LADR Done Right

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Chapter 1

Vector Spaces

1.A $\mathbf{R}^n$ and $\mathbf{C}^n$

1.B Definition of Vector Space

1.C Subspaces

Exercise 12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Suppose U and W are subspaces of V, such that $U \cup W$ is also a subspace and neither is contained in the other. In other words, $U \not\subset W$ and $W \not\subset U$. Let $u \in U - W$ and $w \in W - U$. Since $U \cup W$ is a subspace of V, we have $u + w \in U \cup W$. However,

$$u + w \in U \Rightarrow w = (u + w) - u \in U,$$
$$u + w \in W \Rightarrow u = (u + w) - w \in W,$$

which implies that $u + w$ is in neither U nor W, a contradiction. ■

Chapter 2

Inner Product Spaces

2.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leqslant \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \operatorname{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \left\|\sqrt{2}u\right\| = \sqrt{2}\|u\|$. Since $\|tv\| \leqslant \|v\|$, this implies that $u = 0$. ■

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leqslant \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution. If $\langle u, v \rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geqslant 0.$$

by the Pythagorean Theorem.

If $\|u\| \leqslant \|u + av\|$ for all $a \in \mathbf{F}$, then

$$0 \leqslant \|u + av\|^2 - \|u\|^2 = \overline{a}\langle u, v \rangle + a\overline{\langle u, v \rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u,v \rangle}{\|v\|^2}$ yields $\langle u, v \rangle = 0$. ■

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u-v\|^2 = \langle u-v, u-v\rangle = \|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2 = 0.$$

Hence $u-v=0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leqslant (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

for all positive numbers a,b,c,d.

Proof. By Exercise 6.17(a), we have

$$\begin{aligned} 16 &= \left|\sqrt{a\cdot\frac{1}{a}}+\sqrt{b\cdot\frac{1}{b}}+\sqrt{c\cdot\frac{1}{c}}+\sqrt{d\cdot\frac{1}{d}}\right|^2 \\ &\leqslant (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right). \end{aligned}$$

■

Exercise 12 Prove that

$$(x_1+x_2+\cdots+x_n)^2 \leqslant n\left(x_1^2+x_2^2+\cdots+x_n^2\right)$$

for all positive integers n and all real numbers $x_1,\ldots,x_n$.

Proof. In Exercise 6.17(a), let $y_1=y_2=\cdots=y_n=1$. ■

Exercise 17 Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x,y)\| = \max\{x,y\}$$

for all $(x,y)\in\mathbf{R}^2$.

Counterexample. Let $u=(0,1), v=(1,0)$. Then 6.22 fails. ■

2.B Orthonormal Bases

Exercise F ind a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)\,\mathrm{d}x$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Solution. Let $\varphi(p) = p\left(\frac{1}{2}\right)$ and $\langle p, q\rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x$. Then with the orthonormal basis found in Exercise 5 and the formula in 6.43, we can find that

$$q(x) = -15x^2 + 15x - \frac{3}{2}. \qquad \blacksquare$$