Laplace transform

Linear electrical circuits, linear mechanical devices, and, indeed, *linear systems* of any kind can often be conveniently analyzed using the Laplace transform—the main topic of this section.

The concept of an integral transform requires a high level of abstraction and familiarity with advanced mathematics. This is why in a typical introductory course the Laplace transform is pulled out of a hat and presented as a Calculus tool for solving certain kinds of equations. It is certainly difficult, if not impossible, to discover the Laplace transform without knowing what it is and what it does to linear differential equations with constant coefficients. Therefore, we will start this handout by directly introducing the Laplace transform and some of its basic properties. Once the nature and the utility of the Laplace transform becomes clear, we will examine the circle of ideas which lead to integral transforms and the subject of *operational calculus.*

Definition of the Laplace transform. Let $f = f(t)$ be a function of time. Think of f as the solution of some linear initial value problem with constant coefficients: it could be, say, the charge on the positive plate of a capacitor in a linear RC-circuit, or the displacement of a mass suspended on a spring, etc.

The Laplace transform of f is defined as the following integral:

$$\mathcal{L}[f] = \int_0^\infty e^{-s\,t}\, f(t)\, dt. \tag{1}$$

Henceforth we refer to Equation (1) as the *definition of the Laplace transform.*

Before proceeding further, let us make several general observations:

(1) Since the integral (1) is *definite*, the Laplace transform $\mathcal{L}[f]$ does *not* depend on the variable of integration (time) t.

(2) The integral (1) contains a parameter s which can (and does!) vary: this makes the Laplace transform a function. In the context of differential equations, the “Laplace variable” s does not have a physical meaning—think of it as an abstract independent variable.

(3) The integral (1) is improper[1] and therefore requires special care. Recall that the Calculus interpretation of improper integrals

[1]More precisely, it is an improper integral of the first kind on account of the infinite upper limit.

involves limits:

$$\int_0^\infty e^{-s\,t}\, f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-s\,t}\, f(t)\, dt.$$

If f is well-behaved the truncated integral

$$\int_0^T e^{-s\,t}\, f(t)\, dt$$

will exist for all $T > 0$, however, the limit, as T approaches infinity, may very well fail to exist. For instance, take $f = 1$ and $s = 0$. Then

$$\mathcal{L}[1](0) = \lim_{T\to\infty} \int_0^T 1\, dt \to \infty.$$

On the other hand, if $s = 1$:

$$\mathcal{L}[1](1) = \lim_{T\to\infty} \int_0^T e^{-t}\, dt = \lim_{T\to\infty} (1 - e^{-T}) = 1.$$

Typically, $\mathcal{L}[f]$ will exist for all s exceeding some critical value which depends on the function f that is being transformed; the set of all values of the Laplace variable s for which the Laplace transform $\mathcal{L}[f]$ makes sense is called the *strip of convergence*.

(4) Since integration is a linear operation, so is the Laplace transform. As such, it takes linear combinations of functions into linear combinations of transforms:

$$\mathcal{L}[a\, f + b\, g] = a\, \mathcal{L}[f] + b\, \mathcal{L}[g].$$

Keeping the above observations in mind, let us look at concrete examples of Laplace transforms.

Laplace transform of an exponential. As the first and most important example, let us transform the exponential function $f = e^{k\,t}$. According to the definition of the Laplace transform (see Equation (1)):

$$\begin{aligned}\mathcal{L}[e^{k\,t}] &= \int_0^\infty e^{-s\,t}\, e^{k\,t}\, dt = \lim_{T\to\infty} \int_0^T e^{(k-s)\,t}\, dt \\ &= \lim_{T\to\infty} \frac{e^{(k-s)\,T}}{k-s} - \frac{1}{k-s}.\end{aligned}$$

Now, in order to evaluate the limit, we need to know the sign of $(k-s)$. This leads us to consider three possibilities:

$k - s > 0$**:** The exponential $e^{(k-s)\,T}$ increases to infinity leading to an infinite limit.

$k - s = 0$**:** By L'Hospital's Rule:

$$\lim_{k \to s} \frac{e^{(k-s)\,T} - 1}{k - s} = T,$$

which again leads to an infinite limit since $T \to \infty$.

$k - s < 0$**:** The exponential $e^{(k-s)\,T}$ decays to zero and we are left with

$$0 - \frac{1}{k-s} = \frac{1}{s-k}$$

We conclude that the Laplace transform of $e^{k\,t}$ exists only for $s > k$ and is given by:

$$\mathcal{L}[e^{k\,t}] = \frac{1}{s-k}, \quad s > k. \tag{2}$$

Transform of a simple IVP. Having derived Equation (2) we are now ready to illustrate the use of the Laplace transform. Consider the following simple IVP:

$$\frac{dy}{dt} = -y, \quad y(0) = 3. \tag{3}$$

Applying the Laplace transform to both sides of Equation (3), we get:

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -\mathcal{L}[y]. \tag{4}$$

Notice that on the right-hand side the minus sign was carried out by linearity. However, on the left-hand side of Equation (4) there is an unfamiliar term $\mathcal{L}\left[\frac{dy}{dt}\right]$—the Laplace transform of the derivative.

> The proper way to work with unfamiliar Laplace transforms is to use the definition.

According to Equation (1):

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty e^{-s\,t}\,\frac{dy}{dt}\,dt.$$

What can be done with such an integral?

The presence of the derivative offers a clue. Set

$$u = e^{-s\,t}, \quad dv = \frac{dy}{dt}\,dt = dy.$$

Then integration by parts leads to

$$\int_0^\infty e^{-s\,t}\,\frac{dy}{dt}\,dt = e^{-s\,t}\,y\Big|_0^\infty + s\int_0^\infty e^{-s\,t}\,y(t)\,dt.$$

Observe that the integral on the right is simply $s\,\mathcal{L}[y]$. This is a very promising development! To ensure convergence, we must have $\lim_{t\to\infty} e^{-s\,t}\,y(t) = 0$. Therefore

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\,\mathcal{L}[y] - y_0. \tag{5}$$

In courses on Differential Equations Equation (5) is considered the most important property of the Laplace transform. Indeed, using (5) we can rewrite (4) as an *algebraic* equation in one unknown, namely, $\mathcal{L}[y]$:

$$s\,\mathcal{L}[y] - 3 = \mathcal{L}[y].$$

It then immediately follows that

$$\mathcal{L}[y] = \frac{3}{s+1}. \tag{6}$$

Finally, comparing Equations (6) and (2), we see that the sought solution of IVP (3) is

$$y = 3\,e^{-t},$$

as we well know.

The solution of any linear IVP with constant coefficients can be obtained in the same manner:

(1) Apply the Laplace transform to both sides distributing it over linear combinations and transforming the derivatives using Equation (5) and initial conditions.
(2) From the resulting algebraic equation find the Laplace transform of the solution of the IVP.
(3) Find the actual solution of the IVP by comparing its Laplace transform with known Laplace transforms.

