Advanced Programming Techniques (AdvPT)

Winter Term 2015/16

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Chair for System Simulation

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Assignment 2

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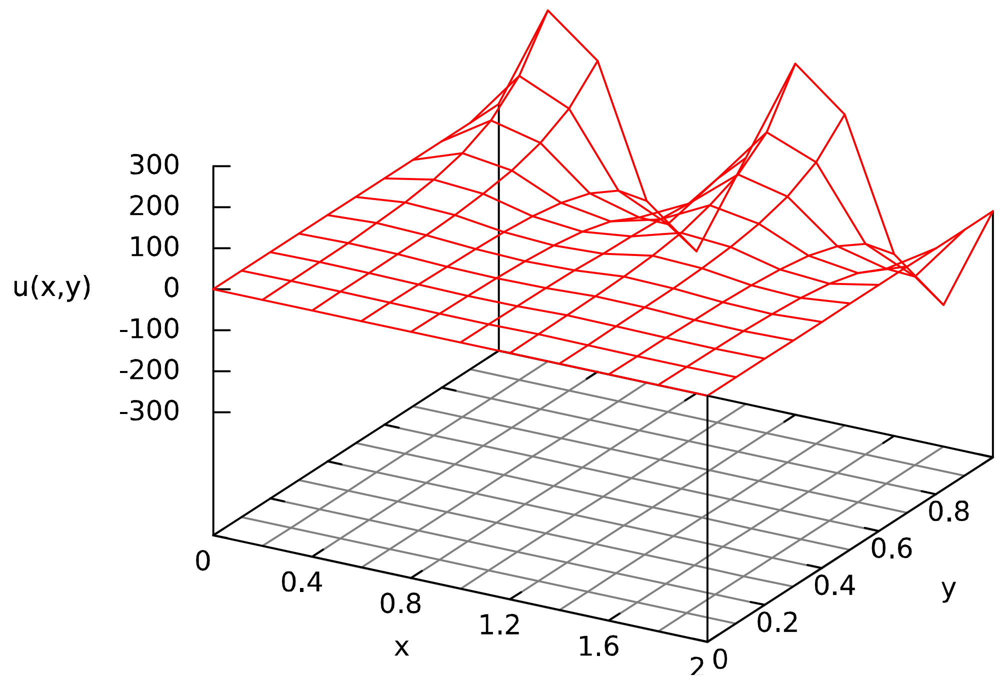
Elliptic PDEs

$$\Delta u(x,y) + k^2 u(x,y) = f(x,y)$$

$$\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) + k^2 u(x,y) = f(x,y)$$

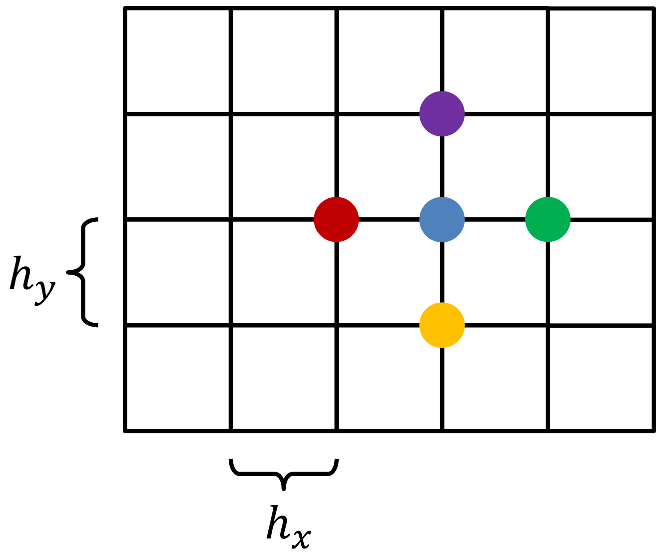


Discretizing Elliptic PDEs

- Discretization using the differential quotient for $\Delta u(x,y)$:

$$\frac{1}{h_x^2}\left[u(x+h_x,y)-2u(x,y)+u(x-h_x,y)\right]$$
$$+\frac{1}{h_y^2}\left[u\left(x,y+h_y\right)-2u(x,y)+u\left(x,y-h_y\right)\right]+k^2u(x,y)=f(x,y)$$

- We are solving the given PDE on a **discretized domain** Ω:



