3. The mystery of Taylor polynomials

In Section 2 we studied IVP of the general form

$$\frac{dx}{dt} = f(t) - x, \quad x(0) = x_0,$$

where the *source term* $f(t)$ was a polynomial of small degree. We found that the substitution $u = f(t) - x$ simplifies such equations. Indeed, if f is a polynomial of degree N, the transformed equation

$$\frac{du}{dt} = f'(t) - u, \quad u(0) = f(0) - x_0$$

is of the same form as the original equation *but* with the source term $f'(t)$ which is a polynomial of order $(N-1)$. This suggests a simple strategy based on repeated substitution which eventually results in a separable equation.

As powerful and versatile as substitutions are, they are not always helpful. This should not come as a surprise if you reflect on your Calculus experience. Come to think of it, in Calculus 2 was it not only a small portion of carefully selected integrals that could be simplified through "obvious" substitutions? Meanwhile, for other integrals the substitutions were not obvious. In fact, you may have learned at the end that there exist integrals such as

$$\int_0^1 e^{-x^2}\, dx$$

which cannot be expressed in terms of elementary functions, so no substitution can convert them to tabular form.

The situation with ODE is very similar. For most ODE substitutions are not obvious or much too complicated to be of practical use. Fortunately, there exist other Calculus techniques for solving ODE and the best part is that you are already familiar with some of the relevant theory. Remember how at the end of Calculus 2 you approximated "impossible" integrals, such as the one above, using Taylor polynomials? The same Taylor theory can be used to solve ODE, as we shall demonstrate after a short review.

3.1. **Taylor polynomials re-explained.**

In order to be able to work with Taylor polynomials, or Taylor series for that matter, you need to know what these things *are*. This is very different from knowing which formulas to use to compute them.

> A Taylor polynomial of degree N centered at $t = t_0$ is a polynomial of degree N which matches the value

of a given function as well as the values of its first N derivatives at t_0—the center of expansion.

For instance, let $f = e^t$ and $t_0 = 0$. At zero, the function f and all of its derivatives have the same value: 1. Therefore, the Taylor polynomial of degree N centered at zero is:

$$p = 1 + t + \frac{t^2}{2} + \ldots + \frac{t^N}{N!}.$$

To see this, let us apply the definition where, for simplicity, we let $N = 2$. We reason as follows. A Taylor quadratic is, first of all, a quadratic polynomial. As such, it is determined by three coefficients. Set $p = a_0 + a_1\, t + a_2\, t^2$. Since the value of p at $t = 0$ must agree with that of $f = e^t$, we must have:

$$p(0) = a_0 = f(0) = 1.$$

So, $a_0 = 1$. To find a_1, equate the derivative of p with that of f *at the center of expansion*:

$$p'(0) = a_1 = f'(0) = 1.$$

This shows that $a_1 = 1$. Finally, to determine a_2, equate the second derivatives at zero:

$$p''(0) = 2\, a_2 = f''(0) = 1.$$

This leads to $a_2 = \frac{1}{2}$ and

$$p = 1 + t + \frac{t^2}{2},$$

which was to be demonstrated.

The process for deriving any Taylor polynomial is exactly the same and you should practice it until you become very comfortable with it. That despite the fact that in Calculus 2 you may have learned the formula:

$$p = \sum_{n=0}^{N} \frac{f^{(n)}(t_0)}{n!}\,(t - t_0)^n.$$

For let us face it, like the majority of students you most likely do not remember that formula and cannot use it confidently. Therefore, practice deriving Taylor polynomials from first principles, by equating derivatives and solving the resulting linear equations. Then the formula will make more sense.

