In lieu of introduction

I often ask students on the first day of classes about their motivation for taking Applied Differential Equations. So far, almost as many times as I have asked that question, the answer has been, overwhelmingly: "Because it is required." Do you know why Math 57 is required?

In case you do not, let me describe in broad strokes what ordinary differential equations are and how you may profit from studying them. That is, how you may profit beyond fulfilling whatever math requirement.

Firstly, a differential equation is a relation between an unknown function and its derivatives. The adjective 'ordinary' signifies that the unknown function depends on a single variable and therefore the derivatives are ordinary Calculus I derivatives. If the unknown is a multivariate function, the derivatives are partial and the differential equation is also called 'partial.' In Math 57 we will study ordinary differential equations (ODE) leaving partial differential equations (PDE) for the sequel (Math 157).

A simple, but nontrivial example of an ODE is the following:

$$\frac{dx}{dt} = k\,x. \tag{1}$$

In words, Equation (1) says that the rate of change of an unknown function $x(t)$ is directly proportional to the function itself. Can you think of a function with that property?

Chances are, you have already seen Equation (1) and perhaps even know how to solve it. Equation (1) and its variants are ubiquitous in science and engineering because of the long list of phenomena that they can model. That list includes, but is not limited to radioactive decay, bacterial growth, charge on a capacitor in an RC-circuit, dependence of atmospheric pressure on altitude, attenuation of an x-ray beam, and many other things. All of which brings us to the first and foremost reason for studying ODE: modeling applications. Most ODE that you will encounter originate as mathematical expressions of various physical laws. Understanding physics, and science in general, is one of the main reasons for requiring [ordinary] differential equations.

Another reason is understanding mathematics. Perhaps not all of mathematics but, certainly, Calculus which historically was developed as a theoretical framework for differential equations. Think of Math 57 as the class where your study of Calculus will finally pay off. In addition, you will learn about Linear Algebra, Numerical Analysis, and get a small taste of nonlinear Dynamics. These are all vast and very important subjects which, I hope, you will continue to pursue after the semester ends.

In summary, if you apply yourself in Math 57, you will (i) develop a better understanding of the physical universe, (ii) make sense of Calculus, and (iii) learn about Linear Algebra, Numerical Analysis, and nonlinear Dynamics. Three very good reasons, would not you say?

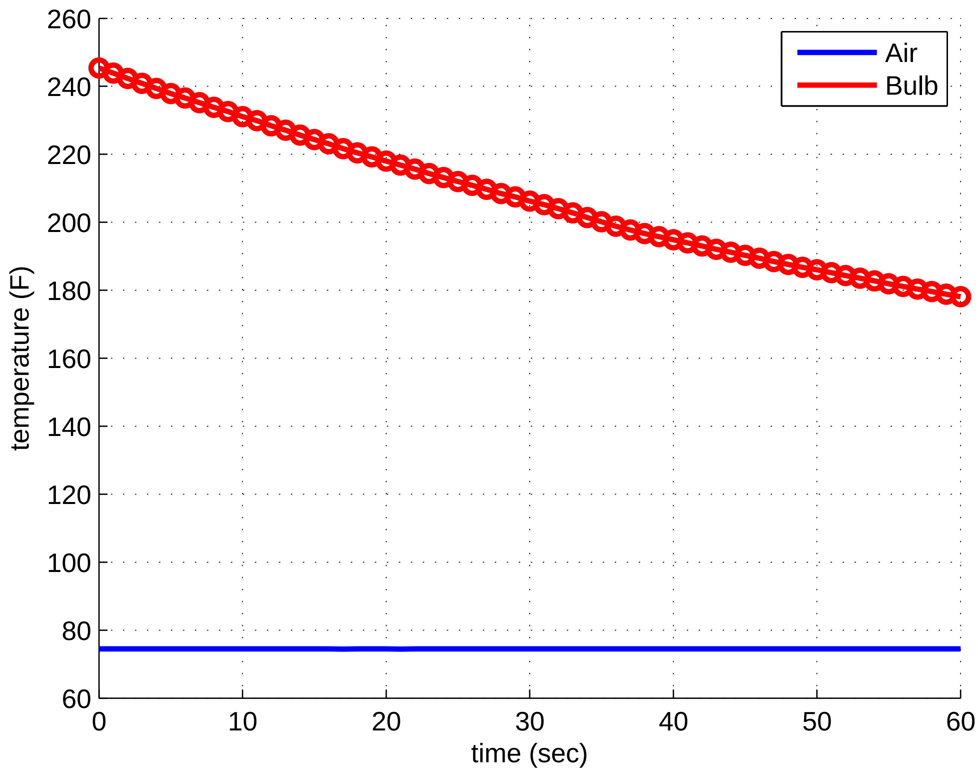

FIGURE 1. What is the temperature of the light bulb after two minutes?

