Problem Set 4 Solutions

Problem 1. *A first example in Fourier Analysis*

We mentioned in class that the set

$$\beta_K = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos Kx, \sin Kx\}$$

is orthogonal in $(C[-\pi,\pi], L^2)$. This gave another (perhaps more convincing) explanation for why β_K is linearly independent in the vector space $C[-\pi,\pi]$. Below you give the formal proof and then explore a peculiar use of this set.

(a) Through direct integral calculations, show that it is indeed the case that

$$\langle 1,1\rangle = 2\pi, \quad \langle 1, \cos nx\rangle = 0, \quad \langle 1, \sin nx\rangle = 0, \quad \langle \cos mx, \sin nx\rangle = 0,$$

$$\langle \cos mx, \cos nx\rangle = \begin{cases} \pi & , \text{ if } m = n, \\ 0 & , \text{ if } m \neq n \end{cases}, \quad \text{and} \quad \langle \sin mx, \sin nx\rangle = \begin{cases} \pi & , \text{ if } m = n, \\ 0 & , \text{ if } m \neq n \end{cases}$$

for all integers $m, n \geq 1$.

Remark - It's essentially a rite of passage for mathematicians and applied mathematicians alike to be able to do this work at least once before they graduate.

This is quite literally a list of computations; so we mention u-substitution is the most useful method for each calculation, but integration by parts is also valid for (iv), (v), and (vi):

$$(i)\ \langle 1,1\rangle = \int_{-\pi}^{\pi} 1\cdot 1\, dx = x|_{x=-\pi}^{\pi} = 2\pi.$$

$$(ii)\ \langle 1, \cos nx\rangle = \int_{-\pi}^{\pi} 1\cdot \cos nx\, dx = \frac{1}{n}\sin nx|_{x=-\pi}^{\pi} = \frac{1}{n}(\sin n\pi - \sin n\pi) = 0.$$

$$(iii)\ \langle 1, \sin nx\rangle = \int_{-\pi}^{\pi} 1\cdot \sin nx\, dx = -\frac{1}{n}\cos nx|_{x=-\pi}^{\pi} = \frac{1}{n}(-\cos n\pi + \cos n\pi)$$

$$= \frac{1}{n}(-(-1)^n + (-1)^n) = 0.$$

$$(iv)\ \langle \cos mx, \sin nx\rangle = \int_{-\pi}^{\pi} \cos mx \sin nx\, dx$$

$$= \begin{cases} \displaystyle\int_{-\pi}^{\pi} \frac{1}{2}(\sin(m+n)x + \sin(m-n)x)\, dx & , \text{ if } m \neq n \\ \displaystyle\int_{-\pi}^{\pi} \sin nx \cos nx\, dx & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right)|_{x=-\pi}^{\pi} \\ \frac{1}{2}\sin^2 nx|_{x=-\pi}^{\pi} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\left(-\frac{(-1)^{m+n}}{m+n} - \frac{(-1)^{m-n}}{m-n}\right) + \left(\frac{(-1)^{m+n}}{m+n} + \frac{(-1)^{m-n}}{m-n}\right)\right] & , \text{ if } m \neq n \\ \frac{1}{2}(0^2 - 0^2) & , \text{ if } m = n \end{cases} = 0.$$

$$(v)\ \langle \cos mx, \cos nx\rangle = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

$$= \begin{cases} \displaystyle\int_{-\pi}^{\pi} \frac{1}{2}(\cos(m+n)x + \cos(m-n)x)\, dx & , \text{ if } m \neq n \\ \displaystyle\int_{-\pi}^{\pi} \cos^2 nx\, dx & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right)\Big|_{x=-\pi}^{\pi} & , \text{ if } m \neq n \\ \frac{1}{2}\left(x + \frac{1}{2n}\sin 2nx\right)\Big|_{x=-\pi}^{\pi} & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} \frac{1}{2}[(0+0) - (0+0)] & , \text{ if } m \neq n \\ \frac{1}{2}[(\pi+0) - (-\pi+0)] & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } m \neq n \\ \pi & , \text{ if } m = n \end{cases}$$

