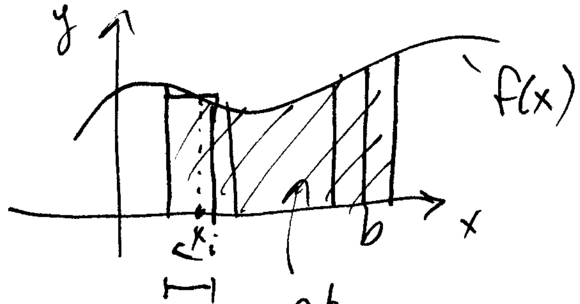
Ch 15.1 - Double, Iterated Integrals

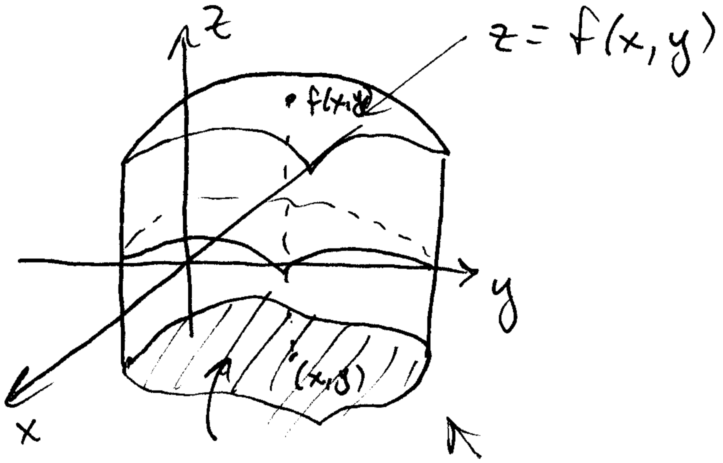


$\underline{\underline{\int_a^b f(x)dx}} = \text{Area}$

$$\text{Area} \cong \sum_{i=1}^{N} f(x_i)\Delta x_i$$

$$\hookrightarrow \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x_i$$

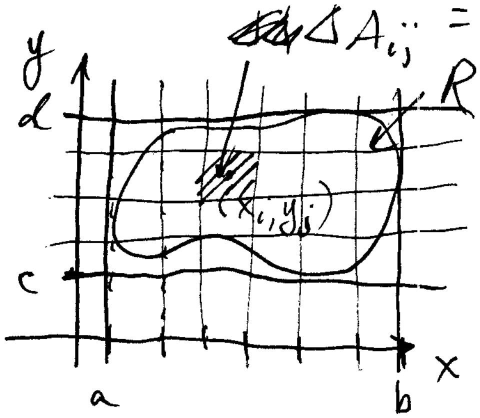
$$:= \int_a^b f(x)dx.$$

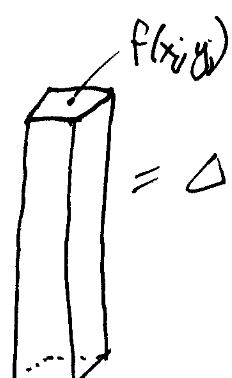


R or D

P = partition on R
= {subintervals} that make up R

$\Delta A_{ij} = \Delta x_i \Delta y_j$





$= \Delta V_{ij}$

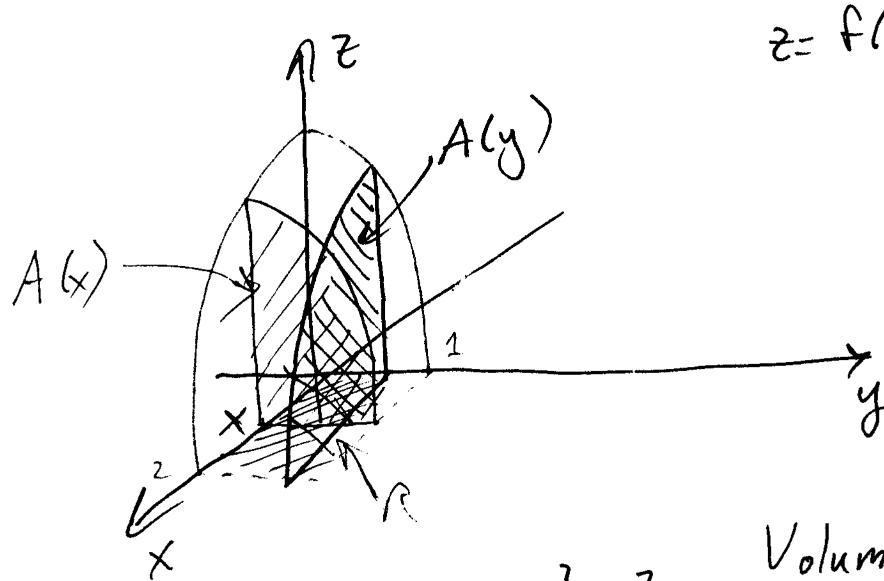
$\|P\|$ = "norm"
= maximum width of any subinterval of P.

$$\text{Volume} \cong \sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i,y_j)\Delta A_{ij} \qquad \boxed{\iint_R f(x,y)\,dA}$$

$$\hookrightarrow \cancel{\lim_{M,N\to\infty}} := \lim_{\|P\|\to 0} \sum_{i,j} f(x_i,y_j)\Delta A_{ij}$$

Evaluating $\iint_R f(x,y)\,dA$

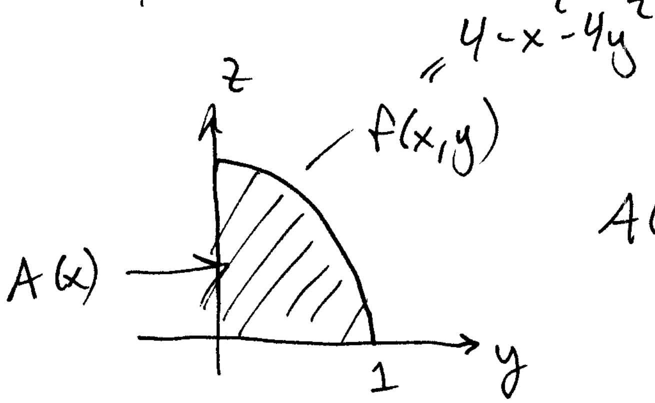
Iterated integrals:



$z = f(x,y) = 4 - x^2 - 4y^2$.

$R = [0,2] \times [0,1]$.

$$\text{Volume} = \int_a^b A(x)\,dx$$



$$A(x) = \int_0^1 [(4-x^2-4y^2) - 0]\,dy$$

$$= \int_0^1 4 - x^2 - 4y^2\,dy$$

$$= \left[4y - x^2y - \tfrac{4}{3}y^3\right]_{y=0}^{1}$$

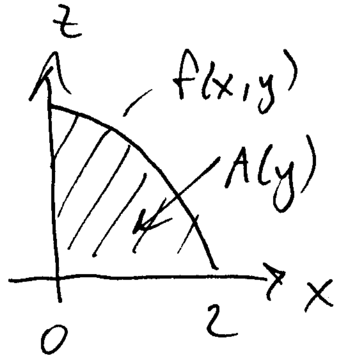
$$= 4 - x^2 - \tfrac{4}{3} = \underline{\tfrac{8}{3} - x^2}.$$

$$\text{Volume} = \int_0^2 A(x)\,dx$$

$$= \int_0^2 \tfrac{8}{3} - x^2\,dx$$

$$= \left[\tfrac{8}{3}x - \tfrac{1}{3}x^3\right]_{x=0}^{2}$$

$$= \frac{16}{3} - \frac{8}{3} = \boxed{\frac{8}{3}}$$



$$A(y) = \int_0^2 \left[(4 - x^2 - 4y^2) - 0\right] dx$$

$$= \left[4x - \tfrac{1}{3}x^3 - 4xy^2\right]_{x=0}^{2}$$

$$= 8 - \tfrac{8}{3} - 8y^2$$

$$= \tfrac{16}{3} - 8y^2$$

$$\text{Volume} = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \tfrac{16}{3} - 8y^2 \, dy$$

$$= \left[\tfrac{16}{3}y - \tfrac{8}{3}y^3\right]_{y=0}^{1}$$

$$= \tfrac{16}{3} - \tfrac{8}{3} = \boxed{8/3}$$

Defining

$$\iint_{\substack{\text{Rect.}\\ =[a,b]\times[c,d]}} f(x,y)\,dA$$

$$= \int_a^b \underbrace{\left[\int_c^d f(x,y)\,dy\right]}_{A(x)} dx$$

$$= \int_c^d \underbrace{\left[\int_a^b f(x,y)\,dx\right]}_{A(y)} dy$$

} Iterated Integrals.

Ex:

$$I = \iint_{[0,1]\times[-\pi/2,\pi/2]} x^2\cos(xy)\, dA \cancel{dy\,dx}$$

$dy\,dx$

$$I = \int_0^1 \left(\int_{-\pi/2}^{\pi/2} x^2 \cos(xy)\, dy \right) dx$$

$$= \int_0^1 \left(x^2 \int_{-\pi/2}^{\pi/2} \cancel{x^2} \cos(xy)\, dy \right) dx$$

$$= \int_0^1 x^2 \left[\frac{1}{x} \sin(xy) \right]_{y=-\pi/2}^{\pi/2} dx$$

$$= \int_0^1 x \left(\sin\left(\tfrac{\pi}{2}x\right) - \sin\left(-\tfrac{\pi}{2}x\right) \right) dx$$

$$= \int_0^1 x \left(2 \sin\left(\tfrac{\pi}{2}x\right) \right) dx$$

I.B.P.

$$= \left[-\frac{4}{\pi} x \cos\left(\tfrac{\pi}{2}x\right) + \frac{8}{\pi^2} \sin\left(\tfrac{\pi}{2}x\right) \right]_{x=0}^{1}$$

$$= -\frac{4}{\pi} \underbrace{\cos(\pi/2)}_{0} + \frac{8}{\pi^2} \underbrace{\sin(\pi/2)}_{1} = \boxed{8/\pi^2}$$

dx dy

$$I = \int_{-\pi/2}^{\pi/2} \left[\int_0^1 x^2 \cos(xy)\, dx \right] dy$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{1}{y} \sin y + \frac{2}{y^2} \cos y - \frac{2}{y^3} \sin y \right] dy$$

$$= 8/\pi^2$$

Warm-up

Evaluate $I = \int_{-2}^{4}\left(\int_{1}^{3} xy + 5 \, dx\right) dy$.

$$= \int_{-2}^{4} \left[\tfrac{1}{2}x^2 y + 5x\right]_{x=1}^{3} dy$$

$$= \int_{-2}^{4} \left[\left(\tfrac{9}{2}y + 15\right) - \left(\tfrac{1}{2}y + 5\right)\right] dy$$

$$= \int_{-2}^{4} (4y + 10)\, dy = 2y^2 + 10y \Big|_{y=-2}^{4}$$

$$= (32 + 40) - (8 - 20)$$

$$= 72 + 12 = \boxed{84}$$

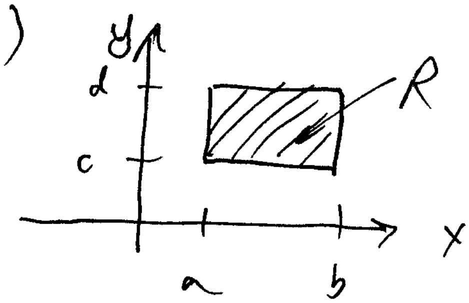
Recall: Over rectangles $R = [a,b] \times [c,d]$.

$$I = \iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

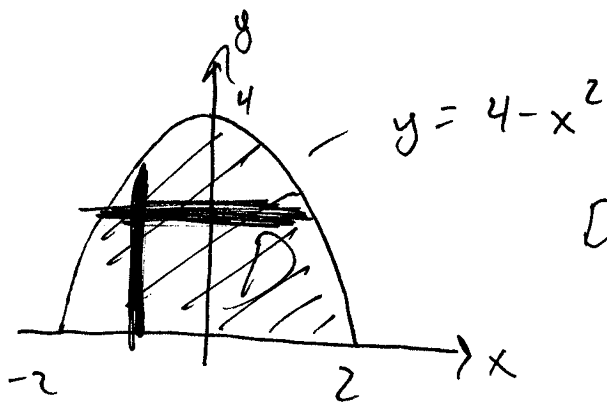
Ex:

(a)



$R : [a,b] \times [c,d] \;\rightarrow\; \begin{cases} a \le x \le b \\ \cancel{y}\, c \le y \le d. \end{cases}$

(b)



Easy

$D : \begin{cases} -2 \le x \le 2 \\ 0 \le y \le 4 - x^2. \end{cases}$

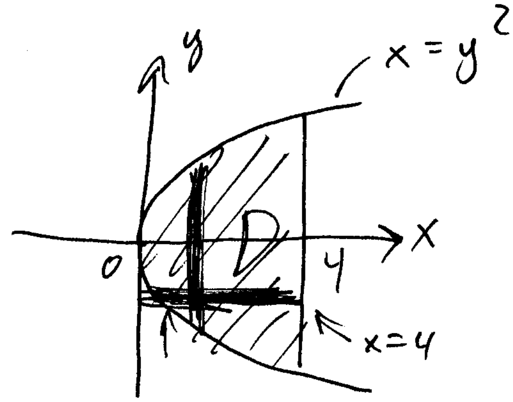
$y = 4 - x^2$

$x^2 = 4 - y$

$x = \pm\sqrt{4-y}$

Harder: $D : \begin{cases} 0 \le y \le 4 \\ -\sqrt{4-y} \le x \le \sqrt{4-y} \end{cases}$

(c)

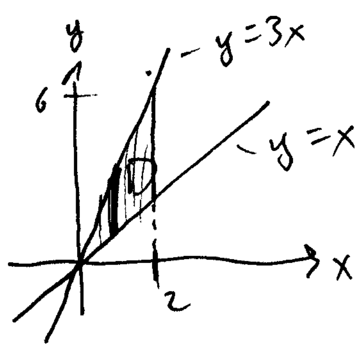


Harder:

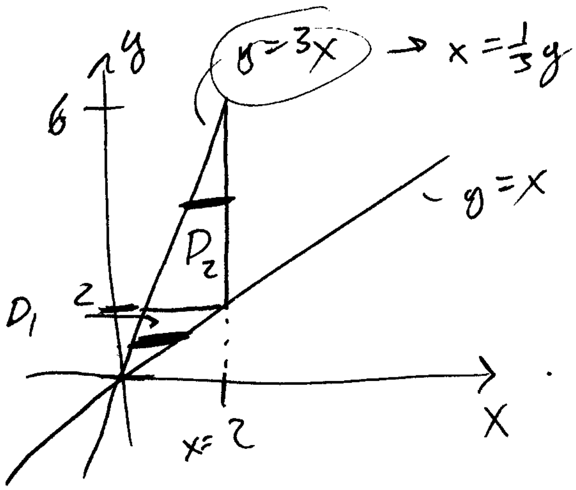
$D : \begin{cases} 0 \le x \le 4 \\ -\sqrt{x} \le y \le \sqrt{x} \end{cases}$

$D : \begin{cases} -2 \le y \le 2 \\ y^2 \le x \le 4 \end{cases}$

(d)



$D : \begin{cases} 0 \le x \le 2 \\ x \le y \le 3x \end{cases}$



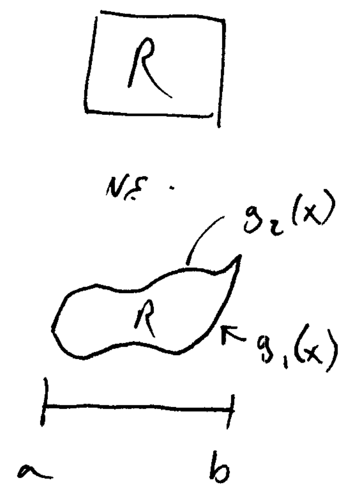
$$D \begin{cases} D_1 : \begin{cases} 0 \le y \le 2 \\ \frac{1}{3}y \le x \le y \end{cases} \\ D_2 : \begin{cases} 2 \le y \le 6 \\ \frac{1}{3}y \le x \le 2 \end{cases} \end{cases}$$

How does this apply to $\iint_R f(x,y)\,dA$?

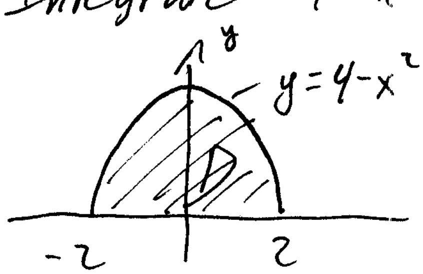
$$\text{Area} \cong \sum_{i,j}^{M,N} \underbrace{f(x_i, y_j)}_{\text{same as before}} \Delta A_{ij}$$

Volume... not Area...

geometry of R changes ΔA_{ij}.



Ex: Integrate $4-x-y = z = f(x,y)$ over



$$I = \iint_D f(x,y)\,dA$$

Easy: $\begin{cases} -2 \le x \le 2 \\ 0 \le y \le 4-x^2 \end{cases}$ $\quad I = \int_{-2}^{2} \int_{0}^{4-x^2} (4-x-y)\,dy\,dx$

$$I = \int_{-2}^{2} \left[\int_{0}^{4-x^2} 4 - x - y \, dy \right] dx$$

$$= \int_{-2}^{2} \left[4y - xy - \tfrac{1}{2} y^2 \right]_{\underline{y=0}}^{y=4-x^2} dx$$

$$= \int_{-2}^{2} \left[4(4-x^2) - x(4-x^2) - \tfrac{1}{2}(4-x^2)^2 \right] dx$$

$$- \tfrac{1}{2}(16 - 8x^2 + x^4)$$

$$= \int_{-2}^{2} \left(8 - 4x + x^3 - \tfrac{1}{2} x^4 \right) dx$$

$$= \left[8x - 2x^2 + \tfrac{1}{4} x^4 - \tfrac{1}{10} x^5 \right]_{x=-2}^{2}$$

$$= \left(16 - 8 + 4 - \tfrac{32}{10}\right) - \left(-16 - 8 + 4 + \tfrac{32}{10}\right)$$

$$\left(12 - 16/5\right) \quad - \left(-20 + 16/5\right)$$

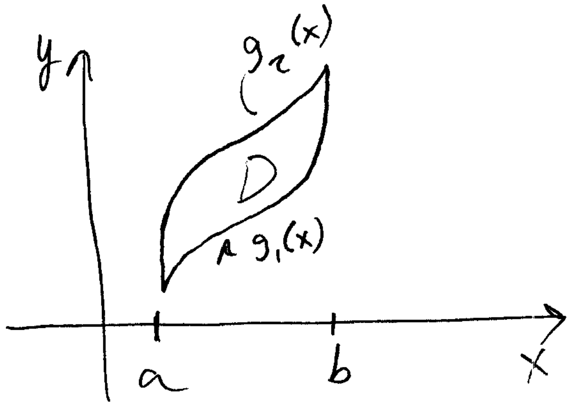
$$= 32 - 32/5 = \tfrac{4}{5} \cdot 32 = \boxed{128/5}$$

How do we do this for $dx\,dy$?

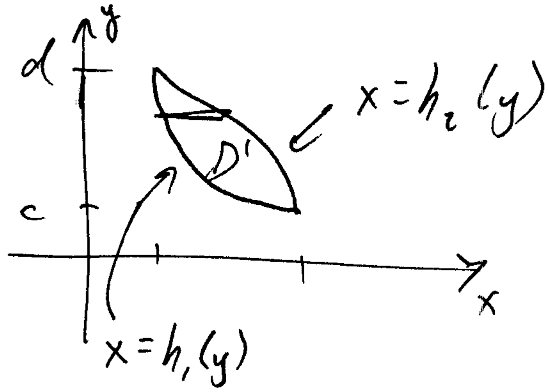
<u>Theorem</u>: (Fubini) If $f(x,y)$ is continuous over the (general) region D, then

$$\iint_D f(x,y)\, dA = \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\, dy\, dx$$

$$= \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx\, dy$$



$$D:\begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{cases}$$

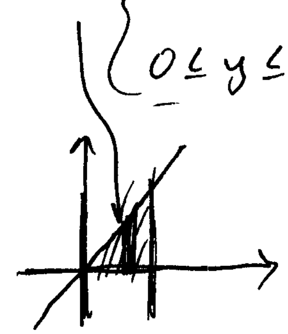


$$D':\begin{cases} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{cases}$$

Ex:

$$I = \int_0^2 \int_0^x (x^2 y - x y^2)\, dy\, dx$$

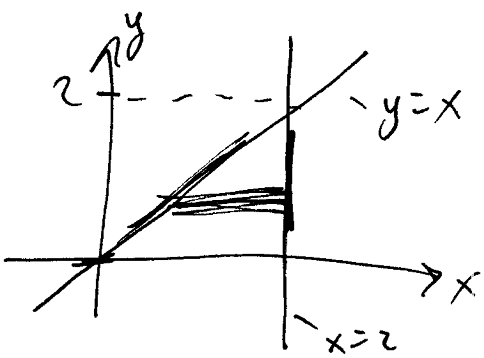
$$D:\begin{cases} 0 \le x \le 2 \\ \underline{0 \le y \le x} \end{cases}$$



$$= \int_0^2 \left[\tfrac{1}{2}x^2y^2 - \tfrac{1}{3}xy^3\right]_{y=0}^{y=x} dx$$

$$= \int_0^2 \overset{\frac{1}{6}x^4}{\left[\cancel{\tfrac{1}{2}x^4 - \tfrac{1}{3}x^4}\right]} dx = \left[\tfrac{1}{30}x^5\right]_{x=0}^{2} = 32/30$$

$$= \boxed{16/15}$$



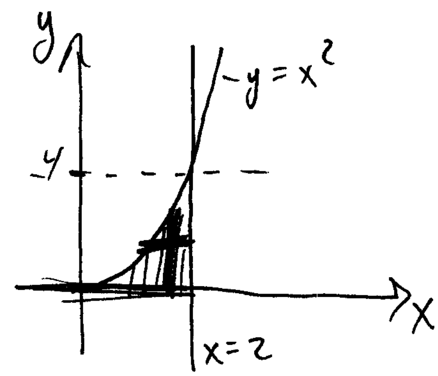
$$I = \int_0^2 \int_y^2 (x^2 y - x y^2)\, dx\, dy$$

$$= \int_0^2 \left[\frac{1}{3}x^3 y - \frac{1}{2}x^2 y^2\right]_{x=y}^{x=2} dy$$

This should be 8/3*y-2y^2+1/6*y^4

$$= \int_0^2 \left(\frac{1}{2}y^4 - \frac{1}{3}y^4\right) dy = \int_0^2 \frac{1}{6}y^4\, dy = \boxed{16/15}$$

Ex: $I = \int_{y=0}^{4}\int_{x=\sqrt{y}}^{x=2} e^{x^3}\, dx\, dy$



$x=\sqrt{y} \Leftrightarrow y = x^2$

$$D: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le x^2 \end{cases}$$

$$= \int_0^2 \left(\int_0^{x^2} e^{x^3}\, dy\right) dx$$

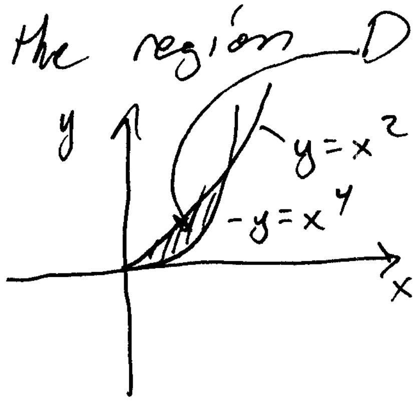
$$= \int_0^2 \left[y e^{x^3}\right]_{y=0}^{y=x^2} dx$$

$$= \int_0^2 x^2 e^{x^3}\, dx = \left[\frac{1}{3}e^{x^3}\right]_{x=0}^{2}$$

$$= \frac{1}{3}e^8 - \frac{1}{3}e^0 = \boxed{\frac{e^8-1}{3}}$$

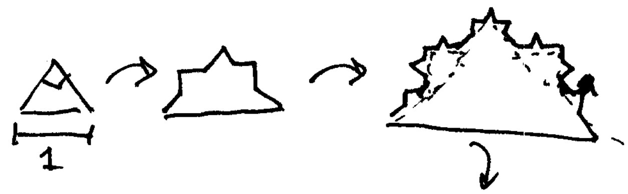
Warm-up

① Describe the region D



~~using~~ in two ways.

② (Argue) "Prove" that the snowflake in the limit



has finite ~~size~~ area.

---

① I : $\begin{cases} 0 \le x \le 1 \\ x^4 \le y \le x^2 \end{cases}$ [redacted] : $\begin{cases} 0 \le y \le 1 \\ \sqrt{y} \le x \le \sqrt[4]{y} \end{cases}$

$x^2 = x^4$

$0 = x^2(x^2-1)$

$= x^2(x+1)(x-1)$

$y = x^2 \Leftrightarrow x = \pm\sqrt{y}$

$y = x^4 \Leftrightarrow x = \pm\sqrt[4]{y}$.

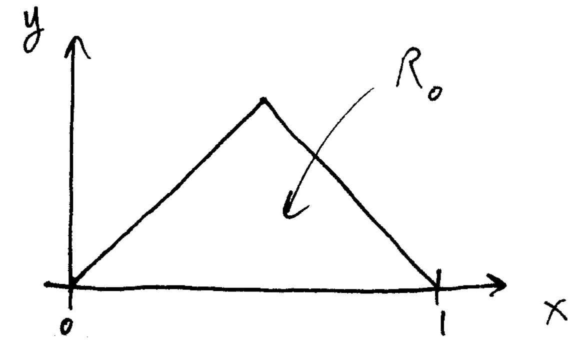
---

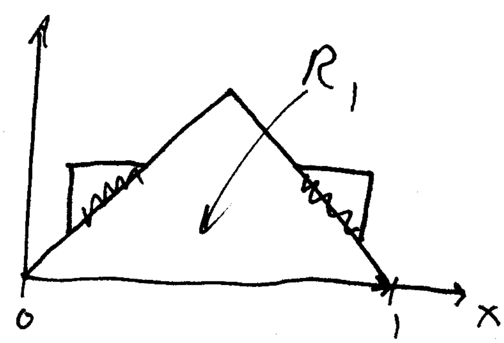
② Idea: The areas of the triangles added decrease geometrically $\sum_{n=1}^{\infty} cr^n < \infty$

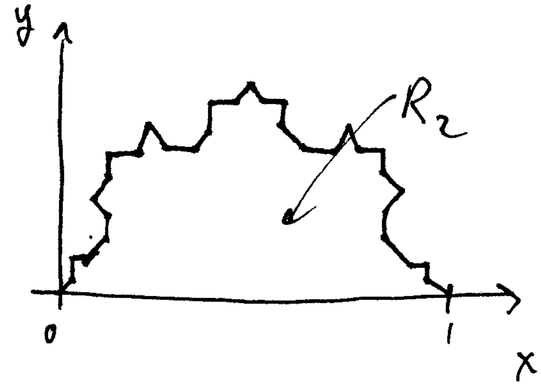
$|r| < 1$.

Problems with definition of $\iint\limits_R f(x,y)\,dA$.

Ex.:







--- R_∞? A Koch snowflake. ... which is a FRACTAL!

Boundary of $R_\infty = R$ is continuous but not easily described.

Question: How can we integrate/define factal boundaries?
Which kinds of regions are we "allowed" to integrate over?

$$\text{Area} = \iint_R dA < \infty$$

how do I evaluate this?

Properties of Double Integrals

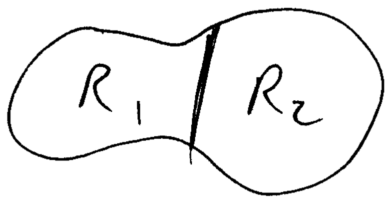
Let $f(x,y)$, $g(x,y)$ be continuous on the region R.

① $$\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

② $$\iint_R f+g\, dA = \iint_R f\, dA + \iint_R g\, dA.$$

③ (a) $$\iint_R f(x,y)\, dA \geq 0 \quad \text{if } f(x,y) \geq 0 \text{ on } R.$$

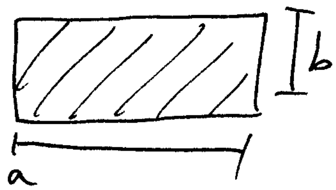
(b) $$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA \quad \text{if } f(x,y) \geq g(x,y) \text{ on } R.$$

④ 

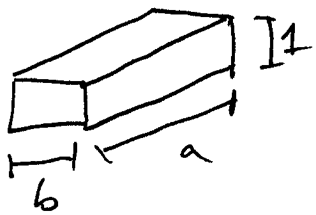
$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA.$$

$R = R_1 \cup R_2$

§ 15.3 – Areas and Average Values



Area $= ab$

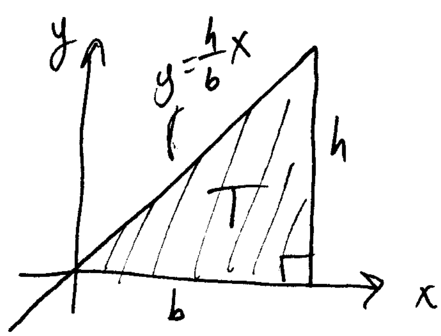


Volume $= a \cdot b \cdot \underline{\underline{1}} = ab$

Using $f(x,y) = 1$,

$$\text{Volume} = \text{Area} = \iint_R dA$$

Ex: Verify some geometry



$$\text{Area} = \iint_T dA$$

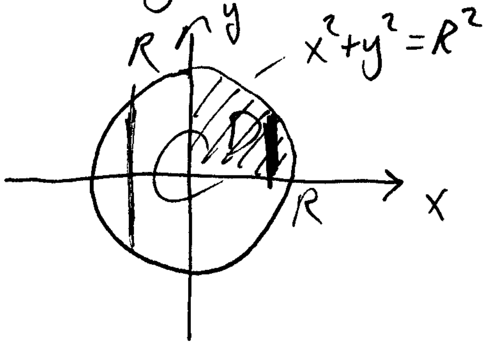
$$= \int_0^b \int_0^{\frac{h}{b}x} dy\, dx$$

$$= \int_0^b \left[y \right]_{y=0}^{\frac{h}{b}x} dx$$

$$= \int_0^b \left[\left(\tfrac{h}{b}x\right) - (0) \right] dx$$

$$= \left[\tfrac{1}{2} \tfrac{h}{b} x^2 \right]_{x=0}^{b} = \tfrac{1}{2} b \cdot h. \checkmark$$

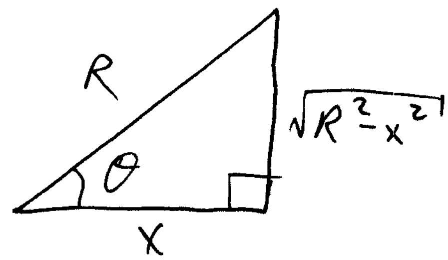
Verify area of a circle :



$$\text{Area}(D) = \iint_D dA$$

$$= \int_0^R \int_0^{\sqrt{R^2-x^2}} dy\, dx$$

$$= \int_0^R \sqrt{R^2-x^2}\, dx$$



$$= \int_{\pi/2}^{0} R\sin\theta \cdot -R\sin\theta\, d\theta$$

$$= R^2 \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= R^2 \int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{R^2}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\theta=0}^{\pi/2}$$

$$= \frac{R^2}{2}\left[\pi/2 - 0\right] = \frac{1}{4}\pi R^2$$

$$R\sin\theta = \frac{\sqrt{R^2-x^2}}{\cancel{R}}$$

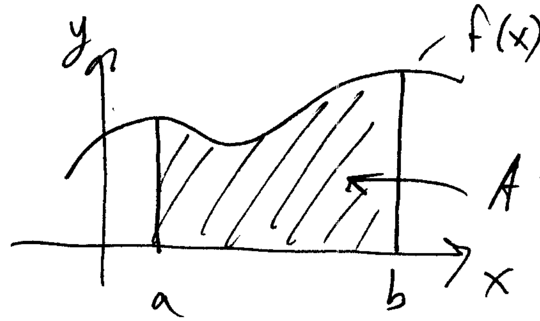
$$R\cos\theta = x \quad \begin{cases} x=0 \Leftrightarrow \theta = \frac{\pi}{2} \\ x=R \Leftrightarrow \theta = 0 \end{cases}$$

$$-R\sin\theta = \frac{dx}{d\theta}$$

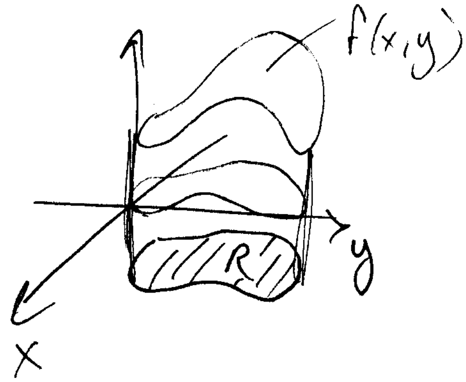
$$dx = -R\sin\theta\, d\theta$$

Average values of f

Recall:



$$f_{avg}[a,b] = \frac{\int_a^b f(x)\,dx}{\int_a^b dx}$$



$$f_{avg}(R) = \frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

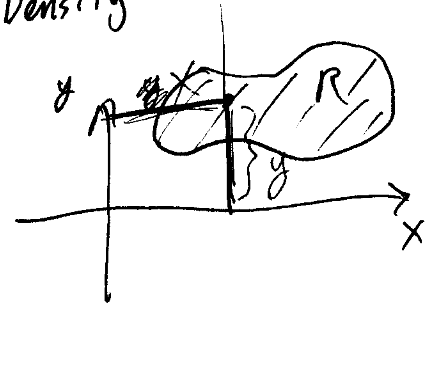
Application: (The section we are skipping: §15.6)

Define $\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$, where

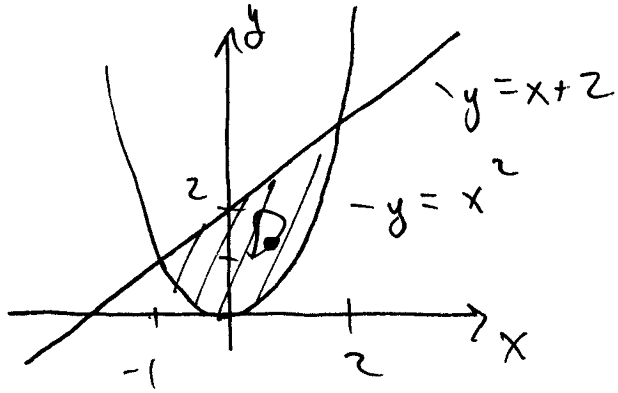
(mass) $M = \iint_R \delta(x,y)\,dA$, $M_y = \iint_R x\,\delta(x,y)\,dA$

$M_x = \iint_R y\,\delta(x,y)\,dA$ } 1^{st} moments of R (wrt. an axis)

Density $= \delta(x,y)$



Ex:



$\delta(x,y) = \delta$ const.

