

LADR Done Right

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Chapter 1

Vector Spaces

1.A \mathbf{R}^n and \mathbf{C}^n

Exercise 7 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Proof. Suppose $\alpha = a + bi$. Let $\beta = -a - bi$. Then

$$\alpha + \beta = (a - a) + (b - b)i = 0,$$

proving existence.

Suppose there exists $\gamma \in \mathbf{C}$ such that $\alpha + \gamma = 0$. Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = \beta,$$

proving uniqueness. ■

1.B Definition of Vector Space

Exercise 6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in \mathbf{R} . Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over \mathbf{R} ? Explain.

Solution. No. If $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is a vector space over \mathbf{R} , we will have

$$1 = 1 + 0 = 1 + (\infty + (-\infty)) = (1 + \infty) + (-\infty) = \infty + (-\infty) = 0,$$

a contradiction.

1.C Subspaces

Exercise 9 A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from \mathbf{R} to \mathbf{R} a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution. No. Let F_p denote the set of periodic functions from \mathbf{R} to \mathbf{R} . If F_p is a subspace of $\mathbf{R}^{\mathbf{R}}$, then $h(x) = \cos x + \sin \pi x \in F_p$ since both $f(x) = \cos x$ and $g(x) = \sin \pi x$ are in F_p . In other words, there exists $p > 0$ such that

$$\cos p - \sin \pi p = 1 = \cos p + \sin \pi p.$$

Hence we have $\cos p = 1$ and $\sin \pi p = 0$. The former implies $p = 2n\pi (n \in \mathbf{Z}_+)$, while the latter implies $p = m (m \in \mathbf{Z}_+)$. However, this means

$$\pi = \frac{m}{2n} \in \mathbf{Q},$$

which is impossible.

Exercise 12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Suppose U and W are two subspaces of V such that $U \cup W$ is also a subspace of V . If $U \subseteq W$, then there is nothing to prove. Otherwise, take $u \in U \setminus W$, $w \in W$ and consider $u + w \in U \cup W$. It cannot be in W , since then will be $u = (u + w) - w \in W$. Therefore, there must be $u + w \in U$, and hence $w = (u + w) - u \in U$, which implies $W \subseteq U$, as desired.

Conversely, suppose one of U and W is contained in the other. Without loss of generality, we assume $U \subseteq W$. Then $U \cup W = W$ is obviously a subspace of V . ■

Exercise 13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

[This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace \mathbf{F} with a field containing only two elements.]

Proof. Necessity is obvious. Namely, suppose U_1, U_2, U_3 are three subspaces of V such that some U_j contains the other two. Then $U_1 \cup U_2 \cup U_3 = U_j$ is also a subspace of V .

To prove sufficiency, suppose $U = U_1 \cup U_2 \cup U_3$ is a subspace of V . Assume that no subspace contains the other two.

First consider the case where there exists a U_i contained by a U_j . Without loss of generality, suppose $U_1 \subseteq U_2$. By applying Exercise 12 twice, we get that $U_1 \cup U_2$ is a subspace of V and that either $(U_1 \cup U_2) \subseteq U_3$ or $U_3 \subseteq (U_1 \cup U_2)$. Both contradict our assumption. So this case cannot hold.

Then consider the case where one of the subspaces is contained in the union of the other two, say $U_1 \subseteq (U_2 \cup U_3)$. Since $U_2 \cup U_3 = U$ is now a subspace of V , Exercise 12 implies that either $U_2 \subseteq U_3$ or $U_3 \subseteq U_2$, which is exactly the previous case. So this case cannot hold either.

Now consider $u_3 = u_1 + u_2$, where $u_1 \in U_1 - (U_2 \cup U_3)$, $u_2 \in U_2 - (U_1 \cup U_3)$. We have $u_3 \notin U_1$: otherwise we deduce $u_2 = u_3 - u_1 \in U_1$, which is impossible. Similarly, we have $u_3 \notin U_2$. Hence $u_3 \in U_3 - (U_1 \cup U_2)$. However, the same reasoning implies $u_1 + u_3 \in U_2 - (U_1 \cup U_3)$, which further implies $2u_1 = (u_1 + u_3) - u_2 \in U_2$ and $u_1 \in U_2$, a contradiction. ■

Chapter 2

Finite-Dimensional Vector Spaces

2.A Span and Linear Independence

Exercise 10 Suppose v_1, \dots, v_m is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, \dots, v_m)$.

