6. Curl and Stokes' Theorem

Many Vector Analysis textbooks (at least a dozen or so that I know of) define curl of a three-dimensional vector field

$$F = f(x,y,z)\,\mathbf{i} + g(x,y,z)\,\mathbf{j} + h(x,y,z)\,\mathbf{k}$$

as the following symbolic determinant:

$$\operatorname{curl}(F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{15}$$

To compute (15), one expands the determinant along the first row and interprets the products of partial derivatives with functions as partial derivatives of functions—that is, as an 'operator product':

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} &= \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g & h \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ f & h \end{vmatrix} \\ &\quad + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f & g \end{vmatrix} = \mathbf{i}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) \\ &\quad - \mathbf{j}\left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) + \mathbf{k}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right). \end{aligned} \tag{16}$$

Equations (15) and (16) may be useful for symbolically computing curl in Cartesian coordinates, yet, as any computational recipe, they make for a terrible mathematical definition:

- It is not clear where curl(F) comes from, what physical significance it has, if any, and why it has the given form.
- The curl of a vector field seems to make sense only in three dimensions. There is no meaningful way to form a determinant such as (15) in two- or four dimensions.
- There is no obvious connection between the curl of a field and other differential operators, such as gradient or divergence.
- Equations (15) and (16) are specific to Cartesian coordinates and cannot be easily adapted to, say, spherical coordinates.
- Equation (15) perverts the standard linear algebra notion of the determinant. Indeed, you are not allowed to interchange rows in a symbolic determinant whereas in a regular determinant such operations cause the change of sign.

I therefore propose that we do *not* follow the textbook route of pulling symbolic determinants out of the blue and making sense of them after the fact. Instead, let us define curl *constructively*, starting with two very big clues: the concept of swirl of a two-dimensional vector field and the Multivariate Calculus construction of gradient.

Recall that we defined swirl as circulation density: that is to say, in a physically meaningful, coordinate-free manner, which at once leads to Green's Theorem. Symbolically,

$$\operatorname{swirl}(F) = \lim_{r\to 0} \frac{\oint_C F \cdot \mathbf{t}\, ds}{\pi\, r^2},$$

where C is the circle of radius r, centered at the point where swirl is to be computed, and oriented counterclockwise. [1] Why cannot we use the same limit in $\mathbb{R}^3$ and simply replace swirl with curl? If you think about it, the reason is the circle C: to define a circle in three-space, one must, in addition to center and radius, specify the circle's orientation.

Although it may not be obvious, we faced a very similar problem in Multivariate Calculus while extending the notion of derivative to multivariate functions. The Calculus I definition of the derivative has a simple geometric meaning—the 'slope of the tangent':

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Unfortunately, in three dimensions, we cannot simply write something like

$$f'(x,y,z) = \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}\ ?$$

The reason is that the denominator $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ can go to zero at different rates depending on how fast the individual increments Δx, Δy, and Δz go to zero; the value of the limit is therefore undefined. To remedy this, we fixed a direction with a unit vector

$$\omega = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}, \quad a^2 + b^2 + c^2 = 1.$$

and defined *directional derivative* (in the direction of ω) via:

$$D_\omega(f)(x,y,z) = \lim_{h\to 0} \frac{f(x+h\,a, y+h\,b, z+h\,c) - f(x,y,z)}{h}$$

Directional derivatives make perfect sense from the point of view of single-variable Calculus. Indeed, to compute D_ω at a point P we restrict the function to the line passing through P and having the unit tangent vector ω; we then compute the one dimensional Calculus I limit

[1] Of course, instead of a circle we can use any other simple positively-oriented loop. And we do, when we need to adapt swirl to, say, polar coordinate system.

of the difference quotient along that line. We showed that directional derivatives were linear combinations of partial derivatives:

$$D_\omega(f) = \frac{\partial f}{\partial x}\, a + \frac{\partial f}{\partial y}\, b + \frac{\partial f}{\partial z}\, c.$$

