Math Problem Set #1
OSM Bootcamp, Dr. Schmedders
John Wilson

Problem 1
3.6 We first proceed to show that:

$$A = \bigcup_{i \in \mathcal{I}} (A \cap B_i)$$

Let $x \in A$. Since $\Omega = \cup_{i \in \mathcal{I}} B_i$ and $x \in \Omega$, it follows that $x \in B_i$ for some i. It follows then that $x \in A \cap B_i$. Thus by definition $x \in \bigcup_{i \in \mathcal{I}}(A \cap B_i)$, and $A \subset \bigcup_{i \in \mathcal{I}}(A \cap B_i)$. Now assume $x \in \bigcup_{i \in \mathcal{I}}(A \cap B_i)$. It then follows that $x \in A$, so we know $\bigcup_{i \in \mathcal{I}}(A \cap B_i) \subset A$. Then $A = \bigcup_{i \in \mathcal{I}}(A \cap B_i)$, as desired.

By Definition 3.1.5 (ii), we can conclude that

$$P(A) = P\left(\bigcup_{i \in \mathcal{I}} (A \cap B_i)\right) = \sum_{i \in \mathcal{I}} P(A \cap B_i)$$

□

3.8 By Propostion 3.1.6 (i), we know that

$$P(\bigcup_{k=1}^{n} E_k) = 1 - P((\bigcup_{k=1}^{n} E_k)^c) = 1 - P(\bigcap_{k=1}^{n} E_k^c)$$

Because of the independence of $\{E_k\}_{k=1}^{n}$, by Definition 3.2.3 we know

$$1 - P(\bigcap_{k=1}^{n} E_k^c) = 1 - \prod_{k=1}^{n} P(E_k^c) = 1 - \prod_{k=1}^{n}(1 - P(E_k))$$

Then

$$P(\bigcup_{k=1}^{n} E_k) = 1 - \prod_{k=1}^{n}(1 - P(E_k))$$

□

3.11 Let C^+ denote the event that the person is a criminal, C^- denote that the person is innocent, and S^+ denote the probability of a person testing positive. Application of Bayes' rule yields

$$P(C^+|S^+) = \frac{P(S^+|C^+)P(C^+)}{P(S^+|C^+)P(C^+) + P(S^+|C^-)P(C^-)}$$

Plugging in the correct probabilities yields

$$P(C^+|S^+) = \frac{1 \cdot (\frac{1}{250000000})}{1 \cdot (\frac{1}{250000000}) + \frac{1}{3000000}(1 - \frac{1}{250000000})} = 0.0118$$

3.12 Consider first the three door case. Note that without switching doors, there is a 1/3 chance of the contestant having selected the correct door and winning the prize. Suppose that they decide to switch. There is a 1/3 chance of them having first selected the correct door, in which case they are guaranteed to lose. There is a 2/3 chance that they selected the incorrect door, in which case they are guaranteed to win. Thus if the contestant switches doors, there is a $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$ chance of winning.

Consider now the 10 door case. The logic is similiar. Without swtiching doors, there is a $\frac{1}{10}$ chance of winning the game. If they do switch doors, there is a $\frac{1}{10}$ chance that they first selected the correct door, in which case they lose. There is a $\frac{9}{10}$ chance that they first selected the incorrect door, in which case they are guaranteed to win. Thus there is a $\frac{1}{10} \cdot 0 + \frac{9}{10} \cdot 1 = \frac{9}{10}$ chance of winning. □

3.16 By Definition 3.3.17 and the linearity of expectation (Theorem 3.3.12), we have

$$Var[X] = E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - E[X]^2$$

□

3.33 Let $B = X_1 + X_2 + \ldots + X_n$, where $X_i \sim Bernoulli(p)$ and X_i is independent from X_j for all $i \neq j$. Then $Var[X] = E[X^2] - E[X]^2 = p + 0(1-p) - (p^2) = p(1-p)$. Having established that $Var[X] = p(1-p)$, we consider the Weak Law of Large Numbers (Theorem 3.6.5). It follows from the Law that for all $\epsilon > 0$, letting X_i be defined as before, that

$$P\left(\left|\frac{B}{n} - p\right| \geq \epsilon\right) = P\left(\left|\frac{X_1 + \ldots + X_n}{n} - p\right| \geq \epsilon\right) \leq \frac{p(1-p)}{n\epsilon^2}$$

