4 Diagonalization

In Section 3.5 we saw that any linear transformation can be represented with a matrix. Consequently, all questions in linear algebra can be reformulated in matrix terms. In general, matrix representation of a linear transformation involves a choice of two different bases and is not unique: different bases lead to different matrices. Theoretically, it does not matter which matrix representation one works with: they are all equivalent. However, in practice, it is much easier to work with *sparse matrices*, that is, with matrices that have lots of zeros. This prompts the question:

> How does one pick bases so that the resulting matrix representation of a linear transformation is as sparse as possible?

Unfortunately, there is no simple answer that would work for all linear transformations. Nevertheless, some very satisfactory results can be quickly established for linear operators to which we turn our attention.

Let T be a linear operator on V: this short sentence means, as you ought to recall, that V is a vector space and T is a linear transformation mapping V *into itself*. Since T maps V into itself, its matrix A can be constructed using only one basis of V, say, $\{v_1, \ldots, v_n\}$. As follows from Section 3.5, in order to find the j-th column of A, we simply need to apply T to the j-th basis vector and expand the result with respect to the basis. Say,

$$T(v_j) = a_{1\,j}\, v_1 + a_{2\,j}\, v_2 + \ldots + a_{n\,j}\, v_n = \sum_{i=1}^{n} a_{ij}\, v_i.$$

Then $A = [a_{ij}]_{i,j=1}^{n}$. That is, A is a square n-by-n matrix with a_{ij} at the intersection of the i-th row and j-th column. Now, if $\{v_1, \ldots, v_n\}$ are chosen randomly, it may very well happen that none of the matrix element a_{ij} are zero: the matrix will be dense. So, let us try to choose $\{v_1, \ldots, v_n\}$ so that some of the matrix elements are zero. In fact, let us try to make the matrix of the operator *diagonal*.

By definition, a diagonal matrix has nonzero elements only on the main diagonal. The general two-by-two diagonal matrix has the form

$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix},$$

whereas a three-by-three diagonal matrix is:

$$\begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}.$$

The pattern should be apparent. Diagonal matrices are as sparse as it gets: only n elements out of n^2 are allowed to be nonzero. More importantly, just about any operation on a diagonal matrix reduces to a simple manipulation of diagonal elements. For instance, to power a diagonal matrix one simply has to power the diagonal elements, e.g.:

$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}^N = \begin{bmatrix} p^N & 0 \\ 0 & q^N \end{bmatrix}.$$

Suppose that we found a basis in which the matrix $A = [a_{ij}]$ representing the operator T is diagonal. That is,

$$a_{ij} = \begin{cases} d_i, & i = j, \\ 0, & i \neq j, \end{cases}$$

or, in compact notation that we will use henceforth:

$$A = \operatorname{diag}(d_1, d_2, \ldots, d_n).$$

What does T do to the basis elements? Evidently, $T(v_j) = d_j\, v_j$. In words, the affect of the operator on the basis vectors is limited to scaling. The converse is also true:

> If we want the matrix of an operator to be diagonal, we have to find a basis whose vectors are scaled by the operator.

That is [almost] all there is to it!

To see that we have indeed arrived at a useful answer, let us look at a two-dimensional example. Let T be the operator on $\mathbb{R}^2$ defined by:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -2\,a + b \\ a - 2\,b \end{bmatrix}. \tag{3}$$

It is easy to see that the matrix of T in *standard* basis is dense:

$$A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

However, let us switch to a new basis composed of the vectors:

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is easy to see (exercise) that

$$\begin{aligned} T(v_1) &= -3\,v_1 \\ T(v_2) &= -v_2 \end{aligned}$$

Consequently, the matrix of T in the new basis is diagonal:

$$A_2 = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} = \text{diag}(-3, -1).$$

Now you are probably wondering: How did we find such a lovely basis? Is it unique? Can every operator be diagonalized? We will answer all of these questions in due time. In the meanwhile, let us convince ourselves that the matrices A_1 and A_2 represent one and the same operator.