1. THE MYSTERY OF THE COOLING LIGHT BULB

Figure 1 displays the temperature of an incandescent light bulb cooling in a room with nearly constant temperature of about 74 degrees. The observations were taken every second for ten minutes yet only the first minute worth of data is shown. Clearly, after an additional minute the red curve will get closer to the blue curve. How would you estimate the temperature of the light bulb at two minutes? Think about that *before* reading the answer below.

1.1. **Mathematical model.** While it is intuitively clear that the temperature of the light bulb will asymptotically approach room temperature not much else can be said about it. At least, not much can be said that can be leveraged into a computation. Therefore, rather than looking at temperature, let us turn our attention to its *rate of change*.

First, it will be helpful to introduce some notation. Let the temperature of the bulb be denoted by x.

> *As a rule, we will always use last letters of the Latin alphabet to denote unknowns; the letter t is an exception because it will be reserved for time. Bear in mind that in this course all unknowns are functions of time. Hence x is synonymous with $x(t)$.*

Next we need a symbol for the temperature of the room which is a *parameter* of the model—a constant presumed to be known.

> *Parameters are usually denoted by first letters of the Latin alphabet or Greek letters.*

Let us denote the temperature of the room by a. Now think about how the rate of cooling dx/dt may depend on x and a. The bulb is cooling because of the *difference* of these temperatures. Therefore, it stands to reason that

$$\frac{dx}{dt} = f(x - a) \tag{2}$$

where f is a function of one variable. Equation (2) is an example of a *first order ODE.*

> *The order of an ODE is the order of the highest derivative it contains.*

We now need to make the right-hand side of Equation (2) more specific. So far, based on our physical intuition, we can only say the following about the function f:

(1) Since there is no cooling when there is no difference in temperatures, we must have $f(0) = 0$.
(2) The signs of $f(x-a)$ and $x-a$ should be opposite (think about this!)

To make further progress, let us assume that f is differentiable. Then our earlier observations imply that the tangent line at the origin must have equation $y = -k\,x$, where k is some positive constant. Therefore, *when the difference of temperatures is small*, we have approximate equality: $f(x - a) \approx k\,(a - x)$. Of course, away from zero the graph of f may be very different from a straight line. Nevertheless, as a first approximation, we make the *linearity assumption*: $f(x-a) = k\,(a-x)$.

> *Even if the process is nonlinear, start with a linear model and add nonlinear features later. Always start with the simplest possible model.*

We now have a *linear* model of the cooling process

$$\frac{dx}{dt} = -k\,(x-a), \tag{3}$$

where the *positive* coefficient of proportionality k has units sec^{-1} (since it converts temperature into its rate of change).

> *All physical constants are positive. Negatives come from the orientation of coordinate axes and other considerations.*

Equation (3) is fully characterized as a first order linear ODE with constant coefficients. Later you will learn that linearity is a very important notion and just about *the* most important attribute of a differential equation. However, understanding linearity will take time. In the meantime, we can solve Equation (3) using calculus since it also happens to be *separable*. We proceed to do that in the next section.

1.2. **Separation of variables.** In order to recover a function from its derivative we need to integrate. However, we cannot integrate x with respect to t because x is so far an unknown function of time. While we cannot integrate x against dt, we can integrate any expression in x with respect to dx. This suggests the following step, called *separation* of variables for obvious reasons:

$$\frac{dx}{x-a} = -k\,dt. \tag{4}$$

Equation (4) is equivalent to (3) yet, unlike the latter, it can be integrated:

$$\int \frac{dx}{x-a} = \int (-k)\,dt. \tag{5}$$

Carrying out integration in (5), we obtain:

$$\ln(x-a) = -k\,t + C. \tag{6}$$

> *Notice that we only added one constant to the right-hand side. In principle, we could add constants to both sides but then they can be combined into one constant.*

Solving (6) for x gives what is called the *general solution*:

$$x = a + C_1\,e^{-k\,t}, \quad (C_1 = e^C) \tag{7}$$

You can (and should) check that Equation (7) satisfies ODE (3) regardless of the value of the constant C_1. Thus a first order ODE has infinitely many solutions which differ by a constant. We now need to select a specific value for the constant of integration which corresponds to our experimental data.