The last step suggests that the success of the scheme heavily depends on our ability to recognize Laplace transforms of elementary functions. We therefore turn our attention to enlarging our table of known Laplace transforms.

Laplace transforms of elementary functions. In principle, finding the Laplace transform of any given function can be accomplished through straightforward integration. However, the straightforward approach can also be very tedious. For instance, the usual Calculus treatment of

$$\mathcal{L}[t^{10}] = \int_0^\infty e^{-s\,t}\,t^{10}\,dt$$

consists of 10-fold integration by parts. Clearly, this is something we would like to avoid! Fortunately, with a bit of trickery almost all elementary Laplace transforms can be derived from Equations (2) as we now demonstrate.

Constants. By linearity, it suffices to transform the constant function $f = 1$. We know from Equation (2) that

$$\mathcal{L}[e^{k\,t}] = \frac{1}{s-k}, \quad s > k.$$

Set $k = 0$. This at once leads to

$$\mathcal{L}[1] = \frac{1}{s}, \quad s > 0. \tag{7}$$

Powers. The Laplace transform of 1, given by Equation (7), states:

$$\int_0^\infty e^{-s\,t}\,dt = \frac{1}{s}.$$

Differentiate both sides with respect to the Laplace variable s:

$$\frac{d}{ds}\int_0^\infty e^{-s\,t}\,dt = -\frac{1}{s^2}.$$

Since t and s are independent variables, the operations of integration and differentiation can be interchanged:[2]:

$$\frac{d}{ds}\int_0^\infty e^{-s\,t}\,dt = \int_0^\infty \frac{\partial}{\partial s}(e^{-s\,t})\,dt = \int_0^\infty e^{-s\,t}\,(-t)\,dt.$$

Therefore

$$\int_0^\infty e^{-s\,t}\,(-t)\,dt = -\frac{1}{s^2},$$

whence follows:

$$\mathcal{L}[t] = \frac{1}{s^2}, \quad s > 0. \tag{8}$$

Now differentiate Equation (8) with respect to s using the same trick of bringing the derivative inside the integral. This leads to the Laplace transform of the square:

$$\mathcal{L}[t^2] = -\frac{d}{ds}\mathcal{L}[t] = \frac{2}{s^3}, \quad s > 0. \tag{9}$$

Successive differentiation produces

$$\mathcal{L}[t^3] = -\frac{d}{ds}\mathcal{L}[t^2] = \frac{6}{s^4}, \quad \mathcal{L}[t^4] = -\frac{d}{ds}\mathcal{L}[t^3] = \frac{24}{s^5}, \ldots,$$

[2]Note that the full derivative $\frac{d}{ds}$ gets replaced with the partial derivative $\frac{\partial}{\partial s}$ inside the integral because the exponential $e^{-s\,t}$ is a function of *two* variables.

which suggests that for all integer n:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0. \tag{10}$$

Since Laplace transform is linear, we can now transform any polynomial.

Products of powers and exponentials. The discussion leading to Equation (10) suggests a general principle for generating Laplace transforms. Suppose we know the Laplace transform of some function f. The derivative of that transform (with respect to s) is

$$\begin{aligned}\frac{d}{ds}\,\mathcal{L}[f] &= \frac{d}{ds}\int_0^\infty e^{-s\,t}\,f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-s\,t}\,f(t)\right)\,dt \\ &= \int_0^\infty e^{-s\,t}\,(-t)\,f(t)\,dt = -\mathcal{L}[t\,f].\end{aligned}$$

In other words, differentiation of the Laplace transform of the function leads to the Laplace transform of its product with t:

$$\frac{d}{ds}\mathcal{L}[f] = -\mathcal{L}[t\,f]. \tag{11}$$

Notice a remarkable duality of the formulas (11) and (5): the Laplace transform converts differentiation with respect to t into multiplication by the Laplace variable s and, conversely, differentiation of the Laplace transform with respect to s corresponds to multiplication of the original by t.

Applying Equation (11) to the Laplace transform of the exponential (2) we get:

$$\mathcal{L}[t\,e^{k\,t}] = -\frac{d}{ds}\mathcal{L}[e^{k\,t}] = -\frac{d}{ds}\frac{1}{s-k} = \frac{1}{(s-k)^2}.$$

Repeated differentiation leads to the general formula:

$$\mathcal{L}[t^n\,e^{k\,t}] = \frac{n!}{(s-k)^{n+1}}, \quad s > 0. \tag{12}$$

Trig functions. Setting $k = i\,\omega$ in Equation (2) formally leads to

$$\mathcal{L}[e^{i\,\omega\,t}] = \frac{1}{s - i\,\omega}.$$

Now using Euler's formula and linearity of the Laplace transform, we can rewrite the left-hand side as

$$\mathcal{L}[e^{i\,\omega\,t}] = \mathcal{L}[\cos(\omega\,t) + i\,\sin(\omega\,t)] = \mathcal{L}[\cos(\omega\,t)] + i\,\mathcal{L}[\sin(\omega\,t)].$$

On the right, multiply and divide by the conjugate of the denominator:

$$\frac{1}{s-i\,\omega} = \frac{s+i\,\omega}{(s-i\,\omega)\,(s+i\,\omega)} = \frac{s}{s^2+\omega^2} + i\,\frac{\omega}{s^2+\omega^2}.$$

We have shown that

$$\mathcal{L}[\cos(\omega\,t)] + i\,\mathcal{L}[\sin(\omega\,t)] = \frac{s}{s^2+\omega^2} + i\,\frac{\omega}{s^2+\omega^2}.$$

Since two complex numbers are equal if and only if their real and imaginary parts match, we must have:

$$\mathcal{L}[\cos(\omega\,t)] = \frac{s}{s^2+\omega^2}, \quad s>0 \tag{13}$$

$$\mathcal{L}[\sin(\omega\,t)] = \frac{\omega}{s^2+\omega^2}, \quad s>0 \tag{14}$$

Transforms of products of trig functions with exponentials and powers are left as exercises at the end of this handout. In the meanwhile, we have enough transform pairs to consider more sophisticated IVP.

Solving linear ODE with constant coefficients. We will now solve several initial value problems of first and second order following the same methodology as in Example (3). To save space we will omit some of the manipulations. Therefore the reader should keep in mind that:

(1) $\mathcal{L}$ is a linear operation, so constant coefficients can be pulled out and sums are transformed into sums.
(2) To transform derivatives one (repeatedly) uses Equation (5).
(3) To simplify complicated fractions one uses partial fraction decomposition.

In this part of the handout we will merely state the partial fraction decompositions (PFD) without explanation. An overview of (PFD) is provided later in the text.

First order IVP with trig right-hand side. Consider

$$\frac{dy}{dt} + y = \sin(t), \quad y(0)=0. \tag{15}$$

This is our earlier example (3) with an added trigonometric forcing term.

