Problem Set 4

Problem 1. *A first example in Fourier Analysis*

We mentioned in class that the set

$$\beta_K = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos Kx, \sin Kx\}$$

is orthogonal in $(C[-\pi, \pi], L^2)$. This gave another (perhaps more convincing) explanation for why β_K is linearly independent in the vector space $C[-\pi, \pi]$. Below you give the formal proof and then explore a peculiar use of this set.

(a) Through direct integral calculations, show that it is indeed the case that

$$\langle 1, 1 \rangle = 2\pi, \quad \langle 1, \cos nx \rangle = 0, \quad \langle 1, \sin nx \rangle = 0, \quad \langle \cos mx, \sin nx \rangle = 0,$$

$$\langle \cos mx, \cos nx \rangle = \begin{cases} \pi & , \text{ if } m = n, \\ 0 & , \text{ if } m \neq n \end{cases}, \quad \text{and} \quad \langle \sin mx, \sin nx \rangle = \begin{cases} \pi & , \text{ if } m = n, \\ 0 & , \text{ if } m \neq n \end{cases}$$

for all integers $m, n \geq 1$.

Remark - It's essentially a rite of passage for mathematicians and applied mathematicians alike to be able to do this work at least once before they graduate.

(b) Briefly explain why the set

$$\hat{\beta}_K = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \ldots, \frac{1}{\sqrt{\pi}} \cos Kx, \frac{1}{\sqrt{\pi}} \sin Kx \right\}$$

is orthonormal in $(C[-\pi, \pi], L^2)$.

(c) Find the orthogonal projection

$$\tilde{f}_K(x) = \sum_{n=0}^{K} a_n \cos(nx) + \sum_{n=1}^{K} b_n \sin(nx)$$

of the function $f(x) = x$ onto β_K. These a_n and b_n coefficients are called the *Fourier coefficients* of $f(x)$. [*Hint*: Making use of the parity (odd/even) of a function makes some integrals trivial in this computation.]

(d) Use MATLAB to plot $\tilde{f}_K(x)$ on $[-\pi, \pi]$ for $K = 1, 5, 10, 20$. (No need to submit source code.) What happens to $\tilde{f}_K(x)$ as K gets larger? Does anything strange or unexpected happen?

(e) Find the squared distance $d(\tilde{f}_K(x), x)^2 = \|\tilde{f}_K(x) - x\|^2 = \langle \tilde{f}_K(x) - x, \tilde{f}_K(x) - x \rangle$. Leave your answer as a sum over the index n. [*Hint*: If you're computing more integrals, you're working too hard.]

(f) Let's validate your MATLAB observations. Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

to show what happens to $d(\tilde{f}_K(x), x)$ as $K \to \infty$? What then do you suggest is the limit function

$$\tilde{f}(x) = \lim_{K \to \infty} \tilde{f}_K(x)$$

in $(C[-\pi, \pi], L^2)$?