Problem Set #[2]

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Problem 3.1
i)

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle) =$$
$$\frac{1}{4}(\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle - \langle x,x\rangle + 2\langle x,y\rangle - \langle y,y\rangle) = \langle x,y\rangle$$

ii)

$$\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{2}(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle) =$$
$$\frac{1}{2}(\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle + \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle) = \|x\|^2 + \|y\|^2$$

Problem 3.2
Expansion of the norms into inner products and then separation by linearity yields

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) =$$
$$\frac{1}{4}(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle + i\langle x,x\rangle + \dots$$
$$\langle x,y\rangle - \langle y,x\rangle + i\langle y,y\rangle - i\langle x,x\rangle + \langle x,y\rangle - \langle y,x\rangle - i\langle y,y\rangle = \langle x,y\rangle$$

Problem 3.3
By definition 3.3.18, we have that given vectors x and y, $cos\theta = \frac{\langle x,y\rangle}{\|x\|\|y\|}$.
i) Note $\sqrt{\int_0^1 x^2 dx} = \frac{1}{\sqrt{3}}$, $\sqrt{\int_0^1 x^{10} dx} = \frac{1}{\sqrt{11}}$, and $\int_0^1 x^6 dx = \frac{1}{7}$. So $\theta = \cos^{-1}\left(\frac{\sqrt{33}}{7}\right)$
ii) We have $\sqrt{\int_0^1 x^4 dx} = \frac{1}{\sqrt{5}}$, $\sqrt{\int_0^1 x^8 dx} = \frac{1}{3}$, and $\int_0^1 x^6 dx = \frac{1}{7}$. So $\theta = \cos^{-1}\left(\frac{3\sqrt{5}}{7}\right)$

Problem 3.8
i) Given $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$, we have the following results: $\langle \cos(t), \cos(t)\rangle = 1, \langle \cos(t), \sin(t)\rangle = 0, \langle \cos(t), \cos(2t)\rangle = 0, \langle \cos(t), \sin(2t)\rangle = 0, \langle \sin(t), \sin(t)\rangle = 1, \langle \sin(t), \cos(2t)\rangle = 0, \langle \sin(t), \sin(2t)\rangle = 0, \langle \cos(2t), \cos(2t)\rangle = 1, \langle \cos(2t), \sin(2t)\rangle = 0, \langle \sin(2t), \sin(2t)\rangle = 1$. Thus the set is orthonormal.
ii) $\|t\| = \sqrt{\langle t,t\rangle} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt} = \frac{\sqrt{6}\pi}{3}$
iii) $proj_X(\cos(3t)) = \sum_{f\in S}\langle f, \cos(3t)\rangle f = 0$. It results that $\cos(3t)$ is orthogonal to this set.
iv) $proj_X t = \sum_{f\in S}\langle f,t\rangle f = 2\sin t - \sin(2t)$

Problem 3.9 Let $\mathbf{x} = [x_1 \quad x_2]^T, \mathbf{y} = [y_1 \quad y_2]^T$ be real valued vectors. Then $\langle R_\theta \mathbf{x}, R_\theta \mathbf{y}\rangle = \cos^2\theta x_1 y_1 - \cos\theta\sin\theta(x_1y_2 + x_2y_1) + \sin^2\theta(x_2y_2 + x_1y_1) + \cos\theta\sin\theta(x_1y_2 +$

$x_2y_1) + \cos^2\theta x_2y_2 = x_1y_1 + x_2y_2 = \langle x, y\rangle$, as desired.

