3. The meaning of divergence

Our derivation of the Heat Equation in two dimensions suggests that, quite generally,

$$\int_{\partial D} F \cdot n \, ds = \iint_D \operatorname{div}(F) \, dA \tag{9}$$

should hold for any vector field F and any domain D in $\mathbb{R}^2$. This is, indeed, the statement of the Divergence Theorem in two dimensions. Of course, in order to compute divergence, one needs to be able to differentiate the components of the field. Furthermore, to ensure that the integral on the right exists, one may wish $\operatorname{div}(F)$ to be continuous and D reasonably "nice". The full statement of the Divergence Theorem therefore includes (i) some natural restrictions on the field: namely, that its components must be *continuously differentiable* within the interior of D; and (ii) some restrictions on the topology of D which are much less obvious.

In this section we will focus on the main idea behind Equation (9) and the true meaning of divergence. To avoid constant disclaimers, we henceforth tacitly assume that all vector fields are continuously differentiable, or C^1 (pronounced "cee-one"), and all domains are simple (with simple boundaries). Now, the word 'simplicity' when applied to shapes should not be taken literally. A simple domain may actually look rather complicated—think if a weirdly shaped ink blot or a polygon with holes. Simplicity does not mean that one can describe a domain with a simple formula or an equation. Rather, we are avoiding infinite domains, like a half plane, and things like fractals. If the domain is not simple, it does not mean that one cannot use the Divergence Theorem. Yet, it can get very tricky and we will postpone such subtleties until later.

3.1. **One-dimensional analogue.**

As a preparation for proving Equation (9), let us consider its one-dimensional analogue—the Fundamental Theorem of (single-variable) Calculus (FTC):

$$f(b) - f(a) = \int_a^b f'(x) \, dx.$$

Divide the interval $[a, b]$ into N subintervals of not necessarily equal length; label the subdivision points $x_1, x_2, \ldots, x_{N-1}$, setting $x_0 = a$ and $x_N = b$; denote by $\Delta x_i = x_i - x_{i-1}$ the length of the i-th subinterval where $i = 1, \ldots, N$. The integral on the right is, by definition, the limit of the Riemann sum:

$$\int_a^b f'(x)\,dx = \lim_{N\to\infty} \sum_{n=1}^{N} f'(x_i^*)\,\Delta x_i,$$

where, as N increases, $\max_{1\le i\le N}(\Delta x_i) \to 0$. Notice that Riemann sums are constructed using arbitrary subdivision points x_i and arbitrary sampling points x_i^*; the only restriction is that the lengths of subintervals must approach zero. We will exploit the flexibility in the choice of the sampling points x_i^*. Instead of sampling randomly, let us purposefully pick x_i^* in accordance with the Mean Value Theorem (MVT). Recall that the latter states that if f is differentiable on $[a, b]$ there exists a point c within the interval such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Let us apply MVT to each subinterval $[x_i, x_{i-1}]$: namely, let us select x_i^* so that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{f(x_i) - f(x_{i-1})}{\Delta x_i},$$

which is possible because of MVT. With that choice of the sampling points, the Riemann sum telescopes:

$$\begin{aligned}\sum_{n=1}^{N} f'(x_i^*)\,\Delta x_i = \sum_{n=1}^{N} \frac{f(x_i) - f(x_{i-1})}{\Delta x_i}\,\Delta x_i &= \sum_{n=1}^{N} (f(x_i) - f(x_{i-1}) \\ &= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \ldots \\ &+ (f(x_N) - f(x_{N-1})) = f(x_N) - f(x_0) = f(b) - f(a).\end{aligned}$$

And the Fundamental Theorem of Calculus follows.

Before returning to two dimensions, let us take stock of what happened on the line. Cast in the language of Vector Analysis, the proof of FTC says that the net flux of one-dimensional field $f(x)\,\mathbf{i}$ across $D = [a, b]$ equals the integral of the divergence $f'(x)$. And this is because we can break up the domain into small subintervals and add up the fluxes across those subintervals:

$$(f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \ldots + (f(x_N) - f(x_{N-1}))$$

Flux is additive. As we add up the fluxes across subintervals, terms cancel: the flux at the right endpoint of $[x_{i-1}, x_i]$ cancels with the flux at the left endpoint of $[x_i, x_{i+1}]$. In the end only two terms survive: fluxes at the endpoints of $[a, b]$. Notice the key role played by MVT:

it allowed us to write divergence at x_i^* as flux across the subinterval $[x_i, x_{i-1}]$ divided by the length of that subinterval:

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{\Delta x_i}.$$

Divergence in one dimension is flux per length, or *flux density*.

3.2. **Proof in two dimensions.** It stands to reason that if flux is additive in one dimension, it should be additive in higher dimensions. And this is, indeed, the case. To see that, consider Figure 1 which shows a rectangle $ABCD$ split into two rectangles with the line EF.

