Problem Set 6

Problem 1. *Procedural proficiency with a system of linear equations*

Consider the linear system in three equations and three unknowns:

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & - & 5y & - & z & = & 5 \\ -x & + & 3y & + & z & = & -2 \end{array}\right. .$$

(a) First, identify the matrix A and the vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{b}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) Write this system of equations as an augmented matrix system.

(c) Row reduce this augmented matrix system to show that there is exactly one solution to this system of equations.

(d) Convert your reduced augmented matrix system back into an equivalent system of equations, and then use back-substitution to compute the unique solution to the original system of equations.

(e) Verify that the solution $\vec{\mathbf{x}}$ that you found in (d) is indeed a solution of the system of equations by computing $A\vec{\mathbf{x}}$ and showing this is equal to the vector $\vec{\mathbf{b}}$.

Problem 2. *Procedural proficiency in computing eigenvalues and eigenvectors*

Consider the matrices

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

(a) Compute the characteristic polynomials for A and B.

(b) Use the characteristic polynomial of A to compute the eigenvalues of A.

(c) Use the eigenvalues of A to compute the representative eigenvectors associated with each eigenvalue.

(d) Verify that each vector $\vec{\mathbf{v}}$ computed in (c) is indeed an eigenvector of A by multiplying $A\vec{\mathbf{v}}$ and showing that the resulting vector is $\lambda\vec{\mathbf{v}}$ for the correct eigenvalue λ of A.

(e) Repeat parts (b), (c), and (d) for the matrix B.

Problem 3. *A connection to second-order ODE*

Recall the second-order ODE

$$ay'' + by' + cy = g(t),$$

where $a \neq 0$, b, and c are all constant and $g(t)$ is the non-homogenous term.

(a) Use a substitution to show that this ODE can be written as a vector equation

$$\frac{d\vec{\mathbf{x}}}{dx} = A\vec{\mathbf{x}} + \vec{\mathbf{G}}(t)$$

for a constant, 2×2 matrix A and vector functions $\vec{\mathbf{x}}(t)$ and $\vec{\mathbf{G}}(t)$.

(b) Compute the characteristic equation of the matrix A, and relate this to the original second-order ODE.