1.3. **Initial condition.** In order to find C_1 in Equation (7) we need to use the initial condition. Let $x(0) = x_0$. Then evaluation at time zero of (7) leads to: $x_0 = a + C_1$. Hence:

$$x = a + (x_0 - a)\, e^{-k\,t}. \tag{8}$$

Equation (8) is the unique solution of the *initial value problem* (IVP) which consists of ODE (3) and the initial condition $x(0) = x_0$. Again, convince yourself that formula (8) is correct by substituting it into the ODE (3) and evaluating it at time zero. Validation is important and should become an ingrained habit!

1.4. **Data analysis.** Having solved the initial value problem, we need to match the solution against the data. You can load the data and plot it using the following snippet of code:

```
load -ascii 'bulb.txt';
t = bulb(:,1);   % time in seconds
ch1 = bulb(:,2); % room temperature   (F)
ch2 = bulb(:,3); % bulb's temperature (F)

figure
hold on
plot(t,ch1,'b-','LineWidth',2)
plot(t,ch2,'r-',t,ch2,'ro','LineWidth',2)
xlabel('time (sec)')
ylabel('temperature (F)')
title('Cooling bulb data')
legend('Air','Bulb')
grid on
```

Note that the the first line assumes that the data is in the main MATLAB directory. If is is not there, use the full path to the file as an argument to `load`.

Having loaded the data, we need to find x_0, a, and k: then we can make the prediction using Equation (8). Sounds like a lot of processing and it would be in a low level language. Luckily in MATLAB all of that is accomplished by five lines of code.

```
% Fit x = a + (x(0)-a)*exp(-k*t)

x0 = ch2(1);                      % initial temperature
a = mean(ch1);                    % room temperature
y = -log((ch2-a)/(x0-a));         % y = k*t
k = (y'*t)/(t'*t);                % least squares estimate of k
```

```
x120 = a + (x0-a)*exp(-k*120);    % estimate of x(120)
```

The initial condition is simply the first element of `ch2`, so the first line is self-explanatory. Room temperature a is not exactly constant due to small fluctuations and noise in the sensor. Therefore we compute a as the average of `ch1` in the second line. The trickiest part is estimating the constant k which requires two steps. First we transform the data by computing

$$y(t) = \ln\left(\frac{x(t) - a}{x_0 - a}\right).$$

Whereas x depends on k exponentially, y is a linear function of k which allows for the use of *linear* regression. We will discuss linear, and other forms of regression in more detail later. What follows now is a brief explanation of how it works.

If the data y had no noise and we knew k then for each time instance t_i we would have, simply, $y_i = k\,t_i$, $i = 0, \ldots, 60$. In other words, the 61-dimensional vectors $\mathbf{y}$ and $k\,\mathbf{t}$ would be identical. Equivalently, the square of the magnitude of the difference

$$g(k) = |\mathbf{y} - k\mathbf{t}|^2 = \sum_{i=0}^{60} (y_i - k\,t_i)^2$$

would be identically zero and we could define k as the root of $g(k) = 0$. However, as it stands, the data y has noise in it and, as a result, no choice of k can make $g(k)$ exactly zero. The idea behind regression analysis is to *minimize* $g(k)$, the *residual*, with respect to k. This leads to the *least squares estimate*:

$$k = \frac{\sum_{i=0}^{60} y_i\,t_i}{\sum_{i=0}^{60} t_i^2}$$

It is precisely this estimate that is computed in the fourth line.

I should also explain that the line

```
k = (y'*t)/(t'*t);                % least squares estimate of k
```

implements the least squares estimate using the *dot product* rather than the `sum` command which you would normally expect. The apostrophes *transpose* column vectors so that they become rows; rows can be multiplied with columns using *matrix multiplication* and the result is exactly the dot product. More on that and MATLAB programming in future handouts.

