Problem Set #5

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Problem 7.1 Consider $x, y \in \text{conv}(S)$. Then for $x_i, y_i \in S,\ x = \lambda_1 x_1 + \cdots + \lambda_k x_k,\ y = \gamma_1 y_1 + \cdots + \gamma_m y_m$. Then we know that $z = \lambda x + (1-\lambda)y = \lambda\lambda_1 x_1 + \cdots + \lambda\lambda_k x_k + (1-\lambda)\gamma_1 y_1 + \cdots + (1-\lambda)\gamma_m y_m$. Observe that the sum of these coefficients is $\lambda(\lambda_1 + \cdots + \lambda_k) + (1-\lambda)(\gamma_1 + \cdots + \gamma_m) = \lambda + 1 - \lambda = 1$. Thus $z \in \text{conv}(S)$, and $\text{conv}(S)$ is a convex set.

Problem 7.2
i) Consider a hyperplane P. Let $x, t \in P$. Then we know that $\langle a, x\rangle = b = \langle a, y\rangle$ for some vector a and scalar b. Then $\langle a, \lambda x + (1-\lambda)y\rangle = \langle a, \lambda x\rangle + \langle a, (1-\lambda)y\rangle = \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle = b$. Thus P is convex.
ii) Consider a half space H. We know that for some vector a and some scalar b, given $x, y \in H,\ \langle a, x\rangle \leq b,\ \langle a, y\rangle \leq b$. So by the logic in part (i), $\langle a, \lambda x + (1-\lambda)y\rangle = \langle a, \lambda x\rangle + \langle a, (1-\lambda)y\rangle = \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle \leq \lambda b + (1-\lambda)b = b$. Thus H is a convex set.

Problem 7.4 Let $p, y \in C$, where $C \subset \mathbb{R}^n$ is a non-empty, closed, and convex set.
i) $\|x = p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle = \langle x-p, x-p\rangle + \langle p-y, p-y\rangle + 2\langle x-p, p-y\rangle$. Expansion and simplification yields that $\|x = p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle = \langle x-y, x-y\rangle = \|x-y\|^2$.
ii) Assume (7.14) holds and $y \neq p$. Then by the positivity of norms, we have $\|x-y\|^2 \geq \|p-y\|^2 + \|x-p\|^2 \Rightarrow \|x-y\|^2 > \|x-p\|^2 \Rightarrow \|x-y\| > \|x-p\|$.
iii) Since $p, y \in C$, we know that $z = \lambda y + (1-\lambda)p \in C$ since C is a convex set. Then by the equality established in (i), we have $\|x-z\|^2 = \|x-p\|^2 + \|p - \lambda y - (1-\lambda)p\|^2 + 2\langle x-p, p-\lambda y - (1-\lambda)p\rangle = \|x-p\|^2 + \langle p - \lambda y - p + \lambda p, p - \lambda y - p + \lambda p\rangle + 2\langle x-p, p-\lambda y - p + \lambda p\rangle = \|x-p\|^2 + \langle \lambda(p-y), \lambda(p-y)\rangle + 2\langle x-p, \lambda(p-y)\rangle = \|x-p\|^2 + \lambda^2\langle p-y, p-y\rangle + 2\lambda\langle x-p, p-y\rangle = \|x-p\|^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2$
iv) Suppose p is the projection of x into C. Then by definition of projection, for $z \in C,\ z \neq p,\ \|x-z\| > \|x-p\|$. First use the definition of z in the formulation of part (iii), and suppose $\lambda = 0$, so y=p. Using the equality established in (i), we immediately have that $0 = 2\langle x-p, 0\rangle$. Now suppose that $\lambda \neq 0$, so $z \neq p$. Then we know that $\|x-z\| > \|x-p\|$, so $\|x-z\|^2 > \|x-p\|^2$ and $0 < \|x-z\| - \|x-p\| = 2\lambda\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2$, so $0 < 2\langle x-p, p-y\rangle + \lambda\|y-p\|^2$.

Problem 7.6