

# Better-Than-Chance Classification for Signal Detection

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## Abstract

[TODO]

## 1 Introduction

A common workflow in neuroimaging consists of fitting a classifier, and estimating its predictive accuracy using cross validation. Given that the cross validated accuracy is a random quantity, it is then common to test if the cross validated accuracy is significantly better than chance using a permutation test. Examples in the neuroscientific literature include Golland and Fischl [2003], Pereira et al. [2009], Varoquaux et al. [2016], and especially the recently popularized *multivariate pattern analysis* (MVPA) framework of Kriegeskorte et al. [2006]. This practice is also observed in very high profile publications in the genetics literature: Golub et al. [1999], Slonim et al. [2000], Radmacher et al. [2002], Mukherjee et al. [2003], Juan and Iba [2004], Jiang et al. [2008].

To fix ideas, we will adhere to a concrete example. In Gilron et al. [2016], the authors seek to detect brain regions which encode differences between vocal and non-vocal stimuli. Following the MVPA workflow, the localization problem is cast as a supervised learning problem: if the type of the stimulus can be predicted from the spatial activation pattern significantly better than chance, then a region is declared to encode vocal/non-vocal information. We call this an *accuracy test*, a.k.a. *class prediction*, or *pattern discrimination*.

This same signal detection task can be also approached as a two-group multivariate test. Inferring that a region encodes vocal/non-vocal information, is essentially inferring that the spatial distribution of brain activations is different given a vocal/non-vocal stimulus. As put in Pereira et al. [2009]:

26 ... the problem of deciding whether the classifier learned to dis-  
 27 criminate the classes can be subsumed into the more general ques-  
 28 tion as to whether there is evidence that the underlying distribu-  
 29 tions of each class are equal or not.

30 A practitioner may then call upon a two-group population test such as  
 31 Hotelling’s  $T^2$  [Anderson, 2003]. Alternatively, if the size of a brain re-  
 32 gion is large compared to the number of observations, so that the spatial  
 33 covariance cannot be fully estimated, then a high dimensional version of  
 34 Hotelling’s test can be called upon, such as in Schäfer and Strimmer [2005]  
 35 or Srivastava [2007]. For brevity, and in contrast to *accuracy tests*, we will  
 36 call any two-sample multivariate tests simply *population tests*, also termed  
 37 *class comparisons*. [TODO: rename to parameter test?]

38 At this point, it becomes unclear which is preferable: a population test or  
 39 an accuracy test? The former with a heritage dating back to Hotelling [1931],  
 40 and the latter being extremely popular, as the 959 citations<sup>1</sup> of Kriegeskorte  
 41 et al. [2006] suggest.

42 The comparison between location and accuracy tests was precisely the  
 43 goal of Ramdas et al. [2016], who compared the  $T^2$  population test to the  
 44 accuracy of *Fisher’s linear discriminant analysis* classifier (LDA). By com-  
 45 paring the rates of convergence of the powers to 1, Ramdas et al. [2016]  
 46 concluded that accuracy and population tests are rate equivalent.

47 Asymptotic relative efficiency measures (ARE) are typically used by statis-  
 48 ticians to compare between rate-equivalent test statistics [van der Vaart,  
 49 1998]. Ramdas et al. [2016] derive the asymptotic power functions of the  
 50 two test statistics, which allows to compute the ARE between Hotelling’s  $T^2$   
 51 (location) test and Fisher’s LDA (accuracy) test. Theorem 14.7 of van der  
 52 Vaart [1998] relates asymptotic power functions to ARE. Using the results of  
 53 Ramdas et al. [2016] we deduce that the ARE is lower bounded by  $2\pi \approx 6.3$ .  
 54 This means that Fisher’s LDA requires at least 6.3 more samples to achieve  
 55 the same (asymptotic) power than the  $T^2$  test. In this light, the accuracy  
 56 test is remarkably inefficient compared to the population test. For compar-  
 57 ison, the t-test is only 1.04 more (asymptotically) efficient than Wilcoxon’s  
 58 rank-sum test [Lehmann, 2009], so that an ARE of 6.3 is strong evidence in  
 59 favor of the population test.

60 Before discarding accuracy tests as inefficient, we recall that Ramdas  
 61 et al. [2016] analyzed a *half-sample* holdout. The authors conjectured that a  
 62 leave-one-out approach, which makes more efficient use of the data, may have  
 63 better performance. Also, the analysis in Ramdas et al. [2016] is asymptotic.  
 64 This eschews the discrete nature of the accuracy statistic, which will be

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<sup>1</sup>GoogleScholar. Accessed on Aug 4, 2016.

65 shown to have crucial impact. Since typical sample sizes in neuroscience are  
 66 not large, we seek to study which test is to be preferred in finite samples?  
 67 Our conclusion will be quite simple: *population tests almost always have more*  
 68 *power than accuracy tests.*

69 Our statement rests upon the observation that with typical sample sizes,  
 70 the accuracy test statistic is highly discrete. Permutation testing with dis-  
 71 crete test statistics are known to be conservative [Hemerik and Goeman,  
 72 2014], since they are insensitive to mild perturbations of the data, and they  
 73 cannot exhaust the permissible false positive rate. The degree of discretiza-  
 74 tion is governed by the number of samples. In our neuroscience example  
 75 from Gilron et al. [2016], the classification is performed based on 40 trials,  
 76 so that the test statistic may assume only 40 possible values. This number  
 77 of examples is not unusual if considering this is the number of trial-repeats,  
 78 or the number of subjects in an neuroimaging study.

