PSTAT160A Stochastic Processes
Section 5

Authors: *Denisse Alejandra Escobar Parra, John Inston*
Date: *November 4, 2025*

Problem 1 - Dobrow 3.29

Consider the Markov Chain (MC) with transition probability matrix

$$P = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \\ 0 & 0.2 & 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}.$$

Identify the communication classes. Classify the states as recurrent or transient, and determine the period of each state.

Solution

Recall that states i, j communicate with one another if $i \leftrightarrow j$, that is there is a non-zero probability of moving from i to j and from j to i. States that communicate with one another form communication classes. To identify the communication classes we can produce the Markov chain state diagram

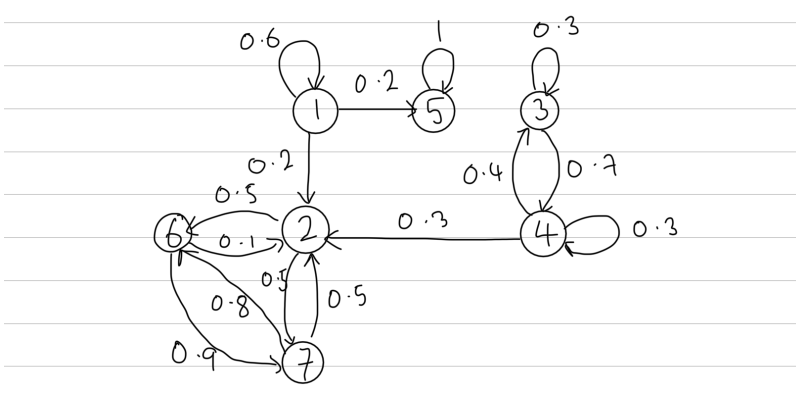

Figure 1: State Diagram

We have the following 4 communication classes: 1. $\{2, 6, 7\}$ 2. $\{3, 4\}$ 3. $\{1\}$ 4. $\{5\}$

State 5 is absorbing and is therefore recurrent. State 1, 3 and 4 are transient as eventually the process will leave them and never return. The states 2, 6 and 7 are recurrent.

The states are all aperiodic (i.e. have period 1).

Problem 2 - Dobrow 3.46

Given a MC with transition matrix P and stationary distribution π, the time reversal is a Markov chain with transition matrix $\tilde{P}$ defined by

$$\tilde{P_{i,j}} = \frac{\pi_j P_{j,i}}{\pi_i},$$

for all $i, j \in \mathcal{S}$.

1. Show that a Markov chain with transition matrix P is reversible if and only if $P = \tilde{P}$.
2. Show that the time reversal Markov chain has the same stationary distribution as the original one.

Solution

(1) *Proof:* The result follows almost immediately. $(\implies)$ Assuming reversibility we have that

$$P_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i} = \tilde{P}_{i,j}.$$

$(\impliedby)$ Assuming that $P = \tilde{P}$ we have that

$$\pi_i P_{i,j} = \pi_i \tilde{P}_{i,j} = \pi_i \frac{\pi_j P_{j,i}}{\pi_i} = \pi_j P_{j,i}.$$

□

(2) *Proof:* Since π is a stationary distribution we have that

$$\pi = \pi P \implies \pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{i,j}.$$

We can therefore compute

$$\begin{aligned}
(\pi \tilde{P})_j &= \sum_{i \in \mathcal{S}} \pi_i \tilde{P}_{i,j} \\
&= \sum_{i \in \mathcal{S}} \frac{\pi_i \cdot \pi_j P_{j,i}}{\pi_i} \\
&= \pi_j \sum_{i \in \mathcal{S}} P_{j,i} \\
&= \pi_j.
\end{aligned}$$

□

Problem 3 - Dobrow 3.52

The board for a modified Snakes and Ladder game is shown in Figure 1 below. The game is played with a tetrahedron (4-sided) die.

1. Find the expected length of the game.
2. Assume that the player is on square 6. Find the probability that they will find themselves at square 3 before finishing the game.

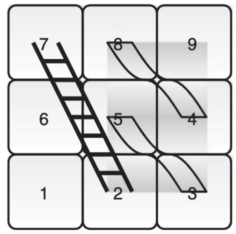

Figure 2: Modified snakes and ladders.

Solution

We label the squares of the board as $0, 1, 2, \dots, 9$, where 0 represents the starting position before the first move, and 9 is the terminal (absorbing) state. The game is played using a fair tetrahedral die, so the moves $+1, +2, +3, +4$ each occur with probability $1/4$. After moving, we apply the ladder or snake shown in the figure.