$u(x,y)\ /\ f(x,y)$

$u(x-h_x,y)$

$u(x+h_x,y)$

$u\left(x,y-h_y\right)$

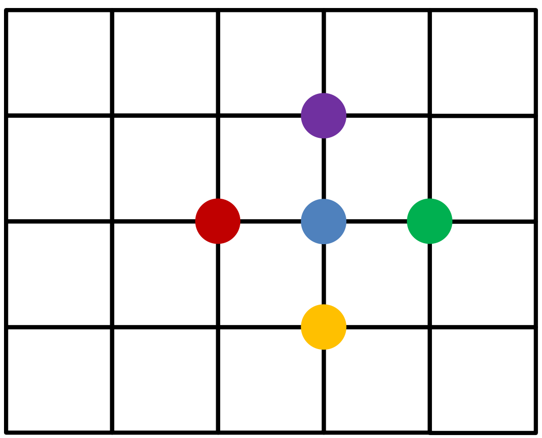
$u\left(x,y+h_y\right)$

Based on the exercise slides of Simulation and Scientific Computing (Simulation und Wissenschaftliches Rechnen - SiWiR)

Discretizing Elliptic PDEs

- For every point $u_{x,y}$ on the grid, we can formulate a linear equation:

$$\frac{1}{h_x^2}\left[u_{x-1,y}+u_{x+1,y}\right]+\frac{1}{h_y^2}\left[u_{x,y-1}+u_{x,y+1}\right]+\left(-\frac{2}{h_x^2}-\frac{2}{h_y^2}+k^2\right)u_{x,y}=f_{x,y}$$



$u_{x,y}$ / $f_{x,y}$

$u_{x-1,y}$

$u_{x+1,y}$

$u_{x,y-1}$

$u_{x,y+1}$

- Formulating this equation for every point of the grid leads to a linear system of equations (LSE): $A\vec{u}=\vec{f}$

Jacobi Method

- The Jacobi method **iteratively** solves the LSE (k represents the number of the iteration) according to the following formula:

$$u^{k+1} = D^{-1}(f - (A - D)u^k)$$

- Or, for each line:

$$u_i^{k+1} = \frac{1}{a_{ii}}\left(f_i - \sum_{j \neq i} a_{ij} u_j^k\right)$$

$\Rightarrow$ This corresponds to solving the i-th equation of the LSE using the unknowns from the **previous** iteration.

Solving Elliptic PDEs

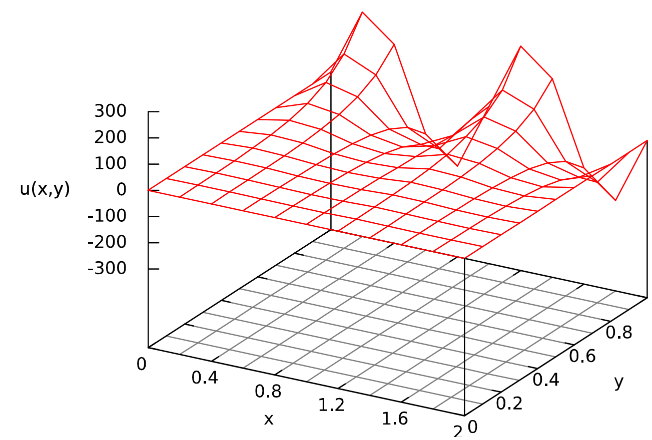
- In order to solve the elliptic partial differential equation …

$$\Delta u(x,y) + k^2 u(x,y) = f(x,y)$$

… no Matrix A must be assembled.

- We just need to know the "stencil":

$$\begin{bmatrix} & \frac{1}{h_y^2} & \\ \frac{1}{h_x^2} & -\frac{2}{h_x^2} - \frac{2}{h_y^2} + k^2 & \frac{1}{h_x^2} \\ & \frac{1}{h_y^2} & \end{bmatrix}$$



A solution of the PDE

- Every grid point $u_{x,y}$ can be computed as follows:

$$u_{x,y} = \frac{1}{\left(\frac{-2}{h_x^2} + \frac{-2}{h_y^2} + k^2\right)} \left(f_{x,y} - \frac{1}{h_x^2}\left[u_{x-1,y} + u_{x+1,y}\right] - \frac{1}{h_y^2}\left[u_{x,y-1} + u_{x,y+1}\right] \right)$$

The End

Questions ?

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