This suggested defining gradient as the vector:

$$\mathrm{grad}(f) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

Once gradient is known, any directional derivative can be quickly computed as the dot product: $D_\omega f = \mathrm{grad}(f) \cdot \omega$. Furthermore, since

$$\mathrm{grad}(f) \cdot \omega = |\,\mathrm{grad}(f)|\, \cos(\theta),$$

where θ is the angle between $\mathrm{grad}(f)$ and ω, the directional derivative attains maximum value $|\,\mathrm{grad}(f)|$ when ω and $\mathrm{grad}(f)$ have the same direction. We can say, therefore, that gradient is the vector pointing in the direction of maximum growth of the function and having length equal to the rate of maximum growth.

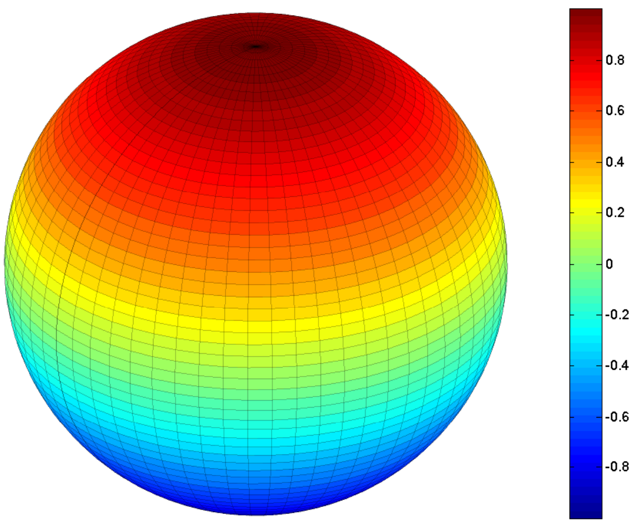


FIGURE 3. Directional derivative of $f = 2x^2 + \frac{1}{2}x\,y + z + 3$ at the origin plotted on the unit sphere

We can visualize directional derivatives and the gradient in three-dimensions as follows. Let us say we are interested in the "slope" of the function

$$f = 2x^2 + \frac{1}{2}x\,y + z + 3$$

at the origin. We can easily compute $D_\omega(f)(0,0,0)$ for various unit vectors ω. Now, ω being a unit vector, always points to some point on the unit sphere. Therefore, the directional derivative $D_\omega(f)(0,0,0)$ can be considered a function on the unit sphere, as long as the function f and the point where it is being differentiated are held fixed. Figure 3 shows the corresponding color plot. Evidently, the directional derivative for this case depends only on latitude. Furthermore, along any meridian the directional derivative changes as cosine of the latitude which, of course, is to be expected from the dot product.

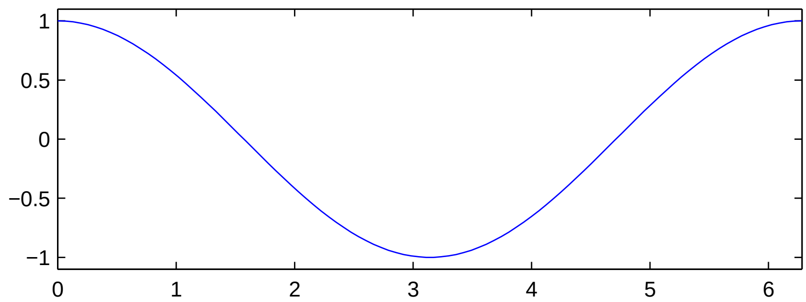


FIGURE 4. Directional derivative of $f = 2x^2+\frac{1}{2}x\,y+z+3$ along the meridian $\theta = 0$

The highest value of the directional derivative for our example is 1 and it is attained at the North Pole. Hence $\mathrm{grad}(f)(0,0,0) = \mathbf{k}$ which is trivial to verify.