Find "center of mass" $cm(\bar{x}, \bar{y})$.

$$M = \iint_D \delta \, dA$$

$$= \delta \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx$$

$$= \delta \int_{-1}^{2} [x+2 - x^2]\, dx$$

$$= \delta \left[\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\right]_{x=-1}^{2}$$

$$= \frac{9}{2}\delta$$

$$M_y = \delta \int_{-1}^{2} \int_{x^2}^{x+2} x \, dy\, dx$$

$$= \delta \int_{-1}^{2} x\left[(x+2) - x^2\right] dx$$

$$= \delta \int_{-1}^{2} -x^3 + x^2 + 2x \, dx$$

$$= \delta \left[-\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 + x^2\right]_{x=-1}^{2}$$

$$= \frac{9}{4}\delta$$

$$M_x = \iint_D \delta \cdot y \, dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} \delta y \, dA$$

$$= \delta \int_{-1}^{2} \left[\tfrac{1}{2}(x^2 + 4x + 4) - \tfrac{1}{2}x^4\right] dx$$

$$= \delta \left[-\tfrac{1}{10}x^5 + \tfrac{1}{6}x^3 + x^2 + 2x\right]_{x=-1}^{2} = \frac{36}{5}\delta$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{9}{4}\delta}{\frac{9}{2}\delta} = \frac{1}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{36/5 \cdot \delta}{9/2 \cdot \delta} = \frac{8}{5}$$

$$\boxed{cm(\bar{x}, \bar{y}) = \left(\tfrac{1}{2}, \tfrac{8}{5}\right)}$$

Let's look a little closer at M

$$dV = dA\,dz = dx\,dy\,dz$$

$$M = \iint_D \delta(x,y)\,dA = \iint_D \int_{z=0}^{z=\delta(x,y)} 1 \cdot dV$$

$$M_y = \iint_D x\,\delta(x,y)\,dA = \iint_D \int_{z=0}^{z=\delta(x,y)} x\,dV$$

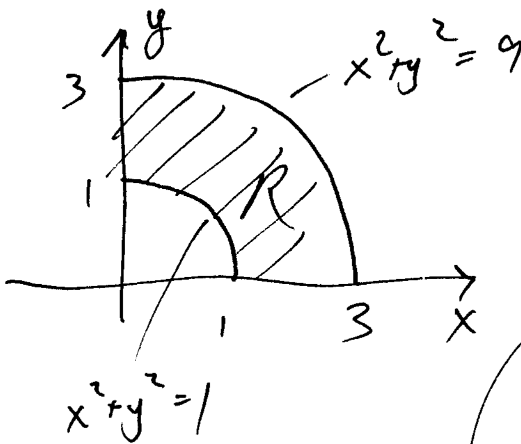
$$M_x = \iint_D y\,\delta(x,y)\,dA = \iint_D \int_{z=0}^{z=\delta(x,y)} y\,dV$$

$$R = \begin{cases} (x,y) \in D \\ 0 \le z \le \delta(x,y) \end{cases}$$

$$\boxed{\begin{aligned} \bar{x} &= x_{avg}(D) = \frac{\iiint_R x\,dV}{\iiint_R dV} \\ \bar{y} &= y_{avg}(D) = \frac{\iiint_R y\,dV}{\iiint_R dV} \end{aligned}}$$

§ 15.4 - Polar Coordinates

Ex:



$$I = \iint_R f(x,y)\, dA$$

Bad

$$R = \begin{cases} 0 \le x \le 1 \\ \sqrt{1-x^2} \le y \le \sqrt{9-x^2} \end{cases}$$

$$\cup$$

$$\begin{cases} 1 \le x \le 3 \\ 0 \le y \le \sqrt{9-x^2} \end{cases}$$

Better!

$$R = \begin{cases} 0 \le \theta \le \pi/2 \\ 1 \le r \le 3 \end{cases}$$

Polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

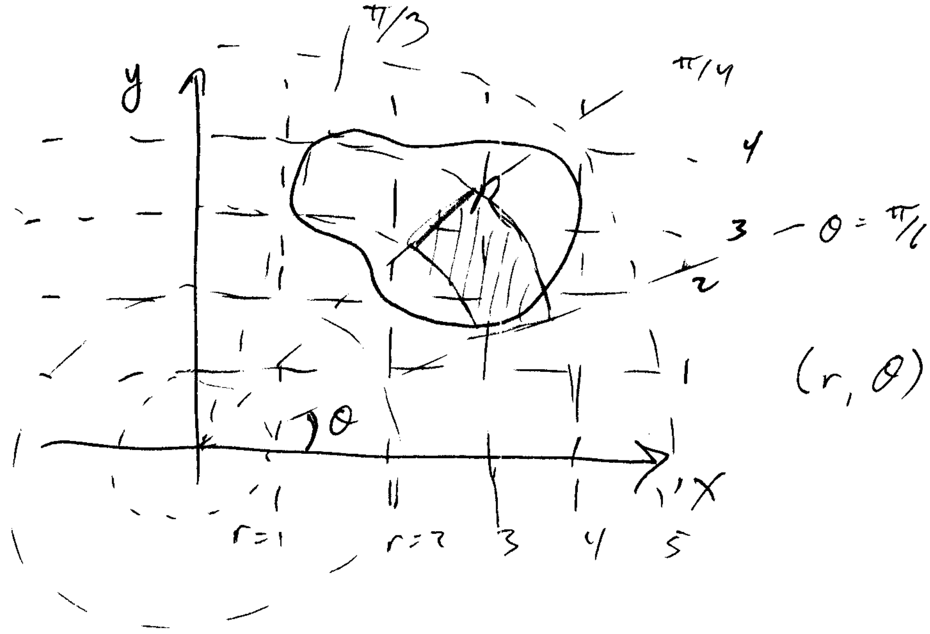
$$\begin{cases} 0 \le \theta \le 2\pi \text{ (preferred)} \\ 0 \le r \text{ (preferred)} \end{cases}$$

$$I = \iint_{R_{pol}} f(r,\theta)\, dr\, d\theta \text{ ?}$$

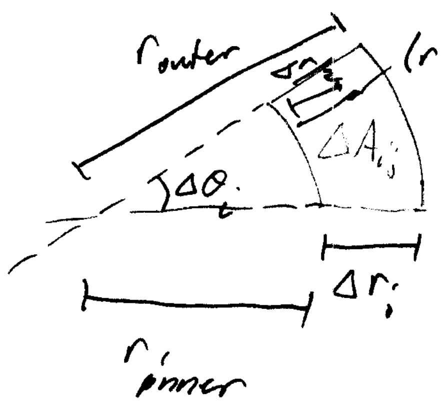
$$I = \iint_{R_{rect}} f(x,y)\, dx\, dy$$

$$= \iint_{R_{pol}} f(r\cos\theta, r\sin\theta) \_\_\_\_$$

Ex:

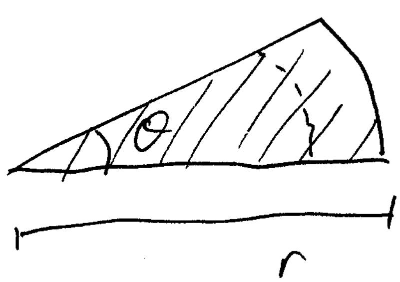


(r, θ)



$$(r^*, \theta^*) = \Big(\overbrace{\tfrac{1}{2}(r_{in} + r_{out})}^{= r_k}, \theta^*\Big)$$

$$\begin{cases} r_{in} = r_k - \Delta r_i/2 \\ r_{out} = r_k + \Delta r_i/2 \end{cases}$$



$$\text{Area} = \pi r^2 \cdot \frac{\theta}{2\pi} = \boxed{\frac{1}{2} r^2 \theta}$$

$$\begin{aligned}
\Delta A_{ij} &= \text{Area (outer sector)} - \text{Area (inner sector)} \\
&= \frac{1}{2} r_{out}^2 \cdot \Delta\theta - \frac{1}{2} r_{in}^2 \cdot \Delta\theta \\
&= \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta - \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta \\
&= \frac{\Delta\theta}{2}\left[\left(r_k^2 + r_k \Delta r + \frac{\Delta r^2}{4}\right) - \left(r_k^2 - r_k \Delta r + \frac{\Delta r^2}{4}\right)\right] \\
&= r_k \cdot \Delta r \, \Delta\theta
\end{aligned}$$

$$I = \lim_{\|P\| \to 0} \sum_{i,j} f(r^*\cos\theta^*, r^*\sin\theta^*) \cdot r_{ij} \Delta r_{ij} \Delta \theta_{ij}$$

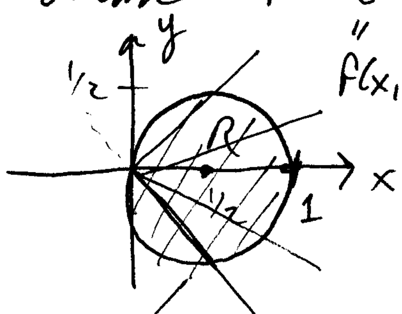
$$:= \boxed{\iint_{R_{pol}} f(r\cos\theta, r\sin\theta) \cdot \underline{\underline{r\,dr\,d\theta}}}$$

Ex: $\text{Area} = \iint_R r\,dr\,d\theta = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} r\,dr\,d\theta$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2}\left[r_2^2(\theta) - r_1^2(\theta)\right] d\theta$$

Ex: Volume of $\underset{\underset{f(x,y)}{\parallel}}{z} = xy$ over R:

$x = r\cos\theta$, $y = r\sin\theta$



$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

$$\text{Volume} = \iint_R f(x,y)\,dA$$

R_{rect}: $\begin{cases} 0 \le x \le 1 \\ -\sqrt{\frac{1}{4} - (x-\frac{1}{2})^2} \le y \le \sqrt{\frac{1}{4} - (x-\frac{1}{2})^2} \end{cases}$

R_{pol}: $\underline{r(\theta) = \cos\theta}$

$\begin{cases} \cancel{-\frac{\pi}{2} \le \theta \le \pi/2} \\ \cancel{0 \le \theta \le \pi} \\ 0 \le r \le \cos\theta \end{cases}$

$$\text{Volume} = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^2 \sin\theta\cos\theta \cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^3 \sin\theta\cos\theta\,dr\,d\theta$$

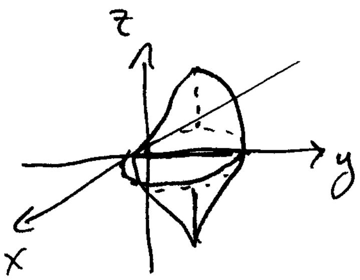
$$= \int_{-\pi/2}^{\pi/2} \left[\frac{1}{4} r^4 \sin\theta \cos\theta \right]_{r=0}^{\cos\theta} d\theta$$

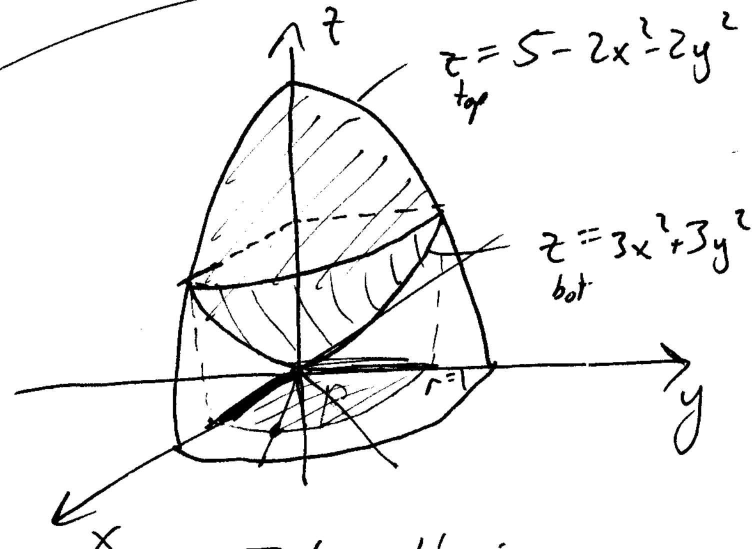
$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4} \cos^5\theta \sin\theta \, d\theta$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= \left[\frac{-1}{24} \cos^6\theta \right]_{\theta = \cancel{\pi/2}\ 0}^{\cancel{\pi/2}\ \pi} = \left[\left(-\frac{1}{24}\right) - \left(-\frac{1}{24}\right) \right] = \boxed{0}$$





Ex: Volume between these functions?

$$\text{Volume} = \iint_R z_{top} - z_{bot} \, dA$$

$$= \int_0^{\pi/2} \int_0^1 (5 - 5r^2) \cdot r \, dr \, d\theta$$

$$= \left(\int_0^{\pi/2} 1 \cdot d\theta \right) \left(\int_0^1 5r - 5r^3 \, dr \right)$$

$$= \left(\frac{\pi}{2} \right) \left[\frac{5}{2} r^2 - \frac{5}{4} r^4 \right]_{r=0}^{1} = \boxed{\frac{5\pi}{8}}$$

Intersection:

$5 - 2r^2 \qquad z_{top} = z_{bottom} \qquad 3r^2$

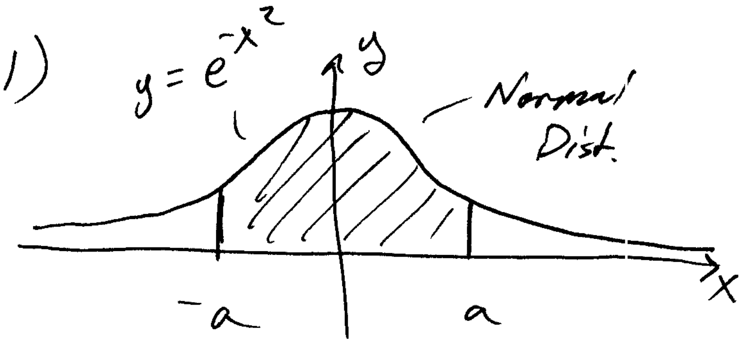
$$5 - 2x^2 - 2y^2 = 3x^2 + 3y^2$$

$$5 = 5(x^2 + y^2)$$

$$\boxed{1 = x^2 + y^2}$$

$$R : \begin{cases} 0 \le \theta \le \pi/2 \\ 0 \le r \le 1 \end{cases}$$

Ex: (Gaussian Integral)



$$G^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx\, dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} \cdot r \, dr \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left[-\frac{1}{2} e^{-r^2}\right]_{r=0}^{\text{"}\infty\text{"}}$$

$$= (2\pi)\cdot\frac{1}{2} = \pi = G^2$$

$$\boxed{G = \sqrt{\pi}}$$

$$\text{Area} = \int_{-a}^{a} e^{-x^2} dx$$

$$\frac{\text{Whole thing}}{\text{Area}} = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= G$$

$y = e^{-x^2}$ // not a PDF

$$\boxed{y = \frac{1}{\sqrt{\pi}} e^{-x^2}}$$ Formula for a normal dist.

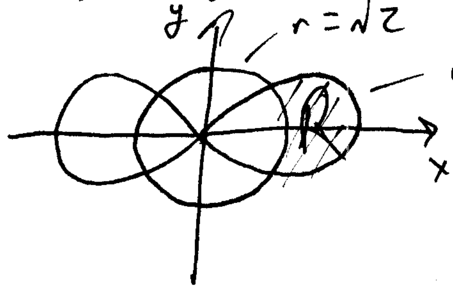
$\mu = 0$, $\sigma = \frac{1}{\sqrt{2}}$

Warm-up

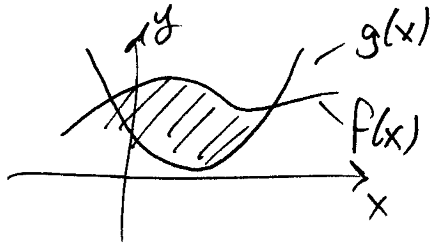
1) Set up an integral that evaluates the volume of the cone $(z-R)^2 = x^2+y^2$, $0 \le z \le R$.

2) Evaluate it!

3) Describe this region:



3

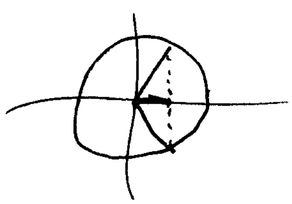


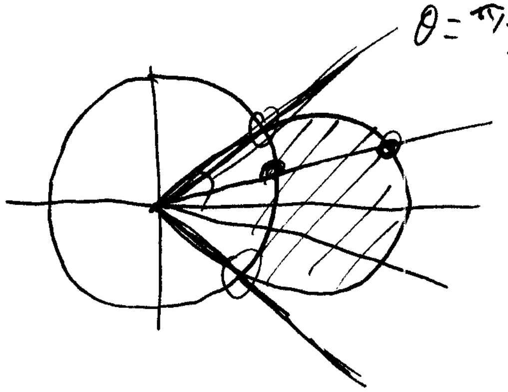
$$r=\sqrt{2} = \sqrt{4\cos 2\theta}$$

$$\frac{2}{4} = \cos 2\theta$$

$$\frac{1}{2}\underbrace{\cos^{-1}\left(\frac{1}{2}\right)} = \theta.$$

$$\theta = \frac{\pi}{6} \text{ or } \frac{11\pi}{6}$$

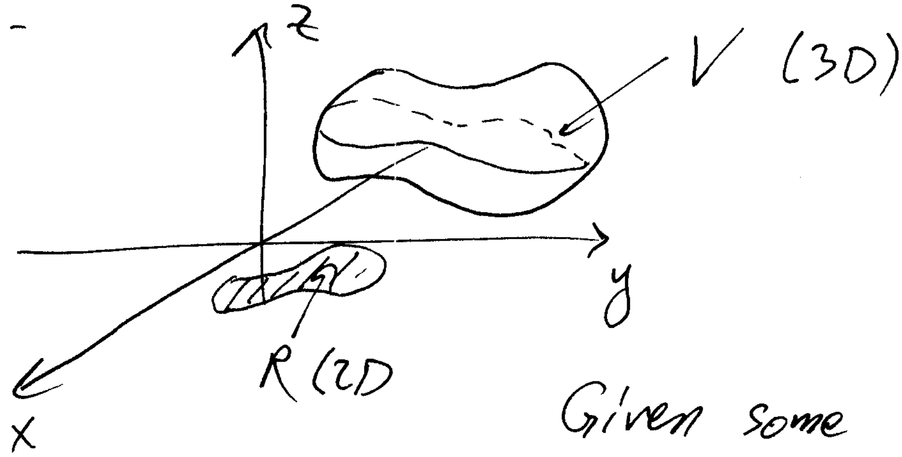




$$R: \begin{cases} -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \\ \sqrt{2} \le r \le \sqrt{4\cos 2\theta} \end{cases}$$

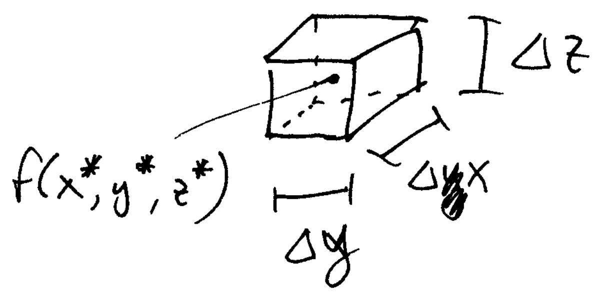
§ 15.5 – Triple Integrals

IDEA –



Given some $f(x,y,z) = w$.

① Cut up V into cubes (tiny):



$$\text{"Volume"} \approx \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*) \cdot \Delta x \, \Delta y \, \Delta z$$

$\Delta A = \Delta x \Delta y$

$\Delta V = \Delta x \Delta y \Delta z$

$$\lim_{\|P\| \to 0} \sum \cdots := \iiint_V f(x,y,z) \, dx \, dy \, dz$$

$$= \iiint_V f(x,y,z) \, dV.$$

Recall: If we can write ~~f(x,y)~~ R as

$$a \le x \le b$$
$$g(x) \le y \le h(x),$$

then $$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

In 3D: V can be written in a few (6) different ways.

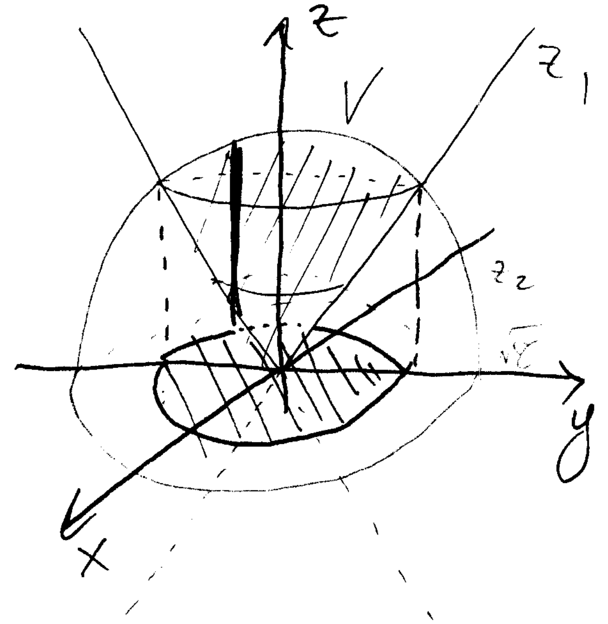
Say $$V: \begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \\ h_1(x,y) \le z \le h_2(x,y) \end{cases}.$$

Evaluating $$\iiint_V f(x,y,z)\,dV := \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Ex: Volume of an ice cream cone:

$$z_2 = \sqrt{8 - x^2 - y^2}$$

$$z_1 = \sqrt{x^2 + y^2}$$



$$\text{Volume} = \iiint_V 1 \cdot dV$$

$$= \iint_R \int_{\sqrt{x^2+y^2}}^{\sqrt{8-(x^2+y^2)}} dz \cdot dA$$

$$= \int_0^{2\pi}\int_0^{2}\int_r^{\sqrt{8-r^2}} dz \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r\left(\sqrt{8-r^2} - r\right) dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(-\frac{1}{3}(8-r^2)^{3/2} - \frac{1}{3}r^3\right)\Big|_{r=0}^{2}$$

$$= 2\pi \cdot \left[\left(-\frac{8}{3} - \frac{1}{3}8\right) - \left(-\frac{1}{3}8^{3/2} - 0\right)\right]$$

$$= \boxed{2\pi\left(\frac{8^{3/2}}{3} - \frac{16}{3}\right)}$$

$$V: \begin{cases} \sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2} \\ (x,y) \in R \end{cases}$$

$$z_1 = z_2$$

$$\sqrt{x^2+y^2} = \sqrt{8-x^2-y^2}$$

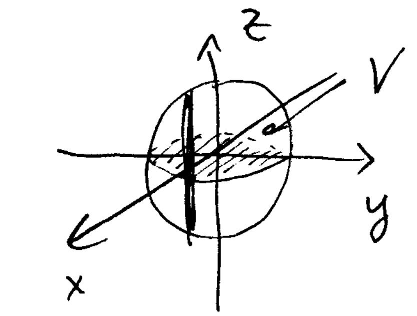
$$(x^2+y^2) = 8 - (x^2+y^2)$$

$$2(x^2+y^2) = 8$$

$$x^2+y^2 = 4$$

$$R: \begin{cases} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{cases}$$

Ex: Volume of a sphere: $x^2+y^2+z^2=R^2$



Volume $= \iiint_V dV$

$z = \pm\sqrt{R^2-x^2-y^2}$

$$V : \begin{cases} -\sqrt{R^2-x^2-y^2} \le z \le \sqrt{R^2-x^2-y^2} \\ 0 \le r \le R \\ 0 \le \theta \le 2\pi \end{cases}$$

$$= \int_0^{2\pi}\int_0^R \int_{-\sqrt{R^2-\underbrace{\cancel{(x^2+y^2)}}_{r^2}}}^{\sqrt{R^2-\underbrace{\cancel{(x^2+y^2)}}_{r^2}}} dz \cdot r\,dr\,d\theta$$

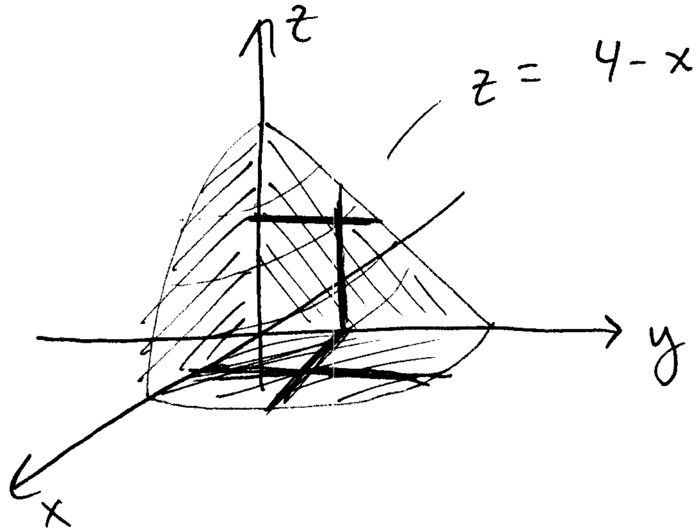
$$= \int_0^{2\pi}\int_0^R 2\sqrt{R^2-r^2}\; r\,\cancel{dz}\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^R 2r\sqrt{R^2-r^2}\,dr\,\cancel{d\theta}\right)$$

$$= 2\pi \cdot \left[-\frac{2}{3}(R^2-r^2)^{3/2}\right]_{r=0}^{R}$$

$$= +\frac{4\pi}{3}\cdot(R^2)^{3/2} = \boxed{\frac{4\pi}{3}R^3}$$

Ex: "Perspectives" (6 of 'em)



dz dy dx:

$$V : \begin{cases} 0 \le z \le 4 - x^2 - y \\ \text{Red} : \begin{cases} 0 \le y \le 4 - x^2 \\ 0 \le x \le 2 \end{cases} \end{cases}$$

$$z = 0 = 4 - x^2 - y$$

$$y = 4 - x^2 \Leftrightarrow x = \pm\sqrt{4 - y}$$

$$y = 0 = 4 - x^2$$

$$x^2 = 4$$

$$x = \pm 2$$

dz dx dy:

$$V : \begin{cases} 0 \le z \le 4 - x^2 - y \\ 0 \le x \le \sqrt{4 - y} \\ 0 \le y \le 4 \end{cases}$$

$\underline{dx\,dy\,dz:}$

$$x^2 = 4 - y - z$$

$$x = \pm\sqrt{4-y-z}$$

$$4 - y - z = 0$$

$$y = 4 - z$$

$$y = 0: \ 4 = z$$

$$V: \begin{cases} 0 \le x \le \sqrt{4-y-z} \\ 0 \le y \le 4 - z \\ 0 \le z \le 4 \end{cases}$$

$\underline{dx\,dz\,dy:}$

$$V: \begin{cases} 0 \le x \le \sqrt{4-y-z} \\ 0 \le z \le 4 - y \\ 0 \le y \le 4 \end{cases}$$

$\underline{dy\,dx\,dz:}$

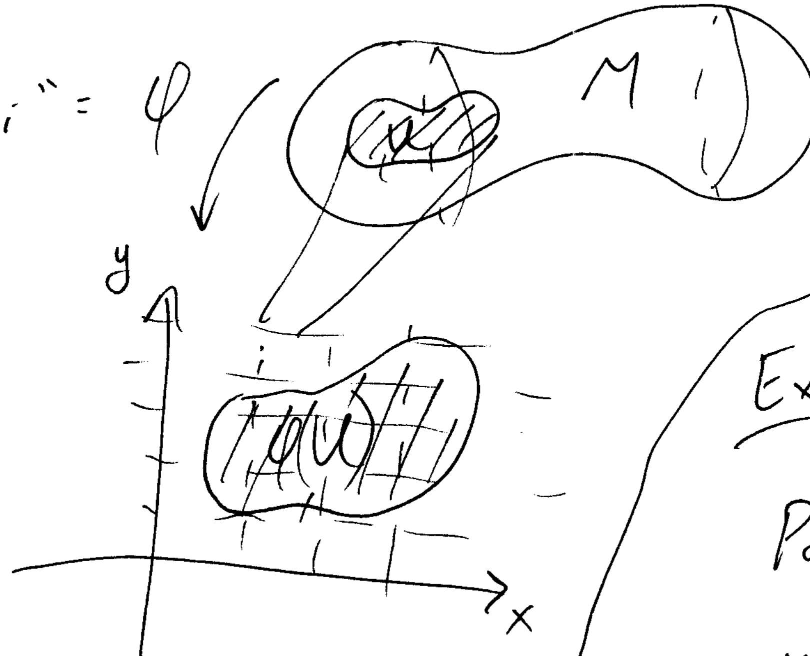
$$V: \begin{cases} 0 \le y \le 4 - x^2 - z \\ 0 \le x \le \sqrt{4-z} \\ 0 \le z \le 4 \end{cases}$$

$\underline{dy\,dz\,dx:}$

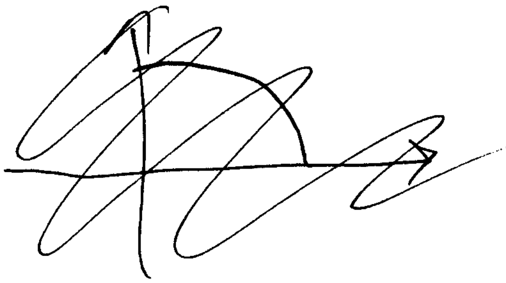
$$V: \begin{cases} 0 \le y \le 4 - x^2 - z \\ 0 \le z \le 4 - x^2 \\ 0 \le x \le 2 \end{cases}$$

§ 15.7 – Cyl / Sph. Coordinates

"phi" = φ



M a manifold



Example~~s~~:

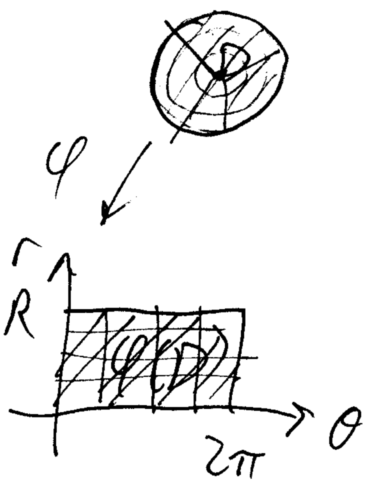
Polar coordinates

$M = \mathbb{R}^2$

Given coords (r, θ),

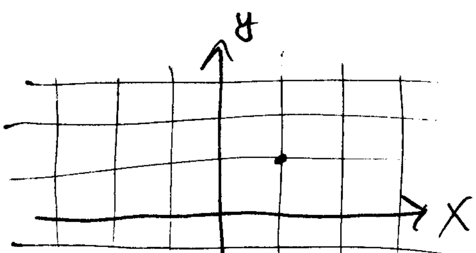
let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

φ^{-1} : Polar domain $\to$ Image in $\mathbb{R}^2$.

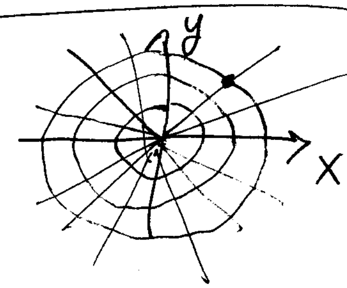


OVERARCHING IDEA :

Changing grid lines



Polar $\to$

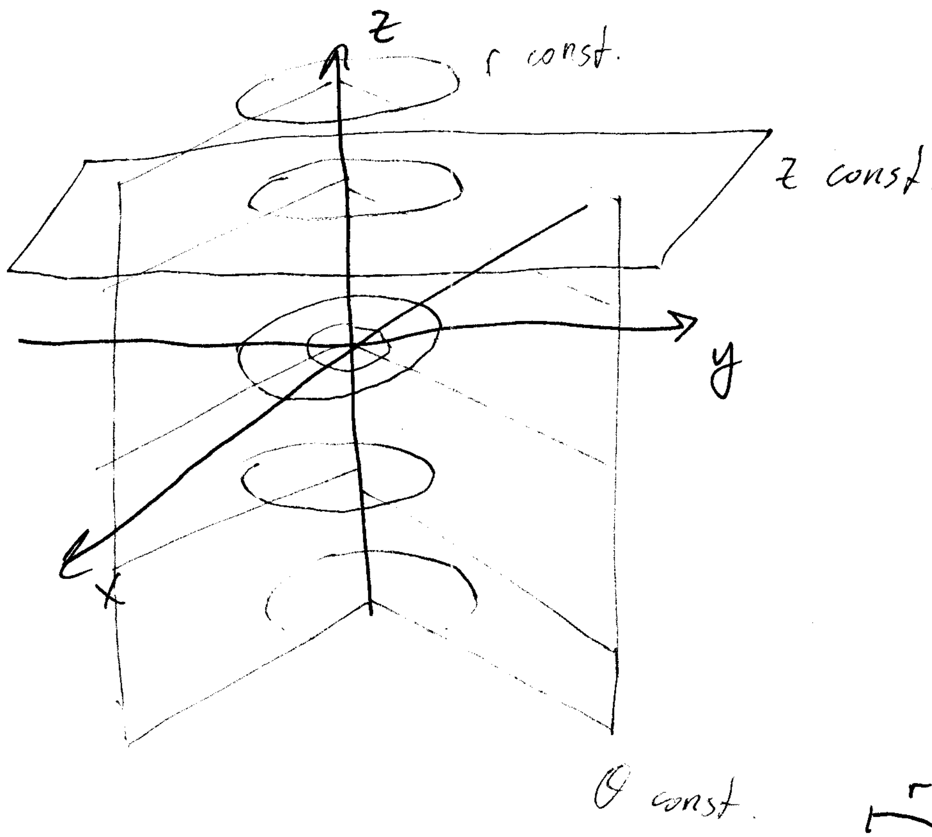


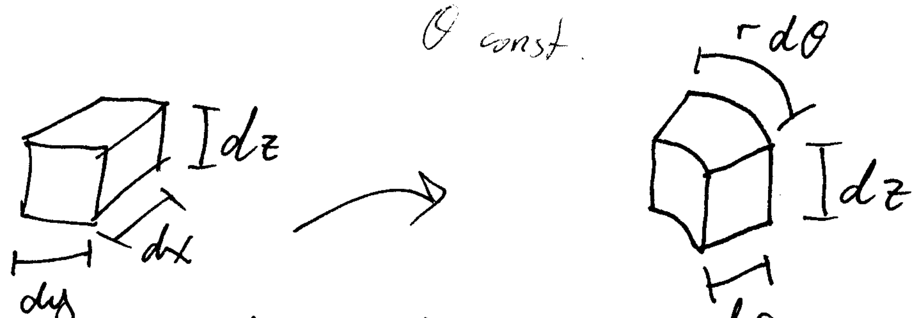
Some common coord. changes of $\mathbb{R}^3$

① <u>Cylindrical</u> : $(x,y,z) \longmapsto (r,\theta,z)$

$$\text{Cyl.} \to \text{Rect} \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{matrix} r \geq 0 \\ 0 \leq \theta \leq 2\pi \\ z \in \mathbb{R} \end{matrix}$$

$$\text{Rect.} \to \text{Cyl.} \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$





$dV = dx\,dy\,dz$

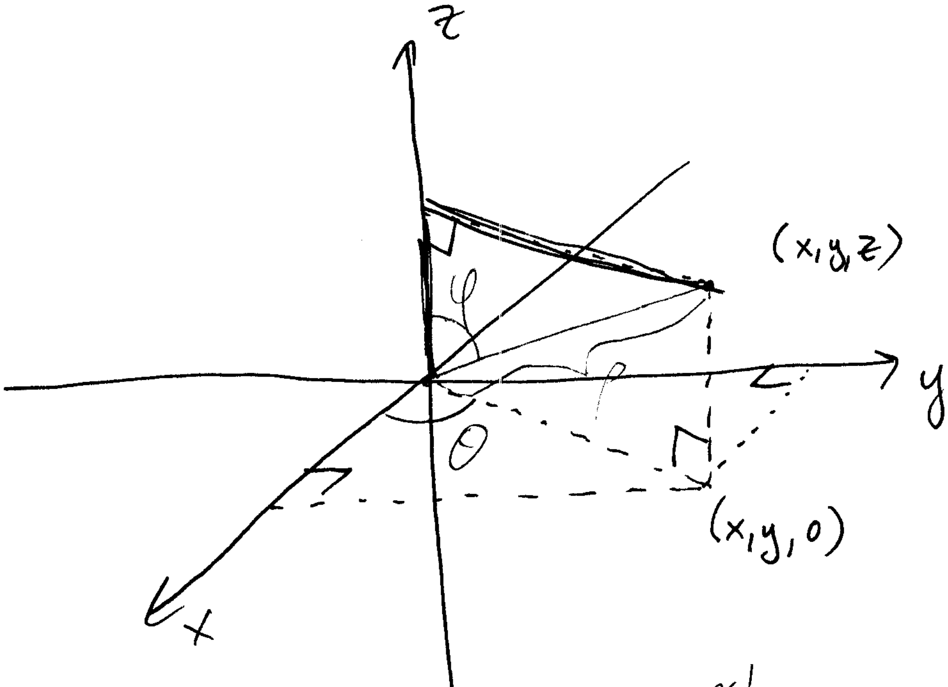
$= r\,dz\,dr\,d\theta$

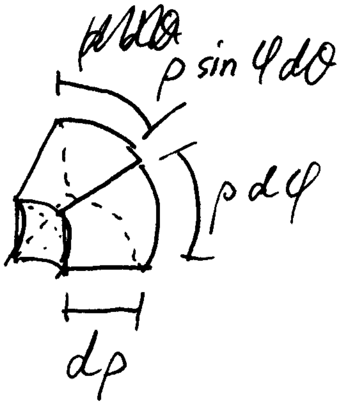
② Spherical : $(x, y, z) \longmapsto (\rho, \theta, \varphi)$

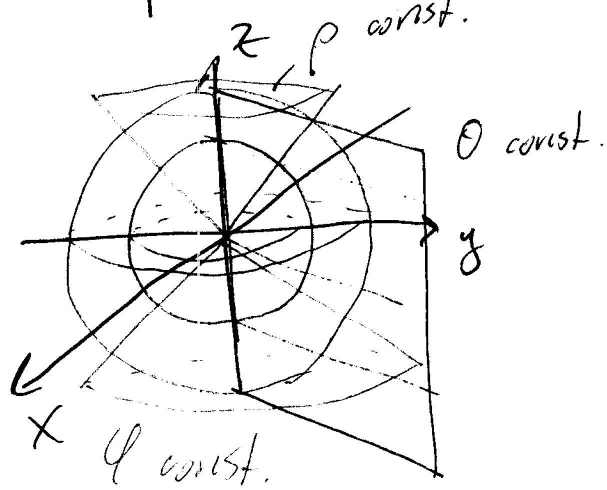
Sph. $\to$ Rect. $\begin{cases} x = (\rho \sin \varphi) \cos \theta \\ y = (\rho \sin \varphi) \sin \theta \\ z = \rho \cos \varphi \end{cases}$

$$\boxed{\begin{array}{l} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}}$$

Rect. $\to$ Sph. $\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(y/x) \quad \text{azimuth} \\ \varphi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \quad \text{polar} \end{cases}$



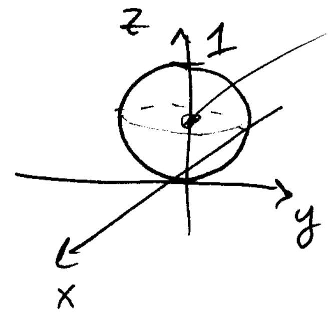




$dV = \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$

Ex: Draw the regions described.

