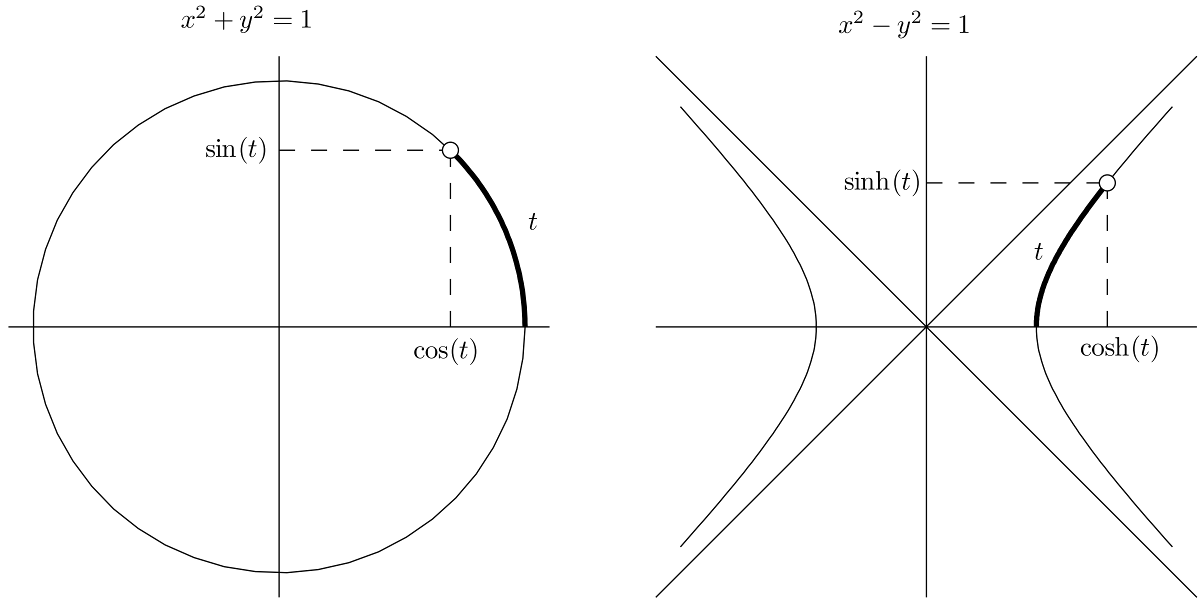

Figure 3: Circular and hyperbolic trigonometries

Trigonometric substitutions

As a standard example of an integral requiring a trigonometric substitution, consider:

$$\int_0^1 \sqrt{1-x^2}\,dx. \tag{7}$$

Geometrically, this integral expresses the area of a quarter circle of unit radius: therefore it must equal $\pi/4$. We will now confirm that using a substitution.

> Which substitution would you use in (7)? Think of one and try it before reading on.

The form of the integrand in (7) suggests two obvious (non-trigonometric) substitutions: $u = 1 - x^2$ and $u = \sqrt{1-x^2}$. However, none of them simplify the integral. For instance, if we set $u = 1 - x^2$ then (7) becomes:

$$-\int_1^0 \sqrt{u}\,\frac{du}{2\sqrt{1-u}}.$$

The new integral is, actually, more complicated than the original one. As an exercise, show that a similar thing happens with the other substitution $u = \sqrt{1-x^2}$.

Why do the obvious substitutions fail? Here the blame can be laid squarely on the differential. For instance, when we substitute $u = 1 - x^2$, we simplify the integrand to $\sqrt{u}$. Meanwhile dx transforms into

$$-\frac{du}{2\sqrt{1-u}}$$

and that is what ruins the outcome. We need to find a substitution that simplifies the radical without overcomplicating the differential.

The correct substitution turns out to be $x = \sin(t)$. Of course, this is not a compete surprise given that the integrand is related to the unit circle. With this substitution

$$\begin{aligned} \sqrt{1-x^2} &= \sqrt{1-\sin^2(t)} = \sqrt{\cos^2(t)} = \cos(t) \\ dx &= d(\sin(t)) = \cos(t)\,dt \end{aligned}$$

while the limits become $\sin^{-1}(0) = 0$ and $\sin^{-1}(1) = \frac{\pi}{2}$. Thus (7) transforms into

$$\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt,$$

which looks eminently reasonable. In order to finish the computation, we need to simplify the square of the cosine. This can be accomplished through the use of the double angle formula:

$$\cos^2(t) = \frac{1+\cos(2\,t)}{2}.$$

We therefore obtain

$$\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\,t)}{2}\,dt = \frac{t}{2} + \frac{\sin(2\,t)}{4}\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{4},$$

as required.