To see that A_1 represents T in the standard basis, we compute the matrix vector product:

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2\,a + b \\ a - 2\,b \end{bmatrix}.$$

This is clearly equivalent to Equation (3): indeed, the matrix A_1 represents the operator T in the standard basis. We now need to replicate this simple computation using A_2.

For convenience, let us set

$$\begin{bmatrix} a \\ b \end{bmatrix} = x.$$

We want to show that $T(x)$ can be computed using matrix multiplication by A_2. Since the components of x are with respect to the standard basis, it makes no sense to multiply x by A_2: we first need to compute the components of x in the basis $\{v_1, v_2\}$. Denote the new components as α and β. By definition (of vector components): $x = \alpha\, v_1 + \beta\, v_2$. We thus need to solve

$$\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix},$$

where a and b—the components in standard (old) basis—are treated as known quantities. The old-fashioned way of solving a vector equation is to convert it to a linear system by equating like components:

$$\alpha + \beta = a, \quad -\alpha + \beta = b.$$

Straightforward elimination of variables then leads to the solution:

$$\alpha = \frac{a - b}{2}, \quad \beta = \frac{a + b}{2}.$$

There is nothing wrong with that approach. However, since we need to practice matrix algebra, let us be more refined. The linear combination $\alpha\, v_1 + \beta\, v_2$ can be written as a matrix-vector product. Therefore the linear system can and should be written in matrix-vector form as:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

It stands to reason that in order to solve a matrix-vector system we somehow need to divide out the matrix. Here we should point out that, in general, one cannot undo matrix multiplication with matrix division. For instance, dividing by a rectangular matrix in `Matlab` generates the `mtimes` error because the lengths of the rows and columns do not match. However, since the matrix of our system

$$V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

is (i) square and (ii) *nonsingular* (which simply means 'invertible'), we can divide it out:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Note that division by V is written as multiplication by V^{-1} (pronounced "V inverse"). As you must have already surmised, by looking at the solution obtained from elimination of variables:

$$V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Before reading on, convince yourself that $V^{-1}V = V\,V^{-1} = I$, where I is the two-by-two *identity matrix*:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is especially easy to do in `Matlab` using the following snippet of code:

```
V = [-2 1; 1 -2];
W = inv(V);          % compute the inverse of V
I = eye(2);          % two-by-two identity matrix
X = W*V;             % should be the same as I
Y = V*W;             % again, should be the same as I
```

We now know that

$$x = \begin{bmatrix} \dfrac{a-b}{2} \\ \dfrac{a+b}{2} \end{bmatrix}$$

in the basis $\{v_1, v_2\}$. Therefore $T(x) = A_2\, x$ is:

$$\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{a-b}{2} \\ \dfrac{a+b}{2} \end{bmatrix} = \begin{bmatrix} -3\,\dfrac{a-b}{2} \\ -\dfrac{a+b}{2} \end{bmatrix}$$

This does not match our earlier result at all! That is because we have found $T(x)$ in the basis $\{v_1, v_2\}$ rather than the standard basis. Since switching from the standard basis to the basis $\{v_1, v_2\}$ amounts to multiplication by V^{-1}, switching back must simply amount to multiplication by V. Indeed,

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3\,\dfrac{a-b}{2} \\ -\dfrac{a+b}{2} \end{bmatrix} = \begin{bmatrix} -2\,a + b \\ a - 2\,b \end{bmatrix},$$

as required.

Try to follow the above example on your own without looking into the handout: this will help you solidify your understanding of matrix representation of linear transformations. In the meantime, we proceed to introduce some terminology which will streamline further discussion of diagonalization.

4.1 Change of basis as a linear transformation

If you followed the above example, you will have noticed that components of a vector can be recomputed in a new basis through matrix multiplication. The process of changing the basis is described by a linear operator! The matrix of that operator is called, unimaginatively, the *change of basis* matrix. In the example, it is the matrix V which we formed by stacking the basis vectors $\{v_1, v_2\}$ column-wise:

$$V = [v_1\; v_2]$$

The way matrix multiplication is set up, which is to say 'row-by-column', the product of a matrix and a vector is a linear combination of columns of the matrix:

$$[v_1\; v_2] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha\, v_1 + \beta\, v_2.$$

Henceforth, bear this little linear algebra tidbit in mind: it will be useful on a number of occasions!

