5. Work, circulation, and circulation density

Recall that (in two dimensions) the work done along the path Γ against the field $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j}$ is given by the following line integral:

$$\int_\Gamma \mathbf{F} \cdot \mathbf{t}\, ds = \int_\Gamma f\, dx + g\, dy.$$

In words, work is the integral of the tangential component of the field taken with respect to arc length. In general, work depends on the path Γ and, in order to compute it, one has to parameterize Γ and integrate. Integration can obviously become difficult for complicated paths. The good news is that for *potential fields* work does not depend on the path and depends only on the field. Recall (from Physics) that a field is called potential if it is given by the gradient of a scalar function:

$$F = \mathrm{grad}(\phi) = \frac{\partial \phi}{\partial x}\,\mathbf{i} + \frac{\partial \phi}{\partial y}\,\mathbf{j}.$$

Potentials are not unique: adding a constant to ϕ does not affect its partial derivatives. For this reason (scalar) potentials are defined up to a constant. Non-uniqueness does not cause any problems because one is normally interested in the *difference* of potential values. For instance, to compute work in the potential case, one simple takes the difference of the values of ϕ at the beginning and terminal points of the path Γ:

$$\int_\Gamma \mathrm{grad}(\phi) \cdot \mathbf{t}\, ds = \int_{(x_0,y_0)}^{(x_1,y_1)} \mathrm{grad}(\phi) \cdot \mathbf{t}\, ds = \phi(x_1, y_1) - \phi(x_0, y_0).$$

This is the Fundamental Theorem of Calculus for line integrals.

From the point of view of Vector Analysis, potential fields deserve special treatment because their components are determined by a single scalar function—the field's potential. It may be very difficult to compute three components of some field in $\mathbb{R}^3$ if they are all independent unknowns. However, if the field is potential, there is only one unknown—the potential; find the potential and you have found the field. You can imagine how convenient that can be in $\mathbb{R}^{15}$! We will now develop a useful criterion for deciding whether a field is potential or not. This requires that we introduce the concept of circulation and the corresponding density.

Circulation of a field is simply work computed along a closed loop:

$$\oint_C \mathbf{F} \cdot \mathbf{t}\, ds.$$

Notice that the sign of circulation depends on the orientation of the loop: if the latter is changed the sign of circulation also flips. By default, all loops in $\mathbb{R}^2$ are oriented counterclockwise unless otherwise noted.

In light of the Fundamental Theorem for line integrals, the circulation of a potential field around *any* loop is zero. Thus, in principle, we can test whether a field is potential by examining its circulation. However, it is clearly inconvenient to have to compute circulation for *all* possible loops. Therefore, let us introduce circulation density, swirl, by analogy with divergence:

$$\operatorname{swirl}(\mathbf{F}) = \lim_{A\to 0} \frac{\oint_C \mathbf{F}\cdot\mathbf{t}\,ds}{A}. \tag{13}$$

Notice that we repeat the same construction that was used previously to define div: compute circulation around a loop, divide by the area enclosed by the loop, and shrink the loop to a point.

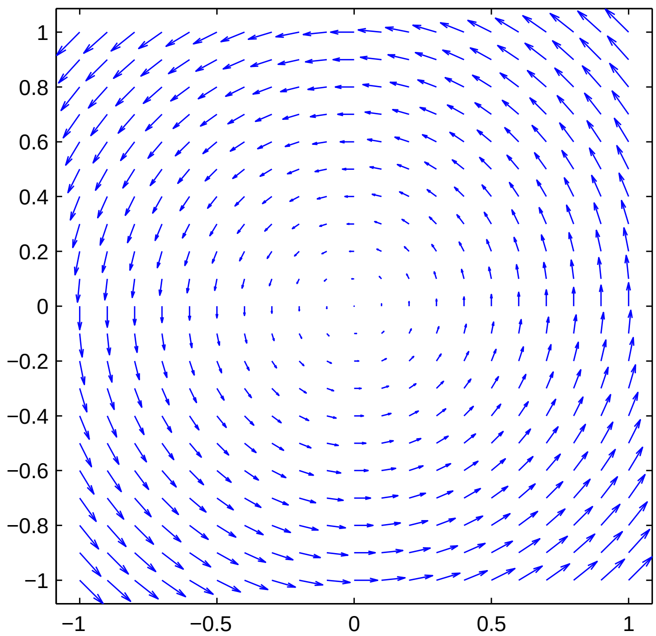

FIGURE 2. The field $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$

In Cartesian coordinates

$$\mathrm{swirl}(f\,\mathbf{i} + g\,\mathbf{j}) = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}.$$

This can be easily seen by computing the limit (13) for a shrinking rectangle (Exercise). For the field $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ shown in Figure 2: $\mathrm{swirl}(\mathbf{F}) = 2$. This explains the term 'swirl'—think of the field in Figure 2 as the velocity of a moving fluid.

