What is an integral?

In class we encountered a function which was very similar to:

$$f(x) = \begin{cases} x^2, & x \neq 1, \\ 3, & x = 1. \end{cases}$$

The question was:

> Does the integral $\int_0^2 f(x)\,dx$ exist? If "Yes" find the value, if "No" explain why the integral fails to exist.

Most students tend to pronounce the integral nonexistent because the integrand f is discontinuous. However the correct answer turns out to be:

$$\int_0^2 f(x)\,dx = \frac{8}{3}.$$

Perplexing, is not it?

Before we go through the reasoning behind the correct answer, let us look at the typical reasons for incorrect responses. The two main reasons can be summarized as follows:

1. In order to integrate f, one has to find F—an antiderivative of f—through some algebraic manipulation. If that cannot be done, the integral does not make sense.

2. Discontinuities are bad.

Now it is certainly true that in order to integrate f *symbolically* one must find F. Furthermore, according to the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

However, not every antiderivative can be computed *symbolically*, in fact, most cannot be. And just because an antiderivative cannot be found through algebraic manipulation does not mean that it does not exist!

As for discontinuities being bad, this misconception comes from differential calculus. If the question were to find the *derivative* of f, the answer would certainly have been "does not exist". Indeed, in order to be differentiable a function must necessarily be continuous. However, as it stands, the question involves the integral. And look as you might for a prohibition against integrating discontinuous functions, you will *not* find it in your calculus textbook. Otherwise throw that textbook away!

On the other hand, the few brave souls who dare to go against the majority vote typically reasone that

$$\int_0^2 f(x)\,dx = \int_0^2 x^2\,dx = \frac{8}{3}$$

on the grounds that

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

and

$$\int_1^1 f(x)\,dx = 0.$$

However this is not precise reasoning. In order to explain the paradox, one has to use the true definition of the definite integral which we now discuss.

Subdivide, sample, and sum

If you read the chapter on integration in, say, Stewart, very slowly and very carefully—which is something that I strongly encourage you to do— you will discover that symbolic computation of the antiderivative is not necessary for finding the definite integral. Contrary to the popular belief, the Fundamental Theorem of Calculus (FTC) does not define the integral. Rather FTC follows from the definition of the integral (and that of the antiderivative) as a profound consequence. Nor are the integrals defined as areas under curves. If they were, we would not be able to compute areas!

The truth of the matter is that integrals are defined as limits of Riemann sums. Symbolically,

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\,\Delta x. \tag{1}$$

Unfortunately, Equation (1) is rather intimidating and difficult to use. No wonder that students quickly forget it after learning about the Fundamental Theorem of Calculus! In order to internalize Equation (1) correctly, think about it as a three-step *limiting process*:

Subdivide Subdivide the interval $[a, b]$ into N subintervals. For simplicity, let us use equal subdivisions so that the length of each subinterval is the same

$$\Delta x = \frac{b-a}{N}.$$

It should be borne in mind, however, that the lengths of subintervals do not have to be the same. What is important is that these lengths go to zero as the number of subintervals increases.

Sample Within each subinterval pick a point. Again, to keep things simple, we will commonly use midpoints although any point will do. Label the points x_1, x_2,..., x_N. Now sample the function at these points. This simply means computing $f(x_1)$, $f(x_2)$, ..., $f(x_N)$.

Sum Finally, form the sum:

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_N)\,\Delta x = \sum_{n=1}^{N} f(x_n)\,\Delta x.$$

This sum is named after Bernhard Riemann, a famous 19-th century German mathematician [1]. The Riemann sum provides an approximation for the integral. The exact value of the integral is obtained by taking the limit of the sum as the number of terms N goes to infinity.

You may recall from Calculus 1 that geometrically the Riemann sum can be interpreted as the sum of areas of rectangles. Let us see how these rectangles approximate the area under

$$f(x) = \begin{cases} x^2, & x \neq 1, \\ 3, & x = 1. \end{cases}$$

[1] You may find it hard to believe that, as important as the idea of the Riemann sum is, it is one of the lesser things that Riemann invented. Among his many mathematical discoveries is Riemannian geometry without which modern physics would be impossible to formulate. Even more famous is the so-called Riemann's Hypothesis which is presently considered one of the most important unsolved problems in mathematics.