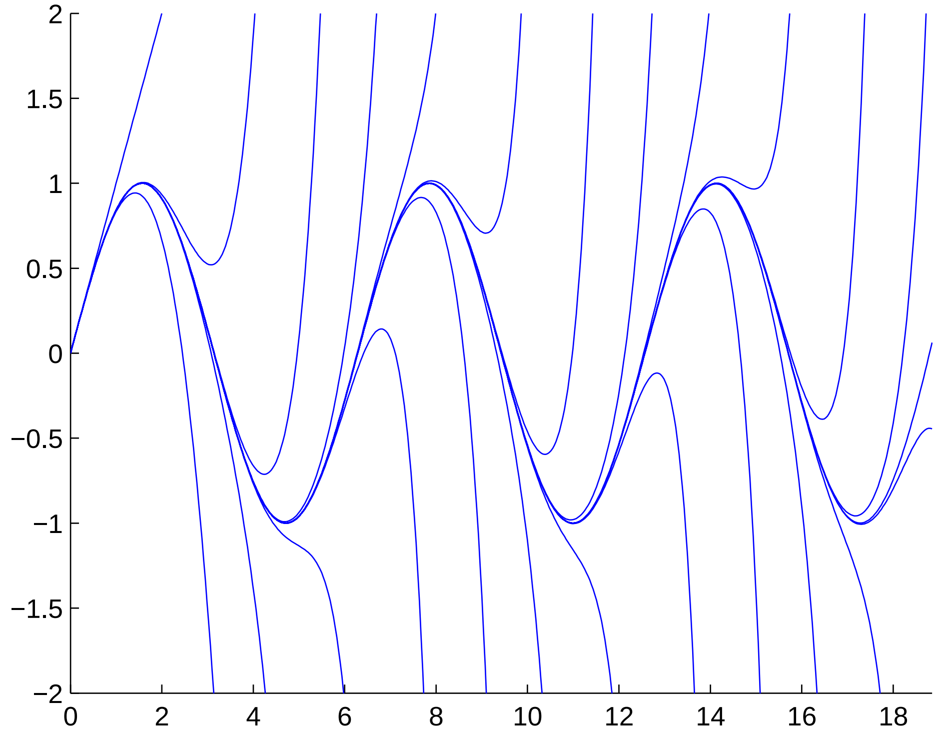

FIGURE 6. The first 50 Taylor polynomials of $y = \sin(t)$ centered at zero. Can you explain why there are only 25 different curves on the plot?

3.2. **Taylor approximation.** It stands to reason that if a polynomial matches the values of a function and its derivatives at some point, it should somehow mimic the behavior of the function *near* that point. In other words, Taylor polynomials can be regarded as *approximations* of the function.

Figure 6 illustrates that using the sine function. The first few Taylor polynomials centered at zero are:

$$p_1(t) = t, \quad p_3(t) = t - \frac{t^3}{6}, \quad p_5(t) = t - \frac{t^3}{6} + \frac{t^5}{120}, \ldots$$

It is visually evident that, as the order of the Taylor polynomial is increased, there is better and better agreement between it and the function. This is often *but not always* the case.

Assuming that the sequence of Taylor polynomials converges to the function f which, again, cannot be always taken for granted, we can replace a complex function with its Taylor polynomial of sufficiently high degree. The idea, of course, is that whatever we cannot do with the function, we can easily do with the polynomial. As an example, consider another "impossible" integral:

$$\int_0^1 \frac{\sin(x)}{x}\,dx.$$

If we replace $\sin(x) \approx x - \frac{x^3}{6}$, we can approximate:

$$\int_0^1 \frac{\sin(x)}{x}\,dx \approx \int_0^1 \frac{x - \frac{x^3}{6}}{x}\,dx = \int_0^1 \left(1 - \frac{x^2}{6}\right)\,dx = \frac{17}{18}.$$

Thus, with minimal effort, we get an answer correct to two decimal places. To get better accuracy, we can use a higher order Taylor polynomial. As an exercise, show that using fifth order Taylor polynomial gives four digits of accuracy.

3.3. **Taylor series.** If the process of Taylor approximation, that is, approximation of a function with Taylor *polynomials*, is taken to its logical conclusion, the result is a Taylor *series*. In general, a series is "simply" a sum with infinitely many terms. Taylor series is a particular series of the general form:

$$s = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}\,(t - t_0)^n.$$

The partial sums of a Taylor series are Taylor polynomials and the value of the series is computed as the limit of polynomial values:

$$s = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(t_0)}{n!}\,(t - t_0)^n.$$

In fact, that is precisely what Figure 6 illustrates. Sounds simple, but it is not. Unlike a sum with finitely many terms, a series can fail to *converge*. That is, the above limit may not necessarily exist. Furthermore, in the case of Taylor series the sum depends on the variable (which we call t) and it is not uncommon for it to exist for some values of t but not for others. Hence, we cannot always write

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}\,(t - t_0)^n$$

because the series on the right may not make sense for some values of t where the left-hand side may be perfectly well-defined. For instance, consider

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

The function on the left makes sense for all $t \neq 1$. However the series on the right exists only for $|t| < 1$; the interval $(-1, 1)$ is called the interval of convergence and the agreement between the series and the function is only on that interval.

