PSTAT160A Stochastic Processes

Section 1 - Solutions

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Problem 1 - Dobrow Q1.1

For the following scenarios identify a stochastic process $\{X_t\}_{t\in I}$ describing: (i) X_t in context, (ii) state space, and (iii) index set. State whether the state space and index set are discrete or continuous.

- 1. From day to day the weather in International Falls, Minnesota is either rain, clear or snow.
- 2. At the end of each year, a 4-year college student either advances in grade, repeats their grade, or drops out.
- 3. Seismologists record daily earthquake magnitudes in Chile. The largest recorded earthquake in history was the Valdivia, Chile earthquake on May 22nd 1960, which had a magnitude of 9.5 on the Richter scale.
- 4. Data is kept on the circumferences of trees in an arboretum. The arboretum covers a two square-mile area. *Assume that the data is for a single time point.*
- 5. Starting Monday morning at 9 a.m., as students arrive to class, the teacher records student arrival times. The class has 30 students and lasts for 60 minutes.
- 6. A card player shuffles a standard deck of cards by the following method: the top card of the deck is placed somewhere in the deck at random. The player does this 100 times to mix up the deck.

Solution

- 1. X_t denotes the weather on day t; discrete state space $\mathcal{S} := \{\text{Rain}, \text{Clear}, \text{Snow}\}$; discrete index set $I := \{0, 1, 2, \dots\}$.
- 2. X_t denotes the student's status in year t; discrete state space $S := \{\text{Drop}, \text{Fresh}, \text{Sophmore}, \text{Junior}, \text{Senior}, \text{Graduate}\};$ discrete index set $I := \{0, 1, 2, \dots\}$.
- 3. X_t denotes the earthquake magnitude on day t; continuous state space $\mathcal{S}:=\mathbb{R}^+$, discrete index set $I:=\{0,1,2,\dots\}.$
- 4. X_i denotes the circumference of trees in an arboretum, continuous state space $\mathcal{S} := \mathbb{R}^+$, continuous index set $I := [0,2] \times [0,2]$ denoting the coordinate location of the tree.
- 5. X_i denotes the lateness of students from the start of class, continuous state space $\mathcal{S}:=[0,60]$, discrete index set $I:=\{1,\ldots,30\}$ denoting the student.
- 6. X_t denotes the ordering of the deck of cards, discrete state space $\#\mathcal{S}=52!$, discrete index set $I:=\{0,1,\ldots,100\}$ denoting number of shuffles.

Problem 2 - Dobrow Q1.2

A regional insurance company insures homeowners against flood damage. Half of their policyholders are in Florida, 30% in Louisiana, and 20% in Texas. Company actuaries give the estimates in Table 1 for the probability that a policyholder will file a claim for flood damage over the next year.

- 1. Find the probability that a random policyholder will file a claim for flood damage next year.
- 2. A claim was filed. Find the probability that the policyholder is from Texas.

Table 1: Probability of Claim for Flood Damage

State	Florida	Louisiana	Texas
Probability of Claim	0.03	0.015	0.02

Solution

For clarity we define the following events: C the policy holder makes a claim; T the holder is from Texas, L the holder is from Louisiana and F policy is from Florida. We assume that events T, L, F are independent (policy holders can only be from one state) and we note that

$$\mathbb{P}(\Omega) = \mathbb{P}(T \cup L \cup F) = \mathbb{P}(T) + \mathbb{P}(L) + \mathbb{P}(F) = 0.5 + 0.3 + 0.2 = 1.$$

For (1) using independence we can compute the weighted average

$$\begin{split} \mathbb{P}(C) &= \mathbb{P}((C \cap F) \cup (C \cap L) \cup (C \cap T) \\ &= \mathbb{P}(C \cap F) + \mathbb{P}(C \cap L) + \mathbb{P}(C \cap T) \\ &= \mathbb{P}(C|F)\mathbb{P}(F) + \mathbb{P}(C|L)\mathbb{P}(L) + \mathbb{P}(C|T)\mathbb{P}(T) \\ &= 0.03(0.5) + 0.015(0.3) + 0.02(0.2) \\ &= 0.0235. \end{split}$$

