## 1. Introduction

A partial differential equation (PDE) is a relation between partial derivatives of an unknown multivariate function. Additionally, a PDE may involve the function itself, as well as independent variables. As a simple example, consider the one-dimensional Heat Equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

Here the unknown function $u$ depends on two variables, time $t$ and position $x$, while $Kk$ is a constant measured in appropriate units. As we will later show, Equation (1.1) can be used to model the diffusion of heat in a thin (one-dimensional) rod—hence the name. In fact, the same Equation (1.1) may be used to describe any type of linear one-dimensional diffusion: for this reason it is often called the Diffusion Equation.

### 1.1. **Initial-Boundary Value Problems.**

Since the Heat (Diffusion) Equation (1.1) describes evolution in time, it must be supplemented with an initial condition: $u(x, 0) = f(x)$. This is very much analogous to the situation in ODE except that now initial conditions depend on space variables. However, specifying initial conditions alone does not ensure that an evolutionary PDE has a unique solution: one also needs *boundary conditions*. These can take many different forms depending on the geometry of the domain and modeling assumptions.

Henceforth think of $u(x, t)$ in (1.1) as the temperature of a thin rod which we identify with an interval $[a, b]$ of the $x$-axis. Then $u(x, 0) = f(x)$ is the initial temperature defined on $[a, b]$. The boundary conditions describe what happens to the temperature at the endpoints of the interval. For instance, the temperature at the endpoints may be given as two functions of time, say, $\alpha(t)$ and $\beta(t)$. Then one has what are called *Dirichlet boundary conditions*. Accordingly, the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad a < x < b, \quad t > 0, \\ &u(a, t) = \alpha(t), \quad u(b, t) = \beta(t), \\ &u(x, 0) = f(x), \end{aligned} \tag{1.2}$$

is known as the Dirichlet initial-boundary value problem (IBVP) for the Heat Equation. We will usually call Equation (1.2) simply the Dirichlet problem for the Heat Equation.

Instead of specifying temperature at the endpoints, one may specify its partial derivative:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad a < x < b, \quad t > 0,\\
&\frac{\partial u}{\partial x}(a,t) = \alpha(t), \quad \frac{\partial u}{\partial x}(b,t) = \beta(t),\\
&u(x,0) = f(x).
\end{aligned} \tag{1.3}$$

Equation (1.3) defines the *Neumann problem* for the Heat Equation. Physically, Neumann conditions have to do with the flux of energy through the boundary. If Neumann conditions are zero, that usually indicates that the boundary is perfectly insulated.

Other common boundary conditions include *Robin conditions* and *periodic conditions*. Robin-type boundary conditions can be thought of as linear combinations of Neumann and Dirichlet. For instance, we may have at $x = a$:

$$\frac{\partial u}{\partial x}(a,t) + c\,u(a,t) = 0$$

and, similarly, at $x = b$. Physically, Robin conditions may mean that the boundary is insulated but not perfectly.

Periodic boundary conditions have the simple form:

$$u(a,t) = u(b,t).$$

These conditions are used when $x = a$ and $x = b$ describe one and the same point: that is, when the interval $[a,b]$ is used to describe a circle.

Boundary conditions are the main reason why PDE are much more complicated than ODE. Varying initial conditions does not change the complexity of an initial value problem. On the other hand, changing boundary conditions may change the problem dramatically. For instance, the Heat Equation on the entire real line is very different from the Heat Equation on a finite interval.

1.2. **Logical organization of the course.** At the core of Mathematical Physics are three *linear* PDE: The Heat (Diffusion) Equation, the Wave Equation, and the Laplace-Poisson Equation. Accordingly, we will spend most of our time studying these PDE using tools from Linear Algebra and Numerical Analysis. Our first step typically will be to convert a linear PDE into a system of linear ODE through discretization. Therefore, as a minimum prerequisite, you need to know how to solve large systems of linear ODE with constant coefficients in MATLAB or similar language.

The following homework is designed to test your prerequisites: you will not have much success with PDE if you do not have a good working knowledge of linear ODE. Therefore, if you find this assignment challenging, you must seek immediate help.

## Homework

(1) The following ODE is often used to model a forced mass-spring system:
$$m\,\frac{d^2u}{dt^2} + r\,\frac{du}{dt} + k\,u = A\,\sin\omega t.$$
Explain why this ODE is classified as *linear nonhomogeneous* and show how linearity is used to construct the general solution. To be absolutely clear, let me emphasize that I know what that general solution is and I am not looking for the formula. Rather I want to see if you can solve the ode using first principles.

(2) Building on the previous exercise, explain why the Heat Equation (1.1) is described as *linear homogeneous*.

(3) Consider the Heat Equation (1.1) with $k = 1$. Find as many particular solutions of that equation as you can. What can you say about the general solution?

(4) Discretization of the Heat Equation (1.1) in the space variable (semi-discretization) produces a system of linear ODE with constant coefficients. The latter can be solved using standard linear algebra methods. This, in fact, will be our standard approach to solving evolutionary PDE. As an illustration, consider the following ODE:
$$\frac{du}{dt} = k\,A\,u, \quad u(0) = u_0. \tag{1.4}$$
Think of it as the result of semi-discretizing the Heat equation. In (1.4) $u$ is a column vector of length $N$; $k = N^2$; $A$ is an $N$-by-$N$ tridiagonal matrix with $(-2)$'s on the main diagonal and 1's on the subdiagonals. E.g., if $N = 4$ then $k = 16$ and the matrix $A$ is given by:
$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$
For definitiveness, let us think of the component $u(n)$ as temperature at the point $x(n) = n/(N+1)$. Thus $u(t)$ is a sampling of continuous temperature on the uniform grid

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x = (1:N)'/(N+1)
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in the interval $[0, 1]$. Compute the solution of Equation (1.4) at time $t = .1$ for $N = 10, 20, 50, 100$ setting $u_0$ to the vector of all 1's in each case. Plot the results on the above grid on the same axes. What are your observations? In particular, what do you think happens in the limit as $N$ approaches infinity? Try to give as quantitative an answer as you can.