9. Three general principles for solving linear ODE with constant coefficients

In this section we consider a forced RLC-circuit which is modeled by the following second order ODE:

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = f(t) \tag{41}$$

Notice how similar Equation (41) is to the model of a mass-spring system: in fact, mathematically, the two are indistinguishable. More precisely, if make the identification

$$\begin{aligned} L &\leftrightarrow m \\ R &\leftrightarrow r \\ \frac{1}{C} &\leftrightarrow k \end{aligned}$$

we can say that an RLC-circuit is equivalent to an mrk-mass-spring system. This is called *electro-mechanical analogy* and is one of the reasons why we chose Equation (41) as the focus of discussion. Another reason is that Equation (41) is the epitome of a linear ODE with constant coefficients. All such ODE can be solved using the same general strategy which we summarize below in the form of three general principles.

9.1. First Principle: Use the Fundamental Theorem.

You are already familiar with the first principle. The general solution of a nonhomogeneous ODE is the sum of the general solution of the homogeneous ODE and a particular solution of the nonhomogeneous ODE. The first component—the general solution of the homogeneous ODE—is often called *complimentary function* because it compliments the particular solution to the general solution. Henceforth we will write the Fundamental Theorem of Math 57 symbolically as:

$$q = q_c + q_p.$$

The "c" subscript stands for 'complimentary' and "p" is for 'particular'.

9.2. Second Principle: Superposition.

Consider the homogeneous equation:

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = 0. \tag{42}$$

Let q_1 and q_2 be any particular solutions of Equation (42). The principle of superposition tells us, simply, that the linear combination

$$q_3 = C_1\,q_1 + C_2\,q_2 \tag{43}$$

is another solution of Equation (42). To see this, substitute (43) into (42). Since differentiation is a linear operation:

$$\begin{aligned} L\,\frac{d^2q_3}{dt^2}+R\,\frac{dq_3}{dt}+\frac{1}{C}\,q_3 &= L\,\frac{d^2}{dt^2}\,(C_1\,q_1+C_2\,q_2)+R\,\frac{d}{dt}\,(C_1\,q_1+C_2\,q_2) \\ &+\frac{1}{C}\,(C_1\,q_1+C_2\,q_2)=C_1\,\left(L\,\frac{d^2q_1}{dt^2}+R\,\frac{dq_1}{dt}+\frac{1}{C}\,q_1\right) \\ &+C_2\,\left(L\,\frac{d^2q_2}{dt^2}+R\,\frac{dq_2}{dt}+\frac{1}{C}\,q_2\right)=C_1\,0+C_2\,0=0. \end{aligned}$$

It should be clear that the principle of superposition applies to any linear combination of particular solutions and that it is an immediate consequence of linearity of Equation (42).

The principle of superposition provides a simple recipe for finding complimentary functions: find enough particular solutions and form their linear combination. By 'enough' we mean that to solve a second order homogeneous ODE one needs two independent particular solutions; for third order equations, one needs three independent particular solutions, and so on.

The principle of superposition can, and should be used to define homogeneity. As you realize, saying that "the right-hand side is zero" is not a good way of explaining why a linear equation is homogeneous. However, consider the set of all solutions of a homogeneous equation where we think of individual solutions as vectors. According to the principle of superposition, this collection of vectors is closed under linear combinations—linear combinations of solutions are, again, solutions. This means that the set of solutions of a homogeneous equation is a *vector space*. Conversely, we can say that an ODE is linear homogeneous *if* its set of solutions forms a vector space. This is the true linear algebra definition of homogeneity.

In linear algebra one proves that each vector space has a basis—a finite collection of vectors which can be used to express any vector in the vector space as a linear combination. For instance, the standard basis of $\mathbb{R}^2$ is the pair of vectors $\{\mathbf{i},\mathbf{j}\}$ and every vector in $\mathbb{R}^2$ can be written as the linear combination $a\,\mathbf{i}+b\,\mathbf{j}$. The coefficients in front of the basis vectors are called *components* of the vector *with respect to the particular basis.* In order to solve (42), we "guess" an exponential solution and, after solving the characteristic equation, arrive at

$$q_c = C_1\,e^{\lambda_1 t}+C_2\,e^{\lambda_2 t}.$$

We can now say that solving the characteristic equation is simply the procedure for finding a basis of the solution space. It works because the ODE has constant coefficients. If it did not, we would need another guess for basis solutions, or we could find them using Taylor series.