As the next example, let us consider

$$\int_0^1 \sqrt{1+4\,x^2}\,dx. \tag{8}$$

This integral expresses the length of the parabola $y = x^2$ on the interval $0 \le x \le 1$. The traditional approach to evaluating (8) commences with the substitution $2\,x = \tan(t)$. The rationale is to create a square under the radical leading to its disappearance:

$$\sqrt{1+4\,x^2} = \sqrt{1+\tan^2(t)} = \sqrt{\sec^2(t)} = \sec(t).$$

Unfortunately, the derivative of the tangent contributes the square of the secant and we end up with

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3(t)\,dt$$

for which there is no obvious trigonometric identity. We will now discuss an alternative to the tangent substitution which involves hyperbolic trigonometry.

Hyperbolic trigonometry

The substitution $x = \sin(t)$ is ideally suited for integrals containing $\sqrt{1-x^2}$ because of the identity $\cos^2(t) + \sin^2(t) = 1$. That identity comes from the equation of the unit circle $x^2 + y^2 = 1$. Figure 3 shows that it is possible to define analogues of the circular cosine and sine using a unit hyperbola $x^2 - y^2 = 1$. Just as in the circular case, shown on the left, we mark an arc of the hyperbola of length t. The endpoint of the arc, shown with a circle, has coordinates dependent on t. Mimicking the circular case, we define *hyperbolic* cosine and sine as:

$$x(t) = \cosh(t), \quad y(t) = \sinh(t).$$

From the equation of the hyperbola $x^2 - y^2 = 1$ follows at once that

$$\cosh^2(t) - \sinh^2(t) = 1.$$

Therefore we may expect the substitution $x = \sinh(t)$ to be applicable to integrals containing $\sqrt{1+x^2}$.

It turns out that the hyperbolic cosine and sine can be expressed in terms of the exponential function as follows:

$$\begin{aligned} \cosh(t) &= \frac{e^t + e^{-t}}{2} \\ \sinh(t) &= \frac{e^t - e^{-t}}{2} \end{aligned}$$

From these equations one can derive all of the hyperbolic identities which either coincide with the circular trigonometric identities or differ from them by a sign. For instance,

$$\begin{aligned} \frac{d}{dt}(\cosh(t)) &= \frac{e^t - e^{-t}}{2} = \sinh(t) \\ \frac{d}{dt}(\sinh(t)) &= \frac{e^t + e^{-t}}{2} = \cosh(t) \end{aligned}$$

Notice the similarity with the circular case. On the other hand, differentiating hyperbolic trigonometric functions is easier because there are no minus signs to keep track of. As another example, consider

$$\begin{aligned} \cosh^2(t) &= \left(\frac{e^t + e^{-t}}{2}\right)^2 = \frac{e^{2t} + 2 + e^{-2t}}{4} \\ &= \frac{1}{2}\left(1 + \frac{e^{2t} + e^{-2t}}{2}\right) = \frac{1 + \cosh(2t)}{2}. \end{aligned}$$

Thus the double angle formula for the hyperbolic cosine is the same as for the circular cosine.

We are now ready to evaluate integral (8). In light of the above discussion we substitute $2\,x = \sinh(t)$. Then

$$\begin{aligned} \sqrt{1+4\,x^2} &= \sqrt{1+\sinh^2(t)} = \sqrt{\cosh^2(t)} = \cosh(t) \\ dx &= \frac{1}{2}\,d(\sinh(t)) = \frac{1}{2}\,\cosh(t)\,dt \end{aligned}$$

while the new limits are $\sinh^{-1}(0) = 0$ and $\sinh^{-1}(2)$. We thus obtain

$$\frac{1}{2}\int_0^{\sinh^{-1}(2)} \cosh^2(t)\,dt$$

which we evaluate using the (hyperbolic) double angle formula:

$$\begin{aligned} \frac{1}{2}\int_0^{\sinh^{-1}(2)} \cosh^2(t)\,dt = \frac{1}{2}\int_0^{\sinh^{-1}(2)} \frac{1+\cosh(2\,t)}{2}\,dt &= \frac{t}{4} + \frac{\sinh(2\,t)}{8}\Bigg|_0^{\sinh^{-1}(2)} \\ &= \frac{\sinh^{-1}(2)}{4} + \frac{\sinh(2\,\sinh^{-1}(2))}{8}. \end{aligned}$$

