Problem Set 4

Problem 1. *Mechanical and electrical energy*
Consider the unforced mass-spring system

$$m\ddot{x} + b\dot{x} + kx = 0, \quad t > 0.$$

We define the *total mechanical energy* of this system to be the function

$$E(t) = \frac{1}{2}m(\dot{x}(t))^2 + \frac{1}{2}k(x(t))^2,$$

which you might recognize as the sum of *kinetic energy* and *spring potential energy.*

(a) Show that $\dot{E}(t) = 0$ in an undamped system. This shows that $E(t)$ is constant over time (this is an instance of the *law of conservation of mechanical energy.*

(b) On the other hand, if there is damping in this system, give a mathematical argument as to why $\dot{E}(t) < 0$ (so that total mechanical energy is *not* conserved over time).

(c) Recall the forced (non-homogeneous) mass-spring system

$$m\ddot{x} + b\dot{x} + kx = g(t),$$

where $g(t)$ is the forcing function. We define the change in total mechanical energy to be the function

$$(\Delta E)(t) = E(t) - E(0) = \int_0^t \dot{E}(s)\, ds.$$

Show that $(\Delta E)(t)$ can be explicitly written as

$$(\Delta E)(t) = \int_0^t \dot{x}(s)g(s) - b(\dot{x}(s))^2\, ds.$$

(d) Recall the RLC-circuit governed by the equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t).$$

Given the discussion above, what should we define as the *total electrical energy* of the circuit? Interpret the two terms in your definition in terms of the charge Q on the capacitor and the current $I = \dot{Q}$ running through it.

Problem 2. *Wait... you can do that?!*

(a) Be sure to look at problems 30 and 31 of the suggested problems. Show that you've read through them by explaining how we can use the *Gamma function* $\Gamma(t)$ to compute $\left(\frac{1}{2}\right)!$; that is, the factorial of $\frac{1}{2}$.

[*Hint*: You may find the u-substitution $u = \sqrt{x}$ helpful in your explanation.]

(b) The only stumbling block in computing $\left(\frac{1}{2}\right)!$ is that we might not know how to compute

$$G = \int_0^\infty e^{-x^2}\,dx.$$

We will show this below, but first use the comparison test to show that G converges.

(c) Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$(2G)^2 = 2\pi \int_0^\infty re^{-r^2}\,dr.$$

[*Hint*: Write $(\int e^{-x^2}\,dx)^2$ as $(\int e^{-x^2}\,dx)(\int e^{-y^2}\,dy)$ and remember that constants can be brought inside integrals.]

(d) Evaluate the improper integral above to show that $G = \frac{1}{2}\sqrt{\pi}$. Explain how this means that

$$\left(\frac{1}{2}\right)! = \sqrt{\pi}.$$