LADR Done Right

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Contents

Chapter 1

Vector Spaces

1.A $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 7 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Proof. Suppose $\alpha = a + bi$. Let $\beta = -a - bi$. Then

$$\alpha + \beta = (a - a) + (b - b)i = 0,$$

proving existence.

Suppose there exists $\gamma \in \mathbf{C}$ such that $\alpha + \gamma = 0$. Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = \beta,$$

proving uniqueness. ∎

1.B Definition of Vector Space

Exercise 6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. No. If $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is a vector space over $\mathbf{R}$, we will have

$$1 = 1 + 0 = 1 + (\infty + (-\infty)) = (1 + \infty) + (-\infty) = \infty + (-\infty) = 0,$$

a contradiction.

1.C Subspaces

Exercise 9 A function $f : \mathbf{R} \to \mathbf{R}$ is called ***periodic*** if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution. No. Let F_p denote the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$. If F_p is a subspace of $\mathbf{R}^{\mathbf{R}}$, then $h(x) = \cos x + \sin \pi x \in F_p$ since both $f(x) = \cos x$ and $g(x) = \sin \pi x$ are in F_p. In other words, there exists $p > 0$ such that

$$\cos p - \sin \pi p = 1 = \cos p + \sin \pi p.$$

Hence we have $\cos p = 1$ and $\sin \pi p = 0$. The former implies $p = 2n\pi (n \in \mathbf{Z}_+)$, while the latter implies $p = m(m \in \mathbf{Z}_+)$. However, this means

$$\pi = \frac{m}{2n} \in \mathbf{Q},$$

which is impossible.

Exercise 12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Suppose U and W are two subspaces of V such that $U \cup W$ is also a subspace of V. If $U \subseteq W$, then there is nothing to proof. Otherwise, take $u \in U \setminus W$, $w \in W$ and consider $u + w \in U \cup W$. It cannot be in W, since then will be $u = (u + w) - w \in W$. Therefore, there must be $u + w \in U$, and hence $w = (u + w) - u \in U$, which implies $W \subseteq U$, as desired.

Conversely, suppose one of U and W is contained in the other. Without loss of generality, we assume $U \subseteq W$. Then $U \cup W = W$ is obviously a subspace of V. ∎

Exercise 13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.
[*This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace* $\mathbf{F}$ *with a field containing only two elements.*]

Proof. Necessity is obvious. Namely, suppose U_1, U_2, U_3 are three subspaces of V such that some U_j contains the other two. Then $U_1 \cup U_2 \cup U_3 = U_j$ is also a subspace of V.

To prove sufficiency, suppose $U = U_1 \cup U_2 \cup U_3$ is a subspace of V. Assume that no subspace contains the other two.

First consider the case where there exists a U_i contained by a U_j. Without loss of generality, suppose $U_1 \subseteq U_2$. By applying Exercise 12 twice, we get that $U_1 \cup U_2$ is a subspace of V and that either $(U_1 \cup U_2) \subseteq U_3$ or $U_3 \subseteq (U_1 \cup U_2)$. Both contradict our assumption. So this case cannot hold.

Then consider the case where one of the subspaces is contained in the union of the other two, say $U_1 \subseteq (U_2 \cup U_3)$. Since $U_2 \cup U_3 = U$ is now a subspace of V, Exercise 12 implies that either $U_2 \subseteq U_3$ or $U_3 \subseteq U_2$, which is exactly the previous case. So this case cannot hold either.

