Almgren and Chriss

Dahiana Riascos, John Sibony, Victor Biaggi

December 2018

1 Trading Set Up

We define a trading trajectory to be a list $x_0, ..., x_N$, where x_k is the number of units that we plan to hold at time t_k. Our initial holding is $x_0 = X$, and liquidation at time T requires $x_N = 0$.
We may equivalently specify a strategy by the "trade list" $n_1, ..., n_N$, where $n_k = x_{k1} x_k$ is the number of units that we will sell between times t_{k1} and t_k . We have

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j$$

Suppose that the initial security price is S_0 , so that the initial market value of our position is XS_0. We distinguish two kinds of market impact. Temporary impact refers to temporary imbalances in supply in demand caused by our trading leading to temporary price movements away from equilibrium. Permanent impact means changes in the "equilibrium" price due to our trading, which remain at least for the life of our liquidation. Thus, we assume that the security price evolves according to the following equation:

$$S_k = S_{k-1} + \sigma \tau^{1/2} \xi_k - \tau\ g(\frac{n_k}{\tau})$$

The ξ_k are draws from independent random variables each with zero mean and unit variance. The h function represents the permanent impact of our trading. The temporary impact can be modeled by introducing another function $h(v)$. Thus, the actual price per share received on sale k is :

$$\tilde{S_k} = S_{k-1} - h(\frac{n_k}{\tau})$$

Thus, the profit resulting from trading along a certain trajectory can be expressed as follows :

$$\sum_{k=1}^{N} n_k \tilde{S_k} = X\ S_0 + \sum_{k=1}^{N} (\sigma \tau^{1/2} \xi_k - \tau\ g(\frac{n_k}{\tau})) x_k - \sum_{k=1}^{N} n_k h(\frac{n_k}{\tau})$$

Thus, the expectation and the variance of our shortfall can be computed as follows :

$$E(x) = \sum_{k=1}^{N} \tau x_k g(\frac{n_k}{\tau}) + \sum_{k=1}^{N} n_k h(\frac{n_k}{\tau})$$

$$V(x) = \sigma^2 \ \sum_{k=1}^{N} \tau x_k^2$$

2 Optimal Trading Strategies

A rational trader will always seek to minimize the expectation of shortfall for a given level of variance of shortfall. Naturally, a trader will prefer a strategy that provides the minimum error in its estimate of expected cost. We dene a trading strategy to be ecient or optimal if there is no strategy which has lower variance for the same or lower level of expected transaction costs, or, equivalently, no strategy which has a lower level of expected transaction costs for the same or lower level of variance.
We may construct efficient strategies by solving the constrained optimization problem :

$$min_{x:V(x)\leq V_0} \ E(x)$$

We solve this constrained optimization problem by introducing a Lagrange multiplier , solving the unconstrained problem :

$$min_x(E(x) + \lambda V(x))$$

If > 0, then $E + \lambda V$ is strictly convex, and this last equation has a unique solution $x(\lambda)$. As varies, $x(\lambda)$ sweeps out the same one-parameter family, and thus traces out an ecient frontier.

3 Close forms in the linear impact functions case

Computing optimal trajectories becomes easy when we take the permanent and temporary impact functions to be linear in the rate of trading. We take :

$$g(v) = \gamma v$$

$$h(\frac{n_k}{\tau}) = \epsilon \ sgn(n_k) + \frac{\eta}{\tau} n_k$$

Where sgn is the sgn function. This linear model for h is often called a quadratic cost model because the total cost incurred by buying or selling n units in a single unit of time is :

$$nh(\frac{n}{\tau}) = \epsilon|n| + \frac{\eta}{\tau} n^2$$

Then, we have :

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \tau^{1/2} \xi_j - \gamma(X - x_k)$$

Then, the permanent impact term becomes:

$$\sum_{k=1}^{N} \tau x_k g(\frac{n_k}{\tau}) = \gamma \sum_{k=1}^{N} x_k n_k = \gamma \sum_{k=1}^{N} x_k (x_{k-1} - x_k)$$

$$= \frac{1}{2}\gamma \sum_{k=1}^{N} (x_{k-1}^2 - x_k^2 - (x_k - x_{k-1})^2) = \frac{1}{2}\gamma X^2 - \frac{1}{2}\gamma \sum_{k=1}^{N} n_k^2$$

With both linear cost models, the expectation of impact costs becomes :

$$E(x) = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{tilde\eta}{\tau} \sum_{k=1}^{N} n_k^2$$

Where

$$\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$$

Note that E is a strictly convex function as long as $\tilde{\eta} > 0$.
There are two obvious trading trajectories that can be examined. The most obvious trajectory is to sell at a constant rate over the whole liquidation period. In this case, we have :

$$n_k = \frac{X}{N},\ x_k = (N-k)\frac{X}{N},\ k = 1, ..., N.$$

Then we have :

$$E = \frac{1}{2} X T g(\frac{X}{T})(1 - \frac{1}{N}) + X h(\frac{X}{T})$$
$$= \frac{1}{2}\gamma X^2 + \epsilon X + (\eta - \frac{1}{2}\gamma\tau)\frac{X^2}{T}$$

And :

$$V = \frac{1}{3}\sigma^2 X^2 T (1 - \frac{1}{N})(1 - \frac{1}{2N})$$

This trajectory minimizes total expected costs, but the variance may be large if the period T is long. Another extreme is to sell our entire position in the rst time step. We then take :

$$n_1 = X,\ n_2 = ... = n_N = 0,\ x_1 = ... = x_N = 0,$$

which give

$$E = X h(\frac{X}{\tau}) = \epsilon X + \eta \frac{X^2}{\tau}$$
$$V = 0$$

This trajectory has the smallest possible variance, equal to zero because of the way that we have discretized time in our model. If N is large and hence is short, then on our full initial portfolio, we take a price hit which can be arbitrarily large. Our purpose is to study optimal trajectories that lie between these two extremes.

