2. The mystery of a discharging capacitor

I hope that you are somewhat familiar with electric circuits and, if so, you may not need the following introduction. Otherwise, read on.

The following section introduces what is called an RC-circuit where 'C' stands for 'capacitor' whose mystery will unfold in Section 2.4; 'R' stands for resistor, by the way—an element present in some form in just about every circuit[1]. Later we will add one more element, called 'inductance.' However, for now, we will look at the simplest possible circuit whose modeling requires ODE.

As a final note, in the discussion that follows we temporarily depart from our usual rules concerning mathematical notation and use the notation that is standard in Circuits. We will return to our usual notational conventions once we are ready to discuss circuits abstractly.

2.1. **Current and voltage.** When thinking about currents and voltages you may find the following analogies useful. Firstly, you can visualize the electric current in a wire as a flow of water or air in a pipe. Actually, the electric current is a flow of charge—usually free electrons in a metal conductor. While charges are physically different from neutral molecules of water and air, their collective behavior is very similar. For instance, air flows from places with high pressure to places with low pressure. Likewise, the electric current flows from places with high voltage to places with low voltage. In Circuits one commonly uses the symbol I for electric current; it is measured in Amperes.

In order to start a current, one needs to create a voltage drop. This can be accomplished with a *voltage source*: a car battery, a wall outlet, or, maybe, a solar panel. A voltage source is an *active element* which is analogous to a pump: it creates "pressure" that pushes the charges. Mathematically, a voltage source is described by a function of time. The common notation is $E = E(t)$; the symbol E being homage to the old-fashioned term *electromotive force*. We should also add that in Circuits one encounters *current sources* as well as voltage sources. However, their mathematics is not substantially different and we will omit them.

2.2. **Resistors.** Now, let us discuss what are called *passive elements* of analogue circuits. Nearly every circuit, with very few exceptions,

[1]The only exception are circuits made of superconductors. These are highly specialized and incredibly expensive circuits in devices like particle accelerators and quantum computers. As you probably know, the expenses are due to the fact that superconductivity can be achieved only at temperatures close to absolute zero.

contains at least one *resistor.* A Fluids analogy for a resistor is a narrowing of a pipe. Just like a narrowing of a pipe obstructs fluid flow, an electric resistor obstructs, or resists electric current. You can also think of resistors as devices for converting electric energy into other forms of energy. An incandescent bulb of Section 1 is a typical example: it converts electrical energy into light and heat. Mathematically, resistors are described by specifying the relation between voltage drop and current. For now, we will only consider *linear* resistors which are described by Ohm's Law:

$$V = R\,I. \tag{9}$$

In words, Ohm's law states that the voltage drop across a resistor is proportional to the current. The coefficient of proportionality is called *resistance* and is measured in Ohms. You may find it easier to internalize Ohm's Law if you think of it as stating that the current is proportional to potential difference (voltage): it certainly makes sense that in order to increase flow one has to apply more pressure.

2.3. **Capacitors.** There are many useful circuits that can be made from resistors and voltage sources alone. Yet the mathematics of such circuits is based on algebra and is usually studied in courses like Linear Algebra (Math 75) and, of course, beginning Circuits. However, many circuits contain additional elements whose modeling requires calculus. For example, a flash in a camera has several *capacitors*; a USB memory stick has several *billion* capacitors—one (or more) for each bit that it can store. This brings us to the main subject of this section.

A capacitor is often compared to a sort of bucket for storing charge. That, however, is a very crude analogy. For one thing, a capacitor is *always* electrically neutral. Also, the operation of a capacitor heavily depends on what is called an *electric field.* A better analogy is as follows. Imagine a hollow sphere which is separated into two equal chambers with a flexible membrane. Suppose we pump air from one chamber into another. The membrane will deform because in one chamber the pressure will be higher than normal while in the other it will be lower than normal: the capacitor is charged. Now if we connect the chambers with a pipe, the air will rush from the high-pressure chamber into the low-pressure chamber: the capacitor will discharge. Something very similar happens in electric capacitors. A non-polarized capacitor consists of two metal plates separated by a dielectric—a substance that does not conduct electricity. If one applies a potential difference to the plates, the electrons migrate from one plate–the one with higher voltage—to the other plate—the one with lower voltage. The resulting charge distribution generates an electric field which permeates the dielectric.

That field is analogous to a flexible membrane. If a charged capacitor is connected to a resistor, it discharges because the field pushes electrons from the negative plate to the positive plate until both plates are neutral.

