Problem Set 2

Problem 1. *Carefully applying axioms*

It is important to make sure that all of our work in applications is properly founded in theory. One of the more common vector spaces we will work in is the space $(\mathcal{M}_{m\times n}(\mathbb{F}), \mathbb{F})$ of $m \times n$ matrices with entries in the field $\mathbb{F}$, where addition of matrices and $\mathbb{F}$-scalar multiples of matrices are defined in the usual, componentwise sense. We will now abbreviate $(\mathcal{M}_{m\times n}(\mathbb{F}), \mathbb{F})$ simply by its set of vectors $\mathbb{M}_{m\times n}(\mathbb{F})$.

(a) Carefully prove that $\mathcal{M}_{m\times n}(\mathbb{F})$ satisfies all 10(!) of the axioms of a vector space. [*Hint*: Much of the work will be taken care of by appealing to the field axioms that $\mathbb{F}$ satisfies.]

(b) Define $\mathcal{M}_n(\mathbb{F}) := \mathcal{M}_{n\times n}(\mathbb{F})$. Show that set $\mathcal{S}_n(\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{F}) \mid A = A^T\}$ of symmetric matrices is a subspace of $\mathcal{M}_n(\mathbb{F})$.

(c) Determine the dimension of $\mathcal{M}_{m\times n}(\mathbb{F})$ by exhibiting (with proof!) a set of matrices that constitutes a basis for this vector space.

(d) In lecture, I mistakenly stated that the set $\mathbb{L}_m(\mathbb{R})$ of $m \times m$ lower unitriangular matrices formed a field. Briefly explain why I was wrong, and offer a way to change $\mathbb{L}_m(\mathbb{R})$ to the field $\mathcal{F}$ of scalars so that we can think of $(\mathbb{M}_{m\times n}(\mathbb{R}), \mathcal{F})$ as a vector space as well. Also comment on how your change might affect the original interpretation of elementary matrices being identified with elementary row operations.

Note: The space $(\mathbb{M}_{m\times n}(\mathbb{R}), \mathbb{L}_m(\mathbb{R}))$ is indeed a legitimate, algebraic object; but, alas, it is not a field. It is considered a *left* $\mathbb{L}_m(\mathbb{R})$*-module*, and it is a generalization of vector spaces. Instead of using fields of scalars, modules use *rings* of scalars; but they largely behave the same as vector spaces. Basically, rings ignore the commutativity of multiplication and existence of multiplicative inverses in the set of scalars (they don't even have to have a unit 1 element!). If you're interested in pursuing this further, feel free to talk to me or take MATH 143 in the Fall and MATH 144 sometime thereafter.

Problem 2. *Sample spaces, huh? Tell me more...*

Because continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ have an infinity of inputs, we can only ever approximate them in the computer. This suggests that we should develop a theory about *sample vectors* and how they relate to the functions they approximate. Ideally we would like to characterize all of the important information about a function using a "small" sample vector - that way we can instead use the sample vector to study the function.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct sample points. Let $F = \{f_1, \ldots, f_n\} \subset C(\mathbb{R})$ be a collection of continuous functions on $\mathbb{R}$, and define

$$\vec{\mathbf{f}}_i = \begin{bmatrix} f_i(x_1) \\ f_i(x_2) \\ \vdots \\ f_i(x_n) \end{bmatrix}$$

to be the i^{th} *sample vector on* S for all $1 \le i \le n$.

(a) Show that F is linearly independent in $C(\mathbb{R})$ if the sample vectors $\vec{\mathbf{F}} = \{\vec{\mathbf{f}}_1, \vec{\mathbf{f}_2}, \ldots, \vec{\mathbf{f}}_n\}$ are linearly independent vectors in $\mathbb{R}^n$.

(b) Show that it is not necessarily the case that the converse holds; that is, find a set of linearly independent functions whose sample vectors are linearly dependent.

(c) Fix an integer $K \geq 1$, and define

$$F_K = \{1, \sin x, \cos x, \sin(2x), \cos(2x), \ldots, \sin(Kx), \cos(Kx)\}.$$

Show that F_K is a set of linearly independent vectors in $C(\mathbb{R})$.

Problem 3. *Every vector space has a basis*

These words are a mantra for most students of theoretical linear algebra, and they favor a rather dramatic claim. We have effectively shown this result for $\mathbb{R}^n$, but we would like to make sure this holds for abstract vector spaces over arbitrary fields. We will show that this is indeed true for the finite-dimensional case and then examine some of the challenges we face in the infinite-dimensional case.

Let $(V, \mathbb{F})$ be an n-dimensional vector space over the field $\mathbb{F}$, and let $(W, \mathbb{F})$ be an infinite-dimensional vector space.

(a) Let $\mathcal{L} \subset V$ be a linearly independent subset of vectors from V. Suppose $\vec{\mathbf{v}} \in V$ doesn't lie in $\mathcal{L}$. Show that $\mathcal{L} \cup \{\vec{\mathbf{v}}\}$ is linearly independent if and only if $\vec{\mathbf{v}}$ is not in $\operatorname{span}(\mathcal{L})$.

(b) Does the statement in (a) hold for W?

(c) Let $\mathcal{S} \subset V$. Show that if $\operatorname{span}(\mathcal{S}) = V$, then there are at least n vectors in $\mathcal{S}$.

(d) Does the statement in (c) hold for W?

(e) Show that any linearly independent set $\mathcal{L}$ of vectors in V can be extended to a basis of V; that is, if $\mathcal{L}$ isn't already a basis for V, we can add vectors to it so that it is.

(f) Does your argument in (e) work for W?

(g) Show that any spanning set $\mathcal{S}$ of V can be reduced to a basis of V; that is, if $\mathcal{S}$ isn't a basis for V, we can remove vectors from it until it is.

(h) Does your argument in (g) work for W?

(i) Use your observations above to explain some things that make it difficult to show that every *infinite-dimensional* vector space has a basis.