

FYP

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January 2025

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1 Introduction

1.1 Cauchy Transform

$$C_{\Gamma}f(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z} dt$$

This is analytic for $z \notin \Gamma$. Define Hilbert Transform to be the limits from the right and the left.

1.2 Orthogonal Polynomials

Family	Notation	Interval	$w(x)$
Legendre	$P_n(x)$	$[-1,1]$	1
Chebyshev (1st)	$T_n(x)$	$[-1,1]$	$(1-x^2)^{-1/2}$
Chebyshev (2nd)	$U_n(x)$	$[-1,1]$	$(1-x^2)^{1/2}$
Ultraspherical	$C_n^{(\lambda)}(x), \lambda > -\frac{1}{2}$	$[-1,1]$	$(1-x^2)^{\lambda-1/2}$
Jacobi	$P_n^{(\alpha,\beta)}(x), \alpha, \beta > -1$	$[-1,1]$	$(1-x)^{\alpha}(1+x)^{\beta}$

2 Log and Stieltjes Transform

In this section we will consider approaches to compute these weakly singular integrals

$$\int_A \log||z-t||f(t)dt \quad \int_A \nabla \log||z-t||f(t)dt$$

$$\mathcal{S}_A f(z) := \int_A \frac{f(t)}{z-t} dt$$

$$\mathcal{L}_A f(z) := \int_A \log(z-t)f(t)dt$$

Depending on the type of area which A is we can begin by approximating f using orthogonal polynomials.

2.1 Transforms across Intervals

We will try to formulate recurrence relations for these transforms across interval $[-1, 1]$. We are looking for looking for $\mathcal{S}_{[-1,1]}f(z)$. Decomposing $f(z) \approx \sum_k f_k P_k(z)$ and writing $S_k(z) := \mathcal{S}_{[-1,1]}P_k(z)$ lets us write:

$$\mathcal{S}_{[-1,1]}f(z) \approx \sum_k f_k S_k(z)$$

This motivates finding fast methods to compute $S_k(z)$. Log kernels are approached similarly letting $L_k(z) := \mathcal{L}_{[-1,1]}P_k(z)$ and looking for recurrence relations.

Stieltjes

Recall recurrence relation of Legendre Polynomials:

$$xP_k(x) = \frac{k}{2k+1}P_{k-1}(x) + \frac{k+1}{2k+1}P_{k+1}(x)$$

Formulate three-term recurrence for their Stieltjes transforms.

$$\begin{aligned} zS_k(z) &= \int_{-1}^1 \frac{zP_k(t)}{z-t} dt \\ &= \int_{-1}^1 \frac{z-t}{z-t} P_k(t) dt + \int_{-1}^1 \frac{tP_k(t)}{z-t} dt \\ &= \int_{-1}^1 P_k(t) dt + \frac{k}{2k+1} \int_{-1}^1 \frac{P_{k-1}(t)}{z-t} dt + \frac{k+1}{2k+1} \int_{-1}^1 \frac{P_{k+1}(t)}{z-t} dt \\ &= 2\delta_{k0} + \frac{k}{2k+1}S_{k-1}(z) + \frac{k+1}{2k+1}S_{k+1}(z) \\ S_0(z) &= \int_{-1}^1 \frac{dt}{z-t} = \log(z+1) - \log(z-1) \end{aligned}$$

We can extend this to work over a square using the recurrence over intervals:

$$\begin{aligned} zS_{k,j}(z) &= z \int_{-1}^1 \int_{-1}^1 \frac{P_k(s)P_j(t)}{z-(s+it)} ds dt \\ &= \int_{-1}^1 zP_j(t) \int_{-1}^1 \frac{P_k(s)}{z-it-s} ds dt \\ &= \int_{-1}^1 (z-it)P_j(t)S_k(z-it) + itP_j(t)S_k(z-it) ds dt \\ &= \int_{-1}^1 P_j(t) \left(\frac{k}{2k+1}S_{k-1}(z-it) + \frac{k+1}{2k+1}S_{k+1}(z-it) + 2\delta_{k0} \right) \\ &\quad + i \left(\frac{j}{2j+1}P_{j-1}(t) + \frac{j+1}{2j+1}P_{j+1}(t) \right) S_k(z-it) ds dt \\ &= \frac{k}{2k+1}S_{k-1,j}(z) + \frac{k+1}{2k+1}S_{k+1,j} \\ &\quad + i \frac{j}{2j+1}S_{k,j-1}(z) + i \frac{j+1}{2j+1}S_{k,j+1} + 4\delta_{j0}\delta_{k0} \end{aligned}$$