Mathematically, capacitors are described by the linear relation:

$$V = \frac{Q}{C}. \tag{10}$$

In words: the voltage drop across a capacitor is proportional to the charge stored on one plate, the charge on the other plate being equal and opposite. Yet, which plate does Q in Equation (10) correspond? To answer that, recall that all physical constants are, by convention, positive. In particular, the *capacitance* C, which is measured in Farads, is positive. Therefore it all depends on how we measure voltage. Say we charge a capacitor with a car battery which supplies *positive* 12 Volts. Then Q corresponds to the positive charge on the "positive" plate.

2.4. ***RC*-circuit.** Having introduced voltage sources, resistors, and capacitors, let us consider the simplest RC-circuit which has one resistor R, one capacitor C, and one voltage source E connected in series. When elements are connected in series, their voltage drops add. Furthermore, according to Kirchhoff's Voltage Law, the sum of voltage drops in any closed loop must match the electromotive force. Therefore:

$$R\,I + \frac{Q}{C} = E. \tag{11}$$

In Equation (11) we know the resistance R, capacitance C, and the electromotive force E. Yet the current and charge on the capacitor are unknown. To close the system, we need to relate I and Q. Since current is the flow of charge, the relation is, simply:

$$I = \frac{dQ}{dt}. \tag{12}$$

Combining Equations (11) and (12) gives a first order ODE for the charge on the capacitor in the form:

$$R\,\frac{dQ}{dt} + \frac{Q}{C} = E. \tag{13}$$

2.5. **Similarity to the cooling light bulb.** Equation (13) can be rearranged as follows:

$$\frac{dQ}{dt} = \frac{1}{R\,C}\,(C\,E - Q)\,. \tag{14}$$

If we now relabel $Q = x$, $(R\,C)^{-1} = k$, and $C\,E = a$, we get

$$\frac{dx}{dt} = k\,(a - x).$$

This is the model we set up in Section 1 to describe the cooling of a light bulb!

If you understood Section 1, you can predict how the capacitor will behave if the voltage source E is constant: it will charge or discharge exponentially, the way an object heats up or cools down in a room with constant temperature. Whereas room temperature is hard to control, we have great flexibility in choosing voltage sources. We will therefore study Equation (14) with various time-dependent voltage sources E, starting with the case when it is a linear function of time. This will give us insight both into circuits and into heating and cooling processes in environments with varying temperature.

Aside note: at this point you may be wondering whether the similarity between a discharging capacitor and a cooling light bulb is a mere coincidence. It is not! Ask your Physics professor about it if we do not get to it in class.

2.6. **An *RC*-circuit driven by ramp voltage.** If you study circuits you will sooner or later come across an IVP which looks as follows:

$$\frac{dx}{dt} = t - x, \quad x(0) = 1. \tag{15}$$

In Circuits Equation (15) could describe an RC-circuit driven by ramp voltage $E = t$; it may also describe an object cooling in an environment whose temperature is rising linearly: $a = t$.

The right panel of Figure 2.6 shows a common guess for the shape of the solution of (15). Most students think that the graph of the solution of (15) cannot cross the "slant asymptote." We will now show that that is false.

You may recall that our first attempts to solve Equation (15) in class were unsuccessful. We tried separation of variables. However, the equation is nonseparable, so we got silly results. When confronted with a nonseparable equation for the first time, students often think that they cannot do anything with it unless they are taught some new method of solution. That logic is entirely wrong!

> *When confronted with a mathematical challenge in Math 57, always take stock of what you know from Math 51, Math 53, and Math 55. Chances are, you had been taught a technique in calculus and now simply need to apply it in a new context.*

