NOTES ON VECTOR ANALYSIS

ALEKSEI BELTUKOV

3. Heat Equation in two dimensions

In this section we derive the Heat Equation for a thermally insulated metal plate by generalizing the arguments used in Section 2 to derive the Heat Equation for a metal rod. Let us denote the thickness of the plate by δ and think of its shape as a simple domain D in the xy-plane. The heat flows within the plate in accordance with Fourier Law—against the gradient of temperature. For simplicity, let us assume that initially the temperature varies only within the xy-plane and is constant along the z-axis. Then the heat flow is two-dimensional and, consequently, the temperature u depends on two spacial variables x and y, and the time t.

Recall from Section 2 that the Heat Equation is a local form of the Law of Conservation of Energy. Local laws are typically derived from global laws. For the one-dimensional rod we applied energy balance to an arbitrary section $[a, b]$. As the reader may have already guessed, in two dimensions one applies the energy balance to an arbitrary patch.

To simplify the Calculus, we will first work with rectangular patches. Once it becomes clear how the arguments work for rectangles, we will broaden them to encompass more general shapes.

3.1. **Energy stored in an arbitrary rectangular patch.** Let R be a rectangular portion of the plate:

$$R = \{(x, y) \in D \mid x_1 \le x \le x_2,\ y_1 \le y \le y_2\},$$

with limits x_1, x_2, y_1, and y_2 chosen in an *arbitrary* manner. Following the same reasoning as in Section 2.1, we write the total amount of heat q contained within R at time t as:

$$q(t) = c\,\rho\,\delta \iint_R u(x, y, t)\,dx\,dy,$$

where, as before, c stands for heat capacity and ρ stands for density.

One advantage of working with rectangles is that one can easily convert double integrals into iterated integrals:

$$q(t) = c\,\rho\,\delta \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} u(x, y, t)\,dx\right] dy. \tag{9}$$

Notice that in Equation (9) we chose a particular order of integration: first with respect to the x-variable and then with respect to the y-variable. The order of integration can be reversed and we will use that, whenever necessary.

3.2. **The change in energy stored in an arbitrary rectangular patch.** Differentiating Equation (9) with respect to time leads to:

$$\frac{dq}{dt} = c\,\rho\,\delta \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x, y, t)\, dx \right] dy. \tag{10}$$

Here, we brought the time derivative inside the integral at once; for explanation, refer to Section 2.2.

3.3. **Flux and the global energy balance.** We now need to express dq/dt in terms of heat flux. Yet, what is flux in two dimensions?

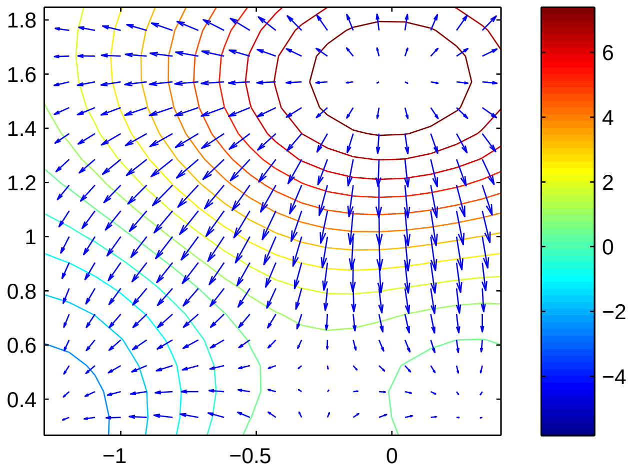

FIGURE 1. Heat flux in two dimensions

Figure 1 provides a visual answer: it shows a contour plot of a typical $2D$ temperature function and the local directions of heat flow. Recall that the lines of constant temperature are called *isotherms*; the colorbar on the right shows their scale. The collection of vectors is the graphic representation of two-dimensional flux: it is an example of a *vector field*. Notice that the heat appears to emanate from a hot spot in the upper right corner and flow towards cold spots in the two lower corners. Furthermore, the direction of the flow is perpendicular, or *orthogonal*, to the isotherms. This is an illustration of the familiar multivariate Calculus fact about the gradient of a function being perpendicular to its level set.

In order to produce Figure 1 one first plots a few isotherms. Then one computes the negative gradient of the temperature, scales it, if necessary, and plots the result as vectors issuing from evenly spaced grid points. Thus, symbolically, flux is a vector-valued function of position and time:

$$\phi(x, y, t) = f(x, y, t)\,\mathbf{i} + g(x, y, t)\,\mathbf{j}.$$

The *components* f and g show, roughly speaking, at what rate the heat flows locally in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. We will now use these components to compute the net flux across the boundary of R.

