FYP

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Contents

1 Introduction

1.1 Solving Inhomogenous PDEs

We will begin by considering an equation in the form of $\mathcal{D}u = f$. The green's function for a particular PDE is defined in a way such that for a green's function $G(x, x')$ given for some differential operator $\mathcal{D}$, we have that $\mathcal{D}G(x, x') = \delta(x, x')$ My project will be to go through how we can quickly and efficiently by decomposing into orthogonal polynomials and then utilising the recurrence properties of these OPs to solve PDEs using a Dynamic Programming approach.

1.2 Cauchy Transform

$$C_\Gamma f(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z} dt$$

This is analytic for $z \notin \Gamma$. Define Hilbert Transform to be the limits from the right and the left.

1.3 Orthogonal Polynomials

Family	Notation	Interval	$w(x)$
Legendre	$P_n(x)$	[-1,1]	1
Chebyshev (1st)	$T_n(x)$	[-1,1]	$(1 - x^2)^{-1/2}$
Chebyshev (2nd)	$U_n(x)$	[-1,1]	$(1 - x^2)^{1/2}$
Ultraspherical	$C_n^{(\lambda)}(x),\ \lambda > -\frac{1}{2}$	[-1,1]	$(1 - x^2)^{\lambda - 1/2}$
Jacobi	$P_n^{(\alpha,\beta)}(x),\ \alpha, \beta > -1$	[-1,1]	$(1 - x)^\alpha (1 - x)^\beta$

2 Log and Stieltjes Transform

In this section we will consider approaches to compute these weakly singular integrals

$$\int_A log||z - t||f(t)dt \qquad \int_A \nabla log||z - t||f(t)dt$$

$$\mathcal{S}_A f(z) := \int_A \frac{f(t)}{z - t} dt$$

$$\mathcal{L}_A f(z) := \int_A log(z - t)f(t)dt$$

Depending on the type of area which A is we can begin by approximating f using orthogonal polynomials.

2.1 Transforms across Intervals

We will try to formulate recurrence relations for these transforms across interval [-1, 1]. We are looking for looking for $\mathcal{S}_{[-1,1]}f(z)$. Decomposing $f(z) \approx \Sigma_k f_k P_k(z)$ and writing $S_k(z) := \mathcal{S}_{[-1,1]}P_k(z)$ lets us write:

$$\mathcal{S}_{[-1,1]}f(z) \approx \Sigma_k f_k S_k(z)$$

This motivates finding fast methods to compute $S_k(z)$. Log kernels are approached similarly letting $L_k(z) := \mathcal{L}_{[-1,1]}P_k(z)$ and looking for recurrence relations.

Stieltjes

Recall recurrence relation of Legendre Polynomials:

$$xP_k(x) = \frac{k}{2k+1}P_{k-1}(x) + \frac{k+1}{2k+1}P_{k+1}(x)$$

Formulate three-term recurrence for their Stieltjes transforms.

$$\begin{aligned}
zS_k(z) &= \int_{-1}^{1} \frac{zP_k(t)}{z-t}dt \\
&= \int_{-1}^{1} \frac{z-t}{z-t}P_k(t)dt + \int_{-1}^{1} \frac{tP_k(t)}{z-t}dt \\
&= \int_{-1}^{1} P_k(t)dt + \frac{k}{2k+1}\int_{-1}^{1} \frac{P_{k-1}(t)}{z-t}dt + \frac{k+1}{2k+1}\int_{-1}^{1} \frac{P_{k+1}(t)}{z-t}dt \\
&= 2\delta_{k0} + \frac{k}{2k+1}S_{k-1}(z) + \frac{k+1}{2k+1}S_{k+1}(z) \\
S_0(z) &= \int_{-1}^{1} \frac{dt}{z-t} = log(z+1) - log(z-1)
\end{aligned}$$

We can extend this to work over a square using the recurrence over intervals:

$$
\begin{aligned}
zS_{k,j}(z) &= z\int_{-1}^{1}\int_{-1}^{1}\frac{P_k(s)P_j(t)}{z-(s+it)}dsdt \\
&= \int_{-1}^{1} zP_j(t)\int_{-1}^{1}\frac{P_k(s)}{z-it-s}dsdt \\
&= \int_{-1}^{1}(z-it)P_j(t)S_k(z-it)+itP_j(t)S_k(z-it)dsdt \\
&= \int_{-1}^{1}P_j(t)(\frac{k}{2k+1}S_{k-1}(z-it)+\frac{k+1}{2k+1}S_{k+1}(z-it)+2\delta_{k0}) \\
&+ i(\frac{j}{2j+1}P_{j-1}(t)+\frac{j+1}{2j+1}P_{j+1}(t))S_k(z-it)dsdt \\
&= \frac{k}{2k+1}S_{k-1,j}(z)+\frac{k+1}{2k+1}S_{k+1,j} \\
&+ i\frac{j}{2j+1}S_{k,j-1}(z)+i\frac{j+1}{2j+1}S_{k,j+1}+4\delta_{j0}\delta k0
\end{aligned}
$$

Log

We can begin by connecting log kernel to the Stieltjes kernel. To do this we define:

$$S_k^{(\lambda)}(z) := \int_{-1}^{1}\frac{C_k^{(\lambda)}(t)}{z-t}dt$$

We let $F(x)=\int_{-1}^{1}f(s)ds$ and apply integration by parts on log transform:

$$
\begin{aligned}
\int_{-1}^{1}f(t)log(z-t)dt &= [-F(t)log(z-t)]_{-1}^{1}-\int_{-1}^{1}\frac{F(t)}{z-t}dt \\
&= log(z+1)\int_{-1}^{1}f(t)dt-\int_{-1}^{1}\frac{F(t)}{z-t}dt
\end{aligned}
$$

3 Polynomial Transforms

We can begin to consider taking these transforms across different geometries. Currently we have a way to find these transforms across [-1,1] but we will be trying to use this to solve other geometries. The first type of geometry we should consider is one where we apply a degree d polynomial transform to the interval:

$$p : [-1,1] \to \Gamma$$

We will show why the solution to a cauchy transform across this interval is as follows:

$$C_\Gamma f(z) = \Sigma_{j=0}^{d} C_{[-1,1]}[f\circ p](p_j^{-1}(z))$$

Where $p_j^{-1}(z)$ are the d pre-images of p. In order to solve this we will use plemelj. There are 3 properties that need to hold for a function $\psi : \Gamma \to \mathbb{C}$ to be a cauchy transform:

$$\begin{gathered} \underset{z\to\infty}{lim} = 0 \\ \psi^+(z) - \psi^-(z) = f(z) \\ \psi \; analytic \; on \; \Gamma \end{gathered} \tag{1}$$

Checking (1).1 we get that $\underset{z\to\infty}{p_j^{-1}}(z) = \infty \implies$

$$\begin{aligned} \underset{z\to\infty}{lim} C_\Gamma f(z) &= \Sigma_{j=1}^d \underset{z\to\infty}{lim} C_{[-1,1]}(f \circ p)(p_j^{-1}(z)) \\ &= \Sigma_{j=1}^d C_{[-1,1]}(f \circ p)(\underset{z\to\infty}{lim} p_j^{-1}(z)) \\ &= \Sigma_{j=1}^d 0 = 0 \end{aligned}$$

Checking (1).2 we need an expression for ψ^+ and ψ^-. Let us begin by saying that we are looking for cauchy transform of point s which happens to lie on Γ. This means that there is a unique root of $t_k := p_k^{-1}(s) \in [-1, 1]$. TODO: Show that $\underset{z\to s^+}{lim} p_k^{-1}(s) = \underset{z\to p^{-1}(s)^+}{lim}$. Taking limits of ψ^+, ψ^- gives us:

$$\begin{aligned} \psi^+(s) &= \underset{z\to s}{lim} C_{[-1,1]}(f \circ p)(p_k^{-1}(z)) \\ &+ \Sigma_{j\neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \\ &= C^+_{[-1,1]}(f \circ p)(p_k^{-1}(z)) \\ &+ \Sigma_{j\neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \end{aligned}$$

We can do a similar thing with ψ^- and putting everything together:

$$\begin{aligned} \psi^+(s) - \psi^-(s) &= C^+_{[-1,1]}(f \circ p)(p_k^{-1}(s)) - C^-_{[-1,1]}(f \circ p)(p_k^{-1}(s)) \\ &= (f \circ p)(p_k^{-1}(s)) = f(s) \end{aligned}$$

In the case where $z \notin \psi, \psi^+ = \psi^-$ which is expected since the area in between is analytic

TODO show that condition (1).3 holds

4 Affine Transformations

Affine transformations can be solved in 2 distinct ways: We will begin by considering the case of solving for a horizontally skewed square with the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha x + \beta y \\ y \end{pmatrix}$$

It can be shown that any affine transformation in the form of $(x, y)^T \to A(x, y)^T$ can be done by taking the above translation and performing scaling and rotations. TODO: Show that this is indeed the case

5 Trapezium Region

We can attempt to generalise the method of stieltjes on a square to work for any given quadrilateral. This can easily be done by computing for a Trapezium, We can use the parameterisation:

$$Q\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} (1+x)(\alpha+\beta y) \\ y \end{pmatrix}$$

