Minimal Distinct Distance Trees

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Abstract

In 1975, Leech introduced the problem of finding trees whose edges can be labeled with positive integers in such a way that the set of distances (sums of weights) between vertices is $\{1, 2, \cdots, \binom{n}{2}\}$, where n is the number of vertices. We refer to such trees as perfect distance trees. More generally, we define a distinct distance tree to be a weighted tree in which the distances between vertices are distinct. In this article, we focus on identifying minimal distinct distance trees. These are the distinct distance trees on n vertices that minimize the maximum distance between vertices. We determine $M(n)$, the maximum distance in a minimal distinct distance tree on n vertices, for $n \leq 10$, and give bounds on $M(n)$ for $n \geq 11$. This includes a determination of all perfect distance trees for $n < 18$. We then consider trees according to their diameter and show that there are no further perfect distance trees with diameter at most 3. Finally, generalizations to graphs, forests and distinct distance sets are considered.

1 Introduction

A *weighted tree* is a tree in which each edge is labeled with a positive integer, called the *weight* of the edge. The *distance* between two vertices in a weighted tree is the sum of the weights on the edges of the unique path connecting the pair. Since each pair of vertices determines a distance, there are a total of $\binom{n}{2}$ distances in a tree (or any connected graph) with n vertices. If all of these distances are distinct, we call the tree a *distinct distance tree.* Define the function $M(n)$ to be the smallest integer so that there exists a distinct distance tree with n vertices and maximum distance $M(n)$. We refer to a distinct distance tree on n vertices as a *perfect distance tree* if the set of distances $\{1, 2, \cdots, \binom{n}{2}\}$ can be achieved. To generalize this notion further, we define a *minimal distinct distance tree* to be a distinct distance tree with maximal distance $M(n)$.

A simple lower bound for $M(n)$ follows immediately from the definitions.

Proposition 1.1. *For all $n \geq 1$, $M(n) \geq \binom{n}{2}$.*

Equality holds in Proposition 1.1 if and only if there is a perfect distance tree on n vertices. Perfect distance trees were introduced by Leech [7] as a generalization of his previous work on the set of distances in a weighted path [6]. The primary aim of this paper is to analyze the function $M(n)$. In particular, we calculate $M(n)$ for $n \leq 10$ and provide some bounds on $M(n)$ for larger n. Moreover, all perfect distance trees for $n < 18$ are determined.

A problem related to perfect distance trees and calculating $M(n)$ was popularized by Golomb [3] who sought the shortest ruler with n marks at integer distances (including the ends) such that the distances between marks are all distinct. Such a ruler is known as a Golomb ruler. Since a Golomb ruler with n marks has $\binom{n}{2}$ pairs of marks, its length cannot be less than $\binom{n}{2}$. A Golomb ruler with length $\binom{n}{2}$ is called a *perfect ruler* and corresponds to a weighted path that is a perfect distance tree. The marks are taken as the vertices which are then connected sequentially by edges to form a path. The weights are the distances between consecutive marks on the Golomb ruler. For example, the Golomb ruler of length 6 with tick marks at 0, 1, 4, and 6 corresponds to the weighted path with weights 1, 3 and 2 shown at the top right of Figure 1. As noted by Leech [7], the only weighted paths that are perfect distance trees are the three shown in the top row of Figure 1.

Proposition 1.2. *For $n > 4$, the path on n vertices, P_n, cannot be labeled to form a perfect distance tree.*

Moreover, Leech performed a hand search and found that the five perfect distance trees shown in Figure 1 are the only ones on 6 or fewer vertices. They remain the only known perfect distance trees.

Perfect distance trees and minimal distinct distance trees have applications in electrical networks, where the weights are interpreted as electrical resistances [1]. When such a network forms a perfect distance tree, the network might be used as an efficient multipurpose resistor, since only $n - 1$ resistors are needed to construct the network, and any of $\binom{n}{2}$ different resistances can be obtained. Using a minimal distinct distance tree minimizes the number of gaps corresponding to resistances that cannot be obtained.

Taylor [10] provides the most general known result on perfect distance trees.

Theorem 1.3 (Taylor's Condition). *If there is a perfect distance tree on n vertices, then n or $n - 2$ is a perfect square.*

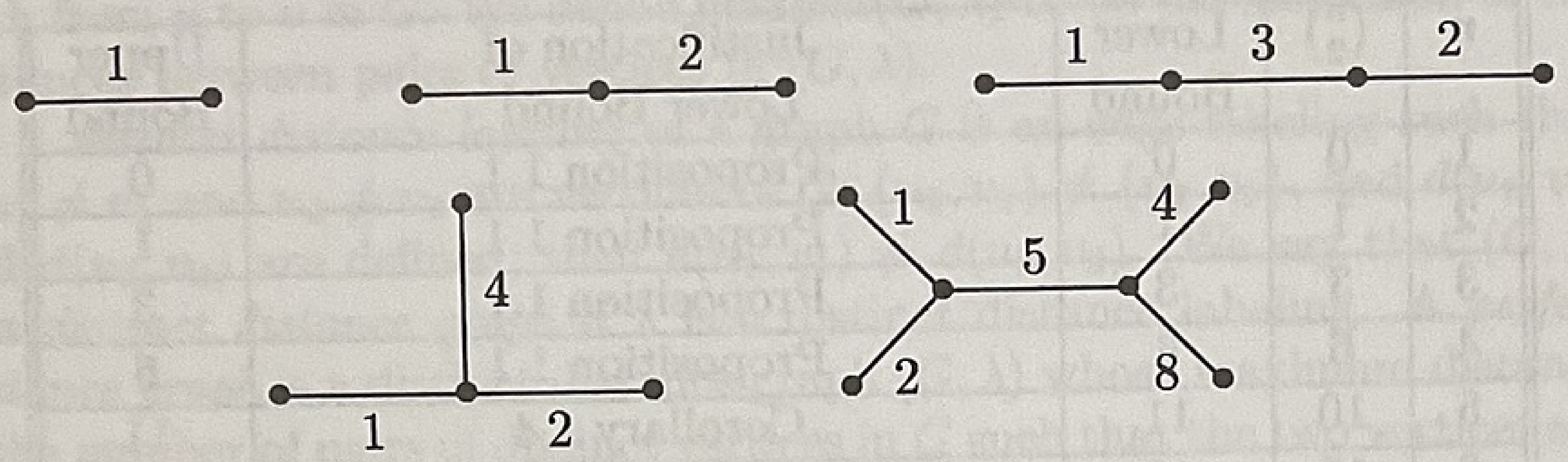

Figure 1: All Known Perfect Distance Trees

Combining Taylor's Condition with Proposition 1.1 gives an immediate corollary.

Corollary 1.4. *If neither n nor $n-2$ is a perfect square, then $M(n) \geq \binom{n}{2}+1$.*

In 1991, Taylor [11] reported that a computer search by Shen Lin showed that there are no perfect distance trees on 9 vertices. He noted that it was unknown whether perfect distance trees on n vertices exist for $n > 9$. In Theorems 1.6 and 2.2, we shall give some generalizations of Taylor's Condition that also improve upon results of Gibbs and Slater [2]. Section 2.2 describes an algorithm we used to show that there are no perfect distance trees on 11 or 16 vertices. A modified version of this algorithm is used to compute $M(n)$ for $n \leq 10$. Also, all minimal distinct distance trees for $7 \leq n \leq 10$ shall be provided. For other n, the bounds on $M(n)$ shown in Table 1 are established.

The upper bounds for $M(n)$ in Table 1 are provided by examples. The examples for $n = 2, 3, 4, 6$ were given by Leech [7]. The examples for $n = 5$ are presented in Figure 2 and include two that are equivalent to Golomb rulers and others that were found by hand. A computer search was used to confirm that these are the only examples for $n = 5$. The examples for $7 \leq n \leq 10$ presented in Figures 4 through 7 are minimal distinct distance trees that were found by computer search. We doubt that the examples we constructed for $11 \leq n \leq 18$ in subsection 2.3 are minimal.

The *diameter* of a tree is the largest possible number of edges in a path connecting two vertices of that tree. All of the known perfect distance trees have diameter at most 3. In Section 2.5, the algorithm of Section 2.2 is used to show that there are no additional perfect distance trees of diameter 3 or less.

In the final section of the paper, we extend this problem to more general graphs. In particular, if we have a forest with k components, where component i is a tree on n_i vertices, then we define a *perfect distance forest*

n	$\binom{n}{2}$	Lower Bound	Justification of Lower Bound	Upper Bound
1	0	0	Proposition 1.1	0
2	1	1	Proposition 1.1	1
3	3	3	Proposition 1.1	3
4	6	6	Proposition 1.1	6
5	10	11	Corollary 1.4	11
6	15	15	Proposition 1.1	15
7	21	22	Corollary 1.4	22
8	28	30	Theorem 2.2 + Computer Search	30
9	36	39	Theorem 2.2 + Computer Search	39
10	45	50	Theorem 2.2 + Computer Search	50
11	55	59	Theorem 2.2 + Computer Search	77
12	66	69	Theorem 2.2 + Computer Search	94
13	78	80	Theorem 2.2 + Computer Search	119
14	91	93	Theorem 2.2 + Computer Search	142
15	105	107	Theorem 2.2 + Computer Search	165
16	120	121	Computer Search	214
17	136	139	Theorem 2.2	254
18	153	153	Proposition 1.1	294

Table 1: Bounds on $M(n)$

to be a weighted forest with set of distances

$$\{1, 2, \cdots, \sum_{i=1}^{k} \binom{n_i}{2}\}.$$

We provide some interesting examples and a generalization of $M(n)$ for forests.

1.1 A Generalization of Taylor's Condition

Before we can generalize Theorem 1.3, we need to extend our definitions and notation from trees to arbitrary graphs. These graphs need not be connected, but can have neither loops nor multiple edges.

Let $G = (V, E)$ be a graph on n vertices and k components. We denote the sizes of the components of G by n_i for $i = 1 \ldots k$. An *edge-labeling* of G is a map $\lambda : E \to Z^+$. The (weighted) *distance*, $d(u, v)$, between vertices u and v in an edge-labeled graph (G, λ) is defined only if u and v are in the same component of G. In that case $d(u, v)$ is the minimum weight of a path from u to v in G. We define $\text{maxdist}(G, \lambda)$ to be the maximum of the distances between pairs of vertices in (G, λ).