Take the Laplace transform of both sides

$$s\,\mathcal{L}[y] + \mathcal{L}[y] = \frac{1}{s^2+1},$$

and solve for $\mathcal{L}[y]$:

$$\mathcal{L}[y] = \frac{1}{(s+1)\,(s^2+1)}. \tag{16}$$

Using partial fraction decomposition, rewrite (16) as

$$\mathcal{L}[y] = \frac{1}{2}\left\{\frac{1}{s+1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}.\right\}$$

Comparison with Equations (2), (14), and (13) shows that the solution of (15) has three terms:

$$y = \frac{1}{2}\left\{e^{-t} + \sin(t) - \cos(t)\right\}. \tag{17}$$

As a quick exercise, parse Equation (17) into y_c and y_p and confirm that the answer is correct.

First order IVP with exponential right-hand side. Let us replace the sine in Equation (15) with an exponential term:

$$\frac{dy}{dt} + y = e^{-t}, \quad y(0) = 0. \tag{18}$$

Recall that previously we had to solve this equation using L'Hôspital's Rule: the simple Guess-and-Check method fails since the right-hand side is a part of the complimentary function y_c.

Following the same steps as before, we transform the IVP

$$s\,\mathcal{L}[y] + \mathcal{L}[y] = \frac{1}{s+1},$$

solve for $\mathcal{L}[y]$

$$\mathcal{L}[y] = \frac{1}{(s+1)^2}, \tag{19}$$

and, finally, identify y by looking at Equation (12):

$$y = t\,e^{-t}. \tag{20}$$

Notice that Equation (20) agrees with our earlier result.

Second order IVP with trig right-hand side. Let us increase the order to two and consider the following IVP:

$$\frac{d^2y}{dt^2} + \omega_0^2\,y = \sin(\omega\,t), \quad y(0) = \frac{dy}{dt}(0) = 0. \tag{21}$$

Think of Equation (21) as describing a frictionless mass-spring system: y is displacement, ω_0 is the natural frequency, the right-hand side is a periodic forcing term with frequency ω.

Applying the Laplace transform to both sides of Equation (21), we get

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + \omega_0^2\,\mathcal{L}[y] = \frac{\omega}{s^2+\omega^2}. \tag{22}$$

In order to transform the second derivative, introduce velocity:

$$\frac{dy}{dt} = v, \quad v(0) = \frac{dy}{dt}(0) = 0.$$

Writing the second derivative of y as the first derivative of v and applying the fundamental identity (5), we get:

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = \mathcal{L}\left[\frac{dv}{dt}\right] = s\,\mathcal{L}[v] - v_0 = s\,\mathcal{L}[v].$$

Since velocity is the first derivative of position, we apply (5) again to get:

$$\mathcal{L}[v] = \mathcal{L}\left[\frac{dy}{dt}\right] = s\,\mathcal{L}[y] - y_0 = s\,\mathcal{L}[y].$$

This shows that, quite generally, the second derivative transforms according to the rule:

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s\,\mathcal{L}\left[\frac{dy}{dt}\right] - \frac{dy}{dt}(0) = s\,\left(s\,\mathcal{L}[y] - y(0)\right) - \frac{dy}{dt}(0).$$

Due to zero initial conditions, we simply get $s^2\,\mathcal{L}[y]$. Consequently, Equation (22) becomes

$$s^2\,\mathcal{L}[y] + \omega_0^2\,\mathcal{L}[y] = \frac{\omega}{s^2 + \omega^2},$$

which, when solved for $\mathcal{L}[y]$, yields:

$$\mathcal{L}[y] = \frac{\omega}{(s^2 + \omega_0^2)\,(s^2 + \omega^2)}.$$

Partial fraction decomposition splits the fraction into the difference

$$\mathcal{L}[y] = \frac{\omega}{\omega^2 - \omega_0^2}\left\{\frac{1}{(s^2 + \omega_0^2)} - \frac{1}{(s^2 + \omega^2)}\right\},$$

which shows that the solution of IVP (21) is given by:

$$y = \frac{\omega}{\omega^2 - \omega_0^2}\left\{\frac{\sin(\omega_0 t)}{\omega_0} - \frac{\sin(\omega t)}{\omega}\right\}. \tag{23}$$

Partial fraction decomposition. Although partial fraction decomposition is typically encountered in Calculus courses, it has nothing to do with Calculus *per se*: it is a purely algebraic technique for writing complicated fractions as sums of simple fractions. Accordingly, let us discuss PFD in the context of algebra, starting with numerical fractions.

Rational numbers. Recall that a *rational number* is a quotient of two integers. We will symbolize this by writing

$$r = \frac{m}{n}, \quad m, n \in \mathbb{Z}, \quad n \neq 0.$$

Explanation of notation: the symbol $\mathbb{Z}$ is shorthand for the set of all integers; the sign $\in$ stands for "element of"; the set of all rational numbers is usually denoted by $\mathbb{Q}$.

If $m > n$, the fraction m/n is called *improper*. Every improper fraction can be written as a sum of an integer and a proper fraction, e.g.:

$$\frac{32}{15} = 2 + \frac{2}{15}.$$

Henceforth we restrict our attention to proper fractions. Furthermore, since the numerator m is allowed to have any sign, we will assume that the denominator n is a *natural number*[3].

Any natural number can be factored into a product of primes, unique up to order[4]. Let us call any fraction $1/p$ with p prime a *prime factor*. The algebraic rule for adding fractions

$$m_1 \frac{1}{p_1} + m_2 \frac{1}{p_2} = \frac{m_1\, p_2 + m_2\, p_1}{p_1\, p_2},$$

shows that from prime factors one can build fractions with compound denominators. However, this also suggests that one can go in the opposite direction. That is, one should be able to split fractions with compound denominators into sums of prime factors. For instance, we should be able to find integers A and B such that

$$\frac{2}{15} = \frac{2}{3 \times 5} = \frac{A}{3} + \frac{B}{5}.$$

The coefficients must satisfy $|A| < 3$ and $|B| < 5$ for the fractions to be proper.

To find A and B in the above partial fraction decomposition we can reason as follows. Let us bring the terms on the left to the common denominator. This results in

$$\frac{2}{15} = \frac{5\,A + 3\,B}{15}.$$

The denominators match, therefore the numerators must match as well. This means that A and B must be integer solutions of:

$$5\,A + 3\,B = 2.$$

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[3] Positive integer

[4] A prime number is a natural number that is divisible only by one and itself: the prime number sequence begins with the terms 1,2,3,5,7,11,13,...

