Problem Set #[2]
OSM Lab, Zachary Boyd
John Wilson

Problem 3.1
i)

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle) =$$
$$\frac{1}{4}(\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle - \langle x,x\rangle + 2\langle x,y\rangle - \langle y,y\rangle) = \langle x,y\rangle$$

ii)

$$\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{2}(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle) =$$
$$\frac{1}{2}(\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle + \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle) = \|x\|^2 + \|y\|^2$$

Problem 3.2
Expansion of the norms into inner products and then separation by linearity yields

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) =$$
$$\frac{1}{4}(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle + i\langle x,x\rangle + \dots$$
$$\langle x,y\rangle - \langle y,x\rangle + i\langle y,y\rangle - i\langle x,x\rangle + \langle x,y\rangle - \langle y,x\rangle - i\langle y,y\rangle = \langle x,y\rangle$$

Problem 3.3
By definition 3.3.18, we have that given vectors x and y, $cos\theta = \frac{\langle x,y\rangle}{\|x\|\|y\|}$.
i) Note $\sqrt{\int_0^1 x^2dx} = \frac{1}{\sqrt{3}}$, $\sqrt{\int_0^1 x^{10}dx} = \frac{1}{\sqrt{11}}$, and $\int_0^1 x^6dx = \frac{1}{7}$. So $\theta = \cos^{-1}\left(\frac{\sqrt{33}}{7}\right)$
ii) We have $\sqrt{\int_0^1 x^4dx} = \frac{1}{\sqrt{5}}$, $\sqrt{\int_0^1 x^8dx} = \frac{1}{3}$, and $\int_0^1 x^6dx = \frac{1}{7}$. So $\theta = \cos^{-1}\left(\frac{3\sqrt{5}}{7}\right)$

Problem 3.8
i) Given $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$, we have the following results: $\langle \cos(t), \cos(t)\rangle = 1, \langle \cos(t), \sin(t)\rangle = 0, \langle \cos(t), \cos(2t)\rangle = 0, \langle \cos(t), \sin(2t)\rangle = 0, \langle \sin(t), \sin(t)\rangle = 1,$ $\langle \sin(t), \cos(2t)\rangle = 0, \langle \sin(t), \sin(2t)\rangle = 0, \langle \cos(2t), \cos(2t)\rangle = 1, \langle \cos(2t), \sin(2t)\rangle = 0, \langle \sin(2t), \sin(2t)\rangle = 1$. Thus the set is orthonormal.
ii) $\|t\| = \sqrt{\langle t,t\rangle} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t^2dt} = \frac{\sqrt{6}\pi}{3}$
iii) $proj_X(\cos(3t)) = \sum_{f\in S}\langle f, \cos(3t)\rangle f = 0$. It results that $\cos(3t)$ is orthogonal to this set.
iv) $proj_X t = \sum_{f\in S}\langle f,t\rangle f = 2\sin t - \sin(2t)$

Problem 3.9 Let $\mathbf{x} = [x_1 \quad x_2]^T, \mathbf{y} = [y_1 \quad y_2]^T$ be real valued vectors. Then $\langle R_\theta\mathbf{x}, R_\theta\mathbf{y}\rangle = \cos^2\theta x_1y_1 - \cos\theta\sin\theta(x_1y_2 + x_2y_1) + \sin^2\theta(x_2y_2 + x_1y_1) + \cos\theta\sin\theta(x_1y_2 +$

$x_2y_1) + \cos^2\theta x_2y_2 = x_1y_1 + x_2y_2 = \langle x, y\rangle$, as desired.

Problem 3.10 i) Let Q be an orthonormal matrix. Then $\langle Qx, Qy\rangle = \langle x, y\rangle \Rightarrow x^HQ^HQy = x^Hy \Rightarrow Q^HQy = y \Rightarrow Q^HQ = I$. By Proposition 3.2.12, we have that Q is invertible, since the field is of finite dimension n. Thus by uniqueness of inverses, we also have $QQ^H = I$, as desired.
ii) Since i) holds, we have $\|Q\mathbf{x}\| = \sqrt{\langle x^HQ^HQx\rangle} = \sqrt{\langle x^Hx\rangle} = \|x\|$.
iii) Note that by i), $Q^{-1} = Q^H$. We have $\langle Q^Hx, Q^Hy\rangle = x^HQQ^Hy = x^Hy = \langle x, y\rangle$, as desired.
iv) Observe that the i^{th} column of Q is equal to Qe_i, where e_i is the i^{th} standard basis vector. Then $\langle Qe_i, Qe_j\rangle = \langle e_i, e_j\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. This establishes orthonormality of the columns.
v) Let Q be orthonormal. Then we know, since $Q^HQ = 1$, that $\det Q^HQ = 1$. So $1 = \det Q^HQ = (\det Q)^2$, so $|\det Q| = 1$.The converse is not true. Take the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix has determinant 1 but the second column is not a unit vector, so it is not an orthonormal matrix.
vi) Let $Q_1, Q_2 \in M_n(\mathbb{F})$ be orthonormal matrices. Then $\langle Q_1Q_2x, Q_1Q_2y\rangle = x^HQ_2^HQ_1^HQ_1Q_2y = x^HQ_2^HQ_2y = x^Hy = \langle x, y\rangle$, showing that the product is orthonormal.