Since our $\alpha, \beta > 0$, our Q is invertible:

$$Q^{-1}\begin{pmatrix} s \\ t \end{pmatrix} \rightarrow \begin{pmatrix} \frac{s}{\alpha+\beta t} - 1 \\ t \end{pmatrix}$$

There are two approaches which were considered which vary in which functions we are using for our bases. The first approach that was attempted would be to take the function approximation as follows:

$$f(x, y) = \Sigma_{k,j} c_{k,j} P_k(x) P_j(y), c_{k,j} \in \mathbb{R}$$

This method is simpler but it can be seen that we would have to be evaluating integrals of orthogonal polynomials outside the $[-1, 1]$ domain in which they are well behaved. This would result in instable results

It is possible to think of the function space here as the bases of functions which we can add together to approximate the function. In the approach above we can represent the bases to be $\{b_{kj}\}_{kj}$ where $b_{kj}(\mathbf{x}) = P_k(x)P_j(y)$ The approach we will focus on here is the approximation following taking an approximation using the function bases as follows:

$$\begin{aligned} f \circ Q(x, y) &= \Sigma_{k,j} c_{k,j} P_k(x) P_j(y),\ c_{k,j} \in \mathbb{R} \\ f \circ Q(\mathbf{x}) &= \Sigma_{k,j} c_{k,j} b_{kj}(\mathbf{x}) \\ Q\ invertible \implies f(\mathbf{x}) &= \Sigma_{k,j} c_{k,j} b_{kj}(Q^{-1}(\mathbf{x})) \end{aligned}$$

This means we have an altered set of basis based on α, β where the new basis can be represented as:

$$\{\tilde{b}_{kj} = b_{kj} \circ Q^{-1}\}_{kj}$$

In this approach we are need to be able to compute

$$s_{kj} := \int_{-1}^{1} (\alpha + \beta t) \int_{-1}^{1} \frac{P_j(t)P_k(s)}{z - it - (\alpha + \beta t)(1 + s)} ds dt$$

As you can see here there is a term in the denominator which is difficult to deal with as it is harder to seperate the s and t terms. In order to begin we come up with a few different rearrangements of this equation:

$$\tilde{s_{kj}} := \int_{-1}^{1} \int_{-1}^{1} \frac{P_j(t)P_k(s)}{z - it - (\alpha + \beta t)(1 + s)} ds dt$$

$$
\begin{aligned}
&= \int_{-1}^{1} \frac{1}{\alpha+\beta t} \int_{-1}^{1} \frac{P_j(t)P_k(s)}{\frac{z-it}{\alpha+\beta t}-1-s} dsdt \\
&=: \int_{-1}^{1} \frac{P_j(t)}{\alpha+\beta t} \int_{-1}^{1} \frac{P_k(s)}{\tilde{z_t}-s} \\
\tilde{s_{kj}} &= \int_{-1}^{1}\int_{-1}^{1} \frac{P_j(t)P_k(s)}{z-\alpha(1+s)-(i+\beta(1+s))t} dsdt \\
&= \int_{-1}^{1} \frac{P_k(s)}{\beta(1+s)+i} \int_{-1}^{1} \frac{P_j(t)}{\frac{z-\alpha(1+s)}{\beta(1+s)+i}-t} dtds \\
&= \int_{-1}^{1} \frac{P_k(s)}{\beta(1+s)+i} \int_{-1}^{1} \frac{P_j(t)}{\tilde{z_s}-t} dtds
\end{aligned}
$$

Here, I've used $\tilde{z_t}$ and $\tilde{z_s}$ to denote different constants although rigorously both are actually two different functions. It is always the case where $\tilde{z_t}, \tilde{z_s}$ denotes $\frac{z-it}{\alpha+\beta t}-1, \frac{z-\alpha(1+s)}{\beta(1+s)+i}$ respectively

It is also very useful to define function s_k, s_j:

$$
\begin{aligned}
s_k(z) &:= \int_{-1}^{1} \frac{P_k(s)}{z-s} ds \\
s_j(z) &:= \int_{-1}^{1} \frac{P_j(t)}{z-t} dt
\end{aligned}
$$

This can be motivated by the tricky recurrent forms for $\tilde{s_{k0}}$. TODO: Show why tricky?

We can recreate s_{kj} using values of $\tilde{s_{kj}}$ by doing the following:

$$
\begin{aligned}
let\ I(k,j,s,t) &:= \frac{P_j(t)P_k(s)}{z-it-(\alpha+\beta t)(1+s)} \\
s_{kj} &= \int_{-1}^{1} (\alpha+\beta t) \int_{-1}^{1} I(k,j,s,t) dsdt \\
&= \int_{-1}^{1} \alpha \int_{-1}^{1} I(k,j,s,t) dsdt + \int_{-1}^{1} \beta t \int_{-1}^{1} I(k,j,s,t) dsdt \\
&= \alpha \tilde{s_{kj}} + \beta \frac{j}{2j+1} \tilde{s_{kj-1}} + \beta \frac{j+1}{2j+1} \tilde{s_{kj+1}} \\
s_{k0} &= \int_{-1}^{1} \alpha \int_{-1}^{1} I(k,0,s,t) dsdt + \int_{-1}^{1} \beta t \int_{-1}^{1} I(k,0,s,t) dsdt \\
&= \alpha \tilde{s_{k0}} + \beta \tilde{s_{k1}}
\end{aligned}
$$

5.1 Recurrences

Now we can go about trying to construct these recurrences. To make it easier notationally to represent these legendre recurrence relations, it is convenient to represent it as the following:

$$\begin{aligned} xP_j(x) &= \frac{j}{2j+1}P_{j-1}(x) + \frac{j+1}{2j+1}P_{j+1}(x) \\ &:= j_- P_{j-1}(x) + j_+ P_{j+1}(x) \end{aligned}$$

It is easiest to begin with a case where:

5.1.1 Case 1: $k, j > 1$

$$\begin{aligned} z\tilde{s_{kj}} &= \int_{-1}^{1} z\frac{P_j(t)}{\alpha+\beta t}s_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} P_j(t)\frac{z-it}{\alpha+\beta t}s_k(\tilde{z}_t)dt + \int_{-1}^{1}\frac{itP_j(t)}{\alpha+\beta t}s_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} P_j(t)\tilde{z}_t s_k(\tilde{z}_t)dt + \int_{-1}^{1} P_j(t)s_k(\tilde{z}_t) + \int_{-1}^{1}\frac{itP_j(t)}{\alpha+\beta t}s_k(\tilde{z}_t)dt \end{aligned}$$

It is useful here to come up with an expression for:

$$\begin{aligned} \int_{-1}^{1} P_j(t)s_k(\tilde{z}_t) &= \int_{-1}^{1}(\alpha+\beta t)\frac{P_j(t)}{\alpha+\beta t}s_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1}\frac{\alpha P_j(t)}{\alpha+\beta t}s_k(\tilde{z}_t)dt + \int_{-1}^{1}\frac{\beta t P_j(t)}{\alpha+\beta t}s_k(\tilde{z}_t)dt \\ &= \alpha\tilde{s_{kj}} + \beta j_-\tilde{s_{kj-1}} + \beta j_+\tilde{s_{kj+1}} \end{aligned}$$

Similarly:

$$\begin{aligned} \int_{-1}^{1} P_k(s)s_j(\tilde{z}_s)ds &= \int_{-1}^{1}(\beta(1+s)+i)\frac{P_k(s)}{\beta(1+s)+i}s_j(\tilde{z}_s)ds \\ &= \int_{-1}^{1}(\beta+i+\beta s)\frac{P_k(s)}{\beta(1+s)+i}s_j(\tilde{z}_s)ds \\ &= (\beta+i)\tilde{s}_{kj} + \beta k_-\tilde{s}_{k-1j} + \beta k_+\tilde{s}_{k+1j} \end{aligned}$$

Decomposing individual elements of the previous equation:

$$\int_{-1}^{1} P_j(t)\tilde{z}_t s_k(\tilde{z}_t)dt = \int_{-1}^{1} P_j(t)(k_- s_{k-1}(\tilde{z}_t) + k_+ s_{k+1}(\tilde{z}_t))dt$$

$$
\begin{aligned}
&= k_-(\alpha \tilde{s_{k-1j}} + \beta j_- \tilde{s_{k-1j-1}} + \beta j_+ \tilde{s_{k-1j+1}}) \\
&+ k_+(\alpha \tilde{s_{k+1j}} + \beta j_- \tilde{s_{k+1j-1}} + \beta j_+ \tilde{s_{k+1j+1}})
\end{aligned}
$$

$$
\int_{-1}^{1} P_j(t) s_k(\tilde{z}_t) = \alpha \tilde{s_{kj}} + \beta j_- \tilde{s_{kj-1}} + \beta j_+ \tilde{s_{kj+1}}
$$

$$
\int_{-1}^{1} \frac{itP_j(t)}{\alpha + \beta t} s_k(\tilde{z}_t) dt = i(j_- \tilde{s_{kj-1}} + j_+ \tilde{s_{kj+1}})
$$

Returning back to our original equation:

$$
\begin{aligned}
z\tilde{s_{kj}} &= \int_{-1}^{1} P_j(t) \tilde{z}_t s_k(\tilde{z}_t) dt + \int_{-1}^{1} P_j(t) s_k(\tilde{z}_t) + \int_{-1}^{1} \frac{itP_j(t)}{\alpha + \beta t} s_k(\tilde{z}_t) dt \\
&= \beta(k_- j_- \tilde{s}_{k-1j-1} + k_- j_+ \tilde{s}_{k-1j+1} + k_+ j_- \tilde{s}_{k+1j-1} + k_+ j_+ \tilde{s}_{k+1j+1}) \\
&+ \alpha(\tilde{s}_{kj} + k_- \tilde{s}_{k-1j} + k_+ \tilde{s}_{k+1j}) \\
&+ (\beta + i)(j_- \tilde{s}_{kj-1} + j_+ \tilde{s}_{kj+1})
\end{aligned}
$$

The decision to have t as the outside integral was completely arbitrary, we could have taken s as the outside integral and we would have reached the exact same recurrence relation. We can see that for a log kernel we would have different relations which we can use in our benefit to skip finding certain base cases.

