PSTAT160A Stochastic Processes

Section 6 - Solutions

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Problem 1 - Dobrow 4.3

Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Assume that X and Y are independent. Use PGFs to find the distribution of $X + Y$.

Solution

The PGF of a random variable following a *Poisson Distribution* with rate λ is given by

$$M_X(s) := \mathbb{E}[s^X] = \exp(\lambda(s-1)).$$

Since X and Y are independent we have that

$$\begin{aligned} M_{X+Y}(s) &= M_X(s) \cdot M_Y(s) \\ &= \exp(\lambda(s-1)) \cdot \exp(\mu(s-1)) \\ &= \exp((\lambda+\mu)(s-1)). \end{aligned}$$

This is the MGF of a Poisson random variable with rate $\lambda + \mu$ which is therefore the distribution of $X + Y$ since the MGF uniquely determines the distribution.

Problem 2 - Dobrow 4.4

If X has a *Negative Binomial Distribution* with parameters r and p, then X can be written as the sum of r i.i.d. *Geometric* random variables with parameter p. Use this fact to find the PGF of X. Then, use the PGF to find the mean and variance of the negative binomial distribution.

Solution

Define a sequence $\{\xi_i\}_{i=1}^r$ of i.i.d. geometric random variables with PMF

$$p_{\xi_1}(k) := \mathbb{P}(\xi_1 = k) = (1-p)^k p; \;\; k \in \mathbb{N}_0,$$

The PGF is given by

$$G_{\xi_1}(s) = \mathbb{E}[s^{\xi_1}] = \frac{p}{1 - s + ps}.$$

From the independence of ξ_i we can therefore compute $X = \sum_{i=1}^{r} \xi_i = r\xi_1 \sim \text{NB}(r, p)$ and the associated PGF

$$\begin{aligned} G_X(s) &= G_{\sum_{i=1}^{r} \xi_1}(s) \\ &= (P_{\xi_1}(s))^r \\ &= \left(\frac{p}{1 - s + ps}\right)^r . \end{aligned}$$

The expectation is given by

$$\mathbb{E}[X] = G_X^{(1)}(0),$$

hence we first apply the Chain Rule to compute

$$\begin{aligned} G_X^{(1)}(s) &= \frac{d}{ds}\left(\frac{p}{1-(1-p)s}\right)^r \\ &= p^r \frac{d}{ds}\left(\frac{1}{1-(1-p)s}\right)^r \\ &= p^r r \left(\frac{1}{1-(1-p)s}\right)^{r-1} \cdot \frac{d}{ds}(1-(1-p)s)^{-1} \\ &= p^r r \left(\frac{1}{1-(1-p)s}\right)^{r-1} \cdot -(1-(1-p)s)^{-2} \cdot \frac{d}{ds}(1-(1-p)s) \\ &= p^r r \left(\frac{1}{1-(1-p)s}\right)^{r-1} \cdot -(1-(1-p)s)^{-2} \cdot -(1-p) \\ &= \frac{p^r r(1-p)}{(1-(1-p)s)^{r+1}} . \end{aligned}$$

Evaluating at $s = 1$ we have that $1 - (1 - p) = p$ hence

$$\mathbb{E}[X] = \frac{r(1-p)}{p} .$$

Similarly, the variance is given by

$$\text{Var}(X) = G_X^{(2)}(1) + G_X^{(1)}(1) - [G_X^{(1)}(1)]^2,$$

therefore we again apply the Chain Rule to compute

$$\begin{aligned} G_X^{(2)}(s) &= \frac{d}{ds}\frac{p^r r(1-p)}{(1-(1-p)s)^{r+1}} \\ &= p^r r(1-p) \cdot \frac{d}{ds}(1-(1-p)s)^{-r-1} \\ &= p^r r(1-p) \cdot -(r+1)(1-(1-p)s)^{-r-2} \cdot \frac{d}{ds}(1-(1-p)s) \\ &= p^r r(1-p) \cdot -(r+1)(1-(1-p)s)^{-r-2} \cdot -(1-p) \\ &= \frac{p^r r(1-p)^2(r+1)}{(1-(1-p)s)^{r+2}} . \end{aligned}$$

