Problem Set 5 Solutions

Problem 1. *Resonance from discontinuous force*

Our goal for this exercise is to show that our mass-spring model can account for what we see in experiments. Specifically thinking about pushing someone on a swing or hitting a punching bag with a baseball bat, we would like to show that our model predicts resonance in the system for specific *discontinuous* forcing functions.

Consider a frictionless mass-spring system at its rest equilibrium modeled by the IVP

$$\begin{cases} m\ddot{x} + kx = g(t) \\ x(0) = 0, \ \dot{x}(0) = 0 \end{cases}.$$

Starting at time $t = 0$, we have the option to repeatedly

(i) hit the (ferromagnetic) mass with a hammer in perfect elastic collisions, or

(ii) turn an electromagnet on and off

with unit strength at regular intervals of T seconds.

(a) Describe and graph a function $g(t)$ that models each of these scenarios (i) and (ii). (Yes, your answer should be given as a series written in sigma-notation.)

We can suggest the following functions:

$$(i)\ g(t) = \sum_{n=0}^{\infty} \delta(t - nT) \quad \text{and} \quad (ii)\ g(t) = \sum_{n=0}^{\infty} (-1)^n u_{nT}(t).$$

These are plotted below in Figure 1.

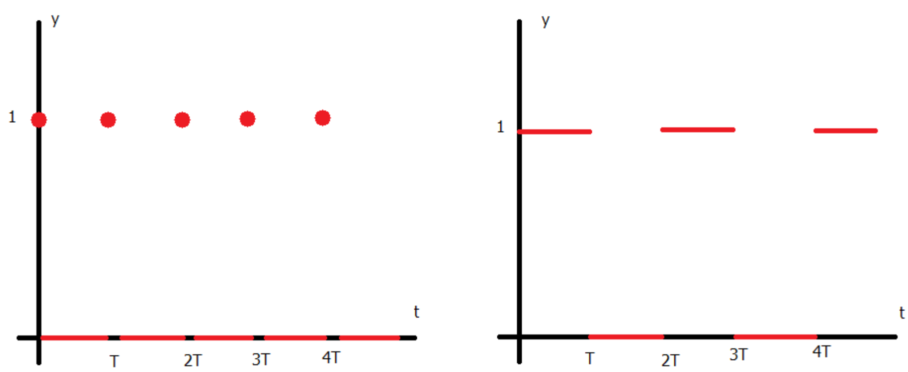

Figure 1: Qualitative sketches of a forcing function $g(t)$ representing scenario (i) [left] and scenario (ii) [right].

For simplicity, let's assume that $m = \frac{1}{4}$ and $k = 1$.

(b) Use the Laplace transform to solve the system in scenario (i) in case when

(1) $T = \pi$.

(2) $T = 3\pi$.

Graph your solution in each case. Comment on the case when $T = \frac{\pi}{2}$?

If we apply the Laplace transform to the left-hand side of the differential equation, we will find that

$$\mathcal{L}\left[\frac{1}{4}y'' + y\right] = \left(\frac{1}{4}s^2 + 1\right)Y(s).$$

The Laplace transform of the right-hand side of the equation depends on the function $g(t)$. In either case, we can solve this transformed equation so that

$$Y(s) = \frac{4G(s)}{s^2 + 4},$$

where $G(s) = \mathcal{L}[g(t)]$.

For scenario (i), we suggested

$$g(t) = \sum_{n=0}^{\infty} \delta(t - nT).$$

We then compute

$$\begin{aligned} G(s) &= \mathcal{L}\left[\sum_{n=0}^{\infty} \delta(t - nT)\right] \\ &= \sum_{n=0}^{\infty} \mathcal{L}[\delta(t - nT)] \\ &= \sum_{n=0}^{\infty} e^{-nTs}. \end{aligned}$$

Then we've shown that the solution in scenario (i) has Laplace transform

$$Y(s) = \sum_{n=0}^{\infty} e^{-Tnt} \frac{4}{s^2 + 4}.$$

Inverting this, we find

$$y(t) = \sum_{n=0}^{\infty} \sin(2(t - nT))u_{nT}(t).$$

When $T = \pi$, this becomes

$$y(t) = \sum_{n=0}^{\infty} (-1)^n \sin(2t))u_{n\pi}(t);$$

and when $T = 3\pi$, this becomes

$$y(t) = \sum_{n=0}^{\infty} (-1)^n \sin(2t)u_{3\pi n}(t).$$

Both of these are plotted below in Figure 2.

In the case when $T = \frac{\pi}{2}$, we see that the system falls into resonance. See Figure 3 below.