And thus we have a 9 point stencil recurrence relation. Given any 8 points we are able to find the final point. Assuming we therefore for some $k, j \geq 2$ we have all $\tilde{s}_{nm}$ for all $n \leq k, m \leq j$, we can compute the value of $\tilde{s}_{k+1j+1}$ since we other values mn centered around kj. Now we need a way of finding the base case, in particular, the case of the two initial rows and columns.

To begin with the computations of the $k = 1, j = 1$ rows/cols, it is useful to prove the following:

$$
\begin{aligned}
zs_0(z) &= \int_{-1}^{1} \frac{z}{z - s} ds \\
&= \int_{-1}^{1} 1 + \frac{s}{z - s} ds \\
&= 2 + s_1(z)
\end{aligned}
$$

5.1.2 Case 2: $k = 1$

For this we are going to assume that we already have values of the following: $\tilde{s}_{kj}$ where both $k \leq 1 \wedge j \leq 1$ as well as for all $k = 0$ and $j = 0$. Computation of these will be another case outlined later. We are able to find a 6 point stencil relation by first beginning with the expansion for $z\tilde{s}_{0j}$

$$
z\tilde{s}_{0j} = \int_{-1}^{1} \frac{zP_j(t)}{\alpha + \beta t} s_0(\tilde{z}_t) dt
$$

$$
\begin{aligned}
&= \int_{-1}^{1} P_j(t)\frac{z-it}{\alpha+\beta t}s_0(\tilde{z}_t)dt + \int_{-1}^{1}\frac{itP_j(t)}{\alpha+\beta t}s_0(\tilde{z}_t)dt \\
&= \int_{-1}^{1} P_j(t)\tilde{z}_t s_0(\tilde{z}_t)dt + \int_{-1}^{1} P_j(t)s_0(\tilde{z}_t)dt + \int_{-1}^{1}\frac{itP_j(t)}{\alpha+\beta t}s_0(\tilde{z}_t)dt \\
&= \int_{-1}^{1} P_j(t)(2+s_1(\tilde{z}_t))dt + \alpha\tilde{s}_{0j} + \beta j_-\tilde{s}_{0j-1} + \beta j_+\tilde{s}_{0j+1} + ij_-\tilde{s}_{0j-1} + ij_+\tilde{s}_{0j+1}
\end{aligned}
$$

$$
\begin{aligned}
&= 4\delta_{0j} + \alpha(\tilde{s}_{0j}+\tilde{s}_{1j}) + \beta j_-(\tilde{s}_{0j-1}+\tilde{s}_{1j-1}) + \beta j_+(\tilde{s}_{0j+1}+\tilde{s}_{1j+1}) \\
&+ ij_-\tilde{s}_{0j-1} + ij_+\tilde{s}_{0j+1} \\
&= 4\delta_{0j} + \alpha(\tilde{s}_{0j}+\tilde{s}_{1j}) + (\beta+i)(j_-(\tilde{s}_{0j-1}+\tilde{s}_{1j-1}) + j_+(\tilde{s}_{0j+1}+\tilde{s}_{1j+1}))
\end{aligned}
$$

5.1.3 Case 3: $j = 1$

We can use a similar approach for expansion on $z\tilde{s}_{k0}$:

$$
\begin{aligned}
z\tilde{s}_{k0} &= \int_{-1}^{1}\int_{-1}^{1}\frac{zP_k(s)}{z-it-(\alpha+\beta t)(1+s)}dsdt \\
&= \int_{-1}^{1}\frac{zP_k(s)}{\beta(1+s)+i}\int_{-1}^{1}\frac{1}{\tilde{z}_s-t}dtds \\
&= \int_{-1}^{1}\frac{zP_k(s)}{\beta(1+s)+i}s_0(\tilde{z}_s)ds \\
&= \int_{-1}^{1}\frac{(z-\alpha(1+s))P_k(s)}{\beta(1+s)+i}s_0(\tilde{z}_s)ds + \int_{-1}^{1}\frac{\alpha(1+s)P_k(s)}{\beta(1+s)+i}s_0(\tilde{z}_s)ds \\
&= \int_{-1}^{1} P_k(s)\tilde{z}_s s_0(\tilde{z}_s)ds + \int_{-1}^{1}\frac{\alpha(1+s)P_k(s)}{\beta(1+s)+i}s_0(\tilde{z}_s)ds \\
&= \int_{-1}^{1} P_k(s)(2+s_1(\tilde{z}_s))ds + \alpha(\tilde{s}_{k0} + k_-\tilde{s}_{k-10} + k_+\tilde{s}_{k+10}) \\
&= (\beta+i)\tilde{s}_{k1} + \beta k_-\tilde{s}_{k-11} + \beta k_+\tilde{s}_{k+11} + \alpha(\tilde{s}_{k0} + k_-\tilde{s}_{k-10} + k_+\tilde{s}_{k+10})
\end{aligned}
$$

5.1.4 Issue with branch cuts

For the following cases we are forced to manipulate integral expressions involving logs. We will be using the log decomposition formula $log(ab) = log(a) + log(b)$. To highlight our issue lets imagine taking the integral:

$$
\int_0^1 f(x)log(abx)dx,\ arg(a) + arg(b) > \frac{\pi}{2}
$$

Here we see that if we try and decompose $log(abx)$ into $log(ax) + log(b)$, we will have to add a correction term since

$$
arg(abx) \neq (arg(ax) + arg(b) = arg(a) + arg(b) > \frac{\pi}{2})
$$

This is because of the branch cut at $\pm\pi$, $arg(abx) = arg(ab) < 0$ It is possible to easily fix this by removing a factor of 2π:

$$\int_0^1 f(x)log(abx)dx = \int_0^1 f(x)log(ax)dx + \int_0^1 f(x)(log(b) - 2\pi)dx$$

Although this is a very simple case which I am just using to illustrate the point we can get a much more complex case when the interval we are taking integrals over crosses over branch cuts. To solve this we find the value w such that $arg(z) + arg(w) = \frac{\pi}{2}$:

$$\int_{c_-}^{c_+} f(x)log(zx)dx, arg(z) + arg(c_-) > \frac{\pi}{2}$$

$$arg(z) + arg(c_+) < \frac{-\pi}{2}$$

$$\int_{c_-}^{c_+} f(x)log(zx)dx = \int_{c_-}^{w} f(x)(log(zx) - 2\pi)dx + \int_{w}^{c_+} f(x)log(zx)dx$$

Before moving to the next two cases we need closed form solutions to the following:

First we define:

$$r_j = \int_{-1}^1 P_j(t)s_0(\tilde{z}_t)dt$$

It is easy to see here that $r_j = s_{0j}$; we are able to very easily find a recurrence relation for s_{0j}:

$$\begin{aligned}
r_j &= \int_{-1}^1 P_j(t)s_0(\tilde{z}_t)dt \\
&= \int_{-1}^1 P_j(t)log(\frac{\tilde{z}_t + 1}{\tilde{z}_t - 1})dt \\
&= \int_{-1}^1 P_j(t)log(\frac{z - it}{z - 2\alpha - (2\beta + i)t})dt \\
&= \int_{-1}^1 P_j(t)log(z - it)dt - \int_{-1}^1 P_j(t)log(z - 2\alpha - (2\beta + i)t)dt \\
&= \int_{-1}^1 P_j(t)log(z - it)dt - \int_{-1}^1 P_j(t)(log(2\beta + i) + log(\frac{z - 2\alpha}{2\beta + i} - t) + C_1(t))dt
\end{aligned}$$

Where $C_1(t)$ is a correction term as a result of branch cuts from splitting up logs which we will derive later.

