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Exercise Sheet 1

Exercise 1: Estimating the Bayes Error (10 + 10 + 10 P)

The Bayes decision rule for the two classes classification problem results in the Bayes error

$$P(\text{error}) = \int P(\text{error}|\boldsymbol{x})\, p(\boldsymbol{x})\, d\boldsymbol{x},$$

where $P(\text{error}|\boldsymbol{x}) = \min\left[\, P(\omega_1|\boldsymbol{x})\, ,\, P(\omega_2|\boldsymbol{x}) \,\right]$ is the probability of error for a particular input $\boldsymbol{x}$. Interestingly, while class posteriors $P(\omega_1|\boldsymbol{x})$ and $P(\omega_2|\boldsymbol{x})$ can often be expressed analytically and are integrable, the error function has discontinuities that prevent its analytical integration, and therefore, direct computation of the Bayes error.

(a) *Show* that the full error can be upper-bounded as follows:

$$P(\text{error}) \leq \int \frac{2}{\frac{1}{P(\omega_1|\boldsymbol{x})} + \frac{1}{P(\omega_2|\boldsymbol{x})}}\, p(\boldsymbol{x})\, d\boldsymbol{x}.$$

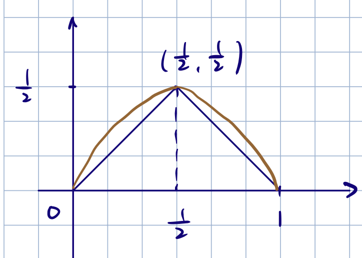
Note that the integrand is now continuous and corresponds to the harmonic mean of class posteriors weighted by $p(\boldsymbol{x})$.

To prove: $P(\text{error}) = \int P(\text{error}|x)\, p(x)\, dx \leq \int \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}}\, p(x)\, dx$

$$\Leftrightarrow \quad P(\text{error}|x) = \min\left[P(\omega_1|x),\, P(\omega_2|x)\right] = \min\left[P(\omega_1|x),\, 1 - P(\omega_1|x)\right] \leq \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}}$$

$$\frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}} = \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{1-P(\omega_1|x)}} = \frac{2}{\frac{1-P(\omega_1|x)+P(\omega_1|x)}{P(\omega_1|x)\left(1-P(\omega_1|x)\right)}} = 2\, P(\omega_1|x)\left(1-P(\omega_1|x)\right)$$

$$P(\text{error}|x) = \min\left[P(\omega_1|x),\, 1-P(\omega_1|x)\right] = \begin{cases} P(\omega_1|x) & 0 \leq P(\omega_1|x) \leq \frac{1}{2} \\ 1-P(\omega_1|x) & \frac{1}{2} < P(\omega_1|x) \leq 1 \end{cases}$$



$$\Rightarrow \quad P(\text{error}|x) \leq 2\, P(\omega_1|x)\left(1-P(\omega_1|x)\right)$$

$$\Rightarrow \quad P(\text{error}) = \int P(\text{error}|x)\, p(x)\, dx \leq \int \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}}\, p(x)\, dx \quad \blacksquare$$

(b) *Show* using this result that for the univariate probability distributions

$$p(x|\omega_1) = \frac{\pi^{-1}}{1+(x-\mu)^2} \quad \text{and} \quad p(x|\omega_2) = \frac{\pi^{-1}}{1+(x+\mu)^2},$$

the Bayes error can be upper-bounded by:

$$P(\text{error}) \leq \frac{2\,P(\omega_1)P(\omega_2)}{\sqrt{1+4\mu^2 P(\omega_1)P(\omega_2)}}$$

(Hint: you can use the identity $\int \frac{1}{ax^2+bx+c}\,dx = \frac{2\pi}{\sqrt{4ac-b^2}}$ for $b^2 < 4ac$.)

$$P(\text{error}) \leq \int \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}} p(x)\,dx$$

$$\leq 2\int \frac{1}{\frac{1}{p(x|\omega_1)\cdot P(\omega_1)} + \frac{1}{p(x|\omega_2)\cdot P(\omega_2)}}\,dx$$

$$\leq 2\int \frac{1}{\frac{1+(x-\mu)^2}{\pi^{-1}\cdot P(\omega_1)} + \frac{1+(x+\mu)^2}{\pi^{-1} P(\omega_2)}}\,dx$$

$$\leq 2\int \frac{P(\omega_1)\cdot P(\omega_2)\cdot \pi^{-1}}{P(\omega_2)\cdot[1+(x-\mu)^2] + P(\omega_1)\cdot[1+(x+\mu)^2]}\,dx$$

$$\leq 2P(\omega_1)P(\omega_2)\pi^{-1} \int \frac{1}{(P(\omega_1)+P(\omega_2))x^2 + 2\mu(P(\omega_1)-P(\omega_2))x + [P(\omega_1)+P(\omega_2)](\mu^2+1)}\,dx$$

$$= 2P(\omega_1)P(\omega_2)\pi^{-1} \int \frac{1}{x^2 + 2\mu(P(\omega_1)-P(\omega_2))\cdot x + (\mu^2+1)}\,dx$$

$$= 2P(\omega_1)P(\omega_2)\cdot\pi^{-1} \frac{2\pi}{\sqrt{4\cdot 1\cdot(\mu^2+1) - 4\mu^2(P(\omega_1)-P(\omega_2))^2}}$$

$$= \frac{2P(\omega_1)P(\omega_2)}{\pi} \cdot \frac{2\pi}{\sqrt{4\mu^2+4 - 4\mu^2(P(\omega_1)+P(\omega_2))^2 + 16P(\omega_1)\cdot P(\omega_2)}}$$

$$= 2P(\omega_1)P(\omega_2) \cdot \frac{1}{1+4P(\omega_1)\cdot P(\omega_2)}$$

$\blacksquare$

(c) *Explain* how you would estimate the error if there was no upper-bounds that are both tight and analytically integrable. Discuss following two cases: (1) the data is low-dimensional and (2) the data is high-dimensional.

