Problem Set #3

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Problem 4.2 We know from a prior excercise that

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\det(D - \lambda I) = -\lambda^3$, the only eigenvalue of D is 0, with algebraic multiplicity 3. Accompanying this eigenvalue is eigenvector $[1\ 0\ 0]^T$. This implies that eigenvalue 0 has geometric multiplicity equal to 1.

Problem 4.4 i) Let

$$A = \begin{bmatrix} a & c+di \\ c-di & b \end{bmatrix}$$

Then the eigenvalues are solutions to $\lambda^2 - (a+b)\lambda + ab - c^2 - d^2 = 0$. To determine completixy of roots, we examine the portion inside the square root of the quadratic formula, here equal to $a^2 + b^2 + 2ab + 4c^2 + 4d^2 = (a+b)^2 + 4c^2 + 4d^2 \geq 0$. Thus A can have only real valued eigenvalues.

ii) Let

$$A = \begin{bmatrix} ai & c+di \\ -c+di & bi \end{bmatrix}$$

Eigenvalues are solutions to $\lambda^2 - (a+b)i\lambda - ab + c^2 + d^2 = 0$. Note that the quadratic formula implies that eigenvalues will begin with $(a+b)i$ and are therefore imaginary.

Problem 4.6 For each $a_{ii} \in \text{diag}(A)$, we know that $\det(A - a_{ii}I) = 0$ since there is a zero in the diagonal and A is upper-triangular or lower-triangular. So thus each a_{ii} is an eigenvalue, and by Proposition 4.1.11 there cannot be any other eigenvalues.

Problem 4.8 i) We simply show that the elements of S are linearly independent since $\text{span}(S) = V$. By 3.2.5, orthogonal sets are linearly independent, and we showed that S is a linearly independent set in exercise 3.8.

ii) Note $D(\sin(x)) = \cos(x)$, $D(\cos(x)) = -\sin(x)$, $D(\sin(2x)) = 2\cos(2x)$, $D(\sin(2x)) = -2\cos(2x)$. So the matrix representation of D in this matrix is

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

iii) One subspace is {cos(x),sin(x)} and the other is {cos(2x),sin(2x)}.

Problem 4.13 We compute the eigenvalues of A. The characteristic equation of A is $\lambda^2 - 1.4\lambda + 0.4$. This solves to eigenvalues equal to 1 and 0.4. The eigenvector associated with eigenvalue 1 is the null space of

$$\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$$

which has as a solution the vector $[2\ 1]^T$. The eigenvector associated with eigenvalue 0.4 is the null space of

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}$$

which has a solution of $[1\ \ -1]^T$. Then

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Problem 4.15 Let A be a semisimple matrix. Then we know A is diagonalizable so there exist P and D such that $A = P^{-1}DP$. Note $f(A) = f(P^{-1}DP) = a_0 I + A_1 P^{-1}DP + \cdots + a_n P^{-1}DP = P^{-1}(a_0 I + a_1 D + \cdots + a_n D^n)P = P^{-1}f(D)P$. Then the elements of $f(D)$ must be the eigenvalues of $f(A)$. $f(D)$ is a diagonal matrix whose elements are $f((\lambda_i)_{i=1}^n)$.

Problem 4.16 i) We use the diagonalization from problem 13, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Note

$$\lim_{k\to\infty} A^k = \lim_{k\to\infty} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

This implies

$$A^k - B = \begin{bmatrix} \frac{.4^k}{3} & \frac{-2\cdot .4^k}{3} \\ -\frac{.4^k}{3} & \frac{2\cdot .4^k}{3} \end{bmatrix}$$

So $\|A^k - B\|_1 = \frac{4}{3}(.4)^k$. We have $\frac{4}{3}(.4)^k < \epsilon \Rightarrow k > \frac{\ln\left(\frac{3}{4}\epsilon\right)}{\ln(.4)}$. Thus $\lim_{n\to\infty} A^n = B$.
ii) We have $\|A^k - B\|_\infty = 0.4^k$. $0.4^k < \epsilon \Rightarrow k > \frac{\ln\epsilon}{\ln 0.4}$
We also have $\|A^k - B\|_{FR} = \frac{10}{9}0.4^k < \epsilon \Rightarrow k > \frac{\ln\left(\frac{9}{10}\epsilon\right)}{\ln(0.4)}$. Thus the answer is the same regardless of the norm.
iii) By Theorem 4.3.12, the eigenvalues are $3 + 5 + 1 = 9$ and $3 + 2 + 0.4^3 = \frac{633}{125}$.