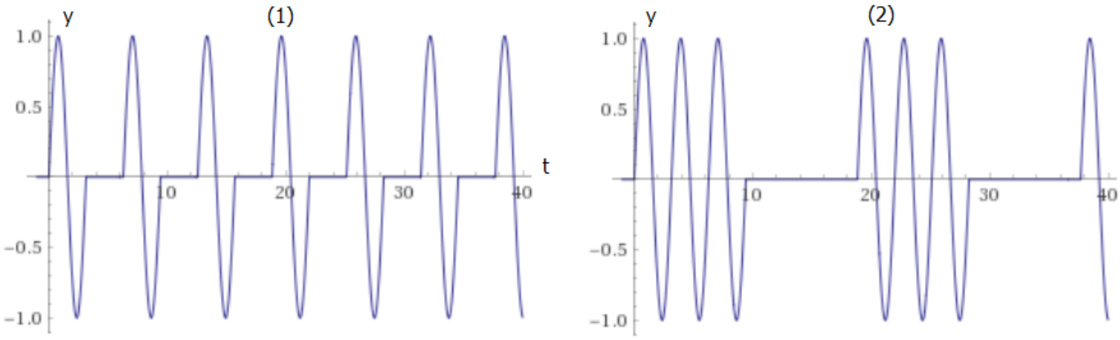

Figure 2: The solution to the IVP in scenario (ii) when (1) $T = \pi$ and $T = 3\pi$.

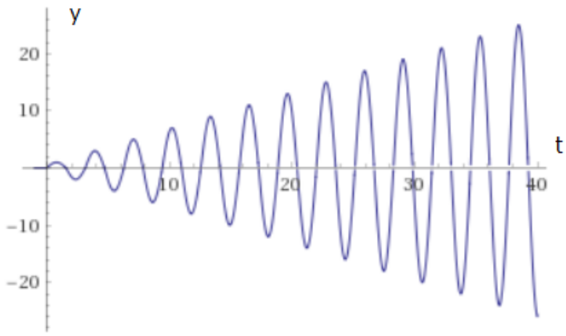

Figure 3: Resonance in the system with a $\frac{\pi}{2}$-periodic impulse function.

(c) Repeat part (b) for scenario (ii) instead, again being sure to comment on the case when $T = \frac{\pi}{2}$.

We use the same setup as before so that

$$Y(s) = \frac{4G(s)}{s^2+4}.$$

In scenario (ii), we suggested

$$g(t) = \sum_{n=0}^{\infty} (-1)^n u_{nT}(t).$$

We then find that

$$\begin{aligned} G(s) &= \mathcal{L}\left[\sum_{n=0}^{\infty} (-1)^n u_{nT}(t)\right] \\ &= \sum_{n=0}^{\infty} (-1)^n \mathcal{L}[u_{nT}(t)] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{e^{-nTs}}{s}. \end{aligned}$$

Then we've shown that the solution in scenario (ii) has Laplace transform

$$Y(s) = \sum_{n=0}^{\infty} (-1)^n e^{-nTs} \frac{4}{s(s^2+4)} = \sum_{n=0}^{\infty} (-1)^n e^{-nTs} \left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

Inverting this, we find

$$y(t)=\sum_{n=0}^{\infty}(-1)^n[1-\cos(2(t-nT))]u_{nT}(t).$$

When $T=\pi$, this becomes

$$y(t)=\sum_{n=0}^{\infty}(-1)^n[1-\cos(2t)]u_{\pi n}(t);$$

and when $T=3\pi$, this becomes

$$y(t)=\sum_{n=0}^{\infty}(-1)^n[1-\cos(2t)]u_{3\pi n}(t).$$

Both of these are plotted below in Figure 4.

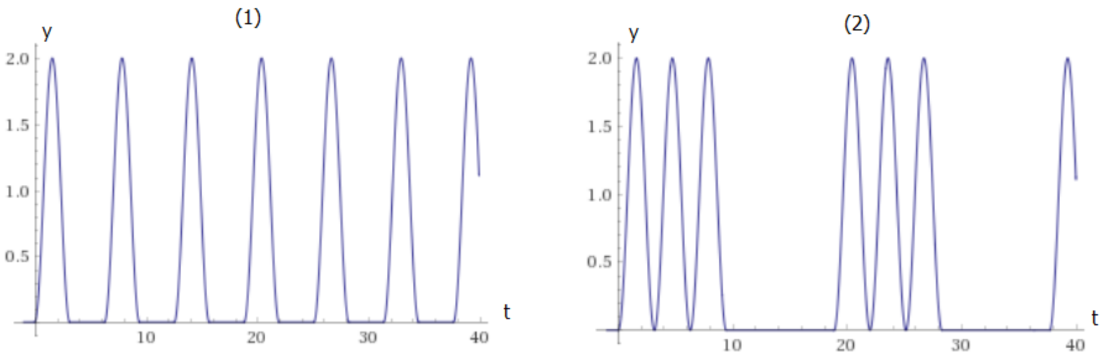

Figure 4: The solution to the IVP in scenario (ii) when (1) $T=\pi$ and (2) $T=3\pi$.