As before, let x be the vector in $\mathbb{R}^2$ whose components in the *standard basis* we denoted with Latin letters a and b. Let ξ be the same vector expanded in the basis $\{v_1, v_2\}$ whose components are Greek letters α and β. In the example, we have found the reciprocal relations:

$$V\,\xi = x, \quad \xi = V^{-1}\,x.$$

These relations hold in general. Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be two bases of one and the same vector space V. To switch from the v-basis to the w-basis, express the w-vectors as linear combinations of v's:

$$w_j = \sum_{i=1}^{n} c_{ij}\,v_i.$$

Then multiply the vector in the v-basis by the inverse of the change of basis matrix $C = [c_{ij}]$. The details are left to exercises at the end of the handout.

4.2 Invertibility

In the process of changing the basis we introduced invertibility through a back door. Since this is an important concept, it will be helpful to formalize it.

Let $T : V \mapsto V$ be a linear operator and let I be the identity operator on V defined by: $I(x) = x$. We say that S is the inverse of T if $S \circ T = I$: the composition of operators is identity. Symbolically, we write $S = T^{-1}$. It can be shown that $S \circ T = I$ implies and is implied by $T \circ S = I$: i.e., if S is the left inverse it must also be the right inverse.

Not every operator is invertible. For instance, the zero operator, which is defined by $T(x) = 0$, clearly does not have an inverse since in this case $T \circ S = 0$ for any S. If an operator fails to be invertible we will call it singular; otherwise we will call it nonsingular. The same language applies to matrices representing operators.

4.3 Similarity

At the start of this chapter we said that all matrix representations of a linear operator are equivalent. There is a more precise term for that: the matrices representing the same operator in different bases are called *similar*.

Definition 7 (Similarity)**.** *Square matrices A and B are said to be similar if there exists an invertible matrix C such that:*

$$A = C\,B\,C^{-1}.$$

In which case we write $A \sim B$. Clearly, all three matrices must have the same dimensions.

As an immediate example, consider the matrices A_1 and A_2. Our computations show that $A_1 \sim A_2$. In fact: $A_1 = V\, A_2\, V^{-1}$. Or, written explicitly:

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$

Using the new language, we can now say that the matrix of T is similar to a diagonal matrix. An even shorter way to express that is to say that T is d*iagonalizable*.

4.4 Eigenvectors and eigenvalues

The preceding discussion shows that in order to diagonalize an operator $T : V \mapsto V$, it suffices to find a special basis $\{v_1, \ldots, v_n\}$ of V with the property: $T(v_j) = \lambda_j\, v_j$. In words, the application of T to basis vectors has the effect of scalar multiplication. There ought to be special language to express this concisely and it is a hybrid of English and German!

Definition 8. *A nonzero vector v is said to be an eigenvector of operator T with eigenvalue λ if $T(v) = \lambda\, v$.*

The German word ‘eigen’ can be loosely translated as ‘belonging to’, ‘proper’, or ‘characteristic of’. Eigenvectors and eigenvalues thus pertain to and characterize linear operators. As a historical curiosity we note that many English speaking mathematicians were at first reluctant to adopt ‘eigen’-terminology and insisted on the use of ‘proper values’, ‘characteristic values’, ‘latent values’, ‘secular values’, and the same for vectors. However the word ‘eigen’ eventually won and is now universally accepted.

The best way to understand eigenvalues and eigenvectors is through numerical experiments. For this reason, I am not going to state and prove standard results: it will be best if you discover them on your own. Instead, here os a list of general remarks which should be helpful with the exercises at the end.

