Extra credit problem

Let T be the Calculus I antiderivative:

$$T(f) = \int_0^x f(t)\,dt.$$

The goal of this exercise is to solve

$$T(f) = g, \tag{1}$$

where the right-hand side g is given on the interval $[0, 1]$; by solving Equation (1) we mean finding f on the interval $[0, 1]$.

From the point of view of Calculus, Equation (1) is solved simply by differentiation:

$$f = T^{-1}(g) = \frac{dg}{dx}.$$

Clearly, the symbolic differentiation will work only if g is specified by a symbolic expression. More likely, however, is the scenario where we are given a sample of values of g and asked for a sample of values of f. Furthermore, it is very likely that the values of g will not be exact due to noise and round-off errors.

Part 1 (25 Points)

Divide the interval $[0, 1]$ into N equal subintervals; denote the subdivision points by $\mathbf{x}$. To simplify notation, let us abbreviate

$$\mathbf{g} = g(\mathbf{x}), \quad \mathbf{f} = f(\mathbf{x}).$$

Denote by A the matrix representation of T corresponding to the left end-point rule of integration [1]. Now, to solve the discretized problem

$$A\,\mathbf{f} = \mathbf{g},$$

form the normal equations

$$A^T A\,\mathbf{f} = A^T\,\mathbf{g},$$

[1] There are many ways to discretize T and the left end-point rule is by no means ideal, yet it will suffice for our purposes.

and consider the eigenvalue decomposition $[V, \Lambda]$ of $A^T A$. Henceforth we assume, and this is important, that the eigenvalues of $A^T A$ are sorted in decreasing order; the eigenvectors $\mathbf{v}_n$ must be sorted accordingly. In class we derived the following formula

$$\mathbf{f} = \sum_{j=1}^{N} \frac{\mathbf{v}_j^T A^T \mathbf{g}}{\lambda_j} \mathbf{v}_j.$$

This formula works well if λ_n are bounded away from zero. If $\lambda_n \to 0$ we can pick $n < N$ and approximate:

$$\mathbf{f} \approx \sum_{j=1}^{n} \frac{\mathbf{v}_j^T A^T \mathbf{g}}{\lambda_j} \mathbf{v}_j. \tag{2}$$

Perform the following investigation in `Matlab` or its equivalent:

1. Set $N = 100$. Form the matrix A and compute the eigen-decomposition $[V, \Lambda]$ of $A^T A$. Use the `sort` command to rearrange the eigen-decomposition so that the eigenvalues form a *decreasing* sequence: $\lambda_1 > \lambda_2 > \ldots$

2. Produce plots of the first nine eigenvectors of V (corresponding to the *largest* eigenvalues). Indicate the eigenvalues in the titles of the plots. You can arrange the plots on one figure using the `subplot` command. If the plots looks highly oscillatory, you may have forgotten to sort the decomposition.

3. Plot the sorted eigenvalues in logarithmic coordinates (`loglog` command). How fast are the eigenvalues decreasing? Make a conjecture.

4. Generate $\mathbf{x}$ using `linspace` command and set $\mathbf{g} = \mathbf{x}$. Notice that $\mathbf{g} = \mathbf{x}$ corresponds to $\mathbf{f} = \mathbf{1}$.

5. Use Equation (2) to find $\mathbf{f}$: first, set $n = N = 100$ and see if it works. Plot on the same figure $\mathbf{f}$ (against $\mathbf{x}$) in red and $\mathbf{1}$ in blue; title the plot “Full reconstruction without noise”.

6. Add a little bit of noise: `g = x + .01*rand(size(x))`. Plot noisy $\mathbf{g}$ against $\mathbf{x}$; title the plot “Noisy data”.

7. Attempt to recover **f** from noisy **g** using all eigenvectors (Equation (2) with $n = N = 100$); Plot the result and the vector of ones on the same figure and title the plot "Full reconstruction with noise".

8. Use Equation (2) with $n = 9$ to compute an approximation of **f**; Plot the result and the vector of ones on the same figure and title the plot "Reconstruction with 9 eigenvectors".

Submit your work in the following format:

1. Explain how you discretize T; Write the matrix A for $N = 4$.

2. Present all of the plots in the order that they were generated. Make sure the plots are titled appropriately.

3. Attach `Matlab` script.

If you wish to submit the problem electronically you must e-mail one PDF file containing explanations, code, and plots `before` the final exam.

Part 2 (25 Points)

As $N \to \infty$ the eigenvalues and eigenvectors of $A^T A$ approach the eigenvalues and *eigenfunctions* of $T^* T$ (the star superscript is the continuous analogue of transposition). It so happens that the eigenvalues and eigenfunctions of $T^* T$ can be found exactly: doing that will double your score.