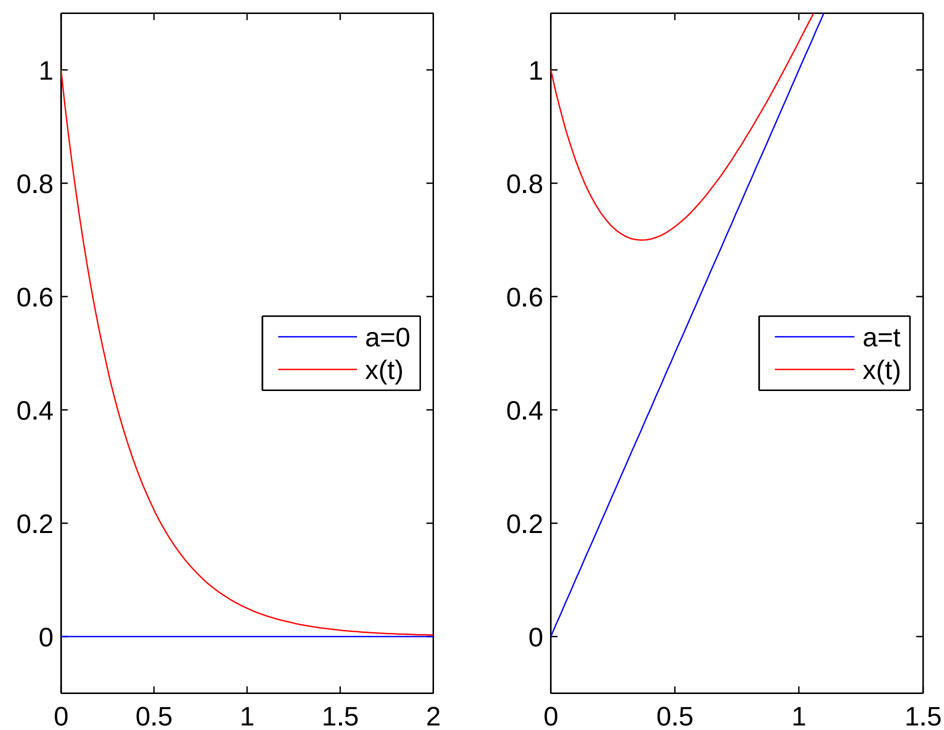

FIGURE 4. The plot on the left shows familiar exponential decay corresponding to constant voltage source/room temperature $a = 0$. The plot on the right shows a common guess for the solution with $a = t$. The guess is erroneous, as our analysis will show.

You had three semesters of Calculus. Surely there ought to be something in Calculus that you can use to solve (15)! There are, in fact, two useful techniques from Calculus II. I'll give you some hints.

How would you work with the following integral:

$$\int_0^1 x\, e^{-x^2}\, dx = ?$$

One approach is to substitute $u = -x^2$. Then the integral becomes

$$\frac{1}{2}\int_0^1 e^{-u}\, du = \frac{1-e^{-1}}{2}.$$

The substitution transforms a difficult integral into a simpler (tabular) integral. Might we use a substitution to transform a difficult nonseparable ODE into a simpler, perhaps separable, ODE?

Equation (15) is nonseparable because the right-hand side is a difference $t - x$. Suppose we substitute $u = t - x$? Come to think of it, there really are no other obvious candidates for a substitution. So, let us work with that idea. Firstly, if we manage to find u then the answer is simply $x = t - u$. Secondly, given the relation between x and u, the rates are related as follows:

$$\frac{dx}{dt} = \frac{d}{dt}(t - u) = 1 - \frac{du}{dt}.$$

This means that Equation (15) transforms into:

$$1 - \frac{du}{dt} = u$$

which is a separable equation with the general solution

$$u = 1 + C\, e^{-t}.$$

(Check it!) From the relation $u = t - x$ we can infer that $u(0) = -x(0) = -1$. Using that as the initial condition for u, we get

$$u = 1 - 2\, e^{-t},$$

which in turn implies

$$x = t - u = t - 1 + 2\, e^{-t}.$$

As an exercise, confirm that the above expression is the solution of IVP (15).

Figure 2.6 shows the solution of IVP (15). Notice that it crosses that the graph of $a = t$. As time goes by, the exponential term becomes vanishingly small:

$$x \approx t - 1, \quad \text{as } t \to \infty.$$

Notice that the solution lags behind $a = t$. In the context of RC-circuits, the magnitude of the lag depends on the product of resistance and capacitance. For that reason, the product $R \times C$ is sometimes called the RC-delay.

Now, let us talk about why most students have difficulty trying the above substitution on their own. The main problem is that they tend to associate substitution with integration whereas in reality the idea of substitution has nothing to do with integration and is much more universal. For example, how would you solve

