Problem Set 4 Solutions

Problem 1. *Mechanical and electrical energy*
Consider the unforced mass-spring system

$$m\ddot{x} + b\dot{x} + kx = 0, \quad t > 0.$$

We define the *total mechanical energy* of this system to be the function

$$E(t) = \frac{1}{2}m(\dot{x}(t))^2 + \frac{1}{2}k(x(t))^2,$$

which you might recognize as the sum of *kinetic energy* and *spring potential energy.*

(a) Show that $\dot{E}(t) = 0$ in an undamped system. This shows that $E(t)$ is constant over time (this is an instance of the *law of conservation of mechanical energy.*

First, we use the chain rule in differentiating $E(t)$ so that

$$\dot{E}(t) = \frac{1}{2}m \cdot 2\dot{x}(t) \cdot \ddot{x}(t) + \frac{1}{2}k \cdot 2x(t) \cdot \dot{x}(t) = \dot{x}(t)\left(m\ddot{x}(t) + kx(t)\right).$$

In the undamped system where $b = 0$, we see that $m\ddot{x} + kx = 0$ so that $\dot{E}(t) = 0$ as required.

(b) On the other hand, if there is damping in this system, give a mathematical argument as to why $\dot{E}(t) < 0$ (so that total mechanical energy is *not* conserved over time).

We use the same derivative calculated above, but this time we have $m\ddot{x} + b\dot{x} + kx = 0$ so that

$$m\ddot{x} + kx = -\dot{x}.$$

This implies that

$$\dot{E}(t) = \dot{x}(t)(m\ddot{x}(t) + kx(t)) = \dot{x}(t)(-b\dot{x}(t)) = -b(\dot{x}(t))^2.$$

Since $b > 0$, we see that $\dot{E}(t) = -b(\dot{x}(t))^2 \leq 0$ for all t. In the case where $\dot{x}(T) = 0$ for some time $t = T$, either (i) the system is at rest or (ii) the system has spring potential energy to force $\dot{x}(t) \neq 0$ immediately after $t = T$. Therefore, the system is always losing mechanical energy and, hence, is not conserved over time.

(c) Recall the forced (non-homogeneous) mass-spring system

$$m\ddot{x} + b\dot{x} + kx = g(t),$$

where $g(t)$ is the forcing function. We define the change in total mechanical energy to be the function

$$(\Delta E)(t) = E(t) - E(0) = \int_0^t \dot{E}(s)\, ds.$$

Show that $(\Delta E)(t)$ can be explicitly written as

$$(\Delta E)(t) = \int_0^t \dot{x}(s)g(s) - b(\dot{x}(s))^2\, ds.$$

We use the fact that

$$m\ddot{x}(t) + kx(t) = g(t) - b\dot{x}(t)$$

along with the definition of ΔE to find that

$$\begin{aligned}(\Delta E)(t) &= \int_0^t \dot{E}(s)\,ds \\ &= \int_0^t \dot{x}(s)(m\ddot{x}(s) + kx(s))\,ds \\ &= \int_0^t \dot{x}(s)(g(s) - b\dot{x}(s))\,ds \\ &= \int_0^t \dot{x}(s)g(s) - b(\dot{x}(s))^2\,ds,\end{aligned}$$

as was to be shown.

(d) Recall the RLC-circuit governed by the equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t).$$

Given the discussion above, what should we define as the *total electrical energy* of the circuit? Interpret the two terms in your definition in terms of the charge Q on the capacitor and the current $I = \dot{Q}$ running through it.

We note here that the equation

$$m\ddot{x} + b\dot{x} + kx = g(t) \quad \text{and} \quad L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t)$$

are virtually identical if we identify

$$x(t) \leftrightarrow Q(t), \quad m \leftrightarrow L, \quad b \leftrightarrow R, \quad k \leftrightarrow \frac{1}{C}, \quad \text{and} \quad g(t) \leftrightarrow V(t).$$

Therefore, we look to the definition of total mechanical energy $E(t)$ from the mass-spring system for inspiration in defining the *total electrical energy* $\mathcal{E}(t)$ of the circuit.
We use the identifications between the two systems to find

$$E(t) = \frac{1}{2}m(\dot{x}(t))^2 + \frac{1}{2}k(x(t))^2 \quad \leftrightarrow \quad \mathcal{E}(t) = \frac{1}{2}L(\dot{Q}(t))^2 + \frac{1}{2C}(Q(t))^2.$$

