LESSON 5: Special Case on the Product of Binomial and Trinomial

Learning Objectives

At the end of the lesson, you will be able to:

1. Recognize special cases in the multiplication of binomials and trinomials that simplify to notable algebraic forms.
2. Apply the special product formula for multiplying a binomial with a trinomial (special case).
3. Identify and solve higher-degree polynomial expressions using structured approaches.

In algebra, certain combinations of binomials and trinomials, when multiplied, lead to surprisingly simplified results due to underlying algebraic structures. These special cases reduce otherwise complex polynomial multiplications to concise forms, significantly easing calculations and providing insightful results crucial for higher mathematical applications.

The term **"special case"** refers to instances where standard algebraic operations—like multiplying a binomial by a trinomial—result in simplified forms due to specific characteristics of the terms involved. These cases often align with well-known algebraic identities, such as the sum or difference of cubes, allowing for straightforward calculations that bypass the typical complexity of polynomial multiplication.

Identifying Whether an Equation is a Special Case

To determine if the multiplication of a binomial and a trinomial is a special case, look for:

- Form and Structure. The trinomial should resemble the expanded form of a binomial squared, minus, or plus any term that completes a square or other recognizable pattern.
- Coefficient Relationships. There may be specific relationships between the coefficients that suggest a simplification (e.g., symmetric coefficients in the trinomial).

Special Product Formula

When the sum of two terms is multiplied by the sum of their squares minus the product of these terms, the result is the sum of their cubes. Similarly, when the difference of two terms is multiplied by the sum of their squares plus the product of these terms, the result is the difference of their cubes.

In symbol,

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$
$$(a-b)(a^2+ab+b^2)=a^3-b^3$$

Step-by-Step Derivation Using the Distributive Property

1. Sum of Cubes

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a(a^2)+\mathrm{a}(-ab)+a(b^2)+b(a^2)+b(-ab)+b(b^2)\\ &= a^3-a^2b+ab^2+a^2b-ab^2+b^3\\ &= a^3+b^3\end{aligned}$$

Here, the middle terms cancel each other out, leaving only the cubes.

2. Difference of Cubes

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a(a^2)+\mathrm{a}(ab)+a(b^2)-b(a^2)-b(ab)-b(b^2)\\ &= a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &= a^3-b^3\end{aligned}$$

Similar cancellation occurs, resulting in the difference of cubes.

Steps to Solve the Product of Binomial and Trinomial (Special Case)

1. Identify the binomial and trinomial involved.
2. Apply the appropriate special product formula based on the structure of the trinomial.
3. Substitute values and simplify the expression.

Example 1: Calculate the value of $(x+1)(x^2-x+1)$.

Solution:

To calculate the product of binomial $(x+1)$ and trinomial (x^2-x+1), we'll use the formula for the Product of Binomial and Trinomial (Special Case):

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$

Here, $a=x$ and $b=1$

Substituting these values into the formula, we get:

$$(x+1)(x^2-(x)(1)+(1)^2)=x^3+1$$

Therefore, the value of $(x+1)(x^2-x+1)$ is $\boldsymbol{x^3+1}$.

Example 2: Calculate the value of $(5x-3)(25x^2+15x+9)$.

Solution:

To calculate the product of binomial $(5x-3)$ and trinomial $(25x^2+15x+9)$, we'll use the formula for the Product of Binomial and Trinomial (Special Case):

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$

Here, $a=5x$ and $b=-3$

Substituting these values into the formula, we get:

$$\begin{aligned}(5x-3)[(5x)^2-(5x)(-3)+(-3)^2] &= (5x)^3+(-3)^3\\ &= 125x^3-27\end{aligned}$$

Therefore, the value of $(5x - 3)(25x^2 + 15x + 9)$ *is* $\mathbf{125x^3 - 27}$.

Example 3: Calculate the value of $(7a + 1)(49a^2 - 7a + 1)$.

Solution:

To calculate the product of binomial $(7a + 1)$ *and trinomial* $(49a^2 - 7a + 1)$*, we'll use the formula for the Product of Binomial and Trinomial (Special Case):*

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$

Here, $a = 7a$ *and* $b = 1$

Substituting these values into the formula, we get:

$$(7a + 1)[(7a)^2 - (7a)(1) + (1)^2] = (7a)^3 + (1)^3$$
$$= 343a^3 + 1$$

Therefore, the value of $(7a + 1)(49a^2 - 7a + 1)$ *is* $\mathbf{343a^3 + 1}$.

In the above examples, the product between the sum/difference of two terms and the sum of their squares plus/minus the product of the given two terms is the sum/difference of their cubes.

SUMMARY

- Special cases occur when the structure of the trinomial complements the binomial in such a way that its product simplifies significantly.
- When the sum of two terms is multiplied by the sum of their squares minus the product of these terms, the result is the sum of their cubes. In symbol,
$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$
- Similarly, when the difference of two terms is multiplied by the sum of their squares plus the product of these terms, the result is the difference of their cubes. In symbol,
$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$