Since $|A| < 3$ the only possible integer values for A are

$$A = -2, -1, 0, 1, 2.$$

Likewise, since $|B| < 5$ the only possible integer values for B are

$$B = -4, -3, -2, -1, 0, 1, 2, 3, 4.$$

This amounts to 45 possibilities of which only $A = 1$ and $B = -1$ works, as can be easily verified. Hence

$$\frac{2}{15} = \frac{1}{3} - \frac{1}{5}.$$

Things complicate slightly if the denominator of a fraction has repeated prime factors. For instance, we cannot write

$$\text{Wrong:} \quad \frac{1}{20} = \frac{1}{2^2\,5} = \frac{A}{2} + \frac{B}{5}.$$

Indeed, $10\,A + 4\,B = 1$ does not have integer solutions because the left side is always even while the right side is odd. The correct decomposition is of the form

$$\text{Correct:} \quad \frac{1}{20} = \frac{A}{2} + \frac{B}{5} + \frac{C}{2^2}.$$

And solving $10\,A + 4\,B + 5\,C = 1$ in integers we can easily find $A = 1$, $B = -1$, and $C = -1$. This shows that fractions such as $1/2^2$ should be treated as prime factors. The full list of prime rational numbers therefore consists of all numbers of the form $1/p^k$, where p is prime and k is a positive integer.

Rational functions. Let us now transfer our understanding of rational numbers to *rational functions*. Recall that a rational function is a quotient of two polynomials:

$$R(s) = \frac{M(s)}{N(s)}.$$

If the degree of the numerator is greater or the same as the degree of the denominator, the rational function is called improper. Any improper rational function can be written as a sum of a polynomial and a proper rational function by means of long division, e.g.:

$$\frac{s^2}{s+1} = s - 1 + \frac{1}{s+1}.$$

Henceforth, to keep the discussion simple, we restrict our attention to proper rational functions [5]. We further assume that the fractions are

[5] Notice that all Laplace transforms we have derived so far are proper fractions

scaled so that the highest order coefficient in the denominator is one:

$$N(s) = s^n + a_{n-1}\, s^{n-1} + \ldots$$

In other words, $N(s)$ is a *monic* polynomial.

Denote by $s_1,\ s_2,\ \ldots,\ s_n$ the complex roots of $N(s)$. The Fundamental Theorem of Algebra (proved by Gauss) states that monic polynomials, just like natural numbers, are built from primes:

$$N(s) = (s - s_1)\,(s - s_2)\ \ldots\ (s - s_n).$$

For instance

$$\begin{aligned} x^2 + 4 &= (x - 2\,i)\,(x + 2\,i) \\ x^3 - 1 &= (x - 1)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)\right), \end{aligned}$$

and so on.

Following the analogy with rational numbers, we call any function of the form

$$\frac{1}{(s-a)^k}, \quad k \text{ is a positive integer,}$$

a *prime factor*. If the complex roots of $N(s)$ are all distinct, PFD has the form:

$$\frac{M(s)}{N(s)} = \frac{C_1}{s - s_1} + \frac{C_2}{s - s_2} + \ldots + \frac{C_n}{s - s_n}.$$

If some root s_k is repeated m_k times, one should introduce the terms

$$\frac{C_{k,1}}{s - s_k} + \frac{C_{k,2}}{(s - s_k)^2} + \ldots + \frac{C_{k,m_k}}{(s - s_k)^{m_k}}.$$

The procedure for finding C_i's—the coefficients in the partial fraction decomposition of rational *functions*—is analogous to the one used for rational numbers. As a simple illustration, let us consider the PFD of

$$\frac{s - 4}{s^2 - 8\,s + 15}.$$

Since $s^2 - 8\,s + 15 = (s - 3)\,(s - 5)$, we can write the fraction as

$$\frac{C_1}{s - 3} + \frac{C_2}{s - 5}.$$

Bring the prime factors to the common denominator and compare the resulting numerator with that of the original fraction:

$$C_1\,(s - 5) + C_2\,(s - 3) = s - 4.$$

Since two polynomials are equal if and only if the like coefficients match, C_1 and C_2 must be chosen so that:

$$\begin{aligned} C_1 + C_2 &= 1 \\ -5\,C_1 - 3\,C_2 &= -4 \end{aligned}$$

Therefore $C_1 = C_2 = \frac{1}{2}$ and

$$\frac{s-4}{s^2 - 8\,s + 15} = \frac{1}{2}\left(\frac{1}{s-3} + \frac{1}{s-5}\right).$$

As another example, consider

$$\frac{1}{(s+1)\,(s^2+1)}.$$

In Calculus courses one writes PFD with two real-valued terms:

$$\frac{A}{s+1} + \frac{B\,s + C}{s^2+1}.$$

Note that the numerator of the second fraction is a linear function because the denominator has degree 2.

The Calculus form of PFD is forced by unfamiliarity with complex numbers: if only real numbers are allowed, quadratics with complex roots must be treated as prime factors. Since we have complex numbers, let us factor $s^2 + 1 = (s-i)\,(s+i)$ and write

$$\frac{1}{(s+1)\,(s-i)\,(s+i)} = \frac{C_1}{s+1} + \frac{C_2}{s+i} + \frac{C_3}{s-i}.$$

Bring the fractions to the common denominator and compare the numerators:

$$\begin{aligned} &C_1(s+i)\,(s-i) + C_2\,(s+1)\,(s-i) + C_3\,(s+1)\,(s+i) \\ &\quad = (C_1 + C_2 + C_3)\,s^2 + ((1-i)\,C_2 + (1+i)\,C_3)\,s + (C_1 - i\,C_2 + i\,C_3) = 1. \end{aligned}$$

Matching the like terms, we arrive at the system of three equations

$$\begin{aligned} C_1 + C_2 + C_3 &= 0 \\ (1-i)\,C_2 + (1+i)\,C_3 &= 0 \\ C_1 - i\,C_2 + i\,C_3 &= 1 \end{aligned}$$

whose solution is:

$$C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{4} + \frac{1}{4}\,i, \quad C_3 = -\frac{1}{4} - \frac{1}{4}\,i.$$

Therefore

$$\frac{1}{(s+1)\,(s^2+1)} = \frac{1}{2}\,\frac{1}{s+1} + \left(-\frac{1}{4}+\frac{1}{4}\,i\right)\,\frac{1}{s+i} + \left(-\frac{1}{4}-\frac{1}{4}\,i\right)\,\frac{1}{s-i}.$$

As a final example of PFD, consider:

$$\frac{1}{s^2\,(s+1)}.$$

Since the factor s is repeated twice, we write PFD in the form:

$$\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+1}.$$

Bringing the fractions to the common denominator and comparing numerators we get

$$A\,s\,(s+1)+B\,(s+1)+C\,s^2=(A+C)\,s^2+(A+B)\,s+A=1.$$

Comparing the like terms, we conclude that $A=1$ which forces $B=C=-1$. Thus

$$\frac{1}{s^2\,(s+1)}=\frac{1}{s}-\frac{1}{s^2}-\frac{1}{s+1}.$$

Summary of PFD. Given a proper[6] rational function

$$R(s)=\frac{M(s)}{N(s)},$$

find the roots of the denominator s_1, s_2, …, s_n. Generally, these will be complex numbers some of which may be repeated. If all roots happen to be distinct, write PFD as

$$R(s)=\frac{C_1}{s-s_1}+\frac{C_2}{s-s_2}+\ldots+\frac{C_n}{s-s_n}.$$

Otherwise, for each root s_k that is repeated $m_k>1$ times, introduce terms

$$\frac{C_{k,1}}{s-s_k}+\frac{C_{k,2}}{(s-s_k)^2}+\ldots+\frac{C_{k,m_k}}{(s-s_k)^{m_k}}.$$

Bring all prime factors in PFD to the common denominator (which is $N(s)$) and set the resulting numerator to $M(s)$—the numerator of $R(s)$. This gives on one side $M(s)$ and on the other a polynomial dependant on the coefficients of PFD. Polynomials match if and only if their like coefficients match. Use this to write a system of linear equations whose solution will give the coefficients of PFD.

