5. Representation of functions with trigonometric sums

In class we saw that the solution of the Heat Equation on the circle is a Fourier series of the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-n^2\,t}\,(a_n\,\cos n\,x + b\,\sin n\,x),$$

where a_n and b_n are given by (for $n > 0$):

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\,\pi} \cos(n\,x)\,f(x)\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\,\pi} \sin(n\,x)\,f(x)\,dx \end{aligned}$$

It is easy to see that the Fourier series satisfies the Heat Equation; also, it is clearly periodic and therefore satisfied periodic boundary conditions. What is not easy is seeing that the initial condition is satisfied. We need to show that:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\,\cos n\,x + b\,\sin n\,x),$$

In words, we need to show that the initial condition f can be represented as a Fourier series.

In this handout we will study the following Fourier representation:

$$f(x) = c_0 + c_1\,\cos(x) + c_2\,\cos(2\,x) + c_3\,\cos(3\,x) + \ldots$$

This series arises when Heat Equation is solved on $[0,\pi]$ with Neumann boundary conditions (zero flux). Before moving on, convince yourself that the solution of the Neumann problem for the Heat Equation on $[0,\pi]$ is

$$u(x,t) = c_0 + c_1\,e^{-t}\,\cos(x) + c_2\,e^{-4\,t}\cos(2\,x) + c_3\,e^{-9\,t}\,\cos(3\,x) + \ldots$$

We will now try to understand how to find c_0, c_1, c_2, c_3, $\ldots$, which are called *Fourier coefficients.*

Most undergraduate textbooks introduce Fourier coefficients in a non-constructive "here-is-how-you-do-it" manner. The student's attention is first directed to a peculiar property of integer harmonics from which the formula for the coefficients can be easily deduced. Some authors even argue that deriving Fourier coefficients otherwise is unfeasible, although most simply pull the necessary property out of a hat, not bothering with motivation. In this section we will take the high road and actually derive the formula for the Fourier coefficients c_n in a constructive manner. As a bonus, we will learn how to relate problems in analysis to corresponding problems in linear algebra which will be very useful to us later.

It stands to reason that the principle behind the derivation of the n-th Fourier coefficient should be largely independent of n. In other words, if we somehow find the formulas for c_0 and c_1, the same approach should yield the formulas for c_2, c_3, and so forth. Let us therefore focus on the case where the function $f(x)$ is a simple sum of just two terms

$$f(x) = c_0 + c_1 \cos(x), \tag{5.1}$$

and let us further assume that instead of Equation (5.1), whose examination immediately reveals c_0 and c_1, we are given f as a list of values: $f_0 = f(0)$, $f_1 = f(h)$, $f_2 = f(2\,h)$, $\ldots$, $f_N = f(N\,h)$, where $h = \frac{\pi}{N}$ and N is a large integer. In practical terms, we can think of f as initial temperature measured with $(N+1)$ sensors located at equidistant points along the rod's length.

Let $\mathbf{x}$ denote the *column* vector $\langle 0, h, 2\,h, \ldots, N\,h\rangle^t$ [9]. Denote by $f(\mathbf{x})$ the corresponding column vector with components $f_n = f(n\,h)$, $n = 0, \ldots, N$: this is the *sampling* of our function $f(x)$. In a like manner, let us introduce $\mathbf{1}$ as the sampling of the constant function 1, and $\cos(\mathbf{x})$ as the sampling of $\cos(x)$. We can now *discretize* Equation (5.1) as follows:

$$f(\mathbf{x}) = \mathbf{c_0}\,\mathbf{1} + \mathbf{c_1}\,\cos(\mathbf{x}). \tag{5.2}$$

Equation (5.2) can also be interpreted as a system of $(N+1)$ equations in unknowns c_0 and c_1:

$$\begin{bmatrix} 1 & \cos(0\,h) \\ 1 & \cos(1\,h) \\ 1 & \cos(2\,h) \\ \vdots & \vdots \\ 1 & \cos(N\,h) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} f(0\,h) \\ f(1\,h) \\ f(2\,h) \\ \vdots \\ f(N\,h) \end{bmatrix}$$

We note that using the vector notation introduced earlier, we can rewrite the system compactly as

$$[\mathbf{1}\,\cos(\mathbf{x})] \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = f(\mathbf{x}),$$

Evidently, in the discrete case, finding the Fourier coefficients amounts to solving a linear system

$$A\,\mathbf{c} = \mathbf{f}$$

[9]The superscript t stands for *transposition* which means interchanging rows and columns in a rectangular array. If the array consists of a single row its transpose is a single column.

in which the coefficient matrix A has dimensions $(N+1)$-by-two and N is potentially very large.