Figure 1 illustrates six Riemann sums with terms that are multiples of five. Notice that the even sums are the same as for the continuous parabola $y = x^2$. For odd number of terms, the discontinuity at $x = 1$ produces one “abnormal” rectangle. That rectangle, however, shrinks to a line segment as the number of subdivisions approaches infinity. In the limit, the contribution from the discontinuity becomes zero and the integral equals the area under $y = x^2$ which is 8/3.

Attaching meaning to formulas

I remarked that Equation (1) which *defines* the definite integral $\int_a^b f(x)\,dx$ is rather intimidating. This, however, does not give you a license to ignore it. In fact, solid understanding of this one equation is central to success in Calculus 2. This understanding can only come from meaning which must be attached. Yet, how does one attach meaning to a mathematical formula?

For answer, take another look at Figure 1. It is obvious that I did not draw it by hand (although I could have). The figure was generated using mathematical software called `Matlab`. In order to make `Matlab` produce the figure I had to write code a snippet of which is shown below:

```
...
x = linspace(a,b,N+1);           % subdivide [a,b] into N subintervals
dx = x(2)-x(1);                  % find delta x
xm = .5*(x(1:end-1)+x(2:end));  % compute midpoints
ym = f(xm);                      % sample f at midpoints
AN = sum(ym)*dx;                 % find the Riemann sum
...
```

The part shown only finds the Riemann sums; the actual code has a few more lines containing commands which plot the function, the rectangles, title the plots, and so on. Notice that the code literally implements the “Subdivide, Sample, and Sum” process and produces meaningful and memorable results.

Now here is where I am going with this. If you can translate a mathematical formula or an equation into working code, that suggests that you understand that formula or equation and have a degree of control over it. Conversely, if the code you write refuses to work or produces silly results, there is a gap in your understanding. Therefore