You will undoubtedly recall the discussion of various convergence tests at the end of Calculus 2. We are not going to repeat it here. Instead, let us try to rely on numerical validation. Each time you derive a sequence of Taylor polynomials approximating some function, produce a composite plot: if it looks like Figure 6 then all is well. Otherwise, if there is apparent divergence, look for a different approach to the problem.

3.4. **Solving ODE using Taylor theory.** To make the discussion concrete, let us consider

$$\frac{dx}{dt} = e^{-t} - x, \quad x(0) = 0. \tag{16}$$

This seemingly contrived equation can be realized using a circuit with two identical capacitors: think of the exponential source term as one capacitor discharging into another. Now, you can easily convince yourself that substituting $u = e^{-t} - x$ is not very helpful. So, let us consider an alternative approach where we *approximate* the unknown rather then compute it exactly.

Set $x = a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3 + \dots$ The numbers a_n are Taylor coefficients that need to be determined. Substituting the expansion into Equation (16) leads to:

$$a_1 + 2\,a_2\,t + 3\,a_3\,t^2 + \dots = e^{-t} - \big(a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3 + \dots\big) \tag{17}$$

Next we could combine the like terms but there is a bit of a problem. The right-hand side of Equation (17) involves an exponential which does not fit very well with the polynomial terms. Is there a way to rewrite the exponential so that it looks like a polynomial? Replace it with its Taylor expansion! Substituting

$$e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \dots$$

into Equation (17), we get:

$$a_1 + 2\,a_2\,t + 3\,a_3\,t^2 + \dots = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \dots - \big(a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3 + \dots\big) \tag{18}$$

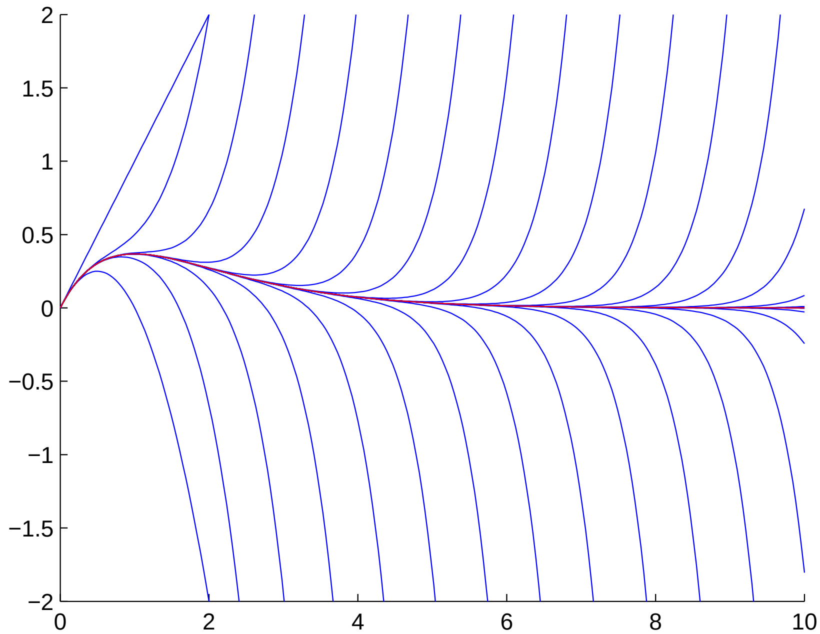

Figure 7. The first 40 Taylor polynomials approximating the solution of Equation (16). The polynomials converge to $x = t\,e^{-t}$.

