9. Fourier transform

Recall that the solution of the following boundary value problem

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\
&\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0,\\
&u(x,0) = f(x),
\end{aligned}
$$

is given by the cosine Fourier series:

$$
u(x,t) = \frac{1}{L}\int_0^L f(y)\,dy + \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(y)\cos\left(\frac{\pi\, n}{L}\,y\right)\,dy\right] e^{-\frac{\pi^2 n^2}{L^2}\,t}\cos\left(\frac{\pi n}{L}\,x\right)
$$

In this section we will consider the limit of the cosine series as $L \to \infty$ which will give us the solution of

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,\\
&\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\\
&u(x,0) = f(x).
\end{aligned}
$$

More importantly, the limit will provide us with an inversion formula for the cosine Fourier transform. This and other Fourier transforms are the main tools of modern analysis; they are widely used to solve ODE, PDE, and many other problems.

Henceforth we assume that the initial condition $u(x,0) = f(x)$ is a continuous function vanishing for all $x > a$; this assumption is for convenience only and will be later dropped. Such stringent restriction on $f(x)$ ensures that it can be integrated against any cosine. Also, the average of f over $[0,L]$ tends to zero as $L \to \infty$:

$$
\lim_{L\to\infty}\frac{1}{L}\int_0^L f(y)\,dy = \lim_{L\to\infty}\frac{1}{L}\int_0^a f(y)\,dy = 0.
$$

Consequently, the limit $\lim_{L\to\infty} u(x,t)$ is given by

$$
\lim_{L\to\infty}\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(y)\cos\left(\frac{\pi\, n}{L}\,y\right)\,dy\right] e^{-\frac{\pi^2 n^2}{L^2}\,t}\cos\left(\frac{\pi n}{L}\,x\right)
$$

Let us interchange the order of operations and write the sum of integrals as the integral of the sum [22]

$$\lim_{L\to\infty}\int_0^L \left[\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{L^2} t}\,\cos\left(\frac{\pi n}{L}\,y\right)\,\cos\left(\frac{\pi n}{L}\,x\right)\,\frac{2}{L}\right]\,f(y)\,dy$$

The rationale for this step is the following: the integral from zero to L in the limit becomes the integral from zero to infinity, hence we can switch our attention to the limit of the bracketed sum. To unclutter the sum, let us introduce

$$\phi(s) = e^{-s^2 t}\,\cos(s\,y)\,\cos(s\,x).$$

Then we can write the sum in terms of ϕ as

$$\sum_{n=1}^{\infty}\phi\left(\frac{\pi\,n}{L}\right)\,\frac{2}{L} = \frac{2}{\pi}\,\sum_{n=1}^{\infty}\phi\left(\frac{\pi\,n}{L}\right)\,\frac{\pi}{L}$$

Let us label $\frac{\pi}{L} = \Delta x$. Then we can recognize the limit as a Riemann sum

$$\frac{2}{\pi}\,\lim_{\Delta x\to 0}\sum_{n=1}^{\infty}\phi(n\,\Delta x)\,\Delta x = \frac{2}{\pi}\,\int_0^{\infty}\phi(s)\,ds.$$

We conclude that in the limit as $L\to\infty$ the solution $u(x,t)$ becomes a double integral:

$$u(x,t) = \int_0^{\infty}\left[\frac{2}{\pi}\,\int_0^{\infty} e^{-s^2 t}\,\cos(s\,y)\,\cos(s\,x)\,ds\right]\,f(y)\,dy$$

Later we will find the exact value of the integral in brackets: it is called the Heat kernel and is some function of x, y, and t. In the meantime, let us rewrite $u(x,t)$ as follows:

$$u(x,t) = \frac{2}{\pi}\,\int_0^{\infty}\cos(x\,s)\,e^{-s^2 t}\,\left[\int_0^{\infty}\cos(s\,y)\,f(y)\,dy\right]\,ds \tag{9.1}$$

Let us introduce the following notation for the cosine Fourier transform and its inverse:

$$\widehat{f}(s) = \int_0^{\infty}\cos(s\,y)\,f(y)\,dy$$

$$\phi^{\vee}(x) = \frac{2}{\pi}\,\int_0^{\infty}\cos(x\,s)\,\phi(s)\,ds$$

Setting $t = 0$ in Equation (9.1) shows that

$$f = \left(\widehat{f}\right)^{\vee}.$$

[22]Interchanging the order of operations is a very common device in analysis

This Fourier inversion formula is very difficult to establish otherwise. Using the "hat" and "check" symbols we can rewrite Equation (9.1) concisely as follows:

$$\boxed{u(x,t) = \left(e^{-s^2\,t}\,\widehat{f}\right)^{\vee}}$$

The formula says that in order to solve the Neumann problem for the Heat equation on $[0,\infty)$ we need to perform the following steps:

(1) Apply the cosine transform to the initial temperature.
(2) Multiply the result by the Gaussian $e^{-s^2\,t}$.
(3) Apply the inverse cosine transform.

