NOTES ON VECTOR ANALYSIS

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1. Introduction

The study of a challenging discipline is greatly facilitated by clear and engaging objectives—the fewer the better. For when the student does not understand the "why" of it, the "how" of it becomes lost on him and the subject turns into a potpourri of strange facts and trivia. Vector Analysis is emphatically not a potpourri of strange facts and trivia: rather it is a challenging and stunningly beautiful part of Mathematical Physics and a bridge between Multivariate Calculus and Differential Geometry.

Historically, Vector Analysis was developed alongside of Electromagnetism, Fluid Dynamics, and Relativity as a mathematical foundation for these disciplines. It includes among its contributors such giants as Maxwell and Einstein, just to name a few. Accordingly, both our context and our main objective will be the derivation of partial differential equations (PDE) of Mathematical Physics. The latter include, but are not limited to, the Heat, Wave, and Laplace-Poisson equations. Understanding these three classical PDE is mandatory for anyone wishing to tread in Maxwell's and Einstein's footsteps as we will eventually do.

Our starting point will be the one-dimensional Heat Equation which we derive in Section 2 using elementary single-variable calculus. The Physics does not change in higher dimensions, however, the mathematics does: Calculus yields the stage to Vector Analysis. Section 2 thus foreshadows similar derivations in two and three dimensions which will bring us directly to the heart of the subject.

2. Heat Equation in one dimension

Consider a thin metal rod of length L and cross-sectional area A that is fully covered in thermal insulation so as to form a closed thermodynamical system. If, initially, the temperature of the rod is constant on perpendicular cross-sections and varies along the length only, it will remain so at all future times. That is, if the initial temperature profile is one dimensional, the flow of thermal energy will also be one-dimensional. Consequently, the temperature u of the rod will depend on two variables: the distance x from one of the ends, which we are free to place at the origin, and time t which, by convention, starts from zero. Our object is to derive a partial differential equation (PDE) describing the evolution of rod's temperature. This PDE, appropriately called the Heat Equation, is one of the most important equations of Mathematical Physics.

For clarity, we will break up the derivation into several logical steps. At the end, the reader's assignment will be to make a tenacious attempt at retracing the same steps in two dimensions. We do not expect the reader to be capable of independently deriving the two-dimensional Heat Equation so early on. Rather the attempt is suggested as an opportunity to raise questions which, when properly formed, lead directly from Calculus to Vector Analysis.

2.1. **Energy stored in an arbitrary section.** From the physical point of view, the Heat Equation, as well as some other PDE which we will encounter later, is an expression of the law of conservation of energy. It is a *local* conservation law, that is, a law that applies pointwise. Typically, local laws are deduced from the more intuitive global laws. In our case, the appropriate first step is to compute the total amount of heat $q(t)$ contained in an *arbitrary* section $[a, b]$ of the rod at a given time t. Notice that we emphasized the word 'arbitrary' as it is going play an important role in a later argument.

To find q, first recall that there is a linear relationship between temperature and heat. If the temperature of an object is uniform, then the amount of heat that it stores is given by the product of the temperature, the object's mass, and a certain positive coefficient called *heat capacitance.* The latter depends on the properties of the material from which the object is made and will be henceforth designated by the symbol c. If the temperature of the object is not uniform, as it is our case, one has to use integration. Let dx be an infinitesimally small portion of $[a, b]$ within which the temperature can be treated as a constant. Denote by ρ the (constant) density of the rod. Then $\rho\, A\, dx$ is the element of mass which stores $dq = c\, u(x, t)\, \rho\, A\, dx$ Joules of heat. The total amount of heat q stored between a and b is the integral $\int_a^b dq$ which, after all constants are pulled out, assumes the form:

$$q(t) = c\, \rho\, A \int_a^b u(x, t)\, dx.$$

Here we would like to stress that, as a definite integral with arbitrarily chosen but fixed limits, q depends on time alone.

2.2. **The change in energy stored in an arbitrary section.** There is very little that can be said about the quantity $q(t)$ itself, yet one can make useful observations about its rate of change. On one hand, since the operations of differentiation with respect to time and integration with respect to position involve independent variables, they can be interchanged as follows:

$$\frac{dq}{dt} = c\,\rho\,A\,\frac{d}{dt}\int_a^b u(x,t)\,dx = c\,\rho\,A\int_a^b \frac{\partial u}{\partial t}(x,t)\,dx. \tag{1}$$

Notice how the full derivative d/dt becomes partial derivative $\partial/\partial t$ when carried inside the integral: the temperature u is a function of two variables, unlike q, which is a functions of time alone.

On the other hand, what makes $q(t)$ vary in the first place? Clearly, this happens because heat enters and leaves the section $[a, b]$ which, due to thermal insulation, it can only do through the boundary cross-sections at $x = a$ and $x = b$. This simple observation leads to a very important idea of *flux*.