A *distinct distance labeling* of a graph G is an edge-labeling such that if $u_1 \neq v_1$ and $u_2 \neq v_2$ are vertices of G, $\{u_1, v_1\} \neq \{u_2, v_2\}$, and $d(u_1, v_1)$ and $d(u_2, v_2)$ are defined, then $d(u_1, v_1) \neq d(u_2, v_2)$. We say that (G, λ) is a *distinct distance graph* if λ is a distinct distance labeling. A *perfect distance graph* is a distinct distance graph (G, λ) whose maximum distance is the number of pairs of distinct vertices in G such that the two vertices are in the same component of G. That is, $\text{maxdist}(G, \lambda) = \Sigma_{i=1}^{k} \binom{n_i}{2}$, and the distances in (G, λ) are the consecutive integers $\{1, 2, 3, \ldots, \text{maxdist}(G, \lambda)\}$. For any graph G, $M(G)$ is the minimum of $\text{maxdist}(G, \lambda)$ over all λ such that (G, λ) is a distinct distance graph.

Finally, a *Taylor coloring* of an edge-labeled graph (G, λ) is a function $t : V \to \{0, 1\}$, where V is the set of vertices of G, such that if x and y are in the same component of G then $d(x, y) \equiv |t(x) - t(y)| \pmod 2$.

Proposition 1.5. *Every edge-labeled forest (F, λ) has a Taylor coloring.*

Proof. Choose a vertex x of F and set $t(x) = 0$. For each vertex y in the component of x, set $t(y) = d(x, y) \bmod 2$. Note that, if y and z are in the component of x and $d(y, z)$ is even, then $t(y) = t(z)$, and, if $d(y, z)$ is odd, then $t(y) \neq t(z)$. Define t on each of the components of F using this same process. □

If t is a Taylor coloring of an edge-labeled graph (G, λ) with n vertices and k components, then, for $i = 1, \ldots, k$ and $j = 0, 1$, we let

$$n_{i,j} = |\{x : x \text{ is in the } i\text{th component and } t(x) = j\}|.$$

The next theorem generalizes Taylor's Condition to weighted graphs. The proof generalizes Taylor's proof for trees to arbitrary graphs that may have multiple components. Applying this theorem to forests will aid us in improving the lower bounds for $M(n)$.

Theorem 1.6 (Generalized Taylor's Condition). *If (G, λ) is an n-vertex perfect distance graph with a Taylor coloring, then there are non-negative integers $a_1, a_2, \ldots, a_k$, where $a_i \equiv n_i \pmod 2$ and $a_i \leq n_i$ for $i = 1, 2, \ldots, k$, such that*

$$a_1^2 + a_2^2 + \cdots + a_k^2 + 2p = n,$$

where $p = \text{maxdist}(G, \lambda) \bmod 2$.

Proof. Let t be a Taylor coloring of (G, λ). In the i-th component of G there are $n_{i,0}n_{i,1}$ odd distances and $\binom{n_{i,0}}{2} + \binom{n_{i,2}}{2}$ even distances as in the proof of Taylor's Theorem. Since (G, λ) is a perfect distance graph, the number of odd distances equals the number of even distances plus p. That is,

$$\Sigma_{i=1}^{k} n_{i,0}n_{i,1} = \Sigma_{i=1}^{k}[\binom{n_{i,0}}{2} + \binom{n_{i,1}}{2}] + p.$$

Expanding and multiplying both sides by 2 we obtain

$$\Sigma_{i=1}^{k} 2n_{i,0}n_{i,1} = \Sigma_{i=1}^{k}(n_{i,0}^2 - n_{i,0} + n_{i,1}^2 - n_{i,1}) + 2p.$$

Rearranging, we obtain

$$n = \Sigma_{i=1}^{k}(n_{i,0} + n_{i,1}) = \Sigma_{i=1}^{k}(n_{i,0}^2 - 2n_{i,0}n_{i,1} + n_{i,1}^2) + 2p$$

$$= \Sigma_{i=1}^{k}(n_{i,0} - n_{i,1})^2 + 2p.$$

Letting $a_i = n_{i,0} - n_{i,1}$ completes the proof. □

The special case of the previous theorem where $k = 1$ implies Taylor's Condition as well as a generalization of Taylor's Condition due to Gibbs and Slater [2] (their Theorem 2).

The next proposition gives a criterion for determining the value of p in the previous theorem if the size and number of components of a perfect distance graph are known.

Proposition 1.7. *Let (G, λ) be an n-vertex perfect distance graph, with q components of odd size, and let $p = maxdist(G, \lambda)$ mod 2. Then, $p = 0$ if and only if $q \equiv n$ (mod 4).*

Proof. By definition

$$p = \frac{\Sigma_{i=1}^{k} n_i(n_i - 1)}{2} \text{ mod } 2.$$

Thus $p = 0$ if and only if

$$\Sigma_{i=1}^{k} n_i^2 - \Sigma_{i=1}^{k} n_i \equiv 0 \text{ (mod 4)}.$$

Since n_i^2 mod $4 = n_i$ mod 2, the result follows. □

The next proposition allows us to calculate $n_{1,0}$ and $n_{1,1}$ for a connected n-vertex perfect distance graph. An equivalent result was noted by Gibbs and Slater [2].

Proposition 1.8. *Suppose t is a Taylor coloring of a connected n-vertex perfect distance graph (G, λ). Let $p = maxdist(G, \lambda)$ mod 2. Then, there exists a nonnegative integer m such that $n = m^2 + 2p$, $n_{1,0} = m(m+1)/2 + p$, and $n_{1,1} = m(m-1)/2 + p$.*

Proof. By Theorem 1.6, $n = m^2 + 2p$ for some nonnegative integer m. It follows that $n_{1,0}+n_{1,1} = n = m^2+2p$. In the proof of Theorem 1.6, we also saw that $n_{1,0}-n_{1,1} = m$. Adding these equations gives $2n_{1,0} = m^2+m+2p$, and subtracting them results in $2n_{1,1} = m^2-m+2p$. The result follows. $\square$

2 Focusing on Trees

In this section, we give some bounds on $M(n)$. We also present the algorithm that we use to determine $M(n)$ for $n \leq 10$ and to show that there are no perfect distance trees for $6 < n < 18$. Examples of weighted trees that meet $M(n)$ for $n \leq 10$ are also presented. Finally, we establish that Figure 1 contains all of the perfect distance trees of diameter 3 or less.

First, consider the smallest minimal distinct distance tree that is not perfect. By Taylor's Condition, it must have 5 vertices, and $M(5) > 10$. A computer search establishes that Figure 2 shows all six examples of distinct distance trees on 5 vertices having maximum distance 11. Thus, $M(5) = 11$. Note that the two paths are equivalent to Golomb rulers of length 11.

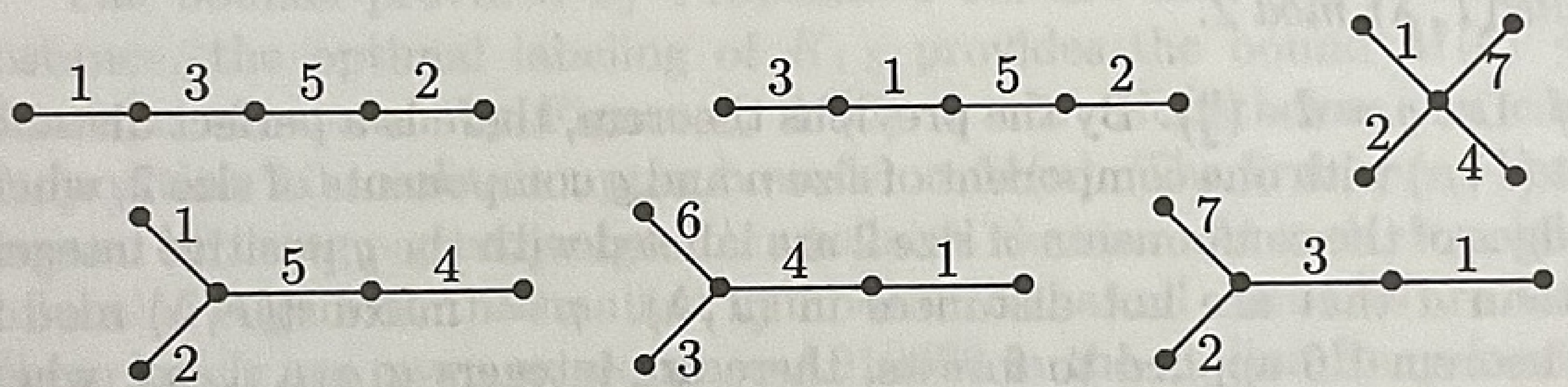

Figure 2: Trees that Meet $M(5) = 11$

2.1 Bounds on $M(n)$

We have seen in Proposition 1.1 and Corollary 1.4 that $M(n) \geq \binom{n}{2}$, in general, and $M(n) \geq \binom{n}{2} + 1$, if neither n nor $n-2$ is a square. Theorem 2.2 of this section will allow us to improve on these lower bounds in some cases. Theorem 2.2 also gives parity conditions that allow us to reduce the number of weighted trees that must be checked when searching for minimal distinct distance trees. In Theorem 2.4, we provide a cubic upper bound on $M(n)$, but conjecture that there is a quadratic upper bound.

First we show that each distinct distance tree is equivalent to a perfect distance forest in which isolated edges are labeled with the distances missing from the tree.

Theorem 2.1. *For any $n > 2$, there is an n-vertex distinct distance tree (T, λ) with maxdist$(T, \lambda) = d$ iff there is a perfect distance forest (F, μ) with one component of size n and $d - \binom{n}{2}$ components of size 2, where $\mu(e) < d$ for each edge e in a component of size 2.*

Proof. Given (T, λ), let F be the disjoint union of T and $d - \binom{n}{2}$ components of size 2. Define μ to equal λ on F and so that the edges of the components of size 2 are labeled with the $d - \binom{n}{2}$ numbers x such that $x < m$ and x is not a distance in (T, λ). Clearly (F, μ) is a perfect distance forest.

Conversely, given (F, μ), let T be the component of size n. Let λ be the restriction of μ to T. Then since $\mu(e) < d$ for each edge e in a component of size 2, maxdist$(T, \lambda) = d$. □

Now we prove a version of Taylor's Condition for distinct distance trees.