Problem 3.10 i) Let Q be an orthonormal matrix. Then $\langle Qx, Qy\rangle = \langle x, y\rangle \Rightarrow x^HQ^HQy = x^Hy \Rightarrow Q^HQy = y \Rightarrow Q^HQ = I$. By Proposition 3.2.12, we have that Q is invertible, since the field is of finite dimension n. Thus by uniqueness of inverses, we also have $QQ^H = I$, as desired.
ii) Since i) holds, we have $\|Qx\| = \sqrt{\langle x^HQ^HQx\rangle} = \sqrt{\langle x^Hx\rangle} = \|x\|$.
iii) Note that by i), $Q^{-1} = Q^H$. We have $\langle Q^Hx, Q^Hy\rangle = x^HQQ^Hy = x^Hy = \langle x, y\rangle$, as desired.
iv) Observe that the i^{th} column of Q is equal to Qe_i, where e_i is the i^{th} standard basis vector. Then $\langle Qe_i, Qe_j\rangle = \langle e_i, e_j\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta. This establishes orthonormality of the columns.
v) Let Q be orthonormal. Then we know, since $Q^HQ = 1$, that $\det Q^HQ = 1$. So $1 = \det Q^HQ = (\det Q)^2$, so $|\det Q| = 1$.The converse is not true. Take the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix has determinant 1 but the second column is not a unit vector, so it is not an orthonormal matrix.
vi) Let $Q_1, Q_2 \in M_n(\mathbb{F})$ be orthonormal matrices. Then $\langle Q_1Q_2x, Q_1Q_2y\rangle = x^HQ_2^HQ_1^HQ_1Q_2y = x^HQ_2^HQ_2y = x^Hy = \langle x, y\rangle$, showing that the product is orthonormal.

Problem 3.11 Suppose x_i is a linear combination of the elements x_j of your basis set with $j < i$. When this happens, the difference $x_i - p_{i-1}$ will equal zero, and division by the norm of this difference will cause the problem to be undefined. Thus each element of the basis set must be linearly independent.

Problem 3.16 i) We provide a counterexample. Consider the matrices Q and R given in Example 3.3.11. Modify Q such that

$$Q = \begin{bmatrix} 1 & -1 & 1 & -\sqrt{2} \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & \sqrt{2} \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

We still have that $A = QR$ and Q is orthonormal, showing that QR decompositions are not unique.
ii) Let $A = Q_1R_1 = Q_2R_2$ and R_1, R_2 have only positive diagonal elements. Then it follows that $R_1R_2^{-1} = Q_1^{-1}Q_2 = Q_1^HQ_2$. Since upper triangularity and orthornormality are closed under matrix multiplication, $R_1R_2^{-1}$ must be an orthonormal, upper triangular matrix with all positive entries along the diagonal. This implies that it must equal the identity matrix. Then $I = Q_1^HQ_2$, and $Q_2^{-1} = Q_2^H = Q_1^H$. Thus $Q_2 = Q_1$. Identical logic shows that $R_2 = R_1$. Thus the QR decomposition is unique here.

Problem 3.17 $A^HAx = A^Hb \Leftrightarrow \widehat{R}^H\widehat{Q}^H\widehat{Q}\widehat{R}x = \widehat{R}^H\widehat{Q}^Hb \Leftrightarrow \widehat{R}^H\widehat{R}x = \widehat{R}^H\widehat{Q}^Hb \Leftrightarrow \widehat{R}x = \widehat{Q}^Hb$

Problem 3.23 Note $\|x\| = \|x+y-y\| \leq \|x-y\| + \|y\|$ by the triangle inequality, so $\|x\| - \|y\| \leq \|x-y\|$. Similar logic yields $\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$. Since $|\|x\| - \|y\||$ is one of these two cases, the result is immediate.