Problem 4.18 By Proposition 4.1.23, A and A^T have the same characteristic polynomial. Then by 4.1.8, they have the same eigenvalues. So for all $\lambda \in \sigma(A), \exists x | A^T x = \lambda x \Rightarrow x^T A = \lambda x^T$.

Problem 4.20 Suppose A is orthonormally similar to B and it Hermetian. Then we know that $B = U^H AU = U^H A^H U = (U^H AU)^H = B^H$. Thus B is Hermetian.

Problem 4.24 Suppose A is Hermetian. Then $x^H Ax = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle} = \overline{x^H A^H x} = \overline{x^H Ax}$. So since $x^H Ax = \overline{x^H Ax}$, the Rayleigh quotient is real valued only. Now suppose A is skew-Hermetian. Then $\langle x, Ax\rangle = x^H Ax = -x^H A^H x = -\langle Ax, x\rangle = -\overline{\langle x, Ax\rangle}$. Thus the Rayleigh quotient is imaginary only.

Problem 4.25 i) Since A is a normal matrix, it has orthonormal eigenvectors. Write

$$Q = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

Note that Q is an orthonormal matrix, so $QQ^H = Q^H Q = I$. Expanding QQ^H, we get that $QQ^H = x_1 x_1^H + \cdots + x_n x_n^H = I$, as desired.
ii) Since $x_1 x_1^H + \cdots + x_n x_n^H = I$, where each x_i is an eigenvector of A, we have $A = AI = A(x_1 x_1^H + \cdots + x_n x_n^H) = Ax_1 x_1^H + \cdots + Ax_n x_n^H = \lambda_1 x_1 x_1^h + \cdots \lambda_n x_n x_n^H$.

Problem 4.27 Let A be positive definite. Define a vector x_i which is all zeroes except for a one in the i^{th} position. Observe that the i^{th} element of the diagonal of A, $A_{ii} = \langle x_i, Ax_i\rangle$. This must be positive due to the positive definiteness of A.

Problem 4.28 Notice that

$$\text{tr}(AB) = \sum_{i=1}^{n} \langle e_i, ABe_i\rangle = \sum_{i=1}^{n} e_i^H ABe_i = \sum_{i=1}^{n} e_i^H Ae_i e_i^H Be_i = \sum_{i=1}^{n} \langle e_i, Ae_i\rangle\langle e_i, Be_i\rangle$$

This must be non-negative because of the positive semi-definiteness of A and B. We also know that

$$\text{tr}(AB) \leq \text{tr}(AB) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \langle e_i, Ae_i\rangle\langle e_j, Be_j\rangle = \text{tr}(A)\text{tr}(B)$$

Problem 4.31 i) Observe that $A^H A$ is Hermetian, and therefore there exists an eigenbasis. This implies that for any vector $x \in \mathbb{F}^n, x = a_1 x_1 + \cdots + a^n x^n$, where each x_i is an eigenvector. Let σ_i be the largest singular value of A. Then $\|A\|_2^2 = sup_{\|x\|=1}\langle Ax, Ax\rangle = \langle x, A^H Ax\rangle = \langle x, a_1 A^H Ax_1 + \cdots + a_n A^H Ax_n\rangle \leq \langle x, \sigma_i^2 a_1 x_1 + \cdots + \sigma_i^2 a_n x_n\rangle = \sigma_i^2\langle x, x\rangle = \sigma_i^2$. Thus $\|A\|_2 \leq \sigma_i$. Note that $\|x_i\|_2 = 1$, so since $\|A\|_2 = \sup_{\|x\|=1}\|Ax\|_2, \|A\|_2 \geq \|Ax_i\|_2 = \sigma_i\|x_i\|_2 = \sigma_i$. Thus $\|A\|_2 = \sigma_i$.
ii) Observe that by properties of the SVD, $A = U\Sigma V^H \Rightarrow A^{-1} = V\Sigma^{-1}U^H$, where $\Sigma_{ij}^{-1} = \frac{1}{\Sigma_{ij}}$. By i), we know that $\|A^{-1}\|_2$ is equal to its largest singular value, which are the elements of Σ^{-1}. The largest of these is $\frac{1}{\sigma_n}$, as desired.
iii) Expanding A using the SVD yields that $A = U\Sigma V^H$. So $A^H = V\overline{\Sigma}U^H$ and $A^T = \overline{V}\Sigma U^T$. Since we know that for any matrix A $\|A\|_2$ is equal to the largest singular value and that the singular values are real, positive numbers found in the diagonal

