

### Problem Set #3

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**Problem 4.2** We know from a prior exercise that

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since  $\det(D - \lambda I) = -\lambda^3$ , the only eigenvalue of  $D$  is 0, with algebraic multiplicity 3. Accompanying this eigenvalue is eigenvector  $[1 \ 0 \ 0]^T$ . This implies that eigenvalue 0 has geometric multiplicity equal to 1.

**Problem 4.4 i)** Let

$$A = \begin{bmatrix} a & c + di \\ c - di & b \end{bmatrix}$$

Then the eigenvalues are solutions to  $\lambda^2 - (a + b)\lambda + ab - c^2 - d^2 = 0$ . To determine complexity of roots, we examine the portion inside the square root of the quadratic formula, here equal to  $a^2 + b^2 + 2ab + 4c^2 + 4d^2 = (a + b)^2 + 4c^2 + 4d^2 \geq 0$ . Thus  $A$  can have only real valued eigenvalues.

ii) Let

$$A = \begin{bmatrix} ai & c + di \\ -c + di & bi \end{bmatrix}$$

Eigenvalues are solutions to  $\lambda^2 - (a + b)i\lambda - ab + c^2 + d^2 = 0$ . Note that the quadratic formula implies that eigenvalues will begin with  $(a + b)i$  and are therefore imaginary.

**Problem 4.6** For each  $a_{ii} \in \text{diag}(A)$ , we know that  $\det(A - a_{ii}I) = 0$  since there is a zero in the diagonal and  $A$  is upper-triangular or lower-triangular. So thus each  $a_{ii}$  is an eigenvalue, and by Proposition 4.1.11 there cannot be any other eigenvalues.

**Problem 4.8 i)** We simply show that the elements of  $S$  are linearly independent since  $\text{span}(S) = V$ . By 3.2.5, orthogonal sets are linearly independent, and we showed that  $S$  is a linearly independent set in exercise 3.8.

ii) Note  $D(\sin(x)) = \cos(x)$ ,  $D(\cos(x)) = -\sin(x)$ ,  $D(\sin(2x)) = 2\cos(2x)$ ,  $D(\sin(2x)) = -2\cos(2x)$ . So the matrix representation of  $D$  in this matrix is

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

iii) One subspace is  $\{\cos(x), \sin(x)\}$  and the other is  $\{\cos(2x), \sin(2x)\}$ .

**Problem 4.13** We compute the eigenvalues of  $A$ . The characteristic equation of  $A$  is  $\lambda^2 - 1.4\lambda + 0.4$ . This solves to eigenvalues equal to 1 and 0.4. The eigenvector associated with eigenvalue 1 is the null space of

$$\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$$

which has as a solution the vector  $[2 \ 1]^T$ . The eigenvector associated with eigenvalue 0.4 is the null space of

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}$$

which has a solution of  $[1 \ -1]^T$ . Then

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

**Problem 4.15** Let  $A$  be a semisimple matrix. Then we know  $A$  is diagonalizable so there exist  $P$  and  $D$  such that  $A = P^{-1}DP$ . Note  $f(A) = f(P^{-1}DP) = a_0I + A_1P^{-1}DP + \dots + a_nP^{-1}DP = P^{-1}(a_0I + a_1D + \dots + a_nD^n)P = P^{-1}f(D)P$ . Then the elements of  $f(D)$  must be the eigenvalues of  $f(A)$ .  $f(D)$  is a diagonal matrix whose elements are  $f((\lambda_i)_{i=1}^n)$ .

**Problem 4.16 i)** We use the diagonalization from problem 13, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Note

$$\lim_{k \rightarrow \infty} A^k = \lim_{k \rightarrow \infty} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

This implies

$$A^k - B = \begin{bmatrix} \frac{.4^k}{3} & \frac{-2 \cdot .4^k}{3} \\ -\frac{.4^k}{3} & \frac{2 \cdot .4^k}{3} \end{bmatrix}$$

So  $\|A^k - B\|_1 = \frac{4}{3}(.4)^k$ . We have  $\frac{4}{3}(.4)^k < \epsilon \Rightarrow k > \frac{\ln(\frac{3}{4}\epsilon)}{\ln(.4)}$ . Thus  $\lim_{n \rightarrow \infty} A^n = B$ .

ii) We have  $\|A^k - B\|_\infty = 0.4^k$ .  $0.4^k < \epsilon \Rightarrow k > \frac{\ln \epsilon}{\ln 0.4}$

We also have  $\|A^k - B\|_{FR} = \frac{10}{9}0.4^k < \epsilon \Rightarrow k > \frac{\ln(\frac{9}{10}\epsilon)}{\ln(0.4)}$ . Thus the answer is the same regardless of the norm.

iii) By Theorem 4.3.12, the eigenvalues are  $3 + 5 + 1 = 9$  and  $3 + 2 + 0.4^3 = \frac{633}{125}$ .