Now consider $u_3 = u_1 + u_2$, where $u_1 \in U_1 - (U_2 \cup U_3)$, $u_2 \in U_2 - (U_1 \cup U_3)$. We have $u_3 \notin U_1$: otherwise we deduce $u_2 = u_3 - u_1 \in U_1$, which is impossible. Similarly, we have $u_3 \notin U_2$. Hence $u_3 \in U_3 - (U_1 \cup U_2)$. However, the same reasoning implies $u_1 + u_3 \in U_2 - (U_1 \cup U_3)$, which further implies $2u_1 = (u_1 + u_3) - u_2 \in U_2$ and $u_1 \in U_2$, a contradiction. ∎

Chapter 2

Finite-Dimensional Vector Spaces

2.A Span and Linear Independence

Exercise 10 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Proof. If $v_1 + w, \ldots, v_m + w$ is linearly dependent, then there exists $a_1, \ldots, a_m$, not all 0, such that

$$\begin{aligned} a_1(v_1 + w) + \cdots + a_m(v_m + w) &= 0, \\ a_1 v_1 + \cdots + a_m v_m &= -(a_1 + \cdots + a_m)w. \end{aligned}$$

Since $v_1 + w, \ldots, v_m + w$ is linearly dependent, $a_1 + \cdots + a_m \neq 0$, and hence

$$w = \frac{-a_1}{a_1 + \cdots + a_m} v_1 + \cdots + \frac{-a_m}{a_1 + \cdots + a_m} v_m \in \operatorname{span}(v_1, \ldots, v_m),$$

as desired. ■

2.B Bases

Exercise 8 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof. Consider an arbitary $v \in V$. Since $V = U \oplus W$, there are unique vectors $u \in U$ and $w \in W$ such that $v = u + w$. Hence there are unique scalars $a_1, \dots, a_m, b_1, \dots, b_n$ such that

$$v = u + w = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n,$$

which, by 2.29, implies that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V. ■

2.C Dimension

Exercise 17 You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = {} & \dim U_1 + \dim U_2 + \dim U_3 \\ & - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_3 \cap U_1) \\ & + \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

Counterexample. Consider the $\mathbf{R}^2$ plane. Let V_1, V_2, V_3 be three distinct lines through the origin. Then LHS = $\dim \mathbf{R}^2 = 2$, while RHS = $1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$.

Chapter 3

Linear Maps

3.A The Vector Space of Linear Maps

Exercise 10 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T \in \mathcal{L}(V, W)$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Proof. Take $u \in U$ such that $Su \neq 0$. Since $U \neq V$, there exists $v \in V \setminus U$. Then

$$T(u - v) + Tv = 0 + 0 = 0 \neq Su = Tu.$$

Hence T is not a linear map. ■

3.B Null Spaces and Ranges

Exercise 29 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V$ is not in $\operatorname{null}\varphi$. Prove that

$$V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}.$$

Proof. For arbitary $v \in V$, since $\varphi(u) \neq 0$, we have

$$\varphi\left(v - \frac{\varphi(v)}{\varphi(u)}u\right) = \varphi(v) - \varphi\left(\frac{\varphi(v)}{\varphi(u)}u\right) = \varphi(v) - \frac{\varphi(v)}{\varphi(u)}\varphi(u) = 0.$$

This implies $v - \frac{\varphi(v)}{\varphi(u)}u = w$ for some $w \in \operatorname{null}\varphi$, and hence

$$v = w + \frac{\varphi(v)}{\varphi(u)}u \in \operatorname{null}\varphi + \{au : a \in \mathbf{F}\}.$$

Therefore, $V = \operatorname{null}\varphi + \{au : a \in \mathbf{F}\}$. Since $\operatorname{null}\varphi \cap \{au : a \in \mathbf{F}\} = \{0\}$, by 1.45, we have $V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}$. ■

3.C Matrices

Exercise 15 Suppose A is an n-by-n matrix and $1 \leqslant j, k \leqslant n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.$$

Proof. By the definition of matrix multiplication, we have

$$\begin{aligned}
(A^3)_{j,k} &= (AA^2)_{j,k} \\
&= \sum_{p=1}^{n} A_{j,p}(A^2)_{p,k} \\
&= \sum_{p=1}^{n} A_{j,p}\sum_{r=1}^{n} A_{p,r}A_{r,k} \\
&= \sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}.
\end{aligned}$$
■

3.D Invertibility and Isomorphic Vector Spaces

3.E Products and Quotients of Vector Spaces

3.F Duality

Chapter 4

Polynomials

Exercise 6 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

Proof. We will prove by contrapositive. Suppose p and p' have a zero in common, say λ. Then by 4.14, we can write $p(z) = (z - \lambda)q(z)$. Hence

$$p'(\lambda) = q(\lambda) + (\lambda - \lambda)q'(\lambda) = 0,$$

which implies $q(\lambda) = 0$. Therefore, p has a zero λ of multiplicity at least 2.