With E(x) and V(x) from their initial expression, and assuming that n_j does not change sign, the combination $U(x) = E(x) + \lambda V(x)$ is a quadratic function of the control parameters $x_1, ..., x_{N1}$; it is strictly convex for $\lambda \geq 0$. Therefore we determine the unique global minimum by setting its partial derivatives to zero. Since :

$$\frac{\partial U}{\partial x_j} = 2\tau(\lambda\sigma^2 x_j - \tilde{\eta}\frac{x_{j-1} - 2x_j + x_{j+1}}{\tau})$$

Then, $\frac{\partial U}{\partial x_j} = 0$ is equivalent to :

$$\frac{1}{\tau^2}(x_{j-1} - 2x_j + x_{j+1}) = \tilde{\kappa}^2 x_j$$

Where

$$\tilde{\kappa} = \frac{\lambda\sigma^2}{\tilde{\eta}}$$

The solutions of this equation can be written as a combination of the exponentials $exp(\pm\kappa t_j), where \kappa$ satisfies:

$$\frac{2}{\tau^2}(cosh(\kappa\tau) - 1) = \tilde{\kappa}$$

The specific solutions with $x_0 = X$ and $x_N = 0$ is a trading trajectory of the form:

$$x_j = \frac{sinh(\kappa(T - t_j))}{sinh(\kappa T)} X$$

and the associated trade list :

$$n_j = \frac{2sinh(\frac{1}{2}\kappa\tau)}{sinh(\kappa T)} cosh(\kappa(T - t_{j-\frac{1}{2}}))X$$

The expectation and variance of the optimal strategy, for a given initial portfolio size X, are then :

$$E(X) = \frac{1}{2}\gamma X^2 + \epsilon X + \tilde{\eta} X^2 \frac{tanh(\frac{1}{2})(\tau sinh(2\kappa T) + 2T sinh(\kappa\tau))}{2\tau^2 sinh^2(\kappa T)}$$

$$V(X) = \frac{1}{2}\sigma^2 X^2 \frac{\tau sinh(\kappa T) cosh(\kappa(T - \tau)) - T sinh(\kappa\tau)}{sinh^2(\kappa T) sinh(\kappa\tau)}$$

4 Solving Almgren-Chris through dynamic programming

4.1 Bellman equation

The problem of minimizing $E(x) + \lambda V(x)$ is equivalent to solving the following Bellman equation:

Terminal condition :

$$v(r,x,s,T-1) = exp(-\gamma[r + x(s - g(\frac{x}{\tau}))])$$

Backward equation:

$$v(r,x,s,t) = min_n E(v(\tilde{x}, \tilde{s}, q-n, t+1))$$

Where

$$\tilde{x} = x + n(s - h(\frac{n}{\tau})),\ \tilde{s} = s + \sqrt{\tau}\sigma\xi_t - g(\frac{n}{\tau})\tau$$

As any combination (x,t) can take any value for the spot price s and revenue r, the implementation suffers from a curse of dimensionality.
A possible apprach consists in simplifying $v(.)$:

$$v(r,x,s,t) = u(x,t)exp(-\gamma(r + xs))$$

The dynamic programming principle on $u(t,x)$ can be written as :
Terminal condition:

$$u(x,T-1) = exp(\gamma x g(\frac{x}{\tau}))$$

Backward equation: The value of $u(x,t)$ is given minimizing over $0 < n < x$ the following quantity:

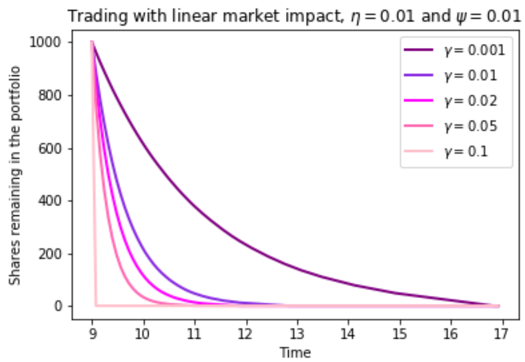
$$u(x-n,t+1)exp(\gamma n g(\frac{n}{\tau}) + \gamma(x-n)\tau h(\frac{n}{\tau}) + \frac{1}{2}\gamma^2(x-n)^2\sigma^2\tau)$$

This change of variable is based on the fact that $r + xs$ is our mark to market at any time t.

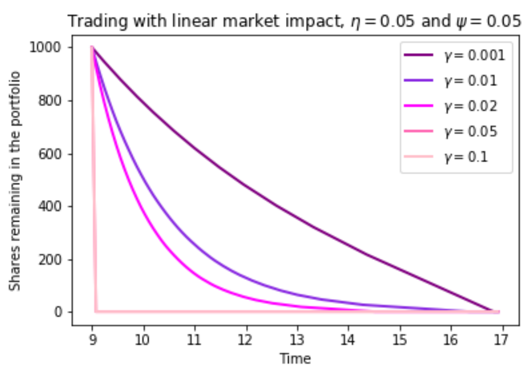
4.2 Experiments

We perform the experiment with the following functions, parameters :

$h(v) = \eta v,\ \eta = 0.01$
$g(v) = \psi v,\ \psi = 0.01$
$\gamma \in \{0.1, 0.05, 0.02, 0.01, 0.001\}$
$\tau = \frac{9}{100}h$ (100 time periods during a 9 hours trading day).



$h(v) = \eta v, \ \eta = 0.05$
$g(v) = \psi v, \ \psi = 0.05$
$\gamma \in \{0.1, 0.05, 0.02, 0.01, 0.001\}$
$\tau = \frac{9}{100}h$



Dynamic programming allows us to model more realistically how portfolio liquidation works, integrate dynamic parameters. However, the curse of dimentionality limits this method. We obtained close for formula in the linear case.

References :

Robert Almgren and Neil Chris.
Optimal execution of portfolio transaction.