Problem 3.24 Norm properties: 1: $\|\mathbf{x}\| \geq 0$, with equality only if $x = 0$. 2: $\|a\mathbf{x}\| = |a|\|\mathbf{x}\|$. 3: $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
i) Note $|\cdot|$ is a norm, so the three properties hold. Then $\|f\|_{L^1} = \int_a^b |f(t)|dt \geq 0$ since the integrand is non-negative. Similarly, $\int_a^b |f(t)|dt = 0 \Leftrightarrow |f(t)| = 0 \Leftrightarrow f = 0$. So property 1 holds. Property 2: $\|af\|_{L^1} = \int_a^b |af(t)|dt = |a|\int_a^b |f(t)|dt = |a|\|f\|_{L^1}$. Property 3: $\|f+g\|_{L^1} = \int_a^b |f(t)+g(t)|dt \leq \int_a^b |f(t)| + |g(t)|dt = \int_a^b |f(t)|dt + \int_a^b |g(t)|dt = \|f\|_{L^1} + \|g\|_{L^1}$
ii) 1: $\|f\|_{L^2} = \left(\int_a^b |f(t)|^2dt\right)^{\frac{1}{2}} \geq 0$ since the integrand is non-negative. Similarly, $\left(\int_a^b |f(t)|^2dt\right)^{\frac{1}{2}} = 0 \Leftrightarrow |f(t)| = 0 \Leftrightarrow f = 0$. So property 1 holds. Property 2: $\|af\|_{L^2} = \left(\int_a^b |af(t)|^2dt\right)^{\frac{1}{2}} = \left(\int_a^b |a|^2|f(t)|^2dt\right)^{\frac{1}{2}} = \left(|a|^2\int_a^b |f(t)|^2dt\right)^{\frac{1}{2}} = |a|\left(\int_a^b |f(t)|^2dt\right)^{\frac{1}{2}} = |a|\|f\|_{L^2}$. Property 3: First observe that by i), both $\|f+g\|_{L^2}$ and $\|f\|_{L^2} + \|g\|_{L^2}$ are positive quantities. Then we can square both sides and show equivalently that $\|f+g\|_{L^2}^2 \leq \|f\|_{L^2}^2 + 2\|f\|_{L^2}\|g\|_{L^2} + \|g\|_{L^2}^2$. See that $\|f+g\|_{L^2}^2 = \int_a^b |f(t)+g(t)|^2dt \leq \int_a^b |f(t)|^2 + 2|f(t)||g(t)| + |g(t)|^2dt = \int_a^b |f(t)|^2dt + 2\int_a^b |f(t)||g(t)|dt + \int_a^b |g(t)|^2dt = \|f\|_{L^2}^2 + 2\|f\|_{L^2}\|g\|_{L^2} + \|g\|_{L^2}^2$. Thus $\|f+g\|_{L^2}^2 \leq \|f\|_{L^2}^2 + 2\|f\|_{L^2}\|g\|_{L^2} + \|g\|_{L^2}^2$ and $\|f+g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$ and 3 holds.
iii) 1: $\|f\|_{L^\infty} = \sup_{x\in[a,b]} |f(t)| \geq 0$ with $\sup_{x\in[a,b]} |f(t)| = 0$ only when $f = 0$, by properties of supremums and norms. 2: $\|cf\|_{L^\infty} = \sup_{x\in[a,b]} |cf(t)| = \sup_{x\in[a,b]} |c||f(t)| = |c|\sup_{x\in[a,b]} |f(t)| = |c|\|f\|_{L^\infty}$. Property 3: $\|f+g\|_{L^\infty} = \sup_{x\in[a,b]} |f(t)+g(t)| \leq \sup_{x\in[a,b]} |f(t)| + \sup_{x\in[a,b]} |g(t)| = \|f\|_{L^\infty} + \|g\|_{L^\infty}$ by supremum and absolute value properties. Thus it is a norm.

Problem 3.26 We show that it is an equivalence relation. Symmetry: suppose $\|\cdot\|_a \approx \|\cdot\|_b$. Then by definition $m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$ which implies that $\frac{1}{M}\|x\|_b \leq \|x\|_a \leq \frac{1}{m}\|x\|_b$, with $0 < \frac{1}{M} \leq \frac{1}{m}$ as desired.
Reflexiveness: We know $1\|x\|_a \leq \|x\|_a \leq 1\|x\|_a$, and thus $\|\cdot\|_a \approx \|\cdot\|_a$.
Transitivity: Suppose $\|\cdot\|_a \approx \|\cdot\|_b$ and $\|\cdot\|_b \approx \|\cdot\|_c$. Then there exist u, U, m, M such that $0 < u \leq U, 0 < m \leq M$ and $m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$ and $u\|x\|_b \leq \|x\|_c \leq U\|x\|_b$. Multiplication yields $mu\|x\|_a \leq u\|x\|_b \leq \|x\|_c \leq U\|x\|_b \leq MU\|x\|_a$, with $0 < mu \leq MU$, as desired. Thus $\|\cdot\|_a \approx \|\cdot\|_c$, and tolopogical equivalence is an equivalence relation.
i) We have $\|\mathbf{x}\|_2 = \|x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n\|_2 \leq \|x_1\mathbf{e}_1\| + \cdots + \|x_n\mathbf{e}_n\| = \sqrt{x_1^2} + \cdots + \sqrt{x_n^2} = |x_1| + \cdots + |x_n| = \|x\|_1$ We proceed to prove the second inequality with the Cauchy-