Now, equating the like terms we get the system of equations:

$$\begin{aligned} a_1 &= 1 - a_0 \\ 2\,a_2 &= -1 - a_1 \\ 3\,a_3 &= \frac{1}{2} - a_2 \\ &\dots \end{aligned}$$

We can find a_0 from the initial condition: $a_0 = x(0) = 0$. It then follows that $a_1 = 1$, $a_2 = -1$, $a_3 = \frac{1}{2}$, and so on. Therefore

$$x(t) = t - t^2 + \frac{t^3}{2} - \frac{t^4}{6} + \dots \tag{19}$$

Using Equation (19) we can plot the first several Taylor polynomials approximating x. Figures 7 and 8 clearly show convergence to the function $x = t\,e^{-t}$. You can check that that is, indeed, the correct solution of Equation (16).

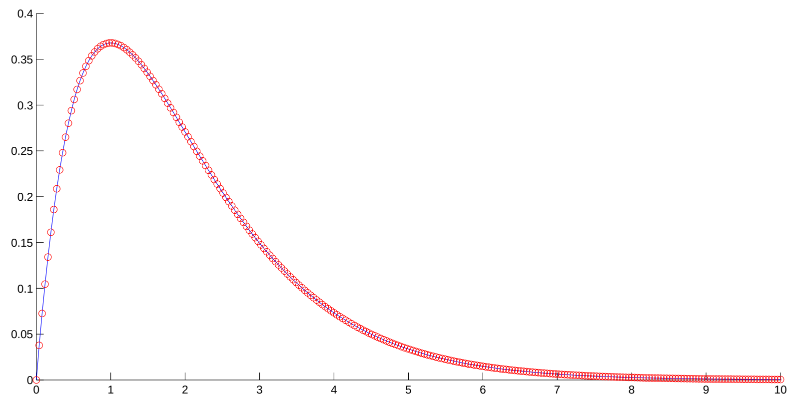

FIGURE 8. The plot of $x = t\,e^{-t}$ (blue line) and its Taylor polynomial of degree 40 (red circles).

Exercises.

(1) Consider the following IVP:

$$\frac{dx}{dt} = \cos(t) + \sin(t) - x, \quad x(0) = 0.$$

(a) Find the Taylor expansion of x.

(b) Plot the first seven Taylor polynomials on the interval $0 < t < 20$. Scale the y-axis (using `ylim` command) to $[-3, 3]$.

(c) What is the actual solution of the IVP? Try to conjecture it by looking at the pattern of Taylor coefficients and the plots.

(2) Repeat the previous exercise with the following IVP:

$$\frac{dx}{dt} = -x^2, \quad x(0) = 1.$$

Comment on the behavior of Taylor polynomials as their order becomes large. Is that what should be expected?

4. Euler's method

Before I tell you what Euler's method is, we need to talk a bit more about Taylor series. Recall that we used these in Section 3 to solve

$$\frac{dx}{dt} = e^{-t} - x, \quad x(0) = 0. \tag{20}$$

The idea was simply to expand all terms in the ODE into Taylor series centered at zero and equate like coefficients. That is, we set

$$\begin{aligned}
x &= a_0 + a_1\,t + a_2\,t^2 + \ldots = \sum_{n=0}^{\infty} a_n\,t^n \\
\frac{dx}{dt} &= a_1 + 2\,a_2\,t + 3\,a_3\,t^2 + \ldots = \sum_{n=1}^{\infty} n\,a_n\,t^{n-1} \\
e^{-t} &= 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n\,t^n}{n!}
\end{aligned}$$

which results in:

$$a_1 + 2\,a_2\,t + 3\,a_3\,t^2 + \ldots = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots - \left(a_0 + a_1\,t + a_2\,t^2 + \ldots\right)$$

We then reason that two power series are equal if and only if their like coefficients match. Hence

$$\begin{aligned}
a_1 &= 1 - a_0 \\
2\,a_2 &= -1 - a_1 \\
3\,a_3 &= \frac{1}{2} - a_2 \\
&\ldots
\end{aligned}$$

or, written concisely,

$$(n+1)\,a_{n+1} = \frac{(-1)^n}{n!} - a_n, \quad n = 0, 1, 2, \ldots$$

This linear system of equations can be solved recursively once a_0 is determined from the initial condition. For zero initial condition, $a_0 = 0$. This leads to $a_1 = 1$, $a_2 = -1$, $a_3 = 1/2$, and so forth. Therefore, the solution of IVP (20) is:

$$x = t - t^2 + \frac{t^3}{2} - \frac{t^4}{6} + \ldots = t\,e^{-t}.$$

The series solution method is, on one hand, insanely powerful. In principle, it can be applied to any ODE of the form

$$\frac{dx}{dt} = f(t, x)$$

as long as the right-hand side is sufficiently smooth; and it can also be applied to higher order ODE. Unfortunately, the series method also has some severe limitations. Firstly, it cannot be applied to an ODE with a non-differentiable right-hand side[2], e.g.:

$$\frac{dx}{dt} = |t-1| - x.$$

Secondly, a Taylor series may not be convergent everywhere where the solution of the IVP exists. For instance, consider

$$\frac{dx}{dt} = -x^2, \quad x(0) = 1.$$

The series representation of the solution

$$x = 1 - t + t^2 - t^3 + \dots$$

converges only for $|t| < 1$. Yet the actual solution

$$x = \frac{1}{1+t}$$

exists for all positive values of t. Finally, even if the series solution converges, it may do so excruciatingly slowly. For instance, suppose the series solution is

$$x = \sum_{n=1}^{\infty} \frac{t^n}{n^{1.01}}.$$

This series converges for $t = 1$. Yet, in order to find the sum to, say, two decimals, one has to add about 10^{400} terms which is, clearly, untenable.

> So, why talk about Taylor series if they have such severe limitations and what does this all have to do with Euler's method?

In regards to the first question, there are times when the series approach works brilliantly, as in the example reviewed at the beginning of this section. That alone makes Taylor theory worthwhile. Also, Taylor series are very common in theoretical investigations. For instance, one can often argue existence and uniqueness of a solution of some wild IVP by constructing its series representation. Finally, and this brings us to the subject of this section, the ideas behind Taylor series are very useful in designing numerical methods for solving IVP. ODE solvers, as they are called, can be quite sophisticated. Yet all ODE solvers are based on the same prototype–the Euler's method, to which we now turn our attention.

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[2] In case you are wondering, such ODE are very common and therefore cannot be easily dismissed.

Suppose we wish to *numerically* solve a first order IVP

$$\frac{dx}{dt} = f(t,x), \quad x(0) = x_0$$

on some interval $[0,T]$. To this end, divide the time interval into N equal subintervals and denote the subdivision points $t_0 = 0, t_1, t_2, \ldots, t_N = T$; it will be convenient to use h to abbreviate the size of the subintervals: $h = T/N$. Our task is to *estimate* the solution of the IVP at these points. That is, we wish to compute $x_n \approx x(t_n)$ for $n = 0, 1, 2, \ldots, N$.

We are given the (exact) initial value $x_0 = x(0)$ and, using the ODE, we can evaluate the initial slope as $\frac{dx}{dt}(0) = f(0,x_0) = f_0$. This means that at the origin, we can use Taylor approximation

$$x(t) \approx x_0 + f_0\, t,$$

which suggests setting $x_1 = x_0 + f_0\, h$. At this point we would like to stress that x_1 only approximates and does not actually equal $x(h)$. Nevertheless, if h is reasonably small, the approximation should be good. Assuming that, we simply repeat the construction of the tangent at $t = h$. The slope is $\frac{dx}{dt}(h) = f(h, x(h)) \approx f(h, x_1) = f_1$. Hence, near $t = h$

$$x(t) \approx x_1 + f_1\,(t-h)$$

which suggests setting $x_2 = x_1 + f_1\, h$. Continuing in this manner we can compute

$$x_{n+1} = x_n + f(t_n, x_n)\, h, \quad n = 0, 1, 2, \ldots, N. \tag{21}$$

Equation (40) is known as Euler's (pronounced "oi-ler's") method.