[6] For improper fractions, apply long division.

Laplace transforms of discontinuous functions. Shifting property. One of the main advantages of the Laplace transform technique is that it allows simple and natural treatment of ODE with discontinuous right-hand sides. As an illustration, consider the following IVP

$$\frac{dy}{dt} + y = f(t), \quad y(0) = 0, \tag{24}$$

where the forcing term is the *unit step* function:

$$f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1 \end{cases} \tag{25}$$

Application of the Laplace transform leads to

$$\mathcal{L}[y] = \frac{1}{s+1}\,\mathcal{L}[f].$$

Here we need to pause and discuss a common mistake which is to distribute the symbol $\mathcal{L}$ over the various cases:

$$\text{Wrong:} \quad \mathcal{L}\left[\begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 0 \end{cases}\right] = \begin{cases} \dfrac{1}{s}, & 0 \le t \le 1, \\ \\ 0, & t > 1 \end{cases}$$

This kind of misconception is rooted in overreliance on symbolic manipulation. It seems "logical" to distribute $\mathcal{L}$ which, clearly, cannot fight back. To avoid this pitfall let us interpret $\mathcal{L}[f]$ graphically.

For instance, let us consider

$$\mathcal{L}[f](1) = \int_0^\infty e^{-t}\, f(t)\, dt.$$

Figure 1 shows that the integrand $e^{-t}f(t)$ has finite area extending from $t = 0$ to $t = 1$. Therefore

$$\mathcal{L}[f](1) = \int_0^\infty e^{-t}\, f(t)\, dt = \int_0^1 e^{-t}\, dt = 1 - e^{-1}.$$

More generally,

$$\begin{aligned} \mathcal{L}[f(t)](s) = \int_0^\infty e^{-s\,t}\, f(t)\, dt &= \int_0^1 e^{-s\,t}\, 1\, dt \\ + \int_1^\infty e^{-s\,t}\, 0\, dt &= -\left.\frac{e^{-s\,t}}{s}\right|_0^\infty \\ &= \frac{1 - e^{-s}}{s}. \end{aligned} \tag{26}$$

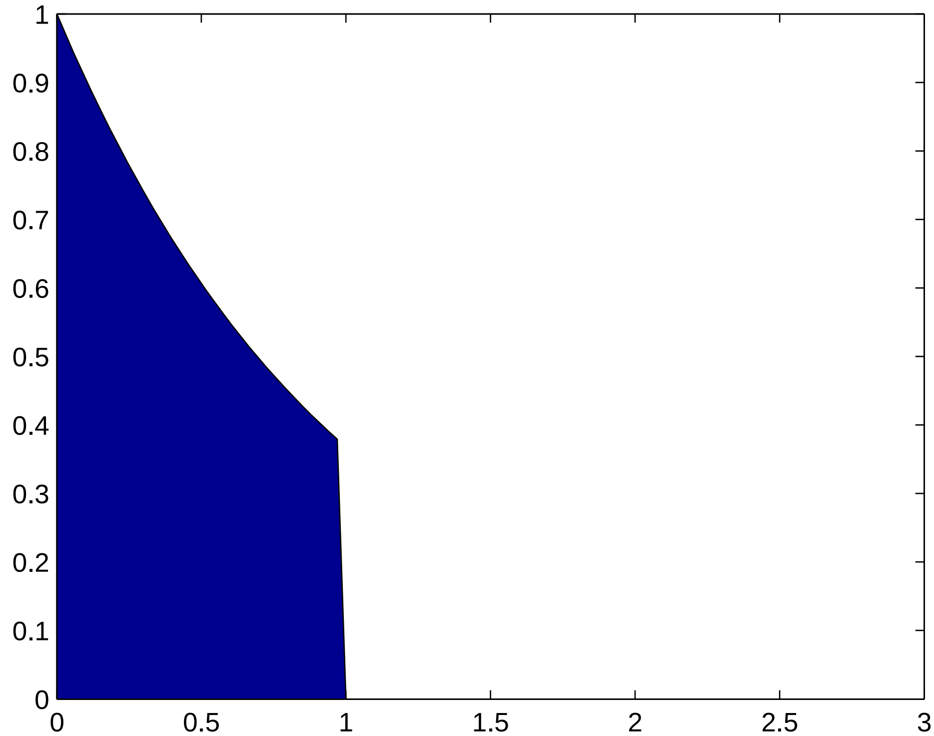

Figure 1. Area under the graph of $e^{-t} f(t)$

We conclude that the Laplace transform of the solution of IVP (24) is given by

$$\mathcal{L}[y] = \frac{1 - e^{-s}}{s\,(s+1)} = \frac{1}{s\,(s+1)} - e^{-s}\,\frac{1}{s\,(s+1)}. \tag{27}$$

Now since

$$\frac{1}{s\,(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

the first term corresponds to the function $1-e^{-t}$. However, the second term has an unfamiliar exponential factor in front. To find out what the second term corresponds to, let us derive the solution of (24) without the use of the Laplace transform.

We can reason as follows. As long as $t < 1$, the IVP states

$$\frac{dy}{dt} + y = 1, \quad y(0) = 0.$$

Consequently, for $t < 1$ the solution is given by $y_1 = 1 - e^{-t}$. When $t > 1$, the ODE changes to

$$\frac{dy}{dt} + y = 0.$$

Therefore, for $t > 1$ the solution must have the form $y_2 = C\,e^{-t}$. Let us think of y as the charge on a capacitor in a linear RC-circuit. For $0 \le t \le 1$ the capacitor is charging starting from zero. At time $t = 1$ the battery is disconnected and the capacitor discharges exponentially. Since the charge varies continuously, we can find C by setting $y_2(1) = y_1(1) = 1 - e^{-1}$. This gives $C = e - 1$. Hence,

$$y = \begin{cases} 1 - e^{-t}, & 0 \le t \le 1, \\ e^{1-t} - e^{-t}, & t > 1, \end{cases} \tag{28}$$

with the graph shown in Figure 2.