Theorem 2.2. *Let (T, λ) be an n-vertex distinct distance tree such that maxdist$(T, \lambda) = d$. Suppose that, of the positive integers less than d that are not distances in (T, λ), u of them are even and v of them are odd. Then, there is an integer a such that $a^2 + 2(u - v + p) = n$, where $p =$ maxdist(T, λ) mod 2.*

Proof. Let $g = d - \binom{n}{2}$. By the previous theorem, there is a perfect distance forest (F, μ) with one component of size n and g components of size 2, where the edges of the components of size 2 are labeled with the g positive integers less than d that are not distances in (T, λ). $p =$ maxdist(F, λ) mod 2. By Theorem 1.6 applied to forests, there are integers $a, a_1, \ldots, a_g$, where $a \equiv n \pmod 2$, $a_i \equiv 0 \pmod 2$, $a \leq n$, $a_i \leq 2$ for $i = 1, 2, \ldots, g$, and

$$a^2 + a_1^2 + a_2^2 + \cdots + a_g^2 + 2p = n + 2g.$$

By examining the proof of Theorem 1.6, we see that $a_i = 2$ if the corresponding edge label is even and $a_i = 0$ if the corresponding edge label is odd. Thus,

$$a^2 + a_1^2 + a_2^2 + \cdots + a_g^2 + 2p = a^2 + 4u + 2p = n + 2(u + v),$$

and the result follows. □

We can use Theorem 2.2 to improve upon our lower bounds in Proposition 1.1 and Corollary 1.4. Corollary 2 in Gibbs and Slater [2] gives a lower bound on the value of $M(n)$ that can be shown to be equivalent to ours. However, we obtain additional information from Theorem 2.2 about the parities of the integers that are not distances in a distinct distance tree on n vertices. For example, Theorem 2.2 implies that $M(10) \geq 47$. To see this, first note that, by Corollary 1.4, $M(10) \geq 46$. If $M(10) = 46$, then Theorem 2.2 provides an integer a such that either $a^2 + 2 = 10$ or $a^2 - 2 = 10$. Since 8 and 12 are not perfect squares, this is impossible. Hence $M(10) \geq 47$. If $M(10) = 47$, then Theorem 2.2 can only be satisfied if $u = 2$ and $v = 0$, giving $a^2 + 6 = 10$. We were therefore able to confine our computer search to the case where the two numbers less than 47 that are not distances in the tree are both even. Using similar parity information from Theorem 2.2 and a computer search, we have shown $M(10) = 50$.

We conclude this subsection by giving upper bounds on $M(n)$. The first follows immediately from the definitions. Recall that a star is the special case of a complete bipartite graph $K_{p,q}$ when $p = 1$. That is, a star on n vertices is a tree with a central vertex attached to $n - 1$ leaves.

Proposition 2.3. *If there is a distinct distance labeling of the star $K_{1,n-1}$, for which the sum of the two largest edge-weights is d, then $M(n) \leq d$.*

Proof. Since $K_{1,n-1}$ is a star, the largest distance d in $K_{1,n-1}$ is the sum of the two largest edge-weights. Since $K_{1,n-1}$ is a tree with n vertices, $M(n) \leq d$. □

The bounds provided by Proposition 2.3 are not best possible. For instance, the optimal labeling of $K_{1,5}$ provides the bound $M(6) \leq 19$. However, as shown by Figure 1, $M(6) = 15$. Nonetheless, we can use Proposition 2.3 to obtain upper bounds on $M(n)$. The first upper bound is derived by using a greedy algorithm to label the star. At each step, label an unlabeled edge with the smallest number such that all distances are distinct. These labels are given as sequence A010672 in the on-line Encyclopedia of Integer Sequences of Sloane and Plouffe [9]. A simple counting argument gives the following explicit bound.

Theorem 2.4. $M(n) \leq 2n - 3 + \binom{n-1}{3} + \binom{n-2}{3} = \frac{1}{3}n^3 - \frac{5}{2}n^2 + \frac{49}{6}n - 8.$

Proof. Using the greedy algorithm to label $K_{1,n-1}$, let e_k be the label on the k^{th} edge. For e_k, we choose the least positive integer that avoids making two equal distances in $K_{1,n-1}$. To do this, we must avoid the $k-1$ previous labels, the $\binom{k-1}{2}$ sums of the previous labels and the $\binom{k-1}{3}$ sums of two previous labels minus one (smaller)previous label. Therefore, $e_k \leq k - 1 + \binom{k-1}{2} + \binom{k-1}{3} + 1 = k + \binom{k}{3}$. By Proposition 2.3, $M(n) \leq e_{n-1} + e_{n-2}$ and the result follows. □

An asymptotically better bound is provided by a result of Graham and Sloane [4, Theorem 1(3)], which implies that there is a distinct distance labeling of $K_{1,n-1}$ for which the sum of the two largest edge-weights is less than $n^2 + O(n^{\frac{36}{23}})$. The exact form of the term $O(n^{\frac{36}{23}})$ depends upon the distribution of the primes. By Proposition 2.3, this gives us a quadratic upper bound on $M(n)$.

Corollary 2.5. $M(n) < 2n^2 + O(n^{\frac{36}{23}})$

2.2 An Algorithm for Growing Trees

The goal of this section is to present an algorithm that constructs all perfect distance trees on any given number of vertices. First, we present a lemma that is used to recursively determine the weights of the edges of the tree. This lemma is also used frequently in Section 2.5. Although we will apply the lemma primarily to trees, it holds in the more general case of a perfect distance forest.

Lemma 2.6. *Let F be a perfect distance forest with n vertices and e edges, and let the weights on the edges of F be $w_1 < w_2 < \cdots < w_e$. For each $1 \leq k \leq e$, w_k is the least positive integer that is not a distance in the subforest of F induced by the edges labeled $w_1, w_2, \ldots, w_{k-1}$.*

Proof. Let m_k be the least positive integer that is not a distance in the subforest of F induced by the edges labeled $w_1, w_2, \ldots, w_{k-1}$. If w_k were less than m_k, then distance w_k would be repeated in F. If w_k were greater than m_k, then we would have $w_j > m_k$ for all $j \geq k$ and m_k could not be a distance in F. Therefore, $w_k = m_k$. □

Note that Lemma 2.6 does not generalize to arbitrary graphs. In fact, the weight on an edge of a perfect distance graph is not determined, in general, by the set of lower weights in the graph. For instance, the three-cycle with edges labeled 1,2,3 and the three-cycle with edges labeled 1,2,4 are both perfect distance graphs.

Algorithm 2.7. Our algorithm for constructing perfect distance trees on n vertices is a depth-first search tree algorithm. The first few branches of the search tree are shown in Figure 3. Each node of the search tree is a weighted forest that is potentially a subforest of a perfect distance tree. The children of each forest are obtained by adding a single edge. The weight of the new edge, as determined by Lemma 2.6, is the least positive integer that is not a distance in the parent forest. The children are forests that can be obtained by adding this weighted edge in one of the following ways:

1. The edge connects a vertex in one tree to a vertex in a second tree of the parent forest. (0 vertices are added)
2. The edge connects a new vertex to an existing vertex in the parent forest. (1 vertex is added)
3. The edge is disjoint from the parent forest. (2 vertices are added)

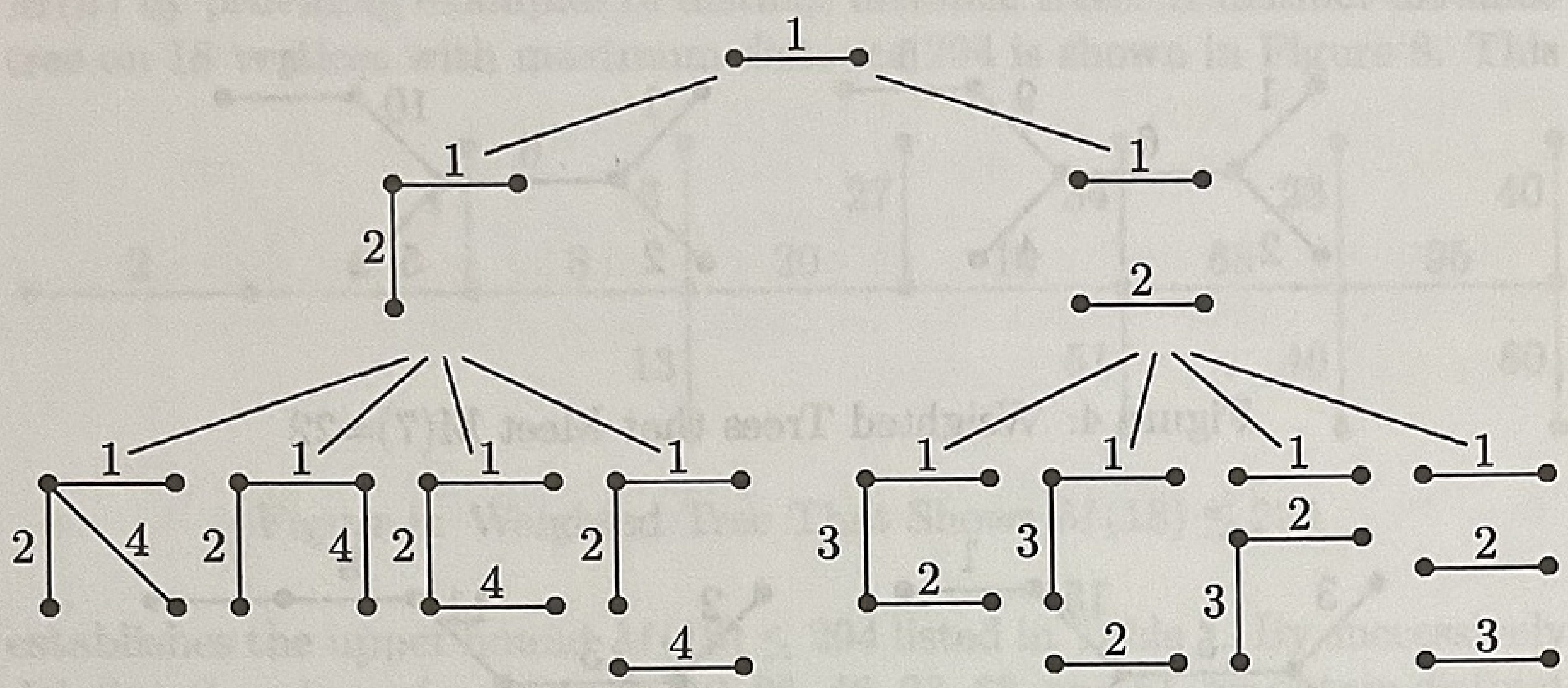

Figure 3: The Search Tree

However, the new weighted edge is only added if it does not cause one of the following conditions to occur:

1. A distance is obtained in two ways.
2. A distance exceeds $\binom{n}{2}$.
3. The number of vertices exceeds n.

Note that no node of the search tree is equivalent to any other node.

