Problem Set 6 Solutions

Problem 1. *Procedural proficiency with a system of linear equations*
Consider the linear system in three equations and three unknowns:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 5y - z = 5 \\ -x + 3y + z = -2 \end{cases}.$$

(a) First, identify the matrix A and the vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{b}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

We identify this system of equations with the matrix-vector system

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -2 \end{bmatrix}.$$

(b) Write this system of equations as an augmented matrix system.

This matrix-vector system is equivalent to the augmented matrix system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -5 & -1 & 5 \\ -1 & 3 & 1 & -2 \end{array}\right].$$

(c) Row reduce this augmented matrix system to show that there is exactly one solution to this system of equations.

We first perform $R2 \to R_2 - 2R1$ and $R3 \to R3 + R1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -9 & -7 & -7 \\ 0 & 5 & 4 & 4 \end{array}\right].$$

To simplify computations, we multiply $R2$ by 5 and $R3$ by 9:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -45 & -35 & -35 \\ 0 & 45 & 36 & 36 \end{array}\right].$$

We finally perform $R3 \to R3 + R2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -45 & -35 & -35 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

(d) Convert your reduced augmented matrix system back into an equivalent system of equations, and then use back-substitution to compute the unique solution to the original system of equations.

The reduced augmented matrix system above is equivalent to the system of equations:

$$\left\{ \begin{array}{rrrrrcr} x & + & 2y & + & 3z & = & 6 \\ & - & 45y & - & 35z & = & -35 \\ & & & & z & = & 1 \end{array} \right. .$$

Substituting from the last equation back up to the first, we see that

$$z = 1 \quad \Rightarrow \quad -45y - 35(1) = -35 \quad \Rightarrow \quad y = 0$$

and then

$$y = 0, z = 1 \quad \Rightarrow \quad x + 2(0) + 3(1) = 6 \quad \Rightarrow \quad x = 3.$$

Therefore, there is exactly one solution to the original system of equations given by

$$\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

(e) Verify that the solution $\vec{\mathbf{x}}$ that you found in (d) is indeed a solution of the system of equations by computing $A\vec{\mathbf{x}}$ and showing this is equal to the vector $\vec{\mathbf{b}}$.

We verify that indeed:

$$A\vec{\mathbf{x}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & -1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(0) + 3(1) \\ 2(3) - 5(0) - 1(1) \\ -1(3) + 3(0) + 1(1) \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -2 \end{bmatrix} = \vec{\mathbf{b}}.$$

Problem 2. *Procedural proficiency in computing eigenvalues and eigenvectors*
Consider the matrices

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

(a) Compute the characteristic polynomials for A and B.

We first see that

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} \quad \text{and} \quad B - \lambda I = \begin{bmatrix} 4-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{bmatrix}.$$

The characteristic polynomials of the matrices A and B are given by the determinants $\det(A - \lambda I)$ and $\det(B - \lambda I)$:

$$\det(A - \lambda I) = (4-\lambda)(4-\lambda) - 1 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$

and

$$\det(B - \lambda I) = (4-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 4-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 3-\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (4-\lambda)[(3-\lambda)(4-\lambda) - 1\cdot 1] - [1(4-\lambda) - 1\cdot 0] = (4-\lambda)[\lambda^2 - 7\lambda + 10]$$
$$= (4-\lambda)(\lambda - 2)(\lambda - 5).$$

(b) Use the characteristic polynomial of A to compute the eigenvalues of A.

The eigenvalues are the roots (zeroes) of the characteristic polynomials:

$$\det(A - \lambda I) = (\lambda - 3)(\lambda - 5) = 0 \quad \Rightarrow \quad \lambda = 3, 5$$

and

$$\det(B - \lambda I) = (\lambda - 2)(\lambda - 4)(\lambda - 5) = 0 \quad \Rightarrow \quad \lambda = 2, 4, 5.$$

(c) Use the eigenvalues of A to compute the representative eigenvectors associated with each eigenvalue.