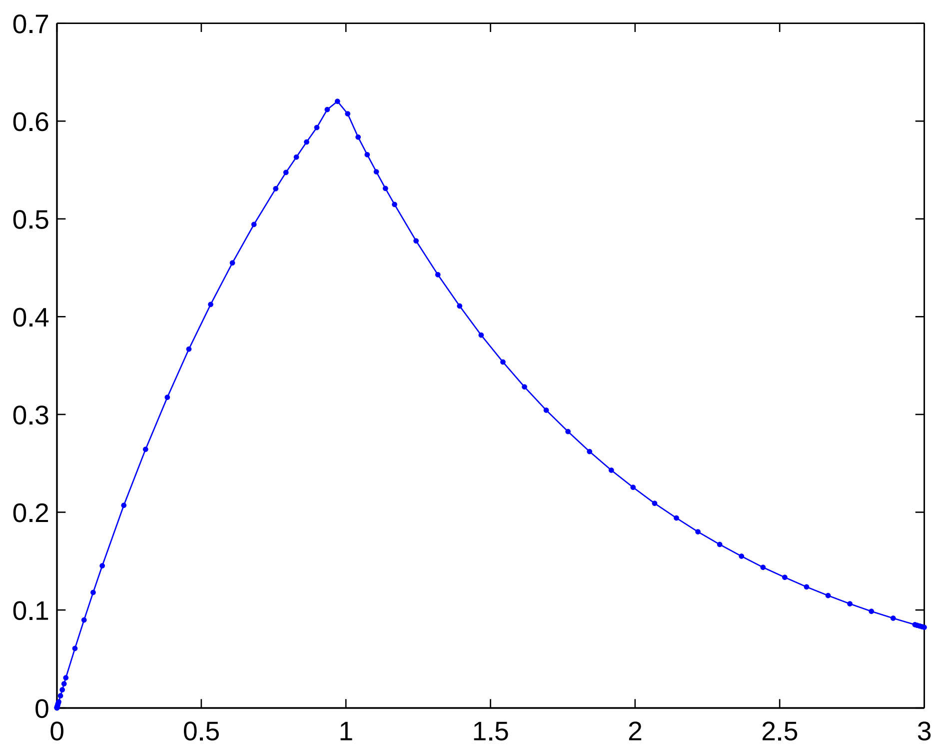

FIGURE 2. Solution of IVP (24).

We now need to reconcile Equation (28) with Equation (27). Isolate the term $1 - e^{-t}$ in Equation (28) as follows:

$$y = 1 - e^{-t} - \begin{cases} 0, & 0 \le t \le 1, \\ 1 - e^{-(t-1)}, & t > 1. \end{cases}$$

Comparison with (27) shows that

$$\mathcal{L}\left[\begin{cases} 0, & 0 \le t \le 1, \\ 1 - e^{-(t-1)}, & t > 1. \end{cases}\right] = e^{-s}\,\frac{1}{s\,(s+1)}.$$

Let us confirm that. The function in brackets is zero on the interval $[0, 1]$; for $t > 1$ it is the translate of $1 - e^{-t}$ by one unit to the right. Therefore, by definition of the Laplace transform:

$$\mathcal{L}\left[\begin{cases} 0, & 0 \le t \le 1, \\ 1 - e^{-(t-1)}, & t > 1. \end{cases}\right] = \int_1^\infty e^{-s\,t}\,\left(1 - e^{-(t-1)}\right)\,dt.$$

Substituting $t - 1 = u$ we get

$$\begin{aligned} \int_0^\infty e^{-s\,(u+1)}\,\left(1 - e^{-u}\right)\,du &= e^{-s}\int_0^\infty e^{-s\,u}\,\left(1 - e^{-u}\right)\,du \\ &= e^{-s}\,\mathcal{L}[1 - e^{-t}] = e^{-s}\,\frac{1}{s\,(s+1)}, \end{aligned}$$

as required.

Quite generally, let f be any function with known Laplace transform $\mathcal{L}[f]$. For $a > 0$, define a shifted function f_a by

$$f_a = \begin{cases} 0, & 0 \le t \le a, \\ f(t-a), & t > a. \end{cases} \tag{29}$$

Using the same logic as above, one can show

$$\mathcal{L}[f_a] = e^{-a\,s}\,\mathcal{L}[f]. \tag{30}$$

Equation (30) is called the *shifting property* of the Laplace transform.

As a quick illustration of the use of Equation (30), let us solve the following IVP:

$$\frac{d^2y}{dt^2} + y = f(t), \quad y(0) = \frac{dy}{dt}(0) = 0, \tag{31}$$

where f is the step function (25). Write the Laplace transform $\mathcal{L}[y]$ as:

$$\mathcal{L}[y] = \frac{1 - e^{-s}}{s\,(s^2+1)} = \frac{1}{s\,(s^2+1)} - \frac{e^{-s}}{s\,(s^2+1)}. \tag{32}$$

Since

$$\frac{1}{s\,(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

the first term corresponds to the function $1-\cos(t)$. The second term, according to Equation (30), is the first term shifted by 1. Therefore

$$y = 1-\cos(t) - \begin{cases} 0, & 0 \le t \le 1, \\ 1-\cos(t-1), & t > 1. \end{cases} \tag{33}$$

Figure 3 confirms our result: the solid line is the plot of Equation (33); the dots are the numerical approximation obtained using Matlab's `ode45` command.

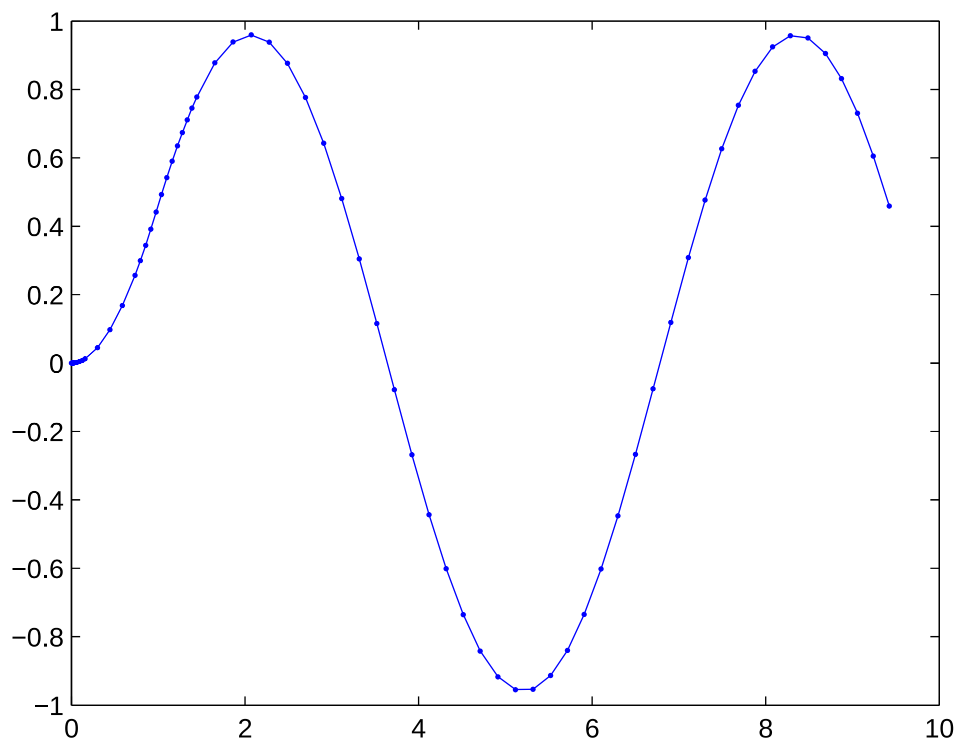

FIGURE 3. Solution of IVP (31).

