

# LADR Done Right

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# Chapter 1

## Vector Spaces

### 1.A $\mathbf{R}^n$ and $\mathbf{C}^n$

**Exercise 7** Show that for every  $\alpha \in \mathbf{C}$ , there exists a unique  $\beta \in \mathbf{C}$  such that  $\alpha + \beta = 0$ .

*Proof.* Suppose  $\alpha = a + bi$ . Let  $\beta = -a - bi$ . Then

$$\alpha + \beta = (a - a) + (b - b)i = 0,$$

proving existence.

Suppose there exists  $\gamma \in \mathbf{C}$  such that  $\alpha + \gamma = 0$ . Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = \beta,$$

proving uniqueness. ■

## 1.B Definition of Vector Space

**Exercise 6** Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbf{R}$ . Define an addition and scalar multiplication on  $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$  as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for  $t \in \mathbf{R}$  define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is  $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$  a vector space over  $\mathbf{R}$ ? Explain.

*Solution.* No. If  $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$  is a vector space over  $\mathbf{R}$ , we will have

$$1 = 1 + 0 = 1 + (\infty + (-\infty)) = (1 + \infty) + (-\infty) = \infty + (-\infty) = 0,$$

a contradiction. ■

## 1.C Subspaces

**Exercise 9** A function  $f : \mathbf{R} \rightarrow \mathbf{R}$  is called *periodic* if there exists a positive number  $p$  such that  $f(x) = f(x+p)$  for all  $x \in \mathbf{R}$ . Is the set of periodic functions from  $\mathbf{R}$  to  $\mathbf{R}$  a subspace of  $\mathbf{R}^{\mathbf{R}}$ ? Explain.

*Solution.* No. Let  $F_p$  denote the set of periodic functions from  $\mathbf{R}$  to  $\mathbf{R}$ . If  $F_p$  is a subspace of  $\mathbf{R}^{\mathbf{R}}$ , then  $h(x) = \cos x + \sin \pi x \in F_p$  since both  $f(x) = \cos x$  and  $g(x) = \sin \pi x$  are in  $F_p$ . In other words, there exists  $p > 0$  such that

$$\cos p - \sin \pi p = 1 = \cos p + \sin \pi p.$$

Hence we have  $\cos p = 1$  and  $\sin \pi p = 0$ . The former implies  $p = 2n\pi (n \in \mathbf{Z}_+)$ , while the latter implies  $p = m (m \in \mathbf{Z}_+)$ . However, this means

$$\pi = \frac{m}{2n} \in \mathbf{Q},$$

which is impossible. ■

**Exercise 12** Prove that the union of two subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces is contained in the other.

*Proof.* Suppose  $U$  and  $W$  are two subspaces of  $V$  such that  $U \cup W$  is also a subspace of  $V$ . If  $U \subseteq W$ , then there is nothing to prove. Otherwise, take  $u \in U \setminus W$ ,  $w \in W$  and consider  $u + w \in U \cup W$ . It cannot be in  $W$ , since then will be  $u = (u + w) - w \in W$ . Therefore, there must be  $u + w \in U$ , and hence  $w = (u + w) - u \in U$ , which implies  $W \subseteq U$ , as desired.

Conversely, suppose one of  $U$  and  $W$  is contained in the other. Without loss of generality, we assume  $U \subseteq W$ . Then  $U \cup W = W$  is obviously a subspace of  $V$ . ■

**Exercise 13** Prove that the union of three subspaces of  $V$  is a subspace of  $V$  if and only if one of the subspaces contains the other two.

*[This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace  $\mathbf{F}$  with a field containing only two elements.]*

*Proof.* Suppose  $A, B, C$  are three subspaces of  $V$  such that  $S = A \cup B \cup C$  is also a subspace of  $V$ .

If  $A \subseteq B$ , then by Exercise 12, one of  $B \subseteq C$

Let  $v \in A - (B \cup C)$  and  $w \in B - A$ . Then  $v + w$  is in neither  $A$  nor  $B$ , and hence is in  $C$ . Consider the vector  $2v$ . For any subspace  $S$ ,  $2v \in S$  if and only if  $v \in S$ . Hence  $2v \in A - (B \cup C)$ , which implies  $2v + w \in C$ . However, this deduces

$$v = (2v + w) - (v + w) \in C,$$

a contradiction.

