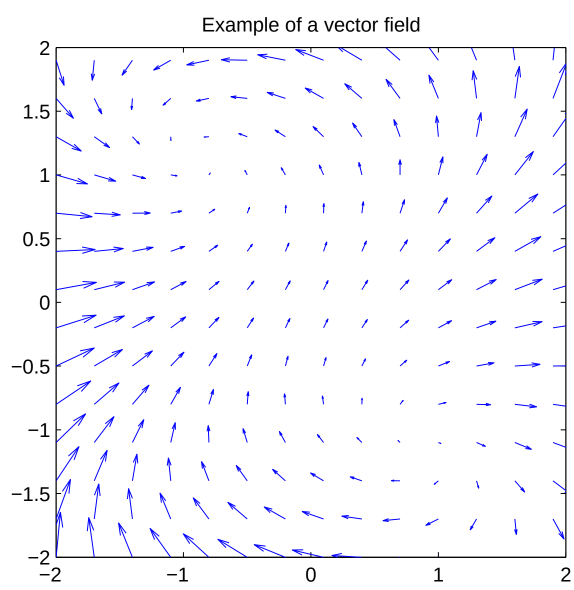
Homework 7 (Due Thursday, July 10)

1. A plane vector field is an assignment of vectors to points in the plane. Figure 1 shows the vector field

$$\mathbf{F} = \left(\frac{1}{2} - x^2 - y^2\right)\,\mathbf{i} + (1 + x\,y)\,\mathbf{j}$$

on the square $-2 \le x, y \le 2$. It was produced with the following code:

```
[x,y] = meshgrid(-2:.3:2);
u = .5 + x.^2 - y.^2;
v = 1 + x.*y;
quiver(x,y,u,v)
axis equal
xlim([-2 2])
ylim([-2 2])
title(’Example of a vector field’)
```



At each point of the grid (x_0, y_0) (generated with `meshgrid`) the code computes components

$$u(x_0, y_0) = \frac{1}{2} - x_0^2 - y_0^2, \quad v(x_0, y_0) = 1 + x_0\, y_0$$

of a vector and then plots the vector starting from (x_0, y_0). Modify the code to plot the following vector fields (on the same square):

(a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$

(b) $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$

2. The gradient of a function $f(x, y)$ is the vector of partial derivatives:

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

In this problem we consider the gradient as a *vector field*.

Let $f(x, y) = x^2 + y^2$. Plot the gradient ∇f (using `quiver`) on some square around the origin. Make sure that the plot is not cluttered and the arrows are neither too big nor too small; this will take some trial and error. Superimpose the contour plot of f on the plot of the vector field (don't forget to set `axis equal`) Record your observation. Repeat the exercise with $f(x, y) = x^2 - y^2$.

3. Let $f(x, y) = x^2 + y^2$ and let

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

In this problem we regard f as a function of one *vector* variable:

$$f(x, y) = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = f(\mathbf{r}).$$

A *directional derivative* of f at $\mathbf{r}_0$ in the direction of $\mathbf{v}$ is defined as:

$$D_{\mathbf{v}}(f)(\mathbf{r}_0) = \lim_{h \to 0} \frac{f(\mathbf{r}_0 + h\,\mathbf{v}) - f(\mathbf{r}_0)}{h}$$

Find an explicit expression for $D_{\mathbf{v}}(f)(\mathbf{r}_0)$ (for $f = x^2 + y^2$). What is the general formula for $D_{\mathbf{v}}(f)$ where f is a nonspecific function of two variables?

4. Let $f = x^2 + y^2$ and

$$\omega = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j}.$$

Notice that ω is a unit vector whose direction depends on t. Compute the directional derivative $D_\omega(f)$ at $(-1, 2)$. Plot the value of the directional derivative as a function of t for $0 \leq t \leq 2\,\pi$. What does the graph look like? Where are the minima and maxima?

5. Find the equation of a plane using the following information:

   (a) The plane passes through $(1, 2, 3)$ and has normal $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

   (b) The plane passes through $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.