Since inverses of non-square matrices cannot be defined[10], we cannot simply write $\mathbf{c} = \mathbf{A^{-1}}\,\mathbf{f}$. In order to find the unknown vector $\mathbf{c}$, we must first transform our $(N+1)$-by-two system into a two-by-two system. The standard way of converting non-square linear systems into square linear systems is to multiply both sides by the transpose matrix A^t:

$$A^t\,A\,\mathbf{c} = \mathbf{A^t}\,\mathbf{f}.$$

The matrix $A^t\,A$ of the new system is not merely two-by-two but is also *symmetric*: that is, it is its own transpose. Symmetric matrices have very useful properties which can be utilized for solving the corresponding linear systems. Therefore the multiplication by the transpose is, in fact, a very natural step which is frequently performed even if the linear system is square to begin with. Now straightforward matrix multiplication leads to the following expression for $A^t\,A$ in which we use the standard sigma notation to abbreviate sums:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \cos(0\,h) & \cos(1\,h) & \cos(2\,h) & \dots & \cos(N\,h) \end{bmatrix} \begin{bmatrix} 1 & \cos(0\,h) \\ 1 & \cos(1\,h) \\ 1 & \cos(2\,h) \\ \vdots & \vdots \\ 1 & \cos(N\,h) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{n=0}^N 1 & \sum_{n=0}^N \cos(n\,h) \\ \sum_{n=0}^N \cos(n\,h) & \sum_{n=0}^N \cos^2(n\,h) \end{bmatrix} = \begin{bmatrix} \mathbf{1}\cdot\mathbf{1} & \mathbf{1}\cdot\cos(\mathbf{x}) \\ \mathbf{1}\cdot\cos(\mathbf{x}) & \cos(\mathbf{x})\cdot\cos(\mathbf{x}) \end{bmatrix}.$$

Similarly, for the vector $A^t\,\mathbf{f}$ on the right-hand side, we get

$$\begin{bmatrix} \sum_{n=0}^N f(n\,h) \\ \sum_{n=0}^N \cos(n\,h)\,f(n\,h) \end{bmatrix} = \begin{bmatrix} \mathbf{1}\cdot\mathbf{f}(\mathbf{x}) \\ \cos(\mathbf{x})\cdot\mathbf{f}(\mathbf{x}) \end{bmatrix}$$

We now have a nice two-by-two system of linear equations for the determination of the two unknowns c_0 and c_1:

$$\begin{aligned} &\begin{bmatrix} \sum_{n=0}^N 1 & \sum_{n=0}^N \cos(n\,h) \\ \sum_{n=0}^N \cos(n\,h) & \sum_{n=0}^N \cos^2(n\,h) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \\ &= \begin{bmatrix} \sum_{n=0}^N f(n\,h) \\ \sum_{n=0}^N \cos(n\,h)\,f(n\,h) \end{bmatrix}. \end{aligned} \tag{5.3}$$

---

[10]The inverse of a square matrix A is defined as the matrix B whose product with A (on either side) is the identity matrix. Since non-square matrices of the same dimensions cannot be multiplied, they cannot be inverted.