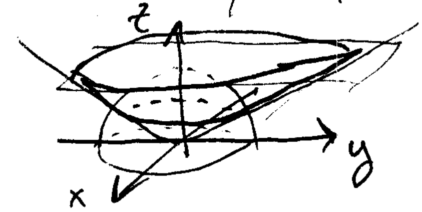
① $R: \begin{cases} 0 \le \rho \le \cos\varphi \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases}$



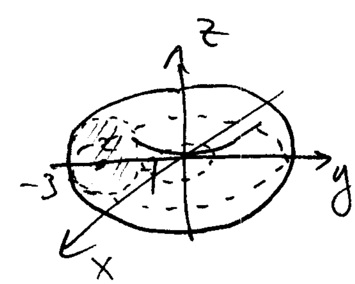
② $R: \begin{cases} 1 \le \rho \le \sec\varphi \\ 0 \le \varphi \le \pi/3 \\ 0 \le \theta \le 2\pi \end{cases}$

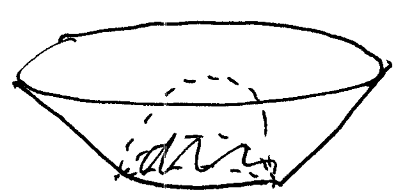
$\rho = 1 \to \rho = \sec\varphi$

$\rho\cos\varphi = 1 = z$



③ $R: \begin{cases} -\sqrt{1-(r-2)^2} \le z \le \sqrt{1-(r-2)^2} \\ 1 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{cases}$





upside-down water bowl

Warm-up

Draw the images for the objects defined by

(a) $r = \text{const.}$ (b) $\rho = \text{const.}$

$\theta = \text{const.}$ $\theta = \text{const.}$

$z = \text{const.}$ $\varphi = \text{const.}$

Also give a written explanation of what they are.

Describe the given regions.

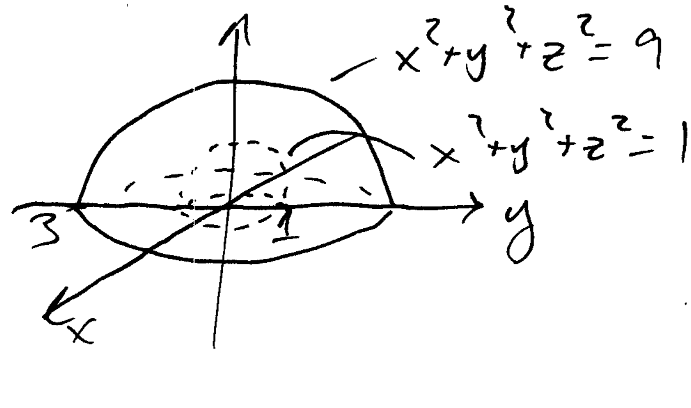
Ex: ① Between $z = \sqrt{27}$, and $x^2+y^2=z^2$.

(Cyl)

② Quarter cylinder : $x^2+y^2=R^2$, over

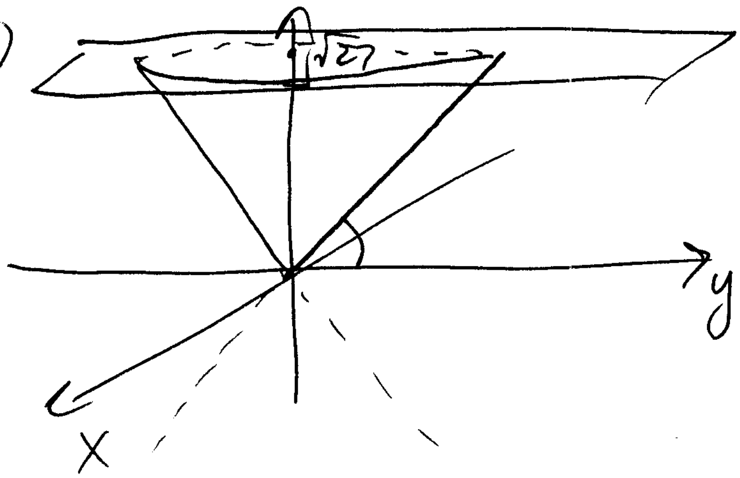
$$D : \begin{cases} 0 \le y \le \sqrt{R^2-x^2} \\ 0 \le x \le R. \end{cases}$$

③ The "slapper pogs":



$z \ge 0.$

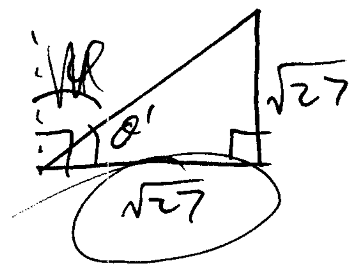
(1)



$$x^2+y^2=r^2$$

$$r^2=z^2$$

$$\underline{z=\pm r}$$



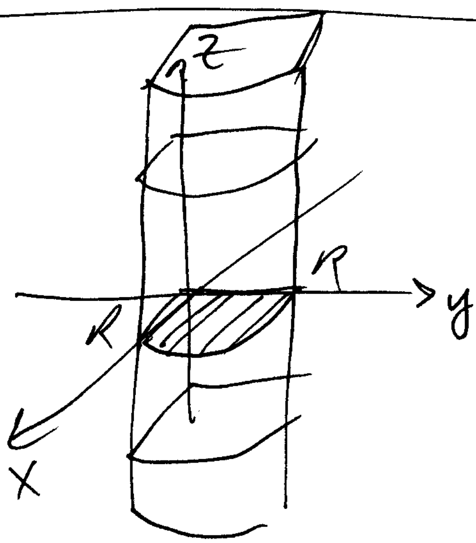
$$\tan\theta' = \frac{\sqrt{27}}{\sqrt{27}} = 1$$

$$\theta' = \pi/4$$

$$\begin{cases} r \le z \le \sqrt{27} \\ 0 \le r \le \sqrt{27} \\ 0 \le \theta \le 2\pi \end{cases}$$

---

(2)



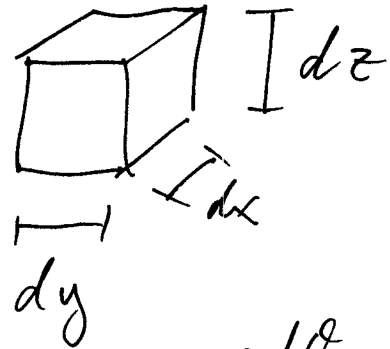
$$\begin{cases} -\infty < z < \infty \\ 0 \le r \le R \\ 0 \le \theta \le \pi/2 \end{cases}$$

---

(3)

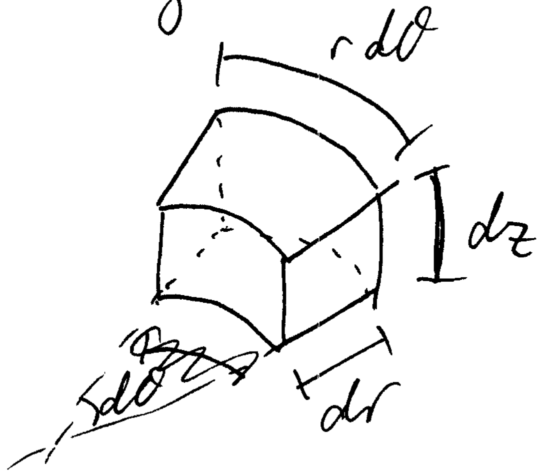
$$\begin{cases} 1 \le \rho \le 3 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/2 \end{cases}$$

"Recall" :



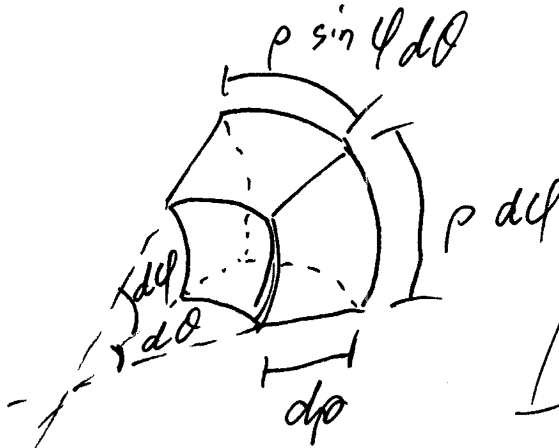
$$dV = dx\,dy\,dz$$

Cyl. :



$$dV = r\,dz\,dr\,d\theta$$

Sph :



$$dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Exactly ~~as~~ an "amplification factor".

Thm:

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V_{cyl.}} f(r,\theta,z)\cdot r\,dz\,dr\,d\theta$$

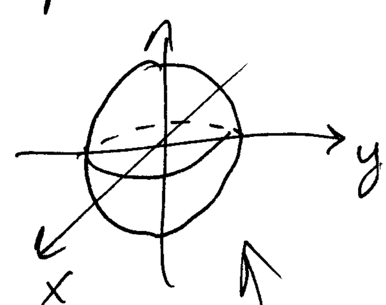
$$= \iiint_{V_{sph.}} f(\rho,\theta,\varphi)\cdot \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta.$$

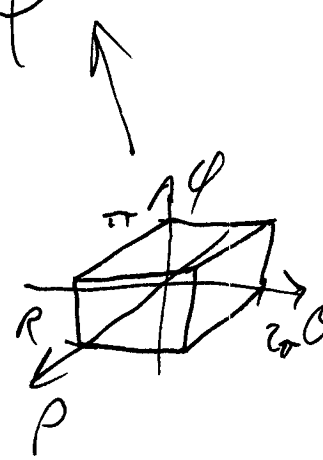
Ex: Volume of sphere: $x^2+y^2+z^2=R^2$.

$\rho^2 = R^2$.

$\rho = R$.

$$V: \begin{cases} 0 \le \rho \le R \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$$





$$\text{Volume} = \iiint_V dV$$

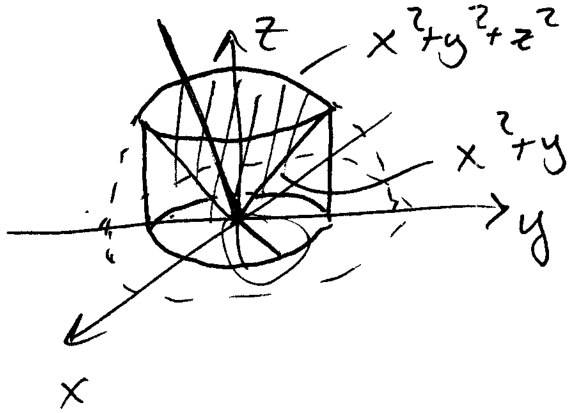
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 1\cdot \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi\,d\varphi\right)\left(\int_0^{R} \rho^2\,d\rho\right)$$

$$= 2\pi \cdot 2 \cdot \left[\tfrac{1}{3}\rho^3\right]_{\rho=0}^{R}$$

$$= \tfrac{4}{3}\pi R^3.$$

Ex: Ice cream cone (revisited)



$x^2+y^2+z^2=8 \to \rho^2 = 8 \to \rho = \sqrt{8}$

$x^2+y^2=z^2 \to \rho^2\sin^2\varphi = \rho^2\cos^2\varphi$

$\sin\varphi = \cos\varphi$

$\varphi = \pi/4$

$$V: \begin{cases} 0 \le \rho \le 2\sqrt{2} \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi/4 \end{cases}$$

$$\text{Volume} = \iiint_V dV$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/4} \sin\varphi \, d\varphi\right)\left(\int_0^{2\sqrt{2}} \rho^2 \, d\rho\right)$$

$$= 2\pi \left(\frac{2-\sqrt{2}}{2}\right)\left(\frac{16\sqrt{2}}{3}\right)$$

$$= \boxed{\frac{16\sqrt{2}\,(2-\sqrt{2})}{3}}$$

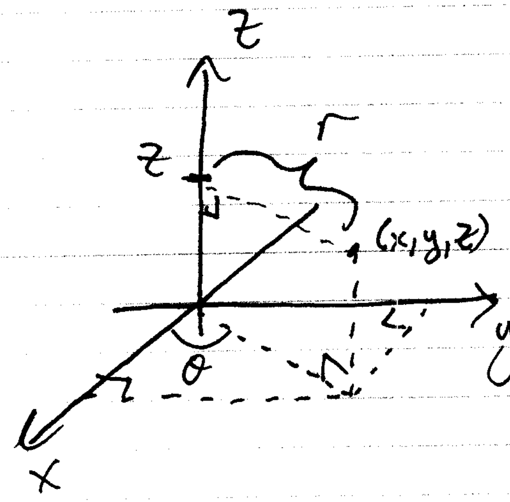
Coordinate change "Cheat Sheet"

Cylindrical:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

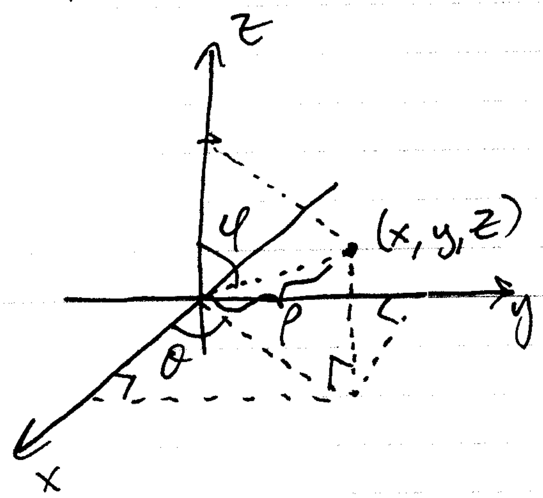
$$\begin{cases} r \geq 0 \\ 0 \leq \theta \leq 2\pi \\ z \in \mathbb{R} \end{cases}$$

$$dV = r\,dz\,dr\,d\theta$$



Spherical:

$$\begin{cases} x = (\rho\sin\varphi)\cos\theta \\ y = (\rho\sin\varphi)\sin\theta \\ z = \rho\cos\varphi \end{cases}$$



$$\begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{cases}$$

$$dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Two Themes

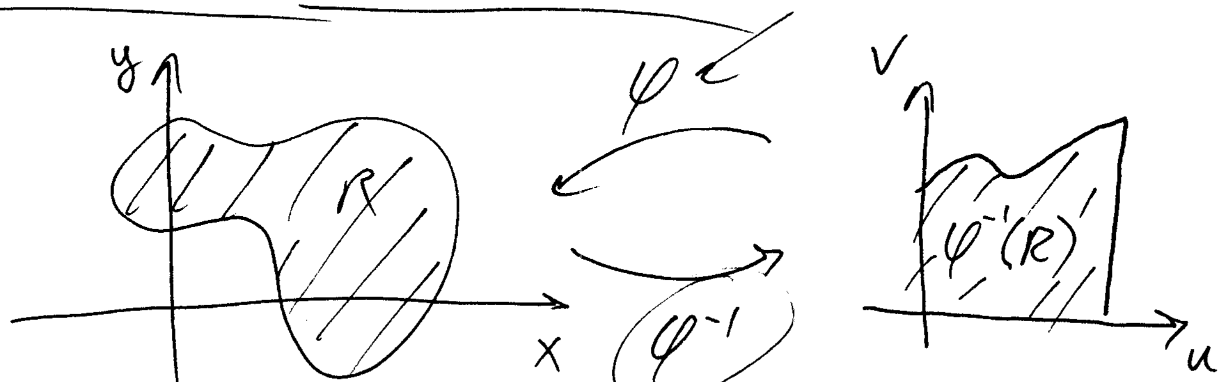
1. Change of Coordinates

2. Exchange of Integrals

---

§ 15.8 – Change of Coords

would like him

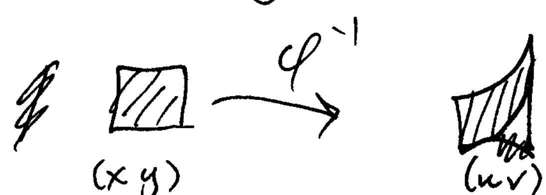


Bad (for some reason)

Better (for some other reason).

Usually given : E.g. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

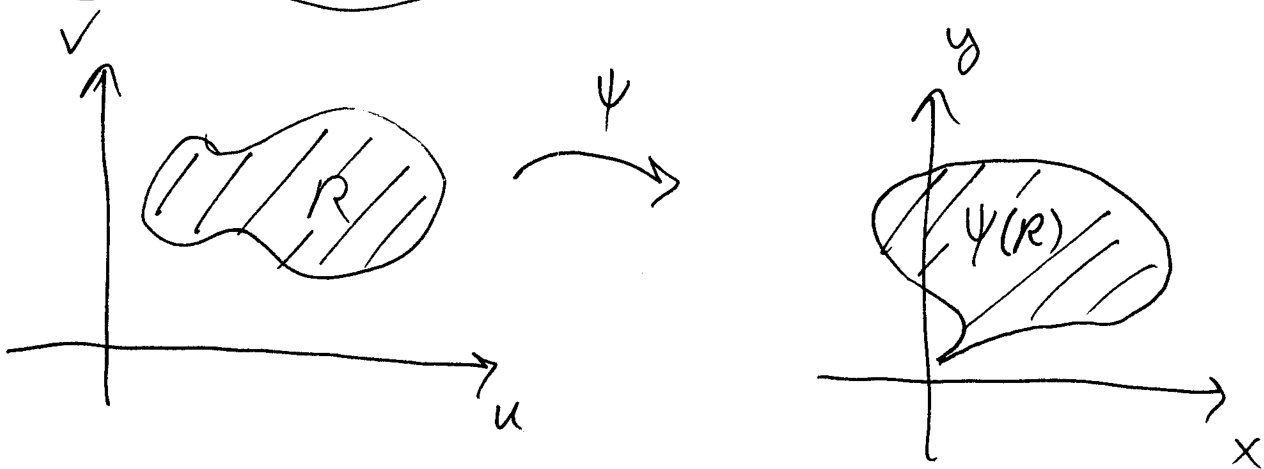
φ^{-1}:



φ:

IDEA: "Pullback" region to better coords.
(rather than "pushforward")

How?



Then $\displaystyle\iint_{\psi(R)} f(x,y)\,dx\,dy$

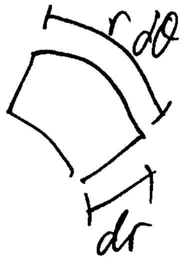
$$= \iint_R f(x(u,v), y(u,v)) \cdot \underline{\hspace{2cm}}\, du\,dv$$

what's this?

Definition: **Jacobian of ψ**

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

Ex: (Polar) $\psi(x,y) = (\overset{x}{\overset{\shortparallel}{r\cos\theta}}, \overset{y}{\overset{\shortparallel}{r\sin\theta}})$



$$\begin{cases} \dfrac{\partial x}{\partial r} = \cos\theta \quad , & \dfrac{\partial x}{\partial \theta} = -r\sin\theta \\[2ex] \dfrac{\partial y}{\partial r} = \sin\theta \quad , & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta)$$

$$= r(\cos^2\theta + \sin^2\theta) = r. \quad \checkmark$$

Theorem:

$$\iint_{\psi(R)} f(x,y)\,dx\,dy = \iint_R f(\psi^{-1}(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot du\,dv$$

No warm-up

Recall: Given coord. transf.

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

and given $I = \iint\limits_{R_{xy}} f(x,y)\,dx\,dy$.

then

Theorem: $\iint\limits_{R_{xy}} f(x,y)\,dx\,dy$

$$= \iint\limits_{R_{uv}} f(g(u,v), h(u,v)) \cdot \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{Jacobian determinant}} du\,dv.$$

Jacobian matrix $= \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

Determinants: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = |A| = ad - bc$.

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ cofactor expansion

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

— = +

— = −

$$\begin{aligned} &aei + bfg + cdh \\ &- ceg - afh - bdi \end{aligned} = \det(A).$$

Ex: (Cyl.) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r \checkmark$$

Ex: (Sph.) $\begin{cases} x = (\rho \sin\varphi)\cos\theta \\ y = (\rho \sin\varphi)\sin\theta \\ z = \rho\cos\varphi \end{cases}$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\overset{\cos}{\cancel{\sin}}\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

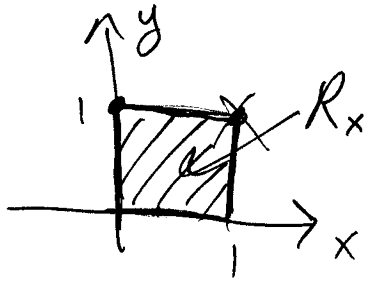
$$= \rho^2\sin\varphi$$

Two reasons to change coordinates

1. The integrand looks bad/awful/terrible/ugly/ like it needs a shower
2. The region over which we're integrating ...sucks

Ex:

$$I = \int_0^1 \int_0^1 \frac{1}{\sqrt{x^2 - y^2}}\, dy\, dx$$ (ignoring "improperness")



$R_{xy} = [0,1] \times [0,1]$.

Perhaps we can make

$$\begin{cases} u = x + y \\ v = x - y \end{cases}$$

How does (u,v) change R_{xy}?

4 functions for ∂R_{xy}: $y=0$, $y=1$, $x=0$, $x=1$.

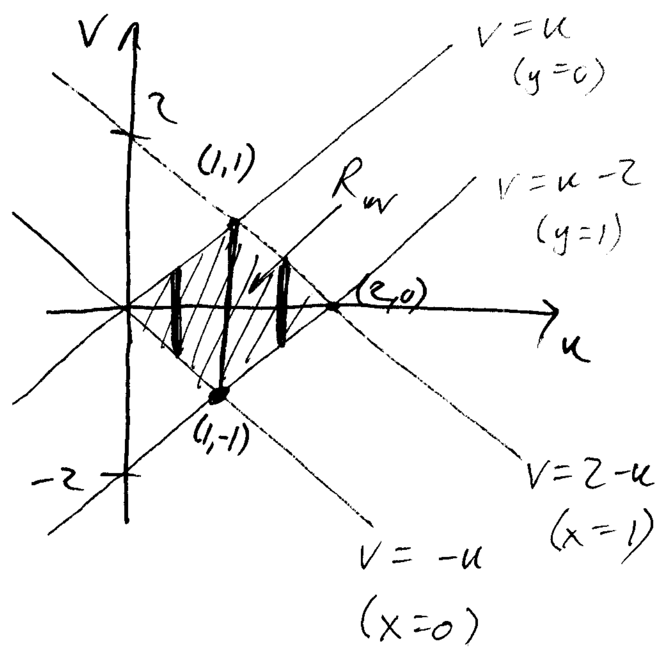
Take
$$\left.\begin{aligned} \frac{u+v}{2} &= \frac{(x+y)+(x-y)}{2} = x \\ \frac{u-v}{2} &= \frac{(x+y)-(x-y)}{2} = y \end{aligned}\right\}$$

$x = 0 \to \frac{u+v}{2} = 0 \to v = -u$

$x = 1 \to \frac{u+v}{2} = 1 \to v = 2-u$

$y = 0 \to v = u$

$y = 1 \to v = u - 2$



$$R_{uv}: \begin{cases} 0 \le u \le 1 \\ -u \le v \le u \end{cases}$$

&

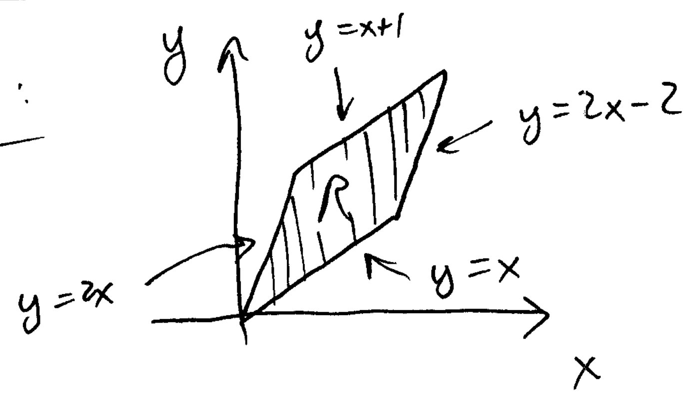
$$\begin{cases} 1 \le u \le 2 \\ u-2 \le v \le 2-u \end{cases}$$

Jacobian : $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$

$$I = \iint_{R_{xy}} \frac{1}{\sqrt{x^2-y^2}}\, dx\, dy = \int_0^1 \int_{-u}^{u} \frac{1}{\sqrt{uv}} \cdot \left|-\tfrac{1}{2}\right| \cdot dv\, du$$

$$+ \int_1^2 \int_{u-2}^{2-u} \frac{1}{\sqrt{uv}} \cdot \left|-1/2\right| \cdot dv\, du$$

Ex:



Evaluate

$$\iint_R xy \cdot dA$$

$y = 2x$, $y = 2x - 2$ $y = x$, $y = x + 1$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

$\underbrace{y - 2x}_{v} = 0$ $\underbrace{y - 2x}_{v} = -2$ $\underbrace{y - x}_{u} = 0$ $\underbrace{y - x}_{u} = 1$

Let $\begin{cases} u = y - x \\ v = y - 2x \end{cases} \rightarrow \begin{cases} x = (y-x) - (y-2x) = u - v \\ y = 2(y-x) - (y-2x) = 2u - v \end{cases}$

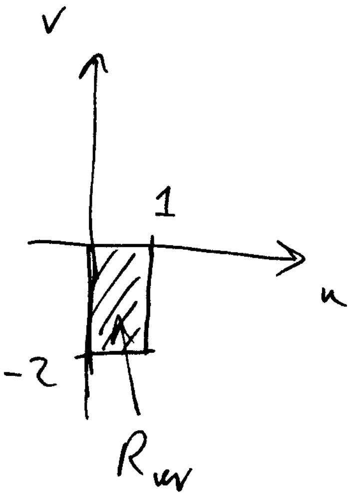
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\| \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \right\| = |-1-(-2)| = |1| = 1 \checkmark$$

So, $$\iint_{R_{xy}} x \cdot y \, dA = \int_{-2}^{0} \int_{0}^{1} (u-v)(2u-v) \cdot 1 \cdot du\, dv$$

$$= \int_{-2}^{0} \int_{0}^{1} (2u^2 - 3uv + v^2)\, du\, dv$$

$$\vdots$$

$$\boxed{7}$$



Quick review:

$$\begin{cases} u = y - x \\ v = y - 2x \end{cases} \longrightarrow \begin{array}{ccc} x & y & \text{const} \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 1 & u \\ -2 & 1 & v \end{array}\right]$$

$R1 \to -2R1$

$R2 \to R2 + R1$

$$\left[\begin{array}{cc|c} 2 & -2 & -2u \\ 0 & -1 & v-2u \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & -2 & -2u \\ -2 & 1 & v \end{array}\right]$$

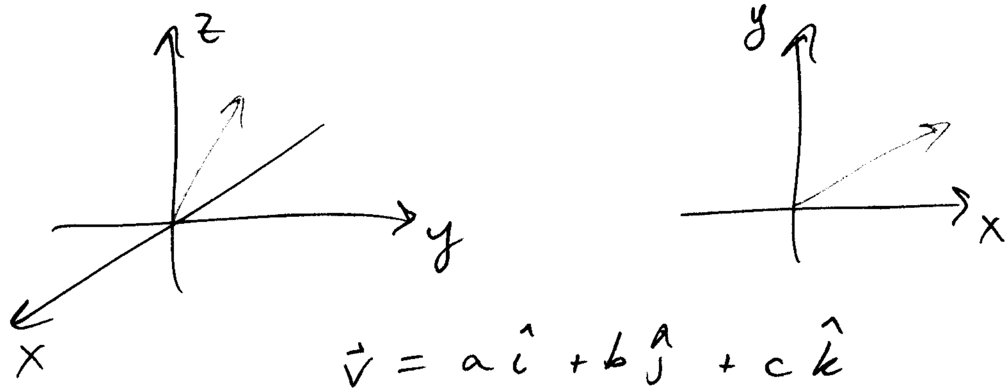
$$\begin{cases} 2x - 2y = -2u \\ \quad\quad\;\; y = 2u - v \end{cases}$$

$$2x - 2(2u - v) = -2u$$
$$2x - 4u + 2v = -2u$$
$$2x = 2u - 2v$$
$$x = u - v$$

Ch. 12 — Basic of (constant) vectors

Skip §12.1

§12.2 — Vectors & their operations



$$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$$
$$= (a, b, c)$$

Operations: Add vectors (componentwise)

Scalar × (distributes const.)

$\vec{v} = (a,b,c)$ // Length: $|\vec{v}| = \sqrt{a^2+b^2+c^2}$
$= \|\vec{v}\|$

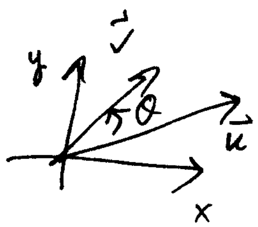
Direct: "Unitize"

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

§12.3 – Dot product

Given $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

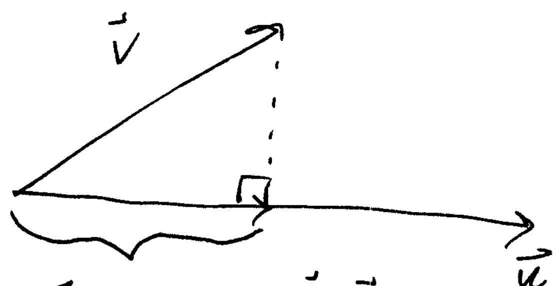
Define : $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$



$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

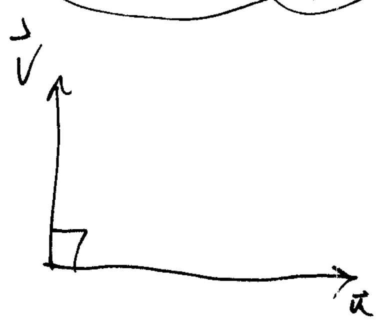
$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

Dot product measures "projectability":



$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u}$$

$$= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \hat{u}$$



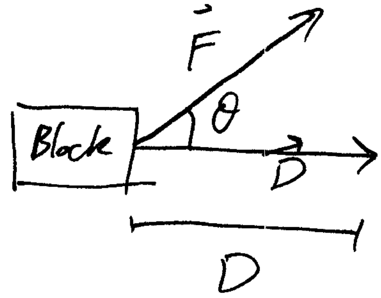
$\text{proj}_{\vec{u}} \vec{v} = \vec{0}$

$\Rightarrow \vec{u} \cdot \vec{v} = 0$

$\Updownarrow$ (orthogonal)

$\vec{u} \perp \vec{v}$

Work:



$$\text{Work} = \vec{F}\cdot\vec{D}$$
$$= |\vec{F}||\vec{D}|\cos\theta$$

Moral: Dot product $(\vec{u}\cdot\vec{v})$ measures how much of $\vec{u}$ points along $\vec{v}$ (and which direction)

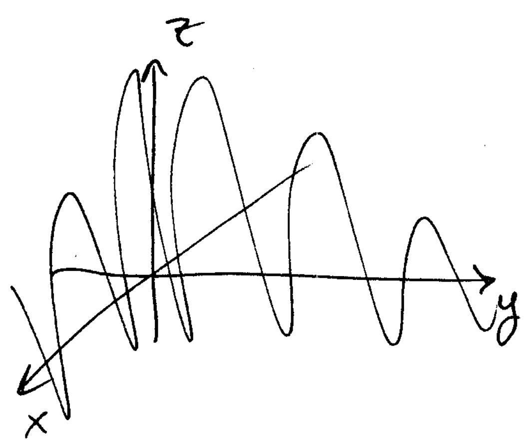
~~Chap~~ §12.4 – Cross product

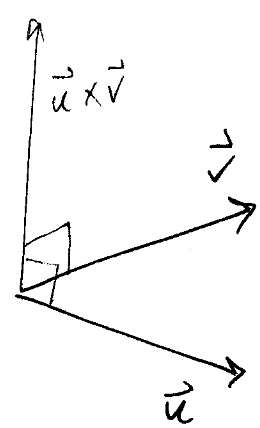
$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

Torque $\vec{F}\times\vec{r}$

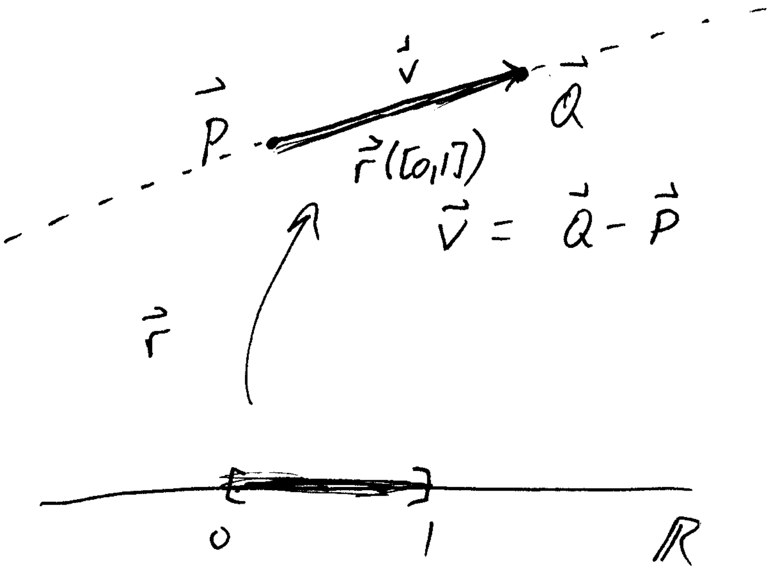
$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1u_2 - u_2v_1)$$





§ 12.5 - Lines



① $\vec{r}(t) = \cancel{Q(t-v)}$

$(1-t)\vec{P} + t\,\vec{Q}$

$t \in \mathbb{R}$

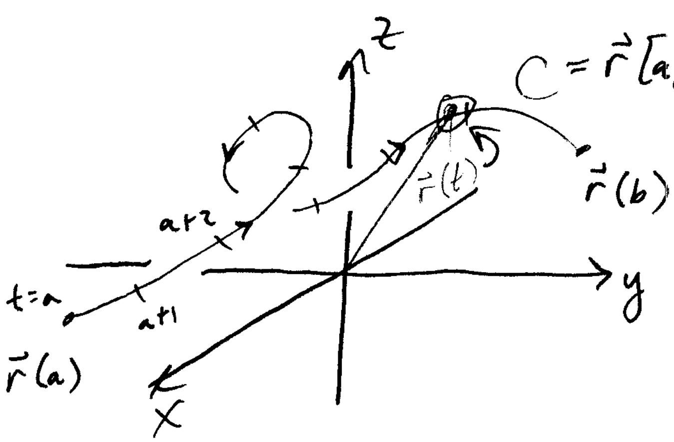
Line segment $\overline{PQ}$ is $\vec{r}([0,1])$

② $\vec{r}(t) = \vec{P} + t\,\vec{v}$

$t \in \mathbb{R}$

---

§ 13.1 - Vector-valued functions and their derivatives

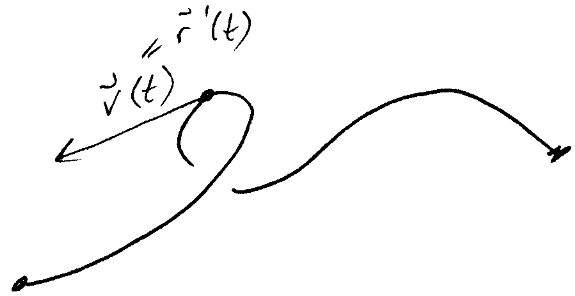


$C = \vec{r}[a,b]$

$$C : \begin{cases} \vec{r}(t) = (f_1(t), f_2(t), f_3(t)) \\ t \in [a,b] \end{cases}$$

Position / Displacement vector

velocity & acceleration



$$\vec{r}\,'(t) = (f_1'(t),\ f_2'(t),\ f_3'(t))$$
$$= \vec{v}(t) \quad \text{(velocity)}$$

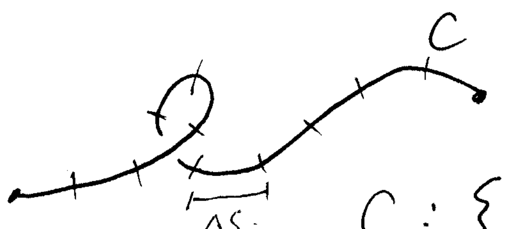
$$\vec{r}\,''(t) = (f_1''(t),\ f_2''(t),\ f_3''(t))$$
$$= \vec{a}(t) = \vec{v}\,'(t).$$
(acceleration)

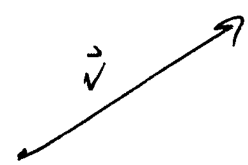
$$\vec{v} = |\vec{v}|\ \hat{v}$$

speed; direction of motion

---

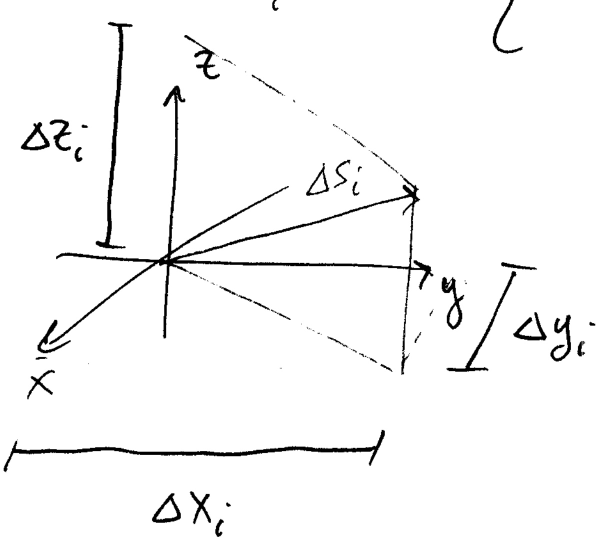
§ 13.3 - Arclength





$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$C: \begin{cases} \vec{r}(t) = (x(t), y(t), z(t)) \\ t \in [a, b] \end{cases}$$



$$\Delta s_i^2 = \Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2 \quad (\ast)$$

$$\text{Length}(C) \approx \sum_{i=1}^{N} \Delta s_i$$

$$\boxed{\lim_{N\to\infty} \sum_{i=1}^{N} \Delta s_i = \int_C ds}$$

$$\iint_R f(x,y)\,dA \quad // \quad \int_C ds = \int_a^b \underline{\qquad}$$

ds, parametrized by $\vec{r}(t)$.

$$C: \begin{cases} \vec{r}(t) = (x(t), y(t), z(t)) \\ t \in [a,b] \end{cases}$$

From $(*)$: $ds^2 = dx^2 + dy^2 + dz^2$ [metric]

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \sqrt{dt^2\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right)}$$

$\vec{v}(t) = \vec{r}\,'(t)$

$= (x'(t), y'(t), z'(t))$

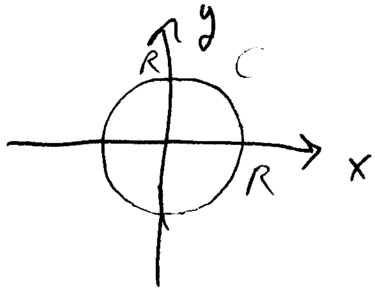
$$= \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{|\vec{v}(t)| = |\vec{r}\,'(t)| = v(t)}\, dt$$

$$\boxed{\int_C ds = \int_a^b v(t)\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt}$$

//

Length of C

Ex: Circumference of a circle:



From polar –

$$\vec{r}_1(t) = \begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ 0 \le \theta \le 2\pi \end{cases} \quad \text{(standard)}$$

$$\vec{r}_3(t) = \begin{cases} x = R\cos(-2\theta) \\ y = R\sin(-2\theta) \\ 0 \le \theta \le \pi \end{cases} \qquad \vec{r}_2(t) = \begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ \pi \le \theta \le 3\pi \end{cases}$$

$$\underset{\vec{r}_1}{\mathrm{Length}}(C) = \int_0^{2\pi} \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} R\, d\theta = 2\pi R$$

$$\underset{\vec{r}_2}{\mathrm{Length}}(C) = \int_{\pi}^{3\pi} R\, d\theta = (3\pi - \pi)R = 2\pi R.$$

$$\underset{\vec{r}_3}{\mathrm{Length}}(C) = \int_0^{\pi} \sqrt{(2R\sin(-2\theta))^2 + (-2R\cos(-2\theta))^2} \cdot d\theta$$

$$= \int_0^{\pi} 2R\, d\theta = \pi \cdot 2R = 2\pi R.$$

Warm-up : Recover the "arclength formula" (MAT 21B) for a curve $y = f(x)$ over $a \le x \le b$ by parametrizing it as $\begin{cases} \vec{r}(t) = (t, f(t)) \\ a \le t \le b \end{cases}$.