5 Comparing trading algorithms

After implementing the Almgren-Chriss model with the Bellman equation and linear cost functions we tried doing it with the original model using the static optimization. Based on the original benchmark used in the paper, Implementation Shortfall, we determine the usual equation to solve:

$$\min\{\mathbb{E}[X] + \lambda \mathbb{V}[X]\}$$

Also, like for the Dynamic Programming part we will use the linear cost functions.
To implement the models in Python we used the Minimize function of scipy.optimize with the Sequential Least Square method as to add the boundaries and constraints that are within our model. The boundaries being between 0 and the total inventory to liquidate. The constraint is that we must liquidate everything, so the sum of the liquidation must be equal to the beginning inventory.
With X the total inventory:

$$\text{bounds} = [0.0, X]$$

$$\text{constraints: } \sum_{k=1}^{N} q_k = X$$

5.1 Implementation Shortfall (IS)

a. Model

IS is a cost-driven algorithm. It is the total transaction cost of a trade. It aims to minimize the difference between the average trade and a certain benchmark. The IS is the difference between the price at the time the decision was taken, the average trade price, and the final execution price.

$$\text{IS\_payoff} = XS_0 - \sum_{k=1}^{N} n_k \widetilde{S_k}$$

Both the expectation and the variance are given and defined by Almgren and Chriss for this benchmark, we have:

$$\mathbb{E} = \sum_{k=1}^{N} \tau x_k g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

$$V[x] = \sigma^2 \sum_{k=1}^{N} \tau {x_k}^2$$

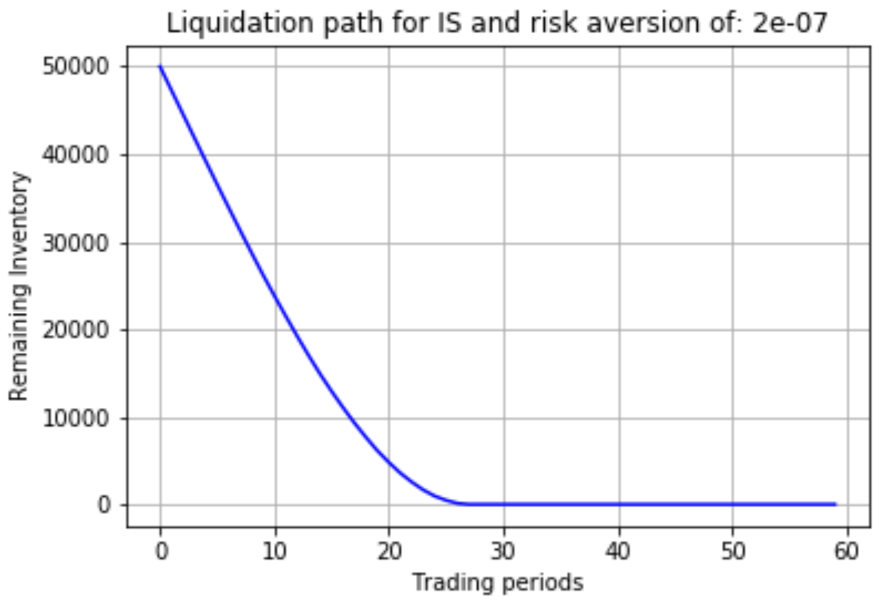
For the expectation we will use the approximation given in case of g and h being linear cost functions. Therefore, we will use that:

$$\mathbb{E} = \frac{1}{2}\alpha X^2 + \varepsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} {n_k}^2$$

$$with\ \tilde{\eta} = \eta - \frac{1}{2}\alpha\tau$$

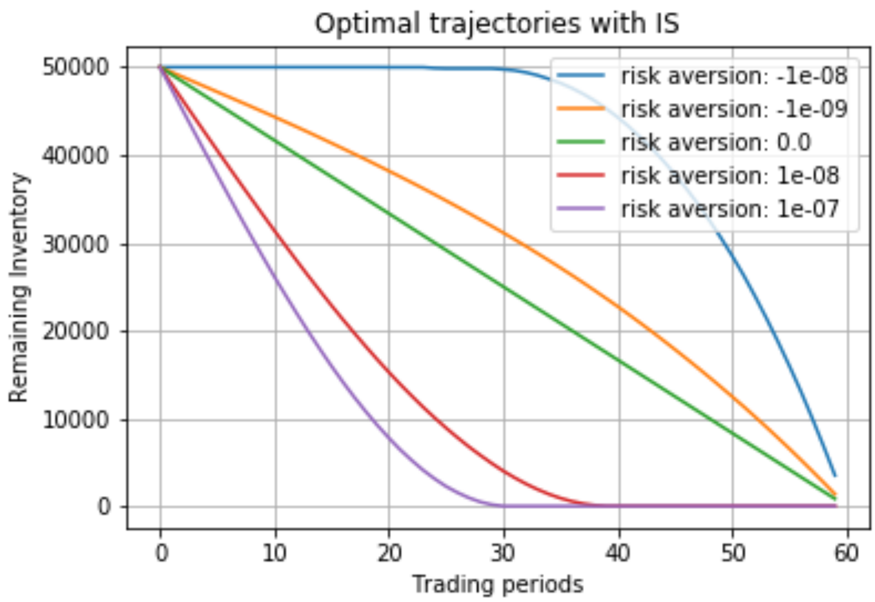
b. Application

An example of liquidation price with the IS:



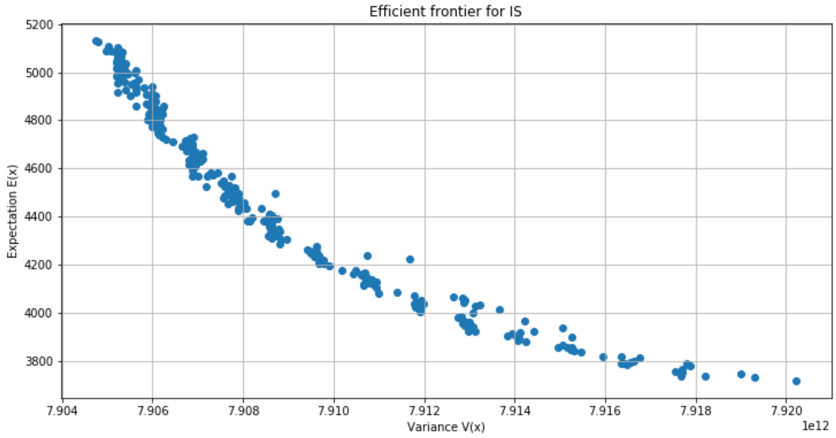
Here we can see that to liquidate a risk averse trader following this strategy would rather liquidate steadily but quickly. This is because the difference between the average price and the execution price is smaller at the beginning of the period than later on.

Various risk aversion coefficients and their optimal trajectories:



Here we observe that there is a positive relationship between risk aversion and faster liquidation. Given the nature of this strategy this was to be expected.
At the contrary, between risk takers and faster liquidation we observe a negative relationship. The more risk taker a trader is and the slower the liquidation will be done, waiting for a potentially better opportunity to come.