Let us start with the left side. Consider a small segment dy on the vertical line $x = x_1$. Across the area $\delta\, dy$ the heat flows *into* the patch (of thickness δ) at the rate $f(x_1, y, t)\,\delta\, dy$. Notice that only the first component of ϕ contributes to the flow across a vertical boundary. The net flux across the side $x = x_1$ is the integral:

$$\int_{y_1}^{y_2} f(x_1, y, t)\,\delta\, dy.$$

Similarly, the net flux across $x = x_2$ is given by

$$-\int_{y_1}^{y_2} f(x_2, y, t)\,\delta\, dy.$$

The minus sign is placed because positive flux across the right side is outward—out of the patch. Proceeding in like manner, we find arrive at the sum of four integrals:

$$\begin{aligned} \frac{dq}{dt} &= \int_{y_1}^{y_2} f(x_1, y, t)\,\delta\, dy - \int_{y_1}^{y_2} f(x_2, y, t)\,\delta\, dy \\ &+ \int_{x_1}^{x_2} f(x, y_1, t)\,\delta\, dx - \int_{x_1}^{x_2} f(x, y_2, t)\,\delta\, dx. \end{aligned} \tag{11}$$

From Equations (10) and (11) now follows that

(12)

$$\begin{aligned} c\,\rho \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x, y, t)\, dx\right] dy &= \int_{y_1}^{y_2} \left[f(x_1, y, t) - f(x_2, y, t)\right] dy \\ &+ \int_{x_1}^{x_2} \left[f(x, y_1, t) - f(x, y_2, t)\right] dx. \end{aligned}$$

The reason for pairing up the integrals on the right will become apparent in the next step.

3.4. **Equation of Continuity.** Using the Fundamental Theorem of Calculus, we can rewrite the bracketed differences on the right-hand side of Equation (12) as integrals:

$$\int_{y_1}^{y_2} \left[f(x_1, y, t) - f(x_2, y, t)\right]\, dy + \int_{x_1}^{x_2} \left[f(x, y_1, t) - f(x, y_2, t)\right]\, dx$$
$$= -\int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y, t)\, dx\right] dy - \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} \frac{\partial g}{\partial y}(x, y, t)\, dy\right] dx.$$

We now have three iterated integrals which can be combined into one (remember that the order of integration can be exchanged):

$$\int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} \left\{c\,\rho\, \frac{\partial u}{\partial t}(x, y, t) + \frac{\partial f}{\partial x}(x, y, t) + \frac{\partial g}{\partial y}(x, y, t)\right\} dx\right] dy = 0.$$

Since the limits x_1, x_2, y_1, and y_2 are arbitrary, the integrand must necessarily be zero:

$$c\,\rho\, \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \tag{13}$$

Equation (13) is the Equation of Continuity in two dimensions.

3.5. **Fourier's Law and the Heat Equation.** According to Fourier's Law:

$$\phi = -K \operatorname{grad} u = -K \left(\frac{\partial u}{\partial x}\,\mathbf{i} + \frac{\partial u}{\partial y}\,\mathbf{j}\right).$$

In words, the components of flux are given by partial derivatives of the temperature. Introducing the constant

$$k = \frac{K}{c\,\rho},$$

as before, we can write the Heat Equation in two dimensions as:

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.\right) \tag{14}$$

4. Flux across arbitrary two-dimensional domains

We derived the Heat Equation in two-dimensions by working with rectangular patches. From the Calculus perspective, rectangles may seem like a "natural" choice. Indeed, it is easy to convert double integrals over rectangles into iterated integrals; also, the flux across a rectangle is a relatively simple sum of four one-dimensional integrals. Conceptually, however, rectangles are no better or worse than triangles, disks, and other shapes. And it is important to be able to compute flux across *any* shape.

4.1. **Limit definition.** Let D be a generic planar domain with boundary ∂D and let $F(x, y) = f(x, y)\,\mathbf{i} + g(x, y)\,\mathbf{j}$ be a generic two-dimensional vector field. Computing flux of F across D is a process very similar to that of finding areas and volumes:

(1) Break up the curve ∂D—the domain of integration—into N "small" segments.
(2) Within each segment pick a point. Label these points (x_i, y_i) where $i = 1, \ldots, N$.
(3) To find the flux across the i-th segment, project the local flow $F(x_i, y_i)$ onto the outward unit normal $n(x_i, y_i)$ and multiply by the length of the segment Δs_i; the total flux is *approximated* by the sum:

$$\sum_{i=1}^{N} F(x_i, y_i) \cdot n(x_i, y_i)\,\Delta s_i.$$

(4) Finally, to find the exact flux, take the limit as $N \to \infty$ and $\Delta s_i \to 0$:

$$\lim_{N\to\infty} \sum_{i=1}^{N} F(x_i, y_i) \cdot n(x_i, y_i)\,\Delta s = \int_{\partial D} F \cdot n\, ds..$$

In words, flux across a domain D is the integral of the *normal component* of the field F over the boundary ∂D where the integration is performed with respect to the arc length measure.