Distance = Rate × Time

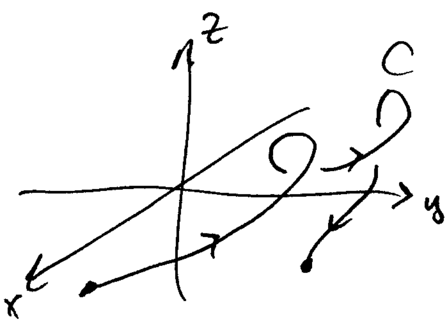
arclength speed parameter

$$\vec{v}(t) = \vec{r}\,'(t) = (1, f'(t))$$

$$v(t) = \sqrt{1^2 + (f'(t))^2}$$

$$\boxed{L = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt}$$

Continue §13.3 – Arc Length



C: $\vec{r}(t)$, $a \le t \le b$

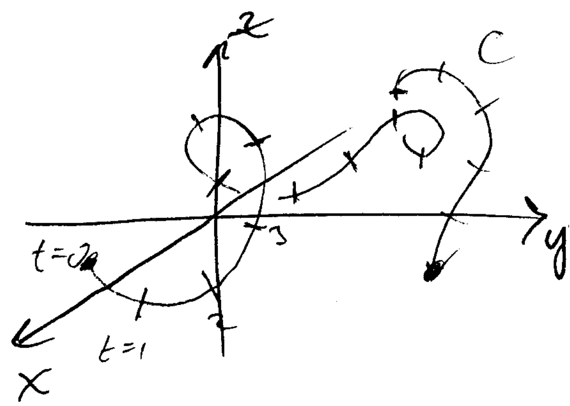
NO ① Do integrals depend on which $\vec{r}(t)$ we use?

Yes ② Is there some "preferred" $\vec{r}(t)$?

Answer ② : Arclength parametrization

function (based $t=t_0$)

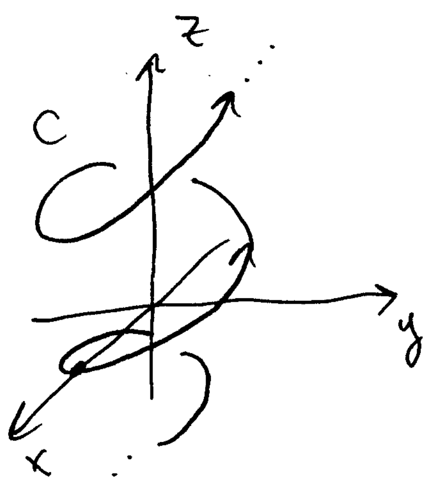
$$s(t) = \int_{t_0}^{t} |\overset{\vec{v}(u) = \vec{r}\,'(u)}{\vec{v}(u)}|\, du$$



something $= \int d(\text{something})$

Ex: Helix (C) $- \begin{cases} \vec{r}(t) = (\cos at, \sin at, bt) & (a,b>0) \\ t \in \mathbb{R} \end{cases}$

Based at $t_0 = 0$ $\quad (\vec{r}(0) = (1,0,0))$.



$$s(t) = \int_0^t v(u)\, du$$

$\vec{r}\,'(t) = (-a \sin at, a \cos at, b)$

$$= \int_0^t \sqrt{a^2+b^2}\, du$$

$v(t) = \sqrt{a^2+b^2}$

$$= t\sqrt{a^2+b^2}$$

$$s(t) = c\,t \quad (c = \sqrt{a^2+b^2})$$

Arclength param : $\quad C : \begin{cases} \vec{r}(t(s)) \\ s \text{ does something} \end{cases}$

Replacing ~~the~~ t by s in some given $\vec{r}(t)$.

Ex: Helix (revisited)

We have one param. $C: \vec{r}(t),\ t \in \mathbb{R}$.

$= (\cos at, \sin at, bt)$

Goal: Replace t w/ s.

Remark: $s(t) = \int_{t_0}^{t} v(u)\, du$

$$\frac{ds}{dt} = v(t)$$

$t(s) = s^{-1}(t)$ // $\left(\vec{r}(t(s))\right)' = \vec{r}\,'(t(s)) \cdot t'(s)$

$\Downarrow$ $\qquad = \vec{r}\,'(t(s)) \cdot \frac{1}{s'(t)}$

$t'(s) = \frac{1}{s'(t)} > 0$ $\qquad \left\| \frac{d}{dt}(\vec{r}(t(s))) \right\| = |\vec{r}\,'(t(s))| \cdot \frac{1}{v(t)}$

$$= \frac{v(t)}{v(t)} = 1.$$

For helix:

Use $s(t) = ct$

$s^{-1}(t) = t(s) = \frac{s}{c}$

$$\vec{r}(t(s)) = \left(\cos \frac{as}{c}, \sin \frac{as}{c}, \frac{bs}{c}\right)$$

for all s $\in \mathbb{R}$.

Q: Does an arclength param. always exist?

A: Yes (if $v(t) > 0$ or ~~$v(t) < 0$~~)

Curves w/ such param. are called regular.

Remark: ① Yes they exist, but they're often intractible for computation.

② This method yields a unit-speed curve from some other param.

Unit tangent vector

Given $\vec{r}(t)$, we know $\vec{v}(t) = \vec{r}\,'(t)$.

Define $\vec{T}(t) = \hat{v}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\vec{v}(t)}{v(t)}$.

Interpretations: ① ~~veloc~~ Velocity at the arclength param.

② Direction of travel in any param.

Ex: Helix (one last time) — $\vec{r}\,'(t) = (-a \sin at, a \cos at, b)$

$v(t) = c$

$\vec{T}(t) = \hat{v}(t) = \left(\frac{-a}{c} \sin at, \frac{a}{c} \cos at, \frac{b}{c} \right)$

<u>Warm-up</u>: Find the tangent vector for the curve

$$C: \begin{cases} \vec{r}(t) = (t, t^2, t^3) \\ t \in \mathbb{R} \end{cases}$$

Use that information to compute the arc length function

$s(t) = \int_0^t v(u)\,du$. <u>What is $s'(t) = \frac{ds}{dt}$</u>?

$$\vec{v}(t) = \vec{r}\,'(t) = (1, 2t, 3t^2) \quad // \quad v(t) = |\vec{r}\,'(t)|$$

$$= \sqrt{1^2 + (2t)^2 + (3t^2)^2}$$

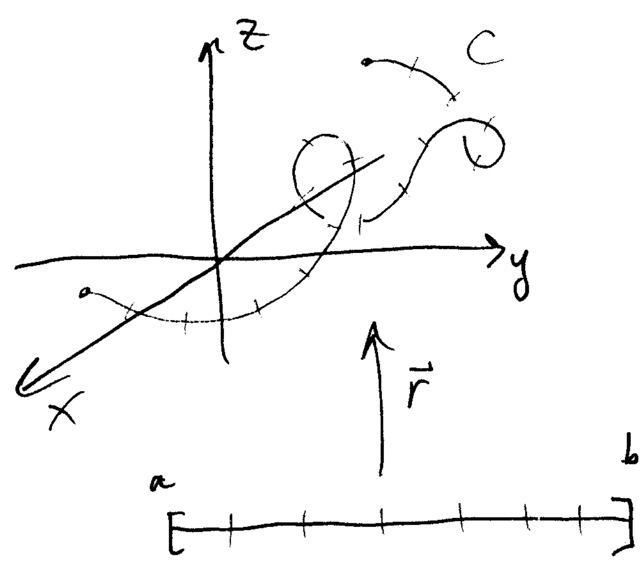
$$= \sqrt{1 + 4t^2 + 9t^4}.$$

$$\boxed{s(t) = \int_0^t \sqrt{1 + 4u^2 + 9u^4}\,du}$$

$$\boxed{s'(t) = \frac{ds}{dt} = v(t)}$$

---

§ 13.4 - Curvature



"Natural" parametrization:

Arclength (s) param.

$\{\vec{r}(s),\ 0 \le s \le T$

<u>Observe</u>: $\dfrac{d\vec{r}}{ds} = \dfrac{d\vec{r}}{dt} \cdot \dfrac{dt}{ds}$ // Chain rule = swaps param's

$$= \frac{d\vec{r}}{dt} \cdot \frac{1}{ds/dt}$$

$$= \frac{d\vec{r}/dt}{v(t)} = \frac{\vec{v}(t)}{v(t)} = \vec{T}(t)$$

Derivative already unit $\Rightarrow$ nice/natural param.

<u>Definition</u>: Curvature of $\vec{r}(s)$:

$$K(s) = \left|\frac{d\vec{T}}{ds}\right|$$ (interpret: how hard the curve curves)

To compute: $\dfrac{d\vec{T}}{ds} = \dfrac{d\vec{T}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{d\vec{T}/dt}{v(t)}$

$$K(t) = \frac{1}{v(t)} \left|\frac{d\vec{T}}{dt}\right|$$

$(|v(t)| = v(t) > 0.$

Ex: Circles (should be "round", right?)

(should have $K = \text{const} > 0$)

$C: \vec{r}(t) = (a\cos bt,\ a\sin bt)$ $(a, b > 0)$

$0 \le t \le 2\pi$

$\vec{r}\,'(t) = (-ab\sin bt,\ ab\cos bt)$

$v(t) = ab$, $\vec{T}(t) = (-\sin bt,\ \cos bt)$.

$\vec{T}\,'(t) = (-b\cos bt,\ -b\sin bt)$

$\left|\frac{d\vec{T}}{dt}\right| = b$, $K = \frac{1}{v(t)}\left|\frac{d\vec{T}}{dt}\right| = \frac{b}{ab} = \frac{1}{a}$.

We know how hard C is turning...

Now where is it turning?

Use direction of $\frac{d\vec{T}}{ds}$ ... Why?

$\vec{T}\cdot\vec{T} = |\vec{T}|^2 = 1^2 = 1$

$\frac{d}{ds}(\vec{T}\cdot\vec{T}) = \frac{d}{ds}(1) = 0$

$2\vec{T}\cdot\frac{d\vec{T}}{ds} = 0/2 = 0$ // $\vec{T} \perp \frac{d\vec{T}}{ds}$

Compute $\widehat{\left(\frac{d\vec{T}}{ds}\right)} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} \left(= \frac{d\vec{T}/ds}{K(s)} \right)$

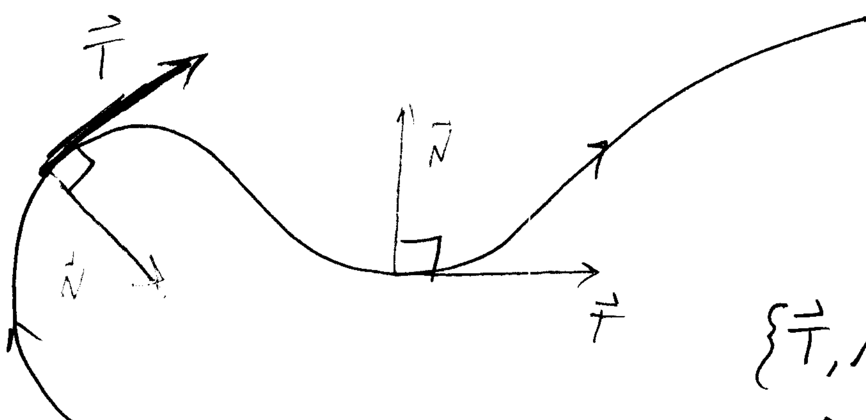
$$= \frac{d\vec{T}/dt \cdot dt/ds}{|d\vec{T}/dt \cdot dt/ds|}$$

$$= \frac{d\vec{T}/dt}{|d\vec{T}/dt|} \cdot \cancel{\frac{dt/ds}{dt/ds}}$$

$$= \widehat{\left(\frac{d\vec{T}}{dt}\right)}$$

Definition: (Unit) Normal vector for $C : \vec{r}(t)$

$$\vec{N} = \widehat{\left(\frac{d\vec{T}}{ds}\right)} = \widehat{\left(\frac{d\vec{T}}{dt}\right)}$$



$\vec{N}$ points in the direction of concavity.

$\{\vec{T}, \vec{N}\}$ define a sort of tangent plane

Definition - Osculating plane

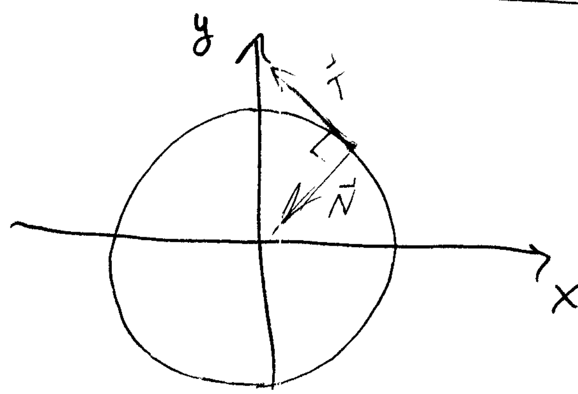
"Kissing" or "Tangent"

Ex: Back to the circle: $\vec r(t) = (a\cos bt, \cancel{a}\sin bt)$

$\vec T(t) = (-\sin bt, \cos bt)$.

$\widehat{(\vec T'(t))}$ ... $\vec T'(t) = (-b\cos bt, -b\sin bt)$

$$\boxed{\vec N(t) = \widehat{(\vec T'(t))} = (-\cos bt, -\sin bt)}$$



If $a=1$,

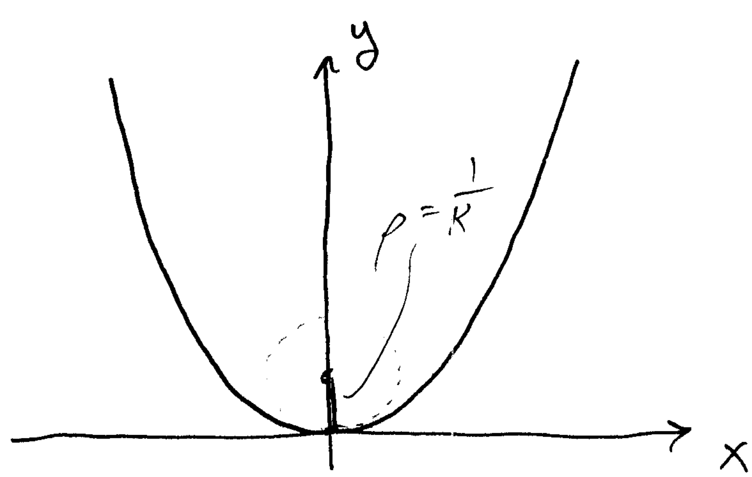
$\vec N = -\vec r$

Osculating plane/circle

Recall: Circle w/ rad R has $\kappa = \frac{1}{R}$.

Define radius of an osculating circle as $\rho = \frac{1}{\kappa}$.

radius of curvature.



Center of o.c.:

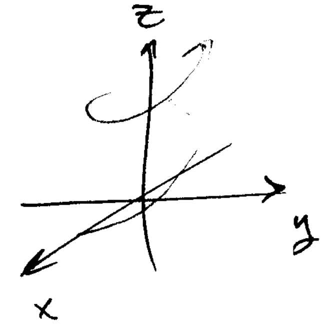
$\vec r(s) + \frac{d\vec T}{ds}$

$= \vec r(s) + \frac{1}{\kappa}\cdot \vec N$

Ex: Helix and his osculating circles

$\vec{r}(t) = (a\cos t, a\sin t, bt)$ $\quad (a, b > 0)$

$t \in \mathbb{R}$



$\vec{r}\,'(t) = (-a\sin t, a\cos t, b)$

$\vec{T}(t) = \left(-\frac{a}{c}\sin t, \frac{a}{c}\cos t, \frac{b}{c}\right)$ // $c = \sqrt{a^2+b^2}$

$\vec{T}'(t) = \left(-\frac{a}{c}\cos t, -\frac{a}{c}\sin t, 0\right)$

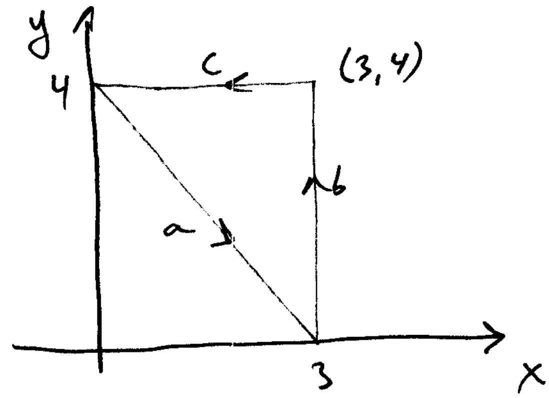
$\widehat{(\vec{T}'(t))} = \vec{N} = (-\cos t, \sin t, 0)$

Define the osculating plane.

$K = \frac{c}{a} \Rightarrow \rho = \frac{a}{c}$

center $= \vec{r}(s) + \rho \frac{d\vec{T}}{ds} = \boxed{\vec{r}(t) + \frac{\rho}{v(t)} \cdot \vec{T}'(t)}$

Warm-up : Parametrize the triangle below.



Counter-clockwise

b) End: (3,4)
Starts: (3,0)
$$\begin{cases} \vec{r}(t) = (3,0) + t(0,4) = (3, 4t) \\ 0 \le t \le 1 \end{cases}$$

c) End: (0,4)
Starts: (3,4)
$$\begin{cases} \vec{r}(t) = (3,4) + t(-3,0) = (3-3t, 4) \\ 0 \le t \le 1 \end{cases}$$

a) End: (3,0)
Starts: (0,4)
$$\begin{cases} \vec{r}(t) = (0,4) + t(3,-4) = (3t, 4-4t) \\ 0 \le t \le 1 \end{cases}$$

§ 13.5 – Components of Acceleration

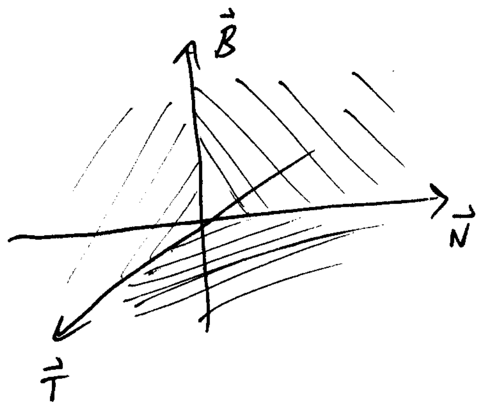
Recall: $\vec{T}$ unit tangent : $\vec{T} = \widehat{(\vec{r}\,')}$

$\vec{N}$ unit normal : $\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{dt} = \widehat{\left(\frac{d\vec{T}}{ds}\right)}$

New vector : $\vec{B} = \vec{T} \times \vec{N}$ unit binormal

Together : $\{\vec{T}, \vec{N}, \vec{B}\}$ are called Frenet frame.

Also called the "moving trihedral."



$\vec{T}, \vec{N}$: Osculating

$\vec{N}, \vec{B}$: Normal

$\vec{B}, \vec{T}$: Binormal

The book calls the Binormal Plane the "Rectifying Plane."

Theme – Make all measurements of a curve C independent of param. used (at least compare ~~it~~ them to a standard).

Breaking up acceleration $\vec{a}(t) = \vec{r}\,''(t)$

Recall: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} = \frac{ds}{dt}\vec{T} \quad (= v(t)\,\vec{T})$

Observe:

$$\frac{d}{dt}\left(\vec{v}(t) = \frac{ds}{dt} \vec{T} \right)$$

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{ds}{dt}\right) \vec{T} + \frac{ds}{dt} \cdot \frac{d}{dt}(\vec{T})$$

$$= \frac{d^2 s}{dt^2} \vec{T} + \frac{ds}{dt}\left(\frac{d\vec{T}}{dt}\right)$$

$$= \frac{d^2 s}{dt^2} \vec{T} + \frac{ds}{dt}\left(\frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}\right)$$

$$= \left(\frac{ds}{dt}\right)' \vec{T} + \left(\frac{ds}{dt}\right)^2 (\kappa \vec{N})$$

$$\vec{a}(t) = \underline{\left(\frac{d^2 s}{dt^2}\right)} \vec{T} + \left[\kappa \left(\frac{ds}{dt}\right)^2\right] \vec{N}$$

Definition - $\vec{a}(t) = a_T \vec{T} + a_N \vec{N}$ where

$$a_T = v'(t) = \frac{d^2 s}{dt^2}$$

$$a_N = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa [v(t)]^2$$

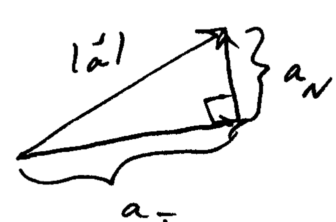
Nifty trick: $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$

$$= (a_T \vec{T} + a_N \vec{N}) \cdot (a_T \vec{T} + a_N \vec{N})$$

$$= a_T^2 \vec{T} \cdot \vec{T} + 2 a_T a_N \vec{T} \cdot \vec{N} + a_N^2 \vec{N} \cdot \vec{N}$$

$$= a_T^2 + a_N^2$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$$



Ex: Circle - $C : \begin{cases} \vec{r}(t) = (\cos t, \sin t) \\ 0 \le t \le 2\pi \end{cases}$

$$\vec{r}\,'(t) = (-\sin t, \cos t) = \vec{T}, \quad \vec{r}\,''(t) = (-\cos t, -\sin t) = \vec{N}$$

$$|\vec{a}| = 1, \quad v(t) = 1, \quad v'(t) = 0.$$

$$a_T = 0.$$

$$a_N = \sqrt{1^2 - 0^2} = 1.$$

$$\boxed{\vec{a} = \vec{N}}.$$

Ex: Parabola - $C : \begin{cases} \vec{r}(t) = (t, t^2) \\ t \in \mathbb{R} \end{cases}$

$$\vec{r}\,'(t) = (1, 2t), \quad \vec{r}\,''(t) = (0, 2). \quad |\vec{a}| = 2$$

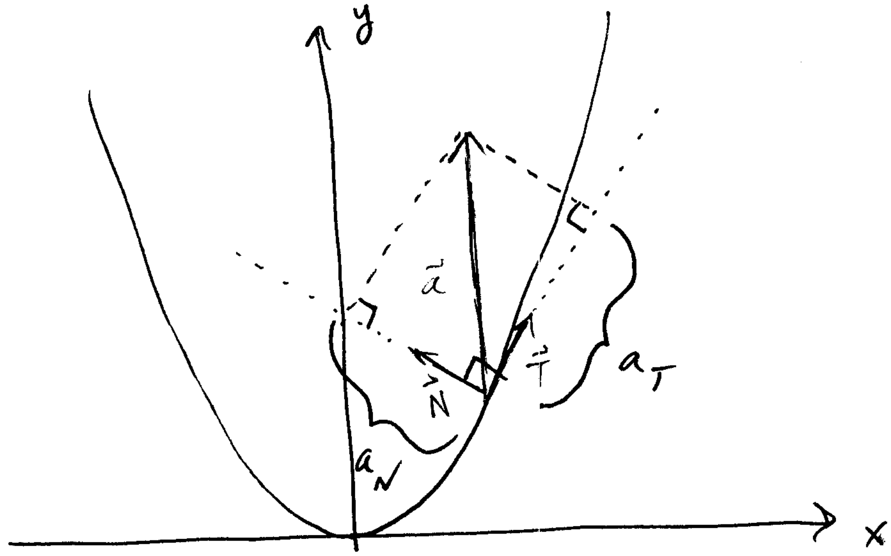
$$a_T = v'(t) \quad \ldots \quad v(t) = \sqrt{1+4t^2}$$

$$= \cancel{\frac{4t}{(1+4t^2)^{3/2}}} = \frac{4t}{\sqrt{1+4t^2}}$$

$$\cancel{a_N = \sqrt{2^2 - \left(\frac{4t}{(1+4t^2)^{3/2}}\right)^2}} = \sqrt{4 - \frac{16t^2}{(1+4t^2)^3}}$$

$$\vec{a}(t) = \left(\frac{4t}{\sqrt{1+4t^2}}\right)\vec{T} + \left(\cancel{\sqrt{4 - \frac{16t^2}{(1+4t^2)^3}}}\right)\vec{N}$$

$$a_N = \sqrt{4 - \frac{16t^2}{1+4t^2}}$$



So what do we need $\vec{B}$ for?

Consider $\dfrac{d\vec{B}}{ds} = \dfrac{d}{ds}(\vec{T} \times \vec{N})$

$$= \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds}$$

$$= (\kappa \vec{N}) \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds}$$

$$= \vec{T} \times \frac{d\vec{N}}{ds} \Rightarrow \frac{d\vec{B}}{ds} \perp \vec{T}$$

Also recall – $\vec{B} \cdot \vec{B} = 1 \Rightarrow 2 \dfrac{d\vec{B}}{ds} \cdot \vec{B} = 0$

$$\frac{d\vec{B}}{ds} \cdot \vec{B} = 0 \Rightarrow \frac{d\vec{B}}{ds} \perp \vec{B}.$$

$\Rightarrow \dfrac{d\vec{B}}{ds} = C_1 \vec{N}$ ... Define $C_1 = -\tau$

Call τ the <u>torsion</u> of the curve C.

$$\frac{d\vec{B}}{ds} = -\tau \vec{N}$$

τ measure how C is moving out of the osculating plane.

Why is this useful?

Two theorems: ① Up to a rigid rotation and translation, ANY curve in $\mathbb{R}^3$ is uniquely determined by the pair of functions κ and τ.

② Given a unit-speed / arclength-parametrized curve C with nonzero curvature κ, then we have

$$\begin{cases} \vec{T}'(s) = \kappa \vec{N} & \text{(i)} \\ \vec{N}'(s) = -\kappa \vec{T} + \tau \vec{B} & \text{(ii)} \\ \vec{B}'(s) = -\tau \vec{N} & \text{(iii)} \end{cases}$$

Frenet formulas

Proof of ②: (i) and (iii) already done! (hooray! ☺)

Can write $\vec{N}' = (\vec{N}'\cdot\vec{T})\vec{T} + (\vec{N}'\cdot\vec{N})\vec{N} + (\vec{N}'\cdot\vec{B})\vec{B}$

All three of $\vec{T}$, $\vec{N}$, and $\vec{B}$ are constant-length vectors.

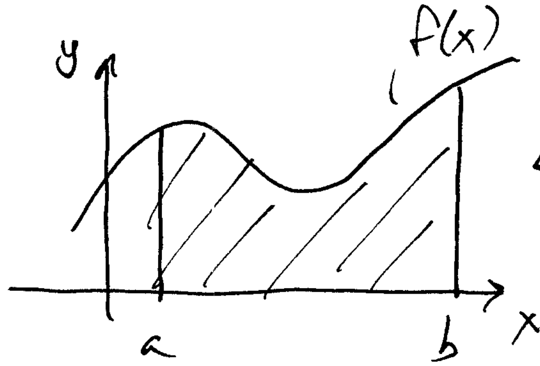
$$\vec{N}\cdot\vec{T} = 0 \Rightarrow \vec{N}'\cdot\vec{T} + \vec{N}\cdot\vec{T}' = 0$$

$$\vec{N}'\cdot\vec{T} = -\vec{N}\cdot\vec{T}' = -\vec{N}\cdot(\kappa\vec{N}) = -\kappa$$

Similarly - get $\vec{N}'\cdot\vec{N} = 0$ and $\vec{N}'\cdot\vec{B} = -\vec{N}\cdot\vec{B}' = -\vec{N}\cdot(-\tau\vec{N}) = \tau$

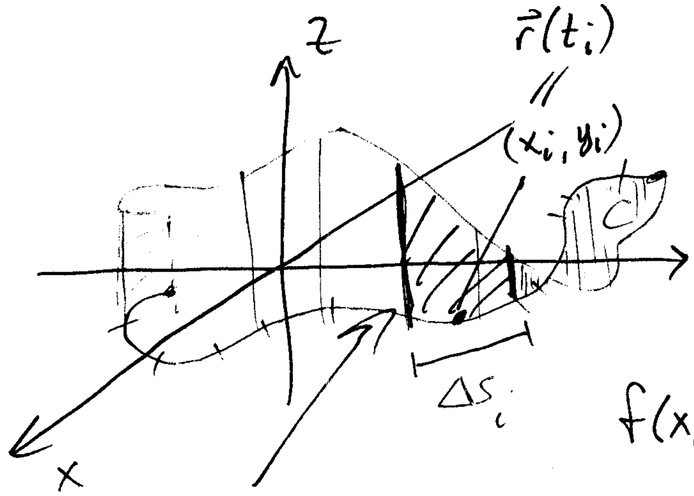
§ 16.1 – Line Integrals

Recall:



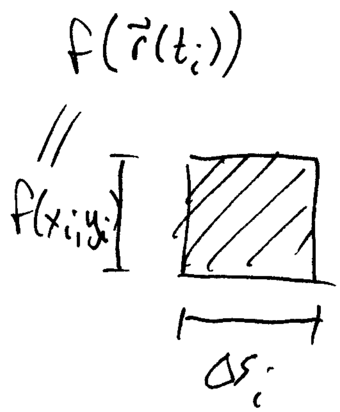
$\text{Area} = \int_a^b f(x)\,dx.$

Instead:



$C: \begin{cases} \vec{r}(t) = (x(t), y(t)) \\ a \le t \le b \end{cases}$

$f(x,y)$ (integrable)



$\Delta A_i = f(\vec{r}(t_i))\,\Delta s_i$ ← approx.

$= f(x_i, y_i)\,\Delta s_i$

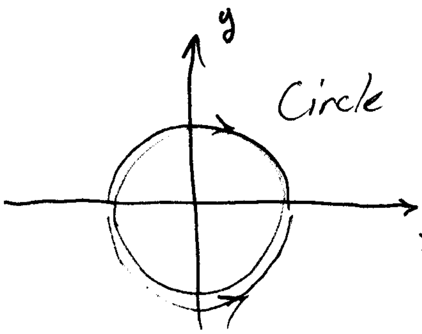
$\hookrightarrow \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i, y_i)\,\Delta s_i$

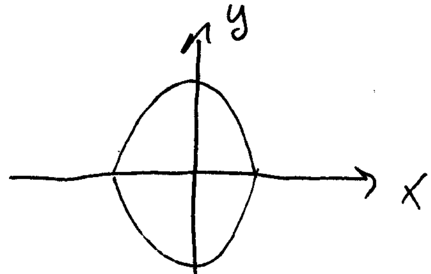
$:= \int_C f(x,y)\,ds$

Computing this:

1. Need a parametrization of C ($\vec{r}(t)$, $a \le t \le b$).
2. Recall $ds = \frac{ds}{dt}\,dt = v(t) = |\vec{r}\,'(t)|$

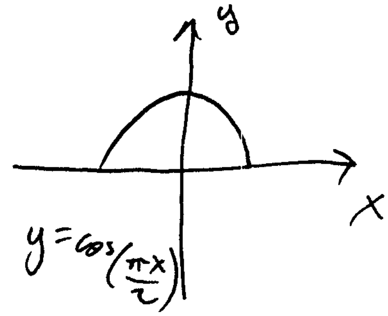
Warm-up: Connect $A = (-1,0)$ to $B = (1,0)$ with a curve(s) in as many ways as you can think of.

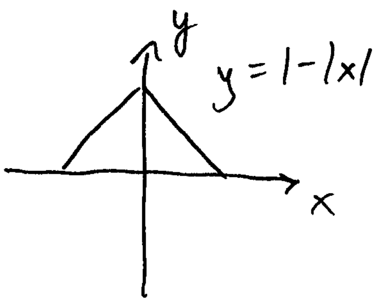


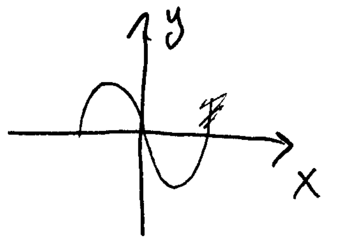


parabola

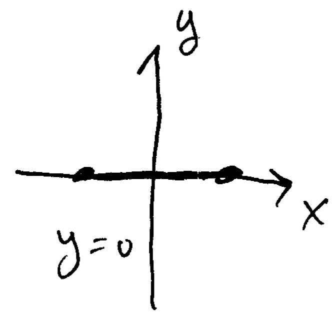
$y = x^2 - 1$

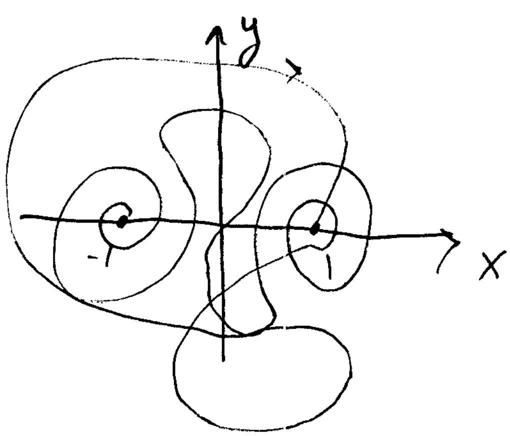






$x^3 - x = x(x+1)(x-1)$

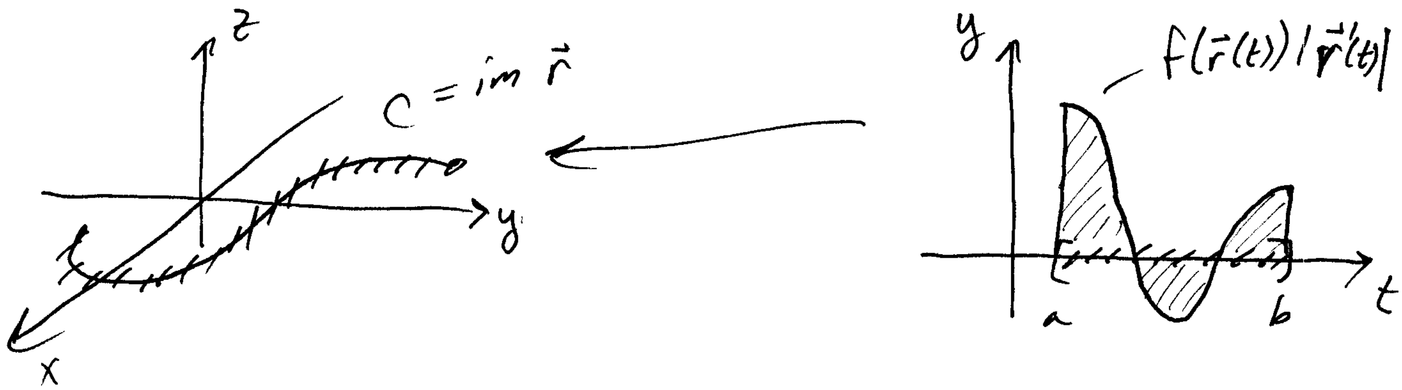




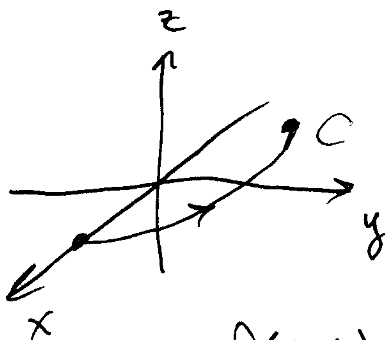
③ **Def^n:**

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\,dt}$$

Pictorially:



Ex: Integrate the function $f(x,y,z) = x^2 + y^2 + z^2$ over the curve $C: \begin{cases} \vec{r}(t) = (\cos t, \sin t, t) \\ 0 \le t \le \pi/2 \end{cases}$.



$$I = \int_C f\,ds = \int_0^{\pi/2} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$f(\vec{r}(t)) = f(\overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t})$$

$$= (\cos t)^2 + (\sin t)^2 + (t)^2$$

$$= 1 + t^2 \ \star$$

$$\vec{r}\,'(t) = (-\sin t, \cos t, 1)$$

$$|\vec{r}\,'(t)| = \sqrt{2} \ \star$$

$$= \int_0^{\pi/2} (1+t^2)\sqrt{2}\,dt$$

$$= \sqrt{2}\left[t + \tfrac{1}{3}t^3\right]_{t=0}^{\pi/2}$$

$$= \boxed{\sqrt{2}\left(\frac{\pi}{2} + \frac{\pi^3}{24}\right)}.$$

Parametrize C in reverse? 0.0

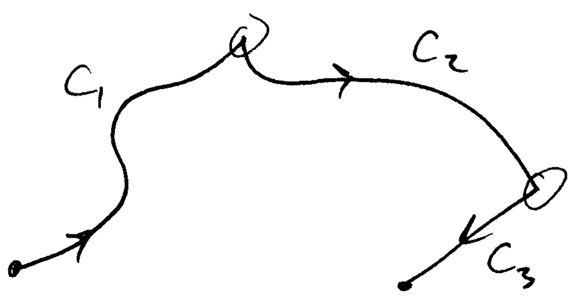
Q: $\displaystyle\int_{-C} f\,ds \overset{?}{=} -\int_{C} f\,ds$

Usually: $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

A: No! $\displaystyle\int_{-C} f\,ds = \int_C f\,ds$

$ds = \left|\frac{d\vec{r}}{dt}\right| dt$ doesn't care about orientation. (dx does!).

