Taylor tricks

In theory, the construction of a Taylor polynomial for a given function at a given point is a straightforward task owing to the general formula:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

However, in practice, the mechanical use of the general formula may be too cumbersome. For instance, suppose we want T_8 for $f(x) = \sin(x^2)$ centered at the origin. It is straightforward to compute:

$$\begin{aligned}
f^{(1)} &= 2\cos(x^2)\,x \\
f^{(2)} &= -4\,\sin(x^2)x^2 + 2\,\cos(x^2) \\
f^{(3)} &= -8\,\cos(x^2)x^3 - 12\,\sin(x^2)x \\
f^{(4)} &= 16\,\sin(x^2)x^4 - 48\,\cos(x^2)x^2 - 12\,\sin(x^2) \\
f^{(5)} &= 32\,\cos(x^2)x^5 + 160\,\sin(x^2)x^3 - 120\,\cos(x^2)x \\
f^{(6)} &= -64\,\sin(x^2)x^6 + 480\,\cos(x^2)x^4 + 720\,\sin(x^2)x^2 - 120\,\cos(x^2) \\
f^{(7)} &= -128\,\cos(x^2)x^7 - 1344\,\sin(x^2)x^5 + 3360\,\cos(x^2)x^3 + 1680\,\sin(x^2)x \\
f^{(8)} &= 256\,\sin(x^2)x^8 - 3584\,\cos(x^2)x^6 - 13440\,\sin(x^2)x^4 \\
&\quad + 13440\,\cos(x^2)x^2 + 1680\,\sin(x^2)
\end{aligned}$$

Yet, notice how, due to repeated application of Chain Rule, the number of terms keeps growing. This is clearly not a sustainable computation. On the other hand, if we evaluate the derivatives at zero and substitute those values into the formula for T_8, we get:

$$T_8 = x^2 - \frac{x^6}{6}$$

Clearly, there ought to be a better way of finding such a simple formula!

Here is the first trick. Start with the expansion of $\sin(t)$ at zero which we know to be:

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

Now set $t = x^2$:

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

Finally, since we want T_8, truncate the series at order 8—delete all terms of order higher than 8. The result is

$$T_8 = x^2 - \frac{x^6}{3!}.$$

Quick and painless! If we want a better approximation, we need to go to degree 10:

$$T_{10} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}.$$

Imagine having to compute the 10-th derivative of $\sin(x^2)$ by hand! The moral of the story is that the Taylor series of $f(x^k)$ centered at zero can be easily computed by composing the series of $f(t)$ with $t = x^k$. That allows one to avoid Chain Rule.

To motivate the next trick, consider the same task of finding T_8 at zero for $x^6 \sin(x)$.

$$\begin{aligned}
f^{(1)} &= 6\,x^5 \sin(x) + x^6 \cos(x)\\
f^{(2)} &= 30\,x^4 \sin(x) + 12\,x^5 \cos(x) - x^6 \sin(x)\\
f^{(3)} &= 120\,x^3 \sin(x) + 90\,x^4 \cos(x) - 18\,x^5 \sin(x) - x^6 \cos(x)\\
f^{(4)} &= 360\,x^2 \sin(x) + 480\,x^3 \cos(x) - 180\,x^4 \sin(x) - 24\,x^5 \cos(x)\\
&\quad + x^6 \sin(x)\\
f^{(5)} &= 720\,x \sin(x) + 1800\,x^2 \cos(x) - 1200\,x^3 \sin(x) - 300\,x^4 \cos(x)\\
&\quad + 30\,x^5 \sin(x) + x^6 \cos(x)\\
f^{(6)} &= 720\, \sin(x) + 4320\,x \cos(x) - 5400\,x^2 \sin(x) - 2400\,x^3 \cos(x)\\
&\quad + 450\,x^4 \sin(x) + 36\,x^5 \cos(x) - x^6 \sin(x)\\
f^{(7)} &= 5040\, \cos(x) - 15120\,x \sin(x) - 12600\,x^2 \cos(x) + 4200\,x^3 \sin(x)\\
&\quad + 630\,x^4 \cos(x) - 42\,x^5 \sin(x) - x^6 \cos(x)\\
f^{(8)} &= -20160\, \sin(x) - 40320\,x \cos(x) + 25200\,x^2 \sin(x) + 6720\,x^3 \cos(x)\\
&\quad - 840\,x^4 \sin(x) - 48\,x^5 \cos(x) + x^6 \sin(x)
\end{aligned}$$

Again the derivatives become very cumbersome as the order increases and yet the final result is, simply: $T_8 = x^7$.