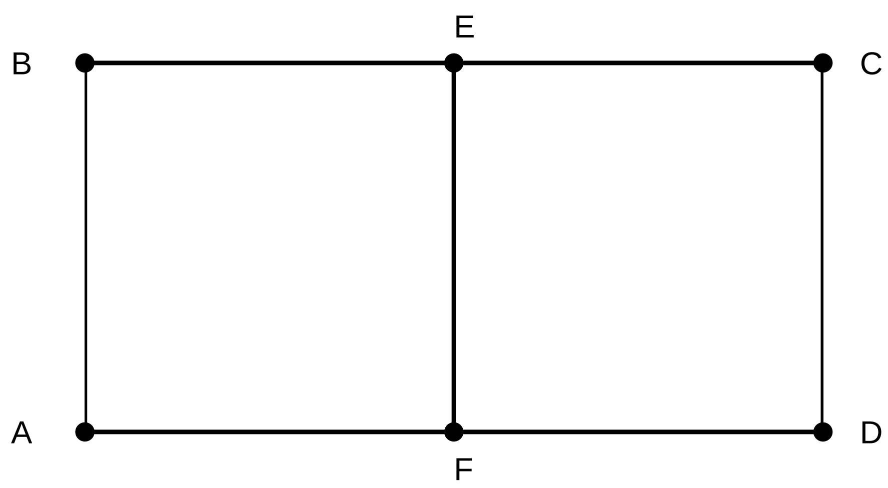

FIGURE 1. Additivity of flux

What is the sum of fluxes across the rectangles $ABEF$ and $DCEF$? The flux across $ABEF$ can be broken up into four terms corresponding to fluxes across the four sides. Same for the flux across $DCEF$. We have thus eight terms in total. Of these, six form the flux across $ABCD$ while the remaining two cancel. The cancelation happens for fluxes across EF because the outward unit normal for $ABEF$ on that side is the *inward* unit normal for $DCEF$.

It is now clear that if a two-dimensional domain D can be tiled with rectangles then the Divergence Theorem holds for D because it holds for rectangles. Since we restrict ourselves to *simple* domains, we can always approximate D with a patchwork of small rectangles $R_1, \ldots, R_N$ even if the boundary ∂D is curved. But then we can approximate flux

$$\begin{aligned}\int_{\partial D} (F \cdot n)\, ds &\approx \sum_{i=1}^{N} \int_{\partial R_i} (F \cdot n)\, ds \\ &= \sum_{i=1}^{N} \iint_{R_i} \operatorname{div}(F)\, dA \approx \iint_D \operatorname{div}(F)\, dA\end{aligned}$$

which, in the limit as $N \to \infty$, yields exact equality. Therefore the Divergence Theorem holds for all simple domains in two dimensions.

As a final note, write the approximation to the flux as

$$\int_{\partial D} (F \cdot n)\, ds \approx \sum_{i=1}^{N} \frac{\int_{\partial R_i} (F \cdot n)\, ds}{\Delta A_i} \Delta A_i,$$

where ΔA_i is the area of the i-th rectangle, assumed very small. Let (x_i^*, y_i^*) be any point in the rectangle R_i. The Divergence Theorem suggests that

$$\operatorname{div}(F)(x_i^*, y_i^*) \approx \frac{\int_{\partial R_i} (F \cdot n)\, ds}{\Delta A_i}.$$

To write the exact equality, we need to shrink the rectangle to a single point. Thus, quite generally,

$$\operatorname{div}(F)(x, y) = \lim_{A \to 0} \frac{\int_{\partial R} (F \cdot n)\, ds}{A}$$

where R is a rectangle containing (x, y) and A is its (shrinking) area. *Divergence is flux density.*

4. Divergence Theorem in three dimensions and consequences

The flux in three dimensions is defined as the integral of the normal component of the field over a (closed) surface—the two-dimensional boundary of a solid D:

$$\iint_{\partial D} (F \cdot n)\, dS.$$

Again, the normal vector n is the outward unit normal and instead of arc length ds we use surface area dS. Since divergence in one- and two dimensions is flux density, we define it as flux density in three dimensions:

$$\operatorname{div}(F)(x, y, z) = \lim_{V \to 0} \frac{\iint_{\partial D} (F \cdot n)\, dS}{V}.$$

That is, we surround a point (x, y, z) with a solid D and compute the above limit as the volume V of the solid shrinks to zero. We can now easily formulate the Divergence Theorem: the flux across the boundary of D must equal the integral of the flux density over D. In symbols,

$$\iint_{\partial D} (F \cdot n)\, dS = \iiint_D \operatorname{div}(F)\, dV. \tag{10}$$

The proof is analogous to the one in two dimensions and is left as an exercise.

