Problem Set #4

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Problem 6.1

$$\begin{aligned} \min \quad & -e^{-w^T x} \\ \text{s.t.} \quad & w^T A w - w^T A y - w^T x \leq -a \\ & y^T w - w^T x = b \end{aligned}$$

Problem 6.5

$$\begin{aligned} \min \quad & -0.07m - 0.05k \\ \text{s.t.} \quad & 4m + 3k \leq 240000 \\ & 2m + k \leq 6000 \end{aligned}$$

where m is the number of milk bottles and k is the number of knobs produced.

Problem 6.6 First observe that

$$\begin{aligned} f_x(x,y) &= 6xy + y + 4y^2 \\ f_y(x,y) &= 3x^2 + 8xy + x \\ f_{xy}(x,y) &= 6x + 8y + 1 \\ f_{xx}(x,y) &= 6y \\ f_{yy}(x,y) &= 8x \end{aligned}$$

Setting the first order derivatives equal to zero yields the following four critical points: $(0,0), (-\frac{1}{3}, 0), (0, -\frac{1}{4}), (-\frac{1}{9}, -\frac{1}{12})$. First consider $(0,0)$. We have that $f_{xx} = f_{yy} = 0$, implying that this point is a saddle point. Now consider $(-\frac{1}{3}, 0)$. We have $f_{xx} = 0, f_{yy} = -\frac{8}{3}, f_{xy} < 0$, so this point is a saddle point. For $(0, -\frac{1}{4})$, we know that $f_{xx} = -\frac{3}{2}, f_{yy} = 0, f_{xx}f_{yy} - fxy^2 = -1$, so this is a saddle point. Now consider $(-\frac{1}{9}, -\frac{1}{12})$. We have $f_{xx} = -\frac{1}{2}, f_{yy} = -\frac{8}{9}, f_{xy} = -\frac{1}{3}, f_{xx}f_{yy} - fxy^2 = \frac{1}{3}$, so the point is a local maximum.

Problem 6.11 We first solve for the unique minimum point of a function $f(x) = ax^2 + bx + c, a > 0$ using Second Order Sufficient conditions. $f'(x) = 2ax + b = 0 \Rightarrow x = -\frac{b}{2a}$. Checking second order conditions shows that $f''(x) = 2a > 0$, so $x = -\frac{b}{2a}$ is an optimal solution. We now demonstrate that Newton's method will immediately converge for such a problem. Note that $x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = x_0 - x_0 - \frac{b}{2a} = -\frac{b}{2a}$, as desired.

Problem 6.14 See the Jupyter notebook included in this same folder titled "Newton's Method.ipynb" for the code.