Problem Set 2

Problem 1. Suppose you've just been hired by the Nimbus Fish Hatchery in Folsom. It is your responsibility to maintain the population of salmon that they breed. Prior work by the hatchery has determined that the population best follows the logistic growth model:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where $N = N(t) \geq 0$ is the number of fish in the pond at time $t \geq 0$, $r > 0$ is the (constant) linear growth rate of the salmon, and $K > 0$ is the (constant) carrying capacity of the lake.

There are two ways that we could harvest fish from the lake:

1. Assuming fish are easier to catch at higher population sizes, we can harvest more fish if there is a larger number of them in the lake.

2. Harvest fish from the lake at a constant rate.

We would like to harvest the most number of fish possible without fishing them to extinction. This value is called the *maximum sustainable yield* for the population.

(a) Devise a scenario where we would chose method 1 over method 2 and a scenario where we would choose method 2 over method 1.

(b) Explain why

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN$$

is a reasonable model for the first harvest method and why

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - h$$

is a reasonable model for the second (assuming $E, h > 0$ are both constant). The first equation is called the *Schaefer model*.

(c) Find all steady-state equilibria for the first model *without solving the differential equation*.

(d) We are in control of the rate E, so assume that we can change this value to anything we'd like. What happens to the stability of your steady-state equilibria in the cases when $E > r$ versus when $E \leq r$.

(e) Now find all steady-state equilibria for the second model, again *without solving the differential equation*.

(f) Again, we are in control of h here. Show that there are zero, one, or two equilibria in this model depending on the value of h. What is the value h_c of h that produces the case where the system has only one equilibrium? (This will be in terms of r and K.)

(g) Remember it is our responsibility to maintain these salmon. What happens if we make a small mistake in choosing our harvest rates E or h? Which method, then, would be considered "safer" to practice?

(h) Back to the original question: What is the maximum sustainable yield for each of these methods?

Problem 2. Consider a 100-gallon brine tank that is initially filled only with 60 gallons of distilled (pure) water. A 0.2 lb/gal brine solution is pumped into the tank at 30 gal/min. The tank is allowed to drain at the same rate so that the water may be treated and recycled for later use.

(a) Develop a model that monitors the amount of salt $w(t)$ in the tank at any given time $t \geq 0$.

(b) What does the Existence and Uniqueness Theorem say about the potential solutions of your model? Be sure to comment on the interval of t over which a solution might exist.

(c) Solve the model in part (a) for $w(t)$.

(d) Use your solution from above to directly compute $w_\infty = \lim_{t\to\infty} w(t)$. This value is called the *steady-state equilibrium* of the system.

(e) Compute the steady-state equilibrium just from the model and not its solution.

(f) Suppose now that the brine pump is incorrectly calibrated so that it pumps at a rate of 35 gal/min instead of the expected 30 gal/min. Repeat parts (a), (b), and (c) for this new setup.

(g) Explain why your model is invalid after a certain point $T \geq 0$ in time. Adapt your model to account for this *branch point* at T so that we can continue modeling the brine tank for $t \geq T$.

(h) We expect a steady-state equilibrium of $w_\infty = 20$ lbs (why?). Show that your model (the function $w(t)$) accounts for this.