Problem Set 3

Problem 1. *How changing the inner product affects* $\mathbb{R}^2$

Remember that vector spaces on their own are merely *algebraic* objects. If we want to impose any geometric features, we need to give them either a norm or an inner product. Since there is such a variety in the inner products that we can use, we can expect a variety of geometries for the same vector space.

(a) First, show that the bracket

$$\langle \vec{\mathbf{v}}, \vec{\mathbf{w}} \rangle = \vec{\mathbf{v}}^T \begin{bmatrix} 4 & -2 \\ -2 & 9 \end{bmatrix} \vec{\mathbf{w}}$$

defines an inner product on $\mathbb{R}^2$.

(b) In the standard Euclidean inner product space $(\mathbb{R}^2, \cdot)$ that uses the dot product $\cdot$, the vectors $\hat{\mathbf{i}} = (1,0)^T$ and $\hat{\mathbf{j}} = (0,1)^T$ have unit length. What is $\|\hat{\mathbf{i}}\|$ and $\|\hat{\mathbf{j}}\|$ in $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$? What multiples of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ actually have unit length in $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$?

(c) In $(\mathbb{R}^2, \cdot)$, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are orthogonal. Are they orthogonal in $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$? Find all vectors that are orthogonal to $\hat{\mathbf{i}}$.

(d) The equation for the unit circle in $(\mathbb{R}^2, \cdot)$ is $x^2 + y^2 = 1$. What is the equation for the unit circle in $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$? What does this unit "circle" actually look like to us?

Note: In lecture we showed that any two norms on a finite-dimensional vector space are *equivalent*. Given what we've just seen in this exercise, we can definitively say that this notion of equivalence does not take into account the geometry of the inner product space. In fact, the equivalence of norms is more of a statement about the *topology* of the space rather than the geometry.

Problem 2. *Direct computation with the 1-norm*

It is often the case that we use the p-norms both on vectors in $\mathbb{R}^n$ and on matrices in $\mathcal{M}_{m\times n}(\mathbb{R})$. Because of how frequently they are used in computational practice, we should (1) show that these norms are equivalent and (2) come up with some simple formulas that allow us to compute these norms quickly. In particular, we can do this for the 1-, 2-, and ∞-norms quite easily.

(a) We showed in class that the 2-norm and the ∞-norm were equivalent on $\mathbb{R}^n$. Show that the 1-norm and 2-norm are also equivalent on $\mathbb{R}^n$.

(b) We have now shown that the 2-norm is equivalent to both the 1- and ∞-norms on $\mathbb{R}^n$. Show how this implies that the 1-norm is equivalent to the ∞-norm on $\mathbb{R}^n$.

(c) We came across a simple way of explaining the ∞-norm on $\mathcal{M}_{m\times n}(\mathbb{R})$ as the maximum absolute row sum of a matrix. Show that for $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, the 1-norm $\|A\|_1$ is the maximum absolute column sum of A. [*Hint*: Try to mimic the proof that $\|A\|_\infty$ is the maximum absolute row sum of A.]

Problem 3. *Coefficients of positive-definite matrices*

We have seen many examples of positive-definite matrices in class, and we have determined that it is hard to tell when an arbitrary symmetric matrix is positive-definite. We might wonder if there is a simple "formula" in terms of the coefficients of the matrix that tells us when it is positive-definite, much like the discriminant of a quadratic polynomial indicating the existence of real roots of the polynomial. There is a reason we haven't seen such a formula...

(a) Let's define

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

to be the form of an arbitrary, symmetric 2×2 matrix. Find a collection of inequalities of a, b, and c that tells us when A is positive-definite.

(b) Do the same for the arbitrary, symmetric 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

(c) Does this method seem like a practical way to check if an $n \times n$ matrix positive-definite when $n \geq 4$?