9.3. **Third Principle: Fourier analysis.** So far, the most problematic part of constructing the solution of (41) has been guessing the particular solution q_p. The guess is usually clear when the right-hand side is an elementary function, like a polynomial or an exponential. However, what if it is some strange function like the *square wave*: $f = \mathrm{sign}(\sin(t))$?

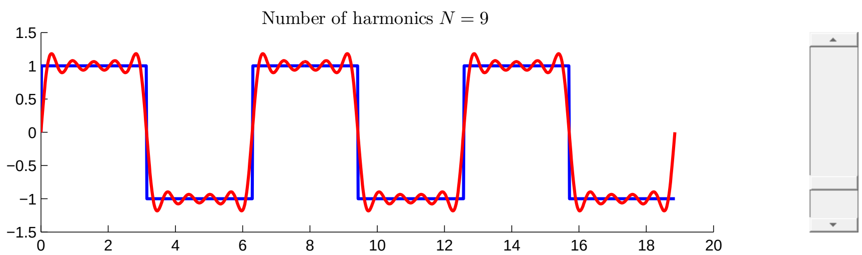

FIGURE 21. Square wave approximated by nine harmonics

Figure 21 shows the graph of the square wave (blue) and the graph of the trigonometric sum (red)

$$\frac{4}{\pi}\left(\sin(t) + \frac{\sin(3\,t)}{3} + \frac{\sin(5\,t)}{5} + \ldots + \frac{\sin(9\,t)}{9}\right)$$

Evidently, there is close agreement. The agreement can be made even closer if we add more sine terms. In fact, it can be shown that the limit

$$\lim_{N\to\infty} \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3\,t)}{3} + \frac{\sin(5\,t)}{5} + \ldots + \frac{\sin((2\,N+1)\,t)}{2\,N+1}\right)$$

is the square wave $f = \mathrm{sign}(\sin(t))$.

More generally, we claim that any periodic function can be represented as a sum of sines and cosines with certain frequencies:

$$f = \sum_n A_n \cos(\omega_n t) + B_n \sin(\omega_n t). \tag{44}$$

Moreover, even if the function is not periodic, the assertion is still true (but the sum needs to be replaced with an integral). In light of this, we will henceforth restrict our right-hand sides to 2π-periodic functions. These functions can be expanded into trigonometric series where sines and cosines have *integer* frequencies. Such series are called *Fourier series*.

We will show how to find Fourier coefficients A_n and B_n in the next section. In the meantime, let us examine the implication of Equation (44). Once the right-hand side of (41) is expanded into a Fourier series, the particular solution can be constructed as a Fourier series:

$$q_p = \sum_n a_n \cos(\omega_n t) + b_n \sin(\omega_n t).$$

To find the relation between (a_n, b_n) and (A_n, B_n) substitute the Fourier expansion of q_p into Equation (41) and compare with (44):

$$\begin{aligned} L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q &= \sum_n (-L\,\omega_n^2)\,(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \\ &+ \sum_n (R\,\omega_n)\,(-a_n \sin(\omega_n t) + b_n \cos(\omega_n t)) \\ &+ \sum_n \frac{1}{C}\,(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \\ &= (\heartsuit) = \sum_n A_n \cos(\omega_n t) + B_n \sin(\omega_n t). \end{aligned}$$

Comparing the coefficients in front of $\cos(\omega_n t)$ and $\sin(\omega_n t)$ on both sides, we arrive at the following system of equations

$$\begin{aligned} \left(\frac{1}{C} - L\,\omega_n^2\right) a_n + R\,\omega_n\,b_n &= A_n, \\ -R\,\omega_n\,a_n + \left(\frac{1}{C} - L\,\omega_n^2\right) b_n &= B_n, \end{aligned}$$

whose solution is given by

$$a_n = \frac{\left(\frac{1}{C} - L\,\omega_n^2\right)\,A_n - R\,\omega_n\,B_n}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2} \qquad b_n = \frac{\left(\frac{1}{C} - L\,\omega_n^2\right)\,B_n + R\,\omega_n\,A_n}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}.$$

We conclude that q_p is given by

$$\begin{aligned} q_p = &\sum_n \frac{\left[\left(\frac{1}{C} - L\,\omega_n^2\right)\,A_n - R\,\omega_n\,B_n\right]}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}\,\cos(\omega_n t) \\ +&\sum_n \frac{\left[\left(\frac{1}{C} - L\,\omega_n^2\right)\,B_n + R\,\omega_n\,a_n\right]}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}\,\sin(\omega_n t) \end{aligned} \tag{45}$$