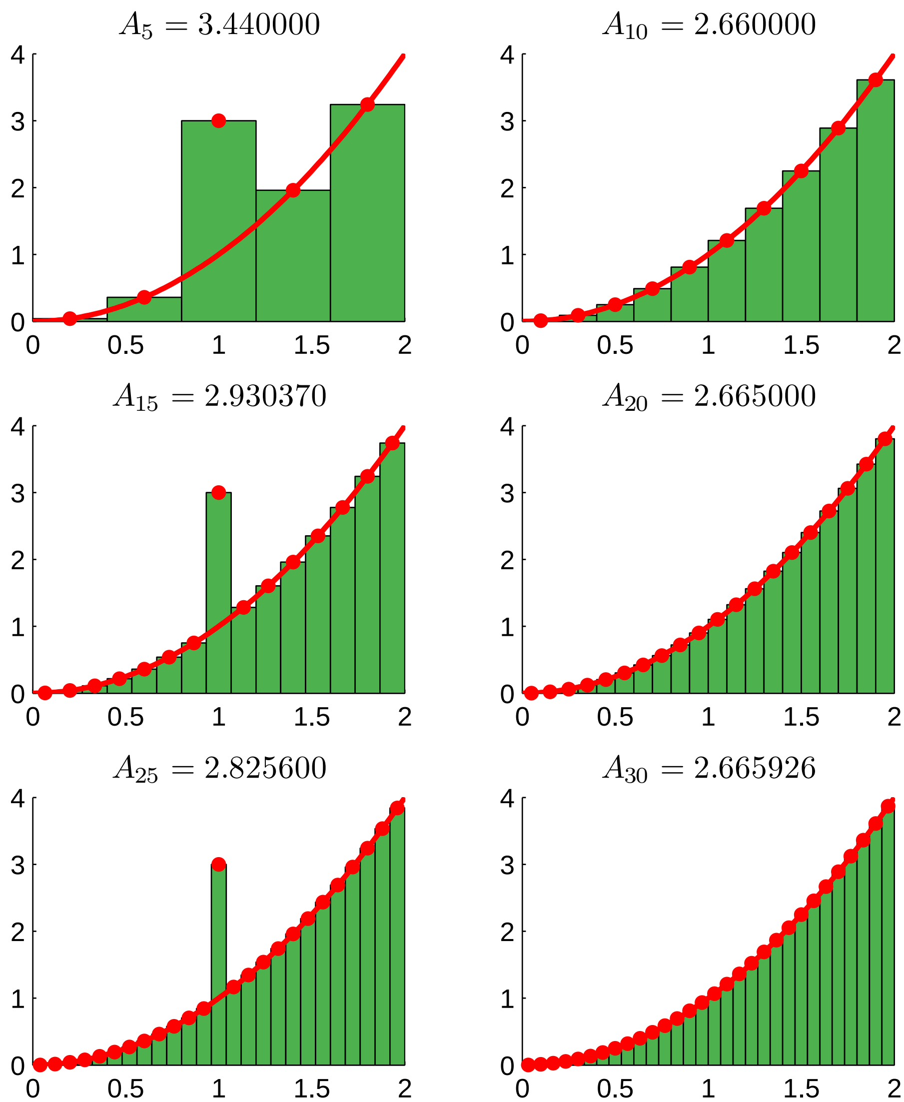

Figure 1: Riemann sums for the "perturbed parabola" $y = f(x)$ with terms that are multiples of five. Notice that the even sums are the same as for $y = x^2$. The odd sums have one "abnormal" term caused by the discontinuity at $x = 1$. However, in the limit, the contribution from that term becomes zero. Therefore the integral exists and equals $8/3$.

> Translate mathematics into working code in order to attach meaning to equations and formulas.

I hope that by the end of the semester you will master enough `Matlab` to reproduce Figure 1 on your own. Chances are, you will never need to integrate by parts, make u-substitutions, or perform partial fraction decomposition after finishing Calculus 2. However it is almost a given that at some point you will need to write a few lines of working code. So start coding, if you have not already.

Exercises

If you encounter any difficulties with this assignment (e.g., `Matlab` does not find your file), please come to office hours. If the difficulties do not get resolved in time, explain that in your homework.

1. Let k be a positive number. Use the definition of the integral to explain why
$$\int_0^1 k\,dx = k.$$

2. Explain, as in the previous exercise, why
$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

3. You now know that a function which is discontinuous at one point can be integrated. Clearly, two, three, or any finite number of discontinuities can be dealt with in the same manner as in the text of this handout. Let
$$f(x) = \begin{cases} 1, & x \neq \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \\ 0, & \text{otherwise} \end{cases}$$
Notice that this function has infinitely many discontinuities near the origin. Does the integral $\int_0^1 f(x)\,dx$ exist? This is a profound and intimidating question. Show that you cannot be easily intimidated.

4. How many terms are in the following sums:

   (a) $\sum_{i=0}^{11} i$

(b) $\sum_{j=1}^{12} j^2$

5. Use `Matlab` to compute the sums in the previous problem.

6. Write a script for approximating the area under $y = f(x)$ on $[0, 2]$ using the Midpoint Rule with N subdivisions. Then use the script to find the error of the Midpoint Rule for $N = 1, 2, 3, \ldots, 100$ for the following functions:

   (a) $f(x) = x^2$

   (b) $f(x) = e^{-x} \sin(10\,x)$

   (c) $f(x) = |x - 1|$

   (d) $f(x) = \text{sign}(x - 1)$

   In the last item the sign function returns $+1$ if the input is positive, -1 if the input is negative, and 0 if the input is zero. In all cases, present the results as loglog plots of the error against the number of subdivisions. Comment on the results. Note: If you cannot find the exact value of the integral symbolically, use the following code:

```
f = @(x) ...;                  % define the function
q = quad(f,0,2);               % use as exact integral
```

7. Find the area of the region formed by intersecting the parabola $y = x\,(1-x)$ with the line $y = m\,x$. First, compute the symbolic expression. Then confirm your answer numerically for a few choices of m.