Problem Set 8

Problem 1. *Two-dimensional analogs of one-dimensional bifurcations*
Recall the three elementary bifurcations in one dimension:

- Saddle-node bifurcation:
$$\dot{x} = r + x^2$$

- Transcritical bifurcation:
$$\dot{x} = rx - x^2$$

- (Supercritical) Pitchfork bifurcation:
$$\dot{x} = rx - x^3$$

We can experiment with these bifurcations in two dimensions by replacing the '$\dot{x}$' terms with "$\ddot{x}$" instead. That is, for each $\dot{x} = f(x; r)$ above, we can set $\ddot{x} = f(x; r)$ instead. For example, we can write extend the saddle-node bifurcation as

$$\begin{cases} \dot{x} & = y \\ \dot{y} & = r + x^2 \end{cases}$$

For each of the bifurcation types above:

(a) Find all fixed points $\vec{\mathbf{x}}^*$ of the two-dimensional system as functions of r. Count how many fixed points there are for each case of $r < 0$, $r = 0$, and $r > 0$.

(b) Construct the Jacobian matrix $J(x, y)$ and compute $J(\vec{\mathbf{x}}^*)$ for each fixed point found above.

(c) Classify the equilibrium type of each fixed point in each case of $r < 0$, $r = 0$, and $r > 0$.

(d) Plot a phase portrait for the system in each case of $r < 0$, $r = 0$, and $r > 0$.

(e) Describe what happens to the fixed points of the system and how their stabilities change as we increase r from negative to positive.