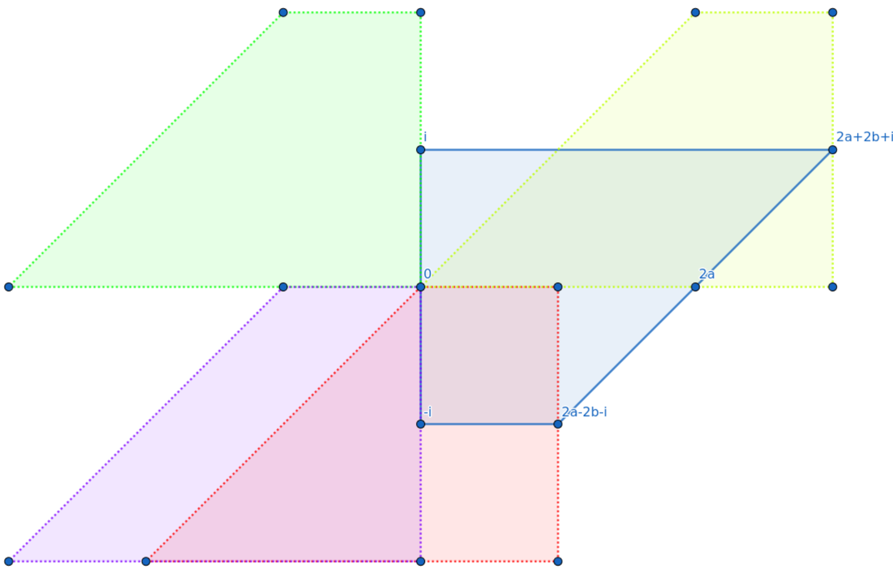
We also need to show that it is valid to split up the other logs and prove that there will not exist any branch cuts. When going through the cases $log(f(t)g(t))$ we are verifying such that for every t:

$$|arg(f(t)) + arg(g(t))| < \pi$$

First we address:

$$log(\frac{z-it}{z-2\alpha-(2\beta+i)t})$$

This problem is potentially susceptible to strange branch cuts but this problem is simplified greatly if we only consider the feasible set of z.



In this diagram we can see various trapeziums drawn around the main trapezium we are integrating over. The right edge and the slanted left edge of each trapezium around the main trapezium are the intervals $z-it, z-2\alpha-(2\beta+i)t, t\in[-1,1]$ respectively. Each trapezium represents what happens if we take our z to be a different corner of the main trapezium. Since our problem is one of taking the stieltjes kernel across a trapezium domain, we are allowed to restrict values of z from not being inside this domain:

$$z \notin \{z : Im(z) \in [-1,1],\ Re(z) \in [0, 2\alpha + 2\beta \cdot Im(z)],\ a \geq b > 0\}$$

Going back to our diagram this helps us see that not taking z from within the triangle ensures that the origin is always "outside" which saves us a few cases of branch cut proofs. We can now consider two cases. let:

$$\begin{aligned}
z_1 &= z - it \\
z_2 &= z - 2\alpha - 2\beta t - it \\
k_t &= 2\alpha + 2\beta t \\
&\implies (Re(z_1) - k_t, Im(z_1)) = (Re(z_2), Im(z_2))
\end{aligned}$$

Case 1 ($Im(z_1) = Im(z_2) > 0$ - Origin lies below):

$$arg(z_1) + arg(z_2^{-1}) = arg(z_1) - arg(z_2)$$

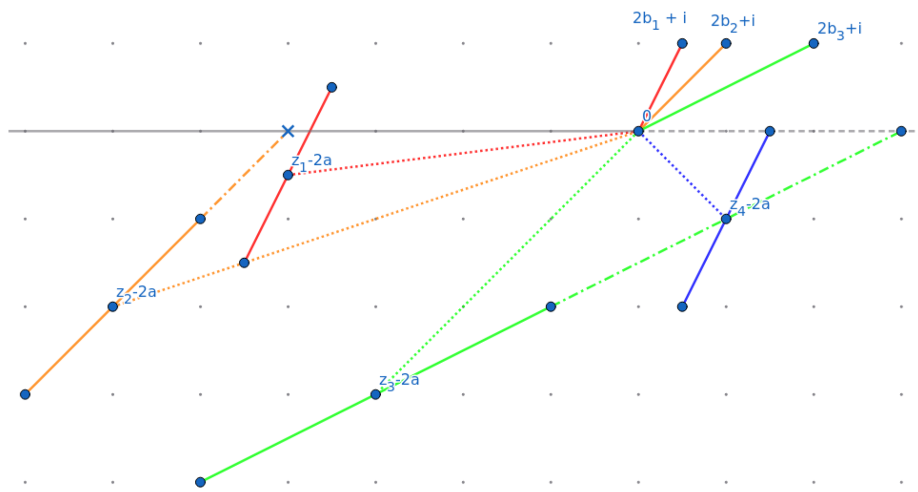
$$arg(z_1) > 0,\ arg(z_2) < \pi \implies arg(z_1) - arg(z_2) > -\pi$$
$$arg(z_1) < arg(z_2) \implies arg(z_1) - arg(z_2) < 0$$

Case 2 ($Im(z_1) = Im(z_2) < 0$ - Origin lies above):

$$arg(z_1) + arg(z_2^{-1}) = arg(z_1) - arg(z_2)$$
$$arg(z_1) < 0,\ arg(z_2) > -\pi \implies arg(z_1) - arg(z_2) < \frac{\pi}{2}$$
$$arg(z_1) > arg(z_2) \implies arg(z_1) - arg(z_2) > 0$$

Next we address

$$log(z_2) = log(z - 2\alpha - (2\beta + i)t) = log(2\beta + i) + log(\frac{z - 2\alpha}{2\beta + i} - t) := log(c_1) + log(z_3)$$



The image above helps illustrate how different intervals of z_2 look like for various values of $2\beta + i$. As $b > 0$, the $arg(2\beta + i) \in (0, \frac{\pi}{2})$. Also we note that in z_3 the interval is flat. This should make sense as we have an interval of angle $2\beta + i$ and we are rotating this interval by the same angle. Therefore the question of branch crossing is reduced down to wether or not the whole interval crosses the line.

Case 1 ($Im(z) > 1$):

$$Im(z) > 1 \implies Im(z_2) > 1,\ arg(z_2) \in (0, \pi)$$
$$arg(2\beta + i) \in (0, \frac{\pi}{2}) \implies arg(z_3) = arg(z_2) - arg(2\beta + i) \in (-\frac{\pi}{2}, \pi)$$

Case 2 ($arg(z - 2\alpha) > -\pi + arg(2\beta + i)$) Rough idea here is that it is never lifted clockwise over the branch cut. Again this will be shown using diagrams Case 3 ($arg(z - 2\alpha) < -\pi + arg(2\beta + i)$) This means the whole interval is lifted over the branch cut.

$$t_1 := min\{max\{\inf_t Im(z - 2\alpha - (2\beta + i)t) < 0, -1\}, 1\}$$

$$
\begin{aligned}
\int_{-1}^{1} f(t)log(z_2)dt = &\int_{-1}^{t} f(t)(log(2\beta + i) + log(z_3))dt \\
&+ \int_{t}^{1} f(t)(log(2\beta + i) + log(z_3) - 2\pi i)dt
\end{aligned}
$$

Case 4 ($z - 2\alpha - 2\beta Im(z) > 0$, $|Im(z)| \leq 1$): This is just another way of saying that when parameterising z_2 with t, the intercept of this line with the real axis is greater than 0 and so does not pass through the branch cut. This means we have a case of rotation that does not cross any branch cut (this is because the rotation will result in a flat line). TODO:Proof for these and diagrams.

Secondly we should also define:

$$
\begin{aligned}
q_k &= \int_{-1}^{1} P_k(s)s_0(\tilde{z}_s)ds \\
&= \int_{-1}^{1} P_k(s)log(\frac{\tilde{z}_s + 1}{\tilde{z}_s - 1})ds = \int_{-1}^{1} P_k(s)log(\frac{z + i - (\alpha - \beta)(1 + s)}{z - i - (\alpha + \beta)(1 + s)})ds \\
&= \int_{-1}^{1} P_k(s)log(z + i - (\alpha - \beta)(1 + s))ds \\
&- \int_{-1}^{1} P_k(s)log(z - i - (\alpha + \beta)(1 + s))ds + \int_{-1}^{1} C_2(s)ds \\
&= \int_{-1}^{1} P_k(s)(log(\alpha - \beta) + log(\frac{z + i}{\alpha - \beta} - 1 - s))ds \\
&- \int_{-1}^{1} P_k(s)(log(\alpha + \beta) + log(\frac{z - i}{\alpha + \beta} - 1 - s))ds + \int_{-1}^{1} C_2(s)ds \\
&= L_k(\frac{z - i}{\alpha - \beta} - 1) - L_k(\frac{z - i}{\alpha + \beta} - 1) + 2\delta_{k0}log(\frac{\alpha - \beta}{\alpha + \beta}) + \int_{-1}^{1} C_2(s)ds
\end{aligned}
$$

In the case where we are attempting to solve over a triangle i.e. $\alpha = \beta$ we have:

$$
\begin{aligned}
q_k &= \int_{-1}^{1} P_k(s)s_0(\tilde{z}_s) \\
&= \int_{-1}^{1} P_k(s)log(z + i - (\alpha - \beta)(1 + s))ds \\
&- \int_{-1}^{1} P_k(s)log(z - i - (\alpha + \beta)(1 + s))ds + \int_{-1}^{1} C_2(s)ds \\
&= \int_{-1}^{1} P_k(s)log(z + i)ds - \int_{-1}^{1} P_k(s)log(z - i - 2\alpha(1 + s))ds + \int_{-1}^{1} C_2(s)ds \\
&= 2\delta_{k0}log(z + i) - \int_{-1}^{1} P_k(s)(log(2\alpha) + log(\frac{z - i}{2\alpha} - (1 + s)))ds + \int_{-1}^{1} C_2(s)ds
\end{aligned}
$$

$$= 2\delta_{k0}(log(z+i) - log(2\alpha)) - L_k(\frac{z-i}{2\alpha} - 1) + \int_{-1}^{1} C_2(s)ds$$