Things become even simpler if instead of trig functions we work with complex exponentials. As follows from Euler's formula

$$\cos(\omega\,t) = \frac{e^{i\,\omega\,t} + e^{-i\,\omega\,t}}{2}, \quad \sin(\omega\,t) = \frac{e^{i\,\omega\,t} - e^{-i\,\omega\,t}}{2\,i}.$$

Consequently, instead of using a trigonometric series, we can represent f as a sum of complex exponentials:

$$f = \sum_n c_n \, e^{i\,\omega_n\,t}.$$

Let $p = L\,\lambda^2 + R\,\lambda + \frac{1}{C}$ be the characteristic polynomial of Equation (41). If the right-hand side f is a sum of complex exponentials, so is the particular solution. In fact, as we showed in class:

$$q_p = \sum_n \frac{c_n}{p(i\,\omega_n)}\, e^{i\,\omega_n\,t}.$$

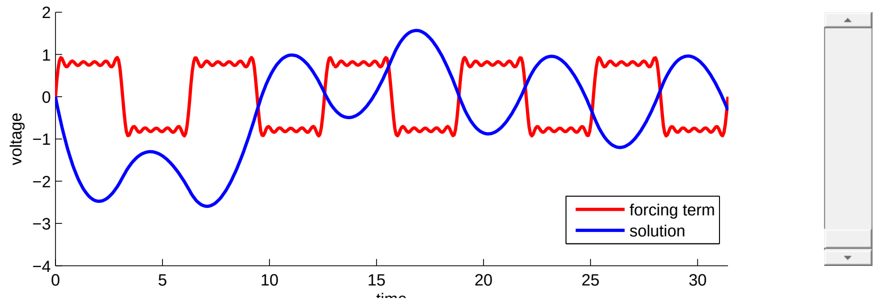

FIGURE 22. RLC-Circuit driven by a square wave

Figure 22 shows a typical response of an RLC-circuit (blue curve) driven by the square wave (red curve). The parameters were chosen to be $R = \frac{1}{4}$, $L = 1$, $C = 10$; the equation for the wave was

$$f = \sum_{n=1}^{6} \frac{\sin((2\,n-1)\,t)}{2\,n-1}.$$

The initial conditions were set to zero.

10. Fourier series

We have claimed that any forcing term can be written in the form

$$\sum_n A_n\,\cos(\omega_n t) + B_n\,\sin(\omega_n t),$$

or, equivalently, as a sum of complex exponentials. We will now formalize this for 2π-periodic functions. The extension to functions of different periods and aperiodic functions is relatively straightforward.

Theorem 2 (Fourier Series)**.** *Let $f(t)$ be 2π-periodic function. Then one can express $f(t)$ as an infinite trigonometric sum of the form:*

$$f(t) = a_0 + \sum_{n=1} \left(a_n \cos(n\,t) + b_n \sin(n\,t)\right), \tag{46}$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(t)\,\cos(n\,t)\,dt, \quad n > 0 \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(t)\,\sin(n\,t)\,dt, \quad n > 0. \end{aligned}$$

Proof. In principle, we need to show two things: (i) that the Fourier coefficients are given as above integrals, and (ii) that if one forms a trigonometric series using the above formulas for coefficients, that series will converge to the function that it is supposed to represent. Proving that latter is outside the scope of this handout and we will therefore contend by showing the former. To this end, assume that Equation (46) is valid. Integrating both sides of (46) from 0 to 2π, we get

$$\begin{aligned} \int_0^{2\pi} f(t)\,dt &= 2\pi\,a_0 + \sum_{n=1} \left(a_n \int_0^{2\pi} \cos(n\,t)\,dt + b_n \int_0^{2\pi} \sin(n\,t)\,dt\right) \\ &= 2\pi\,a_0, \end{aligned}$$

since cosines and sines integrate to zero over their periods. This shows that

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt,$$

as required. Let us now integrate both sides of (46) against $\cos(m\,t)$:

$$\begin{aligned} \int_0^{2\pi} f(t)\,\cos(m\,t)\,dt &= \int_0^{2\pi} \cos(m\,t) \sum_{n=1} \left(a_n \cos(n\,t)\,dt + b_n \sin(n\,t)\right)\,dt \\ &= \sum_{n=1} \left(a_n \int_0^{2\pi} \cos(m\,t)\,\cos(n\,t)\,dt + b_n \int_0^{2\pi} \cos(m\,t)\,\sin(n\,t)\,dt\right). \end{aligned}$$