Piecewise Differentiability

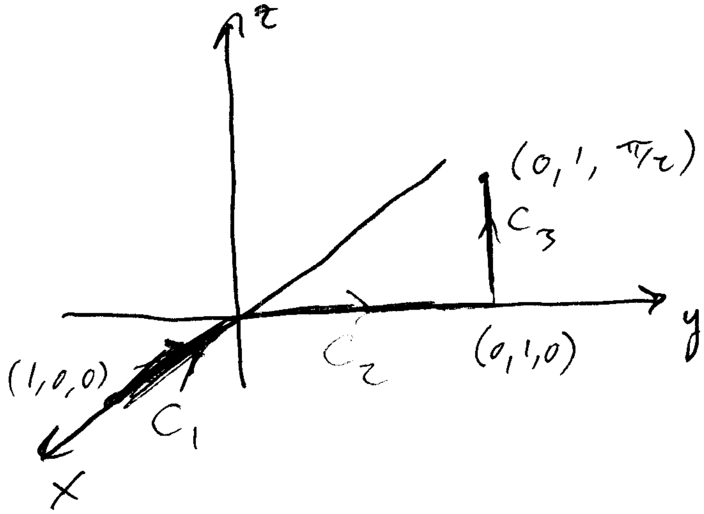


$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds.$$

Write $C = C_1 + C_2 + C_3$.

Generally: $$\int_{C = C_1 + \cdots + C_n} f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds.$$

Ex: Integrate $f(x,y,z) = x^2+y^2+z^2$ along $C = C_1+C_2+C_3$:



$$I = \int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds$$

$$C_1: \vec{r}(t) = (1,0,0) + t(-1,0,0) = (\overset{x}{1-t}, \overset{y}{0}, \overset{z}{0}) \quad // \ 0 \le t \le 1$$

$$C_2: \vec{r}(t) = (0,0,0) + t(0,1,0) = (0,t,0) \quad // \ 0 \le t \le 1$$

$$C_3: \vec{r}(t) = (0,1,0) + t(0,0,\pi/2) = (0,1,\tfrac{\pi}{2}t) \quad // \ 0 \le t \le 1.$$

$$I = \int_0^1 (1-t)^2 \cdot 1\,dt + \int_0^1 t^2 \cdot 1\,dt + \int_0^1 \left[1 + \left(\tfrac{\pi}{2}t\right)^2\right] \tfrac{\pi}{2}\,dt$$

$$= \left[-\tfrac{1}{3}(1-t)^3\right]_0^1 + \left[\tfrac{1}{3}t^3\right]_0^1 + \tfrac{\pi}{2}\left[t + \tfrac{\pi^2}{4} t^3/3\right]_0^1$$

$$= \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{\pi}{2}\left(1 + \pi^2/12\right)$$

$$\left(\ne \sqrt{2}\left(\tfrac{\pi}{2} + \tfrac{\pi^3}{24}\right)\right).$$

Applications:

1. $f(x,y,z) = 1$: Arclength

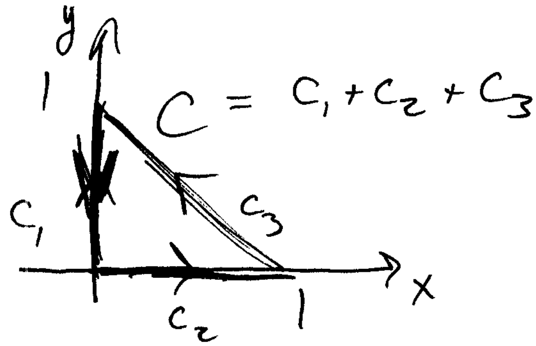
2. $f(x,y,z) =$ Force magnitude : (Unsigned) Work
~~Net~~ TOTAL

3. $f(x,y,z) = \delta(x,y,z) =$ density (linear) : Mass

4. $$f(x,y,z) = \left\{\begin{array}{ll} x\delta & : M_{yz} \\ y\delta & : M_{xz} \\ z\delta & : M_{xy} \end{array}\right\}$$ First moments of of the curve. (relative to coord. planes)

5. $$f(x,y,z) = \left\{\begin{array}{ll} (x^2+y^2)\delta & : I_z \\ (x^2+z^2)\delta & : I_y \\ (y^2+z^2)\delta & : I_x \end{array}\right\}$$ Moments of inertia relative to coord. axes.

<u>Warm-up:</u> Compute $\int_C \sqrt{x+y}\, ds$ where C is given as



$C_1: \vec{r}(t) = (0,1) + t(0,-1) = (0, 1-t), \quad 0 \le t \le 1$

$C_2: \vec{r}(t) = (0,0) + t(1,0) = (t, 0), \quad 0 \le t \le 1$

$C_3: \vec{r}(t) = (1,0) + t(-1,1) = (1-t, t), \quad 0 \le t \le 1$

$$\int_C \sqrt{x+y}\, ds = \int_{C_1} \sqrt{x+y}\, ds + \int_{C_2} \sqrt{x+y}\, ds + \int_{C_3} \sqrt{x+y}\, ds$$

$C_1: \sqrt{x+y} = \sqrt{0+(1-t)} = \sqrt{1-t}, \quad \vec{r}\,'(t) = (0,-1)$

$|\vec{r}\,'(t)| = 1$

$$\int_{C_1} \sqrt{x+y}\, ds = \int_0^1 \sqrt{1-t} \cdot +1\, dt \quad ☆$$

$C_2: \sqrt{x+y} = \sqrt{t}, \quad \vec{r}\,' = (1,0), \; |\vec{r}\,'| = 1 \quad // \int_{C_2} \sqrt{x+y}\, ds$

$$= \int_0^1 \sqrt{t} \cdot 1\, dt \quad ☆$$

$C_3: \sqrt{x+y} = \sqrt{1} = 1, \quad \vec{r}\,' = (-1,1), \; |\vec{r}\,'| = \sqrt{2} \quad // \int_{C_3} \sqrt{x+y}\, ds$

$$= \int_0^1 \sqrt{2}\, dt$$

$$\int_C \sqrt{x+y}\, ds = \int_0^1 \left(\sqrt{1-t} + \sqrt{t} + \sqrt{2}\right) dt$$

--- to be concluded.

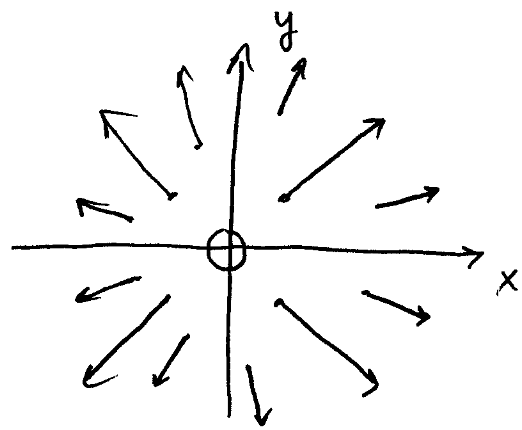
§ 16.2 - Vector field & Line/Work Integrals

Def^n: A vector field in $\mathbb{R}^3$ is a function

$$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 .$$

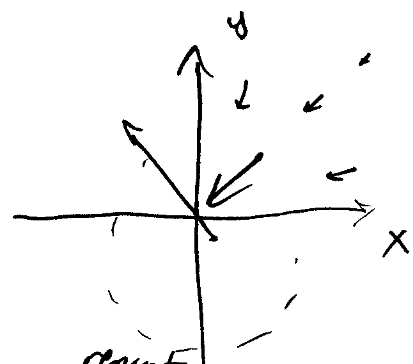
Ex: (Electric field) made by e^-

$$\vec{E}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$



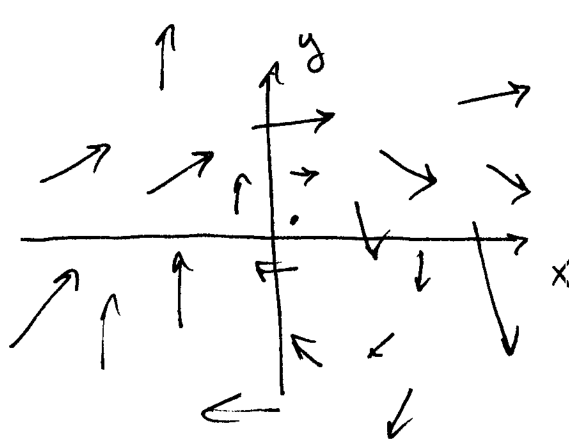
(Gravitational field) made by Earth

$$\vec{G}(x,y) = -\vec{E}$$



(Wind map)

$\vec{W}$ = Data points



Exercise : Sketch the following vect. fields :

① $\vec{F}(x,y) = \langle y, -x \rangle$

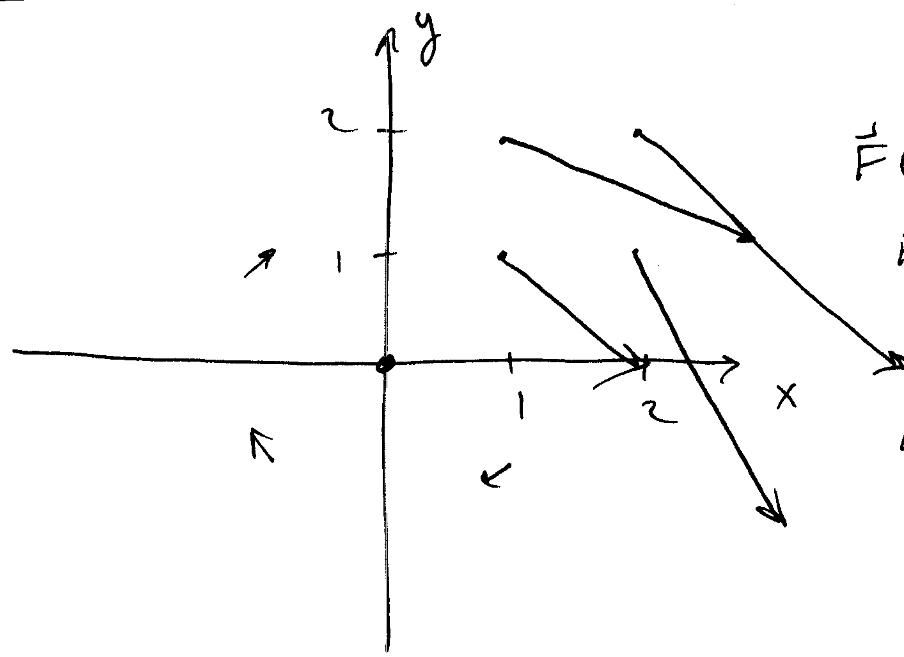
② $\vec{F}(x,y) = \langle x, |x| \rangle$

③ $\vec{F}(x,y) = \langle \sin x, y \rangle$

④ $\vec{F}(x,y) = \langle xy, 1/y^2 \rangle$

---

①



$\vec{F}(1,1) = \overrightarrow{\langle 1,-1 \rangle}$

$\vec{F}(2,1) = \overrightarrow{\langle 1,-2 \rangle}$

$\vec{F}(1,2) = \overrightarrow{\langle 2,-1 \rangle}$

Gradient vector fields

Given a function $f(x,y,z)$, $\nabla f = (f_x, f_y, f_z)$

Ex: $f(x,y,z) = x^2 + y^3 + z^4$

$$\nabla f = (2x, 3y^2, 4z^3)$$

Ex: $f(x,y\cancel{,z}) = x^2 y - y^3$

$$\nabla f = (2xy, x^2 - 3y^2)$$

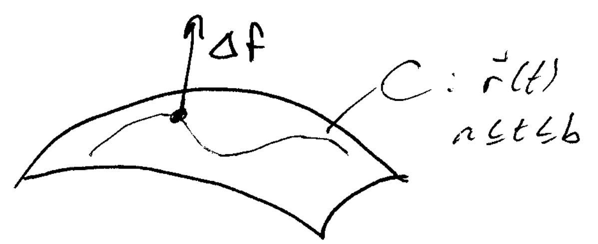
∇f points along direction of steepest ascent/increase in the function f.

$\Rightarrow -\nabla f$ is steepest descent.

Important for ① FTC (vector style)

② Normals to (implicit) surfaces

E.g. $f(x,y,z) = c$ (const)

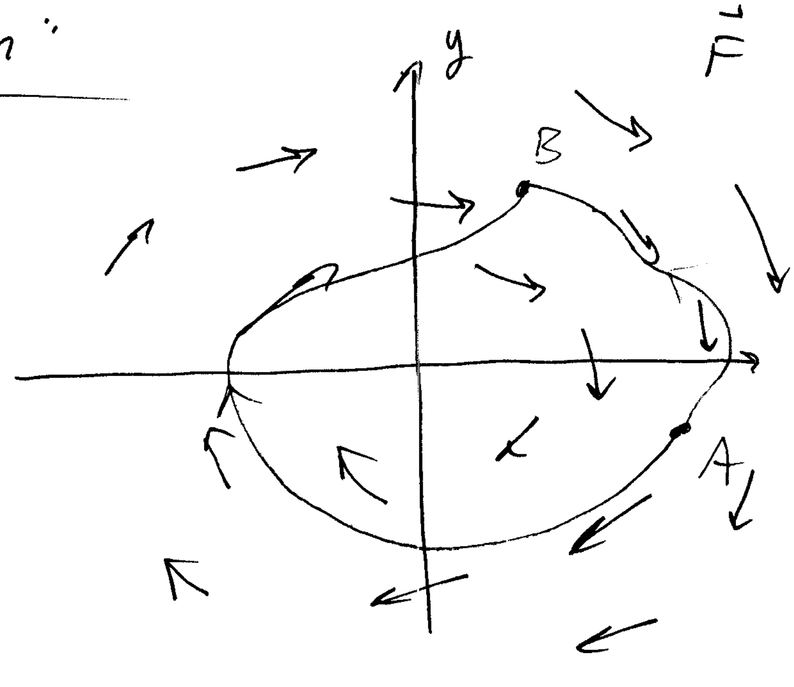


$$f(\vec{r}(t)) = c$$

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \frac{d}{dt}(c)$$

$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0 \Rightarrow \nabla f$ is a normal vector to the surface $f(x,y,z) = c$.

Motivation:



$\longrightarrow$: $W < 0$ particle does work

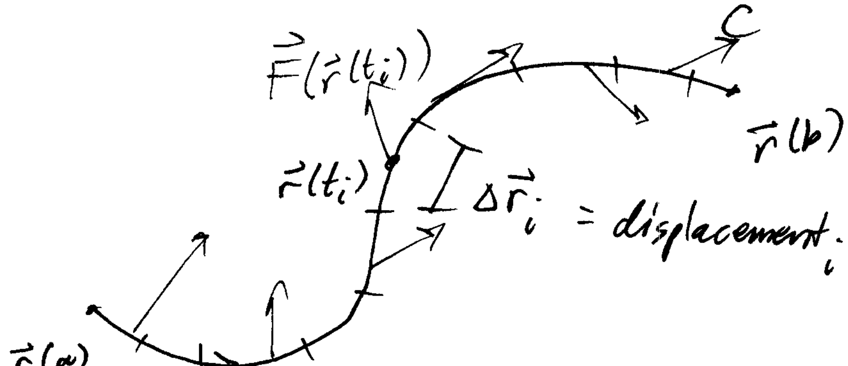
$\longrightarrow$: $W > 0$ field does work on particle.

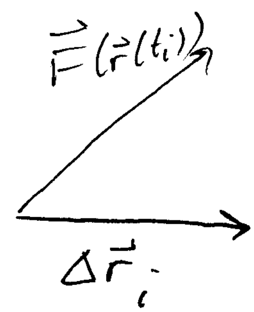
Force field $\vec{F}(x,y)$. Particle moving through $\vec{F}$ experiences it. Must do work to proceed.

Goal: Compute net work particle must do as it travels along paths.

Given force field $\vec{F} = (M, N, P)$ and path $C : \begin{cases} \vec{r}(t) \\ a \le t \le b \end{cases}$.

Find net work done on/by particle along C.





$$\Delta W_i = \vec{F}(\vec{r}(t_i)) \cdot \Delta \vec{r}_i$$

$$\text{Work} \approx \sum_{i=1}^{N} \vec{F}(\vec{r}(t_i)) \cdot \Delta \vec{r}_i$$

$$\xrightarrow{\lim_{N \to \infty}} \boxed{\text{Work} = \int_a^b F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt}$$

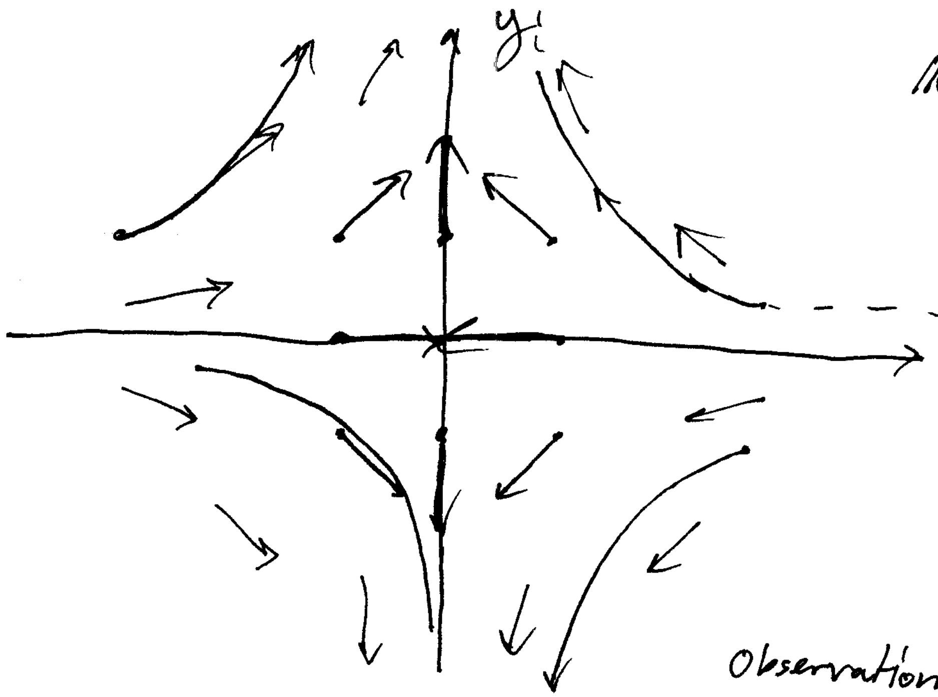
Def^n: The <u>work integral</u> $W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$

$$= \int_C \vec{F} \cdot d\vec{r}$$

Examples next time . . .

Warm-up: Give a qualitative sketch of the vector field

$$\vec{F}(x,y) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$



$\mathbb{R}^2 \setminus \{\vec{0}\}$

Observation:

$$|\vec{F}(x,y)| = \sqrt{\frac{x^2+y^2}{x^2+y^2}} = 1.$$

Continue §16.2

$$\frac{d\vec{r}}{ds} = \vec{T}$$

Recall: $W = \int_C \vec{F} \cdot d\vec{r} = \boxed{\int_c^d \vec{F}(\vec{r}(s)) \cdot \vec{T}\, ds}$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

Let $\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$

$\vec{r}(t) = (x(t), y(t), z(t))$, $a \le t \le b$.

So $W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$

$$= \int_a^b (M, N, P) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

$$\begin{cases} = \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right) dt \\ = \int_C M\,dx + N\,dy + P\,dz. \end{cases}$$

$W = \int_C \vec{F} \cdot \vec{T}\, ds$ // Definition

$= \int_C \vec{F} \cdot d\vec{r}$ // vect. diff. version

For evaluation $\begin{cases} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\,dt & \text{// para. eval. version} \\ = \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right) dt & \text{// para. scal. version} \end{cases}$

$= \int_C M\,dx + N\,dy + P\,dz$ // scalar diff. version

Ex: $\vec{F}(x,y,z) = (x-y^2, y^2-z^3, z^3-x)$ along

$C: \vec{r}(t) = (\underline{t^3}, \underline{t^2}, \underline{t}), \quad 0 \le t \le 1$

$$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C (x-y^2)\,dx + (y^2-z^3)\,dy + (z^3-x)\,dz$$

$$= \int_0^1 \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^1 \vec{F}(t^3, t^2, t)\cdot(3t^2, 2t, 1)\,dt$$

$$= \int_0^1 (t^3-t^4, t^4-t^3, t^3-t^3)\cdot(3t^2, 2t, 1)\,dt$$

$$= \int_0^1 \left[(3t^5-3t^6)+(2t^5-2t^4)+(0)\right]dt$$

$$= \int_0^1 5t^5 - 3t^6 - 2t^4\,dt$$

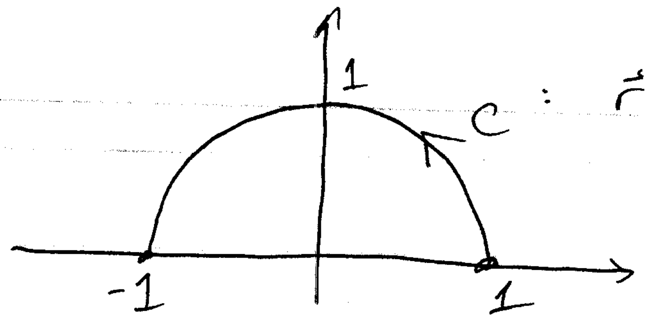
$$= \left[\tfrac{5}{6}t^6 - \tfrac{3}{7}t^7 - \tfrac{2}{5}t^5\right]_0^1 = \tfrac{5}{6} - \tfrac{3}{7} - \tfrac{2}{5}$$

$$= \tfrac{1}{210} > 0.$$

work done on particle

$W < 0$ work done by particle.

Exercise: Compute $W = \int_C y\,dx - x\,dy$ where



$C: \vec{r}(t) = (\cos t, \sin t)$
$0 \le t \le \pi$

Is work being done on or by the particle?

$\vec{F}(x,y) = (y, -x)$

$\vec{r}\,'(t) = (-\sin t, \cos t)$ // $\vec{F}(\vec{r}(t)) = \vec{F}(\cos t, \sin t)$
$= (\sin t, -\cos t)$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = (\sin t, -\cos t) \cdot (-\sin t, \cos t)$$
$$= -1$$

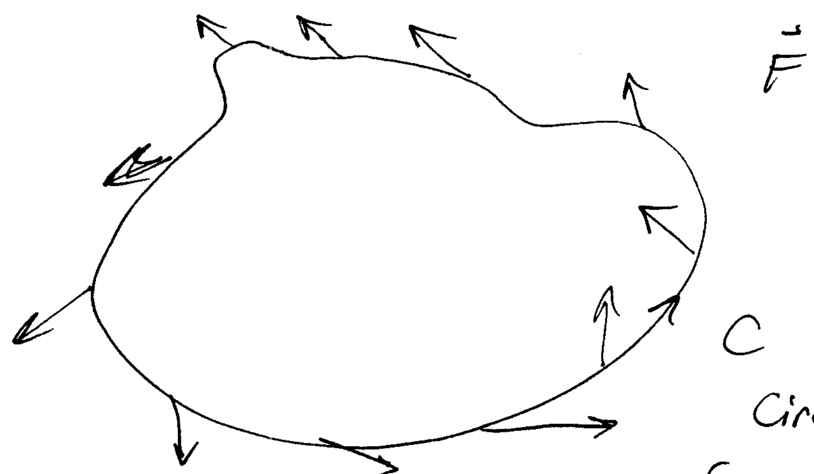
$$\Rightarrow W = \int_0^{\pi} -dt = -\pi < 0$$

Done by particle.

Physical interpretations for planar curves

Def^n: A curve $C: \vec{r}(t),\ a \le t \le b$ is closed if $\vec{r}(a) = \vec{r}(b)$.

Picture:

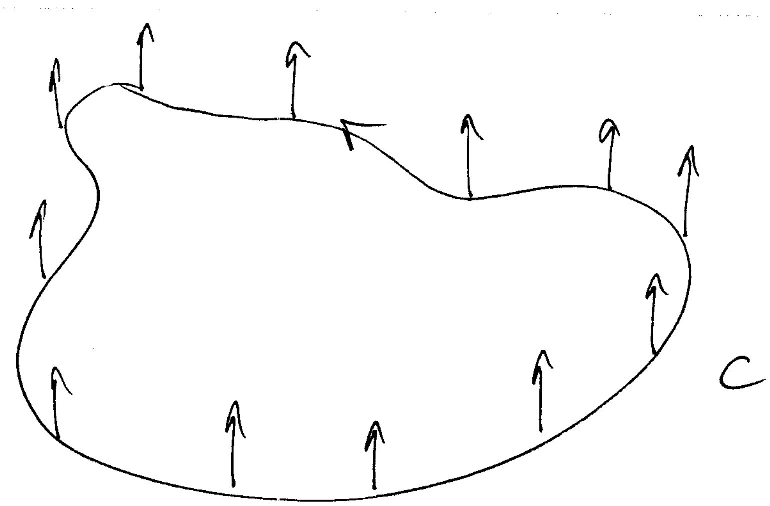


$Circ_C(\vec{F}) > 0$

$[Circ_{-C}(\vec{F}) < 0]$

Def^n: $W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_C M\,dx + N\,dy$ is called the flow of $\vec{F}$ along C. When C is closed, it is called the circulation $Circ_C(\vec{F})$ of $\vec{F}$ around C.

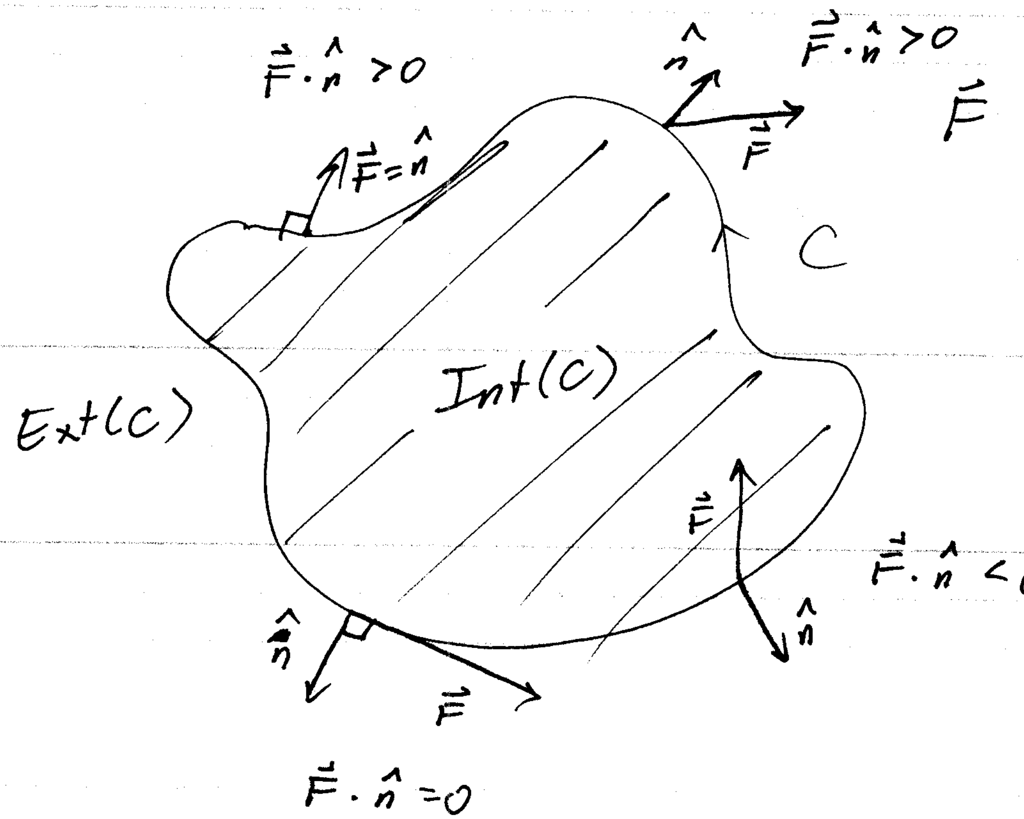
(Net circulation/flow)



$Circ_C(\vec{F}) = 0$

(simple) no overlap in image

Thought: Closed curve in $\mathbb{R}^2$ has an interior $Int(C)$ and an exterior $Ext(C)$. How much of $\vec{F}$ "leaves" the interior of C?



Def^n: The flux of $\vec{F}$ across C given by

$$Flux_C(\vec{F}) = \oint_C \vec{F}\cdot\hat{n}\, ds.$$

Evaluating $\oint_C \vec{F}\cdot\hat{n}\, ds$

Recall: $\vec{r}(t) = (x(t), y(t))$, $a \le t \le b$, $\vec{r}(a) = \vec{r}(b)$.

$\vec{T}(x,y) = \alpha(x', y')$ (α const).

Need $\vec{T}\cdot\vec{n} = \vec{r}\,'\cdot\vec{n} = 0$

$(x', y')\cdot(a,b) = 0$

Choose $\vec{n} = (y', -x')$ $\quad ax' + by' = 0$

$$\vec{F} = (M, N)$$

So $\int_C \vec{F} \cdot \vec{T}\, ds = \text{Circ}_C(\vec{F}) = \int_C M\,dx + N\,dy$.

and $\oint_C \vec{F} \cdot \hat{n}\, ds = \oint_C \vec{F} \cdot d\vec{n} = \oint_C M\,dy - N\,dx$

Warm-up: Define for a vector field $\vec{F} = (M, N, P)$,

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}.$$

Find curl $\vec{F}$ for

① $\vec{F} = (x, y, z)$

② $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

---

① $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = (0-0, 0, 0)$$

② $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = (0, 0, 0)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1\cdot(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

§ 16.3 – Long-winded title of section telling you when line integrals are easy

---

__Seen:__ $\int_C \vec{F} \cdot d\vec{r}$ can suck

__Goal:__ Make it easier to compute

__Problem:__ Can't do this if $\vec{F}$ isn't "nice".

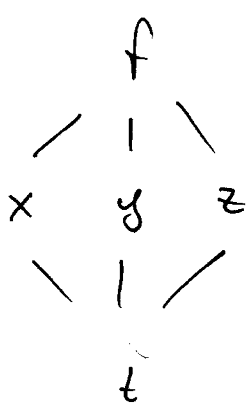
__Question:__ what does a "nice" $\vec{F}$ look like?

IDEA: Make $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$ look like

$$\int_a^b \frac{d}{dt}\left[\underline{f(\vec{r}(t))}\right] dt.$$

Use FTC.

Consider $\frac{d}{dt}\left(f(\vec{r}(t))\right) = \frac{d}{dt}\left(f(x(t), y(t), z(t))\right)$.



$$= f_x \cdot x' + f_y \cdot y' + f_z \cdot z'$$

$$= \underline{(f_x, f_y, f_z)} \cdot (x', y', z')$$

$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$\underline{\text{Def}^{n}}$: If $\vec{F} = \nabla f$ for some <u>function</u> f, then $\vec{F}$ is a <u>gradient field</u>.

<u>Ex:</u> Gravity —

$$\vec{G} = \frac{-Gm_1m_2}{r^2}\,\hat{r} \qquad (r^2 = x^2+y^2+z^2)$$

$$= \nabla\left(\frac{Gm_1m_2}{r}\right)$$

<u>When is $\vec{F}$ a gradient field?</u> $\qquad (\nabla = (\partial_x, \partial_y, \partial_z))$

$$\vec{F} = \nabla f = (f_x, f_y, f_z)$$

What is $\nabla \times \vec{F} = \nabla \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}$

$$= (f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy})$$

Clairaut's Theorem / Equality of Mixed Partial Derivatives

$$= (0, 0, 0) = \vec{0}.$$

In particular, if we want $\vec{F} = \nabla f$, then we need $\nabla \times \vec{F} = \vec{0}$.

For most purposes: If $\nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is a gradient field!

Which $f(x,y,z)$ produces $\vec{F} = \nabla f$? (f is called a potential function)

Ex: Let $\vec{F} = (yz + 2x,\ xz + \cos(y-z),\ xy - \cos(y-z))$.
Find a potential function for $\vec{F} = \nabla f$.

Check $\nabla \times \vec{F} = \vec{0} \Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz+2x & xz+\cos(y-z) & xy-\cos(y-z) \end{vmatrix}$

$$= \big((x+\sin(y-z)) - (x+\sin(y-z)),\ y-y,\ z-z \big)$$

$$= \vec{0}. \checkmark$$

$\vec{F} = \nabla f = (f_x, f_y, f_z)$

So: $\begin{cases} f_x = yz + 2x \rightarrow f = \int f_x\,dx \\ f_y = xz + \cos(y-z) \rightarrow f = \int f_y\,dy \\ f_z = xy - \cos(y-z) \rightarrow f = \int f_z\,dz \end{cases}$

$$f = \int f_x\,dx = \int (yz + 2x)\,dx = xyz + x^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xyz + x^2 + g(y,z)\right) = xz + g_y$$

From 2nd comp. of $\vec{F}$

$$= xz + \cos(y-z)$$

$$g_y = \cos(y-z)$$

Then $g = \int g_y \, dy = \int \cos(y-z) \, dy$

$= \underline{\sin(y-z) + h(z)}$

So $f = xyz + x^2 + \sin(y-z) + h(z)$

$f_z = \frac{\partial}{\partial z}\left(xyz + x^2 + \sin(y-z) + h(z) \right)$

$$\left.\begin{aligned} &= xy - \cos(y-z) + h' \\ &= xy - \cos(y-z) \end{aligned}\right\} \Rightarrow h' = 0$$

From 3^{rd} comp. of $\vec{F}$

$\Downarrow$

$h = C$ (const).

Finally, $\boxed{f(x,y,z) = xyz + x^2 + \sin(y-z)} \cancel{+C}$

Let $C = 0$.

Why is this useful?

Recall - $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$

$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$

$= \int_a^b \frac{d}{dt}\left(f(\vec{r}(t)) \right) dt$

$= f(\vec{r}(t)) \Big|_{t=a}^{b} = f(\vec{r}(b)) - f(\vec{r}(a))$

Ex: Compute $W = \int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is as before

and $C : \vec{r}(t) = (t, t^2, t^3)$, $0 \le t \le 2$.

As usual: $W = \int_0^2 \vec{F}(t, t^2, t^3) \cdot (1, 2t, 3t^2)\, dt$

$= \ldots$ No...

New way?

$$W = \int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0))$$

Recall: $f(x,y,z) = xyz + x^2 + \sin(y-z)$ ...

$$W = f(2,4,8) - f(0,0,0)$$

$$= 2 \cdot 4 \cdot 8 + 2^2 + \sin(4-8) = \boxed{68 + \sin(-4)}$$

Notice! W is independent of the curve choice of C, so long as the curve has the correct initial and terminal points $\vec{r}(a)$ and $\vec{r}(b)$, resp.

So if C is closed, then $W = f(B=A) - f(A) = 0$ (!!)

Question: Is this always the case? That if $\vec{F} = \nabla f$ and C is closed, then $W = \oint_C \vec{F} \cdot d\vec{r} = 0$?

Ex: $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

We showed $\nabla \times \vec{F} = \vec{0}$. $\Rightarrow \vec{F} = \nabla f$!

Then we expect $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed loop C !!

Compute (directly) $\oint_C \vec{F} \cdot d\vec{r}$ when $C : \begin{cases} \vec{r}(t) = (\cos t, \sin t, 0) \\ 0 \le t \le 2\pi \end{cases}$.

$$\vec{F}(\vec{r}(t)) = \left(\frac{-\sin t}{1}, \frac{\cos t}{1}, 0 \right)$$

$$= (-\sin t, \cos t, 0)$$

$$\vec{r}\,'(t) = (-\sin t, \cos t, 0)$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \sin^2 t + \cos^2 t = 1.$$

$$W = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1 \cdot dt = \underline{\underline{2\pi \neq 0}}$$

Warm-up: Compute ~~the~~ a potential function for the vector field $\vec{F}(x,y,z) = (yz + 2x,\ xz - \sin y,\ xy + e^z)$.