First we look for the correction term C_2 in the expression:

$$log(\frac{z_4}{z_5}) = log(z_4) - log(z_5)$$

$$z_4 = z + i - (\alpha - \beta)(1+s),\ z_5 = z - i - (\alpha + \beta)(1+s)$$

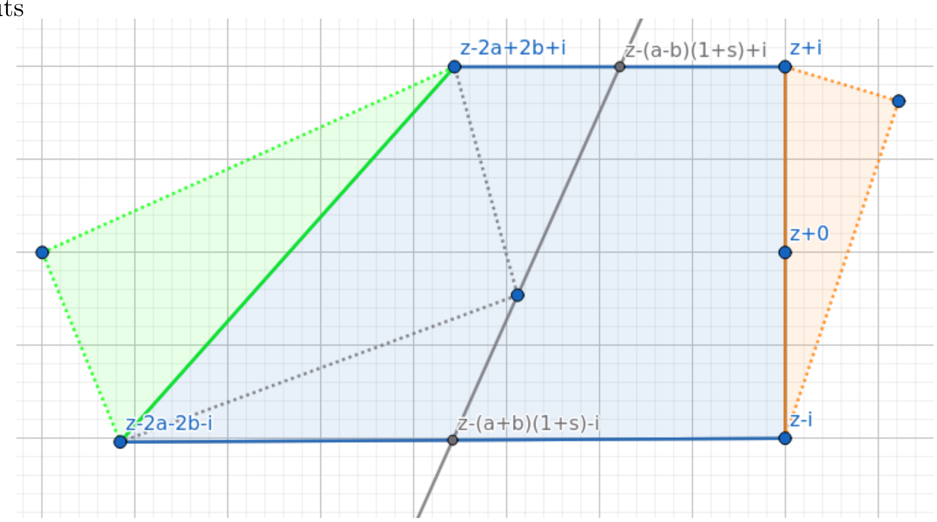
Case 1 ($Im(z) > 1$):

$$\begin{aligned} Im(z) > 1 &\implies 0 < arg(z_4), arg(z_5) < \pi \\ &\implies |arg(z_4) - arg(z_5)| < \pi \end{aligned}$$

Case 2 ($Im(z) < 1$):

$$\begin{aligned} Im(z) < 1 &\implies -\pi < arg(z_4), arg(z_5) < 0 \\ &\implies |arg(z_4) - arg(z_5)| < \pi \end{aligned}$$

For the next 3 cases it is important to understand the geometry of these branch cuts



Here we have a representation of z_4 and z_5 parameterised by $s \in [-1, 1]$. We can imagine as s varies, a line connecting z_4 and z_5 moving leftwards. It will be useful for us later if we are able to take some value of $\tilde{z}$ and find the value s which would a produce a line going through it. To simplify things we can recentre by shifting by z $z \leftarrow (z - z = 0)$ and let $\tilde{z} \leftarrow \tilde{z} - z$. This operation can be undone at the end. Moving the point 0 along with the line towards the left gives us the point $-\alpha(1+s)$ and we can find the gradient to write the following:

$$\begin{aligned} (\beta(1+s) + i)t - \alpha(1+s) &= \tilde{z} =: x + iy \\ \implies t &= y \end{aligned}$$

$$\begin{aligned}\beta(1+s)y - \alpha(1+s) &= x \\ (1+s)(y\beta - \alpha) &= x \\ s &= \frac{Re(\tilde{z})}{Im(\tilde{z})\beta - \alpha} - 1\end{aligned}$$

The usefulness of this formulation is that we are now able to take any arbritary point and find which region it belongs to in the above diagram. Also given an s for a z, we can see that $arg(z_4) - arg(z_5)$ from s onwards is greater than π and passes over the branch cut whereas when it takes on values under s, we have no issues with branch cuts. We have the 3 different cases for where the origin can be relative to our value of z which we can go through one by one:

Case 3 ($Re(z) < 0$):

$$\begin{aligned}arg(z - (\alpha - \beta)(1+s) + i) &> arg(z + i) \\ arg(z - (\alpha + \beta)(1+s) - i) &< arg(z - i) \\ \implies arg(z_4) - arg(z_5) &> arg(z+i) - arg(z-i) \\ &> \pi\end{aligned}$$

Case 4 ($\frac{Re(0-z)}{Im(0-z)\beta - \alpha} > 2 \implies \frac{Re(z)}{\alpha + Im(z)\beta} > 2$):

$$\begin{aligned}arg(z - (\alpha - \beta)(1+s) + i) &< arg(z - 2(\alpha - \beta) + i) \\ arg(z - (\alpha + \beta)(1+s) - i) &> arg(z - 2(\alpha + \beta) - i) \\ \implies arg(z_4) - arg(z_5) &< arg(z - 2(\alpha - \beta) + i) - arg(z - 2(\alpha + \beta) - i) \\ &< \pi\end{aligned}$$

Case 5 ($s + 1 := \frac{Re(z)}{\alpha + Im(z)\beta} \in [0, 2]$):

$$\begin{aligned}s \in [-1, 1] &\implies \\ u \in [0, s+1] &\implies arg(z - (\alpha - \beta)u + i) - arg(z - (\alpha + \beta)u - i) \leq \pi \\ s + 1 < u \leq 2 &\implies arg(z - (\alpha - \beta)u + i) - arg(z - (\alpha + \beta)u - i) > \pi\end{aligned}$$

From these 3 cases, we can come up with a correction using the following:

$$s = max\{0, min\{\frac{Re(z)}{\alpha + Im(z)\beta}, 2\}\} - 1$$

$$\begin{aligned}\int_{-1}^{1} P_k(u) log(\frac{z_4}{z_5}) du &= \int_{-1}^{s} P_k(u)(log(z_4) - log(z_5)) du \\ &+ \int_{s}^{1} P_k(u)(log(z_4) - log(z_5) - 2\pi i) du \\ &= \int_{-1}^{1} P_k(u)(log(z_4) - log(z_5)) du - 2\pi i \int_{s}^{1} P_k(u) du \\ &= \int_{-1}^{1} P_k(u)(log(z_4) - log(z_5)) du - 2\pi i C_{k+1}^{(-1/2)}(s)\end{aligned}$$

$$\int_{-1}^{1} C_2(s)ds = -2\pi i C_{k+1}^{(-1/2)}(s)$$

This gives our final expression for q_k:

$$q_k = L_k(\frac{z-i}{\alpha-\beta} - 1) - L_k(\frac{z-i}{\alpha+\beta} - 1) + 2\delta_{k0} log(\frac{\alpha-\beta}{\alpha+\beta}) - 2\pi i C_{k+1}^{(-1/2)}(s)$$

5.1.5 Case 4: $k = 0$

For this case we are considering how to solve the following:

$$\begin{aligned}\tilde{s}_{0j} &= \int_{-1}^{1} \frac{P_j(t)}{\alpha+\beta t} s_0(\tilde{z}_t)dt \\ &= \int_{-1}^{1} \frac{P_j(t)}{\alpha+\beta t} log(\frac{\tilde{z}_t+1}{\tilde{z}_t-1})dt\end{aligned}$$

This is difficult to find a closed form solution for, especially for all values of j. We will later be forced into solving this problem with the use of dilogarithms, but we can do it in a way where we only have to do this once rather than for every value of j.

We can do the use the fact that we are able to compute all $r_j = s_{0j}$ to help us compute $\tilde{s}_{0j}$. It is easy to see the relation between s_{0j} and $\tilde{s}_{0j}$ as:

$$\begin{aligned}j > 0 : s_{0j} &= \int_{-1}^{1} P_j(t) s_0(\tilde{z}_t)dt \\ &= \int_{-1}^{1} \frac{(\alpha+\beta t)P_j(t)}{\alpha+\beta t} s_0(\tilde{z}_t)dt \\ &= \alpha\tilde{s}_{0j} + \beta(j_-\tilde{s}_{0j-1} + j_+\tilde{s}_{0j+1}) \\ j = 0 : s_{00} &= \alpha\tilde{s}_{00} + \beta\tilde{s}_{01}\end{aligned}$$

Given a value for $\tilde{s}_{00}$ we are therefore easily able to compute all values of $\tilde{s}_{0j}$ recursively.

5.1.6 Case 5: $j = 0$

$$\begin{aligned}\tilde{s}_{k0} &= \int_{-1}^{1} \frac{P_k(s)}{\beta(1+s)+i} s_0(\tilde{z}_s)ds \\ &= \int_{-1}^{1} \frac{P_k(s)}{\beta(1+s)+i} log(\frac{\tilde{z}_s+1}{\tilde{z}_s-1})ds\end{aligned}$$

Again for the same reasons of not being able to find a closed form solution to this we look for a q_k and use this to find $\tilde{s}_{k0}$.

$$k > 0 : q_k = \int_{-1}^{1} P_k(s) s_0(\tilde{z}_s)ds$$

$$
\begin{aligned}
&= \int_{-1}^{1} \frac{(\beta(1+s)+i)P_k(s)}{\beta(1+s)+i} s_0(\tilde{z}_s) ds \\
&= (\beta+i)\tilde{s}_{k0} + \beta(k_-\tilde{s}_{k-10} + k_+\tilde{s}_{k+10}) \\
k = 0 : q_0 &= (\beta+i)\tilde{s}_{00} + \beta\tilde{s}_{10}
\end{aligned}
$$

Given a value of $\tilde{s}_{00}$ we are able to compute all values of $\tilde{s}_{k0}$ recursively. Now the challenge remains to find this $\tilde{s}_{00}$.