What if we now vary the point where $D_\omega f$ is computed? We can construct spherical plots similar to Figure 3 for points on some grid. This produces gradient "droplets" shown in Figure 5. Notice that each individual droplet is a rotated version of Figure 3. Of course, this is not surprising. We know that the directional derivative around each point will depend on the gradient at that point as a dot product. As we move from one point on the grid to the next, the gradient

$$\mathrm{grad}(f)(x,y,z) = \left(4x + \frac{1}{2}\,y\right)\,\mathbf{i} + \frac{1}{2}\,x\,\mathbf{j} + \mathbf{k}$$

changes direction and magnitude. As a result of that, the directional derivative plot seems to rotate as well. The main point of Figure 5 is that the directional derivative is a simple dot product at every point, not just the origin.

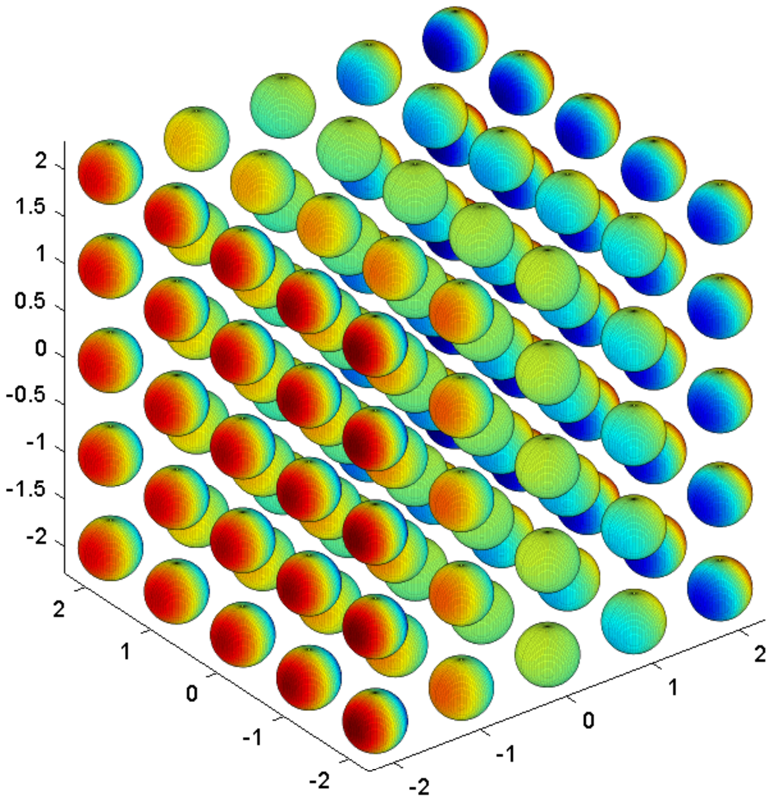


FIGURE 5. Gradient "droplets" for $f = 2x^2 + \frac{1}{2}x\,y + z + 3$

At this point, we could ask: What happens if we change the function? Of course, we already know the answer. We can make f as "complicated" as we want but that will not alter the nature of the plots: they will still look like Figures 3 and 5. So, instead of experimenting with the gradient further, let us return to our discussion of curl. We can specify the orientation of a circle in three dimensions with a unit vector ω. Hence, for a fixed vector field $F = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ and a fixed point in $\mathbb{R}^3$, say the origin, the circulation density can be regarded as a function on the unit sphere. What does the analogue of Figure 3 look like?