For $m > 0$:

$$\int_0^{2\pi} \cos(m\,t)\,\cos(n\,t)\,dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

and

$$\int_0^{2\pi} \cos(m\,t)\,\sin(n\,t)\,dt = 0.$$

Therefore in the above summation only the term

$$a_n \int_0^{2\pi} \cos(m\,t)\,\cos(n\,t)\,dt$$

with $n = m$ is nonzero and, consequently,

$$\int_0^{2\pi} f(t)\,\cos(m\,t)\,dt = \pi\,a_m, \quad m > 0.$$

This is equivalent to

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\,\cos(n\,t)\,dt, \quad n > 0.$$

The remaining identity

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\,\sin(n\,t)\,dt, \quad n > 0,$$

is proven similarly. □

As an illustration of Theorem 2 let us expand the square wave function

$$f(t) = \mathrm{sign}(\sin(t))$$

into a Fourier series. The name of the function is apparent from the graph shown in Figure 21. It is straightforward to show that for the square wave all coefficients $a_n = 0$; thus the Fourier series has only the sine terms. The coefficients b_n are given by:

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\int_0^{\pi} \sin(n\,t)\,dt - \int_\pi^{2\pi} \sin(n\,t)\,dt\right) \\ &= \frac{1}{\pi}\left(-\frac{\cos(n\,t)}{n}\bigg|_0^{\pi} + \frac{\cos(n\,t)}{n}\bigg|_\pi^{2\pi}\right) = \frac{2\,(1-\cos(\pi\,n))}{\pi\,n} \\ &= \frac{2}{\pi\,n} \times \begin{cases} 0, & n \text{ is even}, \\ 2, & n \text{ is odd}. \end{cases} \end{aligned}$$

Thus $f(t)$ is the sum of odd sines:

$$f(t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin((2\,n+1)\,t)}{2\,n+1}. \tag{47}$$

Figure 21 confirms the validity of (47)

11. Analysis of a linear ODE with constant coefficients

We will now show how RLC-circuit equation can be properly analyzed. The same analysis applies to all linear ODE with constant coefficients.

11.1. **Understanding the complimentary function.**

First, we will prove that the complimentary function of (41) decays exponentially as long as $R > 0$ which, in practice, is always the case. This will actually allow us to remove the complimentary function from analysis entirely.

The characteristic equation

$$L\,\lambda^2 + R\,\lambda + \frac{1}{C} = 0$$

has the roots

$$\lambda_{1,2} = -\frac{R}{2\,L} \pm \sqrt{\frac{R^2}{4\,L} - \frac{1}{L\,C}}.$$

Depending on the value of $R^2/(4\,L) - (L\,C)^{-1}$ the following three cases are possible.

11.1.1. *Overdamped case.* If $R^2/(4\,L) > (L\,C)^{-1}$ then the characteristic roots $\lambda_{1,2}$ are distinct real numbers. Since R, L and C are positive, the root

$$\lambda_2 = -\frac{R}{2\,L} - \sqrt{\frac{R^2}{4\,L} - \frac{1}{L\,C}}$$

is clearly a negative number. To see that the other root

$$\lambda_2 = -\frac{R}{2\,L} + \sqrt{\frac{R^2}{4\,L} - \frac{1}{L\,C}}$$

is also negative, recall that two positive numbers compare in the same way as their squares, i.e., $a < b$ implies and is implied by $a^2 < b^2$. Since

$$\frac{R^2}{4\,L} - \frac{1}{L\,C} < \frac{R^2}{4\,L},$$

we have

$$\sqrt{\frac{R^2}{4\,L} - \frac{1}{L\,C}} < \frac{R}{2\,L},$$

and, consequently,

$$\lambda_2 = \sqrt{\frac{R^2}{4\,L} - \frac{1}{L\,C}} - \frac{R}{2\,L} < 0.$$

Now since both λ_1 and λ_2 are negative, the complimentary function

$$q_c = C_1\,e^{\lambda_1\,t} + C_2\,e^{\lambda_2\,t}$$

decays exponentially with t.

