

LADR Done Right

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Chapter 1

Vector Spaces

1.A \mathbf{R}^n and \mathbf{C}^n

Exercise 7 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Proof. Suppose $\alpha = a + bi$. Let $\beta = -a - bi$. Then

$$\alpha + \beta = (a - a) + (b - b)i = 0,$$

proving existence.

Suppose there exists $\gamma \in \mathbf{C}$ such that $\alpha + \gamma = 0$. Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = \beta,$$

proving uniqueness. ■

1.B Definition of Vector Space

Exercise 6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in \mathbf{R} . Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over \mathbf{R} ? Explain.

1.C Subspaces

Exercise 12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Suppose U and W are subspaces of V , such that $U \cup W$ is also a subspace and neither is contained in the other. In other words, $U \not\subset W$ and $W \not\subset U$. Let $u \in U - W$ and $w \in W - U$. Since $U \cup W$ is a subspace of V , we have $u + w \in U \cup W$. However,

$$u + w \in U \Rightarrow w = (u + w) - u \in U,$$

$$u + w \in W \Rightarrow u = (u + w) - w \in W,$$

which implies that $u + w$ is in neither U nor W , a contradiction. ■

Chapter 2

Inner Product Spaces

2.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \text{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \|\sqrt{2}u\| = \sqrt{2}\|u\|$. Since $\|Tv\| \leq \|v\|$, this implies that $u = 0$. ■

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution. If $\langle u, v \rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geq 0.$$

by the Pythagorean Theorem.

If $\|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$, then

$$0 \leq \|u + av\|^2 - \|u\|^2 = \bar{a}\langle u, v \rangle + a\overline{\langle u, v \rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u, v \rangle}{\|v\|^2}$ yields $\langle u, v \rangle = 0$. ■

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 = 0.$$

Hence $u - v = 0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

for all positive numbers a, b, c, d .

Proof. By Exercise 6.17(a), we have

$$\begin{aligned} 16 &= \left| \sqrt{a \cdot \frac{1}{a}} + \sqrt{b \cdot \frac{1}{b}} + \sqrt{c \cdot \frac{1}{c}} + \sqrt{d \cdot \frac{1}{d}} \right|^2 \\ &\leq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right). \end{aligned} \quad \blacksquare$$

Exercise 12 Prove that

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2)$$

for all positive integers n and all real numbers x_1, \dots, x_n .

Proof. In Exercise 6.17(a), let $y_1 = y_2 = \cdots = y_n = 1$. ■

Exercise 17 Prove or disprove: there is an inner product on \mathbf{R}^2 such that the associated norm is given by

$$\|(x, y)\| = \max\{x, y\}$$

for all $(x, y) \in \mathbf{R}^2$.

Counterexample. Let $u = (0, 1), v = (1, 0)$. Then 6.22 fails. ■

2.B Orthonormal Bases

Exercise F Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x) \, dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Solution. Let $\phi(p) = p(\frac{1}{2})$ and $\langle p, q \rangle = \int_0^1 p(x)q(x) \, dx$. Then with the orthonormal basis found in Exercise 5 and the formula in 6.43, we can find that

$$q(x) = -15x^2 + 15x - \frac{3}{2}.$$

■