The purpose of this handout is to provide a brief introduction to linear algebra. We will only look at the connection between linear algebra and linear ODE with constant coefficients. However, that's just the tip of the iceberg. Linear algebra is an extremely powerful abstract framework for studying linear operations *of any kind*. As such, it is indispensable for truly understanding linearity and is worth pursuing independently of ODE.

Matrix-vector ODE. To make the discussion concrete, let us consider the following matrix-vector system:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \tag{92}$$

If you are completely unfamiliar with linear algebra notation then Equation (92) may seem utterly mysterious. Yet, it is just a different way of writing a familiar second order differential equation. To see that, let us convert Equation (92) into a system of *scalar* equations.

From matrix-vector equation to a system of equations. Recall from Multivariate Calculus that the derivative of a vector is the vector of derivatives: therefore, the left-hand side of Equation (92) is, simply,

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix}.$$

For reasons that would be too long to explain in this handout matrix-vector multiplication is defined "row by column". This means that the right-hand side of (92) is the vector:

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (-2)\,x + 1\,y \\ 1\,x + (-2)\,y \end{bmatrix}.$$

Observe that at the core of matrix-vector multiplication is the familiar (from Calculus III) dot product: the first component of the matrix-vector product is the dot product of the first row of the matrix with the vector while the second component of the matrix-vector product is the dot product of the second row of the matrix with the vector.

Matrix-matrix multiplication is defined similarly: the intersection of the i-th row and j-th column of $A\,B$ is the dot product of the i-th row of A with the j-th column of B:

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{array}{cc} & \text{column } j \\ \text{row } i & \begin{bmatrix} a_{i1}\, b_{1j} + \cdots + a_{in}\, b_{nj} \end{bmatrix} \end{array}$$

Of course, the rows of A must have the same length as the columns of B for the product $A\,B$ to make sense. [10]

It is now apparent that Equation (92) equates two vectors:

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2\,x + y \\ x - 2\,y \end{bmatrix}$$

Two vectors are equal if and only if their components match. Hence Equation (92) can be rewritten as the system of first order equations

$$\frac{dx}{dt} = -2\,x + y, \tag{93}$$

$$\frac{dy}{dt} = x - 2\,y, \tag{94}$$

which we now convert into a single ODE of second order.

Elimination of variables in ODE systems. Let us eliminate the unknown y. Using Equation (93), we can express y in terms of x as follows:

$$y = \frac{dx}{dt} + 2\,x. \tag{95}$$

Now substitute the expression (95) into Equation (94): this gives

$$\frac{d^2x}{dt^2} + 2\,\frac{dx}{dt} = x - 2\left(\frac{dx}{dt} + 2\,x\right),$$

which is equivalent to

$$\frac{d^2x}{dt^2} + 4\,\frac{dx}{dt} + 3\,x = 0. \tag{96}$$

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[10] In `Matlab` attempting to multiply matrices with inconsistent dimensions causes the following error:

??? Error using ==> mtimes
Inner matrix dimensions must agree.

Review of the scalar guess-and-check method. Equation (96) is a familiar ODE which we can solve in several different ways. For the purposes of this handout, it is best to "guess" exponential solutions. Accordingly, we set $x = e^{\lambda t}$ which results in

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 3 e^{\lambda t} = 0.$$

Since exponential functions are never zero, we can cancel the exponential terms. Solving the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0,$$

produces two distinct characteristic roots $\lambda_1 = -1$ and $\lambda_2 = -3$. Therefore

$$x = C_1 e^{-t} + C_2 e^{-3t},$$

and, owing to Equation (95):

$$y = \frac{dx}{dt} + 2x = C_1 e^{-t} - C_2 e^{-3t}.$$

Guess-and-check method vectorized. Having found x and y using "old" scalar techniques, let us return to the matrix-vector formalism and try to "vectorize" the scalar guess-and-check method. We now know that in vector form the solution of Equation (92) is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_1 e^{-t} + C_2 e^{-3t} \\ C_1 e^{-t} - C_2 e^{-3t} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}.$$