79 The discretization effect is aggravated if the test statistic is highly concen-  
 80 trated. For an intuition consider the usage of a the *resubstitution accuracy*  
 81 as a test statistic. This statistic simply means that the accuracy is not cross  
 82 validated. If the data is high dimensional, the resubstitution accuracy will be  
 83 very high due to over fitting. In a very high dimensional model, the resubsti-  
 84 tution accuracy will be 1 for the observed data [McLachlan, 1976, Theorem  
 85 1], but also for any permutation. The concentration of resubstitution accu-  
 86 racy near 1, and its discreteness, render this test completely useless, with a  
 87 power tending to 0 for any (fixed) effect size, as the dimension of the model  
 88 grows.

89 To compare the power of accuracy tests and population tests in finite sam-  
 90 ples, we perform a simulation study of a battery of test statistics. We start  
 91 with formalizing the problem in Section 2. The main findings are reported  
 92 in Sections 4 and 5. A discussion follows in Section 6.

## 93 2 Problem setup

94 Let  $y \in \mathcal{Y}$  be a class encoding. Let  $x \in \mathcal{X}$  be a  $p$  dimensional feature vector.  
 95 In our vocal/non-vocal example we have  $\mathcal{Y} = \{-1, 1\}$  and  $p$ , the number of  
 96 voxels in a brain region so that  $\mathcal{X} = \mathbb{R}^{27}$ .

97 Given  $n$  pairs of  $(x_i, y_i)$ , typically assumed i.i.d., a population test amounts  
 98 to testing whether  $x|y = 1$  has the the same distribution as  $x|y = -1$ . I.e.,  
 99 we test if the multivariate voxel activation pattern has the same distribution  
 100 when given a vocal stimulus, as when given a non-vocal stimulus.

An accuracy test amounts to learning a predictive model and testing if its  
 predictions  $y|x$  are better than chance. Denoting a dataset by  $\mathcal{S} := (x_i, y_i)_{i=1}^n$ ,

the a predictor,  $\mathcal{A}_S(x) : \mathcal{X} \rightarrow \mathcal{Y}$ , is the output of a learning algorithm  $\mathcal{A}$  when applied to the dataset,  $\mathcal{A} : \mathcal{S} \rightarrow \mathcal{A}_S(x)$ . The accuracy of predictor  $\mathcal{A}_S(x)$  is defined as the probability of  $\mathcal{A}_S(x)$  making a correct prediction. Denoting by  $\mathcal{P}$  the probability measure of  $(x, y)$ , and by  $\mathcal{P}^n$  the same for the i.i.d sample  $\mathcal{S}$ , then

$$\mathcal{E}_{\mathcal{A}_S(x)} := \mathcal{P}(\mathcal{A}_S(x) = y). \quad (1)$$

The accuracy of an algorithm  $\mathcal{A}$  is defined as the average accuracy, over all possible data sets

$$\mathcal{E}_{\mathcal{A}} := \int_{\mathcal{S}} \mathcal{E}_{\mathcal{A}_S} d\mathcal{P}^n(\mathcal{S}). \quad (2)$$

101 Denoting an estimate of  $\mathcal{E}_{\mathcal{A}_S(x)}$  by  $\hat{\mathcal{E}}_{\mathcal{A}_S(x)}$ , and  $\mathcal{E}_{\mathcal{A}}$  by  $\hat{\mathcal{E}}_{\mathcal{A}}$ , a statistically sig-  
 102 nificant “better than chance” estimate of either, is evidence that the classes  
 103 are distinct. In a typical application, the predictor is not fixed, so that  $\hat{\mathcal{E}}_{\mathcal{A}}$ ,  
 104 and not  $\hat{\mathcal{E}}_{\mathcal{A}_S(x)}$ , will be used for the testing.

105 Two popular estimates of  $\hat{\mathcal{E}}_{\mathcal{A}}$  are the *resubstitution estimate*, and the  
 106 V-fold cross validation (CV) estimate [Hastie et al., 2003].

**Definition 1** (Resubstitution accuracy). The resubstitution accuracy estimator,  $\hat{\mathcal{E}}_{\mathcal{A}}^{resub}$ , is defined as

$$\hat{\mathcal{E}}_{\mathcal{A}}^{Resub} := \frac{1}{n} \sum_{i=1}^n \mathcal{I}\{\mathcal{A}_S(x) = y_i\}, \quad (3)$$

107 where  $\mathcal{I}\{A\}$  is the indicator function of event  $A$ .

108 **Definition 2** (V-fold CV).

## 109 2.1 Candidate Tests

110 The design of a permutation test using  $\hat{\mathcal{E}}_{\mathcal{A}}$ , requires the following design  
 111 choices:

- 112 1. Is  $\hat{\mathcal{E}}_{\mathcal{A}}$  cross validated or not?
- 113 2. For a V-fold cross validated test statistic:
  - 114 (a) Should the data be refolded in each permutation?
  - 115 (b) Should the data folding be balanced (a.k.a. stratified)?
  - 116 (c) How many folds?

117 3. How to estimate  $\hat{\mathcal{E}}_{\mathcal{A}}$ ?

118 We will now address these questions while bearing in mind that unlike  
119 the typical supervised learning setup, we are not interested in an unbiased  
120 estimate of  $\mathcal{E}_{\mathcal{A}}$ , but rather in its mere departure from chance level.

121 **Cross validate or not?** Given our goal, a biased estimate of  $\hat{\mathcal{E}}_{\mathcal{A}}$  is not a  
122 problem provided that bias is consistent over all permutations. The under-  
123 lying intuition is that a permutation test will be unbiased, provided that the  
124 exact same computation is performed over all permutations. We will thus  
125 be considering both cross validated accuracies, and *resubstitution accuracies*,  
126 where the accuracy is evaluated on the training set and not on a holdout.

127 **Balanced folding?** The standard practice when cross validating is to con-  
128 strain the data folds to be balanced, i.e. stratified [e.g. Ojala and Garriga,  
129 2010]. This means that each fold has the same number of examples from  
130 each class. We will report results with both balanced and unbalanced data  
131 foldings, only to discover, it does not really matter.