Schwarz inequality. Define y to be the vector of all ones, but where the sign of each y_i is equal to the sign of x_i. Then by Cauchy-Schwarz, $\|x\|_1 = |\langle x, y\rangle| \leq \|y\|_2\|x\|_2 = \sqrt{n}\|x\|_2$, as desired. Then $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$.
ii) Let x_i be the element of x with greatest magnitude. Then we have that $\|x\|_\infty = |x_i| = \sqrt{x_i^2} \leq \sqrt{x_1^2 + \cdots + x_i^2 + \cdots + x_n^2} = \|x\|_2 \leq \sqrt{x_i^2 + \cdots + x_i^2} = \sqrt{nx_i^2} = \sqrt{n}\|x\|_\infty$.

Problem 3.28 We proceed by definition of the induced norm and by the properties shown in problem 3.26.
i) $\frac{1}{\sqrt{n}}\|A\|_2 = \sup\frac{\|Ax\|_2}{\sqrt{n}\|x\|_2} \leq \sup\frac{\|Ax\|_2}{\|x\|_1} \leq \sup\frac{\|Ax\|_1}{\|x\|_1} = \|A\|_1 \leq \sup\frac{\sqrt{n}\|Ax\|_2}{\|x\|_1} \leq \sqrt{n}\sup\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{n}\|A\|_2$
ii) $\frac{1}{\sqrt{n}}\|A\|_\infty = \sup\frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_\infty} \leq \sup\frac{\|Ax\|_\infty}{\|x\|_2} \leq \sup\frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2 \leq \sup\frac{\|Ax\|_2}{\|x\|_\infty} \leq \sup\frac{\sqrt{n}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty$

Problem 3.29 We show $\sup\frac{\|Rx\|_2}{\|x\|_2} \leq 1$. We have $\sup\frac{\|Ax\|_2}{\|A\|_2\|x\|_2} \leq \sup\frac{\|A\|_2}{\|A\|_2} = 1$. So $\|Rx\|_2 \leq \|x\|_2$. Let $A = I$. Then $\|Ix\|_2 = \|x\|_2$. So $\sup(\|Ax\|_2) \geq \|x\|_2$. Then $\|Rx\|_2 = \|x\|_2$.

Problem 3.30 We show that $\|\cdot\|_S$ has the properties of non-negativity (with equality only if the interior of the norm is zero), scalar preservation, triangle inequality, and the submultiplicative property. We rely on the fact that the matrix norm $\|\cdot\|$ has those properties.
Positivity: $\|A\|_S = \|SAS^{-1}\| = 0 \Leftrightarrow SAS^{-1} = 0 \Leftrightarrow A = 0$, since S is invertible. Thus $\|A\|_S = 0 \Leftrightarrow A = 0$ and similarly $\|A\|_S \geq 0$.
Scalar preservation: $\|cA\|_S = \|ScAS^{-1}\| = |c|\|SAS^{-1}\| = |c|\|A\|_S$
Triangle inequality: $\|A+B\|_S = \|S(A+B)S^{-1}\| = \|SAS^{-1}+SBS^{-1}\| \leq \|SAS^{-1}\|+\|SBS^{-1}\| = \|A\|_S + \|B\|_S$
Submultiplicative property: $\|AB\|_S = \|SABS^{-1}\| = \|SAS^{-1}SBS^{-1}\| \leq \|SAS^{-1}\|\|SBS^{-1}\| = \|A\|_S\|B\|_S$

Problem 3.37 We first establish an orthonormal basis for the space $\mathbb{R}[x; 2]$. Take as a preliminary basis the set $\{1, x, x^2\}$ and apply the Gram-Schmidt orthonormalization process. This yields

$$\|1\| = \int_0^1 1dx = 1$$
$$q_1 = 1$$
$$q_2 = \frac{x - p_1}{\|x - p_1\|}$$
$$p_1 = \langle 1, x\rangle = \int_0^1 xdx = \frac{1}{2}$$
$$\|x - \frac{1}{2}\| = \left(\int_0^1 x^2 - x + \frac{1}{4}dx\right)^{\frac{1}{2}} = \frac{1}{2\sqrt{3}}$$

$$q_2 = 2\sqrt{3}x - \sqrt{3}$$
$$q_3 = \frac{x^2 - p_2}{\|x^2 - p_2\|}$$
$$p_2 = \langle 1, x^2\rangle + \langle 2\sqrt{3}x - \sqrt{3}, x^2\rangle 2\sqrt{3}x - \sqrt{3} = x - \frac{1}{6}$$
$$\|x^2 - x + \frac{1}{6}\| = \left(\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx\right)^{\frac{1}{2}} = \frac{1}{6\sqrt{5}}$$
$$q_3 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$$