(What should you check first?)

Def^n: $df = \nabla f \cdot d\vec{r} = f_x\,dx + f_y\,dy + f_z\,dz$

a differential form $\omega = f_1\,dx + f_2\,dy + f_3\,dz$ is <u>exact</u> if $\omega = df$ for some f.

(Check $\nabla \times \vec{F} = \text{Curl}\,\vec{F} = \vec{0}$).

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz + 2x & xz - \sin y & xy + e^z \end{vmatrix} = \vec{0}.$$

$$\begin{cases} yz + 2x = f_x \\ xz - \sin y = f_y \\ xy + e^z = f_z \end{cases}$$

$\longrightarrow$ $f = \int f_x\,dx$

$= xyz + x^2 + g(y,z)$

$f_y = \cancel{xz} + g_y = \cancel{xz} - \sin y$

$g_y = -\sin y$

$g = \cos y + h(z)$.

$f_z = \cancel{xy} + h'(z) = \cancel{xy} + e^z$

$h(z) = e^z + C$

$$\boxed{f(x,y,z) = xyz + x^2 + \cos y + e^z + C}$$

Recall: We said that for $\vec{F} = \nabla f$ and C closed, we should have $\oint_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = 0$.

Gave $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$ and we found over C: unit circle (ccw) in xy-plane

$$\oint_C \vec{F}\cdot d\vec{r} = 2\pi \neq 0.$$

What went wrong?

---

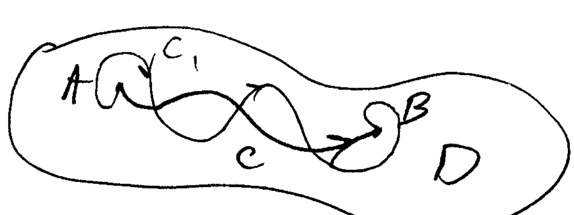
(1) Integrate x first $\rightarrow$ $f = \arctan(y/x)$

y first $\rightarrow$ $f = -\arctan\left(\frac{x}{y}\right)$ . . .

(2) The curve enclosed a hole (z-axis).

Need a stronger condition on $\vec{F}$ than $\nabla f = \vec{F}$.

Def^n: Define $\vec{F}$ over an open region D, and let C be a path in D connecting point A to B in D. If C_1 is any other such curve and $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r}$, then we call $\vec{F}$ conservative over D.



Then $\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$ is path-independent.

Question: When is $\vec{F}$ conservative over a region?

<u>Theorem:</u> If $\vec{F} = (M, N, P)$ w/ $M, N, P \in C^1(D)$ for some open region D, then

$$\left(\vec{F} = \nabla f\right) \overset{\text{equivalent}}{\longleftrightarrow} \left(\vec{F} \text{ conservative over } D\right).$$

diff. on D.

<u>Ex:</u> Potential f's for $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$ are

$$f_1 = \arctan(x/y) \quad \text{and}$$

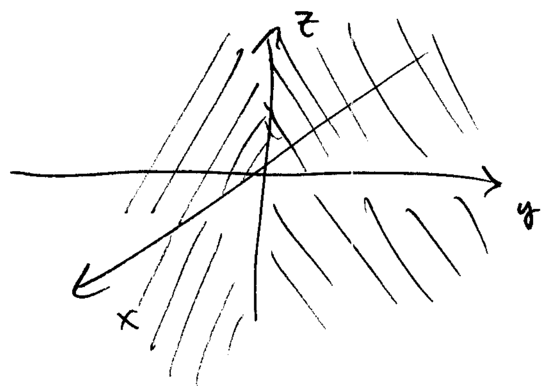
$$f_2 = -\arctan(y/x)$$

Theorem: The following are equivalent:

① $\oint_C \vec{F} \cdot d\vec{r} = 0$ for <u>every</u> closed loop C in D.

② The field $\vec{F}$ is conservative over D.

<u>Ex:</u> C = unit circle xy-plane was some closed loop C where $\oint_C \vec{F} \cdot d\vec{r} \neq 0 \Leftrightarrow \vec{F}$ not conservative over D.



$\{x, y > 0,\ z \in \mathbb{R}\} = D$

$\uparrow$

$\vec{F}$ conservative on D.

Intuition:

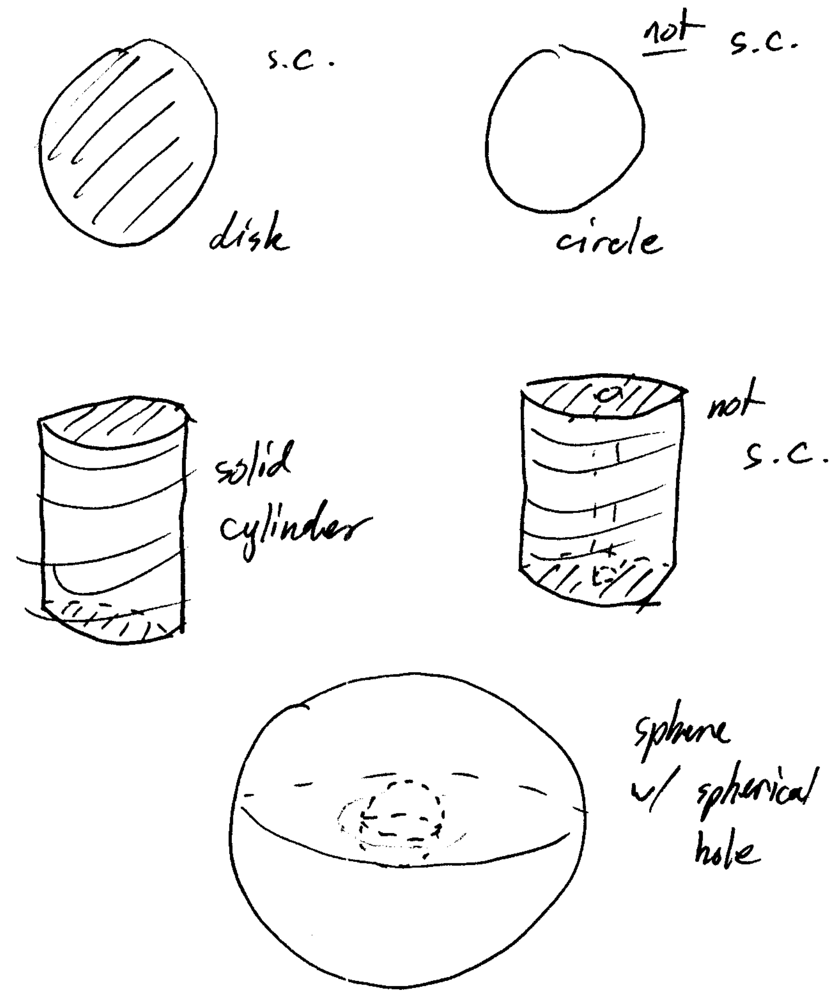
Conservative fields obey "conservation laws."

Vect. fields often come as $\nabla \vec{F} = f$, but the paths C lie in "non-conservative domains D of $\vec{F}$."

They have "holes" or places where $\vec{F}$ not defined.

Def^n: A region D is <u>simply connected</u> if every closed loop in D can be continuously contracted to a point in D.

Ex:



Classical Theorems of Vector Calc

① Green's

$2D \leftrightarrow 1D$

(plane)

② Stokes'

$2D \leftrightarrow 1D$

(space)

③ Gauss'

$3D \leftrightarrow 2D$

(space)

§16.4 – Green's Theorem (in the plane)

~~///~~ Thought: Closed-curve integrals easier than the non-closed-integrals

Needed: Conservative $\bar{F}$ over domain D.

In general: Get neither a gradient field nor a conservative field over D.

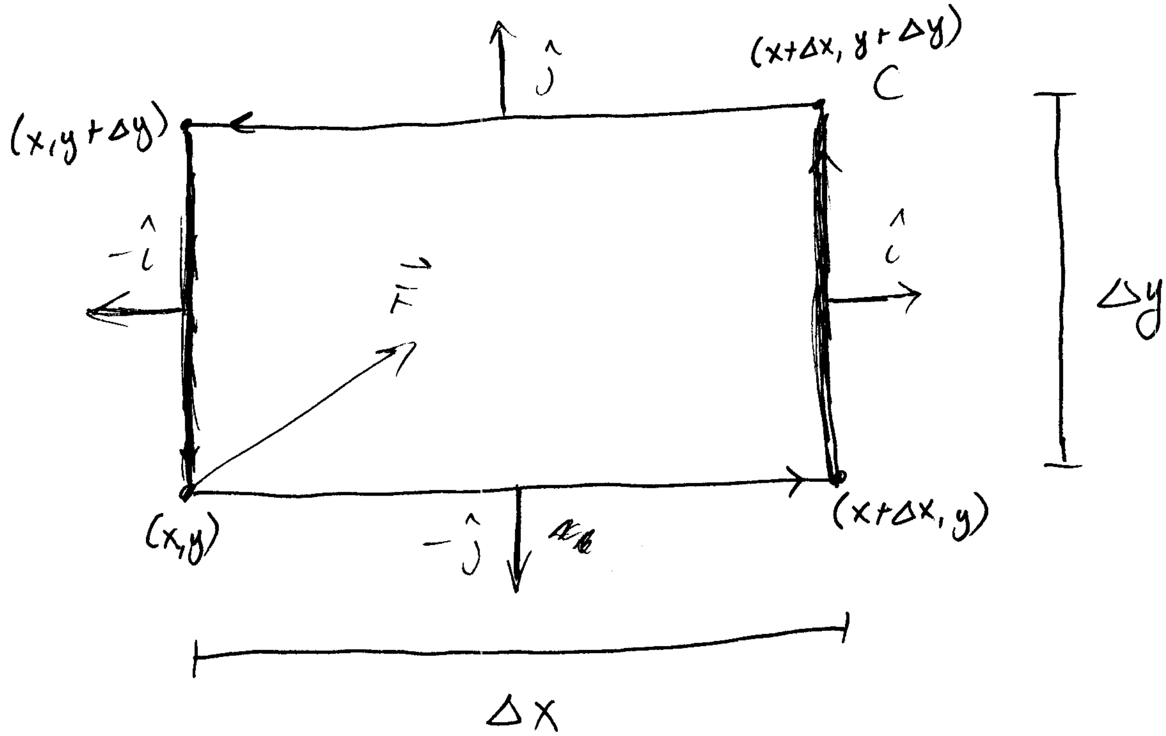
Question: Can we work w/ other vector fields

Answer: Yes, but we need to increase the dimension of the problem to do ~~it~~ so.

IDEA: Exchange "bad" vector field ~~[illegible]~~ in $\oint_C \bar{F} \cdot d\bar{r}$ for a scalar function f in $\iint_D f\, dA$.

Setup: Let $\vec{F} = M\hat{i} + N\hat{j}$, $C = \partial(\text{Rectangle})$.

SMALL



Small rectangle so we can make $\vec{F}$ constant (approx.) in either x or y.

Measure Flux: $\oint_C \underbrace{\vec{F}\cdot\hat{n}\,ds}$

Approximate the flux ~~through~~ across C.

$$\text{Flux} = \vec{F}\cdot\hat{n}\,ds \approx \vec{F}\cdot\underbrace{\hat{n}\,\Delta s}$$

Left: $\vec{F}(x,y)\cdot(-\hat{i}\,\Delta y) = -M(x,y)\,\Delta y$

Right: $\vec{F}(x+\Delta x,y)\cdot(\hat{i}\,\Delta y) = M(x+\Delta x, y)\,\Delta y$

Top: $\vec{F}(x, y+\Delta y)\cdot(\hat{j}\,\Delta x) = N(x,y+\Delta y)\,\Delta x$

Bottom: $\vec{F}(x,y)\cdot(-\hat{j}\,\Delta x) = -N(x,y)\,\Delta x$

Sum : Approx. flux across C :

$$= [M(x+\Delta x, y) - M(x,y)]\Delta y + [N(x,y+\Delta y) - N(x,y)]\Delta x$$

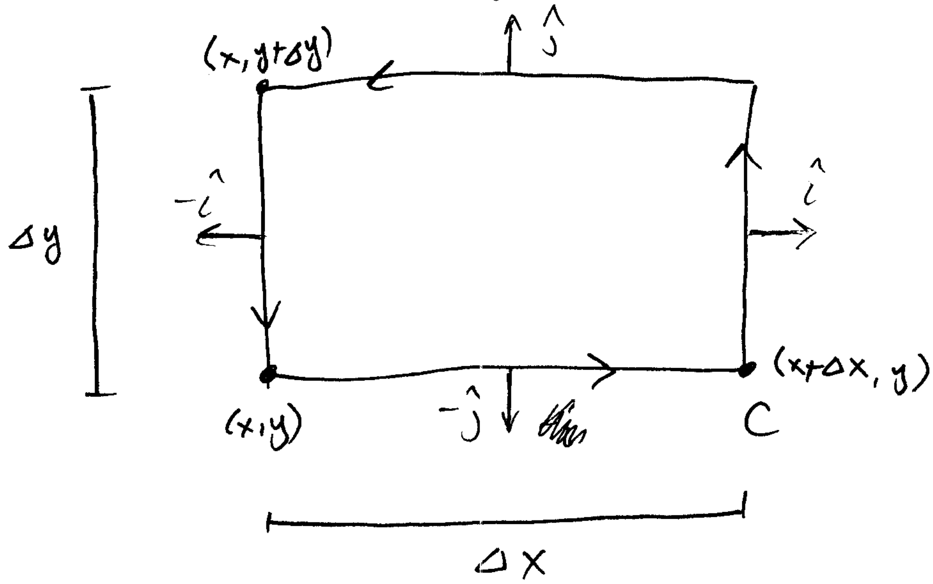
$$= \left[\frac{M(x+\Delta x,y) - M(x,y)}{\Delta x}\right]\Delta x\,\Delta y + \left[\frac{N(x,y+\Delta y) - N(x,y)}{\Delta y}\right]\Delta x\,\Delta y$$

$$\approx [M_x(x,y) + N_y(x,y)]\,\Delta x\,\Delta y$$

Flux density ~~Average flux~~ $= \dfrac{\text{Net flux}}{\text{Area contained}} \approx \dfrac{(M_x + N_y)\Delta x \Delta y}{\Delta x \Delta y}$

$$= M_x + N_y = \operatorname{div} \vec{F}$$

Warm-up: Approximate the flux of the vector field $\vec{F}$ through the (positively oriented) curve C below:



$$\underset{C}{\text{Flux}}(\vec{F}) \approx (M_x + N_y)\,\underline{\underline{\Delta x \Delta y}}$$

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

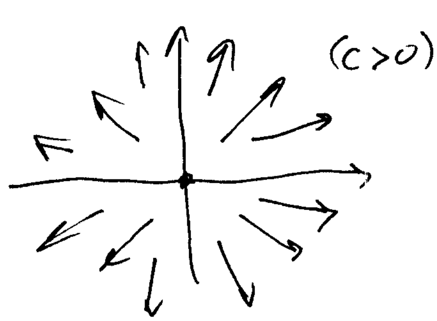
$$\text{"Average Flux"} = \text{Flux density} \approx \frac{\text{Flux}_C(\vec{F})}{\text{Area contained by } C} = M_x + N_y$$

$$= \operatorname{div}(\vec{F})$$

$$= \nabla \cdot \vec{F}$$

Ex: (a) $\vec{F}(x,y) = c(x,y)$ (radial field) $(c \in \mathbb{R})$

$$\operatorname{div} \vec{F} = M_x + N_y = c + c = 2c$$



$c > 0 \Rightarrow$ expanding

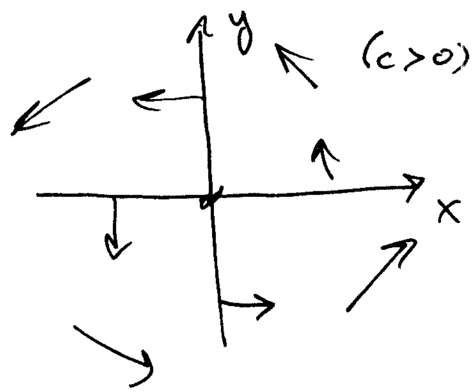
$c < 0 \Rightarrow$ compression

$c = 0 \Rightarrow$ incompressible

(b) $\vec{F}(x,y) = c(-y, x)$ (circulation field) $(c \in \mathbb{R})$

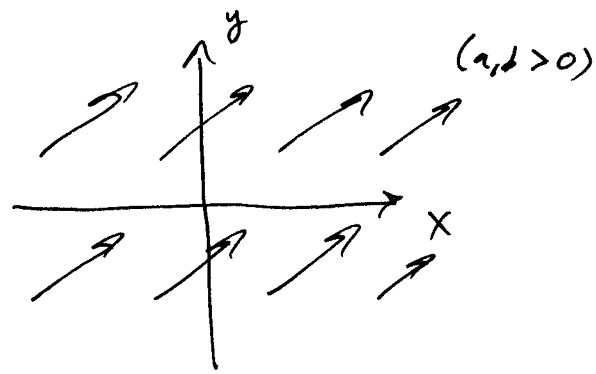
$$\operatorname{div} \vec{F} = M_x + N_y = 0 \Rightarrow \text{incompressible}$$

(everywhere)



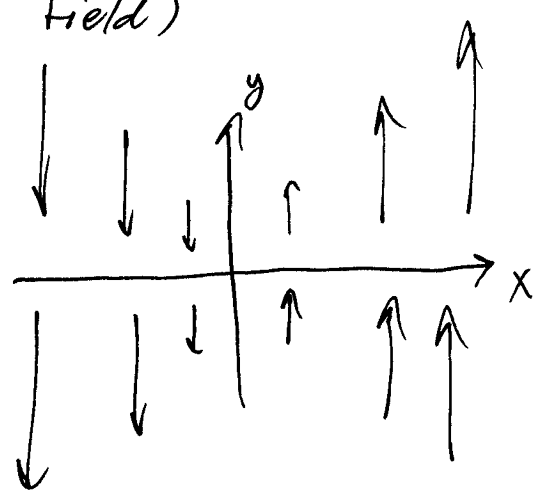
(c) $\vec{F} = (a,b)$ (constant field) $(a,b \in \mathbb{R})$

$$\operatorname{div} \vec{F} = 0$$

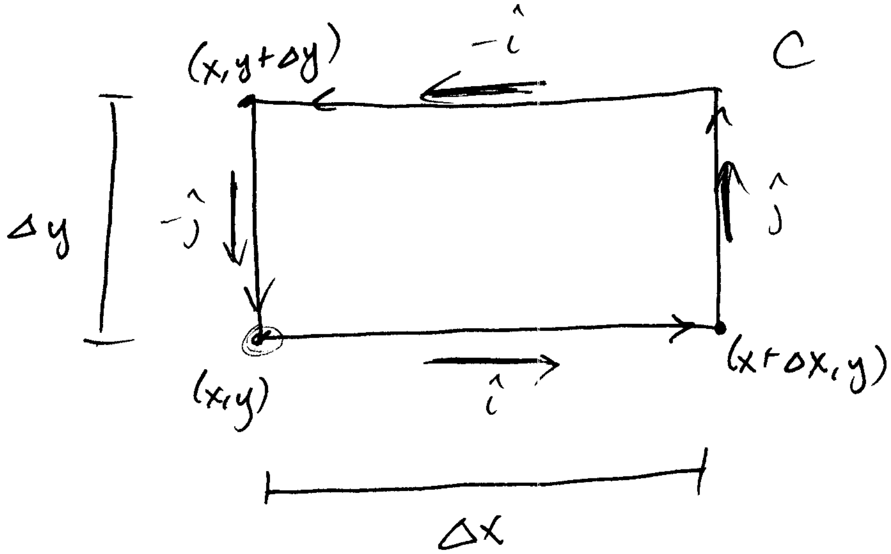


(d) $\vec{F} = (0, x)$ (shear field)

$$\operatorname{div} \vec{F} = 0$$



Same game for circulation:



$$\vec{F} = (M, N) = M\hat{i} + N\hat{j}$$

Circulation $\approx \vec{F} \cdot \vec{T}\,\Delta s$

$$\begin{aligned}
\text{Bottom} &= \vec{F}(x,y) \cdot (\hat{i}\,\Delta x) = M(x,y)\,\Delta x \\
\text{Top} &= \vec{F}(x, y+\Delta y) \cdot (-\hat{i}\,\Delta x) = -M(x, y+\Delta y)\,\Delta x \\
\text{Right} &= \vec{F}(x+\Delta x, y) \cdot (\hat{j}\,\Delta y) = N(x+\Delta x, y)\,\Delta y \\
+\quad \text{Left} &= \vec{F}(x,y) \cdot (-\hat{j}\,\Delta y) = -N(x,y)\,\Delta y
\end{aligned}$$

$$\begin{aligned}
\text{Approx. Circ.} &= [N(x+\Delta x, y) - N(x,y)]\,\Delta y \\
&\quad - [M(x, y+\Delta y) - M(x,y)]\,\Delta x \\
&\approx \left[\frac{\partial N}{\partial x}(x,y)\,\Delta x\right]\Delta y - \left[\frac{\partial M}{\partial y}(x,y)\,\Delta y\right]\Delta x \\
&= [N_x(x,y) - M_y(x,y)]\,\Delta x\,\Delta y
\end{aligned}$$

$$\boxed{\text{Circ. density} = N_x(x,y) - M_y(x,y)}$$

$$= \hat{k} \text{ component of curl } (M, N, 0)$$

$\vec{F}$ in xy-plane.

Ex: (a) $\vec{F} = c(x,y) = (cx)\hat{i} + (cy)\hat{j}$

$\text{curl}\, \vec{F} \cdot \hat{k} = N_x - M_y = 0 \Rightarrow$ irrotational.

(b) $\vec{F} = c(-y, x)$

$\text{curl}\, F \cdot \hat{k} = N_x - M_y = c - (-c) = 2c$

$c > 0$: c.c.w. rot.

$c < 0$: c.w. rot.

$c = 0$: irrotational.

(c) $\vec{F} = (a, b) \rightarrow \text{curl}\, \vec{F} \cdot \hat{k} = 0$

(d) $\vec{F} = (0, x) \rightarrow \text{curl}\, \vec{F} \cdot \hat{k} = 1$

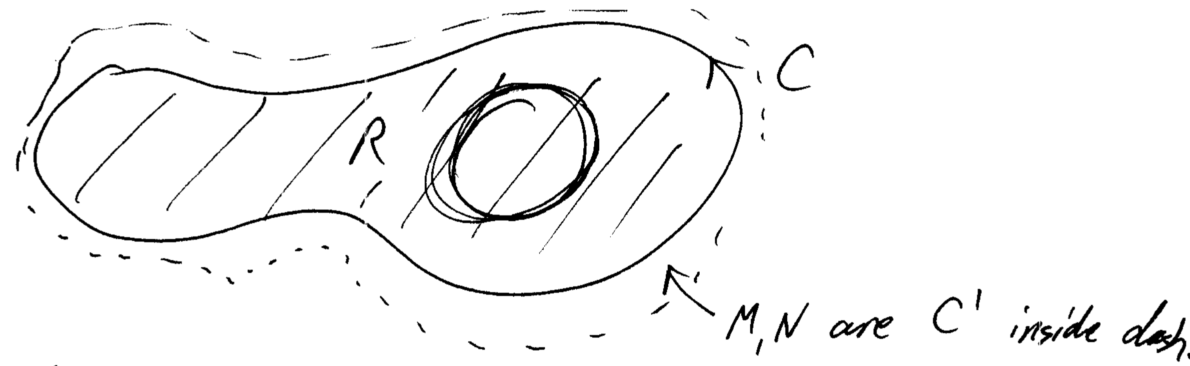
Statements of Green's Theorem

Theorem: Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\hat{i} + N\hat{j}$ where M and N are C^1 function (cont. first partials) in an open region containing R. Then

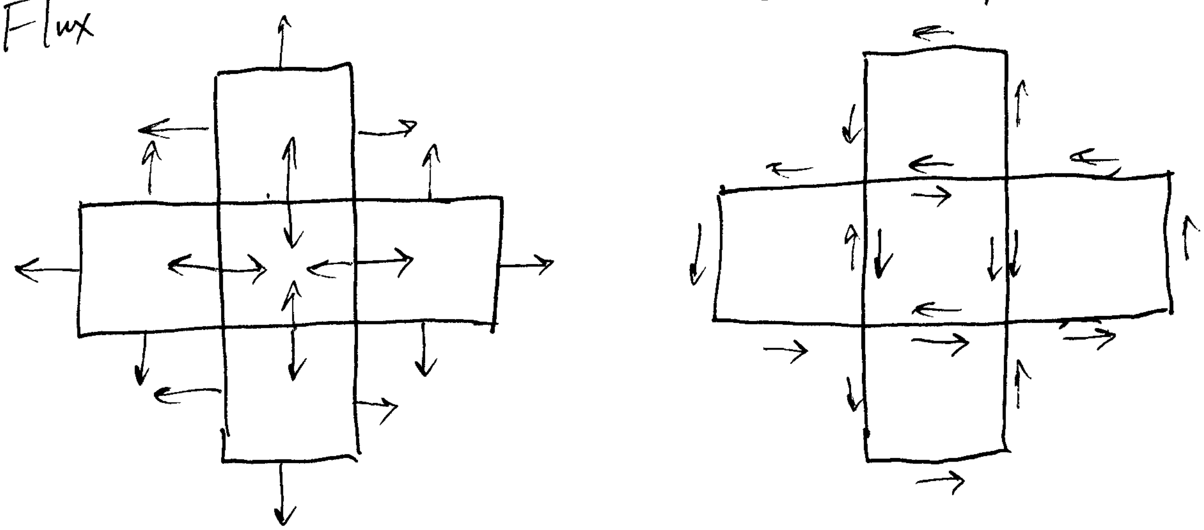
$$\text{Outward flux} = \oint_C \vec{F} \cdot \hat{n}\, ds = \oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\, dA$$
$$= \iint_R \text{div}\, \vec{F} \cdot dA$$

and
$$\text{c.c.w. circ.} = \oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$
$$= \iint_R (\text{curl}\, \vec{F} \cdot \hat{k})\, dA.$$

Picture:



Flux



Ex: $\vec{F}(x,y) = (-y, x)$ over unit circle C.

Directly compute: $\mathrm{Circ}_C(\vec{F}) = \oint_C \vec{F} \cdot \vec{T}\, ds$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} dt = 2\pi \quad (\ast)$$

$$\mathrm{Flux}_C(\vec{F}) = \oint_C \vec{F} \cdot \hat{n}\, ds$$

$$= \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (-\sin t, -\cos t) \cdot (\cos t, -\sin t)\, dt$$

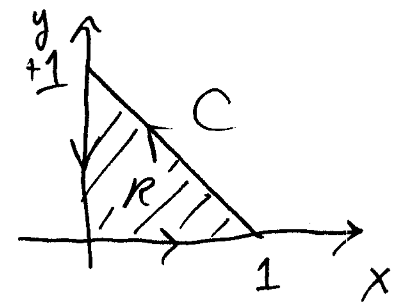
$$= \int_0^{2\pi} 0\, dt = 0. \quad (\ddot{\smile})$$

Green: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \cancel{M}(N_x - M_y)\,dA$

$$= \iint_R 2\cdot dA = 2\iint_R dA = 2\pi\,(A)$$

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA = \iint_R 0\,dA = 0$$

Ex: $\vec{F} = (\sin^2 x,\ \cos^2 y)$, C:



$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \text{curl}\,\vec{F}\cdot\hat{k}\,dA$$

$$= \iint_R (N_x - M_y)\,dA = 0$$

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \text{div}\,\vec{F}\,dA = \iint_R (M_x + N_y)\,dA$$

$$= \iint_R \big(2\sin x\cos x + (-2\cos y\sin y)\big)\,dA$$

$$= \iint_R (\sin 2x - \cancel{\cos}\sin 2y)\,dA$$

$$= \int_0^1 \int_0^{1-x} (\sin 2x - \sin 2y)\,dy\,dx$$

$$= \int_0^1 \Big[y\sin 2x + \tfrac{1}{2}\cos 2y\Big]_{y=0}^{1-x}\,dx$$

$$= \int_0^1 \Big[(1-x)\sin 2x + \tfrac{1}{2}\cos(2(1-x)) - \tfrac{1}{2}\Big]\,dx$$

	u	dv
$+$	x	$\sin 2x$
$-$	1	$-\frac{1}{2}\cos 2x$
$+$	0	$-\frac{1}{4}\sin 2x$

$$= \int_0^1 \left(\sin 2x - x\sin 2x + \frac{1}{2}\cos(2-2x) - \frac{1}{2}\right)dx$$

$$= \Big[-\frac{1}{2}\cos(2x) - \frac{1}{2}x - \frac{1}{4}\cancel{\cos}\sin(2-2x)$$
$$- \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x \Big]_{x=0}^{1}$$

$$= \left[-\frac{1}{2}\cos 2 - \frac{1}{2} - \frac{1}{2}\cos 2 + \frac{1}{4}\sin 2\right]$$
$$- \left[-\frac{1}{2} - \frac{1}{4}\sin 2\right]$$

$$= \boxed{-\cos 2 + \frac{1}{2}\sin 2}$$

Computing area: $\oint_C M\,dy - N\,dx = \iint_R \operatorname{div}(M,N)\,dA.$

$$= \iint_R dA$$

Need $\operatorname{div}\vec{F} = 1$.

Let $\vec{F}(x,y) = (ax, by) \rightarrow \operatorname{div}\vec{F} = a+b = 1$

Use $a=b=\frac{1}{2} \rightarrow \vec{F} = (\frac{1}{2}x, \frac{1}{2}y)$

$$\oint_C \frac{1}{2}x\,dy - \frac{1}{2}y\,dx = \boxed{\frac{1}{2}\oint_C x\,dy - y\,dx}$$

Warm-up: We seen that for any smooth ~~vector field $\vec{F}$~~ function f we have $\nabla \times (\nabla f) = \vec{0}$. Now check that for each smooth vector field $\vec{F}$, we have $\operatorname{div}(\nabla \times \vec{F}) = 0$.

Let $\vec{F} = (M, N, P)$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = (P_y - N_z,\ M_z - P_x,\ N_x - M_y)$$

$$\operatorname{div}(\nabla \times \vec{F}) = (P_{yx} - N_{zx}) + (M_{zy} - P_{xy}) + (N_{xz} - M_{yz})$$

$$= 0 \implies \nabla \times \vec{F} \text{ is } \underline{\text{incompressible}}!$$

---

§16.5 - Surfaces and Area

Parametrizing surfaces

Explicitly : $z = f(x,y)$ / $y = f(x,z)$ / $x = f(y,z)$

Implicitly : $F(x,y,z) = 0$

Parametrically : $\vec{r}(u,v) = (f(u,v), g(u,v), h(u,v))$

$(u,v) \in R \subseteq \mathbb{R}^2$.

u, v called <u>parameters</u>, R called <u>parametrization domain</u> (usually rectangular)

Ex: ① Planes: $Ax + By + Cz = D$ $(A, B, C, D \text{ const})$.

$(C \neq 0)$ $z = \frac{1}{C}(D - Ax - By)$

$$S: \begin{cases} x = u, \quad y = v, \quad z = \frac{1}{C}(D - Au - Bv) \\ \qquad\qquad (u,v) \in \mathbb{R}^2. \end{cases}$$

$$S: \begin{cases} \vec{r}(u,v) = \left(u, v, \frac{1}{C}(D - Au - Bv)\right) \\ (u,v) \in \mathbb{R}^2 \end{cases}$$

② Spheres: $x^2 + y^2 + z^2 = R^2$ $(R > 0)$

Spherical coords w/ $\rho = R$ const.

$$S: \begin{cases} x = (R \sin\varphi) \cos\theta \\ y = (R \sin\varphi) \sin\theta \\ z = R \cos\varphi \end{cases} \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}$$

③ Cylinders: $x^2 + y^2 = R^2$ $(R > 0)$

Cyl. coords w/ $r = R$ const.

$$S: \begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} 0 \leq \theta \leq 2\pi \\ -\infty < z < \infty \quad (z \in \mathbb{R}) \end{array}$$

4 Cones: $x^2+y^2=z^2 \quad (z \geq 0)$

(a) $z=\sqrt{x^2+y^2}$

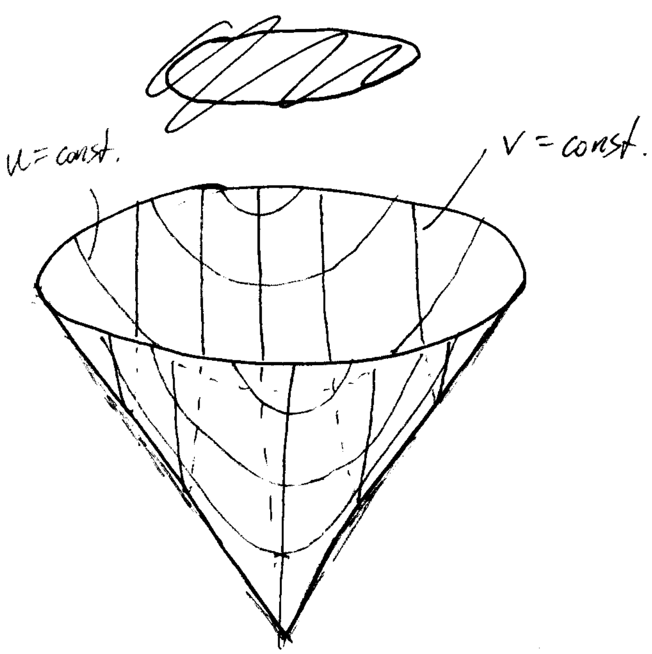
So $\begin{cases} \vec{r}(u,v) = (u, v, \sqrt{u^2+v^2}) \\ (u,v) \in \mathbb{R}^2 \end{cases}$

Cyl. coords

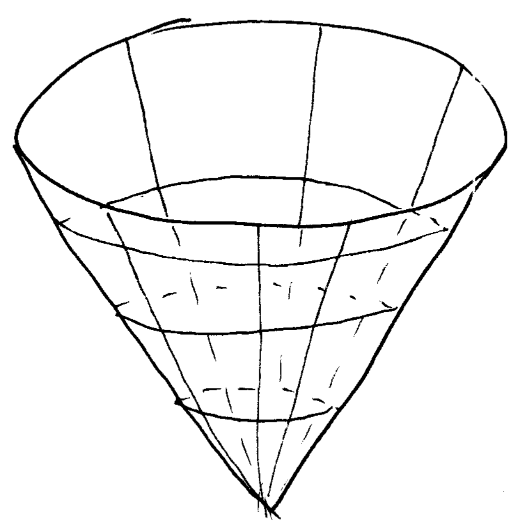
(b) $z^2 = r^2 \rightarrow z = r$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases} \qquad \begin{matrix} 0 \leq r < \infty \\ 0 \leq \theta \leq 2\pi \end{matrix}$

What's the difference?

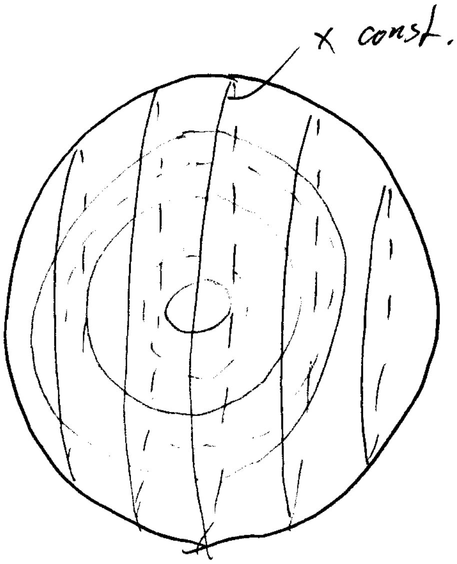


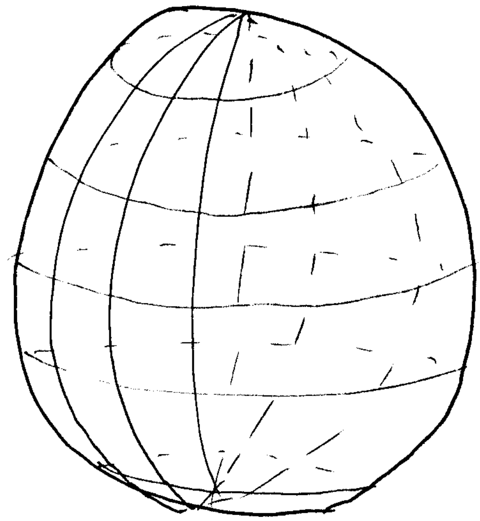
(a) Explicit



(b) "Natural"

E.g. Spheres





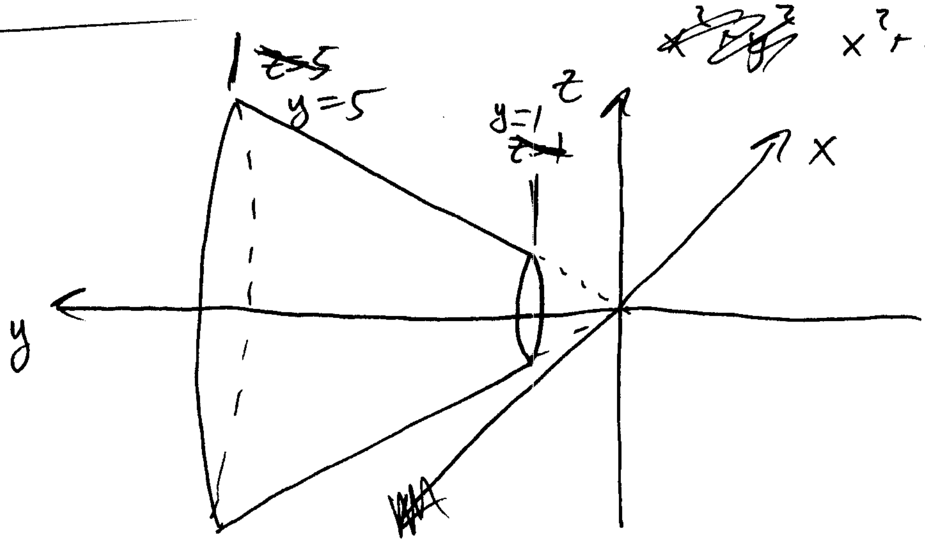
(a) Explicit

$$z = \pm\sqrt{R^2 - x^2 - y^2}$$

(b) "Natural"

spherical coords.