Evaluating at $s = 1$ we obtain

$$\begin{aligned} G_X^{(2)}(1) &= \frac{p^r r(1-p)^2(r+1)}{p^{r+2}} \\ &= \frac{r(1-p)^2(r+1)}{p^2}. \end{aligned}$$

Substituting back into our expression we have

$$\begin{aligned} \operatorname{Var}(X) &= \frac{r(r+1)(1-p)^2}{p^2} + \frac{r(1-p)}{p} - \left[\frac{r(1-p)}{p}\right]^2 \\ &= \frac{r(r+1)(1-p)^2 + rp(1-p) - r^2(1-p)^2}{p^2} \\ &= \frac{r(1-p)\{(r+1)(1-p) + p - r(1-p)\}}{p^2} \\ &= \frac{r(1-p)\{r - rp - p + 1 + p - r + rp\}}{p^2} \\ &= \frac{r(1-p)}{p^2}. \end{aligned}$$

Problem 3 - Dobrow 4.12

A branching process has offspring distribution $\alpha = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$. Find the following:

1. μ;
2. $G(s)$;
3. The extinction probability;
4. $G_2(s)$; and
5. $\mathbb{P}(Z_2 = 0)$.

Solution

Consider a branching process $\{Z_n\}$ defined

$$Z_n := \sum_{i=1}^{Z_{n-1}} Z_{n,i},$$

where $Z_{n,i}$ are i.i.d. with offspring distribution $p_k = \mathbb{P}(Z_{n,i} = k)$ for $k \in \mathbb{N}$ and $Z_0 = 1$. Specifically we have that

$$p_k = \begin{cases} \frac{1}{4} & k \in \{0, 1\} \\ \frac{1}{2} & k \in \{2\} \\ 0 & \text{o.w.} \end{cases}$$

(1) The mean of the offspring distribution is

$$\begin{aligned}\mu &= \mathbb{E}[Z_{n,i}] \\ &= \sum_k k \cdot p_k \\ &= \frac{1}{4} + \frac{2}{2} \\ &= \frac{5}{4}.\end{aligned}$$

(2) To find the generator of one offspring we compute

$$\begin{aligned}G(s) &= \mathbb{E}[s^{Z_{n,i}}] \\ &= \sum_k p_k s^k \\ &= \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2} \\ &= \frac{2s^2 + s + 1}{4}.\end{aligned}$$

(3) To find the extinction probability η we solve

$$\begin{aligned}\eta = G(\eta) &= \frac{2\eta^2 + \eta + 1}{4} \\ \implies 2\eta^2 - 3\eta + 1 &= 0 \\ \implies (2\eta - 1)(\eta - 1) &= 0,\end{aligned}$$

hence $\eta = \frac{1}{2}$ or $\eta = 1$. Since $\mu > 1$ we have that $\eta < 1$ thus $\eta = \frac{1}{2}$.

(4) We can compute

$$\begin{aligned}G_2(s) &= G(G(s)) \\ &= \frac{1}{4}[2G^2(s) + G(s) + 1].\end{aligned}$$

To avoid messy computations we first write

$$\begin{aligned}G^2(s) &= \left(\frac{2s^2 + s + 1}{4}\right)^2 \\ &= \frac{4s^4 + 2s^3 + 2s^2 + 2s^3 + s^2 + s + 2s^2 + s + 1}{16} \\ &= \frac{4s^4 + 4s^3 + 5s^2 + 2s + 1}{16}.\end{aligned}$$

Substituting back into our initial expression we find

$$\begin{aligned}G_2(s) &= \left\{\frac{\frac{4s^4 + 4s^3 + 5s^2 + 2s + 1}{8} + \frac{2s^2 + s + 1}{4} + 1}{4}\right\} \\ &= \frac{4s^4 + 4s^3 + 5s^2 + 2s + 1 + 4s^2 + 2s + s + 8}{32} \\ &= \frac{4s^4 + 4s^3 + 9s^2 + 4s + 11}{32}.\end{aligned}$$

