FYP

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1 Introduction

1.1 Cauchy Transform

$$C_\Gamma f(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}dt$$

This is analytic for $z \notin \Gamma$. Define Hilbert Transform to be the limits from the right and the left.

1.2 Orthogonal Polynomials

Family	Notation	Interval	$w(x)$
Legendre	$P_n(x)$	[-1,1]	1
Chebyshev (1st)	$T_n(x)$	[-1,1]	$(1-x^2)^{-1/2}$
Chebyshev (2nd)	$U_n(x)$	[-1,1]	$(1-x^2)^{1/2}$
Ultraspherical	$C_n^{(\lambda)}(x),\ \lambda > -\frac{1}{2}$	[-1,1]	$(1-x^2)^{\lambda-1/2}$
Jacobi	$P_n^{(\alpha,\beta)}(x),\ \alpha,\beta > -1$	[-1,1]	$(1-x)^\alpha(1-x)^\beta$

2 Log and Stieltjes Transform

In this section we will consider approaches to compute these weakly singular integrals

$$\int_A log||z-t||f(t)dt \qquad \int_A \nabla log||z-t||f(t)dt$$

$$\mathcal{S}_A f(z) := \int_A \frac{f(t)}{z-t}dt$$

$$\mathcal{L}_A f(z) := \int_A log(z-t)f(t)dt$$

Depending on the type of area which A is we can begin by approximating f using orthogonal polynomials.

2.1 Transforms across Intervals

We will try to formulate recurrence relations for these transforms across interval [-1, 1]. We are looking for looking for $\mathcal{S}_{[-1,1]}f(z)$. Decomposing $f(z) \approx \Sigma_k f_k P_k(z)$ and writing $S_k(z) := \mathcal{S}_{[-1,1]}P_k(z)$ lets us write:

$$\mathcal{S}_{[-1,1]}f(z) \approx \Sigma_k f_k S_k(z)$$

This motivates finding fast methods to compute $S_k(z)$. Log kernels are approached similarly letting $L_k(z) := \mathcal{L}_{[-1,1]}P_k(z)$ and looking for recurrence relations.

Stieltjes

Recall recurrence relation of Legendre Polynomials:

$$xP_k(x) = \frac{k}{2k+1}P_{k-1}(x) + \frac{k+1}{2k+1}P_{k+1}(x)$$

Formulate three-term recurrence for their Stieltjes transforms.

$$\begin{aligned}
zS_k(z) &= \int_{-1}^{1} \frac{zP_k(t)}{z-t}dt \\
&= \int_{-1}^{1} \frac{z-t}{z-t}P_k(t)dt + \int_{-1}^{1} \frac{tP_k(t)}{z-t}dt \\
&= \int_{-1}^{1} P_k(t)dt + \frac{k}{2k+1}\int_{-1}^{1} \frac{P_{k-1}(t)}{z-t}dt + \frac{k+1}{2k+1}\int_{-1}^{1} \frac{P_{k+1}(t)}{z-t}dt \\
&= 2\delta_{k0} + \frac{k}{2k+1}S_{k-1}(z) + \frac{k+1}{2k+1}S_{k+1}(z) \\
S_0(z) &= \int_{-1}^{1} \frac{dt}{z-t} = log(z+1) - log(z-1)
\end{aligned}$$

We can extend this to work over a square using the recurrence over intervals:

$$\begin{aligned}
zS_{k,j}(z) &= z\int_{-1}^{1}\int_{-1}^{1} \frac{P_k(s)P_j(t)}{z-(s+it)}dsdt \\
&= \int_{-1}^{1} zP_j(t)\int_{-1}^{1} \frac{P_k(s)}{z-it-s}dsdt \\
&= \int_{-1}^{1} (z-it)P_j(t)S_k(z-it) + itP_j(t)S_k(z-it)dsdt \\
&= \int_{-1}^{1} P_j(t)(\frac{k}{2k+1}S_{k-1}(z-it) + \frac{k+1}{2k+1}S_{k+1}(z-it) + 2\delta_{k0}) \\
&+ i(\frac{j}{2j+1}P_{j-1}(t) + \frac{j+1}{2j+1}P_{j+1}(t))S_k(z-it)dsdt \\
&= \frac{k}{2k+1}S_{k-1,j}(z) + \frac{k+1}{2k+1}S_{k+1,j} \\
&+ i\frac{j}{2j+1}S_{k,j-1}(z) + i\frac{j+1}{2j+1}S_{k,j+1} + 4\delta_{j0}\delta k0
\end{aligned}$$