Proof. If $v_1 + w, \dots, v_m + w$ is linearly dependent, then there exists a_1, \dots, a_m , not all 0, such that

$$\begin{aligned} a_1(v_1 + w) + \dots + a_m(v_m + w) &= 0, \\ a_1v_1 + \dots + a_mv_m &= -(a_1 + \dots + a_m)w. \end{aligned}$$

Since $v_1 + w, \dots, v_m + w$ is linearly dependent, $a_1 + \dots + a_m \neq 0$, and hence

$$w = \frac{-a_1}{a_1 + \dots + a_m}v_1 + \dots + \frac{-a_m}{a_1 + \dots + a_m}v_m \in \text{span}(v_1, \dots, v_m),$$

as desired. ■

2.B Bases

Exercise 8 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that u_1, \dots, u_m is a basis of U and w_1, \dots, w_n is a basis of W . Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V .

Proof. Consider an arbitrary $v \in V$. Since $V = U \oplus W$, there are unique vectors $u \in U$ and $w \in W$ such that $v = u + w$. Hence there are unique scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$v = u + w = a_1u_1 + \dots + a_mu_m + b_1w_1 + \dots + b_nw_n,$$

which, by 2.29, implies that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V . ■

2.C Dimension

Exercise 17 You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned} \dim(U_1 + U_2 + U_3) &= \dim U_1 + \dim U_2 + \dim U_3 \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_3 \cap U_1) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3). \end{aligned}$$

Prove this or give a counterexample.

Counterexample. Consider the \mathbf{R}^2 plane. Let V_1, V_2, V_3 be three distinct lines through the origin. Then $\text{LHS} = \dim \mathbf{R}^2 = 2$, while $\text{RHS} = 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$.

Chapter 3

Linear Maps

3.A The Vector Space of Linear Maps

Exercise 10 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T \in \mathcal{L}(V, W)$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V .

Proof. Take $u \in U$ such that $Su \neq 0$. Since $U \neq V$, there exists $v \in V \setminus U$. Then

$$T(u - v) + Tv = 0 + 0 = 0 \neq Su = Tu.$$

Hence T is not a linear map. ■

3.B Null Spaces and Ranges

Exercise 29 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V$ is not in null φ . Prove that

$$V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}.$$

Proof. For arbitrary $v \in V$, since $\varphi(u) \neq 0$, we have

$$\varphi\left(v - \frac{\varphi(v)}{\varphi(u)}u\right) = \varphi(v) - \varphi\left(\frac{\varphi(v)}{\varphi(u)}u\right) = \varphi(v) - \frac{\varphi(v)}{\varphi(u)}\varphi(u) = 0.$$

This implies $v - \frac{\varphi(v)}{\varphi(u)}u = w$ for some $w \in \text{null } \varphi$, and hence

$$v = w + \frac{\varphi(v)}{\varphi(u)}u \in \text{null } \varphi + \{au : a \in \mathbf{F}\}.$$

Therefore, $V = \text{null } \varphi + \{au : a \in \mathbf{F}\}$. Since $\text{null } \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$, by 1.45, we have $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$. ■

3.C Matrices

3.D Invertibility and Isomorphic Vector Spaces

3.E Products and Quotients of Vector Spaces

3.F Duality

Chapter 4

Inner Product Spaces

4.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \text{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \|\sqrt{2}u\| = \sqrt{2}\|u\|$. Since $\|Tv\| \leq \|v\|$, this implies that $u = 0$. ■

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbb{F}$.

Solution. If $\langle u, v \rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geq 0.$$

by the Pythagorean Theorem.

If $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$, then

$$0 \leq \|u + av\|^2 - \|u\|^2 = \bar{a}\langle u, v \rangle + a\overline{\langle u, v \rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u, v \rangle}{\|v\|^2}$ yields $\langle u, v \rangle = 0$.

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 = 0.$$

Hence $u - v = 0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

for all positive numbers a, b, c, d .

Proof. By Exercise 6.17(a), we have

$$\begin{aligned} 16 &= \left| \sqrt{a \cdot \frac{1}{a}} + \sqrt{b \cdot \frac{1}{b}} + \sqrt{c \cdot \frac{1}{c}} + \sqrt{d \cdot \frac{1}{d}} \right|^2 \\ &\leq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right). \end{aligned} \quad \blacksquare$$

Exercise 12 Prove that

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2)$$

for all positive integers n and all real numbers x_1, \dots, x_n .

Proof. In Exercise 6.17(a), let $y_1 = y_2 = \cdots = y_n = 1$. ■

Exercise 17 Prove or disprove: there is an inner product on \mathbf{R}^2 such that the associated norm is given by

$$\|(x, y)\| = \max\{x, y\}$$

for all $(x, y) \in \mathbf{R}^2$.

Counterexample. Let $u = (0, 1)$, $v = (1, 0)$. Then 6.22 fails.

4.B Orthonormal Bases

Exercise F Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x) \, dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Solution. Let $\varphi(p) = p\left(\frac{1}{2}\right)$ and $\langle p, q \rangle = \int_0^1 p(x)q(x) \, dx$. Then with the orthonormal basis found in Exercise 5 and the formula in 6.43, we can find that

$$q(x) = -15x^2 + 15x - \frac{3}{2}.$$