2.3 Using the Algorithm to Analyze $M(n)$

When no perfect distance tree exists, we modify the algorithm by allowing for gaps and relaxing the condition on the maximum distance. First, we consider the possibility that $M(n) = \binom{n}{2} + 1$ so that the set of distances contains all but one of the integers in the set $S = \{1, 2, ..., \binom{n}{2}, \binom{n}{2} + 1\}$. To handle this, the algorithm first uses the integers $\{2, 3, ..., \binom{n}{2}, \binom{n}{2} + 1\}$, then $\{1, 3, 4, ..., \binom{n}{2}, \binom{n}{2} + 1\}$, and at each stage i skips the integer i. If no weighted tree T with $M(T) = \binom{n}{2}+1$ is constructed, then the algorithm can be set to consider two or more gaps. Using this technique, we implemented our algorithm to construct all of the examples that meet $M(n)$ for $n \leq 10$. These are shown in Figures 4 through 7.

Notice that, for 7 vertices, Taylor's condition rules out the possibility that $M(7) = 21$. For 8 vertices, Taylor's condition gives $M(8) \neq 28$, and a computer search exhaustively establishes that $M(8) \neq 29$. For 9 vertices, a computer search gives that $M(9) \geq 39$. Similarly, for 10 vertices, our bounds show that $M(10) \geq 47$, and a computer search shows that

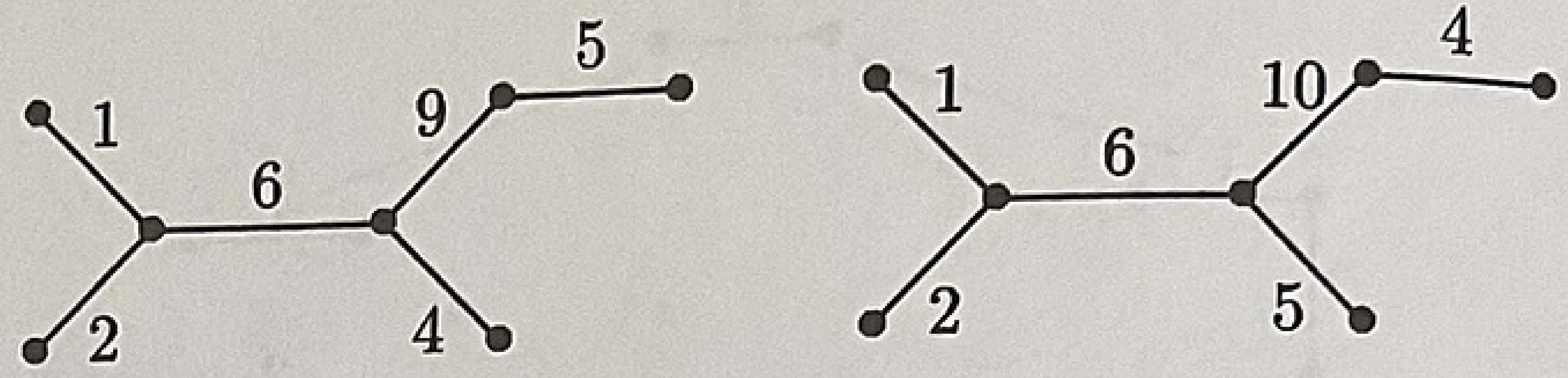

Figure 4: Weighted Trees that Meet M(7)=22

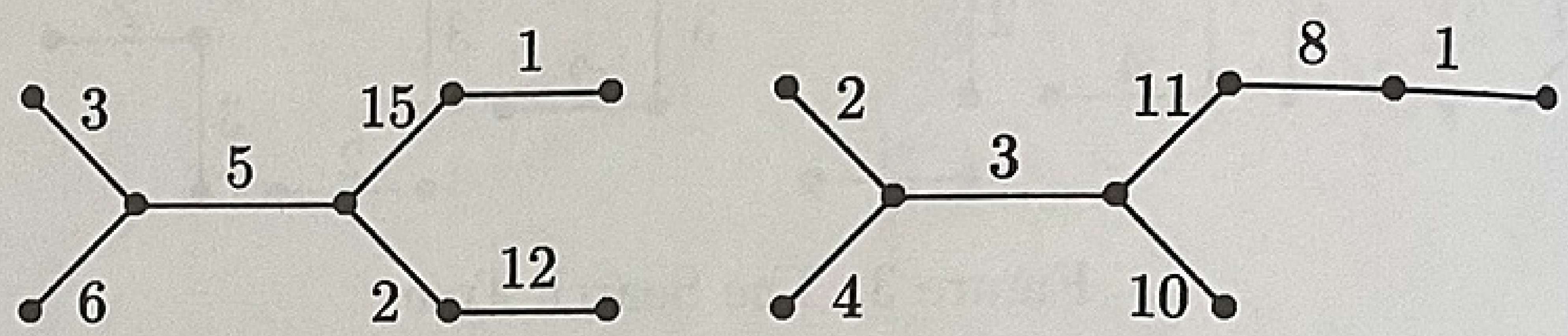

Figure 5: Weighted Trees that Meet M(8)=30

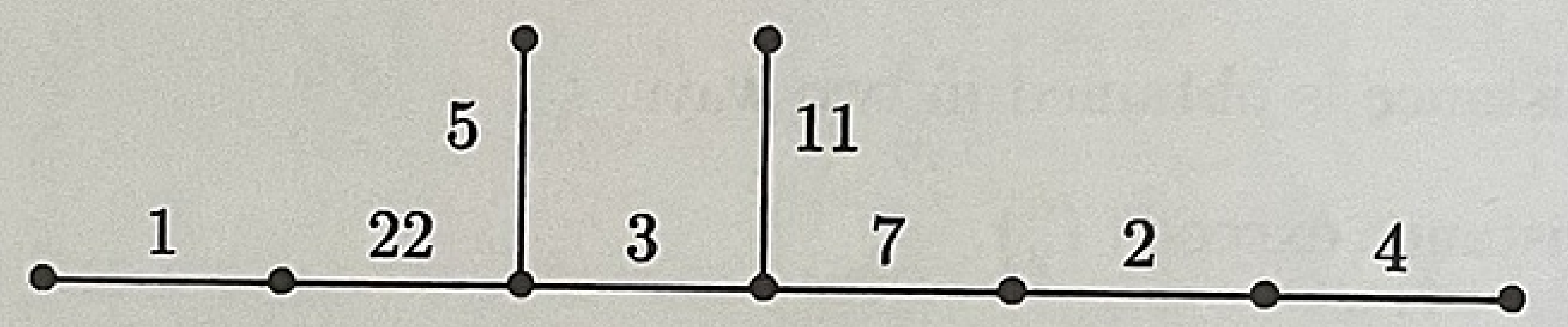

Figure 6: Weighted Trees that Meet M(9)=39

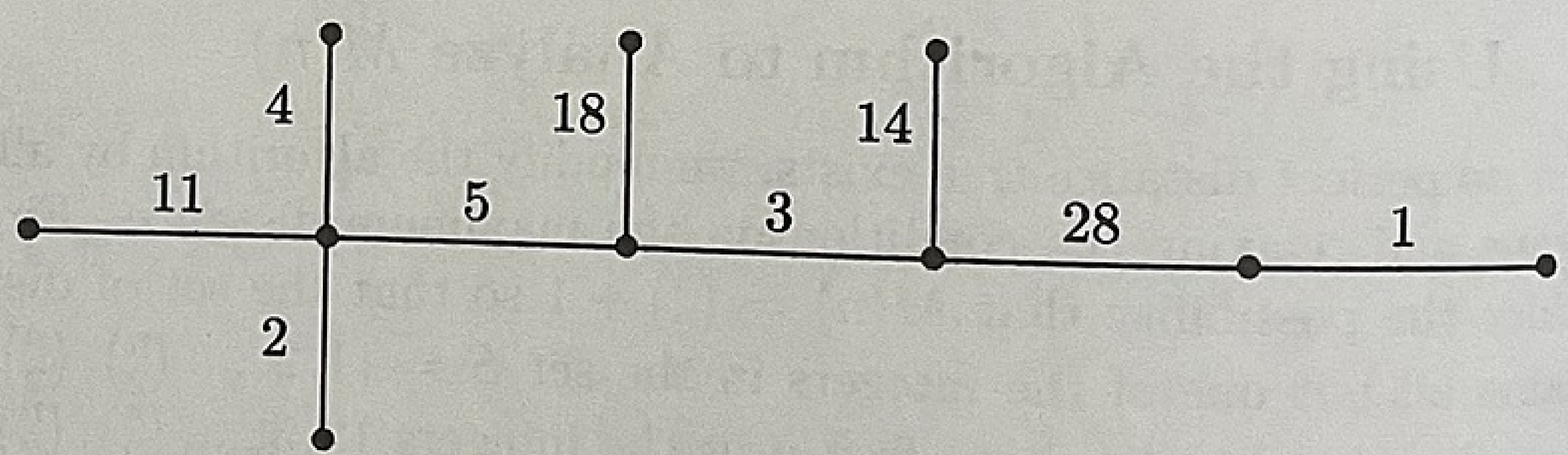

Figure 7: Weighted Trees that Meet M(10)=50

$M(10) \geq 50$. The parity information given by Theorem 2.2 was used to reduce the number of cases that needed to be checked.

For $11 \leq n \leq 18$, we are able to eliminate by computer search the possibilities of relatively few gaps and thereby improve upon the lower bounds in Proposition 1.1 and Corollary 1.4. However, it will take too long for the algorithm to check higher numbers of gaps and find the exact value of $M(n)$. To complement the lower bounds, we give upper bounds for

$M(n)$ by providing examples of distinct distance trees. A distinct distance tree on 18 vertices with maximum distance 294 is shown in Figure 8. This

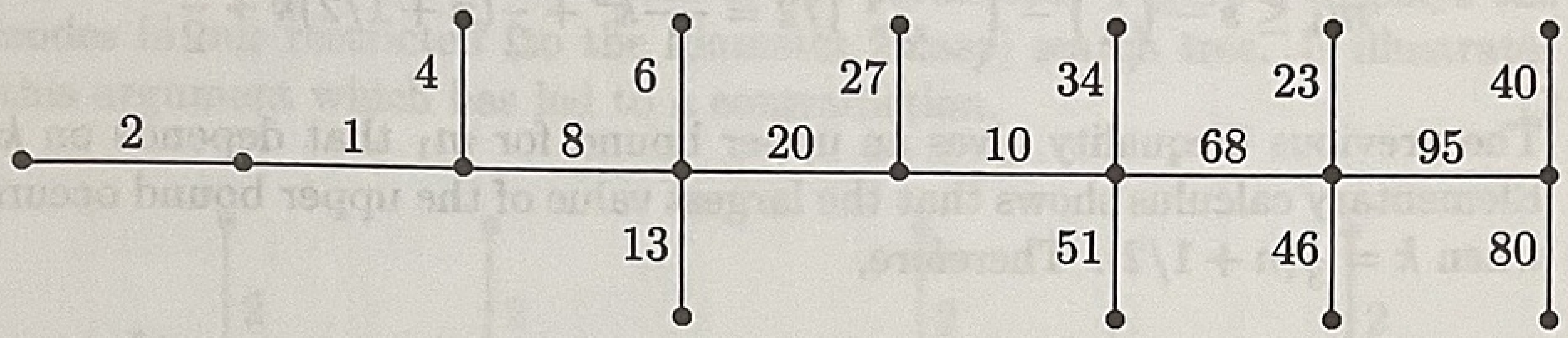

Figure 8: Weighted Tree That Shows $M(18) \leq 294$

establishes the upper bound $M(18) \leq 294$ listed in Table 1. By successively deleting the edges of weights 80, 40, 95, 46, 23, 68, and 51, we obtain distinct distance trees that justify the corresponding upper bounds, for $17 \geq n \geq 11$, listed in Table 1.