132 **Refolding?** The standard practice in neuroimaging is to permute labels  
133 and refold the data after each permutation, so that the balance of the classes  
134 in each fold is preserved. We will adhere to this practice due to its popularity,  
135 even though it can be simplified by permuting features instead of labels, as  
136 done by Golland et al. [2005].

137 **How many folds?** Different authors suggest different rules for the number  
138 of folds. We will be varying the number of folds, and ultimately discover that  
139 the power *decreases with the number of folds*.

**How to estimate accuracy?** Given a predictor for the  $i$ 'th data example<sup>2</sup>,  
 $\mathcal{A}_{\mathcal{S}}(x)$ , a natural test statistic its empirical accuracy

$$\hat{\mathcal{E}}_{\mathcal{A}_{\mathcal{S}}(x)} := \sum_i I(\mathcal{A}_{\mathcal{S}}(x)(x_i) = y_i). \quad (4)$$

Since low accuracies, even 0, are evidence that the classes are separated,  
we can consider the departure from chance level  $|\hat{\mathcal{E}}_{\mathcal{A}_{\mathcal{S}}(x)} - 0.5|$ , as another  
candidate test statistic. For unbalanced classes, chance level is not 0.5, but

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<sup>2</sup>We add the unorthodox observation index  $i$  to  $\mathcal{A}_{\mathcal{S}}(x)$  to accommodate for cross validation methods, where each prediction is made with a different classifier, fit to a slightly different dataset.

rather than the probability of the majority class, we denote by  $\hat{\pi}$ . This suggests the following test statistic  $|\hat{\mathcal{E}}_{\mathcal{A}_S(x)} - \hat{\pi}|$ . Since we will be aggregating these statistics over random data sets where  $\hat{\pi}$  may vary, it seems appropriate to standardize the scale. We thus propose the z-scored accuracy statistic,  $\hat{\mathcal{Z}}_{\mathcal{A}_S(x)}$ :

$$\hat{\mathcal{Z}}_{\mathcal{A}_S(x)} := |\hat{\mathcal{E}}_{\mathcal{A}_S(x)} - \hat{\pi}| / \sqrt{\hat{\pi}(1 - \hat{\pi})}. \quad (5)$$

Table 1 collects an initial battery of tests we will be comparing.

Name	Basis	CV	Accuracy	Parameters
Hotelling	Hotelling	—	—	—
Hotelling.shrink	Hotelling	—	—	—
lda.CV.1	LDA	V-fold	$\hat{\mathcal{E}}$	—
lda.CV.2	LDA	V-fold	$\hat{\mathcal{Z}}$	—
lda.noCV.1	LDA	—	accuracy	—
lda.noCV.2	LDA	—	z-accuracy	—
sd	SD	—	—	—
svm.CV.1	SVM	V-fold	accuracy	cost=1e1
svm.CV.2	SVM	V-fold	accuracy	cost=1e-1
svm.CV.3	SVM	V-fold	z-accuracy	cost=1e1
svm.CV.4	SVM	V-fold	z-accuracy	cost=1e-1
svm.noCV.1	SVM	—	accuracy	cost=1e1
svm.noCV.2	SVM	—	accuracy	cost=1e-1
svm.noCV.3	SVM	—	z-accuracy	cost=1e1
svm.noCV.4	SVM	—	z-accuracy	cost=1e-1

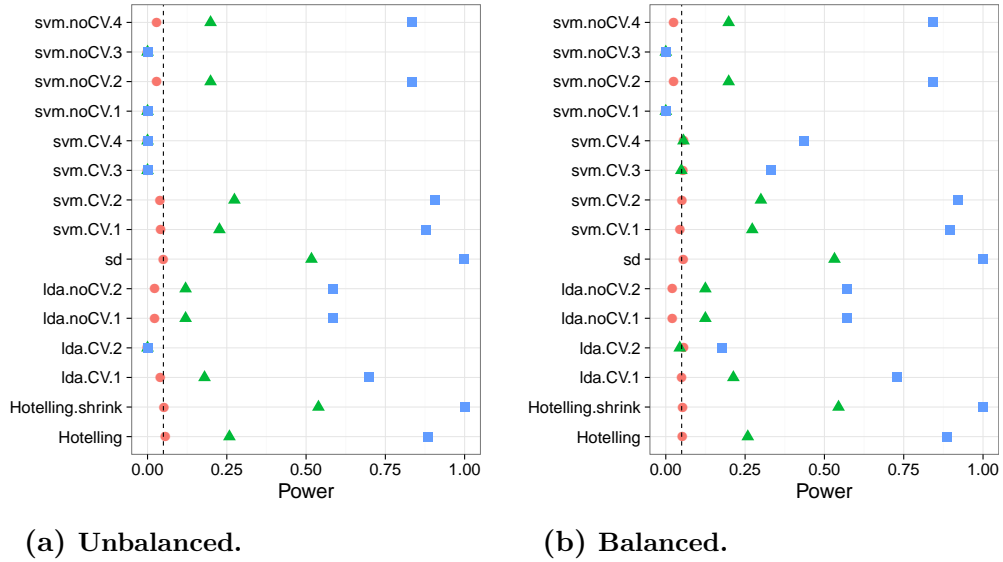
Table 1: This table collects the various test statistics we will be studying. Three are population tests: Hotelling, Hotelling.shrink, and sd. *Hotelling* is the classical two-group  $T^2$  statistic. *Hotelling.shrink* is a high dimensional version with the regularized covariance in Schäfer and Strimmer [2005]. *sd* is another high dimensional version of the  $T^2$ , from Srivastava et al. [2013]. The rest of the tests are variations of the linear SVM, and Fisher’s LDA, with varying accuracy measures, cross validated or not, and varying tuning parameters. For example, *svm.CV.4* is a linear SVM implemented with the *svm* R function, the cost parameter set at 0.1, and using the cross validated z-scored accuracy in Eq. 5. Another example is *lda.noCV.1*, which is Fisher’s LDA, returning the resubstitution accuracy.