5.1.7 Case 6: $k, j = 0, 0$

We are looking for the following:

$$
\begin{aligned}
\tilde{s}_{00} &= \int_{-1}^{1} \frac{s_0(\tilde{z}_t)}{\alpha+\beta t} dt = \int_{-1}^{1} \frac{1}{\alpha+\beta t} log(\frac{\tilde{z}_t+1}{\tilde{z}_t-1}) dt \\
&= \int_{-1}^{1} \frac{1}{\alpha+\beta t} log(\frac{z-it}{z-it-2(\alpha+\beta t)}) dt \\
&= \int_{-1}^{1} \frac{1}{\alpha+\beta t} log(\frac{z-it}{z-2\alpha-(2\beta+i)t}) dt \\
&= \int_{-1}^{1} \frac{log(z-it)}{\alpha+\beta t} - \int_{-1}^{1} \frac{z-2\alpha-(2\beta+i)t}{\alpha+\beta t} dt \\
&= \frac{1}{\beta}(\int_{-1}^{1} \frac{log(z-it)}{\alpha/\beta+t} - \int_{-1}^{1} \frac{z-2\alpha-(2\beta+i)t}{\alpha/\beta+t} dt)
\end{aligned}
$$

We need to be able to solve solutions of the following from:

$$
\int_{-1}^{1} \frac{log(z+ct)}{b+t} dt
$$

Begin by considering decomposing $log(z+ct) \to log(\frac{z}{c}+t) + log(c)$. Rigorously we must show that we do not cross a branch cut for all values of t, however, we can notice that the values of t in the decomposed form is along a horizontal line. We can assume that this interval does not cross over 0 as our integral needs to be defined. Because of this we can say that the whole interval either stays or crosses over the cut. We can therefore add a correction term $(2\pi i)$ depending on $|arg(c) + arg(z/c)|$ and no correction term if this stays under π. We can move forward by denoting as C_3:

$$
\begin{aligned}
\int_{-1}^{1} \frac{log(z+ct)}{b+t} dt &= \int_{-1}^{1} \frac{log(z/c+t)}{b+t} + \frac{log(c)}{b+t} dt + C_3 \\
&= \int_{-1}^{1} \frac{log(z/c+t)}{b+t} dt + log(c)(log(b+1) - log(b-1)) + C_3
\end{aligned}
$$

Now we have an integral in the form $\int_{-1}^{1} \frac{log(a+t)}{b+t} dt$. In the case where $a = b$:

$$
\int_{-1}^{1} \frac{log(b+t)}{b+t} dt = \int_{b-1}^{b+1} \frac{log(t)}{t} dt
$$

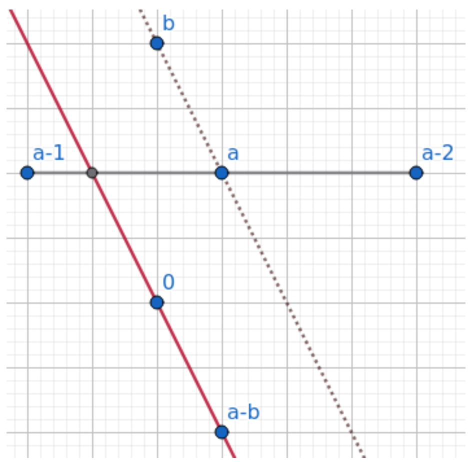
$$u = log(t) \implies = \int_{log(b-1)}^{log(b+1)} udu$$
$$= (log(b+1)^2 - log(b-1)^2)/2$$

Otherwise if $a \neq b$:

$$\int_{-1}^{1} \frac{log(a+t)}{b+t} dt = \int_{b-1}^{b+1} \frac{log(a-b+t)}{t} dt$$

We know go about finding correction terms for: $log(a-b+t) \to log(a-b) + log(1 + \frac{t}{a-b})$ as $a - b \in \mathbb{C}$ We have a branch crossing if for some $t \in [-1, 1]$:

$$|arg(a+t) - arg(a-b)| > \pi$$



Above we can see an example of a, b values which results in a branch crossing over the positive side. The red line denotes the angle at which we rotate when dividing by $a - b$. This means that anything on the left over this line crosses the branch cut whilst anything to the right does not. Same thing applies in the negative version where the interval of a is in the negative case rotated clockwise over the branch cut.Therefore, we first have to check that a cut exists. There are many edge cases here which I will not delve into but the rough idea is to check through the args of $a-1, a+1, a-b$ and make comparisons to determine any crossings. With that we have handled cases where there are either no crossings or where the whole interval has crossed. It is possible to formulate a way using trig functions to find a value of t where $arg(a+t) = \pi - arg(a-b)$ in the positive case and $arg(a+t) = -\pi + arg(a-b)$ in the negative case. let $a-b+w = a+t$ so that we have a w which is a representation of the position of our intercept in the $[b-1, b+1]$ interval. In the case where either the whole interval crosses the branch cut or none of it does we can set it to be the appropriate edge $w \in b-1, b+1$. This is useful since we are then able to represent the integral as follows:

$$\int_{-1}^{1} \frac{log(a+t)}{b+t} dt = \int_{b-1}^{b+1} \frac{log(a-b+t)}{t} dt =: \int_{b-1}^{b+1} \frac{log(d+t)}{t} dt$$

$$
\begin{aligned}
&= \int_{b-1}^{w} \frac{\pm 2\pi i}{t} dt + \int_{b-1}^{b+1} \frac{log(d) + log(1 + t/d)}{t} dt \\
-u = \frac{t}{d}, du = -\frac{dt}{d} : &= \pm 2\pi i log(\frac{w}{b-1}) + log(a-b) log(\frac{b+1}{b-1}) \\
&+ \int_{-\frac{b-1}{a-b}}^{-\frac{b+1}{a-b}} \frac{log(1-u)}{u} du
\end{aligned}
$$

We can try to solve this using the dilogarithm function:

$$
Li_2(z) = -\int_0^z \frac{log(1-u)}{u} du, \; z \in \mathbb{C}
$$

We still have to be careful however since there is a branch cut for the dilogarithm function at $(1, \infty)$ at which $Li_2(z + i0^+) - Li_2(z + i0^-) = 2\pi i log(z)$, $z \in (1, \infty)$. Ignoring edge cases which are less interesting, this can be computed by finding the x-intercept of the interval $[-\frac{b+1}{a-b}, -\frac{b-1}{a-b}]$ and checking is this x intercept is part of the branch cut. We can let v denote the point at which the branch cut is crossed where $v = 1$ if there is no crossing through this cut and then we can write:

$$
-\int_{-\frac{b-1}{a-b}}^{-\frac{b+1}{a-b}} \frac{log(1-u)}{u} du = Li_2(-\frac{b+1}{a-b}) - Li_2(-\frac{b-1}{a-b}) \pm 2\pi i log(v)
$$

Now that we have all the individual elements we can simply plug everything back in to compute a value for $\tilde{s}_{00}$

5.1.8 Recovery of s_{kj}

Now we have a situation where we have all values of $\tilde{s}_k j$ but no values for s_{kj}. The basic idea to recover all s_{kj} was by adding consecutive rows together:

$$
s_{kj} = \int_{-1}^{1} P_j(t) s_k(\tilde{z}_t) dt = \int_{-1}^{1} \frac{\alpha + \beta t}{\alpha + \beta t} P_j(t) s_k(\tilde{z}_t) dt = \alpha \tilde{s}_{kj} + \beta \tilde{s}_{kj+1}
$$

6 Log kernel

Now we move onto solving over the 2d log kernel. This is a more useful version of the stieltjes case as we are able to use this to solve the poisson PDE:

$$
\begin{aligned}
\Delta u(\mathbf{x}) &= f(\mathbf{x}) \; \mathbf{x} \in \Omega \\
u(\mathbf{x}) &= g(\mathbf{x}), \mathbf{x} \in \partial\Omega \\
\partial_n u(\mathbf{x}) &= h(\mathbf{x}), \mathbf{x} \in \partial\Omega
\end{aligned}
$$

In this section we will be trying to solve the following which is similar from the previous Stieltjes kernel:

$$
L_{kj} = \int_{-1}^{1} (\alpha + \beta t) \int_{-1}^{1} P_k(s) P_j(t) log(z - it - (\alpha + \beta t)(1 + s)) ds dt
$$

Again it is useful to simplify this to solve:

$$\tilde{L}_{kj} = \int_{-1}^{1}\int_{-1}^{1} P_k(s)P_j(t)log(z - it - (\alpha + \beta t)(1 + s))dsdt$$

Recovering L_{kj} is as simple as:

$$\begin{aligned} L_{kj} &= \alpha\tilde{L}_{kj} + \beta\int_{-1}^{1}\int_{-1}^{1} tP_j(t)P_k(s)log(z - it - (\alpha + \beta t)(1 + s))dsdt \\ j = 0 \implies L_{k0} &= \alpha\tilde{L}_{k0} + \beta\tilde{L}_{k1} \\ j > 0 \implies L_{kj} &= \alpha\tilde{L}_{kj} + \beta(\frac{j}{2j+1}\tilde{L}_{kj} + \frac{j+1}{2j+1}\tilde{L}_{kj}) \end{aligned}$$