$$(vi)\ \langle \sin mx, \sin nx\rangle = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx$$

$$= \begin{cases} \displaystyle\int_{-\pi}^{\pi} \frac{1}{2}(\cos(m-n)x - \cos(m+n)x)\, dx & , \text{ if } m \neq n \\ \displaystyle\int_{-\pi}^{\pi} \sin^2 nx\, dx & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right)\Big|_{x=-\pi}^{\pi} & , \text{ if } m \neq n \\ \frac{1}{2}\left(x - \frac{1}{2n}\sin 2nx\right)\Big|_{x=-\pi}^{\pi} & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} \frac{1}{2}[(0-0) - (0-0)] & , \text{ if } m \neq n \\ \frac{1}{2}[(\pi-0) - (-\pi-0)] & , \text{ if } m = n \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } m \neq n \\ \pi & , \text{ if } m = n \end{cases}$$

(b) Briefly explain why the set

$$\hat{\beta}_K = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \ldots, \frac{1}{\sqrt{\pi}}\cos Kx, \frac{1}{\sqrt{\pi}}\sin Kx\right\}$$

is orthonormal in $(C[-\pi,\pi], L^2)$.

From part (a), we find

$$\|1\| = \sqrt{2\pi} \quad \text{and} \quad \|\cos nx\| = \|\sin nx\| = \sqrt{\pi}.$$

We then see that the vectors in $\hat{\beta}_K$ are simply the vectors of β_K scaled down by their lengths. This makes each vector in $\hat{\beta}_K$ a unit vector so that $\hat{\beta}_K$ is orthonormal.

(c) Find the orthogonal projection

$$\tilde{f}_K(x) = \sum_{n=0}^{K} a_n \cos(nx) + \sum_{n=1}^{K} b_n \sin(nx)$$

of the function $f(x) = x$ onto β_K. These a_n and b_n coefficients are called the *Fourier coefficients* of $f(x)$. [*Hint*: Making use of the parity (odd/even) of a function makes some integrals trivial in this computation.]

From the orthogonal projection theorem proven in class, we know that the orthogonal projection of $f(x) = x$ onto β_K is given by

$$\tilde{f}_K(x) = \sum_{n=0}^{K} \frac{\langle x, \cos(nx)\rangle}{\|\cos(nx)\|^2} \cos(nx) + \sum_{n=1}^{K} \frac{\langle x, \sin(nx)\rangle}{\|\sin(nx)\|^2} \sin(nx)$$

$$= \frac{\langle x, 1\rangle}{2\pi} + \sum_{n=1}^{K} \frac{\langle x, \cos(nx)\rangle}{\pi} \cos(nx) + \sum_{n=1}^{K} \frac{\langle x, \sin(nx)\rangle}{\pi} \sin(nx).$$

Hence

$$a_0 = \frac{\langle x, 1\rangle}{2\pi}, \quad a_n = \frac{\langle x, \cos(nx)\rangle}{\pi}, \quad \text{and} \quad b_n = \frac{\langle x, \sin(nx)\rangle}{\pi}$$

for $n \geq 1$. We then proceed to compute the inner products of the vectors in β_K with $f(x) = x$.

$$(i)\ a_0 = \frac{\langle x, 1\rangle}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cdot 1 \, dx = 0 \text{ (since } x \text{ is odd)},$$

$$(ii)\ a_n = \frac{\langle x, \cos(nx)\rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \cos(nx) \, dx = 0 \text{ (all due to being odd)},$$

$$(iii)\ b_n = \frac{\langle x \sin(nx)\rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \sin(nx) \, dx = \frac{1}{\pi} \left[-\frac{1}{n} x \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_{x=-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(-\frac{1}{n} \pi (-1)^n + \frac{1}{n^2} \cdot 0 \right) - \left(\frac{1}{n} \pi \cdot (-1)^n + \frac{1}{n^2} \cdot 0 \right) \right] = (-1)^{n+1} \frac{2}{n}.$$

Therefore, the constant term and all cosine terms vanish, and we are left only with the sine terms:

$$\tilde{f}_K(x) = \text{proj}_{\beta_K} f(x) = \sum_{n=1}^{K} (-1)^{n+1} \frac{2}{n} \sin(nx).$$

(d) Use MATLAB to plot $\tilde{f}_K(x)$ on $[-\pi, \pi]$ for $K = 1, 5, 10, 20$. (No need to submit source code.) What happens to $\tilde{f}_K(x)$ as K gets larger? Does anything strange or unexpected happen?

