Almgren-Chriss through Dynamic Programming

Modeling

We want to liquidate a portfolio Q as a sequence of smaller trades $(n_1, \ldots, n_T)$ in a way that solves the optimization problem:

$$\min_{n_1,\ldots,n_T} \mathbb{E}[e^{-\gamma X_T}]$$

where X_T is our cash at the end of the trading session.

After selling an order of size n the price experiences *permanent market impact*, hence it evolves following the equation:

$$S_{t+1} = S_t + \sqrt{\tau}\sigma\xi_t - g\left(\frac{n}{\tau}\right)\tau$$

where ξ_t is a normal random variable $\mathcal{N}(0,1)$.

The effective price (i.e. with *temporary market impact*) is:

$$\widetilde{S}_{t+1} = S_t - h\left(\frac{n}{\tau}\right)$$

State transitions

Our state is represented by the variables:

- Cash x
- Price s
- Inventory q
- Time t

After an trade of size n is executed (assuming we are selling), these variables evolve in the following way:

- Cash $x \mapsto x + n \cdot \left(s - h\left(\frac{n}{\tau}\right)\right)$
- Price $s \mapsto s + \sqrt{\tau}\sigma\xi_t - g\left(\frac{n}{\tau}\right)\tau$
- Inventory $q \mapsto q - n$
- Time $t \mapsto t + 1$

Terminal condition

If at time $T-1$ our cash is x, the price is s, and the remaining inventory is q, because we are constrained to sell everything, we know that the value function of our Dynamic Programming problem is given by:

$$v(x,s,q,T-1) = e^{-\gamma\left(x+q\left(s-h\left(\frac{q}{\tau}\right)\right)\right)}$$

The Bellman Equation

For every other $t \leq T-1$ we have:

$$v(x,s,q,t) = \min_n \mathbb{E}\left[v\left(x+n\left(s-h\left(\frac{n}{\tau}\right)\right), s+\sqrt{\tau}\sigma\xi_t - g\left(\frac{n}{\tau}\right)\tau, q-n, t+1\right)\right]$$

Which we obtain by replacing into the Dynamic Programming equation the State Transition explicited above.

A change of variable

In order to solve this equation it is preferable to perform the following change of variable (which simplifies this particular problem):

$$v(x,s,q,t) = u(q,t)e^{-\gamma(x+qs)}$$

Hence, the Terminal condition for u becomes:

$$u(x,T-1) = e^{\gamma q h\left(\frac{q}{\tau}\right)}$$

You can do the algebra and prove that the Bellman equation for $u(q,t)$ is:

$$u(q,t) = \min_n u(q-n,t+1)e^{\Phi(q,n)}$$

Where $H(q,n)$ is

$$\Phi(q,n) = \frac{\gamma^2}{2}(q-n)^2\sigma^2\tau + \gamma n h\left(\frac{n}{\tau}\right) + \gamma(q-n)g\left(\frac{n}{\tau}\right)\tau$$

The equation you should implement is the equation on $u(q,t)$, not the equation on $v(x,s,q,t)$ which is computationally expensive.