Log

We can begin by connecting log kernel to the Stieltjes kernel. To do this we define:

$$S_k^{(\lambda)}(z) := \int_{-1}^{1} \frac{C_k^{(\lambda)}(t)}{z-t}dt$$

We let $F(x) = \int_{-1}^{1} f(s)ds$ and apply integration by parts on log transform:

$$\begin{aligned}\int_{-1}^{1} f(t)log(z-t)dt &= [-F(t)log(z-t)]_{-1}^{1} - \int_{-1}^{1} \frac{F(t)}{z-t}dt \\ &= log(z+1)\int_{-1}^{1} f(t)dt - \int_{-1}^{1} \frac{F(t)}{z-t}dt\end{aligned}$$

3 Polynomial Transforms

We can begin to consider taking these transforms across different geometries. Currently we have a way to find these transforms across [-1,1] but we will be trying to use this to solve other geometries. The first type of geometry we should consider is one where we apply a degree d polynomial transform to the interval:

$$p : [-1, 1] \to \Gamma$$

We will show why the solution to a cauchy transform across this interval is as follows:

$$C_\Gamma f(z) = \Sigma_{j=0}^{d} C_{[-1,1]}[f \circ p](p_j^{-1}(z))$$

Where $p_j^{-1}(z)$ are the d pre-images of p. In order to solve this we will use plemelj. There are 3 properties that need to hold for a function $\psi : \Gamma \to \mathbb{C}$ to be a cauchy transform:

$$\begin{gathered}\lim_{z\to\infty} = 0 \\ \psi^+(z) - \psi^-(z) = f(z) \\ \psi \ analytic \ on \ \Gamma\end{gathered} \tag{1}$$

Checking (1).1 we get that $\underset{z\to\infty}{p_j^{-1}(z)} = \infty \implies$

$$\begin{aligned}\lim_{z\to\infty} C_\Gamma f(z) &= \Sigma_{j=1}^{d} \lim_{z\to\infty} C_{[-1,1]}(f \circ p)(p_j^{-1}(z)) \\ &= \Sigma_{j=1}^{d} C_{[-1,1]}(f \circ p)(\lim_{z\to\infty} p_j^{-1}(z)) \\ &= \Sigma_{j=1}^{d} 0 = 0\end{aligned}$$

Checking (1).2 we need an expression for ψ^+ and ψ^-. Let us begin by saying that we are looking for cauchy transform of point s which happens to lie on Γ. This means that there is a unique root of $t_k := p_k^{-1}(s) \in [-1, 1]$. TODO: Show that $\lim_{z\to s^+} p_k^{-1}(s) = \lim_{z\to p^{-1}(s)^+}$. Taking limits of ψ^+, ψ^- gives us:

$$\begin{aligned}\psi^+(s) &= \lim_{z\to s} C_{[-1,1]}(f \circ p)(p_k^{-1}(z)) \\ &\quad + \Sigma_{j\neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s)) \\ &= C_{[-1,1]}^{+}(f \circ p)(p_k^{-1}(z)) \\ &\quad + \Sigma_{j\neq k} C_{[-1,1]}(f \circ p)(p_j^{-1}(s))\end{aligned}$$

We can do a similar thing with ψ^- and putting everything together:

$$\begin{aligned}\psi^+(s) - \psi^-(s) &= C^+_{[-1,1]}(f \circ p)(p_k^{-1}(s)) - C^-_{[-1,1]}(f \circ p)(p_k^{-1}(s)) \\ &= (f \circ p)(p_k^{-1}(s)) = f(s)\end{aligned}$$

In the case where $z \notin \psi, \psi^+ = \psi^-$ which is expected since the area in between is analytic