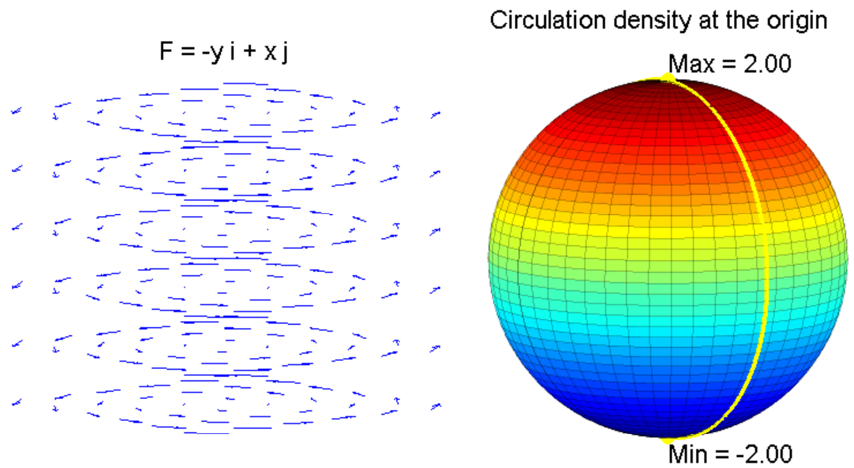
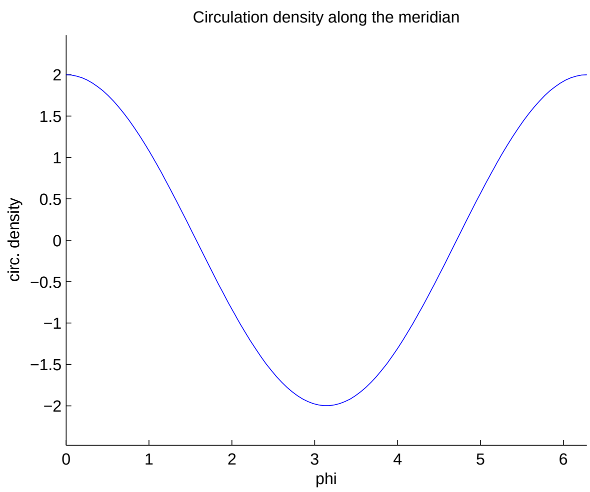


Figure 6 shows the three-dimensional plot of $F = -y\,\mathbf{i} + x\,\mathbf{j}$ and the plot of circulation density at the origin. Amazingly, the plot of circulation density looks exactly like Figure 3. Furthermore, the values of circulation density along the yellow meridian (or any other meridian for that matter) follow the familiar cosine profile:



Notice that the circulation density has the maximum value of 2 at the North Pole ($\omega = \mathbf{k}$) which corresponds to circulation along the equator. Hence,

$$\operatorname{curl}(F)(0,0,0) = 2\,\mathbf{k}.$$

It is not hard to see (exercise) that circulation density of $F = -y\,\mathbf{i} + x\,\mathbf{j}$ has the same form at every point. Therefore

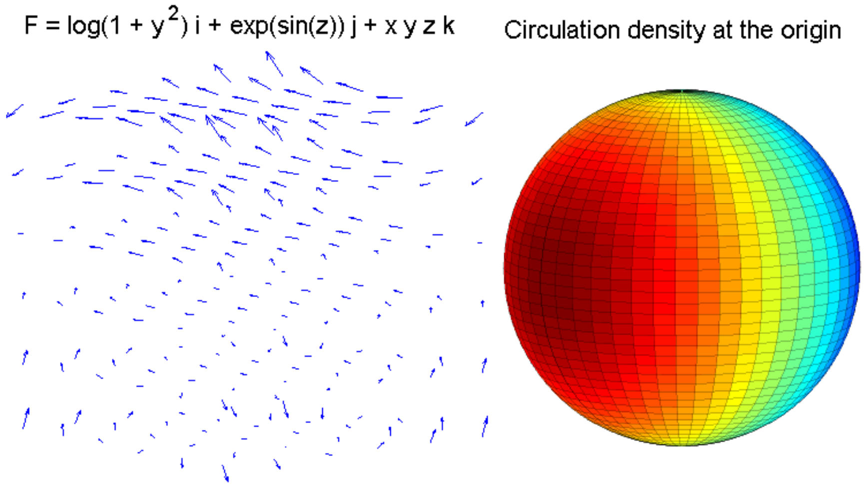
$$\operatorname{curl}(-y\,\mathbf{i} + x\,\mathbf{j}) = 2\,\mathbf{k}, \quad \text{(everywhere in } \mathbb{R}^3).$$