The MATLAB implementation of Euler's method is listed below. Notice that if we do not count comments (lines starting with %) the entire program has only eight lines of very simple code. Comments are very important however, so do not forget to leave them in your programs.

```
function [t,x] = euler(ode,tspan,x0,N)
%     [TOUT,XOUT] = EULER(ODEFUN,TSPAN,X0,N) with TSPAN = [T0 TFINAL]
%     integrates first order ODE x' = f(t,x) from time T0 to TFINAL with
%     initial condition X0. ODEFUN is a function handle. N is the number of
%     subdivisions of TSPAN.  Greater N results in better accuracy but
%     requires more computational time.

t = linspace(tspan(1),tspan(2),N+1);    % subdivide tspan into N subintervals
h = t(2)-t(1);                          % size of the subdivision
x = zeros(size(t));                     % initialize x
x(1) = x0;
for n=2:N
```

```
    x(n) = x(n-1) + h*ode(t(n-1),x(n-1));  % Euler's method
end
```

Let us apply `euler` to IVP (20) where we know the exact solution to be $x = t\,e^{-t}$. The results are shown in Figure 9 which was generated by the following script:

```
f = @(t) exp(-t);
ode = @(t,x) f(t) - x;   % right-hand side of the IVP
tspan = [0 5];
x0 = 0;

tt = linspace(tspan(1),tspan(2),256);
xx = tt.*exp(-tt);       % exact solution for comparison with Euler's method

N = 5*2.^(1:6);
figure
for k=1:6
    [t,x] = euler(ode,tspan,x0,N(k));  % apply Euler with N=5*2^k
    subplot(2,3,k);
    plot(tt,xx,'b-')                   % plot the exact solution
    hold on
    plot(t,x,'b-',t,x,'b.')            % plot Euler's approximation
    xlim(tspan)
    ylim([0 .6])
    title(sprintf('N=%g',N(k)))
end
```

Evidently, as the number of subdivisions increases the output of Euler's method converges to the exact solution. In fact, with 320 subdivisions which corresponds to $h = .015625$ the Euler approximation cannot be visually distinguished from the exact solution.

In Numerical Analysis (Math 110) we prove that, when applied to reasonable IVP's, Euler's method is guaranteed to converge: as the step size h becomes smaller and smaller, the error goes to zero. We could say, therefore, that Figure 9 illustrates typical behavior of Euler's method, yet that would be very misleading! While it is true that Euler's method converges, it does not always do that in an orderly manner illustrated in Figure 9. Nor does it converge quickly. In fact, the convergence of Euler's method is often so slow that it is rarely used in practice. In Math 57 we will use Euler's method purely for conceptual understanding of how numerical ode solvers work. Meanwhile, for serious computations we will usually use MATLAB's routine `ode45`, as in the snippet below:

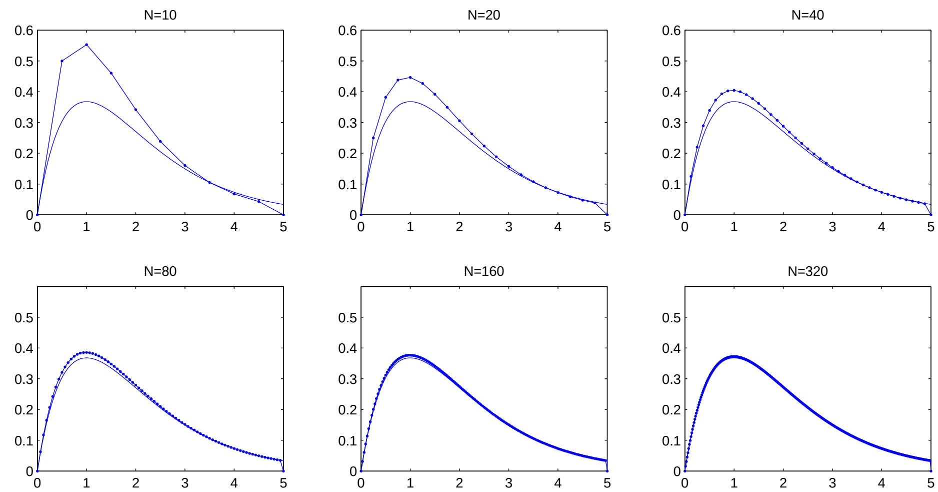

FIGURE 9. Euler's method applied to IVP (20). As the number of subdivisions grows the broken line (Euler's method) converges to the exact solution.

```
[t,x] = ode45(ode,tspan,x0);
```

Notice that `ode45` does not need the number of subdivisions. That is because it actually figures out how to subdivide `tspan` so that the error in the computed solution is less than the default tolerance which can be changed by the user.

