\Big) \\
&= \frac{1}{4}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{13}{48}
\end{aligned}$$

(c) By Bayes Theorem, $\mathbb{P}(X = 2 \mid Y = 2) = \dfrac{\mathbb{P}(Y = 2 \mid X = 2)\mathbb{P}(X = 2)}{\mathbb{P}(Y = 2)} = \dfrac{\frac{1}{2} \cdot \frac{1}{4}}{\frac{13}{48}} = \dfrac{6}{13}$

(d) X is uniformly distributed on $\{1, 2, 3, 4\}$. Hence, $\mathbb{P}(X = 2) = \frac{1}{4}$

(e) $\mathbb{P}(X = 2, Y = 2) = \mathbb{P}(Y = 2 \mid X = 2)\mathbb{P}(X = 2) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

Problem 2 — Dobrow Q1.14

Random variables X and Y have joint density function

$$f(x, y) = 4e^{-2x}, \quad \text{for } 0 < y < x < \infty.$$

(a) Find the conditional density of X given $Y = y$.

(b) Find the conditional density of Y given $X = x$. Describe the conditional distribution.

Solution

(a) To find the conditional density of X given $Y = y$, we use $f_{X|Y}(x \mid y) = \dfrac{f(x, y)}{f_Y(y)}$ where $f_Y(y)$ is the marginal density function of Y.
The marginal density Y is

$$f_Y(y) = \int_y^\infty 4e^{-2x} dx = 2e^{-2y}, \quad y > 0$$

Thus,

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{4e^{-2x}}{2e^{-2y}} = 2e^{-2(x-y)}, \quad x > y > 0$$

(b) To find the conditional density of Y given $X = x$, we use $f_{Y|X}(y \mid x) = \dfrac{f(x, y)}{f_X(x)}$ where $f_X(x)$ is the marginal density function of X.
The marginal density of X is

$$f_X(x) = \int_0^x 4e^{-2x}\, dy = 4x\, e^{-2x}, \quad x > 0.$$

Therefore,

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{4e^{-2x}}{4xe^{-2x}} = \frac{1}{x}, \quad x > 0$$

which is the uniform distribution on $(0, x)$.

Problem 3 - Dobrow Q1.18

From the definition of conditional expectation given an event, show that

$$\mathbb{E}(I_B \mid A) = \mathbb{P}(B \mid A)$$

Solution

By definition (page 22 of the book), we have

$$\mathbb{E}(I_B \mid A) = \frac{\mathbb{E}(I_B I_A)}{\mathbb{P}(A)} = \frac{\mathbb{E}(I_{A\cap B})}{\mathbb{P}(A)} = \frac{\mathbb{E}(I_{A\cap B})}{\mathbb{P}(A)}$$

Notice that

$$\mathbb{E}(I_{A\cap B}) = 1 \cdot \mathbb{P}(A \cap B) + 0 \cdot \mathbb{P}((A \cap B)^c) = \mathbb{P}(A \cap B).$$

Hence,

$$\mathbb{E}(I_B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \mathbb{P}(B \mid A) \quad \square$$

Problem 4 - Dobrow Q1.33

R: Cards are drawn from a standard deck, with replacement, until an ace appears. Simulate the mean and variance of the number of cards required.

Solution

We will proceed with the structure given by the book:

```r
set.seed(2) #Reproducibility
trials<-10000
simlist<-numeric(trials)
#Create the deck of cards
cards <-paste0(
    rep(c("A", "2", "3", "4", "5", "6", "7", "8", "9", "10", "J", "Q", "K"), each = 4),
    rep(c("♠", "♥", "♦", "♣"), times = 13)
  )
# function to check if card is an ace
is_ace <- function(x) {
  x %in% c("A♠", "A♥", "A♦", "A♣")
  }

# repeat simulation
for (i in 1:trials) {
  ace <- 0
  count <- 0 #Counts cards until an ace appears
  while (ace == 0) {
    pick <- sample(cards, 1, replace=TRUE) #We are drawing with replacement
    count <- count + 1
    if (is_ace(pick)==TRUE){
      ace <- 1 #Breaks the while
    }
```

```r
  }
  simlist[i] <- count
}

cat(sprintf("Estimated mean: %.4f \n", mean(simlist)))
```

```
Estimated mean: 13.0261
```

```r
cat(sprintf("Estimated variance: %.4f \n", var(simlist)))
```

```
Estimated variance: 157.3712
```