6.1 Useful rearrangements

Even with this simplification we will see that there are further useful simplifications we can do to uncover the recurrence relations. The motivation is that we need some sort of recurrence to relate to this log transform and a suitable relation to use is $L_k(z) := \int_{-1}^{1} P_k(s)log(z - s)ds$. However, in order to use this we need to bring L_{kj} into a similar format:

$$\begin{aligned} log(z - it - (\alpha + \beta t)(1 + s)) &= log((\alpha + \beta t)\frac{z - it}{\alpha + \beta t} - 1 - s) \\ &= log(\alpha + \beta t) + log(\frac{z - it}{\alpha + \beta t} - 1 - s) \\ &= log(\alpha + \beta t) + log(\tilde{z}_t - s) \end{aligned}$$

Now integrating over s allows us to use the L_k recurrences. Plugging this into our original equation for $\tilde{L}_{kj}$ denoting $b_{kj} = P_k(s)P_j(t)$ for ease:

$$\begin{aligned} \tilde{L}_{kj} &= \int_{-1}^{1}\int_{-1}^{1} b_{kj}log(z - it - (\alpha + \beta t)(1 + s))dsdt \\ &= \int_{-1}^{1}\int_{-1}^{1} b_{kj}(log(\alpha + \beta t) + log(\tilde{z}_t - s))dsdt \\ &= \int_{-1}^{1} 2P_j(t)log(\alpha + \beta t)\delta_{k0}dt + \int_{-1}^{1} P_j(t)L_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} 2P_j(t)log(\alpha + \beta t)\delta_{k0}dt + L^1_{kj}(z) \\ O^1_{kj} :&= \tilde{L}_{kj} - L^1_{kj} = 2\int_{-1}^{1} P_j(t)log(\alpha + \beta t)\delta_{k0}dt \\ &= 2\delta_{k0}\int_{-1}^{1} P_j(t)(log(\beta) + log(\frac{\alpha}{\beta} + t))dt \end{aligned}$$

$$= 2\delta_{k0}(2log(\beta)\delta_{j0} + \begin{cases} \int_{-1}^{1} P_j(t)log(\frac{\alpha}{\beta} - t)dt = L_j(\alpha/\beta), & j \text{ even} \\ \int_{-1}^{1} -P_j(t)log(\frac{\alpha}{\beta} - t)dt = -L_j(\alpha/\beta), & j \text{ odd} \end{cases})$$

Breaking this down gives us the following:

$$O^1_{kj} = \begin{cases} 4log(\beta) + 2L_0(\alpha/\beta), & k, j = 0, 0 \\ 2L_j(\alpha/\beta), & k = 0, j > 0, j \text{ even} \\ -2L_j(\alpha/\beta), & k = 0, j > 0, j \text{ odd} \end{cases}$$

The usefulness of this is that we can now convert between solving recurrences on $\tilde{L}_{kj}$ and L^1_{kj}. We can attempt a similar thing if we want to integrate over the t instead:

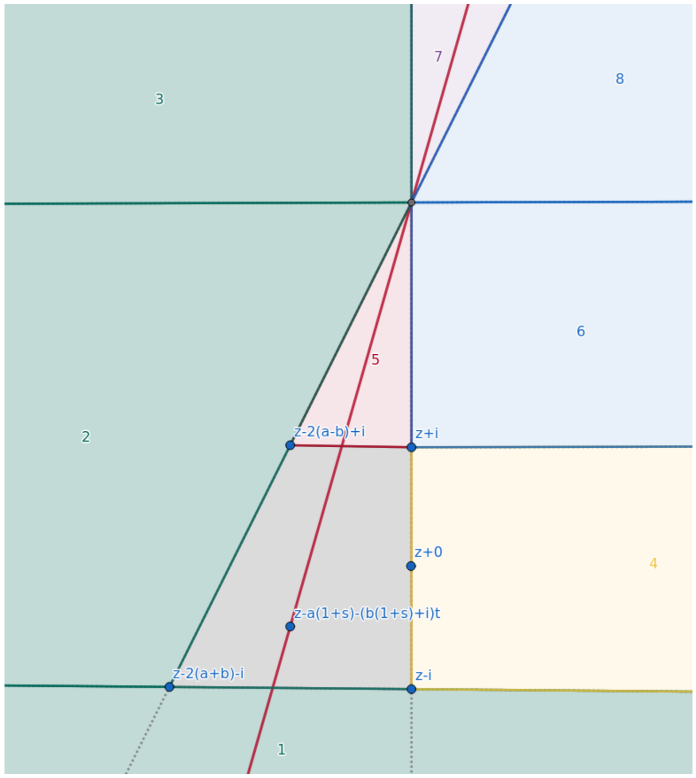
$$\begin{aligned} log(z - it - (\alpha + \beta t)(1 + s)) &= log(z - it - \alpha(1 + s) - t\beta(1 + s)) \\ &= log(z - (\beta(1 + s) + i)t - \alpha(1 + s)) \\ &= log((\beta(1 + s) + i)(\frac{z - \alpha(1 + s)}{\beta(1 + s) + i} - t)) \\ &= log((\beta(1 + s) + i)(\tilde{z}_s - t)) \\ &= log(\beta(1 + s) + i) + log(\tilde{z}_s - t) + C(s, t) \\ &\text{where } C(s, t) \text{ branch correction term} \end{aligned}$$

Plugging back into equation for $\tilde{L}_{kj}$ gives:

$$\begin{aligned} \tilde{L}_{kj} &= l \int_{-1}^{1} \int_{-1}^{1} b_{kj} log(z - it - (\alpha + \beta t)(1 + s))dsdt \\ &= \int_{-1}^{1} \int_{-1}^{1} b_{kj}(log(\beta(1 + s) + i) + C(s, t)) + b_{kj} log(\tilde{z}_s - t)dsdt \\ &= 2\delta_{j0} \int_{-1}^{1} P_k(s)(log(\beta) + log(\frac{\beta + i}{\beta} + s))ds + \int_{-1}^{1} P_k(s)L_j(\tilde{z}_s)ds \\ &+ \int_{-1}^{1} \int_{-1}^{1} b_{kj} C(s, t)dtds \\ &= 4log(\beta)\delta_{j0}\delta_{k0} + 2\delta_{j0}(\begin{cases} L_k(\frac{\beta+i}{\beta}), & k \text{ even} \\ -L_k(\frac{\beta+i}{\beta}), & k \text{ odd} \end{cases}) + L^2_{kj} \\ &+ \int_{-1}^{1} \int_{-1}^{1} b_{kj} C(s, t)dtds \\ O^2_{kj} &:= \tilde{L}_{kj} - L^2_{kj} \end{aligned}$$

This is where we run into an issue of branch cuts. We will see how to deal with these and decompose $log((\beta(1+s)+i)\tilde{z}_s)$. With our initial expression $log(z-it-(\alpha+\beta t)(1+s)$ we can rearrange into $log(z-\alpha(1+s)-(\beta(1+s)+i)t)$.

Fixing s, shows that t parameterises a diagonal line on the complex plane in the direction $\beta(1+s)+i$ centered on $z-(1+s)\alpha$ as you can see from the diagram below.



It is possible to think of changing the value of s as moving the line while it pivots around that intersection. It is simple to compute the value of that intersection and establish cases for computing branch cuts depending on our value of z. We can do this by taking two values s, s' and trying to find t such that the real component is the same:

$$\begin{aligned} Re(z) - \alpha(1+s) - \beta(1+s)t &= Re(z) - \alpha(1+s') - \beta(1+s')t \\ -\alpha s - \beta st &= -\alpha s' - \beta s' t \\ \alpha(s'-s) &= \beta(s-s')t \implies t = -\frac{\alpha}{\beta} \end{aligned}$$

Also, it is neccessary to compute a value of "s" of the origin relative to z to reason about which case we should follow:

$$\begin{aligned} z - \alpha(1+s^*) - (\beta(1+s^*)+i)t &= 0 \\ Im(z) - t = 0 &\implies t = Im(z) \\ Re(z) - (\alpha+\beta t)(1+s^*) &= 0 \end{aligned}$$

$$\frac{Re(z)}{\alpha+\beta t} = \frac{Re(z)}{\alpha+\beta Im(z)} = s^*+1$$

To consider cases of branch cuts, we think about points where we can place the origin relative to this z. Since we cannot have our z in the trapezium, we will restrict ourselves to placing the origin in areas such that the z is outside the trapezium. Therefore we cannot place our origin in this greyed out region.