Problem 3.11 Suppose x_i is a linear combination of the elements x_j of your basis set with $j < i$. When this happens, the difference $x_i - p_{i-1}$ will equal zero, and division by the norm of this difference will cause the problem to be undefined. Thus each element of the basis set must be linearly independent.

Problem 3.16 i) We provide a counterexample. Consider the matrices Q and R given in Example 3.3.11. Modify Q such that

$$Q = \begin{bmatrix} 1 & -1 & 1 & -\sqrt{2} \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & \sqrt{2} \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

We still have that $A = QR$ and Q is orthonormal, showing that QR decompositions are not unique.
ii) TODO

Problem 3.17 $A^HAx = A^Hb \Leftrightarrow \widehat{R}^H\widehat{Q}^H\widehat{Q}\widehat{R}x = \widehat{R}^H\widehat{Q}^Hb \Leftrightarrow \widehat{R}^H\widehat{R}x = \widehat{R}^H\widehat{Q}^Hb \Leftrightarrow \widehat{R}x = \widehat{Q}^Hb$

Problem 3.23 Note $\|x\| = \|x + y - y\| \leq \|x - y\| + \|y\|$ by the triangle inequality, so $\|x\| - \|y\| \leq \|x - y\|$. Similar logic yields $\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$. Since

$|\|x\| - \|y\||$ is one of these two cases, the result is immediate.

Problem 3.24 Norm properties: 1: $\|\mathbf{x}\| \geq 0$, with equality only if $x = 0$. 2: $\|a\mathbf{x}\| = |a|\|\mathbf{x}\|$. 3: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

i) Note $|\cdot|$ is a norm, so the three properties hold. Then $\|f\|_{L^1} = \int_a^b |f(t)|dt \geq 0$ since the integrand is non-negative. Similarly, $\int_a^b |f(t)|dt = 0 \Leftrightarrow |f(t)| = 0 \Leftrightarrow f = 0$. So property 1 holds. Property 2: $\|af\|_{L^1} = \int_a^b |af(t)|dt = |a| \int_a^b |f(t)|dt = |a|\|f\|_{L^1}$. Property 3: $\|f + g\|_{L^1} = \int_a^b |f(t) + g(t)|dt \leq \int_a^b |f(t)| + |g(t)|dt = \int_a^b |f(t)|dt + \int_a^b |g(t)|dt = \|f\|_{L^1} + \|g\|_{L^1}$

ii) 1: $\|f\|_{L^2} = \left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} \geq 0$ since the integrand is non-negative. Similarly, $\left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} = 0 \Leftrightarrow |f(t)| = 0 \Leftrightarrow f = 0$. So property 1 holds. Property 2: $\|af\|_{L^2} = \left(\int_a^b |af(t)|^2 dt\right)^{\frac{1}{2}} = \left(\int_a^b |a|^2|f(t)|^2 dt\right)^{\frac{1}{2}} = \left(|a|^2 \int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} = |a| \left(\int_a^b |f(t)|^2 dt\right)^{\frac{1}{2}} = |a|\|f\|_{L^2}$. Property 3: First observe that by i), both $\|f + g\|_{L^2}$ and $\|f\|_{L^2} + \|g\|_{L^2}$ are positive quantities. Then we can square both sides and show equivalently that $\|f + g\|_{L^2}^2 \leq \|f\|_{L^2}^2 + 2\|f\|_{L^2}\|g\|_{L^2} + \|g\|_{L^2}^2$. See that $\|f + g\|_{L^2}^2 = \int_a^b |f(t) + g(t)|^2 dt \leq \int_a^b |f(t)|^2 + |f(t)||g(t)| + |g(t)|^2 dt = \int_a^b |f(t)|^2 dt + 2\int_a^b |f(t)||g(t)|dt + \int_a^b |g(t)|^2 dt = \|f\|_{L^2}^2 + 2\|f\|_{L^2}\|g\|_{L^2} + \|g\|_{L^2}^2$. Thus $\|f + g\|_{L^2}^2 \leq \|f\|_{L^2}^2 + 2\|f\|_{L^2}\|g\|_{L^2} + \|g\|_{L^2}^2$ and $\|f + g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$ and 3 holds.

iii) 1: $\|f\|_{L^\infty} = \sup_{x \in [a,b]} |f(t)| \geq 0$ with $\sup_{x \in [a,b]} |f(t)| = 0$ only when $f = 0$, by properties of supremums and norms. 2: $\|cf\|_{L^\infty} = \sup_{x \in [a,b]} |cf(t)| = \sup_{x \in [a,b]} |c||f(t)| = |c| \sup_{x \in [a,b]} |f(t)| = |c|\|f\|_{L^\infty}$. Property 3: $\|f + g\|_{L^\infty} = \sup_{x \in [a,b]} |f(t) + g(t)| \leq \sup_{x \in [a,b]} |f(t)| + \sup_{x \in [a,b]} |g(t)| = \|f\|_{L^\infty} + \|g\|_{L^\infty}$ by supremum and absolute value properties.