Let us examine a more complicated field (class example):

$$F = \ln(1 + y^2)\,\mathbf{i} + e^{\sin(z)}\,\mathbf{j} + xyz\,k$$

Figure 6 shows the field F and its circulation density at the origin. Notice that the plot of circulation density for this field is a rotated version of the one for $F = -y\,\mathbf{i} + x\,\mathbf{j}$. The maximum 1 now occurs at $(-1, 0, 0)$. Hence

$$\operatorname{curl}(F)(0,0,0) = -\mathbf{i}.$$



Now it is not hard to see why curl of an arbitrary field is defined by Equations (15) and (16). Let $F = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ be any field in $\mathbb{R}^3$. For simplicity, let's fix the point to be the origin. Our plots show that the circulation density for normal ω is given by the dot product: $\operatorname{curl}(F) \cdot \omega$. To find the components of $\operatorname{curl}(F)$ set ω to the corresponding coordinate vectors. For instance, set $\omega = \mathbf{k}$. Then the circulation density is given by the limit:

$$\lim_{r\to 0} \frac{\oint_C F\mid_{z=0} \cdot \mathbf{t}\, ds}{\pi\, r^2},$$

where C is a circle of radius r centered at the origin and lying in the xy-plane while $F\mid_{z=0}$ is the restriction of the field to the xy-plane:

$$F\mid_{z=0} = f(x,y,0)\,\mathbf{i} + g(x,y,0)\,\mathbf{j} + h(x,y,0)\,\mathbf{k}.$$

Since $\mathbf{t}$ lies in the xy-plane, its dot product with the $\mathbf{k}$-component of $F\mid_{z=0}$ is zero and:

$$\begin{aligned} \operatorname{curl}(F)(0,0,0)\cdot\mathbf{k} &= \operatorname{swirl}(f(x,y,0)\,\mathbf{i} + g(x,y,0)\,\mathbf{j})\mid_{x=0,y=0} \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)(0,0,0). \end{aligned}$$

We can now conclude that, quite generally,

$$\operatorname{curl}(F)\cdot\mathbf{k} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}.$$

Similar reasoning involving other coordinate vectors leads to two more equations

$$\begin{aligned} \operatorname{curl}(F)\cdot\mathbf{i} &= \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \operatorname{curl}(F)\cdot\mathbf{j} &= \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \end{aligned}$$

Hence

$$\operatorname{curl}(F) = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k},$$

which is equivalent to Equation (16).

6.1. **Stokes' Theorem.** Let $\mathbf{F}$ be a field in $\mathbb{R}^3$ and let C be a closed loop lying in the plane with normal $\mathbf{n}$. We assume that the loop is positively oriented: that is, if we look down at the loop from the tip of the normal $\mathbf{n}$ the orientation is counterclockwise. If F is differentiable, we can compute its curl using Equations (15) and (16); dotting the curl with the normal gives circulation density; integrating circulation density gives circulation. To streamline notation, let us regard the loop as the boundary of some planar domain D. Then

$$\iint_D \operatorname{curl}(F)\cdot\mathbf{n}\, dA = \oint_{\partial D} F\cdot\mathbf{t}\, ds. \tag{17}$$

Equation (17) is an instance of Stokes' Theorem for flat (planar) domains.