In principle, we could compute the solution of (5.3) by using the well-known formula for the inverse of a two-by-two matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a\,d - b\,c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

However, our results will be much more informative if we simplify the matrix entries first. In the upper left corner, clearly, we have

$$\sum_{n=0}^{N} 1 = N + 1.$$

We will now simplify the remaining three entries by using the Euler's formula and the formula for the geometric sum. Recall that

$$\cos(x) = \frac{e^{i\,x} + e^{-i\,x}}{2}, \quad \text{(as follows from Euler's formula)}.$$

Also recall that for any $x \neq 1$ the following identity holds:

$$1 + x + x^2 + \ldots + x^{N-1} = \frac{1 - x^N}{1 - x}, \quad \text{(geometric summation)}.$$

Therefore, we can write[11]

$$\begin{aligned}
\sum_{n=0}^{N} \cos(n\,h) &= \sum_{n=0}^{N} \frac{e^{i\,n\,h} + e^{-i\,n\,h}}{2} = \frac{1}{2}\left\{1 + \sum_{n=-N}^{N} e^{i\,n\,h}\right\} \\
&= \frac{1}{2}\left\{1 + e^{-i\,N\,h} \sum_{n=0}^{2\,N} \left(e^{i\,h}\right)^n\right\} \\
&= \frac{1}{2}\left\{1 + e^{-i\,N\,h}\left(\frac{1 - e^{i\,2\,N\,h}}{1 - e^{i\,h}} + e^{i\,2\,N\,h}\right)\right\}
\end{aligned}$$

Now, since $h = \frac{\pi}{N}$, it follows from Euler's formula that

$$\frac{1 - e^{i\,2\,N\,h}}{1 - e^{i\,h}} = \frac{1 - e^{2\,\pi\,i}}{1 - e^{\frac{\pi\,i}{N}}} = 0.$$

Therefore,

$$\sum_{n=0}^{N} \cos(n\,h) = \frac{1}{2}\left\{1 + e^{-i\,N\,h}\,e^{i\,2\,N\,h}\right\} = \frac{1}{2}\left\{1 + e^{\pi\,i}\right\} = 0.$$

[11]In order to avoid the clutter of symbols, I used the same index of summation when manipulating the sigmas. You can check the validity of the manipulations by expanding the sums.

This is a very interesting identity as it holds for every positive integer N. Furthermore, if ω is any nonzero integer and $h = \frac{\pi}{N}$, as before, then a more general identity can be proved using the same technique:

$$\sum_{n=0}^{N} \cos(\omega\, n\, h) = \begin{cases} 0, & \omega \text{ is odd,} \\ 1, & \omega \text{ is even.} \end{cases} \tag{5.4}$$

From Equation (5.4) and the double angle formula

$$\cos^2(x) = \frac{1 + \cos(2\,x)}{2}$$

follows that

$$\sum_{n=0}^{N} \cos^2(n\,h) = \sum_{n=0}^{N} \frac{1 + \cos(2\,n\,h)}{2} = \frac{N+2}{2}.$$

Hence our system (5.3) simplifies to

$$\begin{bmatrix} N+1 & 0 \\ 0 & \frac{1}{2}\,(N+2) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{N} f(n\,h) \\ \sum_{n=0}^{N} \cos(n\,h)\, f(n\,h) \end{bmatrix}$$

from which follows that the requisite Fourier coefficients c_0 and c_1 are *weighted averages* of the data $f(\mathbf{x})$:

$$c_0 = \frac{1}{N+1} \sum_{n=0}^{N} f(n\,h) \tag{5.5}$$

$$c_1 = \frac{2}{N+2} \sum_{n=0}^{N} \cos(n\,h)\, f(n\,h) \tag{5.6}$$

Formulas (5.5) and (5.6) offer two important insights. First, observe that the sums can be recast in terms of the dot product:

$$c_0 = \frac{\mathbf{1} \cdot \mathbf{f}(\mathbf{x})}{\mathbf{1} \cdot \mathbf{1}}, \tag{5.7}$$

$$c_1 = \frac{\cos(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})}{\cos(\mathbf{x}) \cdot \cos(\mathbf{x})} \tag{5.8}$$

It appears, therefore, that Fourier coefficients have something to do with the dot product in $\mathbb{R}^{N+1}$ [12]. Secondly, in the limit as $N \to \infty$ Equations (5.5) and (5.6) produce integrals. For instance, the limiting