1. According to Definition 8 eigenvectors cannot be zero. One reason for this prohibition is that we want to use eigenvectors as bases: you cannot have a zero vector in a basis (why?) Strangely enough, students often think that eigenvalues also cannot be zero. This is not true—eigenvalues can and often are zero. Read Definition 8 (and all other definitions) carefully: the adjective

'nonzero' only applies to the noun 'vector'. If, for some reason, we could not allow zero eigenvalues, we would have explicitly stipulated that in the definition; we will revisit this topic when we discuss *null spaces* of linear transformations.

2. The existence of eigenvalues and eigenvectors is not obvious and is assured only if we allow complex numbers. Which we certainly do. As for uniqueness, the eigenvalues of a linear operator are unique, up to ordering. Furthermore, if $\dim(V) = n$ then $T : V \mapsto V$ always has exactly n complex eigenvalues, some or all of which can be repeated. The non-repeated eigenvalues are called *simple*. Put differently, an eigenvalue is simple if its algebraic multiplicity is one. An eigenvalue repeated twice is called double or having algebraic multiplicity two, and so on. We are being pedantic about calling multiplicity 'algebraic' because there is also geometric multiplicity which will be discussed after the introduction of *subspaces*. The (unordered) set of all eigenvalues of a linear operator is called *spectrum*.

3. Unlike eigenvalues, eigenvectors are not unique. If v is an eigenvector of T with eigenvalue λ and $k \neq 0$ then $w = k\,v$ is another eigenvector of T with the same eigenvalue. Indeed, by linearity, $T(w) = T(k\,v) = k\,T(v) = k\,\lambda\,v = \lambda\,w$. If λ is a *simple* eigenvalue then all of the corresponding eigenvectors are scalar multiples of one and the same vector. We will discuss this in more depth when we define geometric multiplicity.

4. It is often convenient to combine eigenvectors $\{v_1, v_2, \ldots, v_n\}$ of matrix A into another (square) matrix by stacking them as columns, as we did earlier:

   $$V = [v_1\, v_2\, \ldots\, v_n].$$

   Likewise, it is convenient to form a diagonal matrix of eigenvalues:

   $$D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

   The main reason is that we can then write A as the triple product:

   $$A = V\,D\,V^{-1}.$$

   This triple product is an example of *matrix decomposition* called, unsurprisingly, the *eigendecomposition*. In light of Section 4.3, the existence of eigendecomposition shows that the matrix A is similar to the diagonal matrix D. We hasten to add that, unfortunately, at the moment we have no assurance that eigendecomposition always exists. In fact, it does not, and one of our main projects will be discovering what goes wrong and how to fix it.

We close this section with discussion of computational aspects. If you have already had Differential Equations, you ought to know the procedure for finding eigenvalues and eigenvectors of a two-by-two matrix. First, one finds the eigenvalues by computing a certain *determinant*: in the two-by-two case this leads to a quadratic equation called *characteristic equation*. Having found the eigenvalues by solving the characteristic quadratic (in ODE textbooks the eigenvalues are often called *characteristic roots*), one finds the corresponding eigenvectors by solving certain linear two-by-two systems using elimination of variables. The procedure, while simple, is rather tedious, so much so that you have probably never worked with three-by-three or larger matrices. We will examine this procedure in due time. In the meanwhile, use the `eig` command in `Matlab` as in the following snippet:

```
A = [-2 1; 1 -2];
[V,D] = eig(A)

V =

    0.7071    0.7071
   -0.7071    0.7071

D =

    -3     0
     0    -1
```

Notice that `eig(A)` returns two matrices: the diagonal matrix D with eigenvalues on the main diagonal and the matrix V whose columns are the corresponding eigenvectors. The eigenvectors are not the same that we used before because `eig` *normalizes* eigenvectors: i.e., scales them so that they have unit length. Use the above code to verify that, indeed, $A = V\,D\,V^{-1}$ and try it on other matrices. You may also want to read the help file for `eig` to see other examples of its usage.