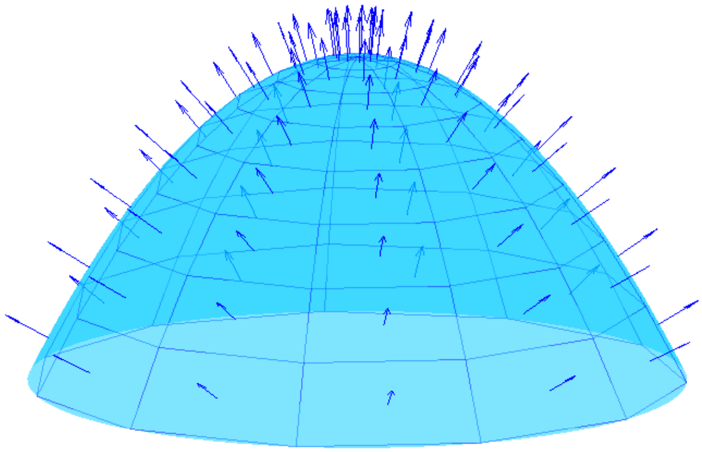


FIGURE 6. An example of a capping surface with normals shown

It is easy to generalize (17) to domains that are not planar. Cap a closed loop in $\mathbb{R}^3$ with any surface. For example, Figure 6 shows a unit circle C capped with a paraboloid of revolution S. Break up the surface S into small pathes. Each patch can be treated as a planar domain: hence the circulation around an individual patch is given by

$$\operatorname{curl}(F) \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the patch's normal and dS is its area. Now think of patches as very small rectangles and imagine adding up all of their circulations. Wherever two patches share an edge, there will be cancelation of terms similar to the one in the Divergence Theorem. Only the edges that are not shared—the ones that lie on the circle—will contribute to the sum. Therefore:

$$\iint_S \operatorname{curl}(F) \cdot \mathbf{n}\, dS = \oint_C F \cdot \mathbf{t}\, ds. \tag{18}$$

This is the full statement of Stokes' Theorem. In Equation (18) S can be any capping surface (imagine a soap film that you can deform in any which way). The loop C can be any simple (no self-intersections) closed loop; it does not have to lie in a plane.

6.2. **Ampere's Circuital Law.** As an application of Stokes' Theorem, let us derive the differential form of Ampere's Circuital Law (in vacuum). The *integral* form of the law states that: the circulation of the magnetic field B around a closed loop C is proportional to the current I passing through the loop. In symbols,

$$\oint_C B \cdot \mathbf{t}\, ds = \mu_0\, I, \tag{19}$$

where the coefficient of proportionality μ_0 is called permeability of free space. To derive the local form of the law, introduce *current density* J. Then the current can be expressed as the integral of its density:

$$I = \iint_S J \cdot \mathbf{n}\, dS. \tag{20}$$

As the surface S, we can take any capping surface[2]. Replacing the current I in Equation (19) with (20) and applying Stokes' Theorem to the line integral, we get:

$$\iint_S \mathrm{curl}(B) \cdot \mathbf{n}\, dS = \mu_0 \iint_S J \cdot \mathbf{n}\, dS.$$

Since this holds true for an arbitrary loop C and an arbitrary capping surface S we must have:

$$\mathrm{curl}(B) = \mu_0\, J.$$

This is a special case of one of Maxwell's equations of magnetostatics.

Exercises.

(1) Suppose that $\mathrm{curl}(F) = 0$. Prove that the field is potential. **Hint:** Imitate similar proof in two dimensions.

(2) Let $F = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ and let C be the unit circle around the origin with normal $\mathbf{j}$. Let S be either of the two half-spheres that cap C. Use F, C, and S to confirm Stokes' Theorem with direct computation.

(3) Let $F = z\, e^x \cos(y)\,\mathbf{i} + x\, e^y \cos(z)\,\mathbf{j} + y\, e^z \cos(x)\,\mathbf{k}$ and let C be the path parameterized by

$$\begin{aligned} x(t) &= 3\cos(t) + \cos(6\,t)\cos(t) \\ y(t) &= 3\sin(t) + \cos(6\,t)\sin(t) \\ z(t) &= \sin(6\,t) \end{aligned}$$

where $0 \le t \le 2\pi$. Use Matlab to compute the circulation of F over C correct to four decimals. Present Matlab code and the value of the circulation as your solution. **Note:** Discretize the

[2] Equation (20) shows what current density $\mathbf{J}$ really is: dot it with the patch's normal and multiply by the area of the patch to get current crossing the patch. Do you see the analogy with gradient and curl?

path and approximate the integral with the Riemann sum. Do not use black box commands such as `quad`.