2.3. **Flux and the global energy balance.** We define the heat flux $\phi(x, t)$ as the instantaneous rate at which the heat passes through a small portion of the cross-section located at x. Note that the units of flux are those of energy divided by area and time: $[\phi] = \mathrm{J}\cdot\mathrm{m}^{-2}\cdot\mathrm{s}^{-1}$. If $\phi > 0$ the heat flows in the positive direction from left to right, whereas if $\phi < 0$ the flow is from right to left. Of course, if $\phi = 0$ there is no flow at all. Since the flow of heat is assumed to be one-dimensional, the net rate at which the heat enters the section $[a, b]$ through the cross-section at $x = a$ is given, simply, by $A\,\phi(a, t)$. Similarly, at $x = b$ the heat leaves $[a, b]$ at the net rate $A\,\phi(b, t)$. The net rate of change of the total amount of heat within the section $[a, b]$ is, therefore, given by:

$$\frac{dq}{dt} = A\,\phi(a,t) - A\,\phi(b,t). \tag{2}$$

We now have two expressions for dq/dt given by Equations (1) and (2) from which follows (after one cancels the constant cross-sectional area A):

$$c\,\rho\int_a^b \frac{\partial u}{\partial t}(x,t)\,dx = \phi(a,t) - \phi(b,t). \tag{3}$$

This is an example of a global law of conservation of energy, applicable to an arbitrary section of the rod.

2.4. **Equation of Continuity.** In order to derive a much more useful local law we need to somehow remove integration in Equation (3). Here, the first clue is the form of the right-hand side, which is reminiscent of the Fundamental Theorem of Calculus (FTC). Using FTC, rewrite the difference of flux values as an integral of the derivative:

$$\phi(a,t) - \phi(b,t) = \int_b^a \frac{\partial \phi}{\partial x}(x,t)\,dx = -\int_a^b \frac{\partial \phi}{\partial x}(x,t)\,dx.$$

Then both sides of (3) can be combined into one integral as follows:

$$\int_a^b \left(c\,\rho\,\frac{\partial u}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) \right)\,dx = 0. \tag{4}$$

Put in general terms, Equation (4) states that the integral of a certain function is zero. Now, if that were all, we would not be in a position to draw any further conclusions. Indeed, there are infinitely many functions whose integral over a given interval $[a,b]$ is zero and there is nothing special about them. However, Equation (4) holds not just for one choice of the limits a and b but for all possible choices of limits drawn from within the interval $[0,L]$. This means that the *integrand* must necessarily be zero:

$$c\,\rho\,\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0. \tag{5}$$

Equation (5) is called Equation of Continuity. It is a *local* form of the law of conservation of thermal energy valid for any $x \in (0,L)$ and any $t > 0$.

2.5. **Fourier's Law and the Heat Equation.** To derive the Heat Equation from Equation (5) we need to relate flux ϕ to the temperature u. As we know from experience, heat flows from places with high temperature to places with low temperature, against the temperature gradient. Mathematically, this is formalized by Fourier Law which in one dimension has the form:

$$\phi = -K\,\frac{\partial u}{\partial x}. \tag{6}$$

Here K is a positive constant called *thermal conductivity*. Combining Equations (5) and (6) leads, after some simple algebra, to

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \tag{7}$$

where we have combined all constants into one positive constant: $k = \frac{K}{c\,\rho}$. Equation (7) is our goal—the one-dimensional Heat Equation.

2.6. **Closing remarks.** Recall from single-variable Calculus that in order to compute the derivative of a function at a point one has to examine the slope of the function in the immediate vicinity of that point. For this reason, the Heat Equation (7) holds only for interior points and positive times. The Heat Equation does not hold at the initial moment $t = 0$. Nor does it hold at the endpoints of the rod $x = a, b$ at any time.

In order to completely describe the temperature of the rod one has to supplement the Heat Equation with initial and boundary conditions. The former is simply the initial temperature profile while the latter describe what happens at the endpoints. In our case, perfect insulation implies zero flux or, equivalently, zero temperature gradient. The full description of the temperature of the rod is thus given by the following Neumann boundary value problem:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L), \quad t > 0,\\
&u(x, 0) = f(x),\\
&\frac{\partial u}{\partial x}(a, t) = \frac{\partial u}{\partial x}(b, t) = 0, \quad t \geq 0.
\end{aligned}
\tag{8}
$$

From this point on, our immediate goal is deriving the analogue of (8) in two dimensions.

Exercises.

(1) First, give three different examples of a function f for which $\int_0^1 f(x)\,dx = 0$: this should be easy. Next, give an example of a function f for which $\int_a^b f(x)\,dx = 0$ for the following intervals $[a, b]$: $[0, 1/2]$, $[0, 1]$, and $[1/3, 2/3]$. Notice that now we impose three conditions on the same function, so finding an example is going to be trickier. Finally, try to prove that $\int_a^b f(x)\,dx = 0$ for all choices of $[a, b]$ from within $[0, 1]$ implies that $f = 0$ on $[0, 1]$.

(2) Consider the Neumann Problem (8) with $L = 2\pi$, $k = 1$ and $f(x) = \cos(2\,x)$. Confirm by direct computation that the solution is given by:

$$u(x, t) = \cos(2\,x)\,e^{-4\,t}.$$

Use software of your choice (e.g., Matlab) to produce an animation of the solution for $0 \leq t \leq 1$. What happens as $t \to \infty$ and why?

(3) Consider a thermally insulated metal plate of small thickness and arbitrary shape. Try to follow the steps in this handout

in an attempt to derive the Heat Equation in two dimensions. Most likely, you will encounter some difficulties (otherwise you do not need to take this course!) Make a note of the difficulties and prepare a list of questions that need to be answered before the two-, and higher dimensional versions of the Heat Equation can be derived.