Problem Set 5

Problem 1. *Resonance from discontinuous force*

Our goal for this exercise is to show that our mass-spring model can account for what we see in experiments. Specifically thinking about pushing someone on a swing or hitting a punching bag with a baseball bat, we would like to show that our model predicts resonance in the system for specific *discontinuous* forcing functions.

Consider a frictionless mass-spring system at its rest equilibrium modeled by the IVP

$$\begin{cases} m\ddot{x} + kx = g(t) \\ x(0) = 0, \ \dot{x}(0) = 0 \end{cases}.$$

Starting at time $t = 0$, we have the option to repeatedly

(i) hit the (ferromagnetic) mass with a hammer in perfect elastic collisions, or

(ii) turn an electromagnet on and off

with unit strength at regular intervals of T seconds.

(a) Describe and graph a function $g(t)$ that models each of these scenarios (i) and (ii). (Yes, your answer should be given as a series written in sigma-notation.)

For simplicity, let's assume that $m = \frac{1}{4}$ and $k = 1$.

(b) Use the Laplace transform to solve the system in scenario (i) in case when

(1) $T = \pi$.

(2) $T = 3\pi$.

Graph your solution in each case. Comment on the case when $T = \frac{\pi}{2}$.

(c) Repeat part (b) for scenario (ii) instead, again being sure to comment on the case when $T = \frac{\pi}{2}$.

Problem 2. *What is $\mathcal{L}[f(t)g(t)](s)$?*

We've pondered in class what the Laplace transform of a product of functions looks like. Ideally, we would have $\mathcal{L}[f(t)g(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$, but this is actually almost never the case. The result we desire requires a bit more finesse to obtain...

(a) Choose functions $f(t)$ and $g(t)$ that show it is possible for

$$\mathcal{L}[f(t)g(t)](s) = F(s)G(s),$$

where F and G are the Laplace transforms of f and g, respectively.

(b) Choose functions $f(t)$ and $g(t)$ that show it is also possible for

$$\mathcal{L}[f(t)g(t)](s) \neq F(s)G(s).$$

It is now clear that our usual method of multiplying together $f(t)$ and $g(t)$ is not necessarily going to produce the result we want. The better question we should ask is

"Given $F(s)$ and $G(s)$ exist, is there a function $h(t)$ where

$$\mathcal{L}[h(t)](s) = F(s)G(s)?''$$

Allow me to propose such a function. We define the *convolution* $f * g$ of the functions $f(t)$ and $g(t)$ to be the integral

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du.$$

(We read $f * g$ as "f convolve g" or "f convolved with g".)

(c) Use the definition of the convolution to directly compute $e^{at} * \sin(bt)$ when $a, b > 0$ are both constants. (Notice that this is not the same as simply multiplying together $f(t)$ and $g(t)$.)

(d) Verify for $f(t) = g(t) = t$ that $\mathcal{L}[(f * g)(t)](s) = F(s)G(s)$.

Remark - It is a general fact that, if $F(s)$ and $G(s)$ exist, then

$$\mathcal{L}[(f * g)(t)](s) = F(s)G(s).$$

This is among one of the stranger and less obvious results we come across in this field. We should think of this convolution integral as a way for us to say that $f * g$ is the result of letting f and g "mix" together. In the past, we've done this using pointwise operations; but now we see the first instance of when those past definitions fail to do what we hope they will.