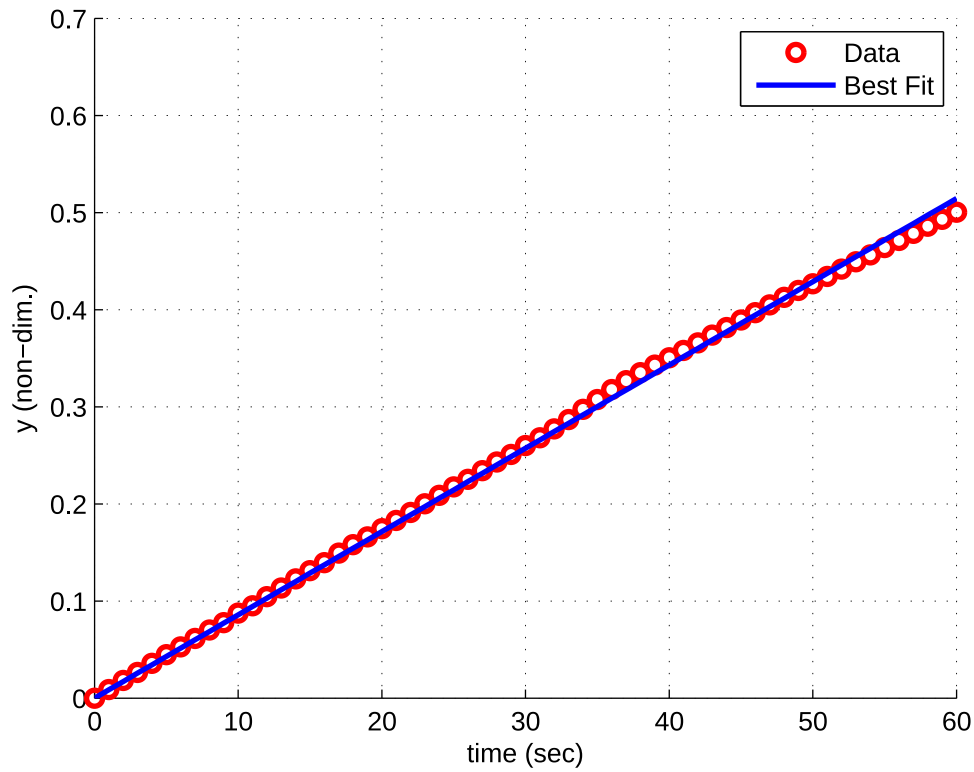

FIGURE 2. Least squares fit

In the meantime, if you run the above code, you will find that the least squares estimate for the constant is $k = 0.0086$ and the corresponding temperature estimate is $x_{120} = 135.5332$.

1.5. **Discussion.** Comparison with the full dataset shows that we found an underestimate. Notice that our least squares fit is very good for the first sixty seconds. However after that the cooling proceeds slower than our theory predicts. As a difficult but very useful exercise, try to refine the model (4) and see if it improves the prediction.

Exercises. The following exercises are designed to test your programming and Calculus skills. Most of them should be simple, however, if you find any of these problems to be strenuous, seek immediate help during office hours. You won't get far without prerequisites in this class!

Please follow the instructions meticulously and try to do all of these problems without help or with bare minimum of help. If you run into