Evidently, the vector solution is a linear combination of two terms which have the general form:

$$\begin{bmatrix} u \\ v \end{bmatrix} e^{\lambda t}$$

This suggests that in order to solve Equation (92) directly, without converting it into a second order scalar ODE, we should "guess":

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} e^{\lambda t}.$$

Checking this guess leads to

$$\begin{bmatrix} u \\ v \end{bmatrix} \left(\lambda e^{\lambda t}\right) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} e^{\lambda t},$$

in which exponentials can be canceled, leading to:

$$\lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{97}$$

Eigenvectors and eigenvalues. It is easy to check that the following *eigenpairs*

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_1 = -1,$$

and

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = -3,$$

satisfy Equation (97). However, let us imagine for a moment that we do not know that. How can we find these solutions?

At first glance this may appear an impossible task. Indeed, Equation (97) is equivalent to a system of two scalar equations, yet we seek three scalar unknowns: λ, u, and v. To find a way out of the difficulty, let us combine all unknowns on one side

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and rewrite the left-hand side as a single matrix-vector product:

$$\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{98}$$

Equation (98) is a simple algebraic rearrangement of Equation (97)[11], yet it is much more amenable to analysis, as we now demonstrate.

Eigenvalues. Let us concentrate on finding the λ's. Suppose that we guess, incorrectly, $\lambda = 0$. Then Equation (98) becomes

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using the following general formula (Exercise) for the inverse of a two-by-two matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a\,d - b\,c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \tag{99}$$

we can write:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Certainly, the pair

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \lambda_3 = 0,$$

[11] If you are new to linear algebra, convince yourself that Equations (97) and (98) are equivalent by carrying out matrix-vector multiplications.

satisfies Equations (97) and (98). However, this is a *trivial* solution from which the general solution cannot be constructed.

Let us make another incorrect guess $\lambda = 1$. The same story repeats—Equation (98) becomes

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the formula (99) for the inverse of a two-by-two matrix forces the solution to be trivial:

$$\begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Clearly, as long as the matrix

$$\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

is *invertible* only trivial solutions are possible:

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{(-2-\lambda)^2 - 1} \begin{bmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Yet, this is something we wish to avoid. Evidently, our only option is to intentionally cause division by zero by setting

$$(-2-\lambda)^2 - 1 = \det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0.$$

Remarkably, this leads to the familiar solutions $\lambda_1 = -1$ and $\lambda_2 = -3$: these are the two *eigenvalues* of the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Eigenvectors. Having found the eigenvalues λ_1 and λ_2, let us find the two corresponding *eigenvectors*.

To find the first eigenvector, set $\lambda = \lambda_1$ in Equation (98). This leads to

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{100}$$

We *cannot* apply Equation (99) because the matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

has zero determinant and is therefore is non-invertible, by design. However, let us rewrite the matrix-vector Equation (100) as a system of scalar equations:

$$\begin{aligned} -u + v &= 0 \\ u - v &= 0 \end{aligned}$$

Note that the scalar equations say the same thing: $u = v$. Since we have only one condition for two unknowns we can set u to any value, say, C_1. Then $v = u = C_1$ and we conclude that Equation (100) has infinitely many solutions of the form:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} C_1 \\ C_1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is easy to check that the pair

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_1 = -1,$$

solves Equations (97) and (98) for any choice of C_1. Since in the process of forming the general solution we scale the particular solutions by arbitrary constants, we can set C_1 to any value other than zero. For instance, we can set $C_1 = 1$ and write

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is an *eigenvector*—one of infinitely many eigenvectors of the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

with the eigenvalue $\lambda_1 = -1$.