Again, notice the analogy with the circular case. The answer, as it stands, can be evaluated in `Matlab` using the `sinh` and `asinh` commands for the hyperbolic sine and its inverse. However, it should be simplified as follows. Firstly, note that for any a:

$$\sinh(2\,\sinh^{-1}(a)) = 2\,\sinh(\sinh^{-1}(a))\,\cosh(\sinh^{-1}(a)) = 2\,a\,\sqrt{1+a^2}.$$

Therefore we can write

$$\int_0^1 \sqrt{1+4\,x^2}\,dx = \frac{\sinh^{-1}(2)}{4} + \frac{\sqrt{5}}{2}$$

which is certainly better than what we had before. Secondly, we can replace hyperbolic arc sine with an expression involving the more familiar logarithmic function. To derive that expression, solve

$$\sinh(y) = \frac{e^y - e^{-y}}{2} = x$$

for y. This gives

$$\sinh^{-1}(x) = \ln(x + \sqrt{1+x^2}).$$

Consequently, the simplest form of the answer (from the point of view of Calculus I) is:

$$\int_0^1 \sqrt{1+4\,x^2}\,dx = \frac{\ln(2+\sqrt{5})}{4} + \frac{\sqrt{5}}{2} \approx 1.4789\ldots$$

We hasten to add that the second simplification, aimed at removing hyperbolic trigonometry, is not really necessary and was done largely as an exercise. It is perfectly acceptable and often desirable to express integrals in terms of inverse hyperbolic functions.

Design of a headlight

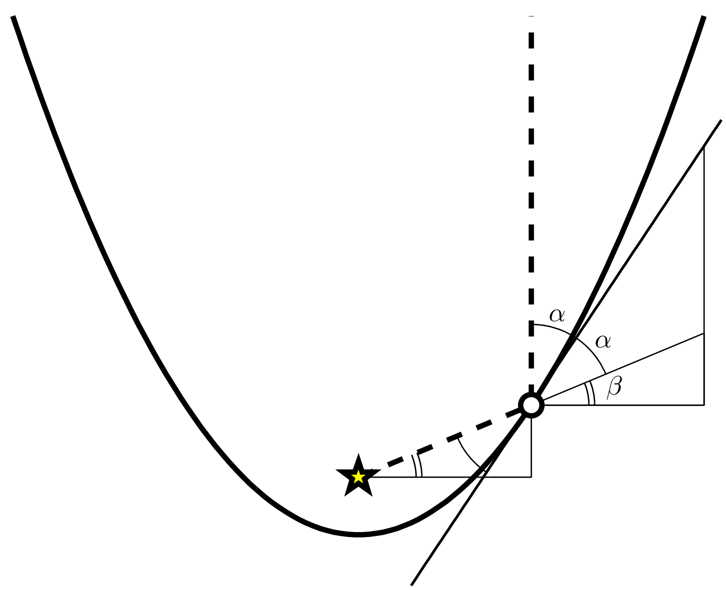

Figure 4: Headlight design.

As another application of hyperbolic trigonometry, let us consider the problem of designing a mirror for an automotive headlight. The purpose of the mirror is to reflect light rays issued by a light bulb in a manner shown in Figure 4: upon reflection all rays become parallel. In the figure the light bulb is marked with a star and a typical ray is shown as a dashed line. The mirror is a surface of revolution obtained by revolving a certain curve around the y-axis. Visually, the curve looks like a parabola and we will show that it is, in fact, a parabola.

Key to our analysis is the law of reflection which, in words, states: the angle of incidence equals the angle of reflection. The angle of incidence is the angle at which the light ray strikes the mirror; the angle of reflection is the angle between the mirror and the reflected ray. In the picture we labeled the angle of incidence and the angle of reflection as α. Notice that in order to measure α we needed to construct the tangent line at the point of incidence which we now give coordinates (x, y). We also introduced β—the angle between the incident ray and the x-axis. Both α and β depend on (x, y) and our goal is to find a curve $y = f(x)$ such that

$$2\,\alpha + \beta = \frac{\pi}{2}.$$

We will now express the angles in terms of the coordinates (x, y) of the point of incidence. Consider the lower right triangle in Figure 4. The end points of the hypothenuse of this triangle are $(0, 0)$ (light bulb) and (x, y) (point of incidence). Therefore the marked angle β is given by:

$$\beta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, to find α, consider the upper triangle. The hypothenuse of that triangle is the tangent at (x, y). Consequently,

$$\alpha + \beta = \tan^{-1}\left(\frac{dy}{dx}\right)$$

and hence

$$\alpha = \tan^{-1}\left(\frac{dy}{dx}\right) - \tan^{-1}\left(\frac{y}{x}\right).$$