The culprit now is the Product Rule. Do you see a way to avoid it? What if we simply multiply the expansion of $\sin(x)$ by x^6? This gives

$$x^6 \sin(x) = x^6 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) = x^7 - \frac{x^{10}}{3!} + \frac{x^{11}}{5!} - \frac{x^{13}}{7!} + \dots$$

If we truncate the series at order 8, we get: $T_8 = x^7$. This matches exactly what we computed earlier.

Quite generally, if we need a Taylor expansion of a product we can compute it by multiplying out Taylor expansions to appropriate order. For instance, to find T_2 at zero for $f(x) = e^x \cos(x)$, we write:

$$\begin{aligned} e^x \cos(x) &= \left(1 + x + \frac{x^2}{2!} + \ldots\right)\left(1 - \frac{x^2}{2!} + \ldots\right) \\ &= 1 + x + \frac{x^2}{2!} - \frac{x^2}{2!} + \text{cubic and higher terms} \end{aligned}$$

Hence in this case $T_2 = 1 + x$.

Basic Taylor series

At this point, we can write Taylor series (at zero) only for three common functions:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \end{aligned}$$

In order to become proficient with Taylor expansions, we need a few more basic expansions.

Geometric series and its consequences

Let x be any number. You may recall the following identity from algebra:

$$1 + x + x^2 + \ldots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$

Or not. Be this as it may, this is the sum of the first N terms of the geometric progression $\{1, x, x^2, x^3, \ldots\}$ and is often called the *geometric formula*. As an exercise, verify it by multiplying both sides by $(1 - x)$. In the meantime, if $|x| < 1$ then we can compute the limit

$$\lim_{N\to\infty} \frac{1 - x^{N+1}}{1 - x} = \frac{1 - \lim_{N\to\infty} x^{N+1}}{1 - x} = \frac{1}{1 - x}.$$

This suggests that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad |x| < 1 \tag{37}$$

The expansion in Equation (37) is commonly called the *geometric series*; it is important to bear in mind that it is only valid for $|x| < 1$. Having added the geometric series to our collection of basic series, what else can we do? It is immediate to derive

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots, \quad |t| < 1$$

This is just another way of writing the geometric series, of course. However, as the next step, let us integrate the resulting series from zero to x. On one hand:

$$\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

On the other hand, since the antiderivative of $(1+t)^{-1}$ is $\ln(1+t)$ we get from the Fundamental Theorem of Calculus:

$$\int_0^x \frac{dt}{1+t} = \ln(1+x) - \ln(1) = \ln(1+x).$$

Therefore

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

which is valid for $|x| < 1$ and, curiously, for $x = 1$. In fact, if we set $x = 1$ we get a familiar looking approximation:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Meanwhile, if $x = -1$ we get the sum which we can write as:

$$-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right)$$

The important series in parentheses is called the *harmonic series*: it is known to diverge to infinity. This should make sense because

$$\lim_{\epsilon \to 0+} \ln \epsilon = -\infty$$

There is one more function that can be easily expanded into a Taylor series using the geometric series: the inverse tangent. The trick is to write $\tan^{-1}(x)$ as the integral:

$$\begin{aligned}\tan^{-1}(x) &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \left(1 - t^2 + t^4 - t^6 + \ldots\right)\, dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\end{aligned}$$

Again, the restriction $|x| < 1$ in the geometric formula (37) limits the range of x in the expansion of inverse tangent to the interval $(-1, 1]$. For the special case $x = 1$ we get Gregory's formula for π:

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Exercises

1. Find the Taylor polynomial of order 10 centered at zero for the following functions:

   (a) $y = \frac{1}{(1-x)^3}$

   (b) $y = x^4 \tan^{-1}(x)$

   (c) $y = e^{-x^2}$

   (d) $y = \sqrt{1+x^2}$

2. Conside the following integral:

   $$\int_0^1 e^{-x^2}\, dx = 0.746824132\ldots$$

   Approximate it using Taylor polynomials of orders 1 through 10. Use the provided value to compute the accuracy of approximation. How quickly do approximations converge?