Now suppose p has a zero λ of multiplicity $k > 1$. Then, again by 4.14, we can write $p(z) = (z - \lambda)^2 q(z)$. Hence

$$p'(\lambda) = 2(\lambda - \lambda)q(\lambda) + (\lambda - \lambda)^2 q'(\lambda) = 0,$$

which implies λ is also a zero of p'. ■

Exercise 7 Prove that every polynomial of odd degree with real coefficients has a real zero.

Proof. The assertion follows directly from 4.17. ■

Exercise 10 Suppose m is a nonnegative integer and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there exist distinct real numbers $x_0, x_1, \ldots, x_m$ such that $p(x_j) \in \mathbf{R}$ for $j = 0, 1, \ldots, m$. Prove that all the coefficients of p are real.

Proof. Consider

$$q(z) = p(z) - \sum_{j=0}^{m} \frac{\prod_{i \neq j}(z - x_i)}{\prod_{i \neq j}(x_j - x_i)} p(x_j).$$

Note that $q(x_j) = 0$ for $j = 0, 1, \ldots, m$. Since q has degree at most m, it must be the zero polynomial. Therefore, all the coefficients of p are real. ■

Chapter 5

Inner Product Spaces

5.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leqslant \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \operatorname{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \left\|\sqrt{2}u\right\| = \sqrt{2}\|u\|$. Since $\|tv\| \leqslant \|v\|$, this implies that $u = 0$. ■

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leqslant \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution. If $\langle u, v\rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geqslant 0.$$

by the Pythagorean Theorem.

If $\|u\| \leqslant \|u + av\|$ for all $a \in \mathbf{F}$, then

$$0 \leqslant \|u + av\|^2 - \|u\|^2 = \overline{a}\langle u, v\rangle + a\overline{\langle u, v\rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u, v\rangle}{\|v\|^2}$ yields $\langle u, v\rangle = 0$.

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u-v\|^2 = \langle u-v, u-v\rangle = \|u\|^2 - \langle u, v\rangle - \langle v, u\rangle + \|v\|^2 = 0.$$

Hence $u - v = 0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leqslant (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

for all positive numbers a, b, c, d.

Proof. By Exercise 6.17(a), we have

$$\begin{aligned} 16 &= \left|\sqrt{a\cdot\frac{1}{a}}+\sqrt{b\cdot\frac{1}{b}}+\sqrt{c\cdot\frac{1}{c}}+\sqrt{d\cdot\frac{1}{d}}\right|^2 \\ &\leqslant (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right). \end{aligned}$$ ■

Exercise 12 Prove that

$$(x_1+x_2+\cdots+x_n)^2 \leqslant n\left(x_1^2+x_2^2+\cdots+x_n^2\right)$$

for all positive integers n and all real numbers $x_1, \ldots, x_n$.

Proof. In Exercise 6.17(a), let $y_1 = y_2 = \cdots = y_n = 1$. ■

Exercise 17 Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = \max\{x, y\}$$

for all $(x, y) \in \mathbf{R}^2$.

Counterexample. Let $u = (0, 1)$, $v = (1, 0)$. Then 6.22 fails.

5.B Orthonormal Bases

Exercise F ind a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)\,\mathrm{d}x$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Solution. Let $\varphi(p) = p\left(\frac{1}{2}\right)$ and $\langle p, q\rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x$. Then with the orthonormal basis found in Exercise 5 and the formula in 6.43, we can find that

$$q(x) = -15x^2 + 15x - \frac{3}{2}.$$