For (2) we compute using Bayes rule

$$\mathbb{P}(T|C) = \frac{\mathbb{P}(T \cap C)}{\mathbb{P}(C)}$$

$$= \frac{\mathbb{P}(C|T)\mathbb{P}(T)}{\mathbb{P}(C)}$$

$$= \frac{0.02(0.2)}{0.0235}$$

$$= 0.170212766.$$

Problem 3 - Dobrow Q1.3

Let B_1, \dots, B_k be a partition of the sample space. For events A and C, prove the law of total probability for conditional probability

$$\mathbb{P}(A|C) = \sum_{i=1}^k \mathbb{P}(A|B_i \cap C)\mathbb{P}(B_i|C).$$

Solution

PROOF: From the definition of conditional probability and noting that $C = \bigcup_i (B_i \cap C)$ through independence of the partition regions we can compute

$$\begin{split} \sum_{i=1}^k \mathbb{P}(A|B_i \cap C) \cdot \mathbb{P}(B_i|C) &= \sum_{i=1}^k \frac{\mathbb{P}(A \cap (B_i \cap C))}{\mathbb{P}(B_i \cap C)} \cdot \frac{\mathbb{P}(B_i \cap C)}{\mathbb{P}(C)} \\ &= \sum_{i=1}^k \frac{\mathbb{P}(A \cap (B_i \cap C))}{\mathbb{P}(C)} \\ &= \frac{\mathbb{P}\left(A \cap \bigcup_{i=1}^k (B_i \cap C)\right)}{\mathbb{P}(C)} \\ &= \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \\ &= \mathbb{P}(A|C). \end{split}$$

Problem 4 - Dobrow Q1.4

(See Problem 2) Among all policyholders who live within five miles of the Atlantic Ocean, 75% live in Florida, 20% live in Louisiana, and 5% live in Texas. For those who live close to the ocean the probabilities of filing a claim increase, as given in Table 2.

Table 2: Table: Probability of claim for those within five miles of Atlantic Coast.

State	Florida	Louisiana	Texas
Probability of Claim	0.10	0.06	0.06

Assume that a policyholder lives within five miles of the Atlantic coast. Use the law of total probability for conditional probability in Exercise 1.3 to find the chance they will file a claim for flood damage next year.

Solution

Using definitions from Problem 2 and further defining event A holder lives within five miles of the Atlantic coast we apply the law of total probability for conditional probability to compute

$$\mathbb{P}(C|A) = \mathbb{P}(C|F \cap A)\mathbb{P}(F|A) + \mathbb{P}(C|L \cap A)\mathbb{P}(L|A) + \mathbb{P}(C|T \cap A)\mathbb{P}(T|A)$$
$$= 0.10(0.75) + 0.06(0.2) + 0.06(0.05)$$
$$= 0.09.$$

Problem 5 - Dobrow Q1.5

Two fair six-sided dice are rolled. Let X_1, X_2 be the outcomes of the first and second die, respectively.

- 1. Find the conditional distribution of X_2 given that $X_1 + X_2 = 7$.
- 2. Find the conditional distribution of X_2 given that $X_1 + X_2 = 8$.

Solution

For (1) from the definition of conditional probability we compute

$$\begin{split} p_{X_2|X_1+X_2=7}(x) &= \mathbb{P}(X_2=x|X_1+X_2=7) \\ &= \frac{\mathbb{P}(X_1+X_2=7|X_2=x)\mathbb{P}(X_2=x)}{\mathbb{P}(X_1+X_2=7)} \\ &= \frac{\mathbb{P}(X_1=7-x)\mathbb{P}(X_2=x)}{\mathbb{P}(X_1+X_2=7)} \\ &= \begin{cases} \frac{\frac{1}{6}\cdot \frac{1}{6}}{36} = \frac{1}{6} & \text{if } x \in \{1,\dots,6\}, \\ 0 & \text{otherwise.} \end{cases} \end{split}$$

For (2) again from the definition of conditional probability we have

$$\begin{split} p_{X_2|X_1+X_2=7}(x) &= \frac{\mathbb{P}(X_1=8-x)\mathbb{P}(X_2=x)}{\mathbb{P}(X_1+X_2=8)} \\ &= \begin{cases} \frac{\frac{1}{6}\cdot\frac{1}{6}}{\frac{5}{36}} &= \frac{1}{5} & \text{if } x \in \{2,\dots,6\}, \\ & \text{otherwise.} \end{cases} \end{split}$$