LADR Done Right

johnsmith0x3f

September 13, 2024

Contents

Chapter 1

Vector Spaces

1.A $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 7 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Proof. Suppose $\alpha = a + bi$. Let $\beta = -a - bi$. Then

$$\alpha + \beta = (a - a) + (b - b)i = 0,$$

proving existence.

Suppose there exists $\gamma \in \mathbf{C}$ such that $\alpha + \gamma = 0$. Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = \beta,$$

proving uniqueness. ∎

1.B Definition of Vector Space

Exercise 6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R}\cup\{\infty\}\cup\{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution. No. If $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ is a vector space over $\mathbf{R}$, we will have

$$1 = 1 + 0 = 1 + (\infty + (-\infty)) = (1 + \infty) + (-\infty) = \infty + (-\infty) = 0,$$

a contradiction. ■

1.C Subspaces

Exercise 9 A function $f : \mathbf{R} \to \mathbf{R}$ is called ***periodic*** if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution. No. Let F_p denote the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$. If F_p is a subspace of $\mathbf{R}^{\mathbf{R}}$, then $h(x) = \cos x + \sin \pi x \in F_p$ since both $f(x) = \cos x$ and $g(x) = \sin \pi x$ are in F_p. In other words, there exists $p > 0$ such that

$$\cos p - \sin \pi p = 1 = \cos p + \sin \pi p.$$

Hence we have $\cos p = 1$ and $\sin \pi p = 0$. The former implies $p = 2n\pi (n \in \mathbf{Z}_+)$, while the latter implies $p = m(m \in \mathbf{Z}_+)$. However, this means

$$\pi = \frac{m}{2n} \in \mathbf{Q},$$

which is impossible. ■

Exercise 12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Suppose U and W are two subspaces of V such that $U \cup W$ is also a subspace of V. If $U \subseteq W$, then there is nothing to proof. Otherwise, take $u \in U \setminus W$, $w \in W$ and consider $u + w \in U \cup W$. It cannot be in W, since then will be $u = (u + w) - w \in W$. Therefore, there must be $u + w \in U$, and hence $w = (u + w) - u \in U$, which implies $W \subseteq U$, as desired.

Conversely, suppose one of U and W is contained in the other. Without loss of generality, we assume $U \subseteq W$. Then $U \cup W = W$ is obviously a subspace of V. ∎

Exercise 13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.
[*This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace* $\mathbf{F}$ *with a field containing only two elements.*]

Proof. Suppose A, B, C are three subspaces of V such that $S = A \cup B \cup C$ is also a subspace of V.

If $A \subseteq B$, then by Exercise 12, one of $B \subseteq C$ and $C \subseteq B$ must hold, and both cases lead to $S \in \{A, B, C\}$.

Otherwise if $B \subseteq A \cup C$, then .

Let $v \in A - (B \cup C)$ and $w \in B - A$. Then $v + w$ is in neither A nor B, and hence is in C. Consider the vector $2v$. For any subspace S, $2v \in S$ if and only if $v \in S$. Hence $2v \in A - (B \cup C)$, which implies $2v + w \in C$. However, this deduces

$$v = (2v + w) - (v + w) \in C,$$

a contradiction.

One direction is obvious. Without loss of generality, suppose A, B, C are three subspaces of V satisfying $A \cup B \subseteq C$. Then $A \cup B \cup C = C$ is a subspace of V. ∎

Chapter 2

Inner Product Spaces

2.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leqslant \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \operatorname{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \left\|\sqrt{2}u\right\| = \sqrt{2}\|u\|$. Since $\|tv\| \leqslant \|v\|$, this implies that $u = 0$. ∎

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leqslant \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution. If $\langle u, v\rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geqslant 0.$$

by the Pythagorean Theorem.

If $\|u\| \leqslant \|u + av\|$ for all $a \in \mathbf{F}$, then

$$0 \leqslant \|u + av\|^2 - \|u\|^2 = \overline{a}\langle u, v\rangle + a\overline{\langle u, v\rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u, v\rangle}{\|v\|^2}$ yields $\langle u, v\rangle = 0$. ∎

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 = 0.$$

Hence $u - v = 0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leqslant (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

for all positive numbers a, b, c, d.

Proof. By Exercise 6.17(a), we have

$$\begin{aligned} 16 &= \left|\sqrt{a \cdot \frac{1}{a}} + \sqrt{b \cdot \frac{1}{b}} + \sqrt{c \cdot \frac{1}{c}} + \sqrt{d \cdot \frac{1}{d}}\right|^2 \\ &\leqslant (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right). \end{aligned}$$

■

Exercise 12 Prove that

$$(x_1 + x_2 + \cdots + x_n)^2 \leqslant n\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)$$

for all positive integers n and all real numbers $x_1, \ldots, x_n$.

Proof. In Exercise 6.17(a), let $y_1 = y_2 = \cdots = y_n = 1$. ■

Exercise 17 Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = \max\{x, y\}$$

for all $(x, y) \in \mathbf{R}^2$.

Counterexample. Let $u = (0, 1)$, $v = (1, 0)$. Then 6.22 fails. ■

2.B Orthonormal Bases

Exercise F ind a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)\,\mathrm{d}x$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

Solution. Let $\varphi(p) = p\left(\frac{1}{2}\right)$ and $\langle p, q\rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x$. Then with the orthonormal basis found in Exercise 5 and the formula in 6.43, we can find that

$$q(x) = -15x^2 + 15x - \frac{3}{2}.$$

■