

FYP

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# 1 Introduction

## 1.1 Cauchy Transform

$$C_\Gamma f(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} dt \quad (1)$$

This is analytic for  $z \notin \Gamma$ . Define Hilbert Transform to be the limits from the right and the left.

## 1.2 Orthogonal Polynomials

Family	Notation	Interval	$w(x)$
Legendre	$P_n(x)$	$[-1,1]$	1
Chebyshev (1st)	$T_n(x)$	$[-1,1]$	$(1-x^2)^{-1/2}$
Chebyshev (2nd)	$U_n(x)$	$[-1,1]$	$(1-x^2)^{1/2}$
Ultraspherical	$C_n^{(\lambda)}(x), \lambda > -\frac{1}{2}$	$[-1,1]$	$(1-x^2)^{\lambda-1/2}$
Jacobi	$P_n^{(\alpha,\beta)}(x), \alpha, \beta > -1$	$[-1,1]$	$(1-x)^\alpha(1-x)^\beta$

# 2 Log and Stieltjes Transform

In this section we will consider approaches to compute these weakly singular integrals

$$\begin{aligned} \int_A \log||z-t||f(t)dt & \quad \int_A \nabla \log||z-t||f(t)dt \\ \mathcal{S}_A f(z) &:= \int_A \frac{f(t)}{z-t} dt \end{aligned} \quad (2)$$

$$\mathcal{L}_A f(z) := \int_A \log(z-t)f(t)dt \quad (3)$$

Depending on the type of area which  $A$  is we can begin by approximating  $f$  using orthogonal polynomials.

## 2.1 Transforms across Intervals

We will try to formulate recurrence relations for these transforms across interval  $[-1, 1]$ . We are looking for looking for  $\mathcal{S}_{[-1,1]}f(z)$ . Decomposing  $f(z) \approx \sum_k f_k P_k(z)$  and writing  $S_k(z) := \mathcal{S}_{[-1,1]}P_k(z)$  lets us write:

$$\mathcal{S}_{[-1,1]}f(z) \approx \sum_k f_k S_k(z)$$

This motivates finding fast methods to compute  $S_k(z)$ . Log kernels are approached similarly letting  $L_k(z) := \mathcal{L}_{[-1,1]}P_k(z)$  and looking for recurrence relations.

## Stieltjes

Recall recurrence relation of Legendre Polynomials:

$$xP_k(x) = \frac{k}{2k+1}P_{k-1}(x) + \frac{k+1}{2k+1}P_{k+1}(x) \quad (4)$$

Formulate three-term recurrence for their Stieltjes transforms.

$$\begin{aligned} zS_k(z) &= \int_{-1}^1 \frac{zP_k(t)}{z-t} dt \\ &= \int_{-1}^1 \frac{z-t}{z-t} P_k(t) dt + \int_{-1}^1 \frac{tP_k(t)}{z-t} dt \\ &= \int_{-1}^1 P_k(t) dt + \frac{k}{2k+1} \int_{-1}^1 \frac{P_{k-1}(t)}{z-t} dt + \frac{k+1}{2k+1} \int_{-1}^1 \frac{P_{k+1}(t)}{z-t} dt \quad (5) \\ &= 2\delta_{k0} + \frac{k}{2k+1}S_{k-1}(z) + \frac{k+1}{2k+1}S_{k+1}(z) \\ S_0(z) &= \int_{-1}^1 \frac{dt}{z-t} = \log(z+1) - \log(z-1) \end{aligned}$$

## Log

We can begin by connecting log kernel to the Stieltjes kernel. To do this we define:

$$S_k^{(\lambda)}(z) := \int_{-1}^1 \frac{C_k^{(\lambda)}(t)}{z-t} dt$$

We let  $F(x) = \int_{-1}^1 f(s)ds$  and apply integration by parts on log transform:

$$\begin{aligned} \int_{-1}^1 f(t) \log(z-t) dt &= [-F(t) \log(z-t)]_{-1}^1 - \int_{-1}^1 \frac{F(t)}{z-t} dt \\ &= \log(z+1) \int_{-1}^1 f(t) dt - \int_{-1}^1 \frac{F(t)}{z-t} dt \end{aligned} \quad (6)$$

## 3 Polynomial Transforms

We can begin to consider taking these transforms across different geometries. Currently we have a way to find these transforms across  $[-1,1]$  but we will be trying to use this to solve other geometries. The first type of geometry we should consider is one where we apply a degree  $d$  polynomial transform to the interval:

$$p : [-1, 1] \rightarrow \Gamma$$

We will show why the solution to a cauchy transform across this interval is as follows:

$$C_\Gamma f(z) = \sum_{j=0}^d C_{[-1,1]}[f \circ p](p_j^{-1}(z)) \quad (7)$$

Where  $p_j^{-1}(z)$  are the  $d$  pre-images of  $p$ . In order to solve this we will use plemelj. There are 3 properties that need to hold for a function  $\psi : \Gamma \rightarrow \mathbb{C}$  to be a cauchy transform:

$$\begin{aligned} \lim_{z \rightarrow \infty} &= 0 \\ \psi^+(z) - \psi^-(z) &= f(z) \\ \psi &\text{ analytic on } \Gamma \end{aligned} \tag{8}$$

Checking (8).1 we get that  $p_j^{-1}(z) = \infty \implies$   
 $z \rightarrow \infty$

$$\begin{aligned} \lim_{z \rightarrow \infty} C_\Gamma f(z) &= \sum_{j=1}^d \lim_{z \rightarrow \infty} C_{[-1,1]}(f \circ p)(p_j^{-1}(z)) \\ &= \sum_{j=1}^d C_{[-1,1]}(f \circ p)(\lim_{z \rightarrow \infty} p_j^{-1}(z)) \\ &= \sum_{j=1}^d 0 = 0 \end{aligned} \tag{9}$$

Checking (8).2 we need an expression for  $\psi^+$  and  $\psi^-$ . Let us begin by saying that we are looking for cauchy transform of point  $s$  which happens to lie on  $\Gamma$ . This means that there is a unique root of  $t_k := p_k^{-1} \in [-1, 1]$ . TODO: Show that  $\lim_{z \rightarrow s^+} p_k^{-1}(s) = \lim_{z \rightarrow p^{-1}(s)^+}$ . Taking limits of  $\psi^+, \psi^-$  gives us:

$$\begin{aligned} \psi^+(s) &= \lim_{z \rightarrow s} C_{[-1,1]}(f \circ p)(p_k^{-1}(z)) \\ &\quad + \sum_{j \neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \\ &= C_{[-1,1]}^+(f \circ p)(p_k^{-1}(s)) \\ &\quad + \sum_{j \neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \end{aligned} \tag{10}$$

We can do a similar thing with  $\psi^-$  and putting everything together:

$$\begin{aligned} \psi^+(s) - \psi^-(s) &= C_{[-1,1]}^+(f \circ p)(p_k^{-1}(s)) - C_{[-1,1]}^-(f \circ p)(p_k^{-1}(s)) \\ &= (f \circ p)(p_k^{-1}(s)) = f(s) \end{aligned} \tag{11}$$

In the case where  $z \notin \psi, \psi^+ = \psi^-$  which is expected since the area in between is analytic

TODO show that condition (8).3 holds