2.4 Bounds on the Weights

The following two theorems give bounds on the largest and second largest weights in a perfect distance tree.

Theorem 2.8. *Let T be a perfect distance tree with n vertices. and let m_1 be the maximum of the weights on the edges of T. Then, $m_1 \leq \lfloor \frac{(4n-1)^2}{48} \rfloor$.*

Proof. Let $s = \binom{n}{2}$, the number of paths in T. Let e be the edge with weight m_1. Removing e from T leaves two disjoint trees: T_1 and T_2. Let d_1 be the maximum distance in T_1 and d_2 be the maximum distance in T_2. Let a and b be the endpoints of edge e such that a is in T_1 and b is in T_2. By the triangle inequality there is a vertex x in T_1 such that the distance from x to a is at least $d_1/2$. Similarly, there is a vertex y in T_2 at a distance at least $d_2/2$ from b. The path in T from x to y passes through e, so the distance from x to y is at least $m_1 + (d_1 + d_2)/2$. Now we get a lower bound for this distance. Let k be the number of vertices in T_1. The number of paths in T_1 is $\binom{k}{2}$. Since T is a perfect distance tree, the distances between pairs of vertices in T_1 are all distinct. Thus $d_1 \geq \binom{k}{2}$. The number of paths in T_2 is $\binom{n-k}{2}$. Therefore, there are $\binom{k}{2} + \binom{n-k}{2}$ paths in $T_1 \cup T_2$. Let P be the path in $T_1 \cup T_2$ of largest weight. Since T is a perfect distance tree, P must have weight at least $\binom{k}{2} + \binom{n-k}{2}$. Without loss of generality, we may assume that P is in T_2. Thus, $d_2 \geq \binom{k}{2} + \binom{n-k}{2}$. Using our lower bounds for d_1 and d_2 we see that the distance from x to y is at least $m_1 + \binom{k}{2} + \binom{n-k}{2}/2$.

Since the maximum distance in T is s, we have

$$m_1 \leq s - \binom{k}{2} - \binom{n-k}{2}/2 = \frac{-3}{4}k^2 + \frac{1}{2}(n+1/2)k + \frac{s}{2}.$$

The previous inequality gives an upper bound for m_1 that depends on k. Elementary calculus shows that the largest value of the upper bound occurs when $k = \frac{1}{3}(n+1/2)$. Therefore,

$$m_1 \leq \frac{-1}{12}(n+1/2)^2 + \frac{1}{2}(n+1/2)\frac{1}{3}(n+1/2) + \frac{s}{2} = \frac{(4n-1)^2}{48}.$$

□

An elementary argument places a bound on the second highest weight of an edge.

Theorem 2.9. *Let T be a perfect distance tree with n vertices. Let m_2 be the second highest weight of an edge of T. Then, $m_2 \leq \lfloor \frac{n^2-n-2}{4} \rfloor$.*

Proof. Let m_1 be the highest weight of an edge of T. Since, there is a path in T that contains the edges of weights m_1 and m_2, we have $m_1 + m_2 \leq \binom{n}{2}$. Since $m_2 < m_1$, it follows that

$$m_2 \leq \left\lfloor \frac{\binom{n}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{n^2-n-2}{4} \right\rfloor. \qquad \square$$

2.5 Trees with Diameter at Most Three

Proposition 1.2 shows that all perfect distance trees with diameter $n-1$, the maximum possible diameter, are included in Figure 1. In fact, all of the known examples of perfect distance trees have diameters less than or equal to three. In this section we use Algorithm 2.7 to show that Figure 1 contains all of the perfect distance trees of diameter at most 3. Clearly, the only perfect distance tree of diameter 1 is the one given in Figure 1. So we start by showing that all the perfect distance trees of diameter 2 are pictured in Figure 1.

Proposition 2.10. *For $n > 4$, the star $K_{1,n-1}$ is not a perfect distance tree.*

Proof. Suppose for some $n > 4$ that $K_{1,n-1}$ has such a labeling. Every pair of edges meets at the central vertex in $K_{1,n-1}$. We use Lemma 2.6 to recursively determine the weights of the edges, starting with $w_1 = 1$ and $w_2 = 2$. Since the sum $w_1 + w_2$ gives the distance 3, we must have

$w_3 = 4$. Now, the sums $w_1 + w_3$ and $w_2 + w_3$ give distances 5 and 6, so $w_4 = 7$. This gives distances 8, 9, and 11, thus $w_5 = 10$. However, $w_1 + w_5 = w_3 + w_4 = 11$, which is not permitted. Figure 9 displays the nodes in our restricted (to the diameter 2 case) search tree. It illustrates this argument which has led to a contradiction. □

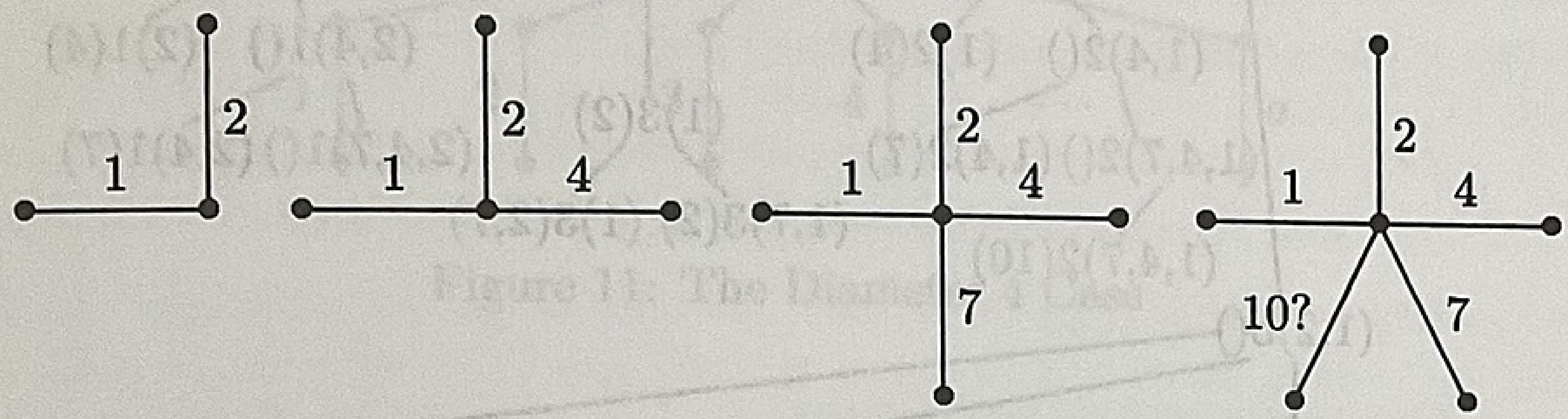

Figure 9: Labeling a Star

We now restrict Algorithm 2.7 to trees of diameter 3. These trees consist of two stars whose central vertices are joined by an edge. We adopt the notation introduced by Leech [7] to denote such a weighted tree by

$$(a_1, a_2, \ldots, a_k)b(c_1, c_2, \ldots, c_m). \tag{2.1}$$

Here, $(a_1, a_2, \ldots, a_k)$ and $(c_1, c_2, \ldots, c_m)$ are the lists of weights in the stars and b is the weight of the edge connecting them. To provide a unique representation we require that $a_1 < \cdots < a_k$, $c_1 < \cdots < c_m$, and $a_1 < c_1$. For example, the perfect distance tree on 6 vertices in Figure 1 is denoted by $(1, 2)5(4, 8)$.

Proposition 2.11. *Let T be a labeled tree on $n > 6$ vertices with diameter 3. Then T cannot be a perfect distance tree.*

Proof. The search tree in Figure 10 shows that T cannot be a perfect distance tree.

The search tree is constructed in stages such that at each stage the weight of the edge added is the least positive integer that is not already a distance in the tree. We first try to assign the weight as an a_i, then as b, and finally as a c_i. Each branch of the search tree stops when we observe that the associated construction cannot be extended to obtain a perfect distance tree. This occurs when the next edge to be added would create two paths of equal distance. For example, in the tree $(1, 7)3(2)$ the next distance needed is 9. If we have $(1, 7, 9)3(2)$ then we have distance 10 twice, namely $1 + 9$ and $7 + 3$. If we have $(1, 7)3(2, 9)$, then we have distance 12 twice, namely $7 + 3 + 2$ and $9 + 3$. Thus, this tree cannot be made into a perfect distance tree. Also note that, in the tree $(1, 2, 4)b(7)$