of Σ in the SVD, we know $\Sigma = \overline{\Sigma}$ and thus $\|A^H\|_2^2 = \|A^T\|_2^2 = \|A\|_2^2$. Observe also that $A^HA = V\overline{\Sigma}U^HU\Sigma V^H = V\Sigma^2V^H$. So $\|A^HA\|_2 = sup_{\|x\|=1}V\Sigma^2V^Hx = \sigma_i^2 = \|A\|_2^2$, where σ_i is the largest singular value of A. Thus they are all equal, as desired.
iv) For notational convenience, we square both sides. Then we have $\|UAV\|_2^2 = \langle UAV, UAV\rangle = V^HA^HU^HUAV = V^HA^HAV = \langle AV, AV\rangle = \langle A, AVV^H\rangle = \langle A, A\rangle = \|A\|_2^2$, as desired.

Problem 4.32 i) $\|UAV\|_F = \sqrt{\text{tr}\,(V^HA^HU^HUAV)} = \sqrt{\text{tr}\,(V^HA^HAV)} = \sqrt{\text{tr}\,(A^HAVV^H)} = \sqrt{\text{tr}\,(A^HA)} = \|A\|_F$
ii) $\|A\|_F = \sqrt{\text{tr}(A^HA)} = \sqrt{\text{tr}(V\Sigma^HU^HU\Sigma V^H)} = \sqrt{\text{tr}(\Sigma^H\Sigma V^HV)} = \sqrt{\text{tr}(\Sigma^H\Sigma)} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$

Problem 4.33 Let $y = \frac{1}{\sigma_i}Ax$, where σ_i is the greatest singular value of A. Then $|y^HAx| = \left|\frac{1}{\sigma_i}x^HA^HAx\right| = \frac{1}{\sigma_i}\|Ax\|_2^2$. Taking the supremum over $\|x\|_2 = 1$ implies that $|y^HAx| = \|A\|_2$. Note also that when x is a normalized eigenvector of A associated with σ_i, $\|x\|_2 = 1, \|y\|_2 = \frac{1}{\sigma_i}\|Ax\| = 1$. As shown is 4.31, the supremum for the 2-norm of A is achieved when x is the unit eigenvector associated with the largest signular value. Thus the equality holds, as desired.

Problem 4.36 Let

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Then

$$A^HA = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Note $\det(A) = -2$ and the singular values of A are $1, 2$, with eigenvalues $\pm\sqrt{2}$.

Problem 4.38 i) $AA^\dagger A = U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^H = U_1\Sigma_1\Sigma_1^{-1}\Sigma_1V_1^H = U_1\Sigma_1V_1^H = A$
ii) $A^\dagger AA^\dagger = V_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H = V_1\Sigma_1^{-1}\Sigma_1\Sigma_1^{-1}U_1^H = V_1\Sigma_1^{-1}U_1^H = A^\dagger$
iii) $(AA^\dagger)^H = (U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H)^H = U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H = AA^\dagger$
iv) $(A^\dagger A)^H = (V_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^H)^H = V_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^H = A^\dagger A$
v) That $AA^\dagger x \in \mathcal{R}(A)$ is obvious. That it is a projection follows since $AA^\dagger AA^\dagger$. To see it is orthogonal, see that $\langle Ax, b - AA^\dagger b\rangle = x^HA^Hb - x^HA^HAA^\dagger b = x^HA^Hb - x^HV_1\Sigma_1^HU_1^HU_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^Hb = x^HV_1\Sigma_1^HU_1^Hb - x^HV_1\Sigma_1^HU_1^Hb = 0$. Thus it must be an orthogonal projection.
vi) By the proof of Theorem 4.6.1, we know that for $x = A^\dagger b, x \in \mathcal{R}(V_1) = \mathcal{R}(A^H)$. We can see that it is a projection because $A^\dagger AA^\dagger A = A^\dagger A$. To check orthogonality of the projection, observe that $\langle A^Hx, b - A^\dagger Ab\rangle = x^HAb - x^HAA^\dagger Ab = x^HAb - x^HAb = 0$.