Having found α and β we can now form the equation of the curve:

$$2\,\tan^{-1}\left(\frac{dy}{dx}\right) - \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2}.$$

Notice that this is an ordinary differential equation. To solve it, we first solve for the derivative. This results in:

$$\frac{dy}{dx} = \tan\left(\frac{\pi}{4} + \frac{1}{2}\,\tan^{-1}\left(\frac{y}{x}\right)\right)$$

Next, using the addition formula for the tangent

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\,\tan(b)}$$

we can bring the ODE to the form:

$$\frac{dy}{dx} = \frac{1 + \tan\left(\frac{1}{2}\,\tan^{-1}\left(\frac{y}{x}\right)\right)}{1 - \tan\left(\frac{1}{2}\,\tan^{-1}\left(\frac{y}{x}\right)\right)}.$$

This heavily suggests the substitution $u = \tan\left(\frac{1}{2}\,\tan^{-1}\left(\frac{y}{x}\right)\right)$ which we now perform. It is clear what happens to the right-hand side. To transform the left-hand side, we first find y as

$$y = x\,\tan\left(2\,\tan^{-1}(u)\right) = x\,\frac{2\,u}{1-u^2}.$$

We then differentiate the result with respect to x treating u as a function of x. This leads to:

$$\frac{dy}{dx} = \frac{2\,u}{1-u^2} + x\,\frac{2\,(1+u^2)}{(1-u^2)^2}\,\frac{du}{dx}$$

We conclude that the ODE for u is

$$\frac{2\,u}{1-u^2} + x\,\frac{2\,(1+u^2)}{(1-u^2)^2}\,\frac{du}{dx} = \frac{1+u}{1-u},$$

which miraculously simplifies to:

$$\frac{du}{dx} = \frac{1-u^2}{2\,x}$$

This, of course, is a separable equation:

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{dx}{x}.$$

The integral on the right leads to natural logarithm. The integral on the left may seem unfamiliar yet we can easily compute it using hyperbolic trigonometry. Recall from Calculus I that

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C.$$

Our integral differs only by the sign which suggests the replacement of inverse tangent with its hyperbolic analogue. This is, indeed, correct (exercise). Therefore:

$$\tanh^{-1}(u) = \frac{1}{2}\,\ln(x) + C = \ln\sqrt{C'\,x}$$

Consequently,

$$\begin{aligned} u = \tanh(\ln\sqrt{C'\,x}) &= \frac{\sinh(\ln\sqrt{C'\,x})}{\cosh(\ln\sqrt{C'\,x})} \\ &= \frac{\sqrt{C'\,x} - \dfrac{1}{\sqrt{C'\,x}}}{\sqrt{C'\,x} + \dfrac{1}{\sqrt{C'\,x}}} = \frac{C'\,x - 1}{C'\,x + 1} \end{aligned}$$

and

$$y = x\,\frac{2\,u}{1-u^2} = \frac{C'}{2}\,x^2 - \frac{1}{2\,C'}.$$

In summary, in order to form the mirror for a headlight, we need to rotate a parabola

$$y = \frac{C'}{2}\,x^2 - \frac{1}{2\,C'} \tag{9}$$

around the y axis. Notice that we have infinitely many choices of parabolas differing by the constant C' which should be positive.[2] As long as the bulb is placed at the origin, which is the focus of the parabola (9), any $C' > 0$ will do.

Exercises

1. Express the inverse hyperbolic tangent in terms of the logarithmic function similarly to how that was done for $\sinh^{-1}$ in the handout.

[2] Try to understand where that restriction comes from—this is a good way to understand the derivation.

2. Use implicit differentiation to find the derivatives of $\tanh^{-1}(x)$ and $\sinh^{-1}(x)$.

3. Evaluate the following integrals using appropriate substitutions:

   (a) $\int_1^2 \sqrt{9 - x^2}\, dx$

   (b) $\int_1^2 \sqrt{9\, x^2 - 1}\, dx$

   In each case, validate your answers using `Matlab`'s `quad`.

4. Consider a heavy chain of length $L > 1$ suspended at points $(0, 0)$ and $(1, 0)$. Let $y = y(x)$ be the equation describing the shape of the chain. It can be shown that with a certain choice of physical units for tension:

$$\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

   Solve this ODE starting with the obvious substitution $u = \frac{dy}{dx}$ followed with separation of variables.