We can generate the MATLAB plots with the following code:

```
clear n x K f F;
close all;

n = 300;
x = linspace(-pi,pi,n+1);
K = 20;

f = zeros(K,n+1);

for ii = 1:K
    f(ii,:) = (-1)^(ii+1)*2/ii*sin(ii*x);
end

F = zeros(1,n+1);

for ii = 1:K
    F = F + f(ii,:);
end

plot(x,F);
title(['Orthogonal projection of f(x) = x onto ',...
    num2str(K),'-truncated Fourier basis']);
xlabel('x');
ylabel(['proj','_{\beta_{',num2str(K),'}}f(x)'],'Interpreter','tex');
```

Please find the plots in Figures 1 and 2 below.

First, we notice that these projections seem to approximate the function $f(x) = x$ in the sense that "the more terms we use, the more the projection looks like $f(x)$". It appears that we are always stuck with $\text{proj}_{\beta_K}(-\pi) = \text{proj}_{\beta_K}(\pi) = 0$, and this doesn't agree with $f(x)$ at $x = \pm\pi$. Perhaps we notice at the ends near $x = \pm\pi$ that the plots never seem to lose the sharp "spike" at the end that appears to overshoot $f(x)$ by a small amount. (This is called the *Gibbs phenomenon*.)

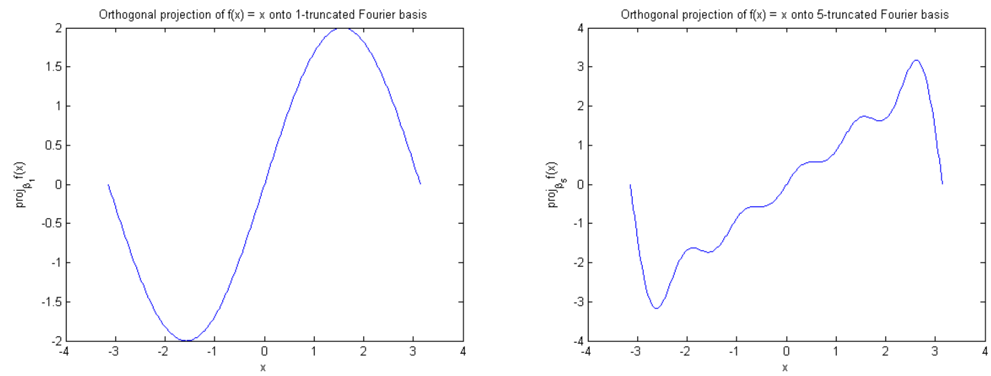

Figure 1: The projection $\text{proj}_{\beta_K} f(x)$ for $K = 1$ and $K = 5$.

(e) Find the squared distance $d(\tilde{f}_K(x), x)^2 = \|\tilde{f}_K(x) - x\|^2 = \langle \tilde{f}_K(x) - x, \tilde{f}_K(x) - x\rangle$. Leave your answer as a sum over the index n. [*Hint*: If you're computing more integrals, you're working too hard.]

Recall that, for any vector $\vec{\mathbf{v}} \in V$ and subset $W \subset V$, we can apply the generalized Pythagorean Theorem to the right triangle formed by $\vec{\mathbf{v}}$,