To find the second eigenvector, we set $\lambda = \lambda_2$ in Equation (98). This results in

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{101}$$

giving infinitely many solutions of the form:

$$\begin{bmatrix} u \\ v \end{bmatrix} = C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Again, to get a *particular* solution we can choose C_2 to be any number other than zero. Let us set $C_2 = 1$ and write the second eigenvector (with eigenvalue $\lambda_2 = -3$) as

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have found two exponential solutions of Equation (92):

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}, \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3\,t}.$$

By linearity, the general solution is the linear combination:

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + C_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3\,t}.$$

This matches the solution of Equation (92) that we found earlier.

Summary. The general solution of the matrix-vector system

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

has the form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} e^{\lambda_1\,t} + C_2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} e^{\lambda_2\,t}.$$

where

$$\mathbf{v}_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

are eigenvectors of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with eigenvalues λ_1 and λ_2, respectively.

Let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

denote the *identity matrix*. To find the eigenvalues, set

$$\det(A - \lambda\,I) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0,$$

and solve the resulting *characteristic equation*. Notice that in the two-by-two case the expansion of the determinant gives a quadratic equation

$$(a-\lambda)\,(d-\lambda) - b\,c = 0,$$

which will have two generally complex roots λ_1 and λ_2.

Henceforth, we assume that the eigenvalues $\lambda_{1,2}$ are distinct[12]. To get an eigenvector corresponding to the first eigenvalue λ_1, find any *nonzero* solution of

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If λ_1 was found correctly, the scalar equations

$$\begin{aligned} (a-\lambda_1)\,u + b\,v &= 0 \\ c\,u + (d-\lambda_1)\,v &= 0 \end{aligned}$$

will be *linearly dependent*—one equation will be a constant multiple of another. A nonzero solution can be obtained by setting $u = 1$ and solving either of the two equations for v; the second eigenvector—the one corresponding to λ_2—can be found in a similar manner.

Matrix-vector solution of the mass-spring system. Recall that the following IVP, describing the displacement of a mass on a spring,

$$m\,\frac{d^2x}{dt^2} + k\,x = 0, \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0. \tag{102}$$

has the solution given by:

$$x(t) = x_0\,\cos(\omega_0\,t) + v_0\,\frac{\sin(\omega_0\,t)}{\omega_0}, \tag{103}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

is the natural frequency of the mass-spring system. As an illustration of the principles discussed earlier, we will solve Equation (102) using Linear Algebra.

From one second order ODE to a system of first order ODE. Application of linear algebra requires that we rewrite Equation (102) in matrix-vector notation. As a first step, let us convert (102) to a system of first order ODE by introducing velocity:

$$\frac{dx}{dt} = v.$$

Differentiating velocity, and using Equation (102), we find:

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = -\frac{k}{m}\,x = -\omega_0^2\,x.$$

[12]To avoid having to go too deeply into linear algebra.

Therefore, Equation (102) is equivalent to the following system:

$$\begin{aligned}\frac{dx}{dt} &= v, \quad v(0) = v_0,\\ \frac{dv}{dt} &= -\omega_0^2\, x, \quad x(0) = x_0.\end{aligned}$$

Matrix-vector form. Aggregate the two scalar unknowns $x(t)$ and $v(t)$ into a single vector:

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}.$$

It is easy to see (Exercise) that, in matrix-vector form, the system of ODE describing the mass-spring system becomes:

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{104}$$

Eigenvalue decomposition. To find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}$$

we form the characteristic equation

$$\det(A - \lambda\, I) = \det\begin{bmatrix} -\lambda & 1 \\ -\omega_0^2 & -\lambda \end{bmatrix} = \lambda^2 + \omega_0^2 = 0.$$

The eigenvalues are purely imaginary

$$\lambda_1 = \omega_0\, i, \quad \lambda_2 = -\omega_0\, i,$$

which is what we expect since the solution should be a simple harmonic with frequency ω_0.