4.1. **Heat Equation in three dimensions.** As an immediate application, let us derive the Heat Equation in three dimensions using the same notation as earlier. We take an arbitrary domain D in three space. The rate of change of the energy content is, on one hand:

$$\frac{d}{dt} \iiint_D c\, \rho\, u\, dV = c\, \rho \iiint_D \frac{\partial u}{\partial t}\, dV.$$

On the other hand, the same rate of change can be found by computing thermal flux across the boundary:

$$c\, \rho \iiint_D \frac{\partial u}{\partial t}\, dV = -\iint_{\partial D} (\phi \cdot n)\, dS.$$

Using the Divergence Theorem (10), rewrite the flux as a volume integral:

$$c\, \rho \iiint_D \frac{\partial u}{\partial t}\, dV = -\iiint_D \operatorname{div}(\phi)\, dV.$$

Since D is arbitrary, integrals can be dropped and we get the Equation of Continuity:

$$c\, \rho \frac{\partial u}{\partial t} = -\operatorname{div}(\phi).$$

Combine the Equation of Continuity with Fourier's Law:

$$\phi = -K \operatorname{grad}(u).$$

Then, after some cosmetic changes, the Heat Equation can be written as:

$$\frac{\partial u}{\partial t} = k\, \Delta u,$$

where $\Delta u = \operatorname{div}(\operatorname{grad}(u))$ is called the Laplace operator.

4.2. **Application to fluid dynamics.** Let v denote the velocity field of a moving fluid (or gas). So, v is a vector field in three dimensions; unless the flow is stead, v also depends on time. If the fluid is compressible (certainly true for gases) its density ρ will also vary in space in time. Consider now an arbitrary (but fixed) domain D. Equation of continuity in the context of fluid dynamics is the expression of mass conservation. The rate of change of mass of D is, on one hand:

$$\iiint_D \frac{\partial \rho}{\partial t}\, dV.$$

On the other hand, the same rate is given by the mass flux across the boundary:

$$\iiint_D \frac{\partial \rho}{\partial t}\, dV = -\iint_{\partial D} (\rho\, v \cdot n)\, dS.$$

Applying the Divergence Theorem and dropping integrals (as in the derivation of the Heat Equation), we arrive at:

$$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho\, v).$$

This is the Equation of Continuity in fluid dynamics. It holds for any fluid or gas flow (in the absence of sources).

If the fluid is incompressible (e.g., water) the Equation of Continuity simply states:

$$\operatorname{div}(v) = 0.$$

So, the divergence of the fluid's velocity field reflects its ability to compress and expand.

4.3. **Gradient.** We defined the divergence of a field as flux density

$$\operatorname{div}(F)(x, y, z) = \lim_{V \to 0} \frac{\iint_{\partial D} (F \cdot n)\, dS}{V}.$$

Notice that this definition does not involve coordinates. However, once the coordinate system is chosen, one can derive the formula for the divergence *adapted to that coordinate system* by taking the limit.

For instance, suppose we use Cartesian coordinates and require a formula for the divergence of the field $F = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ at the point (x_0, y_0, z_0). Let D be a cube centered at (x_0, y_0, z_0), having side length

$2\,\epsilon$. Then, combining the fluxes across opposing faces, leads to:

$$\begin{aligned}\iint_{\partial D} (F \cdot n)\, dS = & \int_{z_0-\epsilon}^{z_0+\epsilon} \left[\int_{y_0-\epsilon}^{y_0+\epsilon} \left(f(x_0+\epsilon, y, z) - f(x_0-\epsilon, y, z)\right) dy \right] dz \\ & + \int_{z_0-\epsilon}^{z_0+\epsilon} \left[\int_{x_0-\epsilon}^{x_0+\epsilon} \left(g(x, y_0+\epsilon, z) - g(x, y_0-\epsilon, z)\right) dx \right] dz \\ & + \int_{y_0-\epsilon}^{y_0+\epsilon} \left[\int_{x_0-\epsilon}^{x_0+\epsilon} \left(h(x, y, z_0+\epsilon) - h(x, y, z_0-\epsilon)\right) dx \right] dy\end{aligned}$$

Since the volume of the cube is $8\,\epsilon^3$ we are led to consider three limits of which the first is:

$$L_1 = \lim_{\epsilon\to 0} \frac{1}{8\,\epsilon^3} \int_{z_0-\epsilon}^{z_0+\epsilon} \left[\int_{y_0-\epsilon}^{y_0+\epsilon} \left(f(x_0+\epsilon, y, z) - f(x_0-\epsilon, y, z)\right) dy \right] dz$$