Lastly, we plot the efficient frontier.



5.2 Time Weighted Average Price (TWAP)

a. Model

The TWAP is an impact-driven algorithm. Meaning that it aims to minimize the market impact. The TWAP benchmark is an average price that takes into account the time evolution of prices during the day. It is a better strategy than putting the full amount to sell/buy all at once. It is an extension of order slicing strategies. It uses a uniformly time-based schedule so that it buys or sells evenly part of the volume both in the morning and in the afternoon session. It is calculated like the mean of prices weighted proportionally to the traded volume.

$$TWAP_{payoff} = \mathrm{X}\frac{\sum_{k=1}^{N} \tau P_k}{\sum_{k=1}^{N} \tau} - \sum_{k=1}^{N} n_k \widetilde{S_k}$$

Here we have $\sum_{k=1}^{N} \tau = T$ so we get:

$$\mathbb{E}[TWAP_p] = \mathbb{E}\left[X\frac{\sum_{k=1}^{N} \tau S_k}{\sum_{k=1}^{N} \tau} - \sum_{k=1}^{N} n_k \widetilde{S_k}\right]$$
$$= \mathbb{E}\left[\frac{X}{T}\sum_{k=1}^{N} S_k - \sum_{k=1}^{N} n_k \widetilde{S_k}\right]$$
$$= \mathbb{E}\left[\frac{X}{T}\sum_{k=1}^{N} S_k - \sum_{k=1}^{N} n_k \widetilde{S_k}\right]$$
$$= \mathbb{E}\left[\frac{X\tau}{T}\sum_{k=1}^{N}(S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k}\zeta_j - \sum_{j=1}^{k}\tau g\left(\frac{n_j}{\tau}\right)) - n_k((S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k-1}\zeta_j - \sum_{j=1}^{k-1}\tau g\left(\frac{n_j}{\tau}\right) - h\left(\frac{n_k}{\tau}\right))\right]$$

$$= \sum_{k=1}^{N}(\frac{X\tau}{T} - n_k)(S_0 - \sum_{j=1}^{k-1}\tau g\left(\frac{n_j}{\tau}\right)) - \frac{X\tau^2}{T} g\left(\frac{n_k}{\tau}\right) + n_k\, h\left(\frac{n_k}{\tau}\right)$$

For the variance:
We know that

$$TWAP_{payoff} = \frac{\sum_{k=1}^{N} \tau S_k}{\sum_{k=1}^{N} \tau}$$
$$= \frac{\tau}{T}\sum_{k=1}^{N}\left(S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k}\zeta_j - \sum_{j=1}^{k}\tau g\left(\frac{n_j}{\tau}\right)\right)$$

$$= \frac{\tau}{T} \sum_{k=1}^{N} \left(\sigma_k \sqrt{\tau} \zeta_k \sum_{j=1}^{k} \tau \right) + a(n_k)$$

And

$$\sum_{k=1}^{N} n_k \widetilde{\tilde{S}_k} = n_k((S_0 + \sigma_k \sqrt{\tau} \sum_{j=1}^{k-1} \zeta_j - \sum_{j=1}^{k-1} \tau g\left(\frac{n_j}{\tau}\right) - h\left(\frac{n_k}{\tau}\right)$$

$$= \sum_{k=1}^{N} \sigma_k \sqrt{\tau} \zeta_k x_k + b(n_k)$$

So we get:

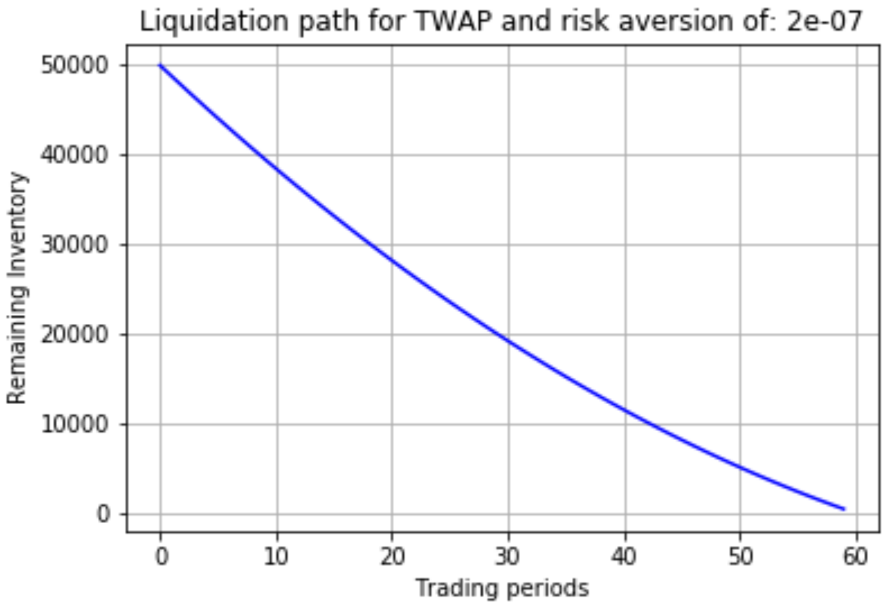
$$\mathbb{V}[TWAP_p] = \mathbb{V}\left[\frac{X}{T} \sum_{k=1}^{N} \left(\sigma_k \sqrt{\tau} \zeta_k \sum_{j=1}^{k} \tau \right) + a(n_k) - \sum_{k=1}^{N} \sigma_k \sqrt{\tau} \zeta_k x_k - b(n_k) \right]$$

$$= \mathbb{V}\left[\frac{X}{T} \sum_{k=1}^{N} \left(\sigma_k \sqrt{\tau} \zeta_k \sum_{j=1}^{k} \tau \right) - \sum_{k=1}^{N} \sigma_k \sqrt{\tau} \zeta_k x_k \right]$$

$$= \mathbb{V}\left[\sum_{k=1}^{N} \left(\sigma_k \sqrt{\tau} \zeta_k \left(\frac{X}{T} \sum_{j=1}^{k} \tau - x_k\right) \right) \right]$$