4.5 Applications of diagonalization

Many different problems involving linear operators can be solved through diagonalization; this will be one of our recurrent themes. The two examples that follow are technically very simple. Yet they epitomize the use of eigendecomposition, so study them carefully.

4.5.1 Fibonacci numbers

Consider the number sequence $\{f_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. Notice that each term is the sum of the preceding two:

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

These are the famous *Fibonacci numbers.*

It is easy to see that the ratio of consecutive Fibonacci numbers approaches a certain limit which we denote:

$$\phi = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}.$$

In fact, this limit is one of the main reasons why Fibonacci sequence is so famous. Fibonacci himself did not know the value of ϕ which we are about the compute. I have no doubt that he would have given much for access to linear algebra which, sadly, he did not have in the 17th century.

Our first move is to form vectors from pairs of consecutive Fibonacci numbers:

$$x_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}, \quad n = 1, 2, 3, \ldots$$

Thus

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix},$$

and so on. Now, by definition, the next Fibonacci number (starting with the third) is the sum of the previous two. Hence:

$$x_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} f_n \\ f_n + f_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = A\, x_{n-1}.$$

We have shown that $x_n = A\, x_{n-1}$: this is called a *recurrence relation.* It now follows that $x_2 = A\, x_1$, $x_3 = A\, x_2 = A^2\, x_1$, $x_4 = A^3\, x_1$, and so on; in general, $x_n = A^{n-1}\, x_1$, or, written explicitly:

$$\begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad n = 2, 3, 4, \ldots$$

Evidently, in order to compute Fibonacci numbers, we need to power a two-by-two matrix.

To compute the powers of A, we use its eigendecomposition: $A = V\,D\,V^{-1}$. Since $V^{-1}\,V = I$ where I is the *identity* matrix:

$$A^2 = V\,D\,\underbrace{V^{-1}\,V}_{\text{cancel}}\,D\,V^{-1} = V\,D^2\,V^{-1}.$$

Similarly,

$$A^3 = A^2\,A = V\,D^2\,\underbrace{V^{-1}\,V}_{\text{cancel}}\,D\,V^{-1} = V\,D^3\,V^{-1},$$

and, more generally:

$$A^{n-1} = V\,D^{n-1}\,V^{-1}.$$

Furthermore, since D is diagonal:

$$D^{n-1} = \operatorname{diag}(\lambda_1, \lambda_2)^{n-1} = \operatorname{diag}(\lambda_1^{n-1}, \lambda_2^{n-1}).$$

We can now write:

$$\begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the matrix whose columns are the eigenvectors of A with corresponding eigenvalues λ_1 and λ_2. To simplify the next step, let us introduce the vector

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then, as you can verify (exercise):

$$\begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} a\,p\,\lambda_1^{n-1} + b\,q\,\lambda_2^{n-1} \\ c\,p\,\lambda_1^{n-1} + d\,q\,\lambda_2^{n-1} \end{bmatrix}.$$

Consequently,

$$\phi = \lim_{n\to\infty} \frac{c\,p\,\lambda_1^{n-1} + d\,q\,\lambda_2^{n-1}}{a\,p\,\lambda_1^{n-1} + b\,q\,\lambda_2^{n-1}}.$$

Assuming that the eigenvalues are distinct, which, indeed, they are, let us order them by magnitude: $|\lambda_1| < |\lambda_2|$. We will call λ_2—the eigenvalue with the greatest magnitude—the *dominant* eigenvalue; the corresponding eigenvector is, naturally, called the *dominant eigenvector*. Having sorted the eigenvalues, we rewrite ϕ as follows:

$$\phi = \lim_{n\to\infty} \frac{c\,p\left(\dfrac{\lambda_1}{\lambda_2}\right)^{n-1} + d\,q}{a\,p\left(\dfrac{\lambda_1}{\lambda_2}\right)^{n-1} + b\,q}.$$