### 3 Controlling the False Positive Rate

Figure 1 demonstrates that all of the tests considered conserve the desired 0.05 false positive rate, up to varying levels of conservatism. This can be

145 seen from the fact that the probability of rejection is no larger than 0.05 in  
 146 the absence of any effect, encoded by a red circle. This is true, in particular  
 147 if: (a) the folds are balanced or not, (b) the tuning parameters of some test  
 148 statistic are varied, (d) the number of folds is varied. We also observe that  
 149 the most conservative tests are the resubstitution accuracy statistics. We  
 150 return to this matter in the Discussion.

*Figure 1:* The power of a permutation test with various test statistics. The power on the  $x$  axis. Effect are color and shape coded. The various statistics on the  $y$  axis. Their details are given in Table 1. Effects vary over 0 (red circle), 0.25 (green triangle), and 0.5 (blue square). Simulation details in Appendix B. Cross-validation was performed with balanced and unbalanced data folding. See sub-captions.



## 151 4 Power

152 Having established that all of the tests in our battery control the false positive  
 153 rate, it remains to be seen if they have similar power— especially when  
 154 comparing population tests to accuracy tests. From the simulation results  
 155 reported in Appendix C we collect the following insights:

- 156 1. population tests have more power than accuracy tests in all our configurations.  
 157
- 158 2. The conservativeness decays as the sample grows (Figures 9a, 9b and  
 159 10a)

- 160 3. For heavy tailed distributions (Figure 8b), the extra power of the loca-  
161 tion test vanishes.
- 162 4. The presence of correlations between coordinates reduces the signal to  
163 noise ratio (SNR), thus reduces power. More importantly, in the pres-  
164 ence of correlations the effect of regularization is amplified, increas-  
165 ing the power difference between regularized and non-regularized test  
166 statistics. Put differently- in low SNR regimes, regularization proves  
167 crucial (Figure 10b).
- 168 5. The z-scoring of the accuracies was introduced to deal with unbalanced  
169 foldings. If the z-scoring has any effect at all, it merely kills power.
- 170 6. Both accuracy and population tests are inappropriate for scale alter-  
171 natives (Figure 8a). This was to be expected and is reported mostly as  
172 a sanity check.
- 173 7. Balanced folding only affects the z-scored accuracy, in the opposite  
174 direction than we anticipated.
- 175 8. Increasing the SVM’s cost parameter, which reduces the number of  
176 support vectors entering the classifier, reduces power.

177 The major insight from simulations is that the use of accuracy tests for  
178 signal detection is underpowered compared to population tests. We now  
179 verify this finding on a neuroimaging dataset.

## 180 5 Neuroimaging Example

181 Figure 2 is an application of both a location and an accuracy test to the data  
182 of Pernet et al. [2015]. The authors of Pernet et al. [2015] collected fMRI  
183 data while subjects were exposed to the sounds of human speech (vocal),  
184 and other non-vocal sounds. Each subject was exposed to 20 sounds of each  
185 type, totaling in  $n = 40$  trials in each scan. The study was rather large and  
186 consisted of about 200 subjects. The data was kindly made available by the  
187 authors at the OpenfMRI website<sup>3</sup>.

188 We perform group inference using within-subject permutations along the  
189 analysis pipeline of Stelzer et al. [2013], which was also reported in Gilron  
190 et al. [2016]. For completeness, the pipeline is described in Appendix A. To  
191 demonstrate our point, we compare the *sd* population test with the *svm.cv.1*  
192 accuracy test.

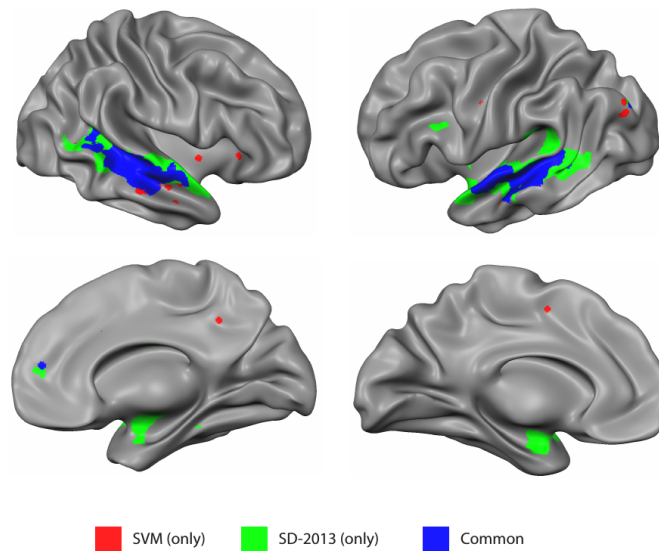
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<sup>3</sup><https://openfmri.org/>



193 In agreement with our simulation results, the population test (*sd*) dis-  
 194 covers more brain regions of interest when compared to an accuracy test  
 195 (*svm.cv.1*). The former discovers 1,232 regions, while the latter only 441, as  
 196 depicted in Figure 2. We emphasize that both test statistics were compared  
 197 with the same permutation scheme, and the same error controls, so that any  
 198 difference in detections is due to their different power.

199 Having established that accuracy tests are typically underpowered for sig-  
 200 nal detection compared to population tests, we wish to identify the conditions  
 201 under which this will occur, and discuss practical implications.



*Figure 2:* Brain regions encoding information discriminating between vocal and non-vocal stimuli. Map reports the centers of 27-voxel sized spherical regions, as discovered by an accuracy test (*svm.cv.1*), and a population test (*sd*). *svm.cv.1* was computed using 5-fold cross validation, and a cost parameter of 1. Region-wise significance was determined using the permutation scheme of Stelzer et al. [2013], followed by region-wise  $FDR \leq 0.05$  control using the Benjamini-Hochberg procedure [Benjamini and Hochberg, 1995]. Number of permutations equals 400. The population test detect 1,232 regions, and the accuracy test 441, 399 of which are common to both. For the details of the analysis see Appendix A and Gilron et al. [2016].