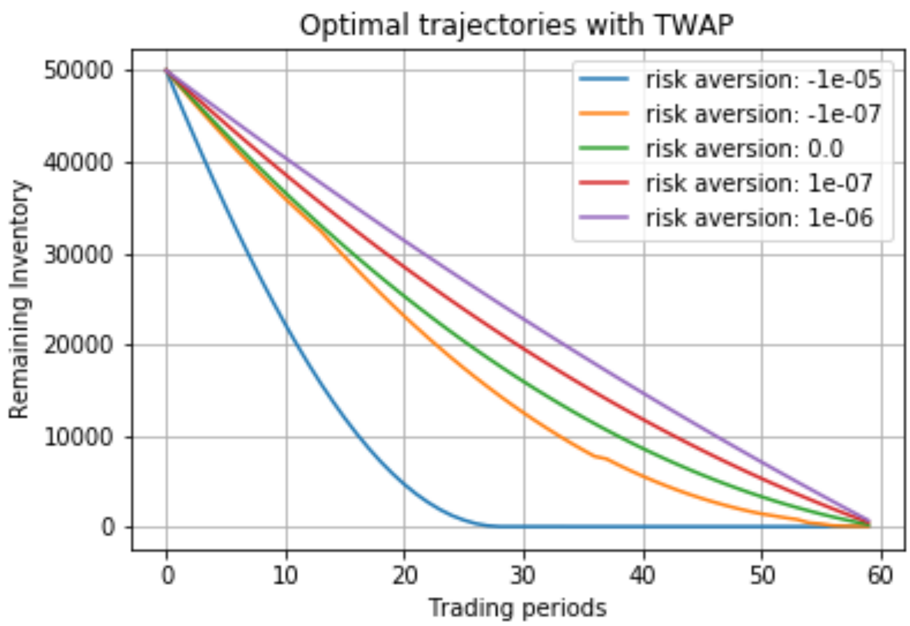
$$= \sum_{k=1}^{N} \tau {\sigma_k}^2 \left(\left(\frac{X}{T} \sum_{j=1}^{k} \tau - x_k\right) \right)^2$$

b. Application



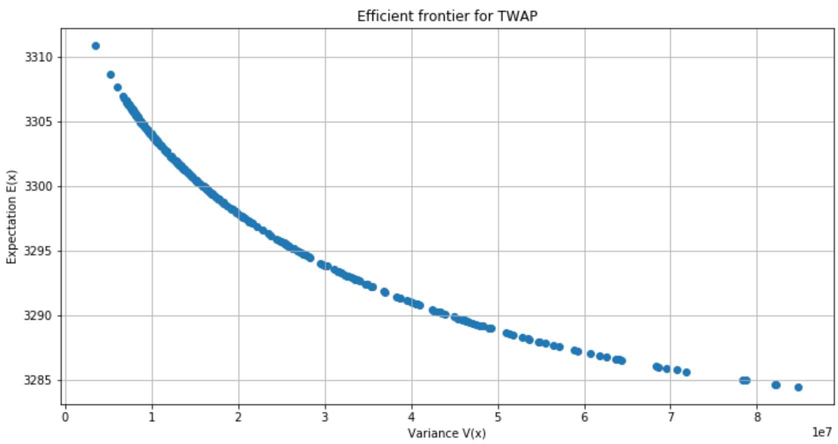
Here we observe that a risk averse trader using this algorithm would steadily liquidate. As the algorithm aims to uniformly sell the inventory through the period it makes sense to have this kind of curve.

Various risk aversion coefficients and their optimal trajectories:



In this graph of optimal trajectories, we can see that a risk adverse agent would rather slowly but steadily sell his inventory over the whole period whereas a risk taker trader would rather quickly liquidate instead of trying to slowly liquidate overtime. A risk taker agent would not bother with waiting to minimize his impact but rather take his chance sooner to liquidate everything.

We plot the efficient frontier.



5.3 Volume Weighted Average Price (VWAP)

a. Model

Similarly, to the TWAP, the VWAP is also an impact-driven algorithm. The VWAP strategy's benchmark is in this case the average price that takes into account the traded volume during a time period.

$$VWAP_{payoff} = X\frac{\sum_{k=1}^{N} V_k P_k}{\sum_{k=1}^{N} V_k} - \sum_{k=1}^{N} n_k \widetilde{S_k}$$

Here we have $\sum_{k=1}^{N} v_k = V$ so we get:

$$\mathbb{E}[VWAP_p] = \mathbb{E}\left[X\frac{\sum_{k=1}^{N} v_k S_k}{\sum_{k=1}^{N} v_k} - \sum_{k=1}^{N} n_k \widetilde{S_k}\right]$$

$$= \mathbb{E}\left[\frac{X}{V}\sum_{k=1}^{N} v_k S_k - \sum_{k=1}^{N} n_k \widetilde{S_k}\right]$$

$$= \mathbb{E}\left[\frac{X}{V}\sum_{k=1}^{N} v_k (S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k}\zeta_j - \sum_{j=1}^{k}\tau g\left(\frac{n_j}{\tau}\right)) - n_k((S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k-1}\zeta_j - \sum_{j=1}^{k-1}\tau g\left(\frac{n_j}{\tau}\right) - h\left(\frac{n_k}{\tau}\right))\right]$$

$$= \sum_{k=1}^{N}(\frac{Xv_k}{V} - n_k)(S_0 - \sum_{j=1}^{k-1}\tau g\left(\frac{n_j}{\tau}\right)) - \frac{Xv_k}{V} g\left(\frac{n_k}{\tau}\right) + n_k\, h\left(\frac{n_k}{\tau}\right)$$

Similarly, to the TWAP we compute the variance of the difference:

$$VWAP_{payoff} = \frac{\sum_{k=1}^{N} v_k S_k}{\sum_{k=1}^{N} V}$$

$$= \frac{1}{T}\sum_{k=1}^{N} v_k \left(S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k}\zeta_j - \sum_{j=1}^{k}\tau g\left(\frac{n_j}{\tau}\right)\right)$$

$$= \frac{1}{T}\sum_{k=1}^{N}\left(\sigma_k\sqrt{\tau}\zeta_k\sum_{j=1}^{k} v_j\right) + a(n_k)$$