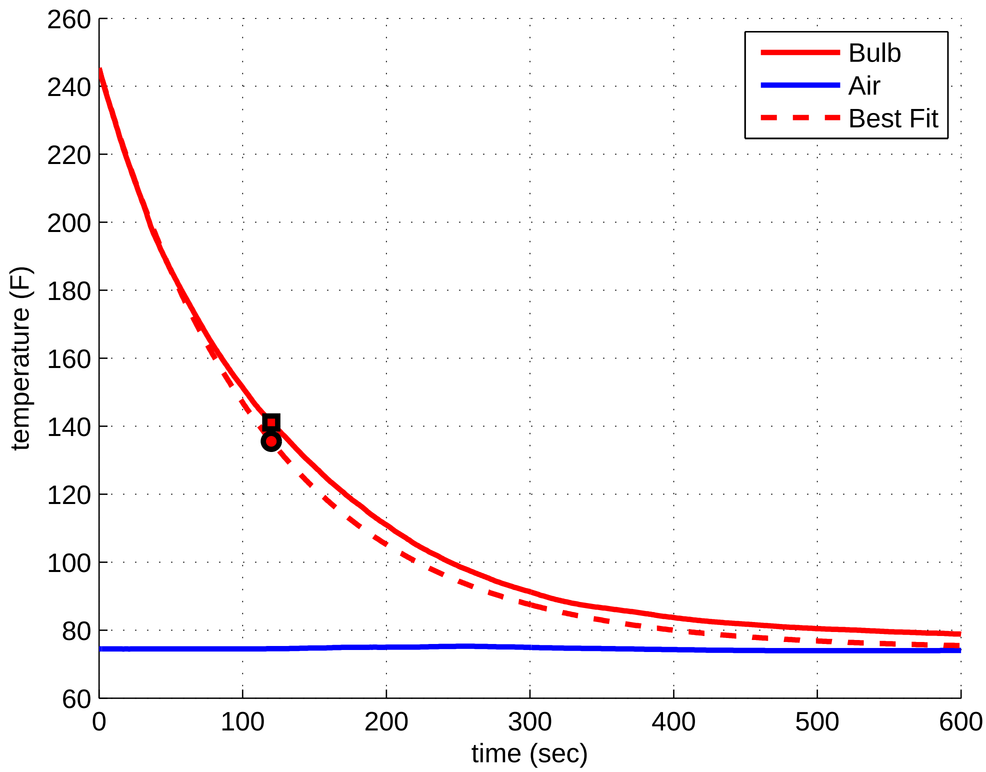

FIGURE 3. Extrapolated temperature vs measured temperature

difficulties, explain them verbally in lieu of solution: do not skip problems or leave blank spaces.

Finally, remember that this homework as any future homework must be emailed as a PDF attachment on the due date.

(1) Download `bulb.txt` from the course web site and reproduce Figures 1 and 2. If you can, try to write the code yourself. If that is too hard, use the code from the handout.
(2) Find the Taylor polynomial of order five centered at zero for the function $y = e^{-\frac{x^2}{2}}$. Produce a MATLAB plot showing the function (solid blue line) and its Taylor polynomial (red circles) on the interval $[0, 2]$. Label the axes of the plot x and y, add a legend, and title the plot "Plot for Problem 2".
(3) Consider the following differential equation:

$$t^2 \frac{d^2x}{dt^2} - t\,\frac{dx}{dt} + x = f(t).$$

Later you may be given the function $f(t)$ and asked to find the unknown $x(t)$, however, right now this may be too hard. Instead, suppose that the solution of the differential equation is

$$x = t\,(3 - \ln(t)) + 2\,\ln(t) + 4.$$

Find $f(t)$. Explain your answer.

(4) The following is known about a mystery function $x(t)$: (i) $x(0) = 1$; (ii) $\frac{dx}{dt}(0) = 2$; and (iii) $\frac{d^2x}{dt^2}(0) = -3$. Use this information to estimate the value $x(1)$. Explain your solution verbally. In particular, explain when one might expect your estimate to produce good accuracy and when your estimate should not be trusted.

(5) Later in the course we will work with trigonometric functions, a lot. This exercise is preparation for that. Consider the function

$$y = 2\,\cos(3\,t) - \sin(3\,t).$$

This is an example of a *simple harmonic* of frequency 3. One can write this function in the form: $A\,\cos(3\,t + \phi)$. Find the *amplitude* A and the *phase* ϕ. Confirm your calculation with a MATLAB plot; be sure to title the plot, label the axes, etc.

(6) Building on the previous exercise, for a positive integer N, set:

$$f_N = \sum_{n=1}^{N} \frac{\sin((2\,n-1)\,t)}{2\,n-1}.$$

Plot f_N for $N = 3, 6, 9, 12$ on the same axes. Describe in words the change in the shape of this function as N becomes large. What do you think will happen in the limit, as N approaches infinity?

(7) Find the coefficients a, b, and c in the partial fraction decomposition:

$$\frac{s^2+1}{s\,(s+1)\,(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}.$$

Confirm your computation with a MATLAB plot. Don't forget that each plot should have a title, labels on axes, etc.

(8) At some point we will run into integrals of the form:

$$L(f) = \int_0^\infty f(t)\,e^{-s\,t}\,dt.$$

The quantity $L(f)$ is called the *Laplace transform* of the function f; it is itself a function of the parameter s. Compute Laplace transforms of the following functions:

(a) $f = t^2$

(b) $f = \sin(t)$

(c) $f = \begin{cases} 1, & \text{if } t < 1; \\ 0, & \text{otherwise.} \end{cases}$

Validate your answers the best way you can.

(9) Find the derivative of

$$y(t) = e^{-\cos(t)}$$

and confirm your answer with an appropriate MATLAB plot.

(10) Imagine a spherical capsule dissolving in water. Suppose that the capsule is losing volume at the rate proportional to its surface area. Try to build an ODE model for this process following the same kind of logic that was used to model the cooling of a light bulb. If you succeed in building the model, use it to find an expression for the radius of the capsule as a function of time. If you cannot form a model, remember to explain your difficulties.