

LADR Done Right

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Chapter 1

Inner Product Spaces

1.A Inner Products and Norms

Exercise 5 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Proof. Suppose V is finite-dimensional. For all $u \in \text{null}(T - \sqrt{2}I)$, we have $Tu = \sqrt{2}u$, and hence $\|Tu\| = \|\sqrt{2}u\| = \sqrt{2}\|u\|$. Since $\|Tv\| \leq \|v\|$, this implies that $u = 0$. ■

Exercise 6 Suppose $u, v \in \mathcal{L}(V)$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbb{F}$.

Solution. If $\langle u, v \rangle = 0$, then

$$\|u + av\|^2 - \|u\|^2 = \|av\|^2 \geq 0.$$

by the Pythagorean Theorem.

If $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$, then

$$0 \leq \|u + av\|^2 - \|u\|^2 = \overline{a}\langle u, v \rangle + a\overline{\langle u, v \rangle} + |a|^2\|v\|^2.$$

Letting $a = -\frac{\langle u, v \rangle}{\|v\|^2}$ yields $\langle u, v \rangle = 0$. ■

Exercise 8 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v \rangle = 1$. Prove that $u = v$.

Proof. Note that

$$\|u - v\|^2 = \langle u - v, u - v \rangle = \|u\|^2 - \langle u, v \rangle - \langle v, u \rangle + \|v\|^2 = 0.$$

Hence $u - v = 0$ by definiteness. ■

Exercise 11 Prove that

$$16 \leq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

for all positive numbers a, b, c, d .

Proof. Let $u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$, $v = (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}})$. Then it follows directly from the Cauchy-Schwarz Inequality. ■