One may surmise that whereas the limit of the Neumann problem leads to the cosine transform, that of the Dirichlet problem will lead to the Fourier sine transform. That is indeed the case. If we redefine the "hat" and "check" operations as

$$\begin{aligned}\widehat{f}(s) &= \int_0^\infty \sin(s\,y)\,f(y)\,dy\\ \phi^{\vee}(x) &= \frac{2}{\pi}\int_0^\infty \sin(x\,s)\,\phi(s)\,ds,\end{aligned}$$

we can then write the solution of the Dirichlet problem

$$\begin{aligned}&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,\\ &u_{x=0} = 0,\\ &u(x,0) = f(x).\end{aligned}$$

in the same format: $u(x,t) = \left(e^{-s^2\,t}\,\widehat{f}\right)^{\vee}$. Just as the Fourier cosine and sine series are special cases of the complex exponential series, the Fourier cosine and sine transforms are special cases of the [complex] Fourier transform which we now define.

9.1. **Complex Fourier transform.** Let $f(x)$ be a square-integrable function:

$$\int_{-\infty}^{+\infty} f^2(x)\,dx < \infty.$$

Henceforth we will say that f belongs to the space $L^2(\mathbb{R}^1)$. The square integrability ensures that the Fourier transform of f exists and has nice properties.[23] We now define the Fourier transform[24] $\widehat{f}(\xi)$ as the integral

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-i\,\xi\,x}\, f(x)\, dx.$$

One is lead to this integral in the solution of the Heat equation for a ring of radius $R \to \infty$. The inverse Fourier transform, unsurprisingly, has the familiar division by 2π:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\,x\,\xi}\, \widehat{f}(\xi)\, d\xi = (\widehat{f})^{\vee}.$$

Notice also that the complex exponential in the inverse Fourier transform does not have the minus sign owing to conjugation in the complex dot product. The solution of the Heat equation on the entire real line with square integrable initial data is given by the limit of the series solution for the ring and is therefore yet again $u(x,t) = \left(e^{-\xi^2\,t}\,\widehat{f}\right)^{\vee}$ where the "hat" and "check" are complex Fourier transforms. We will now derive this formula from scratch. First, however, we need to establish some useful properties of the Fourier transform.

9.2. **Properties of the Fourier transform.** In the context of differential equations, the most useful property of the Fourier transform, other than the Fourier inversion formula $f = \widehat{f}^{\vee}$, is the formula for the transform of the derivative:

$$\widehat{\frac{df}{dx}} = (i\,\xi)\,\widehat{f}$$

It is similar in nature to the analogous formula for the Laplace transform $\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$ and is derived in the same manner using integration by parts:

$$\begin{aligned}\widehat{\frac{df}{dx}} = \int_{-\infty}^{+\infty} e^{-i\,\xi\,x}\, f'(x)\, dx = e^{-i\,\xi\,x}\, f(x)\Big|_{-\infty}^{\infty} \\ + (i\xi) \int_{-\infty}^{+\infty} e^{-i\,\xi\,x}\, f(x)\, dx\end{aligned}$$

[23]Another way to think of square integrability is in terms of lengths. Any vector in a finite-dimensional space has finite length which is the sum of squares of its components. We wish to treat functions as vectors and therefore we would like them to have finite norms.

[24]By the Fourier transform we will henceforth always mean the complex exponential transform

Since $f(x)$ is square integrable, we have

$$\lim_{x \to \pm\infty} f(x) = 0$$

and consequently

$$e^{-i\,\xi\,x}\, f(x)\Big|_{-\infty}^{\infty} = 0.$$

Hence

$$\widehat{\frac{df}{dx}} = (i\xi) \int_{-\infty}^{+\infty} e^{-i\,\xi\,x}\, f(x)\, dx = (i\,\xi)\,\widehat{f}.$$

Another very useful property of the Fourier transform relates to translations:

$$\widehat{f(x+a)} = e^{i\,\xi\,a}\,\widehat{f}.$$

This is left as an exercise.

9.3. **Solution of the Heat equation on the real line.** We now consider the problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ &u(x,0) = f(x) \in L^2(\mathbb{R}) \end{aligned}$$

Recall that the notation $f \in L^2(\mathbb{R})$ means that the integral

$$\int_{-\infty}^{\infty} f^2(x)\, dx$$

is finite: that ensures that the Fourier transform $\widehat{f}$ exists (and is also a square-integrable function).

Applying the Fourier transform in the x-variable to the PDE and using the "derivative property" twice, we obtain an ODE:

$$\frac{d}{dt}\widehat{u} = -\xi^2\,\widehat{u},$$

with initial condition $\widehat{u}_0 = \widehat{f}$. Hence

$$\widehat{u} = e^{-\xi^2\,t}\,\widehat{f},$$

and

$$u(x,t) = \left(e^{-\xi^2\,t}\,\widehat{f}\right)^{\vee},$$

as expected.

9.4. **Exercises.**

(1) Prove the translation property of the Fourier transform: $\widehat{f(x+a)} = e^{i\,\xi\,a}\,\widehat{f}$

(2) Find the Fourier transform of the function

$$f(x) = \begin{cases} -1, & -1 \le x \le 0, \\ +1, & 0 < x \le 1. \end{cases}$$

(3) Solve the Wave Equation using Fourier transform:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x),$$
$$\frac{\partial u}{\partial t}(x,0) = g(x).$$