Problem Set #5

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Problem 7.1 Consider $x, y \in \text{conv}(S)$. Then for $x_i, y_i \in S$, $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$, $y = \gamma_1 y_1 + \cdots + \gamma_m y_m$. Then we know that $z = \lambda x + (1-\lambda)y = \lambda\lambda_1 x_1 + \cdots + \lambda\lambda_k x_k + (1-\lambda)\gamma_1 y_1 + \cdots + (1-\lambda)\gamma_m y_m$. Observe that the sum of these coefficients is $\lambda(\lambda_1 + \cdots + \lambda_k) + (1-\lambda)(\gamma_1 + \cdots + \gamma_m) = \lambda + 1 - \lambda = 1$. Thus $z \in \text{conv}(S)$, and $\text{conv}(S)$ is a convex set.

Problem 7.2
i) Consider a hyperplane P. Let $x, t \in P$. Then we know that $\langle a, x\rangle = b = \langle a, y\rangle$ for some vector a and scalar b. Then $\langle a, \lambda x + (1-\lambda)y\rangle = \langle a, \lambda x\rangle + \langle a, (1-\lambda)y\rangle = \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle = b$. Thus P is convex.
ii) Consider a half space H. We know that for some vector a and some scalar b, given $x, y \in H$, $\langle a, x\rangle \leq b$, $\langle a, y\rangle \leq b$. So by the logic in part (i), $\langle a, \lambda x + (1-\lambda)y\rangle = \langle a, \lambda x\rangle + \langle a, (1-\lambda)y\rangle = \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle \leq \lambda b + (1-\lambda)b = b$. Thus H is a convex set.

Problem 7.4 Let $p, y \in C$, where $C \subset \mathbb{R}^n$ is a non-empty, closed, and convex set.
i) $\|x = p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle = \langle x-p, x-p\rangle + \langle p-y, p-y\rangle + 2\langle x-p, p-y\rangle$. Expansion and simplification yields that $\|x = p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle = \langle x-y, x-y\rangle = \|x-y\|^2$.
ii) Assume (7.14) holds and $y \neq p$. Then by the positivity of norms, we have $\|x-y\|^2 \geq \|p-y\|^2 + \|x-p\|^2 \Rightarrow \|x-y\|^2 > \|x-p\|^2 \Rightarrow \|x-y\| > \|x-p\|$.
iii) Since $p, y \in C$, we know that $z = \lambda y + (1-\lambda)p \in C$ since C is a convex set. Then by the equality established in (i), we have $\|x-z\|^2 = \|x-p\|^2 + \|p-\lambda y-(1-\lambda)p\|^2 + 2\langle x-p, p-\lambda y-(1-\lambda)p\rangle = \|x-p\|^2 + \langle p-\lambda y-p+\lambda p, p-\lambda y-p+\lambda p\rangle + 2\langle x-p, p-\lambda y-p+\lambda p\rangle = \|x-p\|^2 + \langle\lambda(p-y), \lambda(p-y)\rangle + 2\langle x-p, \lambda(p-y)\rangle = \|x-p\|^2 + \lambda^2\langle p-y, p-y\rangle + 2\lambda\langle x-p, p-y\rangle = \|x-p\|^2 + 2\lambda\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2$
iv) Suppose p is the projection of x into C. Then by definition of projection, for $z \in C$, $z \neq p$, $\|x-z\| > \|x-p\|$. First use the definition of z in the formulation of part (iii), and suppose $\lambda = 0$, so y=p. Using the equality established in (i), we immediately have that $0 = 2\langle x-p, 0\rangle$. Now suppose that $\lambda \neq 0$, so $z \neq p$. Then we know that $\|x-z\| > \|x-p\|$, so $\|x-z\|^2 > \|x-p\|^2$ and $0 < \|x-z\| - \|x-p\| = 2\lambda\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2$, so $0 < 2\langle x-p, p-y\rangle + \lambda\|y-p\|^2$. Letting $\lambda = 0$ yields the desired inequality. Parts ii and iv are the required arguments to prove the desired statement.

Problem 7.6 Let f be a convex function, and pick $\mathbf{x}, \mathbf{y}$ such that $f(\mathbf{x}), f(\mathbf{y}) \leq c$. Then for $\lambda \in [0,1]$, $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \leq (\lambda + 1 - \lambda)c = c$. Thus the set is convex.

Problem 7.7 Suppose $\{f_i\}_{i=1}^k$ is a set of convex functions. Then $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) =$

$\sum_{i=1}^{k} \lambda_i f_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \sum_{i=1}^{k} (\lambda\lambda_i f_i(\mathbf{x}) + (1-\lambda)\lambda_i f_i(\mathbf{y})) = \lambda \sum_{i=1}^{k} \lambda_i f_i(\mathbf{x}) + (1-\lambda) \sum_{i=1}^{k} \lambda_i f_i(\mathbf{y}) = \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$. Thus f is a convex function.

Problem 7.13 Let f be convex and bounded above, and suppose by contradiction that f is not constant. Then some element of the derivative of f at some x_0 is nonzero. Suppose without loss of generality that it is the i^{th} element of $Df(x_0)$ that does not equal zero. Let $x = \lim_{c\to\infty} c\mathbf{e}_i$. Then by Theorem 7.2.9, $f(x)$ approaches infinity. Thus f is not bounded above. So it must be constant.

Problem 7.20 Let f and $-f$ be convex. Then $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ and $-f(\lambda x + (1-\lambda)y) \leq -\lambda f(x) - (1-\lambda)f(y)$, and be necessity, $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$. Then f is a linear function, and by 7.4.2, it must be affine.

Problem 7.21 Suppose ϕ is increasing. Then $x > y \Leftrightarrow \phi(x) > \phi(y)$. Suppose $\mathbf{x}^*$ is a local minimizer for the problem of minimizing $\phi(f(\mathbf{x}))$ subject to the constraints. Then $\phi(f(\mathbf{x}^*)) < \phi(f(\mathbf{x}))$ for all feasible x not equal to x^*. Because ϕ is increasing, we know that $f(\mathbf{x}^*) < f(\mathbf{x})$ for all feasible x, and thus $\mathbf{x}^*$ is a local minimizer for the problem of minimizing $f(x)$ subject to given constraints. Similarly, suppose $\mathbf{x}^*$ is a local minimizer for the problem of minimizing $f(\mathbf{x})$ subject to the constraints. Then $f(\mathbf{x}^*) < f(\mathbf{x})$ for all feasible x not equal to x^*, and since ϕ is increasing, $\phi(f(\mathbf{x}^*)) < \phi(f(\mathbf{x}))$ for all feasible x not equal to x^*. Thus the two problems are equivalent.