And

$$\sum_{k=1}^{N} n_k \widetilde{\tilde{S}_k} = n_k((S_0 + \sigma_k\sqrt{\tau}\sum_{j=1}^{k-1}\zeta_j - \sum_{j=1}^{k-1}\tau g\left(\frac{n_j}{\tau}\right) - h\left(\frac{n_k}{\tau}\right)$$

$$= \sum_{k=1}^{N}\sigma_k\sqrt{\tau}\zeta_k x_k + b(n_k)$$

So, we get:

$$\mathbb{V}\left[VWAP_p\right] = \mathbb{V}\left[\frac{X}{V}\sum_{k=1}^{N}\left(\sigma_k\sqrt{\tau}\zeta_k\sum_{j=1}^{k} v_j\right) + a(n_k) - \sum_{k=1}^{N}\sigma_k\sqrt{\tau}\zeta_k x_k - b(n_k)\right]$$

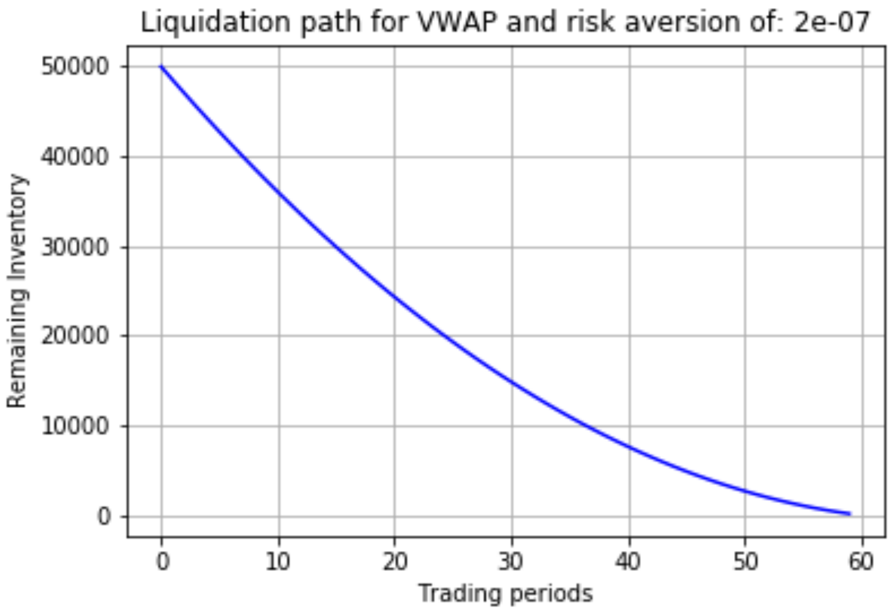
$$= \mathbb{V}\left[\frac{X}{V}\sum_{k=1}^{N}\left(\sigma_k\sqrt{\tau}\zeta_k\sum_{j=1}^{k} v_j\right) - \sum_{k=1}^{N}\sigma_k\sqrt{\tau}\zeta_k x_k\right]$$

$$= \mathbb{V}\left[\sum_{k=1}^{N}\left(\sigma_k\sqrt{\tau}\zeta_k(\frac{X}{V}\sum_{j=1}^{k} v_j - x_k)\right)\right]$$

$$= \sum_{k=1}^{N}\tau{\sigma_k}^2\left((\frac{X}{V}\sum_{j=1}^{k} v_j - x_k)\right)^2$$

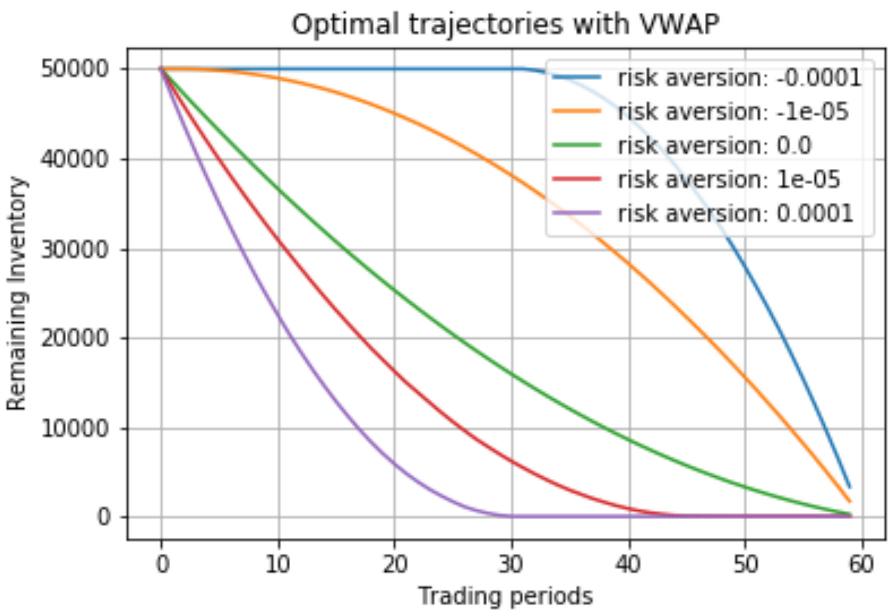
b. Application

An example of liquidation price with the VWAP:



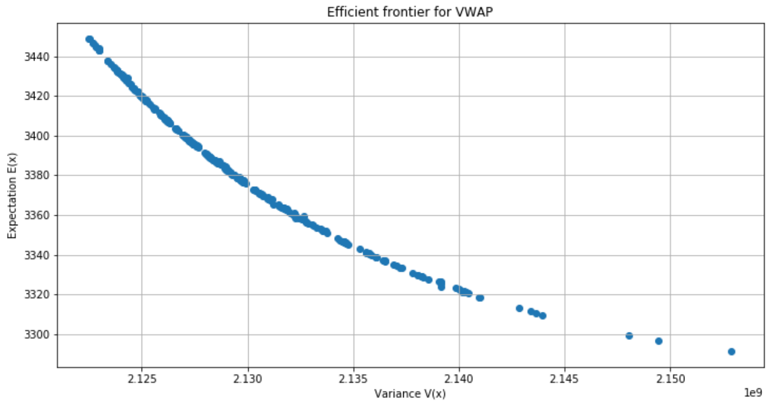
Here we can see that to liquidate a risk averse trader will liquidate his inventory in less tahn half of the period but not in a straight linear way.