## 202 6 Discussion

203 We have set out to understand which of the tests is more powerful: the  
 204 accuracy test or the population test. No amount of simulations can replace

the insight provided by a good closed-form analytic result. The finite sample power of permutation tests is a formidable mathematical problem, so we currently content ourselves with simulations. We have concluded that the population tests are typically preferable. Their high dimensional versions, such as Srivastava [2007] and Schäfer and Strimmer [2005], are particularly well suited for neuroimaging problems such as MVPA. We attribute this to several phenomena: (a) Discretization introduced in finite samples by the accuracy test statistic. (b) Inefficient use of the data for the validation holdout set. (c) Regularization crucial in high dimensional problems.

The presence of heavy tails shrinks the power advantage of the population tests over accuracy tests. Our empirical example suggests that even if the population test does not necessarily dominate the accuracy test in power, empirically, it does have an advantage.

The degree of discretization is governed by the sample size. For this reason, an asymptotic analysis such as Ramdas et al. [2016] may uncover the holdout inefficiency, but will not uncover the discretization effect.

The practical advice for the practitioner, is that for the purpose of signal detection, there is typically a population test that is more powerful than an accuracy test. There is also a good chance that it would be easier to implement, and faster to run, since no cross validation will be involved.

## 6.1 Ease of implementation

A very important consideration is the ease of implementation. The need for cross validation of the accuracy test greatly increases its computational complexity. Moreover, anyone who has actually implemented tests with discrete statistics, will attest they are more prone to programming errors. This is because their unforgiveness to the type of inequalities used. Indeed, mistakenly replacing a weak inequality with a strong inequality in one's program may considerably change the results. This is not the case for continuous test statistics.

## 6.2 Reservations

Some reservations to the generality of our findings are in order. Firstly, not all accuracy tests are concerned with signal detection. Consider brain decoding for machine interfaces, or clinical diagnosis, where the presence of a medical condition is predicted from imaging data [e.g. Olivetti et al., 2012, Wager et al., 2013]. In those examples, the purpose of the test is not to detect a difference between classes, but to actually test the performance of a particular classifier.

Secondly, it may be argued that accuracy tests permits the separation between classes in high dimensions, such as in *reproducing kernel Hilbert spaces* (RKHS) by using non-linear predictors. This is a false argument—accuracy test do not have any more flexibility than population tests. Indeed, it is possible to test for location in the same dimension the classifier is learned. Gretton et al. [2012] is an example where the test for location is performed in the RKHS of the data. It is also possible to test for the equality of two multivariate distributions [TODO: cite vogelstein]. On the other hand, based on our reported neuroimaging example, and others, we find that a population test in the original feature space is indeed a simple and powerful approach to signal detection.

### 6.3 A good accuracy test

For the cases a population test cannot replace an accuracy test, we collect some conclusions and best practices from our simulations. We give particular emphasis in this section to V-fold cross validation due to its popularity, but note that sampling the test set with replacement is actually preferable, as we discuss in Section 6.4.

**Sample size.** The conservativeness of accuracy tests decrease with sample size.

**Permute features.** Permuting features is easier than permuting labels. It allows to preserve balanced folds after a permutation without refolding. Although we did not find a power difference between balanced and unbalanced foldings.

**Use less folds.** For V-fold CV, power decreases as the number of folds increases. This is quite interesting since two phenomena compete as the number of folds increase: (a) the train set is larger so that better accuracies are achievable. (b) The test set is smaller so that the accuracy estimate is more variable. The decrease in power with increase fold number suggests that the latter dominates the former. Put differently: it is easier to detect a small stable departure from chance level, than a large but unstable one.

**Resubstitution accuracy in low dimension.** Resubstitution accuracy is useful in low dimension. In high dimension, the power loss is considerable compared to a cross validated approach. We attribute this to the compounding of discretization and concentration effects: the difference between the

276 sampling distribution of the resubstitution accuracy is simply indistinguish-  
 277 able under the null and under the alternative. In low dimensional problems,  
 278 the discretization is less impactful, and the computational burden of cross  
 279 validation can be avoided by using the resubstitution accuracy. There is  
 280 a fundamental difference between V-folding and resubstitution. The latter  
 281 should not be thought of as the limit of the former.

282 **Regularize** Regularizing the accuracy test proves very useful in high di-  
 283 mensional problems. Put differently: reducing variance by adding some bias  
 284 is very useful to detect better-than-chance classification.

285 **Don't z-score.** There is no gain in z-scoring the accuracy scores. Our  
 286 motivating rational was clearly flawed. [TODO: why?]

## 287 6.4 Smoothing accuracy estimates

288 It may be possible to alleviate the effect of discretization by appropriate  
 289 cross-validation. The discreteness of the accuracy statistic is governed by  
 290 the number of examples in the union (over all validation iterations) of test  
 291 sets. For V-fold CV, for instance, this number is simply the sample size. This  
 292 suggests that the accuracy can be “smoothed” by allowing the test sample to  
 293 be drawn with replacement. The *bootstrap* may seem like a good candidate  
 294 approach since it samples examples with replacement. It does so, however,  
 295 for the train set, and not the test set. An algorithm that samples test sets  
 296 with replacement is the *leave-one-out bootstrap estimator* (bLOO) and its  
 297 derivation– the *0.632 bootstrap estimator* (b0.632) [Hastie et al., 2003, Sec  
 298 7.11].