4.2. **Symbolic computation.** The symbolic computation of flux is best illustrated with an example. Let $F = x\,\mathbf{i} + y\,\mathbf{j}$. As D, let us choose a disk of radius r centered at some fixed point (x_0, y_0) of the plane. Since the radii of a disk are orthogonal to its circumference, the outward unit normal vector is given by:

$$n(x, y) = \frac{(x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j}}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.$$

Consequently, the normal component of F at (x, y) is given by:

$$F \cdot n = \frac{(x - x_0)\,x + (y - y_0)\,y}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.$$

The arc length in Cartesian coordinates is given by:

$$ds = \sqrt{dx^2 + dy^2}.$$

We are thus led to the following formidable-looking integral:

$$\int_{\partial D} \frac{(x-x_0)\,x+(y-y_0)\,y}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\,\sqrt{dx^2+dy^2}.$$

To proceed further, parameterize the boundary circle ∂D as follows:

$$\begin{cases} x(t) = x_0 + r\,\cos(t), \\ y(t) = y_0 + r\,\sin(t), \\ \quad 0 \le t < 2\pi. \end{cases}$$

The standard trigonometric parametrization leads to the following simple expressions

$$\frac{(x-x_0)\,x+(y-y_0)\,y}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = x_0\,\cos(t) + y_0\,\sin(t) + r$$

and

$$ds = \sqrt{dx^2+dy^2} = r\,dt.$$

We thus obtain a simple Calculus I integral:

$$\int_0^{2\pi} (x_0\,\cos(t) + y_0\,\sin(t) + r)\;r\,dt = 2\,\pi\,r^2.$$

Notice that the result is twice the area of the disk.

Other examples of Calculus flux computation can be found in exercises at the end of this handout. In the meantime, we return to the derivation of the Heat Equation in two dimensions.

5. Divergence

Let D be an arbitrary portion of the plate. Using the notation of Section 3, we can express the rate at which the amount of heat in D changes as the double integral of the time derivative of temperature:

$$\frac{dq}{dt} = c\,\rho\,\delta \iint_D \frac{\partial u}{\partial t}\,dx\,dy.$$

On the other hand, in light of Section 4, the same rate is given by:

$$\frac{dq}{dt} = -\delta \int_{\partial D} \phi \cdot n\,ds.$$

The minus sign has to do with the convention of n being the *outward* unit normal: positive flux means heat is leaving D. Combining the two expressions and canceling thickness δ, we get a global conservation law:

$$c\,\rho \iint_D \frac{\partial u}{\partial t}\,dx\,dy = -\int_{\partial D} \phi \cdot n\,ds.$$

We now need to somehow convert the flux integral into a double integral. In Section 3.4 we used the Fundamental Theorem of Calculus to show that

$$\int_{\partial D} \phi \cdot n \, ds = \iint_D \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dx \, dy,$$

as long as D is a rectangle (and $\phi = f\,\mathbf{i} + g\,\mathbf{j}$). Since rectangles are not special, the same should be true for arbitrary domains. Let us denote

$$\operatorname{div}(\phi) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.$$

This operation on vector fields is called *divergence*. It seems possible that

$$\int_{\partial D} \phi \cdot n \, ds = \iint_D \operatorname{div}(\phi) \, dx \, dy \tag{15}$$

should be true for any domain D, rectangular or not; this, in fact, is the statement of the Divergence Theorem in two dimensions.

We will prove the Divergence Theorem in the next section. In the meantime, its application leads to

$$c\,\rho \iint_D \frac{\partial u}{\partial t} \, dx \, dy = -\iint_D \operatorname{div}(\phi) \, dx \, dy$$

and, due to arbitrariness of D, to the equality of the integrands:

$$c\,\rho \frac{\partial u}{\partial t} = -\operatorname{div}(\phi).$$

This is the Equation of Continuity (13) written in traditional vector analysis language. Recall that the next step is to use Fourier's Law: $\phi = -K \operatorname{grad} u$. This almost immediately leads to the Heat Equation

$$\frac{\partial u}{\partial t} = k \operatorname{div} \operatorname{grad} u.$$

The divergence of the gradient, when expanded, is the sum of second partial derivatives which we wrote earlier on the right-hand side of (14). It is such a common and important operation that it has its own name—Laplacian—and its own symbol:

$$\operatorname{div} \operatorname{grad} u = \Delta u.$$

There are other notations that one can use for divergence, gradient, Laplacian, and other vector analysis operations. However, for clarity, we will use the verbal notation in the beginning.

Exercises.

(1) First, guess the Heat Equation in three dimensions. Next derive it by applying energy balance to arbitrary cubes. Finally, attempt the derivation using arbitrary three dimensional domains. Write down your observations.

(2) Let $F = x\,y\,\mathbf{i}$ and let T be the triangle with vertices $(0,0)$, $(0,1)$, and $(3,0)$. Use Calculus to find the flux of F across T in two different ways: using the definition of flux, as the boundary integral of the normal component of the field, and using the Divergence Theorem, as the area integral of the field's divergence. What is your observation?

(3) Let
$$F = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2+y^2)}.$$
Use Matlab, or its equivalent to do the following:
   (a) Plot the field in the rectangle $1 \le x \le 4$, $2 \le y \le 5$. The result should be comparable to Figure 1.
   (b) Compute the flux of F across a disk of radius 1 centered at the following points: $(0,0)$, $(0,.5)$, $(.5,0)$, $(.25,.25)$.
   (c) Repeat the previous exercise with unit disks centered at $(2,0)$, $(0,-2)$, $(1,1)$, and $(2,3)$. Write your observations.