Exercise: Parametrize this frustum of the cone $x^2 + z^2 = y^2$



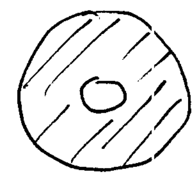
$$S: \begin{cases} x = r\cos\theta \\ y = r \\ z = r\sin\theta \end{cases}$$

$1 \le r \le 5$

$0 \le \theta \le 2\pi$

$$S: \begin{cases} x = u \\ z = v \\ y = \sqrt{u^2+v^2} \end{cases}$$

$1 \le u^2 + v^2 \le 25$



Exercise : Parametrize the ~~ellipse~~ ellipsoid : $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

$(a, b, c > 0)$

$$X = \frac{x}{a}, \quad Y = \frac{y}{b}, \quad Z = \frac{z}{c} \Rightarrow X^2 + Y^2 + Z^2 = 1$$

$$\begin{cases} X = \sin\varphi\cos\theta \\ Y = \sin\varphi\sin\theta \\ Z = \cos\varphi \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$

change back

$$\begin{cases} x = a\sin\varphi\cos\theta \\ y = b\sin\varphi\sin\theta \\ z = c\cos\varphi \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$

<u>Warm-up</u>: Show that if $z = f(x,y)$, then for a function $F(x,y,z) = c$ (const) we have

$$f_x = \frac{-F_x}{F_z} \quad \text{and} \quad f_y = \frac{-F_y}{F_z}.$$

$$F(x, y, f(x,y)) = c$$

$$F_x \cdot 1 + F_y \cdot 0 + F_z \cdot f_x = 0$$

$$f_x = \frac{-F_x}{F_z}.$$

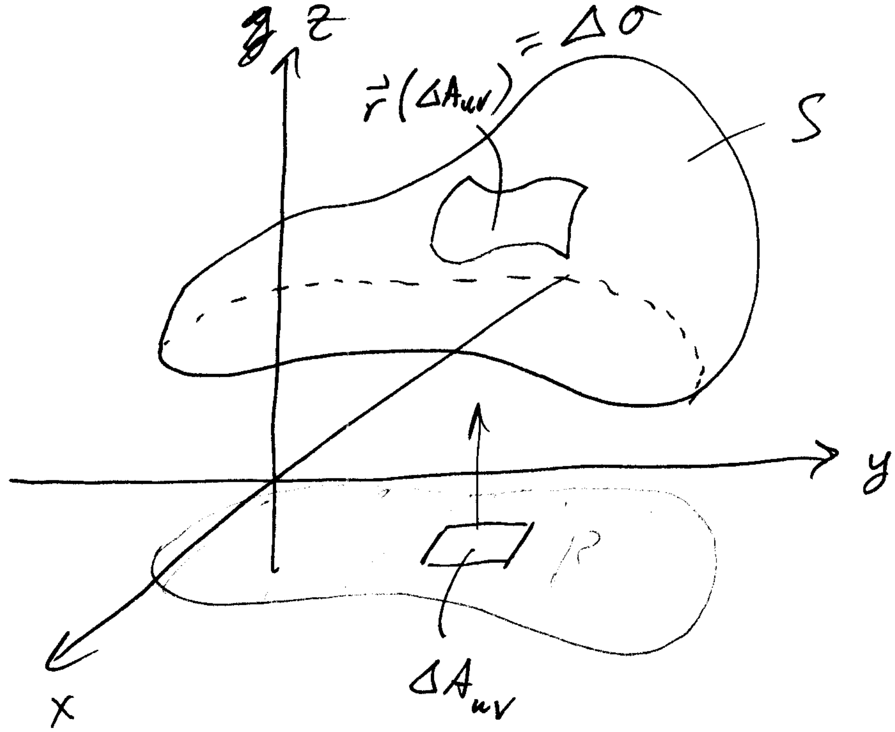
f_y calculated similarly.

<u>Continue §16.5</u>

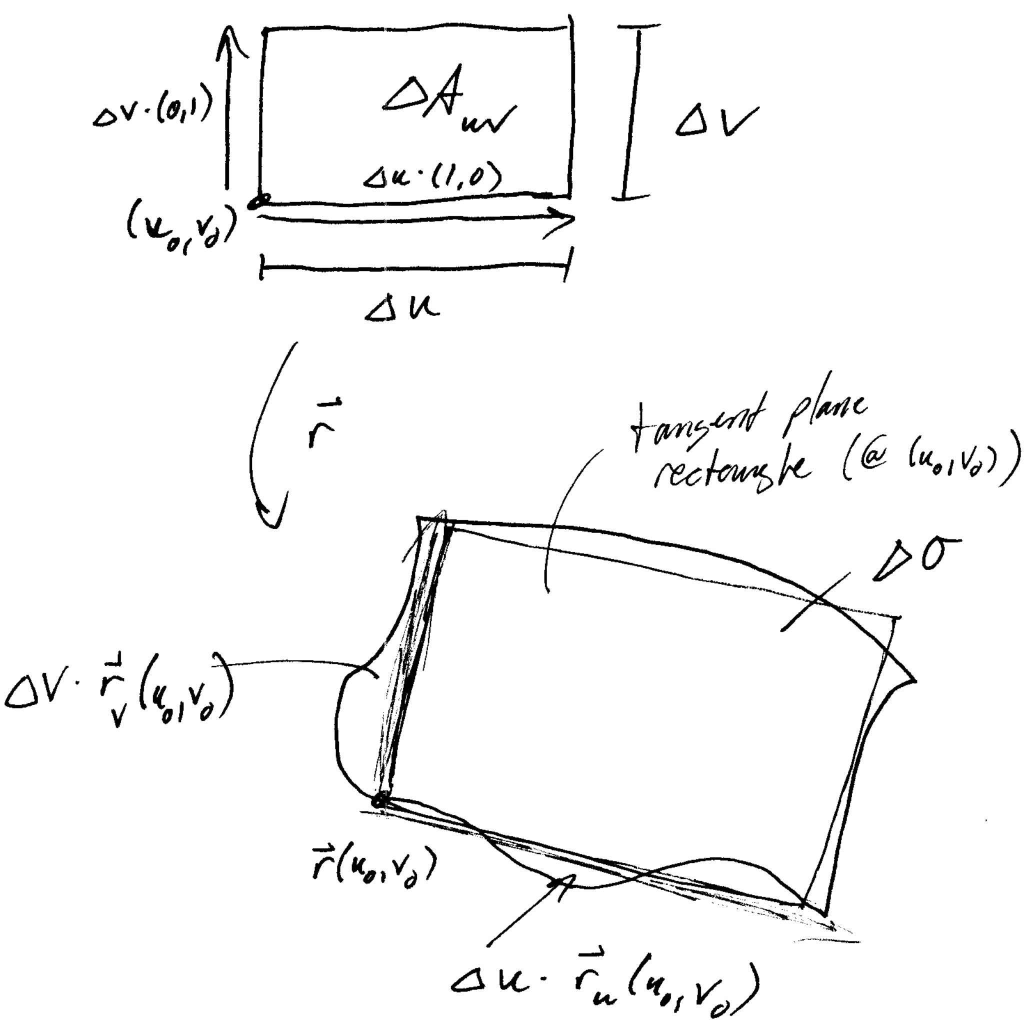
Parametrize surface $S: \vec{r}(u,v) = (f, g, h)$ over domain R.

<u>Def^n</u>: S is regular/smooth if $f, g, h \in C^1(R)$ <u>and</u> $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ on $int(R)$.

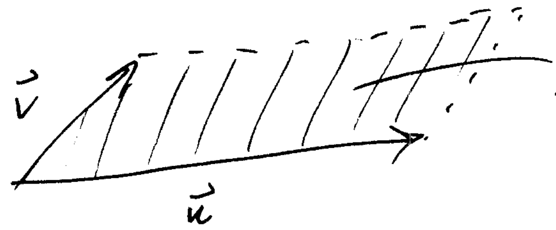
Picture:



Consider a small rectangle in R:



How much area is in $\Delta\sigma$?



Area of parallelogram $= |\vec{u} \times \vec{v}|$

$$\text{Area of } \Delta\sigma = \left|(\Delta v \cdot \vec{r}_v(u_0,v_0)) \times (\Delta u \cdot \vec{r}_u(u_0,v_0))\right|$$

$$\boxed{\Delta\sigma = |\vec{r}_u \times \vec{r}_v (u_0,v_0)|\, \Delta u \Delta v}$$

Taking limit as $\|P(u,v)\| \to 0$, we get

$$\lim_{N\to\infty} \sum_{i=1}^{N} \Delta\sigma_i = \iint_S d\sigma$$

$$\boxed{\text{Define surface area } SA(S) = \iint_S d\sigma}$$

Evaluating $SA(S)$: $\vec{r}(u,v)$ over $(u,v) \in D$.

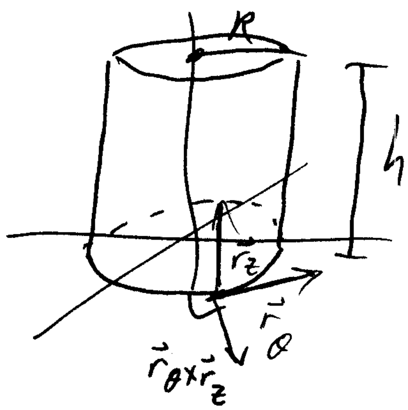
$$\boxed{\iint_S d\sigma = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv}$$

Verify it!

Ex: Surface area of a cylinder: $x^2+y^2=R^2$

$0 \le z \le h$

$R > 0$.



Should be

length $\times$ height $= h$

$= 2\pi R \cdot h$.

$$S: \begin{cases} \vec{r}(\theta, z) = (R\cos\theta, R\sin\theta, z) \\ 0 \le \theta \le 2\pi \\ 0 \le z \le h \end{cases}$$

$$\vec{r}_\theta = (-R\sin\theta, R\cos\theta, 0)$$

$$\vec{r}_z = (0, 0, 1)$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (R\cos\theta, R\sin\theta, 0) \quad \text{(normal to } S\text{)}$$

$$|\vec{r}_\theta \times \vec{r}_z| = \sqrt{(R\cos\theta)^2 + (R\sin\theta)^2 + 0^2} = R.$$

So $SA(S) = \int_0^{2\pi}\int_0^h R \cdot dz\, d\theta = R \cdot \int_0^{2\pi} d\theta \cdot \int_0^h dz$

$= \boxed{2\pi R h}$ ✓

Ex: Spheres - $x^2+y^2+z^2=R^2$ $(R>0)$

$$S: \begin{cases} \vec{r}(\theta,\varphi) = (R\sin\varphi\cos\theta,\ R\sin\varphi\sin\theta,\ R\cos\varphi) \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$$

$$\vec{r}_\theta = (-R\sin\varphi\sin\theta,\ R\sin\varphi\cos\theta,\ 0)$$

$$\vec{r}_\varphi = (R\cos\varphi\cos\theta,\ R\cos\varphi\sin\theta,\ -R\sin\varphi)$$

$$|\vec{r}_\theta \times \vec{r}_\varphi| = R^2\sin\varphi.$$

$$\begin{aligned} SA(S) &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\varphi\, d\varphi\, d\theta \\ &= R^2 \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi}\sin\varphi\, d\varphi \\ &= R^2 \cdot 2\pi \cdot 2 = \boxed{4\pi R^2} \end{aligned}$$

What happens if we can't find $\vec{r}(u,v)$?

Implicit functions: $F(x,y,\cancel{z}) = c$ (const).

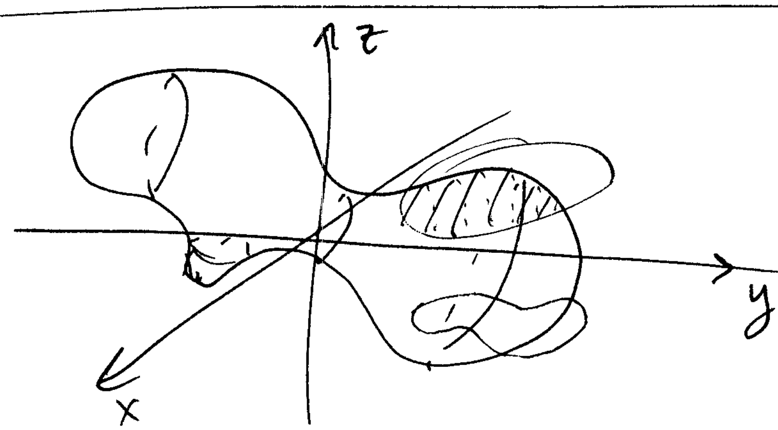
e.g. spheres: $x^2+y^2+z^2=R^2$

$x^3+y^3-z^3=R^2$

$xy^2-x^3=4$ ...

Theorem: Given a surface S (defined $F(x,y,z)=c$), there ~~xxx~~ is a way to find an $\vec{r}(u,v)$ at least locally:

$$z=f(x,y)$$
$$y=f(x,z)$$
$$x=f(y,z)$$



E.g. $\vec{r}(u,v)=(u,v,f(u,v))$ on $(u,v)\in\mathcal{U}$.

Then $\vec{r}_u=(1,0,f_u)$

$\vec{r}_v=(0,1,f_v)$.

Notice that $F(\vec{r}(u,v))=c$

$F(u,v,f(u,v))=c$ (Implicit derivative theorem)

From warm-up, $f_u=\dfrac{-F_x}{F_z}$ and $f_v=\dfrac{-F_y}{F_z}$

Consider $|\vec{r}_u \times \vec{r}_v| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} \right\|$

$$= |(-f_u, -f_v, 1)|$$

$$= \sqrt{1 + (f_u)^2 + (f_v)^2}$$

$$= \sqrt{1 + \left(\frac{-F_x}{F_z}\right)^2 + \left(\frac{-F_y}{F_z}\right)^2}$$

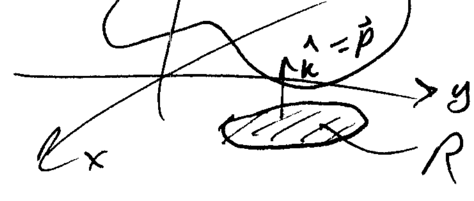
$$= \frac{1}{|F_z|}\sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$= \frac{|\nabla F|}{|F_z|} = \frac{|\nabla F|}{|\nabla F \cdot \hat{k}|}$$

<u>Theorem:</u> The area of a level surface S defined by $F(x,y,z) = C$ over a closed and bounded planar region R is

$$SA(S) = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA$$

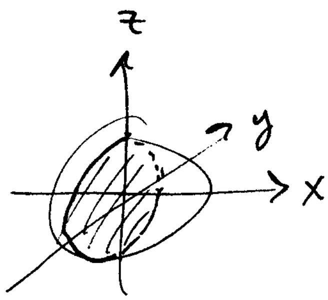
where $\vec{p} = \hat{i}, \hat{j},$ or $\hat{k}$ (whichever is normal to R) and $\nabla F \cdot p \neq 0$ on R.



Ex: sphere again: $F(x,y,z) = x^2+y^2+z^2 = R^2$ $(R>0)$

$$\nabla F = (2x, 2y, 2z)$$

$$\nabla F \cdot \hat{i} = 2x \qquad \nabla F \cdot \hat{j} = 2y \qquad \nabla F \cdot \hat{k} = 2z.$$



$$\text{Area} = \iint_D \frac{|\nabla F|}{|\nabla F \cdot \hat{k}|}\, dA$$

$D =$ disk rad R in yz-plane.

$$= \iint_D \frac{\sqrt{4(x^2+y^2+z^2)}}{|2x|}\, dA$$

$$= \iint_D \cancel{4}\,2\,\frac{R}{|2x|}\, dA$$

$$= \iint_D \frac{\cancel{2}\,R}{\sqrt{R^2-y^2-z^2}}\, dA$$

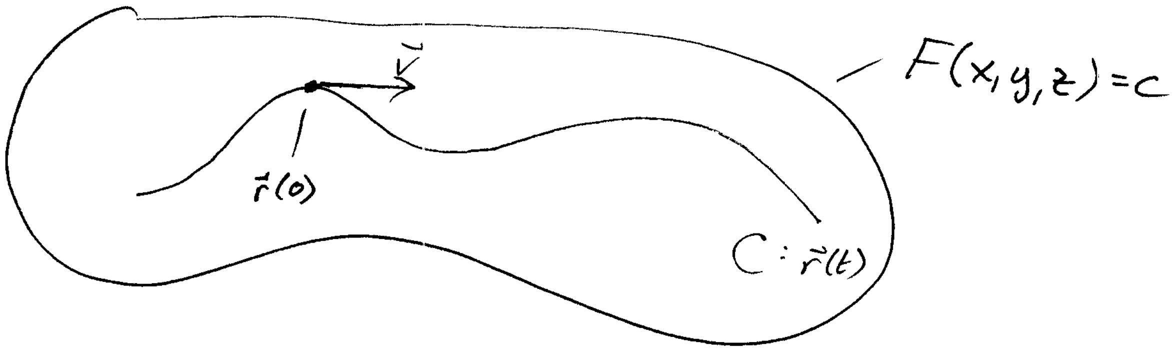
$$= \int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2-r^2}} \cdot r\,dr\,d\theta$$

$$= 2\pi R\left[-2\sqrt{R^2-r^2}\right]_{r=0}^{R}$$

$$= \boxed{4\pi R^2}. \checkmark$$

<u>Warm-up</u>: Show that for an implicit surface $S : F(x,y,z) = c$ the gradient ∇F is normal to S. That is, for each tangent vector on S, ∇F is orthogonal to it.

[Hint: Every tangent vector $\vec{v}$ at $p \in S$ has a curve $\vec{r}(t)$ whose velocity $\vec{r}(0) = \vec{v}$.]



Get to: $\nabla F(\vec{r}(0)) \cdot \vec{r}\,'(0) = 0$

Consider $F(\vec{r}(t)) = c$

$$\frac{d}{dt}\left(F(\vec{r}(t))\right) = \frac{d}{dt}(c)$$

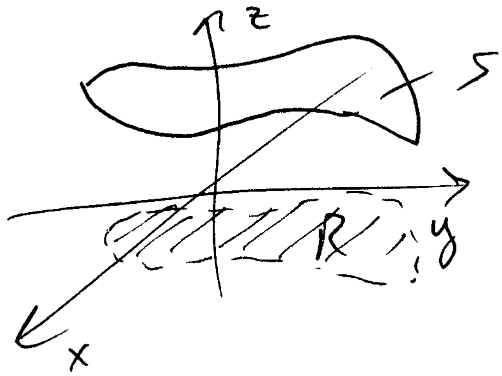
$$\nabla F(\vec{r}(t)) \cdot \vec{r}\,'(t) \Big|_{t=0} = 0$$

$$\nabla F(\vec{r}(0)) \cdot \vec{v} = 0 \quad \checkmark$$

Last remarks from §16.5

Special case of implicit surfaces: $S: F(x,y,z) = c$.

Specifically: Let $z = f(x,y)$ // $[F(x,y,z) = f(x,y) - z]$.
(graph(f))



What is $SA(S)$?

$$SA(S) = \iint_S d\sigma$$

$$= \iint_R |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

graph(f): $\vec{r}(u,v) = (u, v, f(u,v))$
$(u,v) \in R$

So $\vec{r}_u = (1, 0, f_u)$
$\vec{r}_v = (0, 1, f_v)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)$$

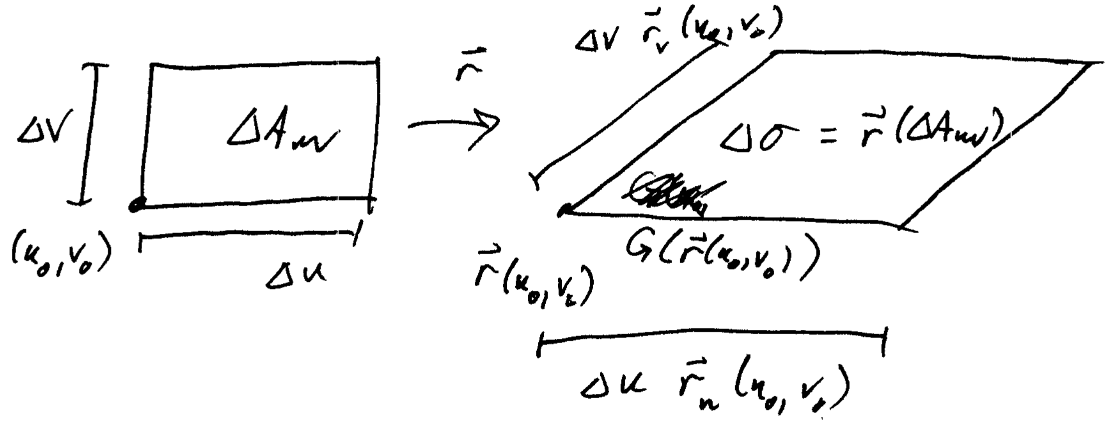
$$|\vec{r}_u \times \vec{r}_v| = \sqrt{f_u^2 + f_v^2 + 1} \Rightarrow SA(S) = \iint_R \sqrt{1 + f_u^2 + f_v^2}\, du\,dv$$

Analogous to: Arclength of $y = f(x)$ on $[a,b]$
$$= \int_a^b \sqrt{1 + (f')^2}\, dx$$

§16.6 – Surface integrals

Want: Analog for $\int_C f(s)\,ds$ ...

Recall:



This is how we define <u>surface integrals</u>

$$\boxed{\iint_S G(x,y,z)\,d\sigma = \iint_{R_{uv}} G(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv}$$

Same formulas apply:

Explicit, Implicit, General

$\sqrt{1+f_u^2+f_v^2}\,du\,dv$

$\left|\dfrac{\nabla F}{\nabla F \cdot \vec{p}}\right| dA$

$|\vec{r}_u \times \vec{r}_v|\,du\,dv$

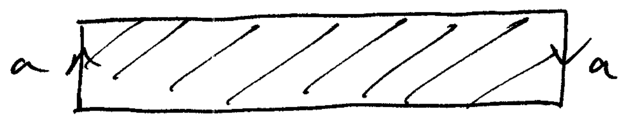
Flux through surfaces

Need a well-defined notion of "across."

Fix: Assign an orientation to S (gives $\hat{n}$).

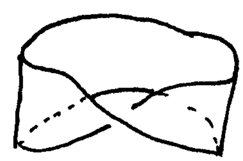
Problem: Not all surfaces have a <u>global</u> notion of "across" / "two-sideness".

E.g.



half-twist and glue a to a

Results in



Möbius strip.

<u>Def^n</u>: A surface S is <u>orientable</u> if there exists a globally defined normal vector field $\hat{n}$ on it.

Otherwise, S is called <u>non-orientable</u>.

We will ignore non-orientable surfaces: No well-defined notion of "area"/ "enclosing volumes".

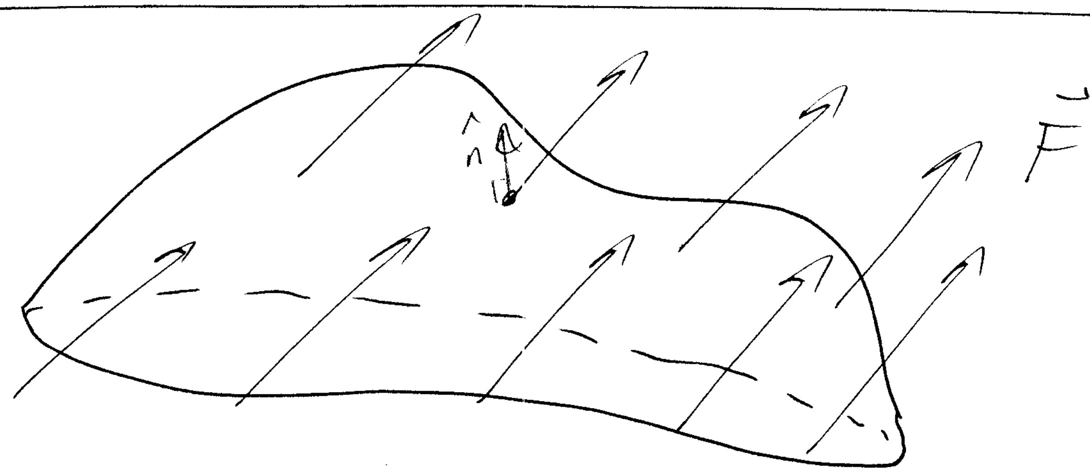
Orientability gives a notion of two sides. Flux will tell us how a vector field penetrates the surface wrt $\hat{n}$.

Finding $\hat{n}$:

① $F(x,y,z) = c \rightarrow \hat{n} = \pm \widehat{(\nabla F)}$

② $\vec{r}(u,v) = (f(u,v), g(u,v), h(u,v)) \rightarrow \hat{n} = \pm \widehat{(\vec{r}_u \times \vec{r}_v)}$

③ $\vec{r}(u,v) = (u, v, f(u,v)) \rightarrow \hat{n} = \pm \widehat{(-f_u, -f_v, 1)}$



How much of $\vec{F}$ points along $\hat{n}$?

Take $\vec{F} \cdot \hat{n} = G(x,y,z)$.

<u>Def^n</u>: The <u>flux</u> of a three-dimensional vect. field $\vec{F}$ across an oriented surface S in the direction of $\hat{n}$ is

$$\text{Flux}_S(\vec{F}) = \iint_S \underbrace{\vec{F} \cdot \hat{n}}_{\text{flux density}} \, d\sigma .$$

(related to divergence)
Gauss theorem

Remark: $\iint_S \underbrace{(\nabla \times \vec{F}) \cdot \hat{n}}_{\text{circulation density}} \, d\sigma$ ... is related
Circulation (a 1D integral)

Ex: Want $\text{Flux}_S(\vec F)$ where

$\vec F = (x,y,z)$ and S is the sphere of radius $R > 0$ at origin.

(Electrical flux of $\vec E$, e-field)

use sph. param.

$$S: \; x^2+y^2+z^2 = R^2 \rightarrow \; = (R^2\sin^2\varphi\cos\theta, \; R^2\sin^2\varphi\sin\theta, \; R^2\sin\varphi\cos\varphi)$$

$$\vec r_\varphi \times \vec r_\theta$$

$$\hat n = \pm(\vec r_\varphi \times r_\theta) = \frac{\pm 1}{R^2\sin\varphi}(\vec r_\varphi \times \vec r_\theta)$$

$$= (\sin\varphi\cos\theta, \; \sin\varphi\sin\theta, \; \cos\varphi)$$

$$= \frac{R(\cdots)}{R}$$

$$= \frac{(x,y,z)}{R} = \hat r \quad \text{(unit radial vect. field)}$$

Then

$$\text{Flux}_S(\vec F) = \iint_S (x,y,z)\cdot \hat r \, d\sigma$$

$$= \iint_S (x,y,z)\cdot \frac{(x,y,z)}{R}\, d\sigma$$

$$= \iint_S \frac{R^2}{R}\, d\sigma = R\iint_S d\sigma$$

$$= R\cdot 4\pi R^2 = \boxed{4\pi R^3} > 0.$$

No warm-up

---

§16.7 - Stokes' Theorem

Generalize Green's theorem

Theorem: Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve $\partial S = C$. Let $\vec{F} = (M, N, P) \in C^1(D \to \mathbb{R}^3)$ (D open contains S).

Then
$$\boxed{\oint_{\partial S = C} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, d\sigma .}$$

To make sense of this, we should interpret $\nabla \times \vec{F}$.

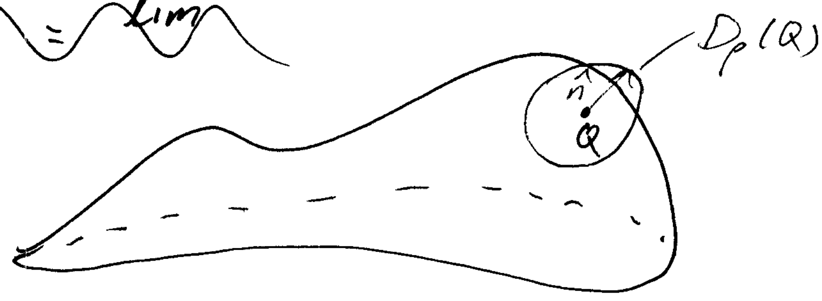
Consider for a function f, we have
$$f(\vec{x}) = \lim_{|D| \to 0} \frac{1}{\operatorname{Vol}(D)} \int_D f(\vec{x}) \, d\vec{x} .$$

"limit of averages = value of function".

Use $(\nabla \times \vec{F}) \cdot \hat{n} \big|_Q = f(Q)$: $\hat{n}$ normal vector field to S, Q is in S.

Goal: Find average over a disk $D_\rho(Q)$.

~~Then~~ $\cancel{(\nabla \times \vec{F}) \cdot \hat{n} \big|_Q = \lim}$



$$(\nabla\times\vec{F})\cdot\hat{n}\Big|_Q = \lim_{\rho\to 0}\frac{1}{\text{Area}(D_\rho(Q))}\iint_{D_\rho(Q)}(\nabla\times\vec{F})\cdot\hat{n}\,d\sigma$$

$$= \lim_{\rho\to 0}\frac{1}{\pi\rho^2}\oint_{\partial D_\rho(Q)}\vec{F}\cdot d\vec{\sigma}$$

$$= \lim_{\rho\to 0}\frac{1}{\pi\rho^2}\,\text{Circ}_{\partial D_\rho(Q)}(\vec{F})$$

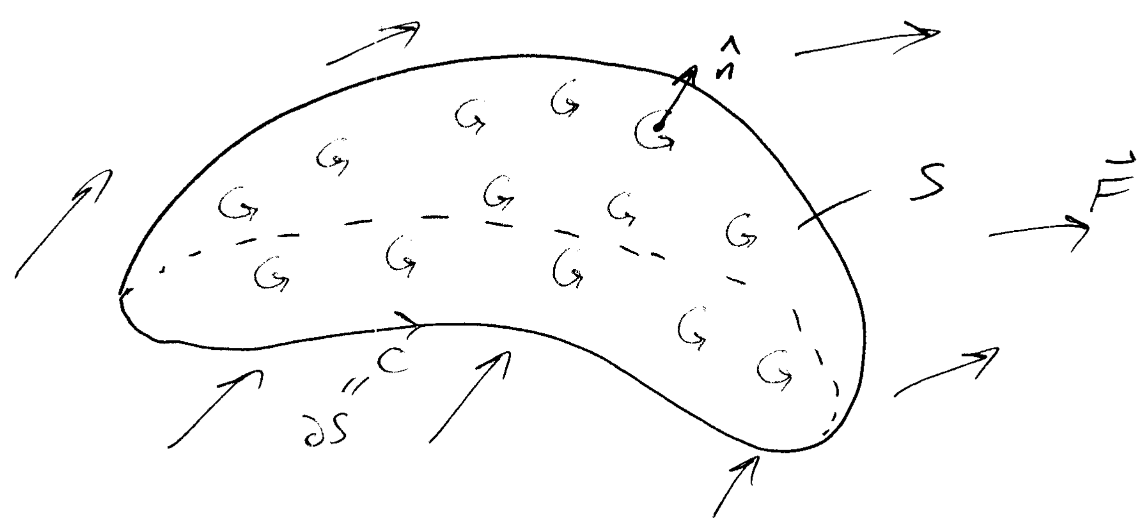
$$\overset{(\lim)}{=} \text{average circulation}$$

$$|\nabla\times\vec{F}|\cancel{|\hat{n}|}^{\,1}\cos\theta = \text{circulation density}$$

<u>Moral:</u> $\nabla\times\vec{F}$ encodes two things

1. $|\nabla\times\vec{F}|$ = greatest circ. density at Q.
2. $\widehat{(\nabla\times\vec{F})}$ = axis of rotation where $(\nabla\times\vec{F})\cdot\hat{n}$ is greatest.

<u>Pictures:</u>

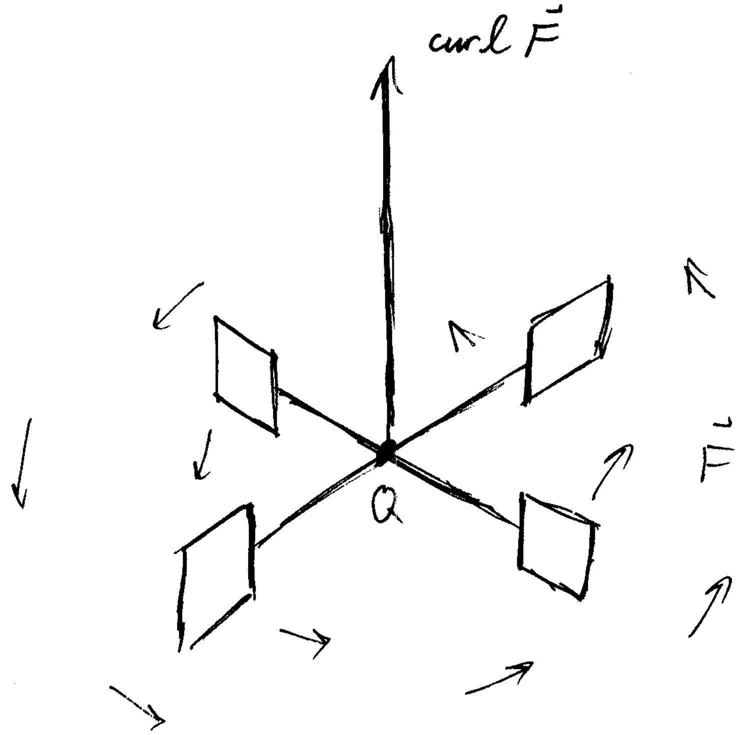


Same cancellations as before in Green's theorem.

$$\iint_D (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma = \oint_{\partial D} \vec{F}\cdot d\vec{r}$$

This measures flux of $\nabla\times\vec{F}$ in S.

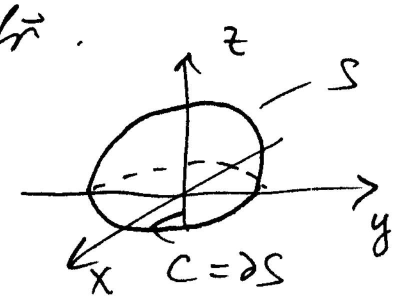
Namely, How much of $\nabla\times\vec{F}$ aligns w/ $\hat{n}$ and how much circulation <u>inside</u> S affects the boundary ∂S.



Ex: Verification - Hemisphere : $x^2+y^2+z^2=R^2,\ z\geq 0.$

~~Vel~~ Field : $\vec{F} = \omega(-y,x,0)$ ($\omega\in\mathbb{R}$)

Show $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}$.



① Give S an orientation:

Inward normal: $\hat{r}$ is normal to S

$$\hat{r} = \frac{(x,y,z)}{R} \rightarrow \text{inward} = -\hat{r} = \frac{-(x,y,z)}{R}.$$

② Find $(\nabla\times\vec{F})\cdot\hat{n}$:

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= (0,0,2\omega)$$

Then $(\nabla\times\vec{F})\cdot\hat{n} = (0,0,2\omega)\cdot\frac{-1}{R}(x,y,z)$

$$= \frac{-2\omega z}{R}$$

③ Find $d\sigma$:

Implicit function: $F(x,y,z) = x^2+y^2+z^2 = R^2$

$|\nabla F| = 2R$, $|\nabla\vec{F}\cdot\hat{k}| = 2|z| = 2z$.

$$d\sigma = \frac{2R}{2z}\,dA = \frac{R}{z}\,dx\,dy$$

④ Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma$:

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \iint_{D_{xy}} \frac{-2\omega z}{R} \cdot \frac{R}{z}\, dx\,dy$$

$$= -2\omega \iint_{D_{xy}} dx\,dy$$

$$= -2\omega \cdot \pi R^2 = \boxed{-2\pi R^2 \omega}$$

ccw circ.

⑤ Check w/ $\oint_C \vec{F} \cdot d\vec{r}$:

$$C: \begin{cases} x = R\cos(\theta) \\ y = R\sin(\theta) \\ z = 0 \end{cases} \qquad 0 \le \theta \le 2\pi$$

$\downarrow$

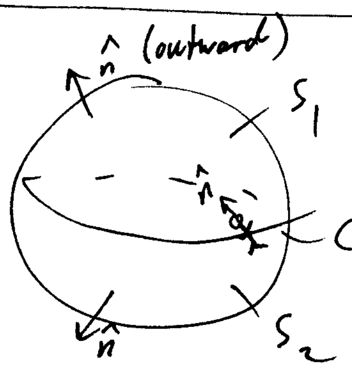
$$C: \begin{cases} x = R\cos\theta \\ y = -R\sin\theta \\ z = 0 \end{cases} \qquad 0 \le \theta \le 2\pi$$

Then $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (+\omega R\sin\theta, \omega R\cos\theta, 0) \cdot (-R\sin\theta, -R\cos\theta, 0)\, d\theta$

$$= \int_0^{2\pi} -\omega R^2 (\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \boxed{-2\pi R^2 \omega}$$

Warm-up: Show that $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = 0$ for any closed surface (i.e. S has no boundary, $\partial S = \emptyset$).