Let us focus on the bracketed integral and denote

$$u(y) = f(x_0+\epsilon, y, z) - f(x_0-\epsilon, y, z).$$

Let U be any antiderivative of u. Then, using FTC and recognizing the Calculus I definition of derivative, we compute:

$$\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{y_0-\epsilon}^{y_0+\epsilon} u(y)\, dy = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \frac{U(y_0+\epsilon) - U(y_0-\epsilon)}{2\,\epsilon} = u(y_0)$$

Hence

$$L_1 = \lim_{\epsilon\to 0} \frac{1}{4\,\epsilon^2} \int_{z_0-\epsilon}^{z_0+\epsilon} \left(f(x_0+\epsilon, y_0, z) - f(x_0-\epsilon, y_0, z)\right) dz,$$

and, iterating the same argument:

$$L_1 = \lim_{\epsilon\to 0} \frac{1}{2\,\epsilon} \left(f(x_0+\epsilon, y_0, z_0) - f(x_0-\epsilon, y_0, z_0)\right) = \frac{\partial f}{\partial x}(x_0, y_0, z_0).$$

By the same token, the limit

$$L_2 = \lim_{\epsilon\to 0} \frac{1}{8\,\epsilon^3} \int_{z_0-\epsilon}^{z_0+\epsilon} \left[\int_{x_0-\epsilon}^{x_0+\epsilon} \left(g(x, y_0+\epsilon, z) - g(x, y_0-\epsilon, z)\right) dx \right] dz$$

leads to the partial derivative of g

$$L_2 = \frac{\partial g}{\partial y}(x_0, y_0, z_0),$$

while

$$L_3 = \lim_{\epsilon\to 0} \frac{1}{8\,\epsilon^3} \int_{y_0-\epsilon}^{y_0+\epsilon} \left[\int_{x_0-\epsilon}^{x_0+\epsilon} \left(h(x, y, z_0+\epsilon) - h(x, y, z_0-\epsilon)\right) dx \right] dy$$

results in $\frac{\partial h}{\partial z}(x_0, y_0, z_0)$. Hence, at any point,

$$\operatorname{div}(F) = L_1 + L_2 + L_3 = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

Let us now turn our attention to the gradient. It stands to reason that there should be a coordinate-free formula that defines it. To derive it, apply the Divergence Theorem on an arbitrary domain D to the field $F = f(x, y, z)\, c$ where c is any constant vector:

$$\iint_{\partial D} (f\, c \cdot n)\, dS = \iiint_D \operatorname{grad}(f) \cdot c\, dV.$$

Since c is constant, we can pull it out

$$c \cdot \iint_{\partial D} f\, n\, dS = c \cdot \iiint_D \operatorname{grad}(f)\, dV,$$

and, since it is also an arbitrarily chosen vector, we conclude that

$$\iint_{\partial D} f\, n\, dS = \iiint_D \operatorname{grad}(f)\, dV. \tag{11}$$

Now divide both sides by the volume V of the domain and shrink the domain to a point. Then the average value of the gradient over D will approach the exact value at that point. Hence

$$\operatorname{grad}(f) = \lim_{V \to 0} \frac{\iint_{\partial D} f\, n\, dS}{V}. \tag{12}$$

Equation (12) defines gradient in a coordinate-free manner. By introducing coordinates and computing the limit for a shrinking solid, one can derive formulas for the gradient in any coordinate system.

4.4. **Archimedes' Principle.** We conclude with an application of Equation (11). Recall that Archimedes' Principle states that a submerged body experiences an upward buoyant force equal to the weight of the displaced liquid. Let D be a solid submerged in water. Align the xy-plane with the water's surface. Then the depth is $(-z)$ and the hydrostatic pressure is given by:

$$P = -\rho\, g\, z.$$

The force on the solid is given by the integral:

$$F = -\iint_{\partial D} P \cdot n\, dS,$$

the minus is because n is the outward normal and the force on dS is directed inward. According to Equation (11):

$$F = -\iiint_D \operatorname{grad}(P)\, dV = \iiint_D (\rho\, g\, \mathbf{k})\, dV = \rho\, g\; V\, \mathbf{k},$$

where V is the volume of D. Notice that we got an upward force whose magnitude $\rho\, g\ V$ is the weight of the displaced water in accordance with Archimedes' Principle.

Exercises.

(1) Derive the formula for divergence in polar coordinates. **Hint:** Write $F = f(r,\theta)\,\mathbf{e}_r + g(r,\theta)\,\mathbf{e}_\theta$ where $\mathbf{e}_r$ is the normalized position vector (unit vector directed away from the origin) and $\mathbf{e}_\theta$ is the orthogonal vector (pointing counterclockwise). Compute flux density taking as D the quadrilateral centered at (r_0, θ_0) (the boundary of D consists of two arcs and two radial line segments).

(2) Find the formula for the gradient in polar coordinates.

(3) Compute the Laplacian in polar coordinates.