□

3.36 Note that the probability of overadmission for the school is one minus the probability of underadmitting. Thus we seek to find $P(Y \leq y)$, where

$$Y = \frac{S_n - n\mu}{\sqrt{n}\sigma}$$

The Central Limit Theorem states that Y approaches the standard normal distribution as n increases. We can treat this problem as a series of Bernoulli trials, with p=0.801 being the probability of success, or that an admitted student enrolls. Note $y = \frac{5500 - 6242 \cdot 0.801}{\sqrt{6242} \cdot \sqrt{(0.801)(1-0.801)}} \approx 15.851$. Since for a standard normal distribution Y, $P(Y \leq 15.851) \approx 1$, the probability of the school over-admitting students is approximately 0. □

Problem 2

Part (a) Let $\Omega = \{1, 2, 3, 4\}$, each with an equal likelihood. Let $A = \{1, 2\}$,

$B = \{1,3\}$, and $C = \{2,3\}$. Then $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4} = P(A)P(B)$, $P(A \cap C) = \frac{1}{4} = P(A)P(C)$, $P(B \cap C) = \frac{1}{4} = P(B)P(C)$, but $P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C) = \frac{1}{8}$.
Part (b) Let $\Omega = \{1,2,3,4,5,6,7,8\}$, each with equal likelihood of occuring. Define $A = \{1,2,3,4\}$, $B = \{2,3,7,8\}$, and $C = \{3,4,5,6\}$. Then the desired conditions are met. □

Problem 3
Given $P(d), d \in \mathcal{D} \equiv \{1,2,3,\ldots,9\}$ with P the distribution defined by Benford's Law,

$$\sum_{d\in\mathcal{D}} P(d) = \log_{10}(1+\frac{1}{1}) + \ldots + \log_{10}(1+\frac{1}{9}) = \log_{10}(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{10}{9}) = \log_{10} 10 = 1$$

Problem 4
Part (a)

$$E(X) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n 2^n = \sum_{n=1}^{\infty} 1 = \infty$$

Part (b)

$$E(\ln X) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \ln 2^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n n \ln 2 = 2\ln 2$$

Problem 5
We first calculate the unit investment value under each of the possible outcomes. Note that since both investment opportunities have the same interest rate, we can ignore that rate in our calculations. Suppose that one country invests entirely in the other's currency, and the conversion rate changes unfavorable to 1:1.25. Then the return per unit of currency is $\frac{1}{1.25}$. If the currency changes favorably to 1.25:1, the return per unit is $\frac{1.25}{1}$. Since either outcome is equally likely, the expected return for one country investing entirely in the other's currency is $.5 \cdot \frac{1}{1.25} + .5 \cdot \frac{1.25}{1} = 1.025$. The expected return for not investing is 1. Thus each country should invest entirely in the other's currency.

Problem 6
Part (a) Since $[0,1]$ and $\mathbb{R}$ have equal cardinality, first define some function $f(x)$ which maps the interval $[0,1]$ to the interval $[1,\mathbb{R})$. Then let X have the Pareto distribution with parameter $x_m = 1$ and $\alpha = 2$. So if g is the pdf of the Pareto distribution, the pdf of X will be $h_X(x) = g(f(x))$. Then by properties of pareto distributions,

$E(X)$ is finite but X has infinite variance, so $E(X^2)$ is infinite, as desired.
Part (b) Define Y be the uniform distribution from 0.855 to 0.865. Y has an expected value of 0.860. Let X be a distribution defined as follows: $f(x) = 6x^5 x \in [0, 1], 0$ otherwise. Then $E[X] = 0.857 < E[Y]$, but $P(X > 0.87 > Y) \approx 0.566$, as desired.
Part (c) Let X, Y, and Z be uniform distributions over the following domains: $X : [-3, 3], Y : [-2, 2], Z : [-1, 1]$. Then the desired characteristics hold.

