

NEML CW

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1 Introduction

It is often very natural to attempt to model data using gaussian distribution. Gaussian distributions are well defined for data points which lie in euclidean space for any number of dimensions. We however know that most data should be modelled as to be taken from a manifold. There are methods to do manifold learning but once we have a manifold the notion of probability distributions need to be altered such that we are defining a probability measure across a manifold. It does not make sense for a region outside the manifold to possess non-zero measure. The Locally Adaptive Normal Distribution (LAND) aims to attempt to do this. Once we have a notion of normal distribution on a manifold given some mean and covariance, We can try and fit distribution to the data using MLE approaches. Finally, we will be able to use the fact that we can fit gaussians to data to be able to perform EM algorithms over data to fit a mixture of LANDS over data. TODO: Write something about how manifold learning is handled i.e. Do we know the manifold which data will be distributed on or is this something that needs to be learnt using graphs?

2 Constructing Metrics

In order to define the idea of distances over manifolds and in order to attempt to capture local behaviour of data, we need to a Riemannian Metric which acts on tangent vectors. The Metric $\mathbf{M}(\mathbf{x})$ gives an inner product $\langle \mathbf{u}, \mathbf{M}(\mathbf{x})\mathbf{v} \rangle$ This allows us to define geodesics on our manifold as the path which minimises the length:

$$\hat{\gamma} = \underset{\gamma}{argmin} \int_0^1 \sqrt{\langle \gamma'(t), \mathbf{M}(\gamma(t))\gamma'(t) \rangle} dt, \gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{y}$$

Once we have $\hat{\gamma}$ it is natural to define our Exponential map given a $\mathbf{v} \in T_{\mathbf{x}}M$ such that $\mathbf{y} = Exp_{\mathbf{x}}(\mathbf{v}) \in M$ where $\hat{\gamma}(t) = Exp_{\mathbf{x}}(t \cdot \mathbf{v})$. We can then begin to define the *Log* mapping as: $Log_{\mathbf{x}}(y) = v \in T_{\mathbf{x}}M$. In order to calculate each

geodesic, we must express solve the equation relating christoffel symbols. This works out to be a second order differential ODE that in practice can be solved numerically.

A generalised view of thinking about the normal distribution is the distribution over a given space such that given a known mean and covariance for which we have maximum entropy. This way of defining the normal distribution allows us to extend to non euclidean spaces. It can be shown that given a Log map, the distribution that satisfies this is as follows:

$$p_M(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \exp(-\frac{1}{2}\langle \text{Log}_{\boldsymbol{\mu}}(\boldsymbol{x}), \boldsymbol{\Sigma}^{-1} \text{Log}_{\boldsymbol{\mu}}(\boldsymbol{x}) \rangle)$$

3 Parametric Inference

4 Extending to Mixture Models