The ladder/snake transitions are:

$$2 \to 7, \qquad 5 \to 3, \qquad \& \qquad 8 \to 4.$$

Notice that for the game to end, a player needs to land exactly on 9. The player who accomplishes this first, wins. The transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(1) The only absorbing state is 9, thus, the transient states are:

$$\mathcal{T} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$$

We write the transition matrix P in canonical absorbing form:

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix},$$

where Q is the submatrix of transitions among transient states. The fundamental matrix is:

$$F = (I - Q)^{-1}.$$

Entry F_{ij} gives the expected number of visits to state j starting from state i before absorption. Therefore, the expected length of the game starting from state 0 is:

$$\mathbb{E}_0[T] = \sum_{j \in \mathcal{T}} F_{0j}.$$

Computing F, we get that

```
P <- matrix(c(
  0,   1/4, 0,   1/4, 1/4, 0,   0,   1/4, 0,   0,
  0,   0,   0,   1/2, 1/4, 0,   0,   1/4, 0,   0,
  0,   0,   0,   1/2, 1/4, 0,   1/4, 0,   0,   0,
  0,   0,   0,   1/4, 1/4, 0,   1/4, 1/4, 0,   0,
  0,   0,   0,   1/4, 1/4, 0,   1/4, 1/4, 0,   0,
  0,   0,   0,   0,   1/4, 0,   1/4, 1/4, 0,   1/4,
  0,   0,   0,   0,   1/4, 0,   1/4, 1/4, 0,   1/4,
  0,   0,   0,   0,   1/4, 0,   0,   1/2, 0,   1/4,
  0,   0,   0,   0,   0,   0,   0,   3/4, 0,   1/4,
  0,   0,   0,   0,   0,   0,   0,   0,   0,   1
), nrow = 10, byrow = TRUE)
dimnames(P) <- list(from = 0:9, to = 0:9)
Q<-P[1:9,1:9]
F<-solve(diag(9)-Q)
a<-F%*%rep(1,9)
a[1,1]
```

```
    0
8.625
```

Hence,

$$\boxed{\mathbb{E}_0[T] = 8.625.}$$

(2) Now we make both 3 and 9 absorbing states giving the new set of transient states:

$$\mathcal{T}' = \{0, 1, 2, 4, 5, 6, 7, 8\}.$$

Again, we write the transition matrix in absorbing form and compute:

$$F = (I - Q)^{-1}, \qquad B = FR,$$

where B is the absorption probability matrix. Row i of B gives the probabilities of being eventually absorbed in each absorbing state when starting from i.

```
Qb<-Q[-4,-4]
Fb<-solve(diag(8)-Qb)
Fb%*%P[-c(4,10),c(4,10)]
```

```
   to
             3        9
 0 0.609375 0.390625
 1 0.687500 0.312500
 2 0.687500 0.312500
 4 0.500000 0.500000
 5 0.250000 0.750000
 6 0.250000 0.750000
 7 0.250000 0.750000
 8 0.187500 0.812500
```

Extracting the row corresponding to state 6, we obtain:

$$\boxed{\mathbb{P}_6(\text{hit 3 before 9}) = \tfrac{1}{4}.}$$

Problem 4 - Dobrow 3.57

In repeated coin flips consider the set of all 3-element patterns

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Which patterns take the longest time, on average, to appear in repeated sampling? Which take the shortest?

Solution

For a given pattern $A = a_1 a_2 a_3$, define states by the longest suffix of the past that matches a prefix of A:

$$S_0 = \emptyset, \quad S_1 = a_1, \quad S_2 = a_1 a_2, \quad S_3 = a_1 a_2 a_3.$$

State S_3 is absorbing. Ordering the states (S_0, S_1, S_2, S_3), let Q be the 3×3 transient block of the transition matrix. The fundamental matrix is

$$F = (I - Q)^{-1} = I + Q + Q^2 + Q^3 + \cdots,$$

and the expected number of flips to first see A (starting in S_0) is

$$\mathbb{E}[T_A] = \sum_{j=0}^{2} F_{0j}.$$

There are only three types of length-3 patterns, determined by whether they overlap with themselves:

1. HHH, TTT
2. HTH, THT
3. HHT, HTT, THH, TTH

By symmetry $H \leftrightarrow T$, each pair in a class has the same expected time. We compute one representative per class.

Class I: Full overlap (we'll do the calculation with HHH)

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad I - Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

$$F = (I-Q)^{-1} = \begin{pmatrix} 8 & 4 & 2 \\ 6 & 4 & 2 \\ 4 & 2 & 2 \end{pmatrix}.$$

Therefore,

$$\mathbb{E}[T_{\text{HHH}}] = 8 + 4 + 2 = 14.$$

By symmetry, $\boxed{\mathbb{E}[T_{\text{TTT}}] = 14}$.

Class II: One-symbol overlap (we'll do the calculation with HTH)

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad I - Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

$$N = \begin{pmatrix} 4 & 4 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

So,

$$\mathbb{E}[T_{\text{HTH}}] = 4 + 4 + 2 = 10.$$

By symmetry, $\mathbb{E}[T_{\text{THT}}] = 10$.

Class III: No self-overlap (we'll do the calculation with HTT)

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad I - Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}.$$

$$N = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

Thus,

$$\mathbb{E}[T_{\text{HTT}}] = 2 + 4 + 2 = 8.$$

By symmetry,

$$\mathbb{E}[T_{\text{HHT}}] = \mathbb{E}[T_{\text{THH}}] = \mathbb{E}[T_{\text{TTH}}] = 8.$$

Problem 5 - Dobrow 3.64

The evolution of forest ecosystems in the United States and Canada is studied in Strigul et al. (2012) using Markov chains. Five-year changes in the state of the forest soil are modeled with a 12-state Markov chain. The transition matrix can be found in the `R`-script file `forest.R`. About how many years does it take for the ecosystem to move from state 1 to state 12?

Solution

```
source("forest.R")

P <- mat
Q<-P[1:11,1:11]
F<-solve(diag(11)-Q)
a<-F%*%rep(1,11)
5*round(a[1,],2)
```

```
[1] 54108.4
```