Various risk aversion coefficients and their optimal trajectories:



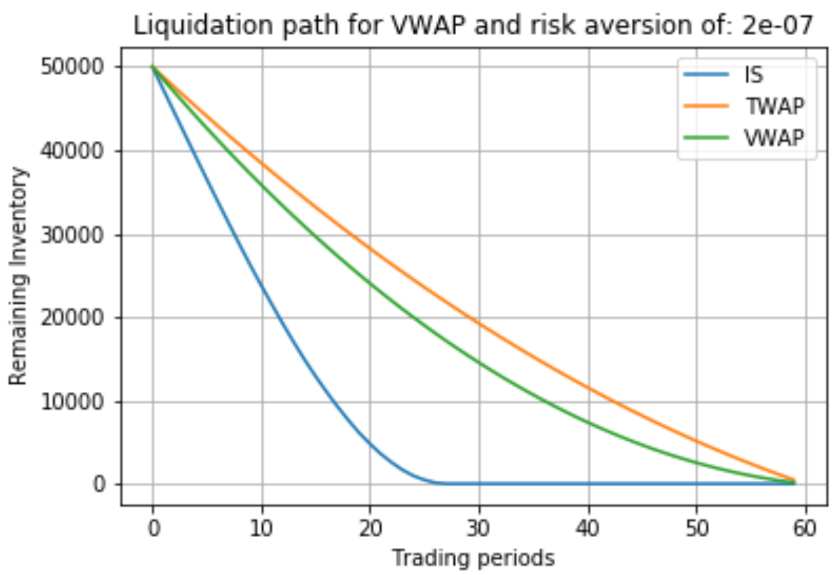
Here we observe that risk averse traders would rather liquidate as soon as possible while trying to minimize its impact in terms of volume. However, a risk taker agent is more likely to wait and liquidate big quantities at the end.

Lastly, we plot the efficient frontier.



5.4 Comparing Results

Comparing optimal trajectories among the 3 strategies implemented we observe that both the TWAP and the VWAP tend to converge more quickly toward the neutral trajectory of liquidation than the IS. Indeed, IS seems to be more sensitive to risk aversion.



We confirm that with plotting the 3 algorithm results with the same risk aversion.

We also observe that the TWAP as an inverse relationship to the other two in terms of liquidation. This is probably due to the fact that it is time sensitive unlike the IS that only care about the cost. Also, here the VWAP is done in a market where the volume is stable so it is not like it would probably behave if there was a historical volume to get some more information into the algorithm.

Overall in all strategies the behavior of the risk averse agent against the liquidation tends to make sense and is coherent when it comes to its risk taker counterpart.

6 Tale of Two Time Scales: Determining Integrated Volatility With Noisy High-Frequency Data

We now focus on the paper written by Zhang, Mykland and Ait-Sahalia (2005), nammed "A Tale of Two Time Scales: Determining Integrated Volatility With Noisy High-Frequency Data".

In the previous article we have made the assumption that prices are following a particular stochastic process with particular parameters. One of these is the volatility of our price that has been assumed to be constant. Numerical examples has been provided but none reflect a particular true prices dataset. How can we calibrate the volatility when we are dealing with high frequency datas ?

Suppose the dynamic of our prices $S_t = exp(X_t)$ follows an GBM process :
$dX_t = \mu_t d_t + \sigma_t dB_t$ that is : $X_t = X_0 + \int_0^T \mu_t dt + \int_0^T \sigma_t dWt$
But when the function σ is deterministic, $\int_0^T \sigma_t dWt$ is a Wiener process, ie a centered Normal law with variance $V = \int_0^T \sigma_t^2 dt$. This quantity is very important since it determines the shape of the volatility at a given interval [0,T]. This is a key quantity that we have to deal with during computation.

Zhang, Mykland and Ait-Sahalia have derived 5 different methods (from naive to robust) to calibrate the integrated volatility for a given dataset of price :

Method A : In this naive method, we compute an integral over time of the squared difference of log-prices. Indeed, by taking $\Delta = \{0 = t_0 < ... < t_n = T\}$ a partition of the interval $[0, T]$ we can easily show by Riemann integral :

$$\lim_{n\to+\infty} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 = \int_0^T \sigma_t^2 dt = V$$

Thus, we can construct an estimator of the integrated volatility defined by :

$$[X,X]_T^{all} = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$$

However, according to the authors using this estimator to compute the integrated volatility on intra-day is a bad idea : by sampling the price very closely (low sampling time interval), you will have few data-points and lots of statistical errors (high variance). If your sampling time interval is very high (high-frequency), then you will take into account lots of noise.

Indeed in high frequency sample, we thus need to correct for microstructure biases (not taking into account the existence of a bid-ask spread for example) since prices can be modelled as :

$$Y_t = X_t + \epsilon_t$$

where X is a latent variable for the "real" unobserved stock prices, Yt the observed latent returns and ϵ some microstructure noise not incorpored in the dynamics of the stock price.

They demonstrate that the estimator $[Y,Y]_T^{all}$ is totally inappropriate. Instead of capturing the true unbiased estimator $[X,X]_T^{all}$, it captures the variance of the white noise :

$$[Y,Y]_T^{all} = 2nVar(\epsilon) + O_p(n^{1/2})$$

Method B : Since $[Y,Y]_T^{all}$ reflect the variance of the market biases due to the high frequency sample, they proposed the same estimator by reducing the frequency. Instead of taking into account all of our sample, we only focus on time-minute datas (between 1 minute to 40 minutes in practice). This is how practitioner used to deal with the integrated volatility estimation. Indeed, one can prove that we have :

$$[Y,Y]_T^{sample} = V + 2n_{sparse}Var(\epsilon) + \sqrt{(4n_{sparse}Var(\epsilon) + \frac{2T}{n_{sparse}} \int_0^T \sigma_t^4 dt)}Z$$

with Z a gaussian law and n_{sparse} is the size of our subsample due to the reduction of the frequency sampling.

Thus, we now obtain an biaised estimator of the true estimator V.

Method C : With the previous method, we have obtained an unbiased estimator but a lot of data has not been reviewed. One way to deal with it is to make an average of different estimator of the method B.