The origin of the Laplace transform. The Laplace transform is an example of a continuous integral transform: that is, $\mathcal{L}$ takes a function of continuous time t into another function of continuous variable s.

The idea of an integral transform can be better understood if one starts with a discrete case. Therefore let us replace continuous time t with discrete time $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\,\Delta t$, $\ldots$; and, for convenience, let us abbreviate $f(t_k) = f_k$.

> Suppose we have some question about an infinite sequence $f_0,\ f_1,\ f_2,\ \ldots$ Is there a general way of transforming such a question into a simpler one?

Let us look at a classic example.

Generating functions. You may have heard of Fibonacci numbers which are usually defined by the *recurrence relation*:

$$f_n = f_{n-1} + f_{n-2}, \quad f_0 = f_1 = 1. \tag{34}$$

Equation (34) simply says that the first two elements of the Fibonacci sequence are 1 while every term starting with the third is the sum of the previous two. Notice how close (34) resembles a second order IVP. This is not a coincidence! Discretization of second order linear homogeneous IVP with constant coefficients leads to equations of this kind. For this reason Equation (34) is classified as the second order linear homogeneous *difference* equation with constant coefficients.

Applying (34) recursively one can easily generate the first several terms of the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

However the general pattern is not easy to find. In fact, it had eluded Fibonacci and his successors for several centuries.

The breakthrough in the quest for the general formula for f_n came with the realization that Fibonacci numbers can be "encoded" as the coefficients of the power series:

$$F(s) = f_0 + f_1\, s + f_2\, s^2 + \ldots = \sum_{n=0}^{\infty} f_n\, s^n. \tag{35}$$

Indeed, suppose we find $F(s)$. Then,

$$f_0 = F(0), \quad f_1 = \frac{dF}{ds}(0), \quad f_2 = \frac{1}{2}\,\frac{dF}{ds^2}(0), \ldots$$

and, generally,

$$f_n = \frac{1}{n!}\frac{d^n}{ds^n}(F(s))\bigg|_{s=0}. \tag{36}$$

Thus the knowledge of *one* power series (35) is equivalent to the knowledge of the *infinitely many* Fibonacci numbers. For this reason, Equation (35) is called the *generating function* of the Fibonacci sequence.

It is much easier to look for one unknown $F(s)$ rather than an infinite sequence of unknowns $\{f_n\}$. And, in fact, $F(s)$ is very easy to find! Apply the recurrence relation (34) to the definition of the generating function (35):

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} f_n\, s^n = f_0 + f_1\, s + \sum_{n=2}^{\infty} f_n\, s^n = 1 + s + \sum_{n=2}^{\infty}(f_{n-1} + f_{n-2})\, s^n \\ &= 1 + s + \sum_{n=2}^{\infty} f_{n-1}\, s^n + \sum_{n=2}^{\infty} f_{n-2}\, s^n = 1 + s + s\, \sum_{n=2}^{\infty} f_{n-1}\, s^{n-1} \\ &+ s^2 \sum_{n=2}^{\infty} f_{n-2}\, s^{n-2} \end{aligned}$$

Expanding the summations, we see that

$$\begin{aligned} \sum_{n=2}^{\infty} f_{n-1}\, s^{n-1} &= f_1\, s + f_2\, s^2 + \ldots = F(s) - f_0 = F(s) - 1 \\ \sum_{n=2}^{\infty} f_{n-2}\, s^{n-2} &= f_0 + f_1\, s + f_2\, s^2 + \ldots = F(s) \end{aligned}$$

Therefore $F(s)$ satisfies

$$F(s) = 1 + s + s\,(F(s) - 1) + s^2\, F(s), \tag{37}$$

and is consequently given by:

$$F(s) = \frac{1}{1 - s - s^2}. \tag{38}$$

Notice the formal similarity between Equation (37) and the Laplace transform of a second order IVP. Notice further that the generating function of the Fibonacci sequence is a familiar looking fraction, which invites partial fraction decomposition.

Denote by

$$s_1 = \frac{1+\sqrt{5}}{2}, \quad s_2 = \frac{1-\sqrt{5}}{2},$$

the roots of the quadratic equation $1 - s - s^2 = 0$. Application of partial fraction decomposition to Equation (38) leads to

$$F(s) = \frac{1}{s_1 - s_2}\left(\frac{1}{s - s_1} - \frac{1}{s - s_2}\right).$$

Rearrange the decomposition as follows

$$F(s) = \frac{1}{s_1 - s_2} \left(\frac{1}{s_2} \frac{1}{1 - \dfrac{s}{s_2}} - \frac{1}{s_1} \frac{1}{1 - \dfrac{s}{s_1}} \right)$$

and use the geometric formula

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

to expand fractions into power series:

$$\begin{aligned} F(s) &= \frac{1}{s_1 - s_2} \left(\frac{1}{s_2} \sum_{n=0}^{\infty} \left(\frac{s}{s_2} \right)^n - \frac{1}{s_1} \sum_{n=0}^{\infty} \left(\frac{s}{s_1} \right)^n \right) \\ &= \sum_{n=0}^{\infty} \frac{s_2^{-(n+1)} - s_1^{-(n+1)}}{s_1 - s_2} \, s^n. \end{aligned}$$

Recall that, by construction, the coefficients of F are Fibonacci numbers. Therefore:

$$f_n = \frac{s_2^{-(n+1)} - s_1^{-(n+1)}}{s_1 - s_2}, \quad \left(s_{1,2} = \frac{1 \pm \sqrt{5}}{2} \right).$$

The lesson to be learned is the following:

> A difficult problem involving an infinite sequence
>
> $$\{f_n\}_{n=0}^{\infty} = \{f_0, f_1, f_2, \ldots\}$$
>
> can be *transformed* into a simpler problem involving the generating function:
>
> $$F(s) = \sum_{n=0}^{\infty} f_n \, s^n.$$

We now have a new and very advanced perspective on Taylor series: they are *transforms*[7] of sequences. Symbolically:

$$\{f_n\}_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} f_n \, s^n$$

As we show in the next section, the Laplace transform can be thought of as a continuous version of the Taylor series.

[7]In certain engineering disciplines Taylor series when viewed as transforms are called the Z-transforms.