We now have an orthonormal basis, and can calculate the adjoint using the formula $\mathbf{y} = \sum_{i=1}^{3} \overline{L(\mathbf{q}_i)}\mathbf{q}_i$. This process yields $y = 180x^2 - 168x + 24$.

Problem 3.38 We seek a matrix representation of D. $D[1] = 0, D[x] = 1, D[x^2] = 2x$, so

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We know that for each column of the adjoint, we can calculate $\langle b_i, D[v]\rangle = \langle c_0 + c_1x + c_2x^2, a_0 + a_1x + a_2x^2\rangle \forall b_i \in \{1, x, x^2\}$, where $v = a_0 + a_1x + a_2x^2$ is an arbitrary polynomial, which will yield a system of equations whose solution will yield the i^{th} column of the matrix representation of the adjoint. Note $\langle c_0 + c_1x + c_2x^2, a_0 + a_1x + a_2x^2\rangle = \int_0^1 (c_0 + c_1x + c_2x^2)(a_0 + a_1x + a_2x^2)dx = \frac{1}{60}(10a_0(6c_0 + 3c_1 + 2c_2) + 5a_1(6c_0 + 4c_1 + 3c_2) + a_2(20c_0 + 15c_1 + 12c_2))$ and $\langle 1, a_1 + 2a_2x\rangle = a_1 + a_2$, $\langle x, a_1 + 2a_2x\rangle = \frac{a_1}{2} + \frac{2a_2}{3}$, and $\langle x, a_1 + 2a_2x\rangle = \frac{a_1}{3} + \frac{a_2}{2}$. Solving this system yields that

$$D^* = \begin{bmatrix} -6 & 2 & 3 \\ 12 & -24 & -26 \\ 0 & 30 & 30 \end{bmatrix}$$

Problem 3.39 i) We have $\langle w, (S+T)(v)\rangle = \langle w, S(v)+T(v)\rangle = \langle w, S(v)\rangle + \langle w, T(v)\rangle = \langle S^*(w), v\rangle + \langle T^*(w), v\rangle = \langle (S^*+T^*)(w), v\rangle$. Also, $\langle w, \alpha T(v)\rangle = \alpha\langle T^*(w), v\rangle = \langle \bar{\alpha}T^*(w), v\rangle$
ii) $\langle w, S^*(v)\rangle = \langle S(w), v\rangle$
iii) $\langle w, S(T(v))\rangle = \langle S^*(w), T(v)\rangle = \langle T^*(S^*(w)), v\rangle$
iv) We show $T^*(T^{-1})^* = I$ and $(T^{-1}) * T^* = I$. Note $\langle w, v\rangle = \langle w, T^{-1}T(v)\rangle = \langle (T^{-1})^*(v), T(v)\rangle = \langle T*(T^{-1})^*(w), v\rangle$. Thus $T*(T^{-1})^* = I$. Also $\langle w, TT^{-1}(v)\rangle = \langle T^*(w), (T^{-1})(v)\rangle = \langle (T^{-1})^*T^*(v), w\rangle$. So $(T^{-1})^* = (T^*)^{-1}$

Problem 3.40 i) $\langle B, AC\rangle = \mathrm{tr}(B^HAC) = \langle A^HB, C\rangle$
ii) $\langle A_2, A_3A_1\rangle = \mathrm{tr}(A_2^HA_3A_1) = \mathrm{tr}(A_1A_2^HA_3) = \langle A_2A_1^H, A_3\rangle = \langle A_2A_1^*, A_3\rangle$
iii) $\langle B, T_A(C)\rangle = \langle B, AC - CA\rangle = \mathrm{tr}(B^HAC - B^HCA) = \mathrm{tr}(B^HAC = AB^HC) = \langle A^HB - BA^H, C\rangle = \langle A^*B - BA^*, C\rangle = \langle T_{A^*}(B), C\rangle$

