Saturday, April 13, 2019 2:57 PM

PS 7 Solutions

① (a) (i) $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \to \det(A-\lambda I) = (2-\lambda)(-2-\lambda) - 3\cdot(-1)$

$= \lambda^2 - 1 = 0$

$\boxed{\lambda = \pm 1}$

$\underline{\lambda = 1:}$ $(A - I)\vec{v} = \vec{0}$

$\begin{bmatrix} 1 & 3 & | & 0 \\ -1 & -3 & | & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + R1} \begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Leftrightarrow \begin{cases} x + 3y = 0 \\ 0 = 0 \end{cases}$

Then $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3y \\ y \end{pmatrix} = y\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\boxed{\vec{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}}$

$\underline{\lambda = -1:}$ $(A + I)\vec{v} = \vec{0}$

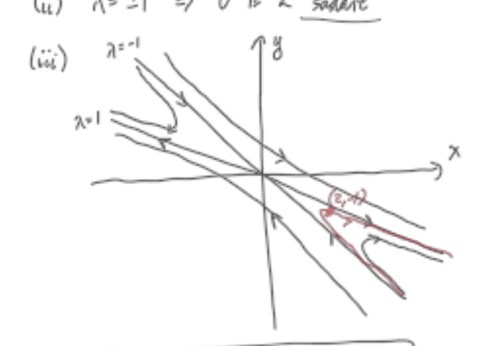
$\begin{bmatrix} 3 & 3 & | & 0 \\ -1 & -1 & | & 0 \end{bmatrix} \xrightarrow{R2 \to \frac{1}{3}R1 + R2} \begin{bmatrix} 3 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Leftrightarrow \begin{cases} 3x + 3y = 0 \\ 0 = 0 \end{cases}$

Then $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\vec{v}_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(ii) $\lambda = \pm 1 \Rightarrow \vec{0}$ is a <u>saddle</u>

(iii)



(iv) $\boxed{\vec{x}(t) = c_1 e^{t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(v) $\vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 e^{0}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{-0}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \begin{bmatrix} -3c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$

$\Rightarrow \begin{cases} -3c_1 + c_2 = 2 \\ c_1 - c_2 = -1 \end{cases} \to c_1 = c_2 - 1$, $\quad -3(c_2 - 1) + c_2 = 2$

$-2c_2 + 3 = 2$

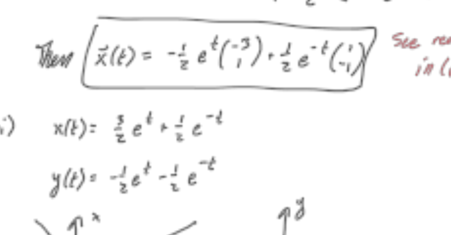
$1 = 2c_2$

$c_1 = -\frac{1}{2} \Leftarrow c_2 = 1/2$

Then $\boxed{\vec{x}(t) = -\frac{1}{2} e^{t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ See red curve in (iii).

(vi) $x(t) = \frac{3}{2}e^{t} + \frac{1}{2}e^{-t}$

$y(t) = -\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$



---

① (b) (i) $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \to \det(A - \lambda I) = (2-\lambda)(2-\lambda) - 1\cdot(-1)$

$= \lambda^2 - 4\lambda + 5 = 0$

$\lambda = \frac{4 \pm \sqrt{16-20}}{2}$

$\boxed{\lambda = 2 \pm i}$

$\underline{\lambda = 2+i:}$

$(A - (2+i)I)\vec{v} = \vec{0}$

$\begin{bmatrix} -i & 1 & | & 0 \\ -1 & -i & | & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + iR1} \begin{bmatrix} -i & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$

$\begin{cases} -ix + y = 0 \\ 0 = 0 \end{cases} \to \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ ix \end{pmatrix} = x\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\boxed{\vec{V}_{2+i} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$

$\Rightarrow \vec{V}_{2-i} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \boxed{\begin{bmatrix} 1 \\ -i \end{bmatrix}}$

(ii) $\lambda = 2 \pm i \begin{cases} \text{complex} \Rightarrow \boxed{\text{spiral}} \\ Re(\lambda) > 0 \Rightarrow \boxed{\text{unstable}} \end{cases}$

(iii) $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(iv) $e^{(2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{2t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$

$= e^{2t}\begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix}$

$= e^{2t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\cdot e^{2t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$\boxed{\vec{x}(t) = e^{2t}\left[c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]}$

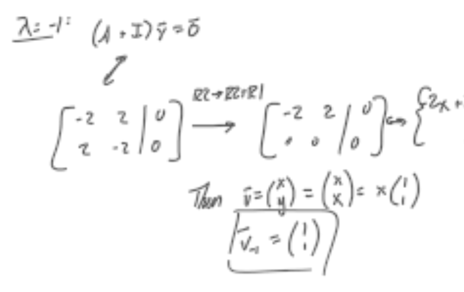
(v) $\vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = e^{2\cdot 0}\left[c_1\begin{pmatrix} 1 \\ -0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Then $\boxed{\vec{x}(t) = e^{2t}\left[2\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]}$. See red curve in (iii)

(vi) $x(t) = e^{2t}(2\cos t - \sin t)$

$y(t) = e^{2t}(-2\sin t - \cos t)$



---

① (c) (i) $A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \to \det(A - \lambda I) = (-3-\lambda)(-3-\lambda) - 2\cdot 2$