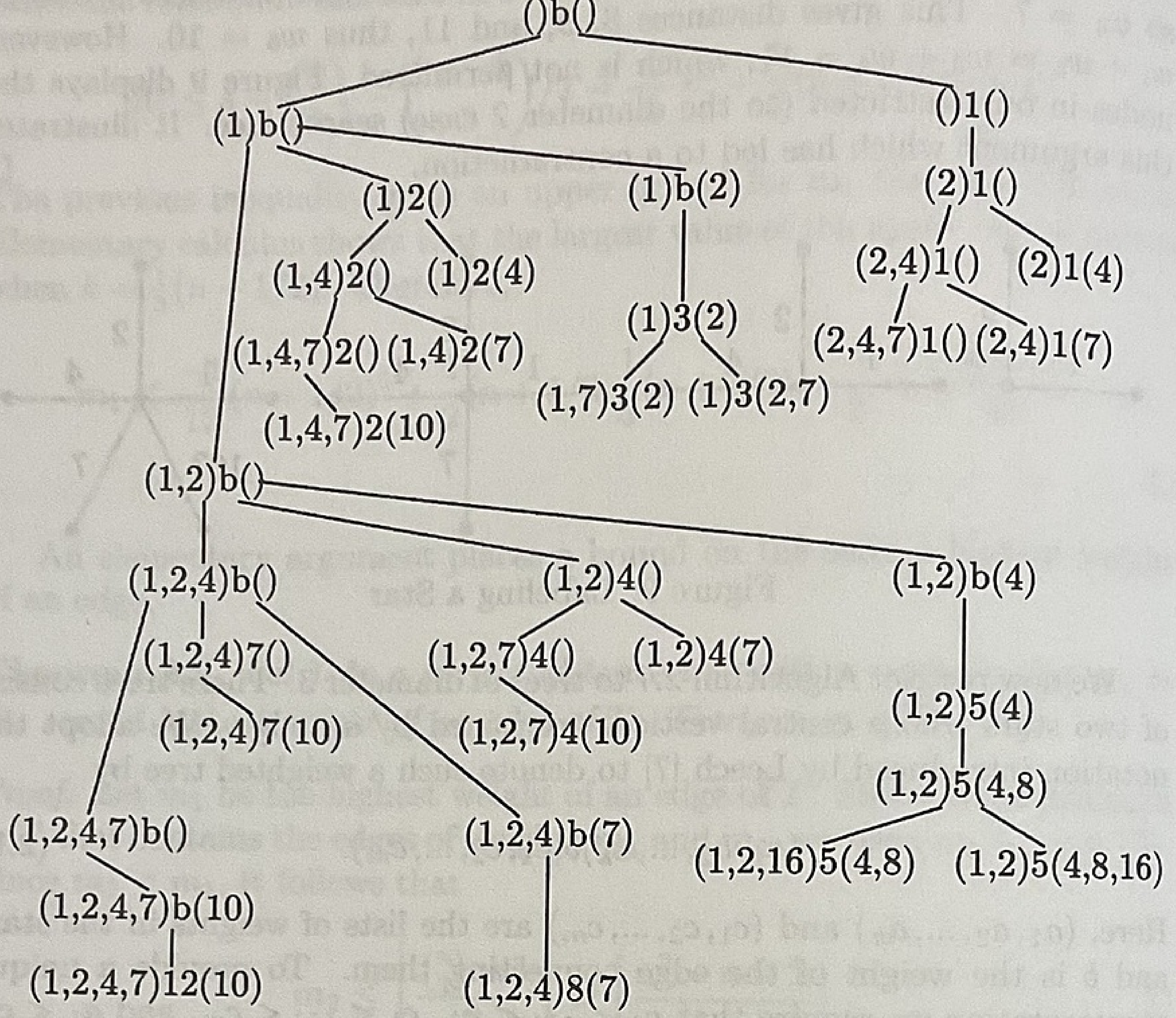

Figure 10: Search Tree for Diameter 3 Trees

the next distance needed is 8. If we have $(1,2,4,8)b(7)$, then it contains the distances $1+b+7$ and $8+b$. Hence, no matter what weight is given to b, we have two equal distances. Similarly, $(1,2,4)b(7,8)$ cannot be a child of $(1,2,4)b(7)$. Hence, $(1,2,4)8(7)$ is the only child of $(1,2,4)b(7)$. It is straightforward to verify that all cases have been considered in the search tree shown in Figure 10. The search tree is finite and contains no perfect distance trees with $n > 6$. □

2.6 Trees with Diameter Four

Propositions 2.10 and 2.11 deal with the diameter 2 and 3 cases, respectively. Hence, it is natural to next consider the case in which the diameter is 4. However, our argument in the diameter 3 case cannot be extended to diameter 4. The problem is that the search tree restricted to the diameter 4 case is infinite. For any integer $a \geq 2$, Figure 11 pictures a weighted tree of diameter 4 on $n = 2a + 1$ vertices that is clearly contained in our search tree. It is straightforward to verify that no two paths in the tree can have the same length. Since a can be arbitrarily large, there are infinitely many examples in our search tree.

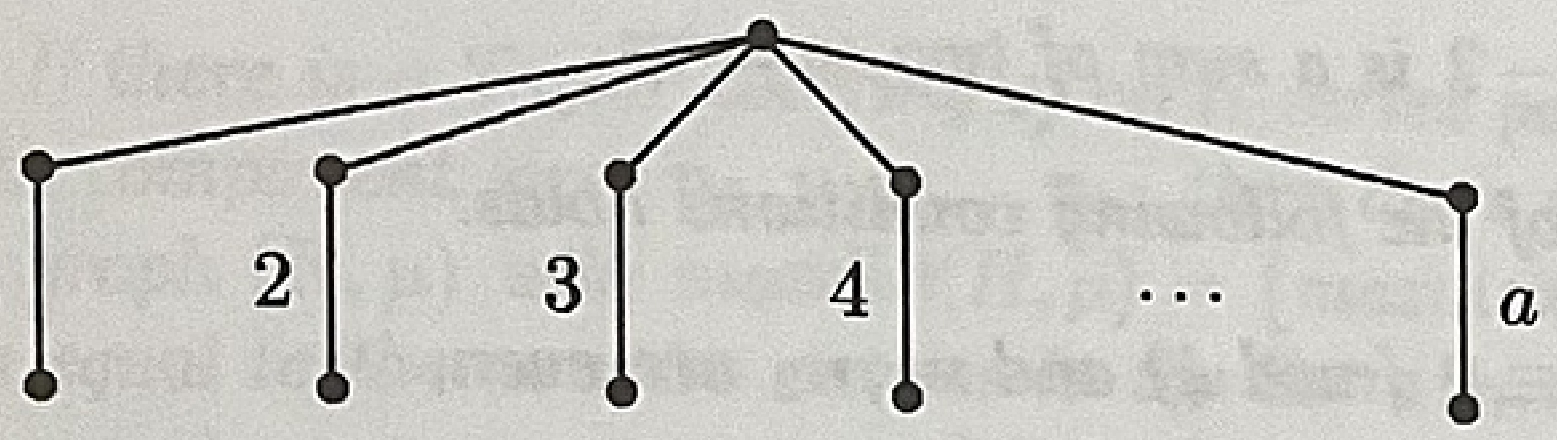

Figure 11: The Diameter 4 Case

3 Graphs and Distinct Distance Sets

In this section we widen our focus to state some results about perfect distance graphs. It is not hard to show that, for every positive integer n, there is a perfect distance graph on n vertices. Gibbs and Slater [2] give one construction to show this. A perhaps simpler construction is to use the set $\{1, 2, \ldots, n\}$ for the vertices of a complete graph on n vertices, and label each edge ab, where we assume that $a > b$, by the function $\lambda(ab) = \binom{a}{2} - b + 1$. It is straightforward to check that this satisfies the definitions.

Although there are perfect distance graphs of all orders, there are still many questions one might ask about their necessary structure. We begin with some applications of Theorem 1.6 to graphs and forests with two components. We then give a procedure for deriving one perfect distance graph from another. Finally, we show how to apply Theorem 1.6 to obtain a theorem on distinct distance sets due to Gibbs and Slater [2].

3.1 Perfect Distance Graphs with Two Components

It follows from Theorem 1.6 that if (G, λ) is a perfect distance graph with n vertices and two components, then n or $n - 2$ must be a sum of two squares. The following theorem gives criteria for determining which of these two alternatives will occur.

Theorem 3.1. *Let (G, λ) be an n-vertex perfect distance graph with two components, one with n_1 vertices and the other with n_2 vertices, such that G has a Taylor coloring.*

(a) If one of the following conditions holds:

(i) $n \equiv 0 \pmod 4$ *and* n_1, n_2 *are odd,*

(ii) $n \equiv 2 \pmod 4$ *and* n_1, n_2 *are even, or*

(iii) $n \equiv 3 \pmod 4$.

then $n - 2$ *is a sum of two squares.*

(b) If one of the following conditions holds:

(iv) $n \equiv 0 \pmod 4$ *and* n_1, n_2 *are even,*

(v) $n \equiv 2 \pmod 4$ *and* n_1, n_2 *are odd, or*

(vi) $n \equiv 1 \pmod 4$.

then n *is a sum of two squares.*

Proof. Under the given conditions, we can deduce the value of p by Proposition 1.7. The result follows by Theorem 1.6. □

It is well known that a positive integer n is representable as a sum of two squares if and only if each of its prime factors of the form $4k+3$ occurs to an even power. Consequently, we call a prime factor of n of the form $4k+3$ that occurs to an odd power a *bad factor*. The following result follows immediately from Theorems 1.6 and 3.1 and generalizes Taylor's Condition to graphs with two components.

Corollary 3.2. *Let* $n = n_1 + n_2$. *If either*

(A) Condition (i), (ii) or (iii) of Theorem 3.1 holds and $n - 2$ *contains a bad factor, or*

(B) Condition (iv), (v) or (vi) of Theorem 3.1 holds and n *contains a bad factor,*

then no perfect distance graph whose components have sizes n_1 *and* n_2 *can have a Taylor coloring.*

Since every edge-labeled forest has a Taylor coloring, we can use Corollary 3.2 to show that there are no two component n-vertex perfect distance forests for certain values of n. For instance, if $n = 14$, then either condition (ii) or condition (v) holds and both $n = 2 \cdot 7$ and $n - 2 = 2^2 \cdot 3$ contain bad factors. If $n = 21$, then condition (vi) holds and $n = 3 \cdot 7$ contains a bad factor. We can also use Corollary 3.2 to determine the parity of n_1 in certain cases. For instance, if we have a two component perfect distance graph with a Taylor coloring and $n = 6$ or $n = 8$, then n_1 must be even; if $n = 12$ then n_1 must be odd.

3.2 Deriving New Perfect Distance Graphs

Given a perfect distance graph, the following theorem enables us to derive a new perfect distance graph with approximately half the maximum distance, but approximately twice the number of components.