6.1.1 Case 1: $Im(z) > 1$

$$\begin{aligned} Im(z) > 1 &\implies arg(z) \in (0,\pi),\ arg(b+(1+s)i) \in (0,\frac{\pi}{2}) \\ &\implies arg(z) - arg(b+(1+s)i) > 0 - \frac{\pi}{2} > -\pi \\ \text{and } & arg(z) - arg(b+(1+s)i) < \pi - 0 < \pi \end{aligned}$$

6.1.2 Case 2: $s^* > 1$ and $Im(z) \in [-\frac{\alpha}{\beta}, 1]$

TODO: Can use proof similar to case 5. In this case we have the origin in the left green region. This means that no paths crosses the branch cut and if at all we cross in the positive real numbers, which trivially lets us infer there is no branch cut issues here

6.1.3 Case 3: $Im(z) < -\frac{\alpha}{\beta}$ and $Re(z) > 0$

TODO: Show proof that there is no branch cut crossing here

6.1.4 Case 4: $Re(z) < 0$ and $Im(z) \in (-1, 1)$

We have that for all s, the integral crosses the branch cut at $t^* = Im(z)$. Since the operation of dividing by $b(1+s)+i$ rotates the integral onto a flat line, we have that for all s, each integral is lifted over the branch cut. Therefore we compensate by the correction: $-2\pi i \mathbb{I}\{t > t^*\}$ Going back to finding out correction term we have:

$$\begin{aligned} \int_{-1}^{1}\int_{-1}^{1} -2\pi i \mathbb{I}\{t > t^*\} P_k(s)P_j(t)dsdt &= -2\pi i \int_{-1}^{1} 2\delta_{k0}P_j(t)\mathbb{I}\{t > t^*\}dt \\ &= -4\delta_{k0}\pi i \int_{t^*}^{1} P_j(t)dt \end{aligned}$$

6.1.5 Case 5: $s^* \in (-1, 1)$ and $Im(z) > -\frac{\alpha}{\beta}$

TODO: Make sure earlier section explains this and link to it Letting x^* be where the line of the integral crosses the real axis:

$$\begin{aligned} Im(z - (\alpha+\beta t)(1+s) - it) = 0 \\ Im(z) - t = 0 \implies t = Im(z) \end{aligned}$$

$$x^* = Re(z) - (\alpha + \beta Im(z))(1+s) = (\alpha + \beta Im(z))(s^* - s)$$
$$Im(z) > -\frac{\alpha}{\beta} \implies (x^* < 0 \iff s^* < s)$$

Given the earlier theorem, we have that if $s > s^*$ we can say that the whole interval for that s has crossed the branch cut and therefore, we compensate with the correction:$-2\pi i \mathbb{I}\{s > s^*\}$ for all t. Yields our final correction term:

$$\int_{-1}^{1}\int_{-1}^{1} -2\pi i \mathbb{I}\{s > s^*\} P_k(s)P_j(t)dtds = -2\pi i \int_{-1}^{1} 2\delta_{j0}\mathbb{I}\{s > s^*\}P_k(s)ds$$
$$= -4\delta\pi i \int_{s^*}^{1} P_k(s)ds$$

6.1.6 Case 6: $Im(z) \in (-\frac{\alpha}{\beta}, -1)$ and $Re(z) < 0$

This is similar to Case 5 except here we can almost think of the origin having an s^* value lower than -1, meaning that all the intervals of s are lifted completely above the branch crossing. We can use the $-2\pi i$ correction for every s, t in our integration:

$$\int_{-1}^{1}\int_{-1}^{1} -2\pi i P_k(s)P_j(t)dsdt = -8\pi i \delta_{k0}\delta_{j0}$$

6.1.7 Case 7: $Im(z) < -\frac{\alpha}{\beta}$ and $s^* \in (-1, 1)$

Also similar to case 5 except reversed. We will begin with the result from case 5:

$$x^* = Re(z) - (\alpha + \beta Im(z))(1+s) = (\alpha + \beta Im(z))(s^* - s)$$
$$Im(z) < -\frac{\alpha}{\beta} \implies (x^* < 0 \iff s^* > s)$$

The correction term here is then $-2\pi i \mathbb{I}\{s < s^*\}$:

$$\int_{-1}^{1}\int_{-1}^{1} -2\pi i \mathbb{I}\{s < s^*\} P_k(s)P_j(t)dtds = -2\pi i \int_{-1}^{1} 2\delta_{j0}\mathbb{I}\{s < s^*\}P_k(s)ds$$
$$= -4\delta\pi i \int_{-1}^{s^*} P_k(s)ds$$

6.1.8 Case 8: $s^* > 1$ and $Im(z) < -\frac{\alpha}{\beta}$

Also similar to case 5:

$$x^* = Re(z) - (\alpha + \beta Im(z))(1+s) = (\alpha + \beta Im(z))(s^* - s)$$
$$> (\alpha + \beta Im(z))(1 - s)$$
$$(\alpha + \beta Im(z)) < 0 \& 1 - s > 0 \implies x^* < 0$$

This means the whole interval must be corrected by $-2\pi i$ for each point and $-8\pi i \delta_{k0}\delta_{j0}$ for the whole interval like in case 6.

6.2 Recurrences

6.2.1 Note on notation

We will be continuing with the convention of representing the 3 term recurrence constants as follows:

$$tP_j(t) = j_- P_{j-1}(t) + j_+ P_{j+1}$$

To add onto this we will be using a recurrence relation of $L_k(z)$ as follows:

$$\begin{aligned} zL_k(z) &= \frac{k-1}{2k+1} L_{k-1}(z) + \frac{k+2}{2k+1} L_{k+1}(z) + \lambda_k(z) \\ &:= k_-^L L_{k-1}(z) + k_+^L L_{k+1}(z) + \lambda_k(z) \\ \lambda_k(z) &= \begin{cases} (z-1)log(z-1) + (z+1)log(z+1), & k = 0 \\ -2/3, & k = 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

6.2.2 Case 1: $k, j > 1$

$$\begin{aligned} z\tilde{L}_{kj} &= \int_{-1}^{1} zP_j(t)L_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} (z - \alpha - (\beta + i)t)P_j(t)L_k(\tilde{z}_t)dt + \int_{-1}^{1} (\alpha + (\beta + i)t)P_j(t)L_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} (\alpha + \beta t)\frac{z - it - (\alpha + \beta t)}{\alpha + \beta t} P_j(t)L_k(\tilde{z}_t)dt + \int_{-1}^{1} (\alpha + (\beta + i)t)P_j(t)L_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} (\alpha + \beta t)P_j(t)\tilde{z}_t L_k(\tilde{z}_t)dt + \int_{-1}^{1} (\alpha + (\beta + i)t)P_j(t)L_k(\tilde{z}_t)dt \\ &= \int_{-1}^{1} (\alpha + \beta t)P_j(t)(k_-^L L_{k-1}(\tilde{z}_t) + k_+^L L_{k+1}(\tilde{z}_t) - \frac{2}{3}\delta_{k1})dt \\ &+ \alpha\tilde{L}_{kj} + (\beta + i)(j_-\tilde{L}_{kj-1} + j_+\tilde{L}_{kj+1}) \\ &= \alpha(k_-^L\tilde{L}_{k-1j} + k_+^L\tilde{L}_{k+1j}) + \beta(k_-^L j_-\tilde{L}_{k-1j-1} + k_-^L j_+\tilde{L}_{k-1j+1} \\ &+ k_+^L j_-\tilde{L}_{k+1j-1} + k_+^L j_+\tilde{L}_{k+1j+1}) + \alpha\tilde{L}_{kj} + (\beta + i)(j_-\tilde{L}_{kj-1} + j_+\tilde{L}_{kj+1}) \\ &- \frac{2\beta}{3}\frac{1}{3}2\delta_{k1}\delta_{j1} \end{aligned}$$

6.2.3 Case 2: $k = 0$

7 Transformation

Up until now, we have been working in very fixed regions using the least amount of degrees of freedom possible for parameterisation. In this section we will explore how to resize and move around our region so we can achieve more complex structures by computing over smaller regions. We will be revisting the stieltjes over a square:

$$s_{kj} = \int_{-1}^{1}\int_{-1}^{1} \frac{P_k(s)P_j(t)}{z-(s+it)} dsdt$$

Let us also recall the way we approach function decomposition onto our function bases:

$$f \circ Q(\mathbf{x}) = \Sigma_{k,j} c_{k,j} b_{kj}(\mathbf{x}),\ b_{kj}(\mathbf{x}) = P_k(\mathbf{x}_0)P_j(\mathbf{x}_1)$$

The transformations which are available to us are operations which we can perform on z which lets us alter the region we are integrating over which is parameterised as $s+it$.

7.1 Translation

If we want to translate the region of interest from $[-1,1]\times[-1,1]$ to $[x-1, x+1], [y-1, y+1]$, we need to change the region to $x(s,t)+x+i(t+y)$ where $x(s,t)$ is a function that takes values from the space of s,t and gives out where s should be based on what shape we are working with. This lets us rewrite the denominator as $(z-x-iy)-(x(s,t)+it)$. Therefore the same recursion rules but with a different value of z gives us this.

7.2 Scaling and Rotation

The kernel of the stieltjes integral is in the form of $z-x(s,t)-it$ in the general case. This means that we can multiply this thie region of $x(s,t)-it$ by $\lambda \in$ The real component of this value contributes to the scaling and the complex component the rotation. We can therefore rewrite the scaled $s_{kj}(z)$ as $\frac{1}{\lambda}s_{kj}(\frac{z}{\lambda})$

8 Larger Meshes

We can add together results from individual triangles to get the result of a convolution over more complicated regions. Given that we are approximating functions to an order of p which results in p^2 basis functions, we have that for a single value of z we have $O(p^2)$ operation. If we naively add up all n shapes, we have $O(np^2)$ and now it has started to increase in size.