5. The art of judicious guessing

With separation of variables, substitutions, and the series method at our disposal, we can solve

$$\frac{dx}{dt} + x = f \tag{22}$$

for a wide variety of right-hand sides f. Table 1 lists some of the general solutions which by now should be familiar.

Notice a simple pattern. Each solution can be viewed as a sum: $x = x_1 + x_2$, where $x_1 = C\,e^{-t}$ and x_2 is a function not far removed from f. Now consider the case

$$\frac{dx}{dt} + x = \sin(t). \tag{23}$$

It seems very likely that the solution of Equation (23) is of the form:

$$x = C\,e^{-t} + \text{some trigonometric terms}$$

f	x
1	$C\,e^{-t} + 1$
t	$C\,e^{-t} + t - 1$
t^2	$C\,e^{-t} + t^2 - 2\,t + 2$
e^{-t}	$C\,e^{-t} + t\,e^{-t}$

TABLE 1. General solutions of ODE (22) for various choices of the forcing term f.

Yet what are these trigonometric terms? For answer, we turn to Figure 10. The plot was generated using the following script:

```
f = @sin;
ode = @(t,x) f(t) - x;
tspan = [0 8*pi];
x0 = 5;
[t,x] = ode45(ode,tspan,x0);

figure
hold on
plot(t,x,'b-',t,x,'b.')
plot(t,f(t),'r-')
xlim(tspan)
```

For small values of t the blue line has the shape of a decaying exponential thus confirming the presence of the term $C\,e^{-t}$. Meanwhile, for large values of t the solution seems to be a horizontally shifted and scaled copy of the sine wave. Therefore, it seems likely that

$$x = C\,e^{-t} + A\,\cos(t) + B\,\sin(t)$$

for some A, B, and C. It is not difficult to find these constants numerically. We have data output by `ode45` which is of the form (t_n, x_n) for $n = 1, \ldots, N$. If our guess is correct, we must have

$$C\,e^{-t_n} + A\,\cos(t_n) + B\,\sin(t_n) = x_n, \tag{24}$$

for all $n = 1, \ldots, N$. To find the constants we need to solve these linear equations. Let