Exercise 2: Bayes Decision Boundaries (15 + 15 P)

One might speculate that, in some cases, the generated data $p(x|\omega_1)$ and $p(x|\omega_2)$ is of no use to improve the accuracy of a classifier, in which case one should only rely on prior class probabilities $P(\omega_1)$ and $P(\omega_2)$ assumed here to be strictly positive.

For the first part of this exercise, we assume that the data for each class is generated by the univariate Laplacian probability distributions:

$$p(x|\omega_1) = \frac{1}{2\sigma}\exp\Big(-\frac{|x-\mu|}{\sigma}\Big) \quad \text{and} \quad p(x|\omega_2) = \frac{1}{2\sigma}\exp\Big(-\frac{|x+\mu|}{\sigma}\Big).$$

where $\mu, \sigma > 0$.

(a) *Determine* for which values of $P(\omega_1), P(\omega_2), \mu, \sigma$ the optimal decision is to always predict the first class (i.e. under which conditions $P(\text{error}|x) = P(\omega_2|x) \;\; \forall\, x \in \mathbb{R}$).

$$P(Err|x) = \min\{P(\omega_1|x), P(\omega_2|x)\} = P(\omega_2|x)$$

$$\Rightarrow \forall x: \quad P(\omega_1|x) \geq P(\omega_2|x)$$

$$\Leftrightarrow \forall x: \quad \frac{P(x|\omega_1)\cdot P(\omega_1)}{P(x)} \geq \frac{P(x|\omega_2)\cdot P(\omega_2)}{P(x)}$$

$$\Leftrightarrow \forall x: \quad \exp\Big(-\frac{|x-\mu|}{\sigma}\Big)\cdot P(\omega_1) \geq \exp\Big(-\frac{|x+\mu|}{\sigma}\Big)\cdot P(\omega_2) \quad \Big|\, \log$$

$$\Leftrightarrow \forall x: \quad -\frac{|x-\mu|}{\sigma} + \log P(\omega_1) \geq -\frac{|x+\mu|}{\sigma} + \log(P(\omega_2))$$

① $x \leq -\mu$: $\quad \dfrac{x-\mu}{\sigma} + \log P(\omega_1) \geq \dfrac{x+\mu}{\sigma} + \log(P(\omega_2))$

$$\Leftrightarrow \log P(\omega_1) - \log P(\omega_2) \geq \frac{2\mu}{\sigma}$$

② $-\mu < x \leq \mu$: $\quad \dfrac{x-\mu}{\sigma} + \log P(\omega_1) \geq \dfrac{-x-\mu}{\sigma} + \log(P(\omega_2))$

$$\Leftrightarrow \log P(\omega_1) - \log P(\omega_2) \geq \frac{-2x}{\sigma}$$

③ $x > \mu$: $\quad \dfrac{-x+\mu}{\sigma} + \log P(\omega_1) \geq \dfrac{-x-\mu}{\sigma} + \log(P(\omega_2))$

$$\Leftrightarrow \log P(\omega_1) - \log P(\omega_2) \geq \frac{-2\mu}{\sigma}$$

$$\Rightarrow \quad \log P(\omega_1) - \log P(\omega_2) \geq \frac{2\mu}{\sigma}$$

(b) *Repeat* the exercise for the case where the data for each class is generated by the univariate Gaussian probability distributions:

$$p(x|\omega_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big) \quad \text{and} \quad p(x|\omega_2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(x+\mu)^2}{2\sigma^2}\Big).$$

where $\mu, \sigma > 0$.

$P(Err|x) = \min\{P(\omega_1|x), P(\omega_2|x)\} = P(\omega_2|x)$

$\Rightarrow \forall x: \quad P(\omega_1|x) \geq P(\omega_2|x)$

$\Leftrightarrow \forall x: \quad \dfrac{P(x|\omega_1)\cdot P(\omega_1)}{P(x)} \geq \dfrac{P(x|\omega_2)\cdot P(\omega_2)}{P(x)}$

$\Leftrightarrow \quad \exp\Big(-\dfrac{(x-\mu)^2}{2\sigma^2}\Big)\cdot P(\omega_1) \geq \exp\Big(-\dfrac{(x+\mu)^2}{2\sigma^2}\Big)\cdot P(\omega_2) \qquad | \log$

$\Leftrightarrow \quad -\dfrac{(x-\mu)^2}{2\sigma^2} + \log P(\omega_1) \geq -\dfrac{(x+\mu)^2}{2\sigma^2} + \log P(\omega_2)$

$\Leftrightarrow \quad \log P(\omega_1) - \log P(\omega_2) \geq \dfrac{(x-\mu)^2-(x+\mu)^2}{2\sigma^2} = -2\mu x$

Since $x \in R$, it follows $-2\mu x \in R$, which $\log P(\omega_1) - \log P(\omega_2)$ cannot always satisfy.