[12] Usually called the *standard* or *Euclidean* dot product

value of c_0 is

$$\begin{aligned}
\lim_{N\to\infty} c_0(N) &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(n\,h) \\
&= \frac{1}{\pi} \lim_{N\to\infty} \frac{N}{N+1} \sum_{n=0}^{N} f\left(\frac{\pi\,n}{N}\right) \frac{\pi}{N} \\
&= \frac{1}{\pi} \int_0^\pi f(x)\,dx,
\end{aligned}$$

which we recognize as the average of the initial temperature. Note that this value of c_0 conforms with our physical intuition: in the limit as $t \to \infty$ the temperature of a perfectly insulated rod must "even out". Similarly,

$$\begin{aligned}
\lim_{N\to\infty} c_1(N) &= \lim_{N\to\infty} \frac{2}{N+2} \sum_{n=0}^{N} \cos(n\,h)\, f(n\,h) \\
&= \frac{2}{\pi} \lim_{N\to\infty} \frac{N}{N+2} \sum_{n=0}^{N} \cos\left(\frac{\pi\,n}{N}\right) f\left(\frac{\pi\,n}{N}\right) \frac{\pi}{N} \\
&= \frac{2}{\pi} \int_0^\pi \cos(x)\, f(x)\,dx,
\end{aligned}$$

which suggests that for the n-th Fourier coefficient (in the continuous case) the formula may have the form

$$c_n = \alpha_n \int_0^\pi \cos(n\,x)\, f(x)\,dx.$$

This we now confirm with a simple computation.

5.1. **Orthogonality of cosines.** Let us assume that we can write

$$f(x) = c_0 + c_1\,\cos(x) + c_2\,\cos(2\,x) + c_3\,\cos(3\,x) + \ldots \tag{5.9}$$

To test our conjectured formula for c_n, we multiply both sides of Equation (5.9) by $\cos(n\,x)$ and integrate from 0 to π. This leads to

$$\begin{aligned}
\int_0^\pi \cos(n\,x)\, f(x)\,dx = c_0 &\int_0^\pi \cos(n\,x)\,dx \\
+ c_1 &\int_0^\pi \cos(n\,x)\ \cos(x)\,dx \\
+ c_2 &\int_0^\pi \cos(n\,x)\ \cos(2\,x)\,dx \\
+ c_3 &\int_0^\pi \cos(n\,x)\ \cos(3\,x)\,dx \ldots
\end{aligned} \tag{5.10}$$

The definite integral on the left-hand side of Equation (5.10) is simply some number depending on n. That number can be readily computed once the function $f(x)$ is specified, as will be illustrated in a later example. On the right-hand side things are much more interesting. At first blush it appears that we have an infinity of terms. Yet, let us examine a typical term

$$c_m \int_0^\pi \cos(n\,x)\,\cos(m\,x)\,dx.$$

Using the trigonometric identity

$$\cos(n\,x)\,\cos(m\,x) = \cos((n+m)\,x) + \cos((n-m)\,x),$$

and assuming that $n \neq m$, we can compute the integral explicitly as

$$c_m \left(\frac{\sin((n+m)\,x)}{n+m}\bigg|_{x=0}^{\pi} + \frac{\sin((n-m)\,x)}{n-m}\bigg|_{x=0}^{\pi} \right) = 0.$$

Hence, in actuality, (5.10) reads:

$$\int_0^\pi \cos(n\,x)\,f(x)\,dx = c_n \int_0^\pi \cos^2(n\,x)\,dx = c_n\,\frac{\pi}{2},$$

whence follows that

$$c_n = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^\pi f(x)\,dx, & \text{if } n = 0, \\[2ex] \dfrac{2}{\pi} \displaystyle\int_0^\pi \cos(n\,x)\,f(x)\,dx, & \text{if } n > 0. \end{cases} \tag{5.11}$$

The remarkable vanishing of the terms on the right-hand side of Equation (5.10) is the consequence of *orthogonality* of cosines which we now explain.