Since $|\lambda_1/\lambda_2| < 1$,

$$\lim_{n\to\infty}\left(\frac{\lambda_1}{\lambda_2}\right)^{n-1} = 0.$$

Consequently,

$$\phi = \frac{d\,q}{b\,q} = \frac{d}{b}.$$

Recall that the numbers b and d form the second column of the matrix V which, by construction, is the eigenvector corresponding to the second (dominant) eigenvalue. Therefore the limit ϕ is simply the ratio of components of the dominant eigenvector of A. It is now a simple matter to compute ϕ in `Matlab`:

```
A = [0 1; 1 1];
[V,D] = eig(A)

V =

  -0.850650808352040   0.525731112119133
   0.525731112119133   0.850650808352040

D =

  -0.618033988749895                   0
                   0   1.618033988749895
```

Evidently, $\lambda_2 = 1.618033988749895$ is the dominant eigenvalue. Hence we continue with:

```
v = V(:,2);    % dominant eigenvector
phi = v(2)/v(1)

phi =

   1.618033988749895
```

Curiously, $\phi = \lambda_2$: the limit equals the dominant eigenvalue. You can check that the exact value of ϕ is

$$\phi = \frac{1+\sqrt{5}}{2}$$

which you may recognize as the *golden ratio*. The surprising emergence of the golden ratio from Fibonacci sequence explains the world's fascination with Fibonacci numbers.

4.5.2 Systems of linear ODE

As our next example, which may be familiar, we consider the following system of ordinary differential equations (ODE):

$$\begin{aligned} \frac{dx_1}{dt} &= -2\,x_1 + x_2, & x_1(0) &= 2, \\ \frac{dx_2}{dt} &= x_1 - 2\,x_2, & x_2(0) &= 0. \end{aligned}$$

In case you have not had ODE yet, the unknowns here are two functions: $x_1(t)$ and $x_2(t)$; you can think of the independent variable t as time. What we know is how the rates of change of the unknowns are related to the functions themselves: this is what ODE are, in general. Evidently, in this case, the relationship is linear, so these are linear ODE. Note that we are also given the values of the unknowns at time zero: these are called *initial conditions*.

The reason why this system of equations is hard is that the unknowns are coupled together. As it stands, we cannot solve the equations individually. However, we can decouple the equations using—you guessed it!—diagonalization. Let

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

In matrix form, the system of ODE can be written as:

$$\frac{dx}{dt} = A\,x, \quad x(0) = x_0.$$

Here by dx/dt we mean the vector of derivatives; if you do not know this from Calculus 3, think about why that makes sense. Using the eigendecomposition of A, we can write:

$$\frac{dx}{dt} = V\,D\,V^{-1}\,x, \quad x(0) = x_0.$$

Now multiply both sides by V^{-1} on the left. Since V is a constant matrix, multiplication by V^{-1} can be interchanged with differentiation:

$$V^{-1}\,\frac{dx}{dt} = \frac{d}{dt}(V^{-1}\,x) = D\,V^{-1}\,x, \quad x(0) = x_0.$$

It should now be apparent that we can greatly simplify things with the simple substitution: $y = V^{-1}\,x$. Clearly, once we find y, we can find $x = V\,y$. Now the ODE for y is:

$$\frac{dy}{dt} = D\,y, \quad y(0) = V^{-1}\,x_0 = y_0.$$

Let us write that as a system of ODE. As we already know

$$D = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}.$$

To simplify formulas somewhat, we will set

$$V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

instead of using the output of `eig` (which differs by the factor of $1/\sqrt{2}$). Then, as you can easily confirm

$$y_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

and we are led to the system:

$$\begin{aligned} \frac{dy_1}{dt} &= -3\,y_1, & y_1(0) &= 1, \\ \frac{dy_2}{dt} &= -y_2, & y_2(0) &= 1. \end{aligned}$$

These ODE are decoupled and can be solved individually. In fact, you should recognize these equations from Calculus as two instances of exponential decay. Hence

$$y_1 = e^{-3\,t}, \quad y_2 = e^{-t},$$

or, in vector form,

$$y = \begin{bmatrix} e^{-3\,t} \\ e^{-t} \end{bmatrix}.$$

It is now a simple matter to find x:

$$x = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + e^{-t} \\ -e^{-3t} + e^{-t} \end{bmatrix}.$$

We conclude that

$$x_1 = e^{-3t} + e^{-t}, \quad x_2 = -e^{-3t} + e^{-t},$$

which you can easily verify.