We might recognize this better if we use the fact that current $I = \dot{Q}(t)$:

$$\mathcal{E}(t) = \frac{1}{2}L(I(t))^2 + \frac{1}{2}\frac{(Q(t))^2}{C}.$$

The first term, representing "electrical kinetic energy", tells us how much energy is stored by the inductor at time t; and the second term, representing *electrical potential energy*, tells us how much electrical energy is stored on the capacitor at time t. It is not a prerequisite of the course to know this, but the voltage $\mathcal{V}(t) = \frac{Q(t)}{C}$ across the capacitor can then find its way into this expression as well:

$$\mathcal{E}(t) = \frac{1}{2}L(I(t))^2 + \frac{1}{2}Q(t)\mathcal{V}(t).$$

Any of these expressions for $\mathcal{E}(t)$ is an acceptable.

Problem 2. *Wait... you can do that?!*

(a) Be sure to look at problems 30 and 31 of the suggested problems. Show that you've read through them by explaining how we can use the *Gamma function* $\Gamma(t)$ to compute $\left(\frac{1}{2}\right)!$; that is, the factorial of $\frac{1}{2}$.

[*Hint*: You may find the u-substitution $u = \sqrt{x}$ helpful in your explanation.]

First, we define the Gamma function

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx$$

for any $p > -1$ (since this is the domain for p where the improper integral converges).

In Exercise 30 of section 6.1, it is shown that $\Gamma(n+1) = n!$ for any integer $n \geq 0$. This suggests that it is possible to extend the notion of *factorial* to non-integer values like $\frac{1}{2}$. Specifically, we would like to define

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2} \, dx.$$

It might look like this integral is improper, but if we make the substitution $u = \sqrt{x}$, we find

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} \frac{1}{u} \cdot 2u \, du = 2 \int_0^\infty e^{-u^2} \, du$$

so that the integrand is no longer discontinuous at $x = u^2 = 0$. Therefore, we need to know the value of the improper integral

$$G = \int_0^\infty e^{-x^2} \, dx$$

in order to calculate $\left(\frac{1}{2}\right)!$.

(b) The only stumbling block in computing $\left(\frac{1}{2}\right)!$ is that we might not know how to compute

$$G = \int_0^\infty e^{-x^2} \, dx.$$

We will show this below, but first use the comparison test to show that G converges.

We know that $e^{-x} \geq e^{-x^2} > 0$ for all $x \geq 1$ either by direct computation or by inspection of their graphs. The Comparison Test theorem tells us that G converges if

$$\int_1^\infty e^{-x} \, dx$$

converges. We showed in class that $\int_0^\infty e^{-x} \, dx = 1$, so we conclude indeed that G must converge.

(c) Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$(2G)^2 = 2\pi \int_0^\infty re^{-r^2}\,dr.$$

[*Hint*: Write $(\int e^{-x^2}\,dx)^2$ as $(\int e^{-x^2}\,dx)(\int e^{-y^2}\,dy)$ and remember that constants can be brought inside integrals.]

First, notice that

$$2G = 2\int_0^\infty e^{-x^2}\,dx = \int_{-\infty}^\infty e^{-x^2}\,dx$$

because e^{-x^2} is an even function. Then we use the hint to show that

$$(2G)^2 = \left(\int_{-\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\,dy\right) = \int_{-\infty}^\infty \int_{-\infty} \infty e^{-(x^2+y^2)}\,dx\,dy$$

since the integral in x is simply a constant that can be brought inside the integral in y.

To compute

$$(2G)^2 = \int_\infty^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\,dy,$$

we use polar coordinates to convert to the domain $0 \le \theta \le 2\pi$ and $0 \le r$ so that

$$(2G)^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2} \cdot r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^\infty re^{-r^2}\,dr,$$

as was to be shown.

(d) Evaluate the improper integral above to show that $G = \frac{1}{2}\sqrt{\pi}$. Explain how this means that

$$\left(\frac{1}{2}\right)! = \sqrt{\pi}.$$

We use the u-substitution $u = r^2$ to find

$$(2G)^2 = 2\left(\frac{1}{2}\int_0^\infty e^{-u}\,du\right) = 2\pi \cdot \frac{1}{2} = \pi.$$

This shows that $G = \frac{1}{2}\sqrt{\pi}$.

From our explanation in part (a), we see that

$$\left(\frac{1}{2}\right)! = \Gamma(\frac{1}{2}) = 2G = \sqrt{\pi}.$$