IDEA: Split S into $S_1 \cup S_2$, use Stokes on each S_1, S_2.

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \oint_{\partial S} \vec{F} \cdot d\vec{r} \quad \text{(Stokes)}$$

If I can write $S = S_1 \cup S_2$, what are ∂S_1 and ∂S_2.

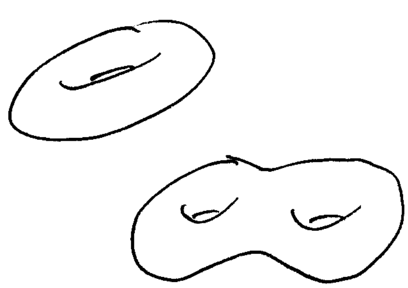
$\partial S_1 = C^+$ (ccw)

$\partial S_2 = C^-$ (cw)

$$\iint_S \cdots = \iint_{S_1} \cdots + \iint_{S_2} \cdots$$

$$= \oint_{C^+} \vec{F} \cdot d\vec{r} + -\oint_{C^+} \vec{F} \cdot d\vec{r}$$

$$= 0. \quad \blacksquare$$



---

§16.7 – continued

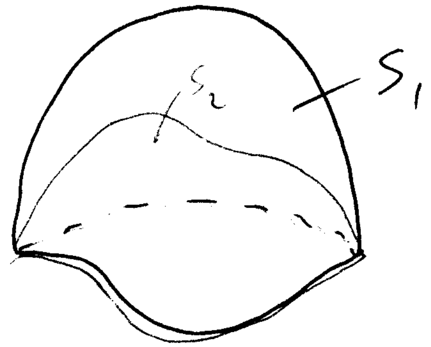
Recall: Stokes' theorem says

$$\underbrace{\iint_S (\text{curl}\, \vec{F}) \cdot \hat{n}\, d\sigma}_{\text{flux of curl through } S} = \underbrace{\oint_{C = \partial S} \vec{F} \cdot d\vec{r}}_{\text{circulation on boundary } S.}$$

Some consequences

① Suppose we have two surfaces S_1 & S_2 w/ same ∂. $\partial S_1 = \partial S_2 = C$

Then $$\iint_{S_1} (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma = \oint_{\partial S_1 = C = \partial S_2} \vec{F}\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma.$$



Ex: Last time, we used $\vec{F} = \omega(-y, x, 0)$ on
$S: x^2+y^2+z^2=R^2,\ z\geq 0.$

Found $$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma = \oint_{\partial S} \vec{F}\cdot d\vec{r} = -2\pi R^2\omega < 0$$

(inward)

(outward flux) / ccw circ.)

Stokes's says $$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,d\sigma = \iint_D \text{curl}\,\vec{F}\cdot\hat{n}\,d\sigma$$

$$\hat{n}_D = -\hat{k}$$

So $$\iint_D (0,0,2\omega)\cdot(0,0,-1)\,d\sigma$$
$$= \iint_D -2\omega\,d\sigma = -2\omega\iint_D d\sigma$$
$$= -2\omega\pi R^2$$

(2) Recall that $\nabla \times (\nabla f) = \vec{0}$ for any smooth function f.

Stokes requires a surface w/ boundary.

Use a surface w/ one ∂-component : A DISK!

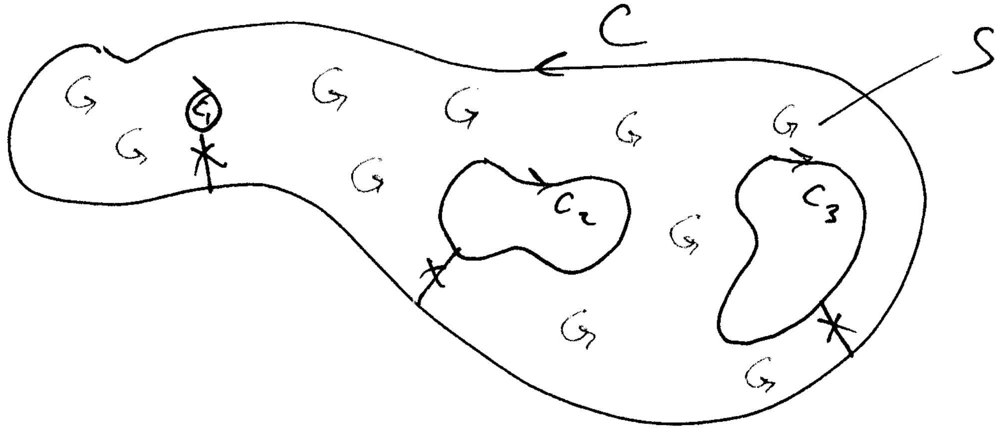
Disks are simply connected.

$$\text{So} \quad \oint_{\partial S} \vec{F} \cdot d\vec{r} = \oint_{\partial S} (\nabla f) \cdot d\vec{r} = \iint_S (\nabla \times \nabla f) \cdot \hat{n}\, d\sigma$$

$$= \iint_S \vec{0} \cdot \hat{n}\, d\sigma = 0.$$

What about punctures??? O.o

(3) Stokes over punctured surfaces:

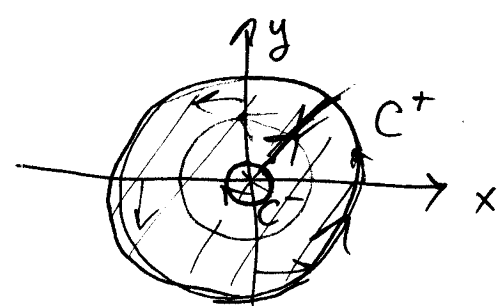


$\partial S = C_1 \cup C_2 \cup C_3 \cup C$

$$\text{Then} \quad \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \sum_{k=1}^{N} \oint_{C_k} \vec{F} \cdot d\vec{r} + \oint_C \vec{F} \cdot d\vec{r}$$

Special case : if $\vec{F} = \nabla f$ over non-simply connected D,

Then



$$0 = \iint_S (\nabla \times \nabla f) \cdot \hat{n}\, d\sigma = \oint_{C^+} \nabla f \cdot d\vec{r} + \oint_{C^-} \nabla f \cdot d\vec{r}$$

§ 16.8 – Divergence Theorem (Gauss' Theorem)

Divergence: $\vec{F} = (M, N, P)$, $\text{div}\,\vec{F} = \nabla \cdot \vec{F}$

Recall: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\boxed{\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}}.$$

Green's Theorem interpretation: div $\vec{F}$ measures flux density.

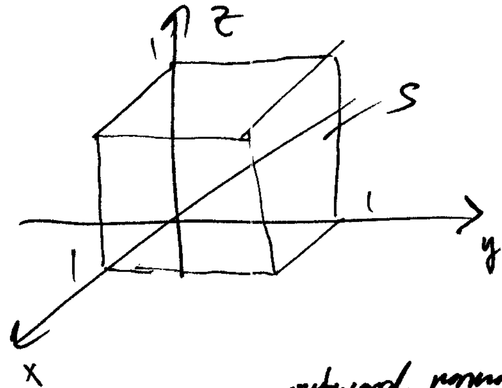
Recall: $\oint_{\partial D} M\,dy - N\,dx = \iint_D \text{div}(M,N)\,dA$

Theorem: Let $\vec{F} \in C^1_{\mathbb{R}^3 \to \mathbb{R}^3}(\mathring{S})$, S closed (no bdy) piecewise smth

Then
$$\boxed{\iint_{S=\partial D} \vec{F} \cdot \hat{n}\,d\sigma = \iiint_D \nabla \cdot \vec{F}\,dV}$$ ✱ ✱ ☺

Same interpretation: Flux across boundary surface = net divergence within it.

Ex: Outward flux of $\vec{F} = (x, y, z)$ through the cube $[0,1]^3$.



To compute $\iint_S \vec{F} \cdot \hat{n}\, d\sigma$ (outward normal): Need 6 separate computations.

6 faces (all w/ $[0,1]^2$)

$$\begin{cases} x = 0 : & \hat{n} = -\hat{i} & // & \vec{F} \cdot \hat{n} = -x \\ x = 1 : & \hat{n} = \hat{i} & // & = x \\ y = 0 : & \hat{n} = -\hat{j} & // & = -y \\ y = 1 : & \hat{n} = \hat{j} & // & = y \\ z = 0 : & \hat{n} = -\hat{k} & // & = -z \\ z = 1 : & \hat{n} = \hat{k} & // & = z \end{cases}$$

$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \cancel{\iint_{x=0} -x\, d\sigma}^{\,0} + \iint_{x=1} d\sigma = 1+1+1 = \boxed{3}$$

$$+ \cancel{\iint_{y=0} -y\, d\sigma}^{\,0} + \iint_{y=1} d\sigma$$

$$+ \cancel{\iint_{z=0} -z\, d\sigma}^{\,0} + \iint_{z=1} d\sigma$$

$$\iint_{S=\partial D} \vec{F} \cdot \hat{n}\, d\sigma = \iiint_{D=\text{cube interior}} \operatorname{div} \vec{F}\, dV = \iiint_D (1+1+1)\, dV = 3 \iiint_D dV = \boxed{3}$$

Next time: Applications!!

① Electric flux (Gauss' Law of e. flux)

$$\left(\iint_{\partial D} \vec{E} \cdot \hat{n}\, d\sigma = q/\varepsilon_0\right)$$

② Conservation of mass / fluid equations $\left(\nabla \cdot \vec{F} + \frac{\partial \delta}{\partial t} = 0\right)$

⋮

Warm-up: Show that the function

$$u(x,t) = c_1 \sin(x-vt) + c_2 \sin(x+vt)$$

(where $v>0$ is constant) satisfies the Partial Differential Equation

c_1, c_2 also constant

$$u_{tt} = v^2 u_{xx}. \quad \text{(Wave equation).}$$

If x is a length/spatial variable and t is a temporal variable (time), interpret physically what v is.

$$u_x = c_1 \cos(x-vt) + c_2 \cos(x+vt)$$

$$u_{xx} = -c_1 \sin(x-vt) - c_2 \sin(x+vt) = -u(x,t).$$

$$u_t = -c_1 v \cos(x-vt) + c_2 v \cos(x+vt)$$

$$u_{tt} = -c_1 v^2 \sin(x-vt) - c_2 v^2 \sin(x+vt) = -v^2 u(x,t).$$

$$v^2 \cdot u_{xx} = -v^2 u = u_{tt} \quad \checkmark$$

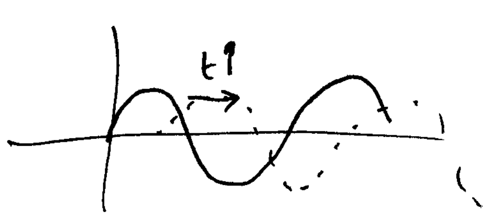
For v: Need $[x-vt]$ = length

$$\underbrace{\cancel{[x]}}_{\text{length}} - [v]\underbrace{\cancel{[t]}}_{\text{time}}$$

$$[v] \downarrow \frac{\text{length}}{\text{time}}$$

v is a speed.

of what???



Sine waves, centered at

$x - vt = 0$

$x = vt.$

Ex: (Electric flux of a point charge)

Let $\vec{F} = \frac{1}{\rho^3}(x, y, z) = \frac{1}{\rho^2}\hat{r}$ // $\rho^2 = x^2+y^2+z^2$.

Let $D: \; 0 < a^2 \le x^2+y^2+z^2 \le b^2$. ($b > a > 0$)

Compute $\iint\limits_{\partial D} \vec{F}\cdot\hat{n}\, d\sigma$ for either $\hat{n}$ vector field.

Gauss' theorem: $\iint\limits_{\partial D} \vec{F}\cdot\hat{n}\, d\sigma = \iiint\limits_{D} \operatorname{div}\vec{F}\, dV$

Then find $\operatorname{div}\vec{F} = \ldots$

$$\vec{F} = (M, N, P) \to \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{\rho^3}\right) = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x}$$

Note: $\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{x}{\rho}$

So ... $\frac{\partial \rho}{\partial y} = y/\rho$, $\frac{\partial \rho}{\partial z} = z/\rho$.

Then $\frac{\partial M}{\partial x} = \rho^{-3} - 3x\rho^{-4}\cdot x/\rho = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}$

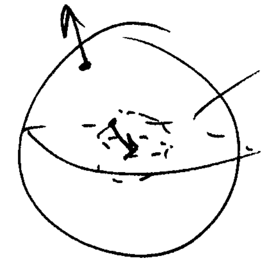
Finally, $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{3}{\rho^3} - \frac{3(x^2+y^2+z^2)}{\rho^5}$

$$= \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0.$$

(Because $\vec{0} \notin D$)

So $\iint\limits_{\partial D} \vec{F}\cdot\hat{n}\, d\sigma = \iiint\limits_{D} \operatorname{div}\vec{F}\, dV = \underline{0}$.

$$\iint_{\partial D} \vec{F} \cdot \hat{n} \, d\sigma$$



$$0 = \iint_{S_a} \vec{F} \cdot (-\hat{n}) \, d\sigma + \iint_{S_b} \vec{F} \cdot \hat{n} \, d\sigma$$

$S_b : \hat{n}$

$S_a : -\hat{n}$.

$$\iint_{S_a} \vec{F} \cdot \hat{n} \, d\sigma = \iint_{S_b} \vec{F} \cdot \hat{n} \, d\sigma \quad !!!$$

Compute this for either S_a or S_b.

Directly: $\displaystyle\iint_{S_b} \vec{F} \cdot \hat{n} \, d\sigma = \iint_{S_b} \frac{1}{\rho^2} \hat{r} \cdot \hat{n} \, d\sigma = \frac{1}{\rho^2} \iint_{S_b} d\sigma = \frac{1}{b^2} \cdot 4\pi b^2 = \boxed{4\pi}$

$(\hat{r} = \hat{n})$ $(\rho = b)$

Leads to: Gauss Law $-$ $\displaystyle\iint_{\partial D} \vec{E} \cdot \hat{n} \, d\sigma = q/\varepsilon_0$.

$$\vec{E} = \frac{-q}{4\pi\varepsilon_0 \rho^2} \hat{r} = \frac{-q}{4\pi\varepsilon_0} \vec{F} \quad \left(\text{Coulomb's law, } k = \frac{q}{4\pi\varepsilon_0}\right)$$

Find $\displaystyle\iint_{\partial D} \vec{E} \cdot \hat{n} \, d\sigma = \frac{-q}{4\pi\varepsilon_0} \cdot 4\pi = -q/\varepsilon_0$.

In general, don't need two spheres or ∂D to be a sphere at all ... Left to reader...

Maxwell's Equations

Point form:

(1) $\nabla \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$

(2) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(3) $\nabla \cdot \vec{D} = \rho$

(4) $\nabla \cdot \vec{B} = 0$

Vector form:

$$\oint_C \vec{H} \cdot d\vec{r} = \iint_S \left(\vec{J}_c + \frac{\partial \vec{D}}{\partial t}\right) \cdot \hat{n}\, d\sigma$$

$$\oint_C \vec{E} \cdot d\vec{r} = \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot \hat{n}\, d\sigma$$

$$\iint_S \vec{D} \cdot \hat{n}\, d\sigma = \iiint_D \rho\, dV$$

$$\iint_S \vec{B} \cdot \hat{n}\, d\sigma = 0$$

$\vec{J} = \sigma \vec{E}$ σ = electric conductivity

$\vec{D} = \varepsilon_0 \vec{E}$ ε_0 = dielectric permittivity

$\vec{B} = \mu_0 \vec{H}$ μ_0 = mag. permeability

$\vec{M} = \chi \vec{H}$ χ = mag. susceptibility

Constitutive relations

(Hooke's law for E & M)

$\vec{E}$ = e-field

$\vec{H}$ = mag. field

$\vec{J}$ = current

$\vec{D}$ = elec. disp.

$\vec{B}$ = mag. induct.

$\vec{M}$ = mag. polarization

(1) Ampère's

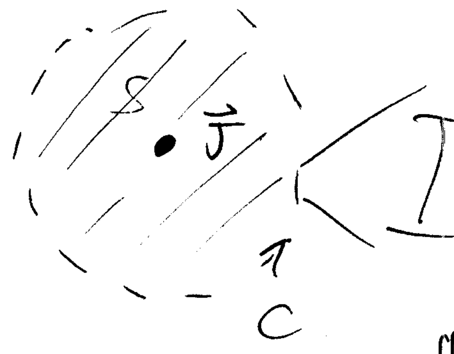
(2) Faraday's

(3) Gauss'

(4) "No magnetic monopoles"

(1)

(Assume $\vec{E}$ const.)



$\int \vec{H}\cdot\Delta\vec{r} \rightarrow |\vec{J}| = j = \oint_C \vec{H}\cdot d\vec{r}$

current density

$\iint (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma$

Consider $\nabla\times\vec{H} = \lim_{\rho\to 0} \frac{\oint_C \vec{H}\cdot d\vec{r}}{\pi\rho^2}\hat{n} = \vec{J}$.

$\frac{\partial\vec{E}}{\partial t} \neq \vec{0} \Rightarrow \vec{D}$ current arises Lenz's Law.

(current opposes changing $\vec{E}$ field)

So $$\boxed{\oint_C \vec{H}\cdot d\vec{r} = \iint_S \left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot\hat{n}\,d\sigma}$$

(2) Faraday: Moving mag. field generates an e-field.

$\vec{E} = -\frac{\partial\vec{u}}{\partial t}$ for some $\vec{u}$.

Let $\vec{u} = \underline{\Psi}$ be the magnetic "potential".

Then $\nabla\times\vec{E} = -\nabla\times\frac{\partial\Psi}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\Psi) := -\frac{\partial}{\partial t}(\vec{B})$

So $\oint_C \vec{E}\cdot d\vec{r} = \iint_S (\nabla\times\vec{E})\cdot\hat{n}\,d\sigma = \iint_S \left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot\hat{n}\,d\sigma$.

③ Recall:

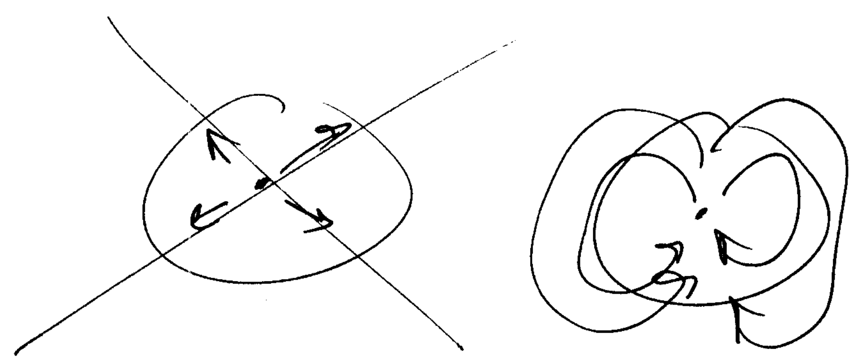
$$\operatorname{div} \vec{D} = \lim_{|V| \to 0} \frac{\iiint_V \operatorname{div} \vec{D}\, dV}{|V|} = \lim_{|V| \to 0} \frac{\iint_{S=\partial V} \vec{D} \cdot \hat{n}\, d\sigma}{|V|}$$

$$= \lim_{|V| \to 0} \frac{\frac{1}{\varepsilon_0} \iint_S \varepsilon_0 \vec{D} \cdot \hat{n}\, d\sigma}{|V|}$$

$$= \frac{q/\varepsilon_0^2}{|V|} = \rho_b .$$

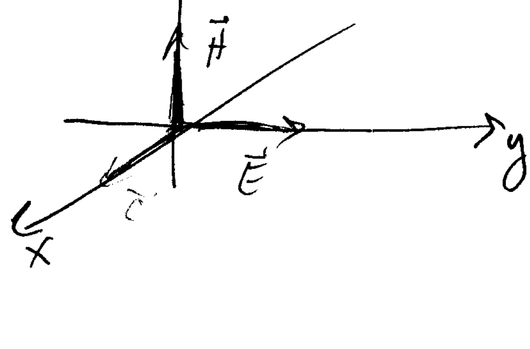
④ Taken as theoretical fact:

$$\iint_S \vec{B} \cdot \hat{n}\, d\sigma = \iiint_V \operatorname{div} \vec{B}\, dV = 0.$$



Warm-up showed that for $k^2 u_{xx} = u_{tt}$, k was the speed of a wave.

For E&M, consider the setup:



$$\vec{E} = E(x,t)\, \hat{j}$$
$$\vec{H} = H(x,t)\, \hat{k}$$

$$\vec{J} = \vec{0}$$
$$Q = 0$$

Faraday:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial B}{\partial t}\,\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & E & 0 \end{vmatrix} = (0, 0, E_x)$$

So $\quad \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$

Ampère:

$$\nabla \times \vec{H} = \cancel{-\frac{\partial B}{\partial x}\hat{j}} \; -\frac{\partial \vec{B}}{\partial x}\,\hat{j}$$

$$\vec{J} = \vec{0} \quad \Rightarrow \quad = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\,\hat{j}$$

Notice:

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial B}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right)$$

$$= -\frac{\partial}{\partial t}\left(-\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$E_{tt} = \frac{1}{\mu_0 \varepsilon_0} E_{xx}$$

$v^2 =$

$$v = c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}$$

speed of light

$\sim 3.0 \times 10^8$ m/s

MAT 21D - Notes on Differential Forms

Kevin Lamb

April 6, 2015

1 Introduction and Goal

We have seen that there are many different ways to realize one integral of a given dimension as an integral of the next dimension higher. Some key examples include Green's, Stokes', and Gauss's theorems as well as the fundamental theorem of Calculus. Since there is a common theme between these theorems, our gut instinct tells us that there may be some fundamental theorem underlying all of them. Our intuition serves us correctly in this case, and the tool we need in order to unify these theorems is the language of differential forms.

Differential forms are what are called *functionals*. There is a vast and well understood theory behind these functionals, but all we need to know is that they are functions that take in a certain object and spit out a number. Mantra: Functionals eat things and spit out numbers. A good example that we are very familiar with is the (single-dimensional) definite integral over a fixed interval. It eats continuous functions and returns a number which we interpret as the area under the curve of its graph over the given interval. So what do differential forms eat? Tangent vectors. So at their core, differential forms are just a fancy (weighted) dot product of sorts. We will omit a formal definition of differential forms in favor of just being satisfied with some examples.

Example 1.1. Functions $f\colon \mathbb{R}^3 \to \mathbb{R}$ are automatically differential forms. They are called *zero-forms* as they pertain to evaluations at single (zero-dimensional) points.

Example 1.2. We have gotten familiar with line elements:

$$\omega = M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz.$$

These are called *one-forms* as they pertain to integrating over one-dimensional objects (i.e., curves).

Example 1.3. We have also become acquainted with flux and surface integrals whose integrands may be (eventually) simplified to

$$\sigma = M(x,y,z)\,dy \wedge dz + N(x,y,z)\,dz \wedge dx + P(x,y,z)\,dx \wedge dy.$$

These are called *two-forms* because we use them to integrate over two-dimensional objects (i.e., surfaces). The order above is very particular and intentional - it will be explained below.

Example 1.4. Finally, we come back to the thing that started it all:

$$dV = f(x, y, z)\, dx \wedge dy \wedge dz.$$

This is called (you guessed it) a *three-form* as we use it to integrate over three-dimensional objects. Within the context of $\mathbb{R}^3$, three-forms are also called *volume forms* because they are the greatest-dimensional, nonzero differential forms that we may work with.

Remark 1.1. It is important to note here that one will never encounter forms that mix different dimensions of wedge products. For example, we will never see something of the form $dx + dy \wedge dz$ as this makes no sense. The idea of why this is the case: Think of the dimensions of the wedge products as being how many tangent vectors we need to feed the form. We need to feed it the same amount across all summands, and the example here shows that dx would need only one vector whereas the second would require two.

Our main goal here is to study differential forms within the context of Euclidean space of three dimensions - that is, $\mathbb{R}^3$. Suffice it to say that the theory of these objects go much deeper and more generally than we will discuss. We only care about creating a theory that allows us to explain the fundamental theorem of Calculus within the context that we have studied in this class.

2 The Algebra of Differential Forms

The funny thing about differential forms is that we can multiply them together to create new forms. However, this multiplication is not the same as what we are used to in a few respects. First of all, this multiplication is not closed - we call it *wedge product* (denoted with a $\wedge$ - pronounced "wedge"). Namely, if we multiply two one-forms together, we obtain not a one-form but a two-form. Secondly, this multiplication is not commutative. That is, like in the regime of matrices, "a times b is not necessarily the same as b times a." In fact, this multiplication is some non-commutative that switching orders of multiplication actually forces us to insert a negative sign. Let's illustrate these oddities with a few examples.

Example 2.1. Nothing strange happens for the multiplication of two zero-forms, $f(x, y, z)$ and $g(x, y, z)$. It goes exactly as one would think it goes:

$$f(x, y, z) \wedge g(x, y, z) = f(x, y, z)g(x, y, z)$$

as usual.

Example 2.2. We need to know what happens when we multiply certain one-forms together. We adopt a term from linear algebra to aid us in our description of differential forms. We say that the set $\{dx, dy, dz\}$ forms a *basis* for all differential forms over $\mathbb{R}^3$. This means that any differential form we want to talk about can be represented using a linear combination of dx, dy, and dz or some wedge product of them. So if we know how multiplication of these few differential forms works, then we know how multiplcation works for all differential forms.

To start, let's consider the basics of multiplying one-forms. These are the building blocks for other forms, so they actually create building blocks for higher-dimensional forms:

$$dx \wedge dy = -dy \wedge dx, \quad dy \wedge dz = -dz \wedge dy, \quad \text{and } dx \wedge dz = -dz \wedge dx.$$

This is the *anticommutativity* or *skew-symmetry* described above. Switching the order of multiplcation of any two one-forms within a higher-dimensional form forces us to multiply by a -1. Because switching the order of multiplcation merely induces a negative sign, we can use a basis for two-forms given by $\{dx \wedge dy, dx \wedge dz, dy \wedge dz\}$.

Using this fact, what can be said about $dx \wedge dx$, $dy \wedge dy$, and $dz \wedge dz$? We leave this to the reader. Also notice here that we have indeed demonstrated here that a product of one-forms will create a two-form.

Example 2.3. Let it now be stated that the wedge product is homogeneous (in functions) and distributive as one would hope/expect. Let $\alpha = f_1,\, dx + f_2\, dy + f_3\, dz$ and $\omega = g_1\, dx + g_2\, dy + g_3\, dz$. Let's multiply them with $\wedge$:

$$\begin{aligned}
\alpha \wedge \omega &= (f_1\, dx + f_2\, dy + f_3\, dz) \wedge (g_1\, dx + g_2\, dy + g_3\, dz) \\
&= f_1 g_1\, dx \wedge dx + f_1 g_2\, dx \wedge dy + f_1 g_3\, dx \wedge dz + \\
&\quad f_2 g_1\, dy \wedge dx + f_2 g_2\, dy \wedge dy + f_2 g_3\, dy \wedge dz + \\
&\quad f_3 g_1\, dz \wedge dx + f_3 g_2\, dz \wedge dy + f_3 g_3\, dz \wedge dz \\
&= (f_1 g_2 - f_2 g_1)\, dx \wedge dy + (f_2 g_3 - f_3 g_2)\, dy \wedge dz + (f_1 g_3 - f_3 g_1)\, dx \wedge dz.
\end{aligned}$$

This is the most general form of multiplication of two one-forms. For a specific example, take

$$(x\, dy + \sin(y+z)\, dz) \wedge ((1+y^2)\, dx - 4\, dy)$$

$$= (x\, dy) \wedge ((1+y^2)\, dx) + (x\, dy) \wedge (-4\, dy) + (\sin(y+z)\, dz) \wedge ((1+y^2)\, dx) + (\sin(y+z)\, dz) \wedge (-4\, dy)$$

$$= x(1+y^2)\, dy \wedge dx - 4x\, dy \wedge dy + \sin(y+z)(1+y^2)\, dz \wedge dx - 4\sin(y+z)\, dz \wedge dy$$

$$= -x(1+y^2)\, dx \wedge dy - \sin(y+z)(1+y^2)\, dx \wedge dz + 4\sin(y+z)\, dx \wedge dz.$$

Example 2.4. What happens when we multiply a one-form and a two-form? Let's just look at the basis elements again:

$$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz.$$

The wedge product is associative (we can move parentheses pairwise) and creates a form of dimension 1+2. Namely, we will get a volume form! There are six different ways to permute the one-forms inside $dx \wedge dy \wedge dz$, and some are equivalent while others are opposites. We see that these equivalences are

$$dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy$$

and

$$dx \wedge dz \wedge dy = dy \wedge dx \wedge dz = dz \wedge dy \wedge dx = -dx \wedge dy \wedge dz.$$

Definition 2.1. Properties of the wedge product $\wedge$:

1. Zero-forms act as constants:

$$f(x,y,z) \wedge \alpha = f(x,y,z)\alpha.$$

2. Homogeneous with respect to function coefficients:

$$\alpha \wedge (f(x,y,z)\beta) = (f(x,y,z)\alpha) \wedge \beta = f(x,y,z)\alpha \wedge \beta.$$

3. Associativity:
$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$
4. Distributivity:
$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma.$$
5. Anticommutativity:
$$\alpha \wedge \beta = -\beta \wedge \alpha.$$
If the reader hasn't checked by now, this property implies that there are no nonzero differential forms (over $\mathbb{R}^3$) of dimension greater than 3. Please check it now if you haven't already.

We note here that the bases mentioned above, we order the differential forms of dimensions two and three in alphabetical order in x, y, and z. That is, instead of using $dy \wedge dx$ as a basis element, we will conventionally use $dx \wedge dy$ (with the exception that we will for simplicity take $dz \wedge dx$ instead of $dx \wedge dy$).

3 Exterior Differentiation

Perhaps the most surprising notion about differential forms is that we can define a meaningful derivative operator on them. We call this *exterior differentiation*, and the operator is (not surprisingly) denoted d. Instead of talking about it, it might be best to work out an example or two first.

Definition 3.1. For a differentiable function $f(x, y, z)$, we can define a one-form df that comes from it:
$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$
The one-form df is called the *differential* of the function f - it is also called the *total derivative* of f as it includes every partial derivative that f has to offer. Notice its striking similarity to the gradient operator ∇f.

Example 3.1. For a specific example of the above defintion, let's take $f(x, y, z) = xz^2 + \cos(z - y)$. Then
$$\begin{aligned} df &= f_x\,dx + f_y\,dy + f_z\,dz \\ &= z^2\,dx + \sin(y - z)\,dy + (2xz - \sin(z - y))\,dz. \end{aligned}$$

Now that we have determined how to differentiate a function, we can now define how to differentiate arbitrary differential form. If $\omega = M\,dx + N\,dy + P\,dz$ and $\sigma = M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$, then we define
$$d\omega = dM \wedge dx + dN \wedge dy + dP \wedge dz$$
and
$$d\sigma = dM \wedge dy \wedge dz + dN \wedge dz \wedge dx + dP \wedge dx \wedge dy.$$

It is important to notice that each of these operations takes a given form to a form of one dimension higher. This is exactly why we have no formula for the derivative of three-forms (for any three-form, what is its derivative?). The interesting thing to note is what these derivatives look like for arbitrary functions.

Example 3.2. *Where the defintion of* $\nabla \times (M, N, P)$ *comes from.* Let ω be defined as in the previous statement. Let's examine what its derivative looks like in the standard basis of two-forms:

$$\begin{aligned} d\omega &= dM \wedge dx + dN \wedge dy + dP \wedge dz \\ &= (M_x\, dx + M_y\, dy + M_z\, dz) \wedge dx + (N_x\, dx + N_y\, dy + N_z\, dz) \wedge dy \\ &\quad +(P_x\, dx + P_y\, dy + P_z\, dz) \wedge dz \\ &= (M_z\, dz \wedge dx - M_y\, dx \wedge dy) + (N_x\, dx \wedge dy - N_z\, dy \wedge dz) \\ &\quad +(-P_x\, dz \wedge dx + P_y\, dy \wedge dz) \\ &= (P_y - N_z)\, dy \wedge dz + (M_z - P_x)\, dz \wedge dx + (N_x - M_y)\, dx \wedge dy. \end{aligned}$$

The components of this derivative align EXACTLY(!) with the curl vector field $\nabla \times (M, N, P)$. This gives our notion of derivative some foundation as to why it might be useful. The Calculus and algebra behind differential forms naturally yields this strange thing that came out of nowhere before.

Example 3.3. *Where the definition of* $\nabla \cdot (M, N, P)$ *comes from.* Let σ be defined as the two-form before the previous example. We can play the same game with the derivative as in the previous example if we merely organize the derivative into a single component of $dx \wedge dy \wedge dz$.

$$\begin{aligned} d\sigma &= dM \wedge dy \wedge dz + dN \wedge dz \wedge dx + dP \wedge dx \wedge dy \\ &= (M_x\, dx + M_y\, dy + M_z\, dz) \wedge dy \wedge dz + (N_x\, dx + N_y\, dy + N_z\, dz) \wedge dz \wedge dx \\ &\quad (P_x\, dx + P_y\, dy + P_z\, dz) \wedge dx \wedge dy \\ &= M_x\, dx \wedge dy \wedge dz + N_y\, dy \wedge dz \wedge dx + P_z\, dz \wedge dx \wedge dy \\ &= (M_x + N_y + P_z)\, dx \wedge dy \wedge dz. \end{aligned}$$

This coefficient before $dx \wedge dy \wedge dz$ is EXACTLY(!) the formula for $\nabla \cdot (M, N, P)$. Again, the Calculus and algebra of differential forms naturally spits out something that we defined (almost) arbitrarily before.

Hopefully these two examples have shown the usefulness of our definition of differential forms and their derivatives. There is one more fact that proves useful only much later on in the study of what is called *De Rham cohomology* (look this one up - it's cool!). We will leave the verification of this computation for the reader (we would have to do it for at least three different cases): It is a well-known fact that $d\,d\omega = 0$ for any differential form ω. That is, d^2 is identically the zero function 0. As a hint as to what to expect, this computation can literally be translated into "mixed partial derivatives are equal" (and will cancel with one another in the computation).

We can summarize the properties of the exterior derivative as follows:

1. Derivative of a 0-form:
$$df = f_x\, dx + f_y\, dy + f_z\, dz.$$
2. Linearity:
$$d(\alpha + \beta) = d\alpha + d\beta.$$
3. Leibniz property:

   If α is a k-form and β is an ℓ-form, then
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k d\alpha \wedge d\beta.$$
4. Consecutive derivatives don't matter:
$$d(d\alpha) = d^2\alpha = 0.$$

4 The Generalized Stokes' Theorem

Finally! We are here at the end of the quarter, and there has to be some way to tie everything together. We have seen from Green's, Stokes', and Gauss's theorems that there should be some unifying theme to Vector Calculus. Now that we have defined and gotten familiar with differential forms and their derivatives, we have the necessary tool to do so.

Recall from the previous section that the gradient, curl, and divergence operators are exactly the results of exterior differentiation. That is, for a differential form ω, its derivative $d\omega$ has ω for an "antiderivative" of sorts. If we recall the (original) fundamental theorem of Calculus,

$$\int_I df = \int_a^b f'(x)\, dx = f(b) - f(a),$$

we are expressing the integral of a derivative df on an interval I as evaluating one of its antiderivatives on the boundary of that interval. We can rewrite this theorem as

$$\int_I df = \text{“} \int_{\partial I} f \text{”}.$$

In its fuller generality, we take would have (if a and b are the initial and terminal points of the curve C)

$$\int_C \nabla f(\vec{r}(t)) \cdots d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

We can rewrite this as

$$\int_C df = \int_{\partial C} f.$$

To see this idea more transparently, consider Stokes' theorem:

$$\iint_S \nabla \times \vec{F} \cdot \vec{n}\, d\sigma = \int_{\partial S} \vec{F} \cdot d\vec{r}.$$

If we take $\omega = \vec{F} \cdot (dx, dy, dz)$, then this equation can be written as

$$\int_S d\omega = \int_{\partial S} \omega.$$

Did you miss it? Let's do it again and take a look at Gauss's theorem:

$$\iiint_V \nabla \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot \vec{n}\, d\sigma.$$

If we take $\omega = \vec{F} \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy)$, then we can rewrite this as

$$\int_V d\omega = \int_{\partial V} \omega.$$

That is, **the integral of a differential form over a region is the integral of its antiderivative over the boundary of the region**!

We compile this information into the following theorem:

Theorem 4.1. (Generalized Stokes' Theorem) Let D be a k-dimensional region in $\mathbb{R}^n$, and let ω be a differential $(k-1)$-form defined over the boundary ∂D of the region D. Then

$$\int_D d\omega = \int_{\partial D} \omega.$$

This is exactly the formulation of the fundamental theorem of Calculus in higher dimensions. It combines the areas of Analysis, Topology, and Algebra into one elegant theorem that completely describes the inverse relationship that integration has to differentiation.

There are no examples of using this theorem as we have already seen it used in full detail. Each side of this equation is evaluated by means of parametrizing the given manifolds and then proceeding using the multiple integration theory of Euclidean space.