$$\mathbf{p} = \begin{bmatrix} C \\ A \\ B \end{bmatrix}$$

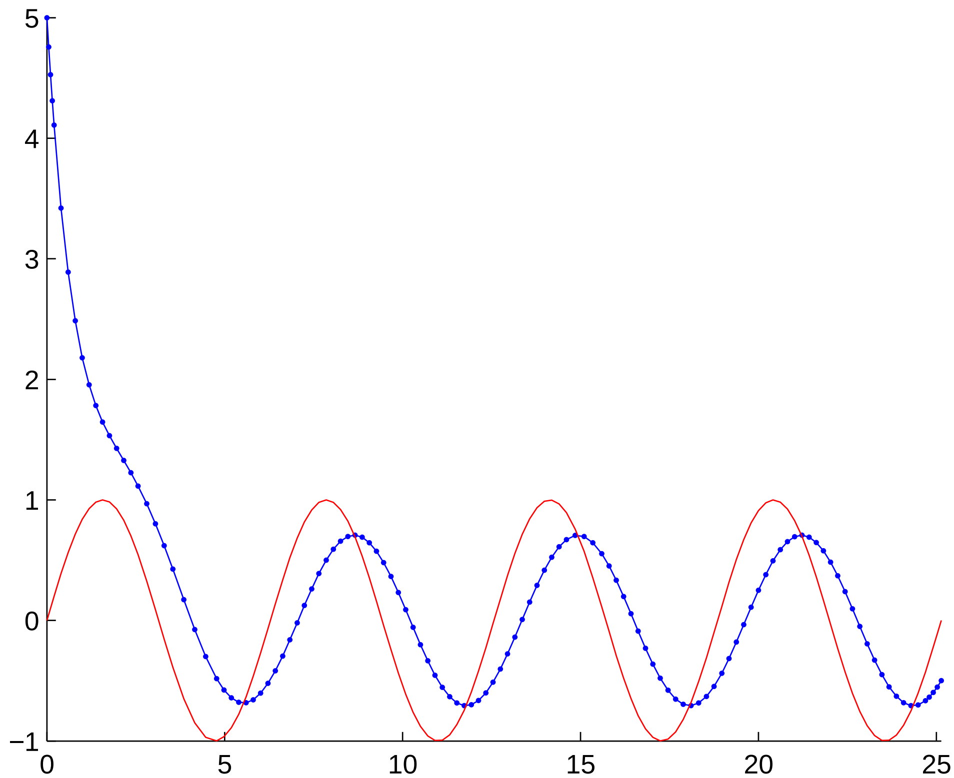

Figure 10. Numerical solution of ODE (23) (blue curve) with initial condition $x(0) = 5$; the red curve is the plot of $f = \sin(t)$.

be the vector whose components are the unknowns A, B, and C; we will use $\mathbf{t}$ and $\mathbf{x}$ to denote *column* vectors with components t_n and x_n, respectively; also, let

$$M = \left[e^{-\mathbf{t}} \ \cos(\mathbf{t}) \ \sin(\mathbf{t})\right]$$

be the $N \times 3$ matrix whose n-th row is $[e^{-t_n} \ \cos(t_n) \ \sin(t_n)]$. Using matrix-vector multiplication, we can rewrite Equation (24) as

$$M\,\mathbf{p} = \mathbf{x}.$$

This matrix-vector system can now be solved using Matlab's `linsolve` command:

```
M = [exp(-t) cos(t) sin(t)];
p = linsolve(M,x)
```

```
p =

   5.4998
  -0.5001
   0.5001
```

Evidently, the solution of Equation (23) with initial value $x(0) = 5$ is

$$x = \frac{11}{2}\, e^{-t} - \frac{1}{2}\,\cos(t) + \frac{1}{2}\,\sin(t).$$

The general solution of ODE (23) is now readily seen to be

$$x = C\, e^{-t} - \frac{1}{2}\,\cos(t) + \frac{1}{2}\,\sin(t).$$

There are two morals to this section. Firstly, when, in the future, we encounter an RC-circuit driven by a harmonic voltage

$$\frac{dx}{dt} + k\, x = a\,\cos(\omega\, t) + b\,\sin(\omega\, t), \tag{25}$$

we can *guess* the solution to be

$$x = C\, e^{-k\, t} + A\,\cos(\omega\, t) + B\,\sin(\omega\, t).$$

To find the constants A and B, we simply need to substitute our guess into ODE (25). This results in

$$-A\,\omega\,\sin(\omega\, t) + B\,\omega\,\cos(\omega\, t) + k\, A\,\cos(\omega\, t) + k\, B\,\sin(\omega\, t) = a\,\cos(\omega\, t) + b\,\sin(\omega\, t),$$

which in turn implies that

$$\begin{aligned} k\, A + \omega\, B &= a \\ -\omega\, A + k\, B &= b. \end{aligned}$$

Once these equations are solved for A and B, the remaining constant C can be found from the initial condition.

The second moral is that it is really useful to look at numerical data when formulating guesses. If an RC-circuit is driven by some strange voltage f, and it is not obvious what to guess—look at numerical data.

Exercises.

(1) Use Euler's method with $N = 10, 20, 40, 80, 160, 320$ subdivisions to solve the following IVP on the interval $[0, 20]$:

$$\frac{dx}{dt} + 20\, x = \cos(t) + 20\,\sin(t), \quad x(0) = 0.$$

Generate a plot similar to Figure 9 and study it (Hint: Don't use `ylim` command—let MATLAB scale axes automatically). Write down your observations. What do you think is the lesson here?

(2) Use `ode45` to solve

$$\frac{dx}{dt} + 2\,x = t + \sin(t), \quad x(0) = 10,$$

on the interval $[0, 50]$. Study the plot and formulate a reasonable conjecture about the formula for x. Confirm or debunk your conjecture by working with the data output by `ode45`.

(3) Make a judicious guess about the solution of

$$\frac{dx}{dt} + 5\,x = 3\,e^{-2\,t}, \quad x(0) = 0.$$

Check the guess by substituting it into the ODE. Validate your work with appropriate MATLAB plots.