Problem 7
Part (a) We examine the pdf of Y. We have

$$f_Y(x) = \frac{1}{2} f_X(x) + \frac{1}{2} f_X(-x) = \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{-(x)^2}{2}} dx \right) + \dots$$
$$\frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{-(-x)^2}{2}} dx \right) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x)^2}{2}} dx$$

This is the standard normal distribution. Thus Y is a standard normal distribution.
Part (b) True. Note $|Y| = |XZ| = |X|$. This holds since $|Z| = 1$. Since $|X| = |Y|, P(|X| = |Y|) = 1$.
Part (c) Independence of X and Y would suggest that $P(Y < x|X > z) = P(Y < x)$ for some x, z. Note, however, that $P(Y < 0|X > 1) < 0.5 = P(Y < 0)$. Thus X and Y are not independent.
Part (d) True. $Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) = E(X^2 Z) = E(X^2)E(Z) = 0$, since X and Z are independent, and X and Y have expected value equal to zero.
Part (e) False. The X and Y distributions described in this problem provide a counterexample to his proposition.

Problem 8
First we consider the random variable $M = \max\{X_1, X_2, \dots, X_n\}$. The cdf $F(x) = P(M \leq x)$ is simply equal to x in the case that $n = 1$, by properties of uniform distributions. Since each distribution is independent and identically distributed, the probability that M remains above x is equal to x^n. Thus $F(x) = x^n$. The find the pdf, we differentiate the cdf with respect to x. We have $f(x) = nx^{n-1}$. Expected value is calculated as follows:

$$E(M) = \int_0^1 xnx^{n-1} dx = \int_0^1 nx^n dx = \frac{n}{n+1} x^{n+1}|_{x=0}^1 = \frac{n}{n+1}$$

Now we consider the cdf of m, with $m = \min\{X_1, X_2, \dots, X_n\}$. The cdf $F(x) = P(m \leq x) = 1 - P(m \geq x) = 1 - (1 - P(X_i \leq x)) \forall i \in \{1, \dots, n\} = 1 - (1 - x)^n$. Differentiation with respect to x yields the pdf $f(x) = n(1 - x)^{n-1}$. First noting that $(1 - x)^n = (1 - x)^{n-1} - x(1 - x)^{n-1}$ and therefore $x(1 - x)^{n-1} = (1 - x)^{n-1} - (1 - x)^n$, calculations for expected value are:

$$E(m) = \int_0^1 nx(1-x)^{n-1}dx = n\int_0^1 (1-x)^{n-1} - (1-x)^n dx =$$

$$\left((1-x)^n + \frac{n}{n+1}(1-x)^{n+1}\right]_0^1 = 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

Problem 9

Part (a) We use the Central Limit Theorem. Given $Y = \frac{S_n - n\mu}{\sqrt{n}\sigma}$, for the amount of good states to vary by more than twenty below has $y = -1.2649$. This has a p-value of approximately 0.1038. The probability of exceeding 520 is the same. Thus there is a $1 - 2 \cdot 0.1038 = 0.7924$ of differing from 500 by at most 2%.

Part (b) Again we use the Weak Law of Large numbers. Call p the sample proportion of good states. The weak law states that

$$P\left(|p - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}$$

where $\mu = .5$. Setting $\epsilon = .005$ and $\sigma = .5$ and setting $\frac{\sigma^2}{n\epsilon^2} = 0.01$ yields n=1,000,000. Thus the minimum required size is 1,000,000.

Problem 10

First note that $e^{\theta x}$ is a convex fuction for all values of x and θ. Thus we can apply Jensen's Inequality. We know that $E[e^{\theta X}] \geq e^{\theta E[X]}$. Since $E[X] < 0, \theta \neq 0$, and $e^{\theta E[X]} \leq 1$, $\theta E[X] < 0$, and $\theta > 0$.