Recall that two vectors in $\mathbb{R}^n$ are said to be orthogonal if their dot product is zero. Recall further that in the discrete case the formulas for c_0 and c_1 can be written in terms of the dot product as evidenced by Equations (5.7) and (5.8). This suggests that we could define the so-called L^2 *inner product* of continuous functions on $[0, \pi]$ as the following integral:

$$f(x) \cdot g(x) = \int_0^\pi f(x)\,g(x)\,dx.$$

Indeed, if we sample our functions then, up to a positive scalar multiple[13], the approximation of the L^2 continuous inner product is the usual dot product[14] of vectors in $\mathbb{R}^n$. With the "dot product of functions" in

[13] The grid size.

[14] Also called L^2 inner product

place, we can speak of *orthogonality of functions*. In particular, we can reinterpret the vanishing of certain integrals in (5.10) as orthogonality of cosines:

$$\cos(n\,x)\cdot\cos(m\,x) = 0, \quad \text{if } n \neq m$$

In other words, the cosines with integer frequencies form an *orthogonal* basis just like the familiar vectors **i**, **j**, and **k** do in $\mathbb{R}^3$. However, the "lengths" or rather *norms*[15] of the cosines are not unit. Quite generally, the square of the norm of a function is the dot product of the function with itself:

$$\|f(x)\|^2 = f(x)\cdot f(x) = \int_0^\pi f(x)^2\,dx.$$

For cosines, we get

$$\|\cos(n\,x)\|^2 = \int_0^\pi \cos^2(n\,x)\,dx = \begin{cases} \dfrac{1}{\pi}, & n = 0, \\[2ex] \dfrac{2}{\pi}, & n > 0. \end{cases}$$

Using the L^2 inner product, we can rewrite Equation (5.11) very elegantly as

$$c_n = \frac{\cos(n\,x)\cdot f(x)}{\cos(n\,x)\cdot\cos(n\,x)}, \quad n \geq 0 \tag{5.12}$$

5.2. **Example.** We now illustrate our results with a concrete example. Let

$$f(x) = \frac{1}{2}\pi\,x^2 - \frac{1}{3}\pi\,x^3.$$

[15] L^2 norms generalize the concept of length. The usual length of a vector in $\mathbb{R}^n$ in functional analysis is also called L^2 (vector) norm.

Using Equation (5.12) and the definition of the inner product, we compute

$$
\begin{aligned}
c_0 &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{12}\,\pi^3 \\
c_1 &= \frac{2}{\pi}\int_0^{\pi} \cos(x)\,f(x)\,dx = -8\,\frac{1}{\pi} \\
c_2 &= \frac{2}{\pi}\int_0^{\pi} \cos(2\,x)\,f(x)\,dx = 0 \\
c_3 &= \frac{2}{\pi}\int_0^{\pi} \cos(3\,x)\,f(x)\,dx = -\frac{8}{81}\,\frac{1}{\pi} \\
c_4 &= \frac{2}{\pi}\int_0^{\pi} \cos(4\,x)\,f(x)\,dx = 0 \\
c_5 &= \frac{2}{\pi}\int_0^{\pi} \cos(5\,x)\,f(x)\,dx = -\frac{8}{625}\,\frac{1}{\pi} \\
c_6 &= \frac{2}{\pi}\int_0^{\pi} \cos(6\,x)\,f(x)\,dx = 0 \\
c_7 &= \frac{2}{\pi}\int_0^{\pi} \cos(7\,x)\,f(x)\,dx = -\frac{8}{2401}\,\frac{1}{\pi}
\end{aligned}
$$

It is easy to spot the pattern: all even coefficients beginning with c_2 are zero

$$c_2 = c_4 = c_6 = \ldots = 0$$

while the odd coefficients are given by the formula

$$c_{2\,n-1} = -\frac{8}{(2\,n-1)^4}\,\frac{1}{\pi}, \qquad n = 1, 2, 3, \ldots$$

Therefore

$$\frac{1}{2}\,\pi\,x^2 - \frac{1}{3}\,\pi\,x^3 = \frac{\pi^3}{12} - \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2\,n-1)\,x)}{(2\,n-1)^4}$$

Notice how rapidly the Fourier coefficients go to zero[16]. This suggests that we could approximate our function with just a few terms. Indeed, Figure 3 shows that three terms

$$f(x) \approx \frac{\pi^3}{12} - \frac{8}{\pi}\,\cos(x) - \frac{8}{\pi}\,\frac{\cos(3\,x)}{81}$$

is already sufficient for most practical purposes. In Figure 3 the solid line is the graph of the polynomial $f(x)$ while the dots are the values computed using the third order trigonometric approximation.