**Problem 4.18** By Proposition 4.1.23,  $A$  and  $A^T$  have the same characteristic polynomial. Then by 4.1.8, they have the same eigenvalues. So for all  $\lambda \in \sigma(A)$ ,  $\exists x | A^T x = \lambda x \Rightarrow x^T A = \lambda x^T$ .

**Problem 4.20** Suppose A is orthonormally similar to B and it Hermetian. Then we know that  $B = U^H A U = U^H A^H U = (U^H A U)^H = B^H$ . Thus B is Hermetian.

**Problem 4.24** Suppose A is Hermetian. Then  $x^H A x = \langle x, A x \rangle = \overline{\langle A x, x \rangle} = \overline{x^H A^H x} = x^H A x$ . So since  $x^H A x = \overline{x^H A x}$ , the Rayleigh quotient is real valued only. Now suppose A is skew-Hermetian. Then  $\langle x, A x \rangle = x^H A x = -x^H A^H x = -\langle A x, x \rangle = -\overline{\langle x, A x \rangle}$ . Thus the Rayleigh quotient is imaginary only.

**Problem 4.27** Let A be positive definite. Define a vector  $x_i$  which is all zeroes except for a one in the  $i^{th}$  position. Observe that the  $i^{th}$  element of the diagonal of A,  $A_{ii} = \langle x_i, A x_i \rangle$ . This must be positive due to the positive definiteness of A.

**Problem 4.28** Notice that

$$\text{tr}(AB) = \sum_{i=1}^n \langle e_i, A B e_i \rangle = \sum_{i=1}^n e_i^H A B e_i = \sum_{i=1}^n e_i^H A e_i e_i^H B e_i = \sum_{i=1}^n \langle e_i, A e_i \rangle \langle e_i, B e_i \rangle$$

This must be non-negative because of the positive semi-definiteness of A and B. We also know that

$$\text{tr}(AB) \leq \text{tr}(AB) + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \langle e_i, A e_i \rangle \langle e_j, B e_j \rangle = \text{tr}(A) \text{tr}(B)$$

**Problem 4.32 i)**  $\|U A V\|_F = \sqrt{\text{tr}(V^H A^H U^H U A V)} = \sqrt{\text{tr}(V^H A^H A V)} = \sqrt{\text{tr}(A^H A V V^H)} = \sqrt{\text{tr}(A^H A)} = \|A\|_F$

**ii)**  $\|A\|_F = \sqrt{\text{tr}(A^H A)} = \sqrt{\text{tr}(V \Sigma^H U^H U \Sigma V^H)} = \sqrt{\text{tr}(\Sigma^H \Sigma V^H V)} = \sqrt{\text{tr}(\Sigma^H \Sigma)} = \sqrt{\sigma_1^2 + \dots + \sigma_r^2}$

**Problem 4.36** Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$A^H A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note  $\det(A) = -1$  and the singular values of A are 1.

**Problem 4.38 i)**  $AA^\dagger A = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = U_1 \Sigma_1 \Sigma_1^{-1} \Sigma_1 V_1^H = U_1 \Sigma_1 V_1^H = A$

**ii)**  $A^\dagger A A^\dagger = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = V_1 \Sigma_1^{-1} \Sigma_1 \Sigma_1^{-1} U_1^H = V_1 \Sigma_1^{-1} U_1^H = A^\dagger$

**iii)**  $(AA^\dagger)^H = (U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H)^H = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = AA^\dagger$

**iv)**  $(A^\dagger A)^H = (V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H)^H = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = A^\dagger A$

**v)** That  $AA^\dagger x \in \mathcal{R}(A)$  is obvious. That it is a projection follows since  $AA^\dagger AA^\dagger = AA^\dagger$ . To see it is orthogonal, see that  $\langle Ax, b - AA^\dagger b \rangle = x^H A^H b - x^H A^H A A^\dagger b = x^H A^H b - x^H V_1 \Sigma_1^H U_1^H U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H b = x^H V_1 \Sigma_1^H U_1^H b - x^H V_1 \Sigma_1^H U_1^H b = 0$ . Thus it must be

an orthogonal projection.

**vi)** By the proof of Theorem 4.6.1, we know that for  $x = A^\dagger b, x \in \mathcal{R}(V_1) = \mathcal{R}(A^H)$ . We can see that it is a projection because  $A^\dagger A A^\dagger A = A^\dagger A$ . To check orthogonality of the projection, observe that  $\langle A^H x, b - A^\dagger A b \rangle = x^H A b - x^H A A^\dagger A b = x^H A b - x^H A b = 0$ .