TODO show that condition (1).3 holds

4 Affine Transformations

An affine transformation here is taken to be in the form:

$$f(z) = \alpha z + \beta \bar{z} + \gamma$$

We must understand what it means to take integrals across paths using parameterisation $z : \mathbb{R} \to \mathbb{C}$:

$$\int_\gamma f(z)dz = \int_{-1}^{1} f(z(t))z'(t)dt$$

Note this requires the existence of $z'(t)$ and is slightly delicate to use in the case of taking a conjugate. We can begin by making a few reparameterisations and by taking derivatives with respect to s:

$$\begin{aligned}z(s) &= \alpha s + \beta \bar{s} + \gamma \\ z_t(s) &= z(s+it) = \alpha(s+it) + \beta(s-it) + \gamma \\ &= (\alpha+\beta)s + (\alpha-\beta)it + \gamma \\ z_t'(s) &= \alpha + \beta\end{aligned} \tag{2}$$

This lets us write the integral taken across $\gamma_t : \{z_t(s) | t \in [-1,1]\}$:

$$\int_{\gamma_t} \frac{f(\zeta)}{z-\zeta} d\zeta = \int_{-1}^{1} \frac{f(z_t(s))}{z - z_t(s)} z_t'(s) ds = (\alpha+\beta) \int_{-1}^{1} \frac{f(z_t(s))}{z - z_t(s)} ds$$

Putting this with the equation for 2d Stieltjes transform:

$$S_{k,j} := \int_{-1}^{1} P_j(t)(\alpha+\beta) \int_{-1}^{1} \frac{P_k(z_t(s))}{z - z_t(s)} ds dt \tag{3}$$

$$\begin{aligned}z - z_t(s) &= z - (\alpha-\beta)it - \gamma - (\alpha+\beta)s \\ &= (\alpha+\beta)\left(\frac{z - \gamma - it(\alpha-\beta)}{\alpha+\beta} - s\right) \\ &=: (\alpha+\beta)(\tilde{z}_t - s),\ \tilde{z}_t = \frac{z - \gamma - it(\alpha-\beta)}{\alpha+\beta}\end{aligned} \tag{4}$$

Plugging (3) into (4):

$$zS_{k,j} = \int_{-1}^{1} zP_j(t) \int_{-1}^{1} \frac{P_k(z_t(s))}{\tilde{z}_t - s} dsdt$$
$$S_k(t)(z) := \int_{-1}^{1} \frac{P_k(z_t(s))}{z - s} ds$$

It is useful to come with a 3-term recurrence for $S_k(t)$ for some fixed t:

$$\begin{aligned}
zS_k(t)(z) &= \int_{-1}^{1} \frac{zP_k(z_t(s))}{z - s} ds \\
&= \int_{-1}^{1} P_k(z_t(s))ds + \int_{-1}^{1} \frac{sP_k(z_t(s))}{z - s} ds \qquad (5) \\
z_t(s)P_k(z_t(s)) &= \frac{k}{2k+1} P_{k-1}(z_t(s)) + \frac{k+1}{2k+1} P_{k+1}(z_t(s)) \\
(2) \rightarrow (5)& \\
&= ((\alpha+\beta)s + (\alpha-\beta)it + \gamma)P_k(z_t(s)) \\
&= (\alpha+\beta)sP_k(z_t(s)) + ((\alpha-\beta)it + \gamma)P_k(z_t(s)) \\
\int_{-1}^{1} \frac{sP_k(z_t(s))}{z - s} ds &= \frac{1}{\alpha+\beta} \int_{-1}^{1} \frac{z_t(s)P_k(z_t(s))}{z - s} ds - \frac{(\alpha-\beta)it + \gamma}{\alpha+\beta} \int_{-1}^{1} \frac{P_k(z_t(s))}{z - s} ds \\
&= \frac{1}{(\alpha+\beta)(2k+1)} \left(\int_{-1}^{1} \frac{kP_{k-1}(z_t(s))}{z - s} ds + \int_{-1}^{1} \frac{(k+1)P_{k+1}(z_t(s))}{z - s} ds \right) \\
&\quad - \frac{(\alpha-\beta)it + \gamma}{\alpha+\beta} \int_{-1}^{1} \frac{P_k(z_t(s))}{z - s} ds \qquad (6) \\
(6) \rightarrow (5)& \\
zS_k(t)(z) &= 2\delta_{k0} + \frac{kS_{k-1}(t)(z) + (k+1)S_{k+1}(t)(z)}{(\alpha+\beta)(2k+1)} - \frac{(\alpha-\beta)it + \gamma}{\alpha+\beta} S_k(t)(z)
\end{aligned}$$