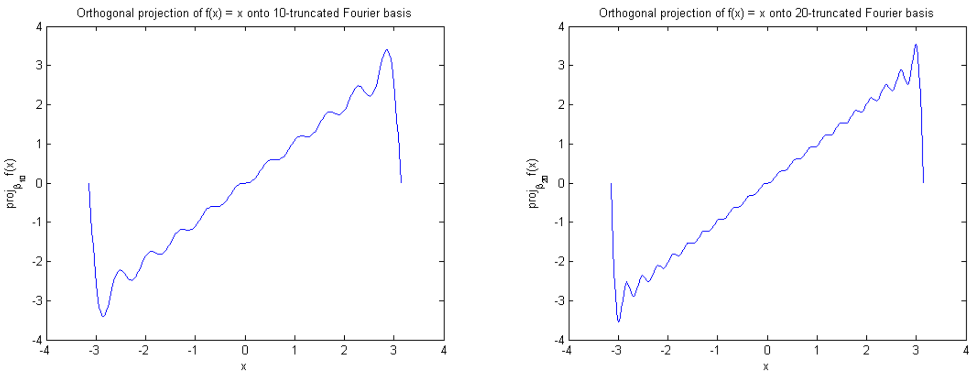

Figure 2: The projection $\text{proj}_{\beta_K} f(x)$ for $K = 10$ and $K = 20$.

$\vec{\mathbf{u}} = \text{proj}_W \vec{\mathbf{v}}$, and $\vec{\mathbf{w}} = \vec{\mathbf{v}} - \vec{\mathbf{u}}$ to see that

$$\|\vec{\mathbf{v}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{w}}\|^2.$$

In our context, we have $\vec{\mathbf{v}} = f(x)$ and $\vec{\mathbf{u}} = \tilde{f}_K(x)$ so that

$$f(x) = \tilde{f}_K(x) + [f(x) - \tilde{f}_K(x)]$$

and, hence,

$$\|f(x)\|^2 = \|\tilde{f}_K(x)\|^2 + \|f(x) - \tilde{f}_K(x)\|^2$$

$$= \sum_{n=1}^{K} a_n^2 \|\sin(nx)\|^2 + \|f(x) - \tilde{f}_K(x)\|^2 = 4\pi \sum_{n=1}^{K} \frac{1}{n^2} + \|f(x) - \tilde{f}_K(x)\|^2$$

Therefore, we have

$$\|f(x) - \tilde{f}_K(x)\|^2 = \|f(x)\|^2 - 4\pi \sum_{n=1}^{K} \frac{1}{n^2\pi} = \frac{2\pi^3}{3} - 4\pi \sum_{n=1}^{K} \frac{1}{n^2},$$

since $\|f(x)\|^2 = \int_{-\pi}^{\pi} x \cdot x \, dx = \frac{1}{3} x^3 |_{x=-\pi}^{\pi} = \frac{2\pi^3}{3}$.

(f) Let's validate your MATLAB observations. Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

to show what happens to $d(\tilde{f}_K(x), x)$ as $K \to \infty$? What then do you suggest is the limit function

$$\tilde{f}(x) = \lim_{K \to \infty} \tilde{f}_K(x)$$

in $(C[-\pi, \pi], L^2)$?

In the limit as $K \to \infty$, notice that the partial sum in $\|f(x) - \tilde{f}_K(x)\|^2$ computed above becomes a series:

$$\|f(x) - \tilde{f}_K(x)\|^2 \to \frac{2\pi^3}{3} - 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^3}{3} - 4\pi \cdot \frac{\pi^2}{6} = 0.$$

Therefore, the approximation is indeed getting better and better with the addition of more vectors to β_K. In particular, in the limit as $K \to \infty$, $f(x)$ and $\tilde{f}(X)$ are identical! We then suggest that our orthogonal projection simply becomes a series in this limit so that

$$x = f(x) = \tilde{f}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx).$$

Remark: We have just computed the *Fourier series* of $f(x) = x$ on $[-\pi, \pi]$. Think of it as a kind of generalization of a Taylor series (which uses polynomials as approximation rather than trigonometric functions). This still isn't quite correct, though, seeing as we haven't addressed the issue of $\tilde{f}(\pm\pi) = 0 \neq f(\pm\pi)$ or the Gibbs phenomenon noted above. Perhaps even more disturbing is the fact that this limit function $\tilde{f}(x)$ is *discontinuous*! This means that our limit function doesn't actually belong to $C[-\pi, \pi]$...

These observations should be quite unsettling, but the discussion of these topics will be saved for another time.

Have a nice Spring Break! You've earned it.