PS 3 Solutions

① $(\sec t)y'' - ty' + (\ln t)y = \frac{t}{t^2-9}$

(a) First, note that the domain of t is restricted:

i) $t \neq \frac{\pi}{2} + \pi k$ for all integers k (from $\sec t$)

ii) $t > 0$ (from $\ln t$)

iii) $t \neq \pm 3$ (from $\frac{t}{t^2-9}$)

Putting the ODE in standard form yields

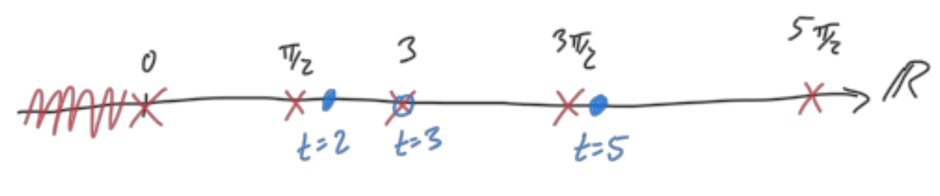
$$y'' + (t\cos t)y' + (\ln t \cdot \cos t)y = \frac{t\cos t}{t^2-9}.$$

This equation offers no further restrictions on the continuity of

$$p(t) = -t\cos t,\quad q(t) = \ln t \cdot \cos t,\quad \text{or}\quad g(t) = \frac{t\cos t}{t^2-9}.$$

Therefore, $p, q, \& g$ are continuous over the domains given by (i), (ii), and (iii) simultaneously.

Also note that $t=2$ is in the interval $(\frac{\pi}{2}, 3)$.



The Existence & Uniqueness Theorem then guarantees a unique solution to the IVP on the interval $(\frac{\pi}{2}, 3)$.

(b) Taking $t=5$ instead, the only difference from part (a) is the interval where the solution exists. We note that $\frac{3\pi}{2} \approx 4.7 < t=5 < \frac{5\pi}{2} \approx 7.8$ so that there is a unique solution to the IVP on the interval $(\frac{3\pi}{2}, \frac{5\pi}{2})$.

(c) Since $t=3$ occurs at a point of discontinuity of $g(t)$, the Existence & Uniqueness Theorem guarantees nothing about the existence or uniqueness of a solution to the IVP. It could exist. It could be unique. We just don't know. The Theorem is inconclusive.

---

② $\begin{cases} \frac{dy}{dt} = \frac{-3t^2+4t-2}{2(y-1)} =: f(t,y) \\ y(0) = 2 \end{cases}$ $\quad (y-1)^2 = -t^3+2t^2-2t+1$

$\underline{y_{actual}(t) = 1 + \sqrt{-t^3+2t^2-2t+1}}$

(a) $\begin{cases} y_{n+1} = y_n + f(t_n, y_n)\cdot h \\ t_{n+1} = t_n + h = t_0 + h(n+1) \end{cases}$

$t_0 = 0,\ y_0 = 2$

$t_1 = \frac{1}{4},\ y_1 = 2 + f(0,2)\cdot\frac{1}{4} = 2 + (-1)\cdot\frac{1}{4} = \frac{7}{4}$

$t_2 = \frac{1}{2},\ y_2 = \frac{7}{4} + f(\frac{1}{4}, \frac{7}{4})\cdot\frac{1}{4} = \frac{7}{4} + (-0.19791\overline{6}) = 1.55208\overline{3}$

$t_3 = \frac{3}{4},\ y_3 \approx 1.552 + f(\frac{1}{2}, 1.552)\cdot\frac{1}{4} \approx 1.382272013$

$t_4 = 1,\ y_4 \approx 1.3823 + f(\frac{3}{4}, 1.3823)\cdot\frac{1}{4} \approx 1.157464815$

$|y_0 - y_{actual}(0)| = 0$

$|y_1 - y_{actual}(\frac{1}{4})| \approx 0.0306247498$

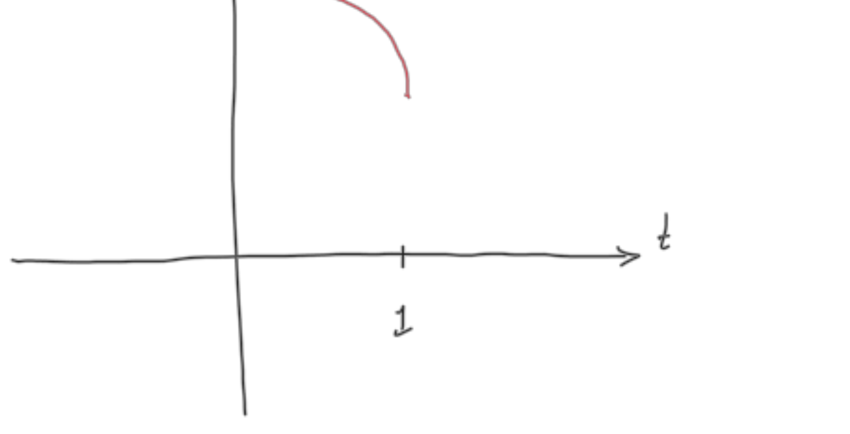
$|y_2 - y_{actual}(\frac{1}{2})| \approx 0.0602891024$

$|y_3 - y_{actual}(\frac{3}{4})| \approx 0.0684218969$

$|y_4 - y_{actual}(1)| \approx 0.1574648146$

(b) We expect this error to decrease as we decrease our step size h. To refine our approximation, perhaps set $h = \frac{1}{10}$ instead.

(c)



Bad idea to approximate past $t=1$ since the function does not exist there. It makes no sense to approximate what doesn't exist.

(d) This was undetectable from the ODE. The only indication of a problem was that we can't let $y=1$. The (non-linear) Existence & Uniqueness Theorem says that a solution exists, but it doesn't specify for what interval of t it exists.

---

③ (a) $y'' - 6y' + 9y = 0$

Characteristic: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0 \rightarrow r = 3$

General form: $y(t) = c_1 e^{3t} + c_2 \cdot t e^{3t}$

(b) $y'' + 4y = 0$

Characteristic: $r^2 + 4 = 0$

$r = \pm 2i$

General form: $y(t) = c_1 \cos 2t + c_2 \sin 2t$

(c) $y'' + 5y' + 6y = 0$

Characteristic: $r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0 \rightarrow r = -2, -3$

General form: $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$

(d) $y'' + y' + y = 0$

Characteristic: $r^2 + r + 1 = 0$

$r = \frac{-1 \pm \sqrt{1-4}}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

General form: $y(t) = e^{-t/2}\left(c_1 \cos\frac{\sqrt{3}}{2}t + c_2 \sin\frac{\sqrt{3}}{2}t\right)$