**Definition 3** (bLOO). Denoting by  $C^{(i)}$  the index set of bootstrap samples,  
 $b$ , where observation  $i$  is not in the train set, and by  $\mathcal{A}(\mathcal{S})^b$  the classifier  
 fitted to the  $b$ 'th bootstrap training sample, then the *leave-one-out bootstrap*  
 estimate is defined as:

$$\mathcal{E}_{bLOO} := \frac{1}{n} \sum_{i=1}^n \frac{1}{|C^{(i)}|} \sum_{b \in C^{(i)}} I(\mathcal{A}(\mathcal{S})^b(x_i) = y_i).$$

Equivalently, denoting by  $S^{(b)}$  the indexes of observations,  $i$ , that are not in  
 the bootstrap train sample  $b$ ,

$$\mathcal{E}_{bLOO} = \frac{1}{B} \sum_{b=1}^B \frac{1}{|S^{(b)}|} \sum_{i \in S^{(b)}} I(\mathcal{A}(\mathcal{S})^b(x_i) = y_i).$$

**Definition 4** (b0.632). Denoting by  $\mathcal{E}_{resub}$  the resubstitution accuracy estimate, the b0.632 accuracy estimator,  $\mathcal{E}_{0.632}$ , is defined as

$$\mathcal{E}_{0.632} := 0.368 \mathcal{E}_{resub} + 0.632 \mathcal{E}_{bLOO}.$$

Simulation results reported in Figure 3 with naming conventions in Table 2. It can be seen that selecting test sets with replacement does increase the power, when compared to V-fold cross validation, but still falls short from the power of population tests. It can also be seen that power increases with the number of bootstrap replications, itself reducing the level of discretization. The type of bootstrap, bLOO versus b0.632, does not change the power.

Name	Basis	Type	B	Accuracy	Parameters
lda.Boot.1	LDA	b0.632	10	accuracy	—
lda.Boot.2	LDA	bLOO	10	accuracy	—
svm.Boot.1	SVM	b0.632	10	accuracy	cost=1e1
svm.Boot.2	SVM	bLOO	10	accuracy	cost=1e1
svm.Boot.3	SVM	b0.632	50	accuracy	cost=1e1
svm.Boot.4	SVM	bLOO	50	accuracy	cost=1e1

Table 2: The same as Table 1 for bootstrapped accuracy estimates. bLOO and b0.632 are defined in definitions 3 and 4 respectively.  $B$  denotes the number of Bootstrap samples.

## 6.5 High dimensional classifiers

Inspecting Figure 1a (for instance), it can be seen that Hotelling’s  $T^2$  test has similar power to accuracy tests. It should thus be argued that the real advantage of the population tests is due to their adaptation to high dimension by regularization (*sd* and *Hotelling.shrink*), and not only to discretization. To study this, we call upon several regularized classifiers, designed for high dimensional problems. In the spirit of the regularized covariance of *Hotelling.shrink*, we try an  $l_2$  regularized svm Friedman et al. [2010], and shrinkage based LDA [Pang et al., 2009, Ramey et al., 2016]. In the spirit of the diagonalized covariance of *sd*, we try a diagonalized LDA [Dudoit et al., 2002], which can be thought of a method intersecting Fisher’s LDA and Naive Bayes.

Simulation results reported in Figure 4 with naming conventions in Table 3. It can be seen that regularizing a classifier in high dimension, just



*Figure 3: Bootstrap*— The power of a permutation test with various test statistics. The power on the  $x$  axis. Effect are color and shape coded. The various statistics on the  $y$  axis. Their details are given in tables 1 and 2. Effects vary over 0 (red circle), 0.25 (green triangle), and 0.5 (blue square). Simulation details in Appendix B.

like a parameter test, improves power. It can also be seen that (regularized) parameter tests are still more powerful than (regularized) accuracy tests. This was to be expected, since we already saw in (e.g. Figure 1a) that the unregularized parameter test, *Hotelling*, is slightly more powerful than the regularized accuracy test, *svm.CV.1* for instance.

We can compound regularization in this section with the bootstrapping from Section 6.4, to improve finite sample power of the accuracy tests. This is done in the *svm.highdim.2* test, which still falls short from the power of the location tests, but is a much more powerful accuracy test than the original non-regularized, V-fold validated, version of *svm.CV.1*.

Name	Basis	CV	Accuracy	Parameters
svm.highdim.1	SVM	V-fold	accuracy	cost=1e-1
svm.highdim.2	SVM	B=50	accuracy	cost=1e-1
lda.highdim.1	LDA	V-fold	accuracy	—
lda.highdim.2	LDA	V-fold	accuracy	—
lda.highdim.3	LDA	V-fold	accuracy	—

Table 3: The same as Table 1 for regularized (high dimensional) predictors. *svm.highdim.1* is an  $l_2$  regularized SVM Friedman et al. [2010]. *svm.highdim.2* is the same with b0.632 instead of V-fold cross validation. *lda.highdim.1* is the Diagonal Linear Discriminant Analysis of Dudoit et al. [2002]. *lda.highdim.2* is the High-Dimensional Regularized Discriminant Analysis of Ramey et al. [2016]. *lda.highdim.3* is the Shrinkage-based Diagonal Linear Discriminant Analysis of Pang et al. [2009].