$= \lambda^2 + 6\lambda + 5$

$= (\lambda+1)(\lambda+5) = 0$

$\boxed{\lambda = -1, -5}$

$\underline{\lambda = -1:}$ $(A + I)\vec{v} = \vec{0}$

$\begin{bmatrix} -2 & 2 & | & 0 \\ 2 & -2 & | & 0 \end{bmatrix} \xrightarrow{R2 \to R2 + R1} \begin{bmatrix} -2 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Leftrightarrow \begin{cases} 2x + 2y = 0 \\ 0 = 0 \end{cases}$

Then $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\vec{v}_{-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\underline{\lambda = -5:}$ $(A + 5I)\vec{v} = \vec{0}$

$\begin{bmatrix} 2 & 2 & | & 0 \\ 2 & 2 & | & 0 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 2 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Leftrightarrow \begin{cases} 2x + 2y = 0 \\ 0 = 0 \end{cases}$

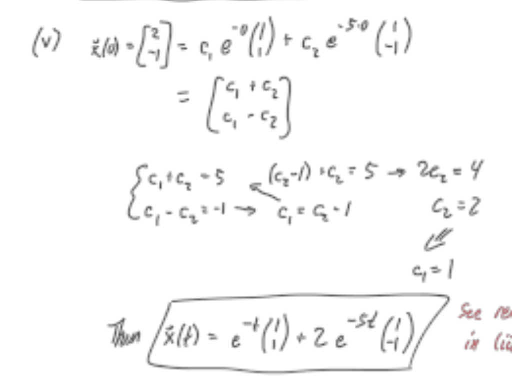
Then $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\vec{v}_{-5} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(ii) $-1, -5$ both $< 0 \Rightarrow \boxed{\text{stable}}$

$-1, -5$ both real $\Rightarrow \boxed{\text{node}}$

(iii)



(iv) $\boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(v) $\vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 e^{-0}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5\cdot 0}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$

$\begin{cases} c_1 + c_2 = 5 \\ c_1 - c_2 = -1 \end{cases} \to c_1 = c_2 - 1$, $\quad (c_2 - 1) + c_2 = 5 \to 2c_2 = 4$

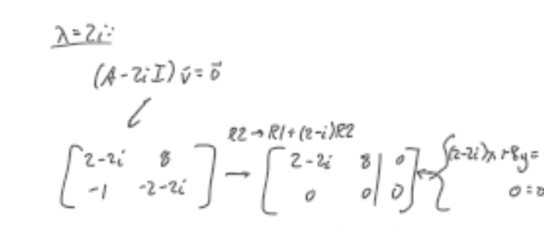
$c_2 = 2$

$c_1 = 1$

Then $\boxed{\vec{x}(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ See red curve in (iii)

(vi) $x(t) = e^{-t} + 2e^{-5t}$

$y(t) = e^{-t} - 2e^{-5t}$



---

① (d) (i) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix} \to \det(A - \lambda I) = (2-\lambda)(-2-\lambda) - 8\cdot(-1)$

$= \lambda^2 + 4 = 0$

$\boxed{\lambda = \pm 2i}$

$\underline{\lambda = 2i:}$

$(A - 2iI)\vec{v} = \vec{0}$

$\begin{bmatrix} 2-2i & 8 \\ -1 & -2-2i \end{bmatrix} \xrightarrow{R2 \to R1 + (2-i)R2} \begin{bmatrix} 2-2i & 8 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Leftrightarrow \begin{cases} (2-2i)x + 8y = 0 \\ 0 = 0 \end{cases}$

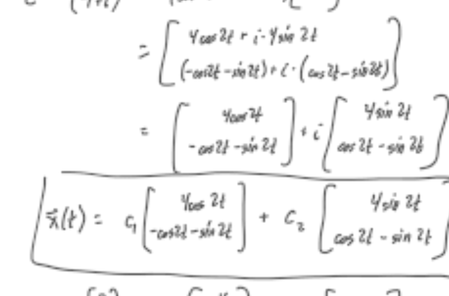
Then $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (-\frac{1}{4} + \frac{1}{4}i)x \end{pmatrix}$

$= \frac{1}{4}x\begin{pmatrix} 4 \\ -1+i \end{pmatrix}$

So $\boxed{\vec{V}_{2i} = \begin{pmatrix} 4 \\ -1+i \end{pmatrix}}$ and $\boxed{\vec{V}_{-2i} = (\overline{\vec{V}_{2i}}) = \begin{pmatrix} 4 \\ -1-i \end{pmatrix}}$

(ii) $\lambda = \pm 2i: \begin{cases} \text{Complex} \Rightarrow \text{spiral} \\ Re(\lambda) = 0 \Rightarrow \underline{\text{neutral}} \text{ stability} \end{cases} \Rightarrow \boxed{\text{neutral spiral}}$ or $\boxed{\text{center}}$

(iii) $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



(iv) $e^{2it}\begin{pmatrix} 4 \\ -1+i \end{pmatrix} = (\cos 2t + i\cdot\sin 2t)\begin{bmatrix} 4 \\ -1+i \end{bmatrix}$

$= \begin{bmatrix} 4\cos 2t + i\cdot 4\sin 2t \\ (-\cos 2t - \sin 2t) + i\cdot(\cos 2t - \sin 2t) \end{bmatrix}$

$= \begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix} + i\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}$

$\boxed{\vec{x}(t) = c_1\begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix} + c_2\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}}$

(v) $\vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{cases} 4c_1 = 2 \\ -c_1 + c_2 = -1 \end{cases} \to \begin{matrix} c_1 = 1/2 \\ c_2 = -1/2 \end{matrix}$

Then $\boxed{\vec{x}(t) = \frac{1}{2}\begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}}$ See red curve in (iii)

(vi) $\begin{cases} x(t) = 2\cos 2t - 2\sin 2t \\ y(t) = -\cos 2t \end{cases}$