Let $G^{(k)} = \{t_{k-1}, t_{k-1+K}, t_{k-1+2K}, ..., t_{k-1+MK}\}$, k=1, ...K being a partition of our total data such that each of the K set have a frequency of K. Then our average estimator is defined by :

$$[Y,Y]_T^{(average)} = \frac{1}{K}\sum_{k=1}^{K}[Y,Y]_T^{(k)}$$

where $[Y,Y]_T^{(k)}$ is the estimator of the method B using the set of sample $G^{(k)}$. Then Zhang, Mykland and Ait-Sahalia proved that its biais is :

$$[Y,Y]_T^{average} = V + 2\overline{n}Var(\epsilon) + \sqrt{(4\frac{\overline{n}}{K}Var(\epsilon) + \frac{4T}{3\overline{n}}\int_0^T \sigma_t^4 dt)}Z$$

where $\overline{n} = \frac{n-K}{K}$. Increasing K will reduce the first part of the bias but it will also increase the part with the integrale (due to the factor $\frac{1}{n}$). Thus there is a tradeoff in the optimal choice of K that we can theorically compute.

Method D : The "first-best approach" combines both the first naive estimator $[Y,Y]_T^{all}$ and the average of subsampled estimators $[Y,Y]_T^{average}$ and gives the following estimator :

$$\widehat{[Y,Y]}_T = [Y,Y]_T^{(average)} - \frac{\bar{n}}{n}[Y,Y]_T^{(all)}$$

To reduce the biais, one should considered this little transformartion :

$$\widehat{[Y,Y]}_T^{(bias-adjusted)} = (1 - \frac{\bar{n}}{n})^{-1}\widehat{[Y,Y]}_T$$

Then the authors demonstrated that we obtain the following error :

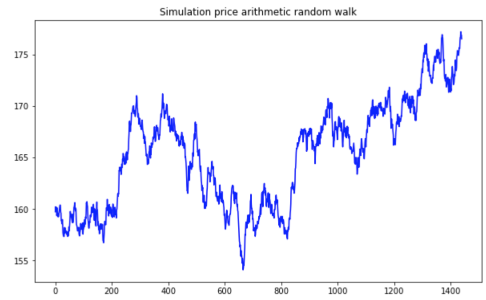
$$[Y,Y]_T^{average} = V + \frac{1}{n^{1/6}}\sqrt{(\frac{8}{c^2}Var(\epsilon)^2 + \frac{c4T}{3}\int_0^T \sigma_t^4 dt)}Z$$

where $c = \frac{K}{n^{2/3}}$. AN optimal choice of c (and thus of K) is defined by :

$$c = (\frac{T}{12Var(\epsilon)^2}\int_0^T \sigma_t^4 dt)^{-1/3}$$

However since we cannot compute it due to the integral part, we have to make an approximation. Numerical approximation has been provided in the article. We have tried to use it to compute c optimal.

Backtesting : We have imported a dataset contening Facebook spot prices on 2018-10-04 with a frequency of 1 minute.



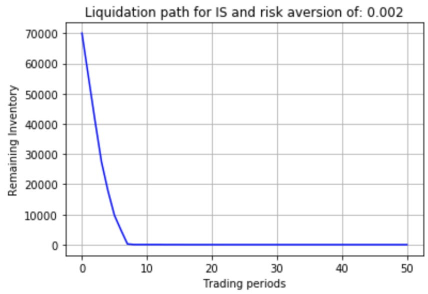
Since the Facebook price can be modelized with a random walk, Almgren-Chriss framework applied.

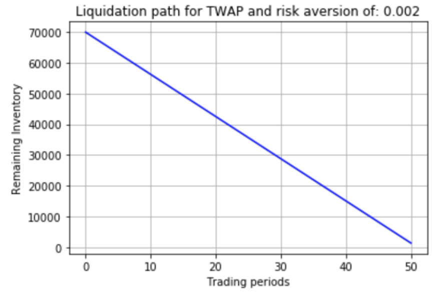
To determine the optimal execution according to IS, TWAP and VWAP benchmark, we choose these hyperparameters :

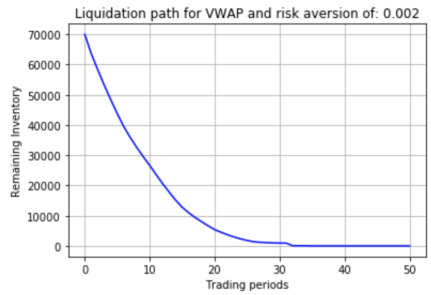
X = 70000,
S0 = 160.3,
N = 50,
τ = 1/50(one day),
r = 2*10^{-3},
ϵ = 0.0625
Then we calibrate with the dataset these parameters :

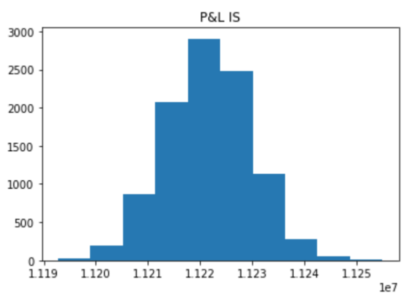
$\gamma = \frac{spread}{(0.1V_{total})} = 4.5 * 10^{-10}$,
σ = 0.74 (average method),
$\eta = \frac{spread}{(0.01V_{total})} = 4.5 * 10^{-9}$

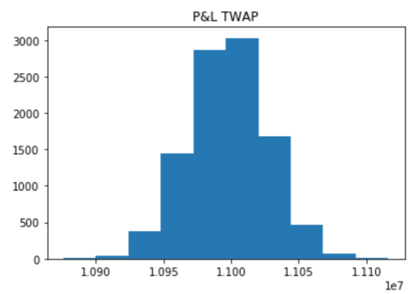
Then we computed the optimal execution shares with these parameters for IS, TWAP and VWAP. We simulated 10 000 times prices with market impact for IS, TWAP and VWAP and we computed the P&L of these 3 benchmark :

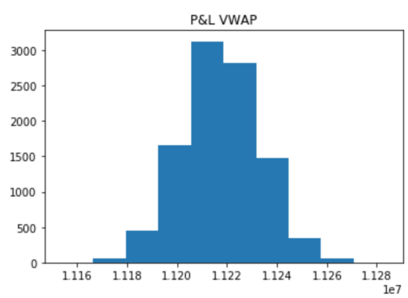












We observe that VWAP and IS strategy are more profitable than the TWAP strategy. We can explain that by the uniform execution of the TWAP strategy for the risk $r = 2 * 10^{-3}$. This risk is too low for the TWAP strategy to be important, hence it will liquidate shares much later.