Now, if the field is potential, the circulation around any loop is zero and consequently the circulation density is zero:

$$\mathbf{F} = \mathrm{grad}(\phi) \Rightarrow \mathrm{swirl}(\mathbf{F}) = 0.$$

We will now prove the much more useful converse statement: if circulation density is (identically) zero then the field is potential. In symbols:

$$\mathrm{swirl}(\mathbf{F}) = 0 \Rightarrow \mathbf{F} = \mathrm{grad}(\phi).$$

To prove that a field $\mathbf{F}$ is potential, one ideally wants to construct the potential ϕ. This is the approach we are going to take. We are given some field $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j}$ with zero swirl:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}.$$

If the field is, indeed, potential then work is given by the difference of potential values. Let Γ be the straight line segment connecting the origin $(0,0)$ with (x,y); parameterize it as $(x\,t, y\,t)$ where $0 \leq t \leq 1$. Then, if our assertion is true:

$$\phi(x,y) = \phi(0,0) + \int_0^1 \left(f(x\,t, y\,t)\,x + g(x\,t, y\,t)\,y\right)\,dt.$$

Recall that scalar potentials are defined up to a constant. We can choose the constant so that $\phi(0,0) = 0$. This suggests that, if the field $\mathbf{F}$ is potential, its scalar potential is given by:

$$\phi(x,y) = \int_0^1 \left(f(x\,t, y\,t)\,x + g(x\,t, y\,t)\,y\right)\,dt.$$

We will now prove that by direct computation. Differentiating the candidate for potential with respect to x, we get:

$$\frac{\partial \phi}{\partial x} = \int_0^1 \left(f(x\,t, y\,t) + \frac{\partial f}{\partial x}(x\,t, y\,t)\,x\,t + \frac{\partial g}{\partial x}(x\,t, y\,t)\,y\,t\right)\,dt$$

We now use the assumption that $\mathrm{swirl}(\mathbf{F}) = 0$ to replace $\frac{\partial g}{\partial x}$ with $\frac{\partial f}{\partial y}$:

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \int_0^1 \left(f(x\,t, y\,t) + \frac{\partial f}{\partial x}(x\,t, y\,t)\,x\,t + \frac{\partial f}{\partial y}(x\,t, y\,t)\,y\,t \right)\,dt \\ &= \int_0^1 \frac{d}{dt}\left(t\,f(x\,t, y\,t)\right)\,dt = f(x,y),\end{aligned}$$

as required. Proving that

$$\frac{\partial \phi}{\partial y} = g(x,y)$$

is similar and left as an exercise. Since the components of the field are partial derivatives of

$$\phi(x,y) = \int_0^1 \left(f(x\,t, y\,t)\,x + g(x\,t, y\,t)\,y\right)\,dt,$$

the function ϕ is the scalar potential of the field.

The Swirl Theorem. As the reader may have already guessed, there must be an analogue of the Divergence Theorem. Replace the flux and divergence (flux density) in the latter with circulation and swirl (circulation density). This leads to:

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{t}\,ds = \iint_D \mathrm{swirl}(\mathbf{F})\,dA. \tag{14}$$

The proof of Equation (14) is the same as the proof of the Divergence Theorem in two dimension and is therefore omitted. Notice that in Calculus III notation Equation (14) reads:

$$\oint_{\partial D} f\,dx + g\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dA.$$

This is the same as Green's Theorem!

Although the Swirl Theorem says the same thing as the Divergence Theorem in two dimensions, the concept of swirl is useful for quickly deciding wether a field is potential. Furthermore, in three dimensions circulation and circulation density are not related in the same way as flux and flux density. The three-dimensional circulation density of a vector field in $\mathbb{R}^3$ is another vector field called 'curl'; the corresponding theorem, named after Stokes, is quite different in nature from the three-dimensional Divergence Theorem.

Exercises.

(1) Derive the formula for swirl($\mathbf{F}$) in Cartesian coordinates.
(2) Repeat Exercise 1 in polar coordinates.
(3) Show that $\mathbf{F} = 3\,(x^2y^2 + y)\,\mathbf{i} + (2x^3y + 3\,x)\,\mathbf{j}$ is potential by computing the scalar potential ϕ.
(4) Let $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ be a (smooth) vector field in $\mathbb{R}^3$ and let C be a circle of radius r centered at the origin and having unit normal $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ (that is, C lies in the plane which passes through the origin and has unit normal $\mathbf{n}$). Compute the limit
$$\lim_{A\to 0} \frac{\oint_C \mathbf{F}\cdot\mathbf{t}\,ds}{A}$$
where C is shrunk to the center while being positively oriented.
(5) Write a `Matlab` function that takes as its inputs functions f, g, and h, a unit vector $\mathbf{n}$, and a positive number r returning the circulation of $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ around the circle of radius r centered at the origin with normal $\mathbf{n}$. Test it for $\mathbf{F} = x^2\,\mathbf{i} + x\,z\,\mathbf{j} + y^2\,z\,\mathbf{k}$, $r = 1$ and $\mathbf{n} = \langle 1, 1, 1\rangle/\sqrt{3}$.