11.1.2. *Critical damping.* If $R^2/(4\,L) = (L\,C)^{-1}$ the characteristic roots $\lambda_{1,2}$ are the same:

$$\lambda_1 = \lambda_2 = -\frac{R}{2\,L}.$$

The complimentary function is then the product of a linear polynomial and a decaying exponential:

$$q_c = (C_1 + C_2\,t)\,e^{-\frac{R\,t}{2\,L}}.$$

As $t \to \infty$ the linear polynomial approaches infinity but the exponential term goes to zero at a much faster exponential rate. Therefore q_c approaches zero exponentially as before.

11.1.3. *Underdamped case.* It remains to consider the case where $R^2/(4\,L) < (L\,C)^{-1}$ and the characteristic roots are complex:

$$\lambda_{1,2} = -\frac{R}{2\,L} \pm \omega_0\,i, \quad \omega_0 = \sqrt{\frac{1}{L\,C} - \frac{R^2}{4\,L}}.$$

Using Euler's formula, we can write complimentary function as

$$q_c = C_1\,e^{(-\frac{R}{2\,L}+\omega_0\,i)\,t} + C_2\,e^{(-\frac{R}{2\,L}-\omega_0\,i)\,t} = e^{-\frac{R\,t}{2\,L}}\,(A\,\cos(\omega_0 t) + B\,\sin(\omega_0 t)).$$

This is a harmonic with an exponentially decaying amplitude: so, yet again, q_c decays exponentially.

Since q_c decays exponentially, it does not have appreciable affect on the steady performance of the circuit. For this reason, it is called a *transient* and is not considered in engineering analysis. Also, since the constants related to initial conditions reside in q_c, the initial conditions are not part of the analysis either. A stable system—the one with exponentially decaying q_c—quickly "forgets" its initial state.

11.2. **Analysis of the particular solution.** The analysis of the particular solution hinges entirely on our ability to write it as a sum of harmonics. These harmonics are shifted and scaled harmonics present in the forcing term f. As we have shown, the shifting and scaling of harmonics depends on their frequency. Therefore, in order to understand the steady performance of the circuit (41), or any similar system, we need to understand how scaling (gain) and shifting (phase shift) depend on frequency.

11.3. **Amplitude amplification.** Formula (45) for the particular solution of (41) shows that every harmonic component of f is mapped into a harmonic component of the same frequency in q_p according to the rule:

$$A_n\,\cos(\omega_n t) + B_n\,\sin(\omega_n t) \mapsto \frac{\left[\left(\frac{1}{C} - L\,\omega_n^2\right)\,A_n - R\,\omega_n\,B_n\right]\,\cos(\omega_n t) + \left[\left(\frac{1}{C} - L\,\omega_n^2\right)\,B_n + R\,\omega_n\,a_n\right]\,\sin(\omega_n t)}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2} \tag{48}$$

Equation (48) says that harmonic signals change their amplitudes and phases as they pass through an RLC-circuit. Let us compute the amplification factor for the amplitude of the harmonic signal as a function of the frequency ω_n. Recall that the amplitude of a linear combination of sine and cosine

$$A_n\,\cos(\omega_n\,t) + B_n\,\sin(\omega_n\,t)$$

is the number $\sqrt{A_n^2 + B_n^2}$. Therefore the square of the amplitude of the right-hand side of (48) is given by

$$\left(\frac{\left(\frac{1}{C} - L\,\omega_n^2\right)\,A_n - R\,\omega_n\,B_n}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}\right)^2 + \left(\frac{\left(\frac{1}{C} - L\,\omega_n^2\right)\,B_n + R\,\omega_n\,A_n}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}\right)^2$$

Bringing the two fractions to a common denominator and simplifying, we get

$$\frac{A_n^2 + B_n^2}{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}$$

which means that a signal with amplitude $\sqrt{A_n^2 + B_n^2}$ and frequency ω_n is mapped into the signal with amplitude

$$\frac{\sqrt{A_n^2 + B_n^2}}{\sqrt{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}}$$

and hence the amplitude magnification factor F is given by

$$G(\omega_n) = \frac{1}{\sqrt{\left(\frac{1}{C} - L\,\omega_n^2\right)^2 + R^2\,\omega_n^2}} = \frac{1}{\sqrt{L^2\,(\omega_0^2 - \omega_n^2)^2 + R^2\,\omega_n^2}}.$$