We first compute the eigenvectors for A. For each eigenvalue λ, we compute all solutions $\vec{\mathbf{v}}$ to $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$.
For $\lambda = 3$, we have

$$(A - 3I)\vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Leftrightarrow \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right].$$

If $\vec{\mathbf{v}} = (x, y)^T$ (transpose), we find that this augmented matrix system implies that $y = -x$ so that

$$\vec{\mathbf{v}} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

An eigenvector for $\lambda = 3$ is given by

$$\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda = 5$, we have

$$(A - 5I)\vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Leftrightarrow \quad \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right].$$

If $\vec{\mathbf{v}} = (x, y)^T$ (transpose), we find that this augmented matrix system implies that $y = x$ so that

$$\vec{\mathbf{v}} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

An eigenvector for $\lambda = 5$ is given by

$$\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We now compute the eigenvectors for B:
For $\lambda = 2$, we have

$$(B - 2I)\vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

If $\vec{\mathbf{v}} = (x, y, z)^T$ (transpose), we find that this augmented matrix system implies that $y = -2z$ and $x = -y - z = -(-2z) - z = z$ so that

$$\vec{\mathbf{v}} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

An eigenvector for $\lambda = 2$ is given by

$$\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

For $\lambda = 4$, we have

$$(B - 4I)\vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

If $\vec{\mathbf{v}} = (x, y, z)^T$ (transpose), we find that this augmented matrix system implies that $y = 0$ and $x = y - z = (0) - z = -z$ so that

$$\vec{\mathbf{v}} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

An eigenvector for $\lambda = 4$ is given by

$$\vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 5$, we have

$$(B - 5I)\vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

If $\vec{\mathbf{v}} = (x, y, z)^T$ (transpose), we find that this augmented matrix system implies that $y = z$ and $x = 2y - z = 2(z) - z = z$ so that

$$\vec{\mathbf{v}} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

An eigenvector for $\lambda = 5$ is given by

$$\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(d) Verify that each vector $\vec{\mathbf{v}}$ computed in (c) is indeed an eigenvector of A by multiplying $A\vec{\mathbf{v}}$ and showing that the resulting vector is $\lambda\vec{\mathbf{v}}$ for the correct eigenvalue λ of A.

Consider for A that

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4(1)+1(-1) \\ 1(1)+4(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and that

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1)+1(1) \\ 1(1)+4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that indeed these vectors are eigenvectors for A.

Consider for B that

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1)+1(-2)+0(1) \\ 1(1)+3(-2)+1(1) \\ 0(1)+1(-2)+4(1) \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(-1)+1(0)+0(1) \\ 1(-1)+3(0)+1(1) \\ 0(-1)+1(0)+4(1) \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix} = 4\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1)+1(1)+0(1) \\ 1(1)+3(1)+1(1) \\ 0(1)+1(1)+4(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

so that indeed these vectors are eigenvectors for B.

(e) Repeat parts (b), (c), and (d) for the matrix B.

See part (b), (c), and (d) above.

Problem 3. *A connection to second-order ODE*

Recall the second-order ODE

$$ay'' + by' + cy = g(t),$$

where $a \neq 0$, b, and c are all constant and $g(t)$ is the non-homogenous term.

(a) Use a substitution to show that this ODE can be written as a vector equation

$$\frac{d\vec{\mathbf{x}}}{dx} = A\vec{\mathbf{x}} + \vec{\mathbf{G}}(t)$$

for a constant, 2×2 matrix A and vector functions $\vec{\mathbf{x}}(t)$ and $\vec{\mathbf{G}}(t)$.

First consider that

$$y'' = \frac{1}{a}[g(t) - cy - by'].$$

If we make the substitution

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{where} \quad x_1 = y \quad \text{and} \quad x_2 = y',$$

then $y'' = \frac{1}{a}[g(t) - cx_1 - bx_2]$ so that

$$\dot{\vec{\mathbf{x}}} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{a}[g(t) - cx_1 - bx_2] \end{bmatrix} = \begin{bmatrix} x_2 \\ -cx_1 - bx_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g(t)}{a} \end{bmatrix}.$$

We then see

$$\dot{\vec{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g(t)}{a} \end{bmatrix} = A\vec{\mathbf{x}} + \vec{\mathbf{G}}(t),$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{G}}(t) = \begin{bmatrix} 0 \\ \frac{g(t)}{a} \end{bmatrix}.$$

(b) Compute the characteristic equation of the matrix A, and relate this to the original second-order ODE.

Once again, the characteristic polynomial is $\det(A - \lambda I)$, so we merely compute

$$\det(A - \lambda I) = \det \begin{bmatrix} 0 - \lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{bmatrix}$$

$$= -\lambda \left(-\frac{b}{a} - \lambda \right) - 1 \cdot \left(-\frac{c}{a} \right) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}.$$

Therefore, the characteristic equation of A is a scaled version of the characteristic equation

$$ar^2 + br + c = 0$$

of the original second-order ODE.