Since the eigenvalues are complex, the corresponding eigenvectors are also complex. The components of the first eigenvector—the one corresponding to $\lambda_1 = \omega_0\, i$ must satisfy

$$\begin{bmatrix} -\omega_0\, i & 1 \\ -\omega_0^2 & -\omega_0\, i \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is equivalent to the system:

$$\begin{aligned} -\omega_0\, i\, u_1 + v_1 &= 0\\ -\omega_0^2\, u_1 - \omega_0\, i\, v_1 &= 0.\end{aligned}$$

Note that the second equation is the first equation multiplied by $(-\omega_0\, i)$: the two equations are linearly dependent, as they should be. To find a nonzero solution, let us set $u_1 = 1$. Then $v_1 = \omega_0\, i\, u_1 = \omega_0\, i$. The first eigenpair is therefore:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \omega_0\, i \end{bmatrix}, \quad \lambda_1 = \omega_0\, i.$$

In a similar manner the second eigenpair is found to be:

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\omega_0\, i \end{bmatrix}, \quad \lambda_2 = -\omega_0\, i$$

We conclude that the general solution of Equation (104) is given by

$$\begin{bmatrix} x \\ v \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ \omega_0\, i \end{bmatrix} e^{\omega_0\, i\, t} + C_2 \begin{bmatrix} 1 \\ -\omega_0\, i \end{bmatrix} e^{-\omega_0\, i\, t}, \tag{105}$$

where C_1 and C_2 are complex constants which remain to be found.

Satisfying initial conditions. To find the constants of integration, evaluate both sides of Equation (105) at $t = 0$. This leads to:

$$\begin{aligned} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} &= C_1 \begin{bmatrix} 1 \\ \omega_0\, i \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -\omega_0\, i \end{bmatrix} = \begin{bmatrix} C_1 + C_2 \\ \omega_0\, i\, C_1 - \omega_0\, i\, C_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ \omega_0\, i & -\omega_0\, i \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \end{aligned}$$

Above, we recast the sum of vectors as a matrix-vector product so that we could find the constants using matrix inversion. Using Equation (99), we can now write

$$\begin{aligned} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \omega_0\, i & -\omega_0\, i \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \\ &= \frac{1}{-2\,\omega_0\, i} \begin{bmatrix} -\omega_0\, i & -1 \\ -\omega_0\, i & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{-\omega_0\, i\, x_0 - v_0}{-2\,\omega_0\, i} \\ \dfrac{-\omega_0\, i\, x_0 + v_0}{-2\,\omega_0\, i} \end{bmatrix} \end{aligned}$$

Therefore, the complex solution of IVP (104) is given by:

$$\begin{aligned} \begin{bmatrix} x \\ v \end{bmatrix} &= \frac{-\omega_0\, i\, x_0 - v_0}{-2\,\omega_0\, i} \begin{bmatrix} 1 \\ \omega_0\, i \end{bmatrix} e^{\omega_0\, i\, t} \\ &+ \frac{-\omega_0\, i\, x_0 + v_0}{-2\,\omega_0\, i} \begin{bmatrix} 1 \\ -\omega_0\, i \end{bmatrix} e^{-\omega_0\, i\, t} \end{aligned} \tag{106}$$

Let us examine the first component of (106):

$$x = \frac{-\omega_0\, i\, x_0 - v_0}{-2\,\omega_0\, i}\, e^{\omega_0\, i\, t} + \frac{-\omega_0\, i\, x_0 + v_0}{-2\,\omega_0\, i}\, e^{-\omega_0\, i\, t}. \tag{107}$$

In order to compare Equations (107) and (103), rewrite the former as:

$$x = x_0\, \frac{e^{\omega_0\, i\, t} + e^{-\omega_0\, i\, t}}{2} + v_0\, \frac{e^{\omega_0\, i\, t} - e^{-\omega_0\, i\, t}}{2\,\omega_0\, i}.$$

Using Euler's formula: $e^{\omega_0\, i\, t} = \cos(\omega_0\, t) + i\, \sin(\omega_0\, t)$ leads to Equation (103), as required.