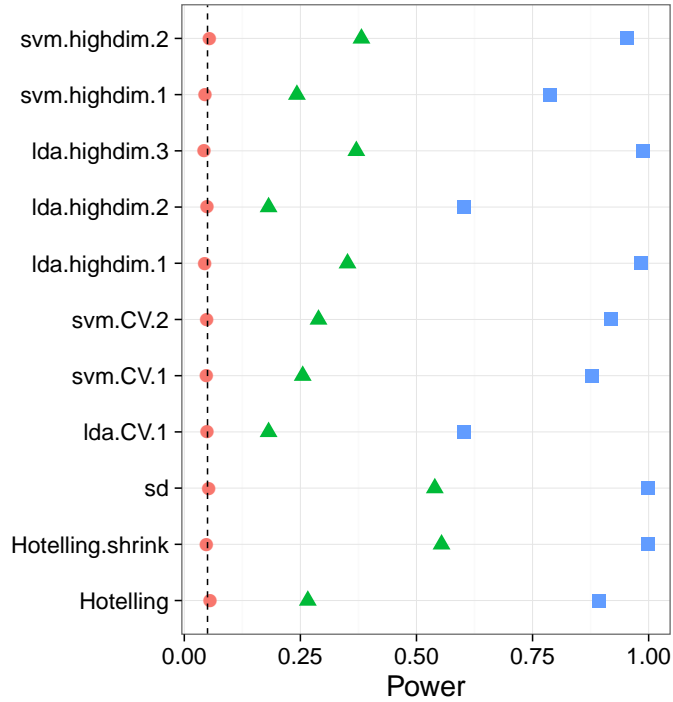


Figure 4: **HighDim Classifier**— The power of a permutation test with various test statistics. The power on the  $x$  axis. Effect are color and shape coded. The various statistics on the  $y$  axis. Their details are given in tables 1 and 3. Effects vary over 0 (red circle), 0.25 (green triangle), and 0.5 (blue square). Simulation details in Appendix B.

## 332 6.6 Related Literature

333 Ojala and Garriga [2010] study the power of two accuracy tests: one test-  
 334 ing the “no signal” null hypothesis, and the other testing the “independent  
 335 features” null hypothesis. They perform an asymptotic analysis, and a sim-  
 336 ulation study. They also apply various classifiers to various data sets. Their  
 337 emphasis is the effect of the underlying classifier on the power, and the po-  
 338 tential of the “independent features” test for feature selection. This is a very  
 339 different emphasis from our own.

340 Olivetti et al. [2012] and Olivetti et al. [2014] looked into the problem of  
 341 choosing a good accuracy test. They propose a new test they call an *indepen-*  
 342 *dence test*, and demonstrate by simulation that it has more power than other  
 343 accuracy tests, and can deal with non-balanced data sets. We did not include  
 344 this test in the battery we compared, but we note the following: (a) The in-  
 345 dependence test of Olivetti et al. [2012] relies on a discrete test statistic. It  
 346 may thus be improved with the methods discussed in this section, before the  
 347 application of Olivetti et al. [2012]’s independence test. (b) In contrast with  
 348 the underlying motivation of Olivetti et al. [2012]’s independence test, we  
 349 did not find that balancing the data folds is crucial for an accuracy test.

350 Golland et al. [2005] study accuracy tests using simulation, neuroimaging  
 351 data, genetic data, and analytically. Their analytic results formalize our in-  
 352 tuition from Section 1 on the effect of concentration of the accuracy statistic:  
 353 The finite Vapnik–Chervonenkis (VC) dimension requirement [Golland and  
 354 Fischl, 2003, Sec 4.3] prevents the permutation p-value from (asymptotically)  
 355 concentrating near 1. Like ourselves, they also find that the power increases  
 356 with the size of the test set (Figure 4, middle). This is seen in their Figure 4,  
 357 where the size of the test-set,  $K$ , governs the discretization. Since they per-  
 358 mutate features, not labels, then all their permutation samples are balanced,  
 359 and there is no issue of refolding.

360 Golland et al. [2005] simulate the power of accuracy tests by sampling  
 361 from a Gaussian mixture family of models, and not from a location family  
 362 as our own simulations. Under their model  $(x_i|y_i = 1) \sim p\mathcal{N}(\mu_1, I) +$   
 363  $(1 - p)\mathcal{N}(\mu_2, I)$  and  $(x_i|y_i = -1) \sim (1 - p)\mathcal{N}(\mu_1, I) + p\mathcal{N}(\mu_2, I)$ . Varying  $p$   
 364 interpolates between the null distribution ( $p = 0.5$ ) and a location shift model  
 365 ( $p = 0$ ). We now perform the same simulation as Golland et al. [2005], after  
 366 parameterizing  $p$  so that  $p = 0$  corresponds to the null model, and in the  
 367 same dimensionality as our previous simulations We find that also in this  
 368 mixture class of models a population test has more power than an accuracy  
 369 test (Figure 5).



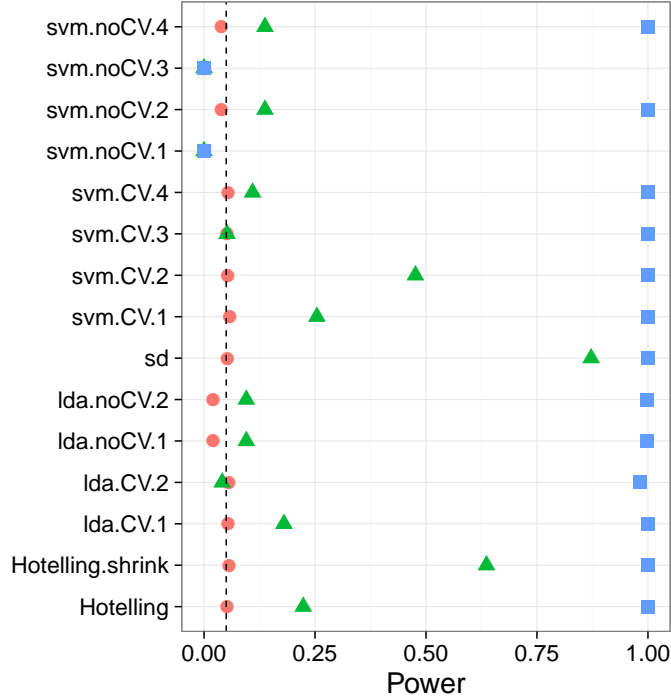


Figure 5: **Mixture**—  $\mathbf{x}_i = \chi_i \mu + \eta_i$ ;  $\chi_i = \{-1, 1\}$  and  $\text{Prob}(\chi_i = 1) = (1/2 - p)^{y_i^*} (1/2 + p)^{1-y_i^*}$ .  $\mu$  is a  $p$ -vector with  $3/\sqrt{p}$  in all coordinates. The effect,  $p$ , is color and shape coded and varies over 0 (red circle),  $1/4$  (green triangle) and  $1/2$  (blue square).