Homework

1. Consider the following two bases of $\mathbb{R}^2$:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

   Suppose that $x \in \mathbb{R}^2$ has components

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

   when expanded in basis $\mathcal{B}_1$. Find the components of x in basis $\mathcal{B}_2$. Provide validation of your answer.

2. Consider the following linear operator on $\mathbb{R}^2$:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a + 2\,b \\ 3\,a + 4\,b \end{bmatrix}$$

   Find the matrices representing T in the two bases from the previous problem. Again, validate your answers.

3. Give three different examples of singular two-by-two matrices. What can you say about the columns of the matrices in each case? Does this statement generalize to three-by-three matrices?

4. Consider the following linear operator on $\mathbb{R}^2$:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a\,x + b\,y \\ c\,x + d\,y \end{bmatrix}$$

   Find the inverse $S = T^{-1}$, assuming that it exists. Explain what may go wrong in your solution which will prevent S from existing (in which case T is singular).

5. `Matlab`'s command `plot` plots a complex number $x + y\,i$ as a point (x, y). If `Z` is an array of complex numbers, the command

```
plot(Z,'b.')
```

   will plot the elements of the array as blue dots.

   Compute the eigenvalues of

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   for fifty equispaced values of θ lying in the interval $[0, 2\pi]$. This will produce one hundred complex numbers which you can store in an array. Plot these numbers as points and try to explain the resulting pattern.

6. Say, `A` is an n-by-m matrix. In `Matlab` the command `A'` produces an m-by-n matrix whose rows are columns of `A`. The operation of interchanging rows and columns is called *transposition*.

   Consider matrices of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

   where a, b, and c are real numbers. These matrices remain unchanged under transposition and for this reason are called *symmetric*. Use `randn` to generate a few (say, six) random symmetric two-by-two matrices and compute their eigenvalues. What is true about the eigenvalues of symmetric matrices that is not true about eigenvalues of non-symmetric matrices?

7. Compute the eigenvalues and eigenvectors of

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   Can this matrix be diagonalized? Explain your answer.

8. Find a nonzero two-by-two matrix one of whose eigenvalues is zero. What can you say about a two-by-two matrix if both of its eigenvalues are zero?

9. A square matrix is called *upper triangular* if all of its entries below the main diagonal are zero. Generate several three-by-three upper triangular matrices and find their eigenvalues. What can be said about eigenvalues of upper triangular matrices in general?

10. Let A and B be square matrices of the same dimension. The matrix $[A, B] = A\,B - B\,A$ is called the *commutator* of A and B. The reason for the name is that $[A, B] = 0$ when $A\,B = B\,A$, that is, when matrices commute. Generally, however, $[A, B] \neq 0$. Find three different pairs of commuting two-by-two matrices and examine their eigenvectors. If $[A, B] = 0$, what can you say about the eigenvectors of A and B?

11. Generate two similar matrices (similar in the sense of Section 4.3) and find their eigenvalues. Repeat the process a few times. What can you say about eigenvalues of similar matrices?

12. If $A = V\,D\,V^{-1}$ what is A^{-1}? Confirm your answer with a `Matlab` experiment.

13. Let $x_1 = 1$, $x_2 = 1$, $x_n = 3\,x_{n-1} + 2\,x_{n-2}$ for $n \geq 3$. Mimicking the Example 4.5.1, compute the limit:

$$\psi = \lim_{n \to \infty} \frac{x_{n+1}}{x_n}.$$

14. Solve the following system of ODE by following Example 4.5.2:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, & x_1(0) &= 1, \\ \frac{dx_2}{dt} &= -2\,x_1 - 3\,x_2, & x_2(0) &= -1. \end{aligned}$$

Don't forget to validate your answer.