Problem 3.44 $(\rightarrow)$ Suppose $Ax = b$ has a solution $x \in \mathbb{F}^n$. Let $y \in \mathcal{N}(A^H)$. Then $\langle y, b\rangle = \langle y, Ax\rangle = y^HAx = \langle A^Hy, x\rangle = \langle 0, x\rangle = 0$. $(\leftarrow)$ Let $y \in \mathcal{N}(A^H)$, and assume $\langle y, b\rangle \neq 0$. Then by the Fundamental Subspaces Theorem, $y \in \mathcal{R}(A)^\perp$. Then given any $b \in \mathcal{R}(A)$, by definition of $\mathcal{R}(A)^\perp$, $\langle y, b\rangle = 0$. Since we assume that $\langle y, b\rangle \neq 0$, $\mathcal{R}(A) = \emptyset$, and $Ax = b$ has no solution.

Problem 3.45 Define a linear transformation $L : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ to be $L(A) = A + A^T$. Then $\mathcal{N}(L) = \{A \in M_n(\mathbb{R}) : A + A^T = 0\} = \{A \in M_n(\mathbb{R}) : A = -A^T\} = \mathrm{Skew}_n(\mathbb{R})$. Note $(A + A^T)^T = A + A^T$, so $\mathcal{R}(L) \subset \mathrm{Sym}_n(\mathbb{R})$. Also, for $A \in \mathrm{Sym}_n(\mathbb{R})$, $A = A^T$ so $A = \frac{1}{2}A + \frac{1}{2}A^T$, so $\mathrm{Sym}_n(\mathbb{R}) \subset \mathcal{R}(L)$ and $\mathrm{Sym}_n(\mathbb{R}) = \mathcal{R}(L)$. Note also that $\mathrm{tr}(BA) = \mathrm{tr}(AB) \Leftrightarrow \mathrm{tr}(A^TB^T) = \mathrm{tr}(AB) \Leftrightarrow \mathrm{tr}(A^TB) + \mathrm{tr}(A^TB^T) = \mathrm{tr}(A^TB) + \mathrm{tr}(AB) \Leftrightarrow \mathrm{tr}(A^T(B+B^T)) = \mathrm{tr}((A+A^T)B) \Leftrightarrow \langle A, L(B)\rangle = \langle L(A), B\rangle$. So $L = L^*$. Then by the Fundamental Subspaces Theorem, we know that $\mathrm{Sym}_n(\mathbb{R})^\perp = \mathcal{R}(L)^\perp = \mathcal{N}(L^*) = \mathcal{N}(L) = \mathrm{Skew}_n(\mathbb{R})$, as desired.

Problem 3.46 i) Let $x \in \mathcal{N}(A^HA)$. That $Ax \in \mathcal{R}(A)$ is immediate. We also know that $A^HAx = 0$. This directly implies that $Ax \in \mathcal{N}(A^H)$.
ii) Let $x \in \mathcal{N}(A^HA)$. By part i) we know that $x \in \mathcal{N}(A^H)$ and $x \in \mathcal{R}(A)$. By the fundamental subspaces theorem, these two spaces are orthogonal and therefore Ax=0. Then $x \in \mathcal{N}(A)$. Let $x \in \mathcal{N}(A)$. Then $Ax = 0 \Rightarrow A^HAx = 0$. So $x \in \mathcal{N}(A^HA)$ and $\mathcal{N}(A^HA) = \mathcal{N}(A)$.
iii) By the Rank Nullity Theorem, since A and A^HA operate on the same space, $\mathrm{rank}(A) + \mathrm{nullity}(A) = \mathrm{rank}(A^HA) + \mathrm{nullity}(A^HA)$. In ii) we proved that $\mathcal{N}(A^HA) = \mathcal{N}(A)$, so $\mathrm{nullity}(A) = \mathrm{nullity}(A^HA)$. It directly follows that $\mathrm{rank}(A) = \mathrm{rank}(A^HA)$.
iv) Suppose that A has all linearly independent columns. Then by necessity, since there are n columns, A is of rank n. So by iii) A^HA is also of rank n. Since A^HA has n columns and is square, it is full rank and therefore is non-singular.