Euler's method and generating functions: take one. Let us now look at an example that is much closer to differential equations. Consider the following simple IVP:

$$\frac{dy}{dt} = -y, \quad y(0) = y_0.$$

To solve it numerically we discretize the continuous time t and look for approximations to $y_n = y(n\,\Delta t)$. The simplest way to obtain such discrete approximations is through Euler's method. Recall that one simply replaces the continuous derivative with the difference quotient

$$\frac{dy}{dt}(n\,\Delta t) \approx \frac{y_n - y_{n-1}}{\Delta t},$$

which, after some algebra, leads to the recurrence relation:

$$y_n = (1 - \Delta t)\, y_{n-1}. \tag{39}$$

Following the principles used to find Fibonacci numbers, let us solve Equation (39) by introducing the generating function:

$$Y(s) = \sum_{n=0}^{\infty} y_n\, s^n. \tag{40}$$

To find the equation for $Y(s)$, multiply both sides of Equation (39) by s^n and sum from $n = 1$ to infinity:

$$\sum_{n=1}^{\infty} s^n\, y_n = (1 - \Delta t) \sum_{n=1}^{\infty} s^n y_{n-1}. \tag{41}$$

On the left, we get $Y(s) - y_0$. Meanwhile, on the right, we can write

$$\sum_{n=1}^{\infty} s^n y_{n-1} = s \sum_{n=1}^{\infty} s^{n-1} y_{n-1} = s\,Y(s).$$

Hence the equation for $Y(s)$ is:

$$Y(s) - y_0 = (1 - \Delta t)\, s\, Y(s),$$

Consequently, the generating function for the Euler's approximating sequence is:

$$Y(s) = \frac{y_0}{1 - (1 - \Delta t)\, s}. \tag{42}$$

Upon expansion into series, Equation (42) gives the correct values

$$y_n = (1 - \Delta t)^n\, y_0, \quad n = 0, 1, 2, \ldots$$

Everything is in order except that in the limit as $\Delta t \to 0$ Equation (42) stops being useful. Indeed,

$$\lim_{\Delta t \to 0} Y(s) = \frac{y_0}{1-s},$$

which implies: $\lim_{\Delta t \to 0} y_n = y_0$ for each n. On one hand this makes sense for

$$\lim_{\Delta t \to 0} y_n = \lim_{\Delta t \to 0} y(n\,\Delta t) = y(0).$$

On the other hand, as we know from the discussion of Euler's method, as $\Delta t \to 0$ the discrete approximations y_n approach the continuous solution $y(t)$. This suggests that we should modify the generating function (40) so that its limit is more interesting.

Euler's method and generating functions: take two. Set $s = 1$. Then Equation (40) reads

$$Y(1) = \sum_{n=0}^{\infty} y_n = \sum_{n=0}^{\infty} y(n\,\Delta t).$$

This gives the limit

$$\lim_{\Delta t \to 0} Y(1) = \lim_{\Delta t \to 0} \sum_{n=0}^{\infty} y(n\,\Delta t) = \sum_{n=0}^{\infty} \lim_{\Delta t \to 0} y(n\,\Delta t) = \sum_{n=0}^{\infty} y(0)$$

which is either zero, if $y(0) = 0$, or does not exist, if $y(0) \neq 0$—not very interesting. However, note that if we multiply Equation (40) by Δt, we get a much more interesting limit of a *Riemann sum*:

$$\lim_{\Delta t \to 0} Y(1)\,\Delta t = \lim_{\Delta t \to 0} \sum_{n=0}^{\infty} y(n\,\Delta t)\,\Delta t = \int_0^{\infty} y(t)\,dt = \mathcal{L}[y](0).$$

The path to the Laplace transform is now clear: the generating function must be scaled by the time step so that it looks like a Riemann sum. Unfortunately, for $s \neq 1$ the limit

$$\lim_{\Delta t \to 0} \sum_{n=0}^{\infty} s^n\, y(n\,\Delta t)\,\Delta t$$

cannot be written as an integral on account of the term s^n. To turn the summation into a *Riemann sum* we must make the term s^n look like a function of $n\,\Delta t$. The easiest way to accomplish that is to set

$$s = e^{-u\,\Delta t}. \tag{43}$$

Then

$$s^n = e^{-u\,n\,\Delta t}$$

and Equation (40) becomes a Riemann sum

$$Y(u) = \sum_{n=0}^{\infty} e^{-u\,n\,\Delta t}\, y_n\, \Delta t.$$

which in the limit produces the Laplace transform:

$$\lim_{\Delta t \to 0} Y(u) = \lim_{\Delta t \to 0} \sum_{n=0}^{\infty} e^{-u\,n\,\Delta t}\, y_n\, \Delta t = \int_0^{\infty} e^{-u\,t}\, y(t)\, dt = \mathcal{L}[y](u).$$

We note that the minus sign in Equation (43) is for convenience only: it makes the strips of convergence include positive infinity rather than negative infinity.

Exercises.

(1) Use the methods of this section to compute the Laplace transforms of the following functions:
   (a) $f_1 = (1+t)^2$
   (b) $f_2 = e^{a\,t}\, \sin(b\,t)$
   (c) $f_3 = \begin{cases} 1, & 0 \le t \le 1, \\ -1 & 1 < t \le 2, \\ 2, & t > 2, \end{cases}$

(2) Find the functions from their Laplace transforms:
   (a) $\mathcal{L}[y_1] = \dfrac{1}{s\,(s+1)\,(s+3)}$
   (b) $\mathcal{L}[y_2] = \dfrac{2+s}{s^2 + 2\,s + 5}$
   (c) $\mathcal{L}[y_3] = \dfrac{e^{-s}}{s}$

(3) Solve the following initial value problems using the Laplace transform. Confirm each solution with a Matlab plot showing the function on the interval $0 \le t \le 5$.
   (a) $\dfrac{dy}{dt} + 3\,y = t, \quad y(0) = 1.$
   (b) $\dfrac{d^2y}{dt^2} + 2\,\dfrac{dy}{dt} = f, \quad y(0) = \dfrac{dy}{dt}(0) = 0$, where f is the unit step function given by Equation (25).
   (c) $\dfrac{d^2y}{dt^2} + 4\,y = \sin(2\,t), \quad y(0) = 3, \quad \dfrac{dy}{dt}(0) = -1.$

(4) Let

$$R_\epsilon(s) = \frac{1}{(s-\epsilon)\,(s+\epsilon)\,(s+1)}$$

Find the partial fraction decomposition of R_ϵ and show that in the limit as $\epsilon \to 0$ it approaches the partial fraction decomposition of

$$R_0 = \frac{1}{s^2\,(s+1)}.$$

(5) Use the Laplace transform to solve the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= v, \quad v(0) = 0, \\ \frac{dv}{dt} &= -x, \quad x(0) = 1. \end{aligned}$$

Hint: Start by applying the Laplace transform to each equation.