We remarked earlier that it is often easier to work with complex exponentials rather than with trig functions. If we write f as a sum of complex exponentials, these will be mapped into corresponding components of q_p according to the rule:

$$c_n\,e^{i\omega_n\,t} \mapsto \frac{1}{p(i\,\omega_n)}\,c_n\,e^{i\omega_n\,t},$$

where p is the characteristic polynomial. We can say that the complex gain is $\frac{1}{p(i\,\omega_n)}$ and it stands to reason that $G(\omega_n)$ is related to that number. It turns out that the relation is, simply:

$$G(\omega_n) = \frac{1}{|p(i\,\omega_n)|},$$

where $|\cdot|$ denotes the modulus of a complex number: $|a + b\,i| = \sqrt{a^2 + b^2}$.

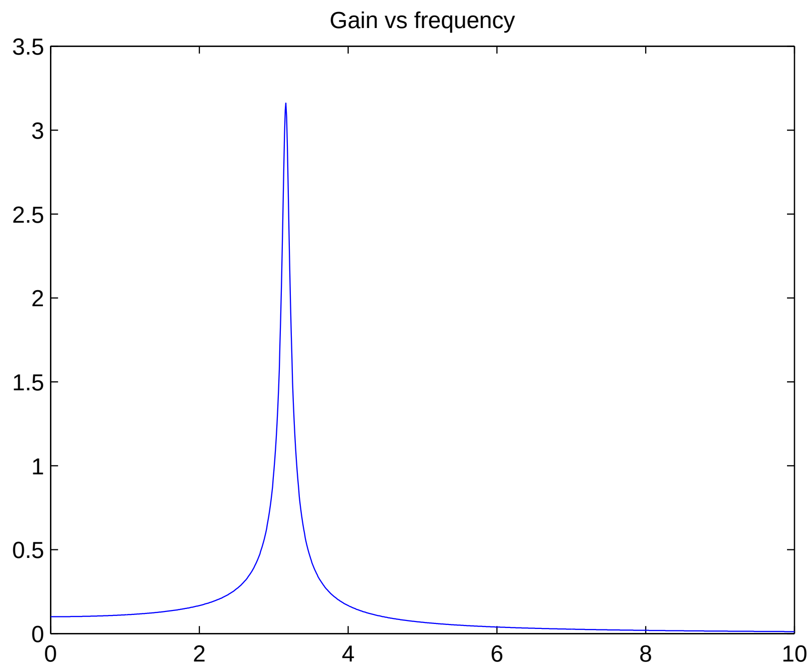

FIGURE 23. Gain for the RLC-circuit with $R = .1$, $L = 1$, and $C = 10$

Figure 23 shows a typical plot of the gain G versus frequency ω for an underdamped RLC-cicruit. Notice that G goes to zero quadratically as ω goes to infinity: any RLC-circuit acts as a low-pass filter. It is also evident from the figure that harmonics with frequency near 3.2 will be amplified threefold while other harmonics will be dampened. This is an example of the phenomenon called *resonance* and is the working principle of the radio: the sum of harmonics received by an antenna is fed through an RLC-circuit "tuned" to some frequency ω_0. The output of the circuit, is completely dominated by the components with frequencies closest to ω_0 which are sent to the speaker.

The gain vs frequency is one plot that is commonly found in any engineering analysis involving ODE (linear or not!) Another common plot is phase shift vs frequency which is left as an exercise.

11.4. **Exercises.**

(1) Let $p(\lambda) = a\,\lambda^2 + b\,\lambda + c$ and set $G(\omega) = 1/|p(i\omega)|$. Show that the gain $G(\omega)$ has a unique global maximum and find the corresponding resonant frequency.

(2) Use Equation (48) to find the phase shift induced by an RLC-circuit. (Hint: write the left-hand side of (48) in the form $a\,\cos(\omega\,t-\phi)$ and the right-hand side in the form $A\,\cos(\omega\,t-\Phi)$. The phase shift is then the difference $\Phi-\phi$.) Create a plot of the phase shift for $R = .1$, $L = 1$, and $C = .1$. Bonus points if you find a formula for the phase shift in terms of the characteristic polynomial.

(3) Solve Equation (41) with $f = \sin^{-1}(\sin(x))$ and zero initial conditions. The plot of f is called the sawtooth wave. First derive the general formula for the solution (using Fourier analysis). Then plot the solution for $R = .1$, $L = 1$, $C = 10$. Present the derivation, the plot, and the code that produces the plot.