(5) To compute $\mathbb{P}(Z_2 = 0)$ we have that

$$\begin{aligned}\mathbb{P}(Z_2 = 0) &= \frac{G_2^{(0)}(0)}{0!} \\ &= \frac{11}{32}.\end{aligned}$$

Problem 4 - Dobrow 4.18

Consider a branching process with offspring distribution

$$\alpha = (p^2, 2p(1-p), (1-p)^2); \;\; 0 < p < 1.$$

The offspring distribution is binomial with parameters 2 and $1 - p$. Find the extinction probability η.

Solution

Consider the new branching with offspring distribution

$$p_k = \begin{cases} p^2 & k = 0 \\ 2p(1-p) & k = 1 \\ (1-p)^2 & k = 2 \\ 0 & \text{o.w.} \end{cases}$$

First we compute

$$\begin{aligned}\mu &= 2p(1-p) + 2(1-p)^2 \\ &= 2p - 2p^2 + 2 - 4p + 2p^2 \\ &= 2 - 2p.\end{aligned}$$

Thus $\mu = 2 - 2p > 1 \implies p < \frac{1}{2}$ is the supercritical case, otherwise $\eta = 1$. When $p < \frac{1}{2}$ to find the extinction probability we first find the offspring generator as

$$\begin{aligned}G(s) &= \mathbb{E}[s^{Z_{n,i}}] \\ &= \sum_k s^k \cdot p_k \\ &= p^2 + 2ps(1-p) + s^2(1-p)^2 \\ &= p^2 + 2ps - 2p^2 s + s^2 - 2ps^2 + s^2 p^2.\end{aligned}$$

We can then compute the extinction probability by solving

$$\begin{aligned}&\eta = G(\eta) \\ &\eta = p^2 + 2p\eta(1-p) + \eta^2(1-p)^2 \\ &0 = \eta^2(1-p)^2 + [2p(1-p) - 1]\eta + p^2.\end{aligned}$$

From the quadratic formula we have that

$$
\begin{aligned}
\eta &= \frac{1-2p(1-p) \pm \sqrt{(2p(1-p)-1)^2 - 4(1-p)^2p^2}}{2(1-p)^2} \\
&= \frac{1-2p(1-p) \pm (2p-1)}{2(1-p)^2}.
\end{aligned}
$$

Case 1:

$$
\begin{aligned}
\eta &= \frac{1-2p+2p^2+2p-1}{2(1-p)^2} \\
&= \frac{p^2}{(1-p)^2} \\
&= \left(\frac{p}{1-p}\right)^2.
\end{aligned}
$$

Case 2:

$$
\begin{aligned}
\eta &= \frac{1-2p+2p^2-2p+1}{2(1-p)^2} \\
&= \frac{(p-1)^2}{(1-p)^2} \\
&= 1.
\end{aligned}
$$

We therefore summarize by giving the extinction probability as

$$
\eta = \begin{cases} \left(\frac{p}{1-p}\right)^2 & 0 < p \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq p < 1. \end{cases}
$$

Problem 5 - Dobrow 4.30

`R`: Simulate the branching process in Problem 3 (Dobrow 4.12). Use your simulation to estimate the extinction probability e.

Solution

We simulate the process in `R` as follows:

```
# branching process simulation
bp_sim <- function(n,k=c(0,1,2),p=c(0.25,0.25,0.5)){
    z_i = 1
    for(i in 1:n){
        if(z_i == 0){
            return(0)
            break
        } else {
            xi = sample(k,z_i,T,p)
            z_i = sum(xi)
```

```
        }
    }
    if(z_i==0){
        return(0)
    } else {
        return(1)
    }
}
```

We can then use Monte-Carlo approximation to estimate the extinction probability:

```
set.seed(49)
test = replicate(n=1000,bp_sim(n=50))
mean(test)
```

```
[1] 0.509
```