$$x^4 - 3\,x^2 + 2 = 0?$$

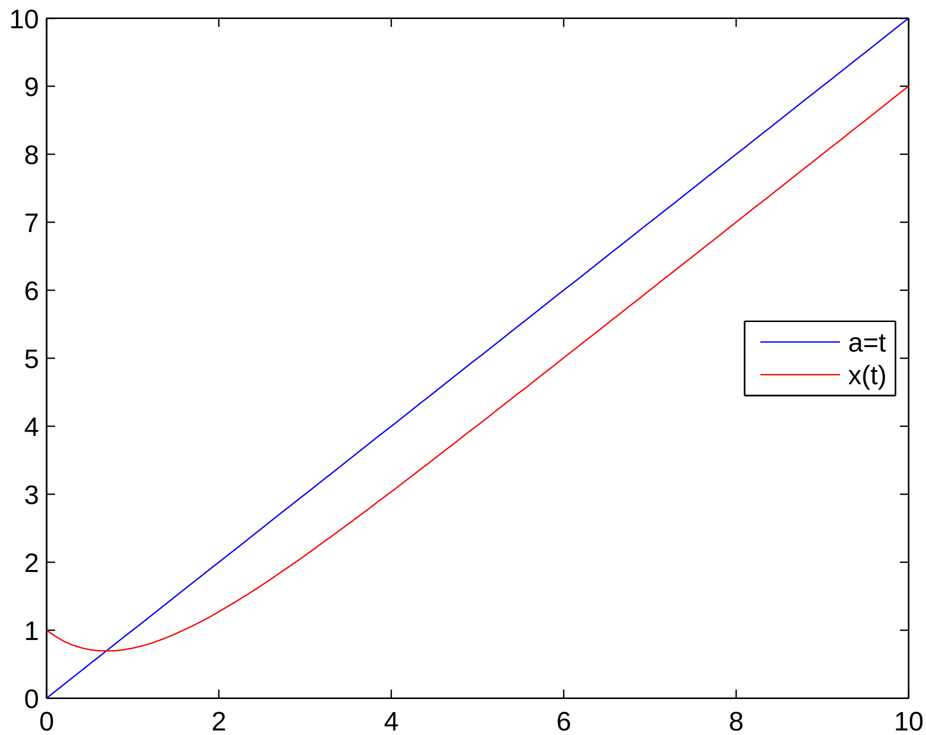

FIGURE 5. Solution of IVP (15).

Unless you happen to know the formula for the roots of a quartic (which I doubt), the only approach is reduction to a quadratic equation by substitution $u = x^2$. Integration in Calculus II is merely the most recent and probably the most extensive context for the use of substitutions.

> *Assume that everything in Calculus was preparation for Math 57 and practice applying Calculus techniques in this new context.*

Earlier we said that there were at least two Calculus techniques that can be used for solving Equation (15). You now know that one of them is substitution. Think about other things your learned in Calculus II that may be pertinent. If you forgot, the following exercises should help.

Exercises.

(1) Let $E = a + b\,t$. Treating a, b, R, C, and Q_0 as known parameters, solve the following IVP:

$$\frac{dQ}{dt} = \frac{1}{R\,C}\,(C\,E - Q), \quad Q(0) = Q_0.$$

Validate your solution.

(2) Solve

$$\frac{dx}{dt} = f(t) - x, \quad x(0) = 0$$

for the following choices of f:

(a) $f(t) = t^2$
(b) $f(t) = t^3$
(c) $f(t) = t^2 + t^3$

Validate the answer in each case and illustrate it with a MATLAB plot on the interval $0 \le t \le 5$. Explain in words what these answers have in common. What can you say about the general case

$$f = a_0 + a_1\,t + a_2\,t^2 + \ldots + a_N\,t^N?$$

(3) Use MATLAB to find the point of intersection of the red and the blue curves in Figure 5. Present your code and its output. Also, comment on the accuracy of the solution: how many digits do you expect to be correct?

(4) At the end of a standard Calculus II course one typically studies Taylor theory which can be used, among other things, to compute integrals. You may imagine that if a Calculus technique can be used to simplify integration, it may be applicable to solving ODE.

(a) Let $J = \int_0^1 e^{-x^2}\,dx$. Use Taylor polynomials to approximate the value of J correct to three decimals. If you do not know how to do that, review Calculus II.
(b) Pretend that you do not know the solution of Equation (15). Try to find its Taylor polynomial of degree four using the same basic principles as in the first part of this exercise. Remember to formulate questions if you run into difficulties.

(5) Consider the following IVP:

$$\frac{dx}{dt} + x = f(t), \quad x(0) = 5.$$

If you got this far, you most likely understand how to solve the IVP when f is either a constant or a linear function. Try to leverage that understanding into the solution of the IVP with a piecewise continuous right-hand side:

$$f(t) = \begin{cases} 2\,t, & \text{when } 0 \le t \le 1, \\ 2, & \text{for } t > 1. \end{cases}$$

It is not as terrifying as it may seem but, if you run into trouble, write about it!

(6) This exercise is unrelated to circuits and is given as modeling practice. Recall that the last problem on the previous handout asked you to model dissolution of a capsule. The law was stated as follows: the rate at which the volume is lost is proportional to surface area. Download `capsule.txt` from the course web site. The data has two columns: the first column is time in seconds, the second is the volume of a dissolving spherical capsule. Use the data and your model to estimate the time it takes for the capsule to fully dissolve.