Theorem 3.3. *If there is a Taylor coloring of an n-vertex perfect distance graph (G, λ) with component sizes $n_1, n_2, \ldots, n_k$, then there is an n-vertex perfect distance graph (H, μ) with maxdist$(H, \mu) = \lfloor$maxdist$(G, \lambda)/2\rfloor$ and component sizes equal to the nonzero entries in the list $n_{1,0}, n_{1,1}, n_{2,0}, n_{2,1}, \ldots, n_{k,0}, n_{k,1}$, where $n_i = n_{i,0} + n_{i,1}$, for each $1 \leq i \leq k$.*

Proof. We define (H, μ) as follows. The vertices of H are the vertices of G. If e is an edge of G and $\lambda(e)$ is even, then e is an edge of H and $\mu(e) = \lambda(e)/2$. If $e_1, e_2, \ldots, e_j$ are the edges of a path from x to y in G and $\lambda(e_i)$ is odd iff $i = 1$ or $i = j$, then there is an edge e from x to y in H and $\mu(e) = d_{(G,\lambda)}(x, y)/2$. It is not hard to see that there is a path from x to y in H iff $d_{(G,\lambda)}(x, y)$ is even. Thus, the sizes of the components of H are the nonzero numbers in the sequence $n_{1,0}, n_{1,1}, n_{2,0}, n_{2,1}, \ldots n_{k,0}, n_{k,1}$. Furthermore, the shortest path in G from x to y corresponds to a path in H from x to y, where the path in H has half the length of the path in G. The path in H is determined as follows. Since the path in G has even length, it must contain an even number of odd edges. Consecutive pairs of these odd edges and the intervening even edges correspond to single edges of H. Now we show $d_{(H,\mu)}(x, y) = d_{(G,\lambda)}(x, y)/2$. Consider any path from x to y in H of length s. Each edge of this path corresponds to an edge or path in G of twice the length, so we get a corresponding trail in G of length $2s$. By the definition of distance, $2s \geq d_{(G,\lambda)}(x, y)$. It follows that $d_{(H,\mu)}(x, y) = d_{(G,\lambda)}(x, y)/2$. Hence, the distances in (H, μ) are $\{1, 2, \ldots \text{maxdist}(H, \mu)\}$. That is, (H, μ) is a perfect distance graph. □

In Theorem 3.3, we must assume that (G, λ) has a Taylor coloring, but it is not necessarily true that (H, μ) will have a Taylor coloring. Since every weighted forest has a Taylor coloring, we can derive a new perfect distance graph from any perfect distance forest. The new perfect distance graph may or may not be a forest. For instance, we may start with a minimal distinct distance tree on 7 vertices and apply Theorem 2.1 to obtain an equivalent distinct distance forest on 9 vertices. Then, applying Theorem 3.3, we derive a new distinct distance forest with two isolated vertices. Removing the isolated vertices, we obtain a distinct distance forest that is equivalent to one of the minimal distinct distance trees on 5 vertices. This is shown in Figure 12. On the other hand, if we apply this procedure to either of the minimal distinct distance trees on 8 vertices, as shown in Figure 13, we obtain a new perfect distance forest. However, it is not

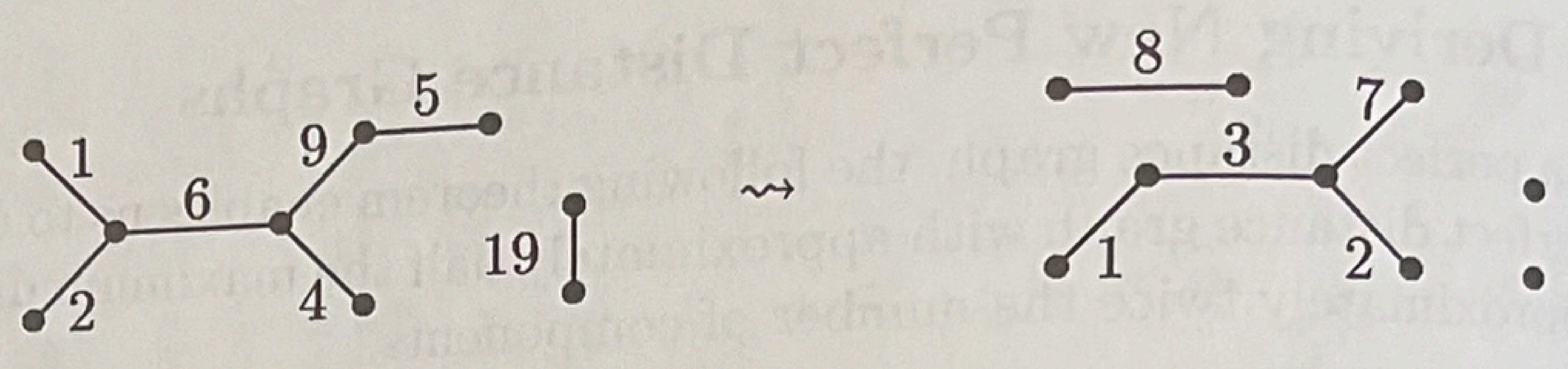

Figure 12: Derivation of a Distinct Distance Tree

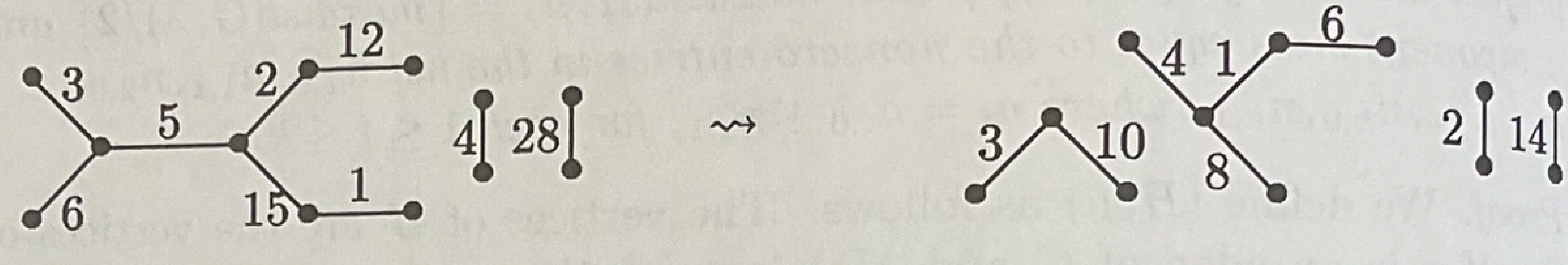

Figure 13: Derivation of a Distinct Distance Forest, but not a Distinct Distance Tree

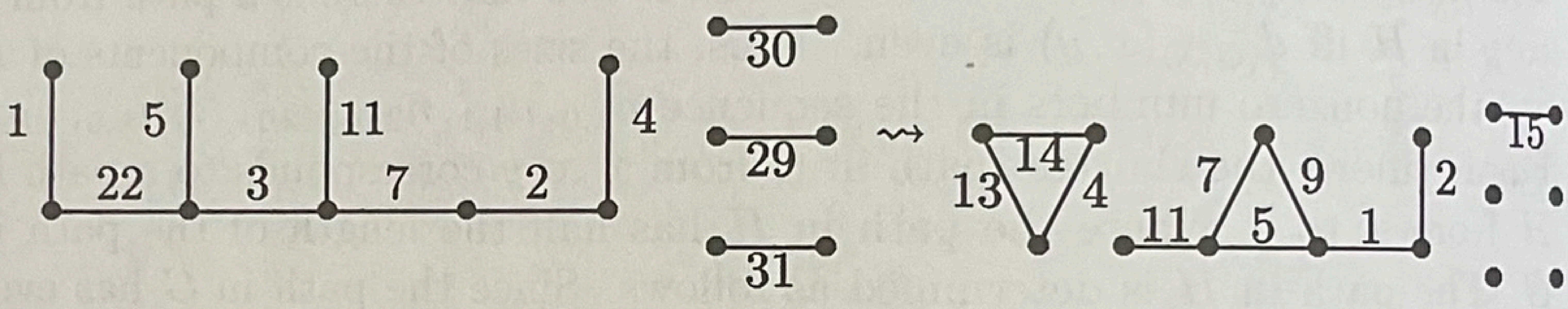

Figure 14: Derivation that is not a Forest

equivalent to a distinct distance tree since it has more then one component of size greater than 2. Furthermore, if we apply the above procedure to the minimal distinct distance tree on 9 vertices, as shown in Figure 14, the resulting perfect distance graph is not a forest and does not have a Taylor coloring.

3.3 Perfect Distance Forests

In the previous section, Figures 12, 13, and 14 provide examples of perfect distance forests, each of which possesses one or more isolated edges. Any distinct distance tree with maximum distance M forms a component of a perfect distance forest where all other components are isolated edges. These edges are labeled with the positive integers less than M not obtained on the original tree. We now consider what other types of forest can be labeled as a perfect distance forest. Figure 15 displays a few examples of perfect distance forests that do not have an isolated edge. We include some partially answered questions regarding the forests in this figure.

1. What is the smallest number of components needed to form a perfect

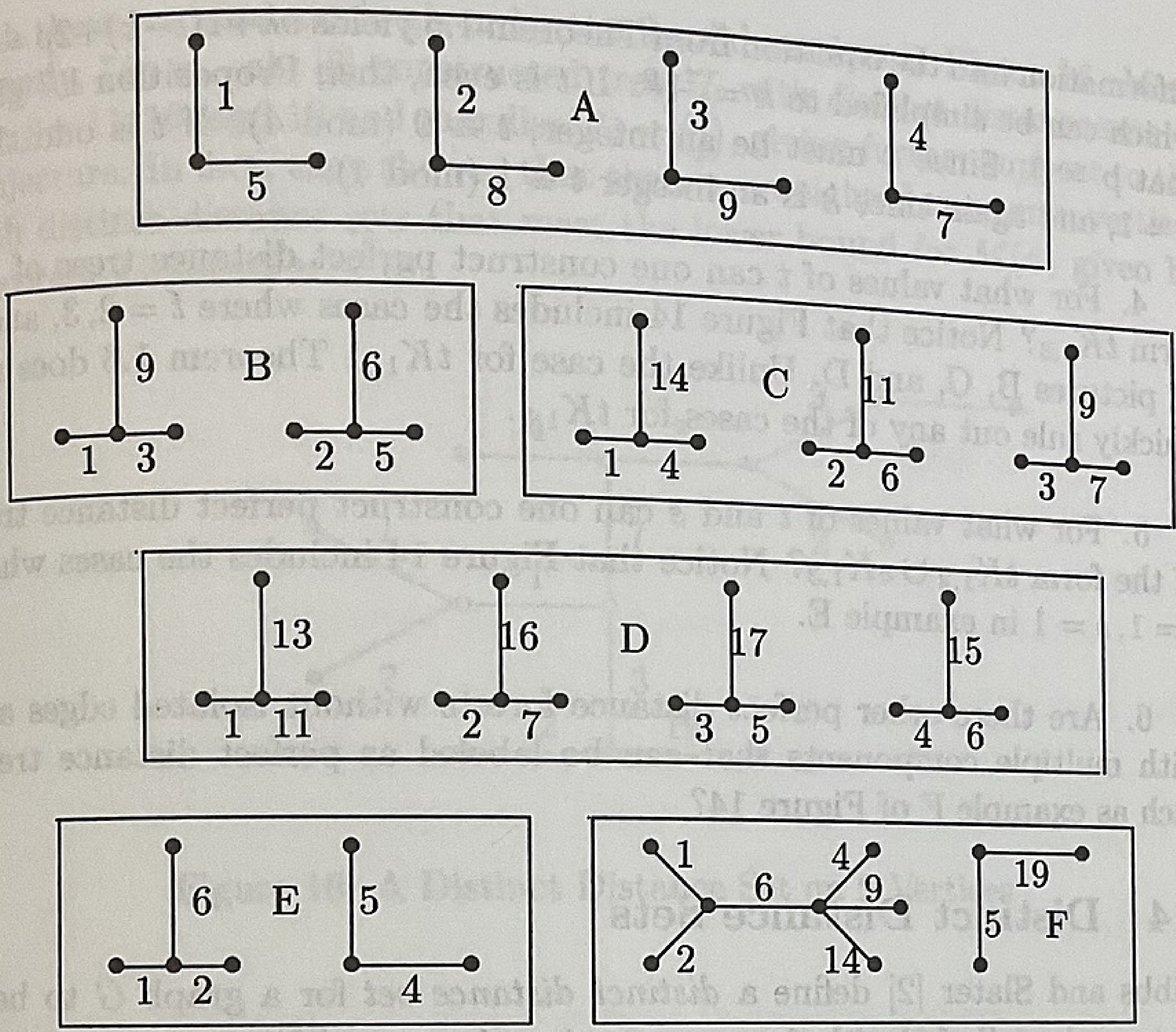

Figure 15: Perfect Distance Forests

distance forest with largest weighted distance d?