In the case when $T=\frac{\pi}{2}$, we see that the system again falls into resonance. See Figure 5 below.

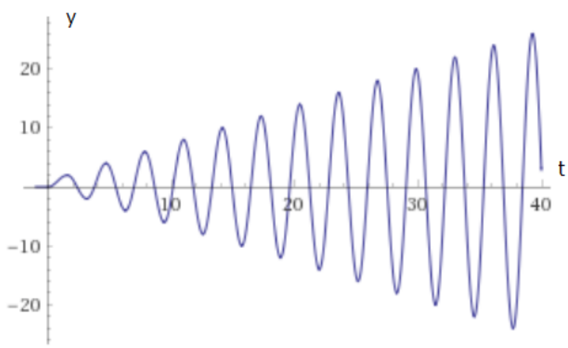

Figure 5: Resonance in the system with a $\frac{\pi}{2}$ alternating step function.

Problem 2. *What is $\mathcal{L}[f(t)g(t)](s)$?*

We've pondered in class what the Laplace transform of a product of functions looks like. Ideally, we would have $\mathcal{L}[f(t)g(t)] = \mathcal{L}[f(t)] \cdot \mathcal{L}[g(t)]$, but this is actually almost never the case. The result we desire requires a bit more finesse to obtain...

(a) Choose functions $f(t)$ and $g(t)$ that show it is possible for

$$\mathcal{L}[f(t)g(t)](s) = F(s)G(s),$$

where F and G are the Laplace transforms of f and g, respectively.

Choose $f(t) = 0$ and let $g(t)$ be an arbitrary function whose Laplace transform exists. Then, of course, we can have the trivial case where

$$0 = \mathcal{L}[0] = \mathcal{L}[0 \cdot g(t)] = 0 \cdot G(s).$$

(b) Choose functions $f(t)$ and $g(t)$ that show it is also possible for

$$\mathcal{L}[f(t)g(t)](s) \neq F(s)G(s).$$

Choose $f(t) = g(t) = 1$. Then we see that

$$\frac{1}{s} = \mathcal{L}[1 \cdot 1] \neq \mathcal{L}[1]\mathcal{L}[1] = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}.$$

It is now clear that our usual method of multiplying together $f(t)$ and $g(t)$ is not necessarily going to produce the result we want. The better question we should ask is

"Given $F(s)$ and $G(s)$ exist, is there a function $h(t)$ where

$$\mathcal{L}[h(t)](s) = F(s)G(s)?''$$

Allow me to propose such a function. We define the *convolution* $f * g$ of the functions $f(t)$ and $g(t)$ to be the integral

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du.$$

(We read $f * g$ as "f convolve g" or "f convolved with g".)

(c) Use the definition of the convolution to directly compute $e^{at} * \sin(bt)$ when $a, b > 0$ are both constants. (Notice that this is not the same as simply multiplying together $f(t)$ and $g(t)$.)

From the definition, we see that

$$\begin{aligned}
(e^{at} * \sin(bt))(t) &= \int_0^\infty e^{a(t-u)} \sin(bu)\,du \\
&= \int_0^t e^{at}e^{-au} \sin(bu)\,du \\
&= e^{at} \int_0^t e^{-au} \sin(bu)\,du \\
&= e^{at} \left[\frac{e^{-au}}{b^2+t^2}(t\sin(bu) + b\cos(bu)) \right]_{u=0}^t \\
&= \frac{be^{at-t^2}}{b^2+t^2} \left[be^{t^2} - t\sin(bt) - b\cos(bt) \right].
\end{aligned}$$

(d) Verify for $f(t) = g(t) = t$ that $\mathcal{L}[(f * g)(t)](s) = F(s)G(s)$.

First, we compute $f * g$ as

$$\begin{aligned}
(t * t)(t) &= \int_0^t (t-u)u\,du \\
&= \int_0^t tu - u^2\,du \\
&= \left[\frac{1}{2}tu^2 - \frac{1}{3}u^3\right]_{u=0}^t \\
&= \frac{1}{6}t^3.
\end{aligned}$$

We then see immediately that

$$\mathcal{L}[t * t] = \mathcal{L}\left[\frac{1}{6}t^3\right] = \frac{1}{6}\frac{3!}{s^4} = \frac{1}{s^2} \cdot \frac{1}{s^2} = \mathcal{L}[t]\mathcal{L}[t].$$

Remark - It is a general fact that, if $F(s)$ and $G(s)$ exist, then

$$\mathcal{L}[(f * g)(t)](s) = F(s)G(s).$$

This is among one of the stranger and less obvious results we come across in this field. We should think of this convolution integral as a way for us to say that $f * g$ is the result of letting f and g "mix" together. In the past, we've done this using pointwise operations; but now we see the first instance of when those past definitions fail to do what we hope they will.