Problem 3.47 i) $P^2 = A(A^HA)^{-1}A^HA(A^HA)^{-1}A^H = A(A^HA)^{-1}A^H = P$

ii)$P^H = A((A^HA)^{-1})^H)A^H = A(A^HA)^{-1}A^H = P$
iii) By the results of 2.14(i), we know that rank(P) $\leq \min\{\text{rank(A)}, \text{rank}((A^HA)^{-1}A^H)\}$ $\leq$ rank(A). Let $y \in \mathcal{R}(A)$. So $y = Ax$ for some x. Then $P(y) = A(A^HA)^{-1}A^HAx = Ax$, implying that $y \in \mathcal{R}(P)$ Then $\mathcal{R}(A) \subset \mathcal{R}(P)$, which means that rank(A) $\leq$ rank(P). Thus rank(A)=rank(P)=n.

Problem 3.48 i) $P(A+cB) = \frac{1}{2}(A+cB+A^T+cB^T) = \frac{1}{2}(A+A^T) + \frac{c}{2}(B+B^T) = P(A)+cP(B)$
ii) $P(P(A)) = \frac{1}{4}(A+A^T+A+A^T) = \frac{1}{4}(2A+2A^T) = \frac{1}{2}(A+A^T) = P(A)$
iii) $\langle A, P(B)\rangle = \frac{1}{2}\text{tr}(A^T(B+B^T)) = \frac{1}{2}\text{tr}(A^TB+A^TB^T) = \frac{1}{2}\text{tr}(BA) + \frac{1}{2}\text{tr}(A^TB) = \frac{1}{2}\text{tr}(AB+A^TB) = \langle P(A), B\rangle$
iv) Let $A \in \mathcal{N}(P)$. Then $P(A) = 0 \Rightarrow A + A^T = 0 \Rightarrow A = -A^T \Rightarrow A \in \text{Skew}_n(\mathbb{R})$. Similarly, $A \in \text{Skew}_n(\mathbb{R}) \Rightarrow A = -A^T \Rightarrow \frac{1}{2}(A+A^T) = 0 \Rightarrow A \in \mathcal{N}(P)$. So $\mathcal{N}(P) = \text{Skew}_n(\mathbb{R})$.
v) Let $A \in \mathcal{R}(P)$. Then $A = \frac{1}{2}(B+B^T) \Rightarrow A^T = \frac{1}{2}(B+B^T) = A \Rightarrow A \in \text{Sym}_n(\mathbb{R})$. Similarly, $A \in \text{Sym}_n(\mathbb{R}) \Rightarrow A = A^T \Rightarrow A = \frac{1}{2}(A+A^T) \Rightarrow A \in \mathcal{R}(P)$. So $\mathcal{R}(P) = \text{Sym}_n(\mathbb{R})$.
vi) $\|A = P(A)\|_F^2 = \text{tr}\left((A-P(A))^T(A-P(A))\right) = \text{tr}\left(A^TA - A^TP(A) - P(A)^TA - P(A)^TP(A)\right) = \frac{1}{4}\text{tr}\left(4A^TA - 2A^T(A+A^T) - 2(A+A^T)A + (A+A^T)(A+A^T)\right) = \frac{1}{4}\text{tr}(A^TA - A^TA^T - AA + AA^T) = \frac{1}{4}\left(\text{tr}(A^TA) - \text{tr}(A^TA^T) - \text{tr}(AA) + \text{tr}(AA^T)\right) = \frac{1}{2}\left(\text{tr}(A^TA) - \text{tr}(A^2)\right)$.
So $\|A-P(A)\|_F = \sqrt{\frac{\text{tr}(A^TA)-\text{tr}(A^2)}{2}}$

Problem 3.50 The problem can be written as $Ax = b$, where

$$A = \begin{bmatrix} x_1^2 & y_1^2 \\ x_2^2 & y_2^2 \\ \vdots & \vdots \\ x_n^2 & y_n^2 \end{bmatrix}, x = \begin{bmatrix} r \\ s \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The normal equations are $A^TAx = A^Tb$. This is simplified to

$$\begin{bmatrix} \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^2y_i^2 \\ \sum_{i=1}^n x_i^2y_i^2 & \sum_{i=1}^n y_i^4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n y_i^2 \end{bmatrix}$$