LADR Done Right

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Chapter 1

Inner Product Spaces

1.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leqslant \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \operatorname{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \left\|\sqrt{2}u\right\| = \sqrt{2}\|u\|$. Since $\|tv\| \leqslant \|v\|$, this implies that $u = 0$. ■

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leqslant \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution. If $\langle u, v\rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geqslant 0.$$

by the Pythagorean Theorem.

If $\|u\| \leqslant \|u + av\|$ for all $a \in \mathbf{F}$, then

$$0 \leqslant \|u + av\|^2 - \|u\|^2 = \overline{a}\langle u, v\rangle + a\overline{\langle u, v\rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u,v\rangle}{\|v\|^2}$ yields $\langle u, v\rangle = 0$. ■

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u-v\|^2 = \langle u-v, u-v\rangle = \|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2 = 0.$$

Hence $u - v = 0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leqslant (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

for all positive numbers a, b, c, d.

Proof. Let $u = \left(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right)$, $v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right)$. Then it follows directly from the Cauchy-Schuarz Inequality. ■

Exercise 17 Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x,y)\| = \max\{x,y\}$$

for all $(x,y) \in \mathbf{R}^2$.