## 6.7 Epilogue

Given all the above, we find the popularity of accuracy tests quite puzzling. We believe this is due to a reversal of the inference cascade. Researchers first fit a classifier, and then ask if the classes are any different. Were they to start by asking if classes are any different, and only then try to classify, then population tests would naturally arise as the preferred method. As put by Ramdas et al. [2016]:

The recent popularity of machine learning has resulted in the extensive teaching and use of prediction in theoretical and applied communities and the relative lack of awareness or popularity of the topic of Neyman-Pearson style hypothesis testing in the computer science and related “data science” communities.

And more simply by Frank Harrell in the CrossValidated Q&A site<sup>4</sup>:

<sup>4</sup><http://stats.stackexchange.com/questions/17408/how-to-assess-statistical-significance-of-the-accuracy-of-a-classifier>.

383 ... your use of proportion classified correctly as your accuracy  
384 score. This is a discontinuous improper scoring rule that can be  
385 easily manipulated because it is arbitrary and insensitive.

## 386 **7 Acknowledgments**

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## 511 A Analysis pipeline

512 Here is the analysis pipeline of Stelzer et al. [2013] we for the auditory data in  
 513 Gilron et al. [2016]. Denoting by  $i = 1, \dots, I$  the subject index,  $v = 1, \dots, V$   
 514 the voxel index, and  $s = 1, \dots, S$  the permutation index. Since regions<sup>5</sup> are  
 515 centered around a unique voxel, the voxel index  $v$  also serves as a unique  
 516 region index. Algorithm 1 computes a region-wise test statistic, which is  
 517 compared to its permutation null distribution computed by Algorithm 2.

**Algorithm 1:** Compute a group parametric map.

**Data:** fMRI scans, and experimental design.  
**Result:** Brain map of group statistics:  $\{\bar{T}_v\}_{v=1}^V$

```

1 for  $v \in 1, \dots, V$  do
518 2   for  $i \in 1, \dots, I$  do
3      $T_{i,v} \leftarrow$  test statistic for subject  $i$  in a region centered at  $v$ .
4    $\bar{T}_v \leftarrow \frac{1}{I} \sum_{i=1}^I T_{i,v}$ .
```

**Algorithm 2:** Compute a permutation p-value map.

**Data:** fMRI scans of 20 subjects, experimental design.  
**Result:** Brain map of permutation p-values:  $\{p_v\}_{v=1}^V$

```

519 1 for  $s \in 1, \dots, S$  do
2   permute labels;
3    $\bar{T}_v^s \leftarrow$  parametric map
```

---

<sup>5</sup>*searchlight* or *sphere* in the MVPA parlance

## 520 B Simulation Details

521 The following details are common to all the reported simulations, unless  
522 stated otherwise in a figure’s caption. The R code for the simulations can be  
523 found in [TODO].

524 Each simulation is based on 4,000 replications. In each replication, we  
525 generate  $n$  i.i.d. samples from a shift model  $\mathbf{x}_i = \mu \mathbf{y}_i^* + \eta_i$ . Where  $y_i^* = \{0, 1\}$   
526 is the class of subject  $i$  in dummy coding. Recalling that  $y_i = \{-1, 1\}$  is the  
527 class in effect coding, then clearly  $y_i = 2y_i^* - 1$ . The noise is distributed as  
528  $\eta_i \sim \mathcal{N}_p(0, \Sigma)$ . The sample size  $n = 40$ . The dimension of the data is  $p = 23$ .  
529 The covariance  $\Sigma = I$ . Effects, i.e. shifts  $\mu$ , are equal coordinate  $p$ -vectors  
530 with coordinates that vary over  $\mu \in \{0, 1/4, 1/2\}$ .

531 Having generated the data, we compute each of the test statistics in Ta-  
532 ble 1. For test statistics that require data folding, we used 8 folds. We then  
533 compute a permutation p-value by permuting the class labels, and recomput-  
534 ing each test statistic. We perform 400 such permutations. We then reject  
535 the  $\mu_i = 0$  null hypothesis if the permutation p-value is smaller than 0.05.  
536 The reported power is the proportion of replication where the permutation  
537 p-value falls below 0.05.



## C Simulation Results

Figure 6: Simulation details in Appendix B except the changes in the sub-captions.



Figure 7: Simulation details in Appendix B except the changes in the sub-captions.



Figure 8: Simulation details in Appendix B except the changes in the sub-captions.



Figure 9: Simulation details in Appendix B except the changes in the sub-captions.

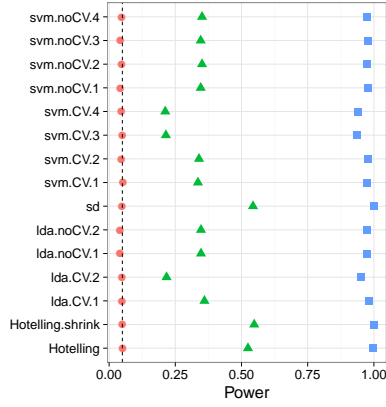


(a) Low-Dimension— False positive rates for  $n = 40$ .



(b) High-Dimension— False positive rates for  $n = 400$ .

Figure 10: Simulation details in Appendix B except the changes in the sub-captions.



(a) High-Dimension, local alternative—  
 $n = 400$ ,  
 $\mu \in \frac{1}{\sqrt{10}} \times \{0, 1/4, 1/2\}$ .



(b) AR(1) dependence—  
 $\Sigma_{k,l} = \rho^{|k-l|}; \rho = 0.8$ .