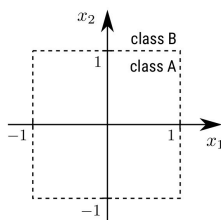**Exercise 1: Designing a Neural Network (25 P)**

We would like to implement a neural network that classifies data points in $\mathbb{R}^2$ according to decision boundary given in the figure below.

We consider as an elementary computation the *threshold neuron* whose relation between inputs $(a_i)_i$ and output a_j is given by

$$z_j = \sum_i a_i w_{ij} + b_j \qquad a_j = 1_{z_j > 0}.$$

(a) *Design* at hand a neural network that takes x_1 and x_2 as input and produces the output "1" if the input belongs to class A, and "0" if the input belongs to class B. *Draw* the neural network model and *write down* the weights w_{ij} and bias b_j of each neuron.
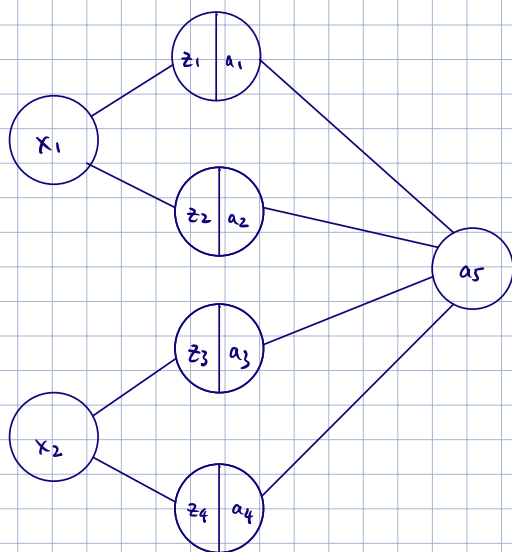
$w_{11} = 1 \qquad b_{11} = 1$

$w_{12} = -1 \qquad b_{11} = 1$

$w_{23} = 1 \qquad b_{23} = 1$

$w_{24} = -1 \qquad b_{24} = 1$

$w_{16} = 1$

$w_{26} = 1$

$w_{36} = 1$

$w_{46} = 1$

$b_5 = -3.1$ (The point is in the area, if and only if its output before going threshold is equal to 4)

Exercise 2: Backward Propagation (5 + 20 P)

We consider a neural network that takes two inputs x_1 and x_2 and produces an output y based on the following set of computations:

$$z_3 = x_1 \cdot w_{13} + x_2 \cdot w_{23} \qquad z_5 = a_3 \cdot w_{35} + a_4 \cdot w_{45} \qquad y = a_5 + a_6$$
$$a_3 = \tanh(z_3) \qquad a_5 = \tanh(z_5)$$

$$z_4 = x_1 \cdot w_{14} + x_2 \cdot w_{24} \qquad z_6 = a_3 \cdot w_{36} + a_4 \cdot w_{46}$$
$$a_4 = \tanh(z_4) \qquad a_6 = \tanh(z_6)$$

(a) *Draw* the neural network graph associated to this set of computations.
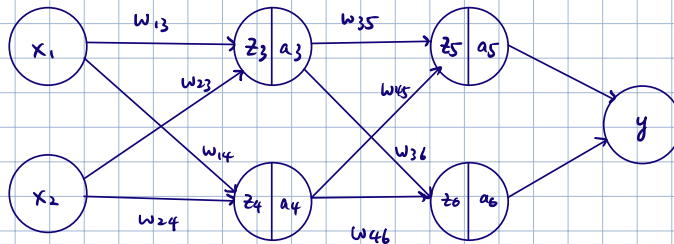
(b) *Write* the set of backward computations that leads to the evaluation of the partial derivative $\partial y / \partial w_{13}$. Your answer should avoid redundant computations. Hint: $\tanh'(t) = 1 - (\tanh(t))^2$.

$$\frac{\partial y}{\partial w_{13}} = \frac{\partial y}{\partial a_5} \cdot \frac{\partial a_5}{\partial z_5} \cdot \frac{\partial z_5}{\partial a_3} \cdot \frac{\partial a_3}{\partial z_3} \cdot \frac{\partial z_3}{\partial w_{13}} + \frac{\partial y}{\partial a_6} \cdot \frac{\partial a_6}{\partial z_6} \cdot \frac{\partial z_6}{\partial a_3} \cdot \frac{\partial a_3}{\partial z_3} \cdot \frac{\partial z_3}{\partial w_{13}}$$

$$= 1 \cdot (1 - \tanh(z_5))^2 \cdot w_{35} \cdot (1 - \tanh(z_3))^2 \cdot x_1$$

$$+ 1 \cdot (1 - \tanh(z_6))^2 \cdot w_{36} \cdot (1 - \tanh(z_3))^2 \cdot x_1$$

$$= (1 - \tanh(z_3))^2 \cdot x_1 \cdot \left[(1 - \tanh(z_5))^2 + (1 - \tanh(z_6))^2 \right]$$

$$= (1 - a_3^2) \cdot x_1 \cdot \left[(1 - a_5^2) + (1 - a_6^2) \right]$$