Log

We can begin by connecting log kernel to the Stieltjes kernel. To do this we define:

$$S_k^{(\lambda)}(z) := \int_{-1}^1 \frac{C_k^{(\lambda)}(t)}{z-t} dt$$

We let $F(x) = \int_{-1}^1 f(s)ds$ and apply integration by parts on log transform:

$$\begin{aligned} \int_{-1}^1 f(t) \log(z-t) dt &= [-F(t) \log(z-t)]_{-1}^1 - \int_{-1}^1 \frac{F(t)}{z-t} dt \\ &= \log(z+1) \int_{-1}^1 f(t) dt - \int_{-1}^1 \frac{F(t)}{z-t} dt \end{aligned}$$

3 Polynomial Transforms

We can begin to consider taking these transforms across different geometries. Currently we have a way to find these transforms across $[-1,1]$ but we will be trying to use this to solve other geometries. The first type of geometry we should consider is one where we apply a degree d polynomial transform to the interval:

$$p : [-1, 1] \rightarrow \Gamma$$

We will show why the solution to a cauchy transform across this interval is as follows:

$$C_{\Gamma} f(z) = \sum_{j=0}^d C_{[-1,1]}[f \circ p](p_j^{-1}(z))$$

Where $p_j^{-1}(z)$ are the d pre-images of p . In order to solve this we will use plemelj. There are 3 properties that need to hold for a function $\psi : \Gamma \rightarrow \mathbb{C}$ to be a cauchy transform:

$$\begin{aligned} \lim_{z \rightarrow \infty} \psi &= 0 \\ \psi^+(z) - \psi^-(z) &= f(z) \\ \psi &\text{ analytic on } \Gamma \end{aligned} \tag{1}$$

Checking (1).1 we get that $\lim_{z \rightarrow \infty} p_j^{-1}(z) = \infty \implies$

$$\begin{aligned} \lim_{z \rightarrow \infty} C_{\Gamma} f(z) &= \sum_{j=1}^d \lim_{z \rightarrow \infty} C_{[-1,1]}(f \circ p)(p_j^{-1}(z)) \\ &= \sum_{j=1}^d C_{[-1,1]}(f \circ p)(\lim_{z \rightarrow \infty} p_j^{-1}(z)) \\ &= \sum_{j=1}^d 0 = 0 \end{aligned}$$

Checking (1).2 we need an expression for ψ^+ and ψ^- . Let us begin by saying that we are looking for cauchy transform of point s which happens to lie on Γ . This means that there is a unique root of $t_k := p_k^{-1}(s) \in [-1, 1]$. TODO: Show that $\lim_{z \rightarrow s^+} p_k^{-1}(s) = \lim_{z \rightarrow p^{-1}(s)^+}$. Taking limits of ψ^+, ψ^- gives us:

$$\begin{aligned} \psi^+(s) &= \lim_{z \rightarrow s} C_{[-1,1]}(f \circ p)(p_k^{-1}(z)) \\ &\quad + \sum_{j \neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \\ &= C_{[-1,1]}^+(f \circ p)(p_k^{-1}(z)) \\ &\quad + \sum_{j \neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \end{aligned}$$

We can do a similar thing with ψ^- and putting everything together:

$$\begin{aligned}\psi^+(s) - \psi^-(s) &= C_{[-1,1]}^+(f \circ p)(p_k^{-1}(s)) - C_{[-1,1]}^-(f \circ p)(p_k^{-1}(s)) \\ &= (f \circ p)(p_k^{-1}(s)) = f(s)\end{aligned}$$

In the case where $z \notin \psi$, $\psi^+ = \psi^-$ which is expected since the area in between is analytic

TODO show that condition (1).3 holds

4 Affine Transformations

Affine transformations can be solved in 2 distinct ways: We will begin by considering the case of solving for a horizontally skewed square with the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \alpha x + \beta y \\ y \end{pmatrix}$$

It can be shown that any affine transformation in the form of $(x, y)^T \rightarrow A(x, y)^T$ can be done by taking the above translation and performing scaling and rotations. TODO: Show that this is indeed the case