One direction is obvious. Without loss of generality, suppose  $A, B, C$  are three subspaces of  $V$  satisfying  $A \cup B \subseteq C$ . Then  $A \cup B \cup C = C$  is a subspace of  $V$ . ■

## Chapter 2

# Inner Product Spaces

### 2.A Inner Products and Norms

**Exercise 5** Suppose  $T \in \mathcal{L}(V)$  is such that  $\|Tv\| \leq \|v\|$  for every  $v \in V$ . Prove that  $T - \sqrt{2}I$  is invertible.

*Proof.* Suppose  $V$  is finite-dimensional. For all  $u \in \text{null}(T - \sqrt{2}I)$ , we have  $Tu = \sqrt{2}u$ , and hence  $\|Tu\| = \|\sqrt{2}u\| = \sqrt{2}\|u\|$ . Since  $\|Tv\| \leq \|v\|$ , this implies that  $u = 0$ . ■

**Exercise 6** Suppose  $u, v \in \mathcal{L}(V)$ . Prove that  $\langle u, v \rangle = 0$  if and only if

$$\|u\| \leq \|u + av\|$$

for all  $a \in \mathbb{F}$ .

*Solution.* If  $\langle u, v \rangle = 0$ , then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geq 0.$$

by the Pythagorean Theorem.

If  $\|u\| \leq \|u + av\|$  for all  $a \in \mathbb{F}$ , then

$$0 \leq \|u + av\|^2 - \|u\|^2 = \bar{a}\langle u, v \rangle + a\overline{\langle u, v \rangle} + |a|^2\|v\|^2.$$

Letting  $a = -\frac{\langle u, v \rangle}{\|v\|^2}$  yields  $\langle u, v \rangle = 0$ . ■

**Exercise 8** Suppose  $u, v \in V$  and  $\|u\| = \|v\| = 1$  and  $\langle u, v \rangle = 1$ . Prove that  $u = v$ .

*Proof.* Note that

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 = 0.$$

Hence  $u - v = 0$  by definiteness. ■

**Exercise 11** Prove that

$$16 \leq (a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

for all positive numbers  $a, b, c, d$ .

*Proof.* By Exercise 6.17(a), we have

$$\begin{aligned} 16 &= \left| \sqrt{a \cdot \frac{1}{a}} + \sqrt{b \cdot \frac{1}{b}} + \sqrt{c \cdot \frac{1}{c}} + \sqrt{d \cdot \frac{1}{d}} \right|^2 \\ &\leq (a + b + c + d) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right). \end{aligned}$$

■

**Exercise 12** Prove that

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2)$$

for all positive integers  $n$  and all real numbers  $x_1, \dots, x_n$ .

*Proof.* In Exercise 6.17(a), let  $y_1 = y_2 = \cdots = y_n = 1$ . ■

**Exercise 17** Prove or disprove: there is an inner product on  $\mathbf{R}^2$  such that the associated norm is given by

$$\|(x, y)\| = \max\{x, y\}$$

for all  $(x, y) \in \mathbf{R}^2$ .

*Counterexample.* Let  $u = (0, 1)$ ,  $v = (1, 0)$ . Then 6.22 fails. ■

## 2.B Orthonormal Bases

**Exercise F** Find a polynomial  $q \in \mathcal{P}_2(\mathbf{R})$  such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x) \, dx$$

for every  $p \in \mathcal{P}_2(\mathbf{R})$ .

*Solution.* Let  $\varphi(p) = p\left(\frac{1}{2}\right)$  and  $\langle p, q \rangle = \int_0^1 p(x)q(x) \, dx$ . Then with the orthonormal basis found in Exercise 5 and the formula in 6.43, we can find that

$$q(x) = -15x^2 + 15x - \frac{3}{2}. \quad \blacksquare$$