2. Can one construct a perfect distance forest in which the smallest component has n vertices?

3. For what values of t can one construct perfect distance trees of the form $tK_{1,2}$? Notice that Figure 14 includes the cases where $t = 4$ as picture A. Moreover, the cases where $t \equiv 2, 3 \pmod 4$ can be eliminated as a corollary to Theorem 1.6.

Corollary 3.4 (Theorem 1.6). *If $tK_{1,2}$ can be labeled as a perfect distance forest, then $t \equiv 0, 1 \pmod 4$.*

Proof. Each component i in $tK_{1,2}$ is a copy of $K_{1,2}$. Consider a coloring of component i. If the labels are all even, then $a_i = n_{i,0} - n_{i,1} = 3$ as defined in the proof following Theorem 1.6. If the integer labels on this component are either both odd or one odd and one even, then it follows that $|a_i| = 1$. Therefore, $a_i{}^2 = 9$ or $a_i{}^2 = 1$.

Let k represent the number of components that are labeled with two even labels, so there are $t - k$ components labeled otherwise. Inserting this information into the equation from Theorem 1.6 yields $9k+1(t-k)+2p = 3t$, which can be simplified to $k = \frac{t-p}{4}$. If t is even, then Proposition 1.7 gives that $p = 0$. Since k must be an integer, $t \equiv 0 \pmod 4$. If t is odd then $p = 1$, and again since k is an integer $t \equiv 1 \pmod 4$. □

4. For what values of t can one construct perfect distance trees of the form $tK_{1,3}$? Notice that Figure 14 includes the cases where $t = 2, 3$, and 4 as pictures B, C, and D. Unlike the case for $tK_{1,2}$, Theorem 1.6 does not quickly rule out any of the cases for $tK_{1,3}$.

5. For what values of t and s can one construct perfect distance trees of the form $tK_{1,2} \cup sK_{1,3}$? Notice that Figure 14 includes the cases where $t = 1, s = 1$ in example E.

6. Are there other perfect distance forests without isolated edges and with multiple components that can be labeled as perfect distance trees, such as example F of Figure 14?

3.4 Distinct Distance Sets

Gibbs and Slater [2] define a *distinct distance set* for a graph G to be a vertex subset S of G with the property that there are $\binom{|S|}{2}$ distinct distances between pairs of vertices in S. Although they consider unweighted graphs, we can consider weighted graphs with integer weights by replacing each edge of weight w by a path of length w. While we do not analyze distinct distance sets in detail, some of our results can be applied to the problem of distinct distance sets. For instance, our results yield a generalization of Taylor's Condition for distinct distance sets from [2] that is equivalent to the following corollary of Theorem 1.6.

Corollary 3.5. *If there is a weighted graph G with a Taylor coloring such that G contains a distinct distance set of size j and $maxdist(G) = \binom{j}{2}$, then either j or $j-2$ is a perfect square.*

Proof. Let H be the complete graph whose vertices are the j members of the distinct distance set. Label each edge xy in H with the distance from x to y in G. Thus, H is a perfect distance graph and the result follows from Theorem 1.6. □

Gibbs and Slater [2] have conjectured that the converse to Corollary 3.5 holds. For instance, although there are no perfect distance trees on 9 vertices, Taylor [11] constructs a tree of *weighted diameter* 36 with a distinct distance set of size 9. A slightly different distinct difference set with these parameters, is given by the 9 solid vertices in Figure 16. More recently, Lin et. al. [8] constructed trees T with distinct distance sets of size $j = 11, 16$, and 18 and $\text{maxdist}(T) = \binom{j}{2}$, giving further support to the conjecture. In fact, they showed that there are weighted trees on n vertices with distinct distance sets that meet the lower bound for $M(n)$ given by Theorem 2.2 for all $n \leq 18$.

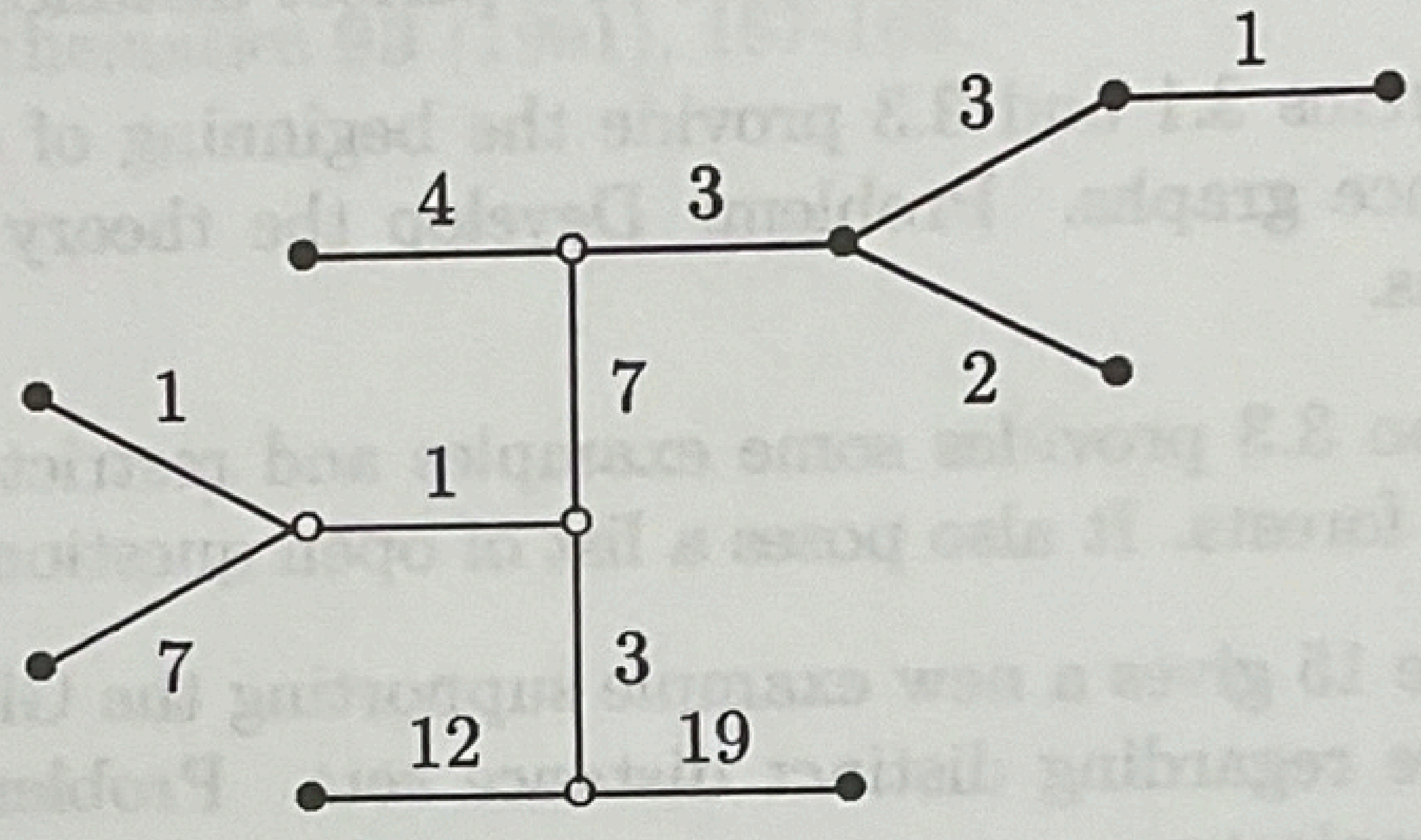

Figure 16: A Distinct Distance Set on 9 Vertices

4 Summary and Open Problems

We conclude with a summary of the results contained in this paper together with some associated open problems.

1. The only perfect distance trees with $n \leq 24$ vertices are those given in Figure 1. Problem: Are there any other perfect distance trees?

2. The values of $M(n)$ for $1 \leq n \leq 10$ are shown in Table 1. Problem: Determine $M(n)$ for $n > 10$.

3. Lower bounds on $M(n)$ for $n \leq 17$ (beyond the obvious lower bound given by Proposition 1.1) were determined using Theorem 2.2 and computer searches, and are given in Table 1. Problem: Improve these lower bounds for $n > 10$.

4. By Graham and Sloane [4], there is a quadratic upper bound on $M(n)$ of the form $2n^2 + O(n^{\frac{36}{23}})$. Problem: Improve this upper bound.

5. The graph in Figure 8 provides examples that improve the upper bound on $M(n)$ for $11 \leq n \leq 18$. Problem: Find further examples that improve those bounds.

6. Bounds on the maximum weight and second highest weight in a perfect distance tree are given by Theorems 2.7 and 2.8. Problem: Find additional constraints on the weights in a perfect distance tree.

7. Proposition 2.10 shows that any unknown perfect distance tree must have (unweighted) diameter greater than 3. Problem: Find additional restrictions on the structure of a perfect distance tree.

8. Theorems 3.1 and 3.3 provide the beginning of a theory for perfect distance graphs. Problem: Develop the theory of perfect distance graphs.

9. Section 3.3 provides some examples and restrictions on perfect distance forests. It also poses a list of open questions.

10. Figure 15 gives a new example supporting the Gibbs and Slater conjecture regarding distinct distance sets. Problem: Verify or refute that conjecture.

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