Nonhomogeneous equations. Let us add a forcing term to IVP (102):

$$m\, \frac{d^2x}{dt^2} + k\, x = A\, \sin(\omega\, t), \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0. \tag{108}$$

Then the matrix-vector form of the ODE in (104) changes to:

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \frac{A}{m}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(\omega\, t), \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{109}$$

By linearity, the general solution of this nonhomogeneous equation is the sum of the complimentary function and any particular solution:

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x \\ v \end{bmatrix}_c + \begin{bmatrix} x \\ v \end{bmatrix}_p.$$

The complimentary function is the solution of the homogeneous equation given by (106):

$$\begin{bmatrix} x \\ v \end{bmatrix}_c = C_1\begin{bmatrix} 1 \\ \omega_0\, i \end{bmatrix} e^{\omega_0\, i\, t} + C_2\begin{bmatrix} 1 \\ -\omega_0\, i \end{bmatrix} e^{-\omega_0\, i\, t}. \tag{110}$$

The particular solution can be guessed as

$$\begin{bmatrix} x \\ v \end{bmatrix}_p = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\cos(\omega\, t) + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\sin(\omega\, t).$$

To find the vector coefficients in front of the trigonometric terms, substitute the guess into (109)

$$\begin{aligned} -\omega &\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\sin(\omega\, t) + \omega\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\cos(\omega\, t) \\ &= \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\cos(\omega\, t) + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\sin(\omega\, t)\right) \\ &+ \frac{A}{m}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(\omega\, t), \end{aligned}$$

collect the like terms

$$\left(\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} - \omega \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) \cos(\omega\, t)$$
$$+ \left(\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + \omega \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \frac{A}{m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \sin(\omega\, t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and set the [vector] coefficients to zero, as usual. This leads to two linear algebraic systems:

$$\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} - \omega \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} + \omega \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = -\frac{A}{m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From the first system follows

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix},$$

which allows the elimination of one vector variable:

$$\frac{1}{\omega} \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}^2 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \omega \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = -\frac{A}{m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Recall that matrices are multiplied row by column:

$$\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} = \begin{bmatrix} -\omega_0^2 & 0 \\ 0 & -\omega_0^2 \end{bmatrix} = -\omega_0^2\, I.$$

Therefore:

$$\frac{1}{\omega} \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}^2 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \omega \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = -\frac{\omega_0^2}{\omega} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \omega \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$
$$= \left(-\frac{\omega_0^2}{\omega} + \omega\right) \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = -\frac{A}{m} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which at once gives

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \frac{A}{m} \frac{\omega}{\omega_0^2 - \omega^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consequently,

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \frac{A}{m} \frac{1}{\omega_0^2 - \omega^2} \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{A}{m} \frac{1}{\omega_0^2 - \omega^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the particular solution of Equation (109) is given by:

$$\begin{bmatrix} x \\ v \end{bmatrix}_p = \frac{A}{m} \frac{\omega}{\omega_0^2 - \omega^2} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(\omega\, t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{\sin(\omega\, t)}{\omega}\right).$$

Collecting our results, we conclude that the general solution of Equation (109) is the following sum:

$$\begin{bmatrix} x \\ v \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ \omega_0\, i \end{bmatrix} e^{\omega_0\, i\, t} + C_2 \begin{bmatrix} 1 \\ -\omega_0\, i \end{bmatrix} e^{-\omega_0\, i\, t}$$
$$+ \frac{A}{m} \frac{\omega}{\omega_0^2 - \omega^2} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(\omega\, t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{\sin(\omega\, t)}{\omega} \right).$$

Finding the constants C_1 and C_2 from initial conditions is the same process as before, however, since it is algebraically involved we omit it.

Exercises.

(1) Consider the following matrix-vector IVP:

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{111}$$

(a) Solve Equation (112) using linear algebra: that is, using the eigenvalues and eigenvectors of the matrix. Find the latter by hand and then check your algebra using the `eig` command. Do not forget to check the solution of the ODE using `ode45` either.

(b) Write Equation (112) as a system of two ODE, eliminate y, and solve the resulting second order equation for x. Compare the result with the matrix-vector solution.

(2) Use linear algebra to solve the following nonhomogeneous IVP:

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{112}$$

Confirm your computations using MATLAB.