[16]Later we will show that the rate at which Fourier coefficients decrease has to do with the smoothness of the function

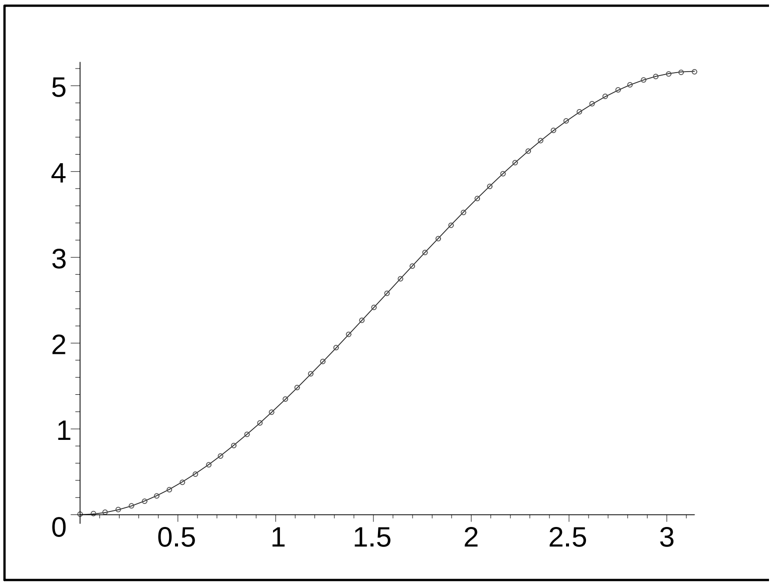

FIGURE 3. Third order approximation of $f(x)$

6. EXERCISES

(1) Suppose that

$$f(\mathbf{x}) = \mathbf{c_0}\,\mathbf{1} + \mathbf{c_1}\,\cos(\mathbf{x}) + \mathbf{c_2}\,\cos(\mathbf{2\,x})$$

where $\mathbf{x}$ is the sampling of $[0,\pi]$ with grid size π/N. (i) Find an exact formula for c_2. Note that it is NOT going to be

$$c_2 = \frac{\cos(2\,\mathbf{x})\cdot\mathbf{f}(\mathbf{x})}{\cos(2\,\mathbf{x})\cdot\cos(\mathbf{2\,x})}$$

as one might expect. If rigorous proof is too difficult, substitute it with computational evidence collected using MATLAB.

(ii) Show that in the limit as $N\to\infty$

$$c_2 \to \frac{2}{\pi}\int_0^{\pi}\cos(2\,x)\,f(x)\,dx.$$

(2) Let $\mathbf{v} = \mathbf{a}\,\mathbf{i} + \mathbf{b}\,\mathbf{j} + \mathbf{c}\,\mathbf{k}$ be an arbitrary vector in $\mathbb{R}^3$. Express the components a, b, and c in terms of v using the dot product.

(3) Repeat the above exercise with the standard basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ replaced by an orthogonal basis $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$. Keep in mind that $\mathbf{e_i}$'s are not required to have unit length.

(4) Consider the space of all linear polynomials $V = \{a + b\,x \mid a, b \in \mathbb{R}\}$ on $[0, 1]$ equipped with the L^2 inner product

$$f \cdot g = \int_0^1 f(x)\,g(x)\,dx.$$

Let $f = 1$. Find the unique $g \in V$ which has unit norm and is orthogonal to f. Show that $\{f, g\}$ form an orthogonal basis of V.

7. Separation of variables

In this section we present a Calculus solution of the following problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, && (7.1) \\ \left.\frac{\partial u}{\partial x}\right|_{x=0} &= \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, && (7.2) \\ u(x,0) &= u_0(x). && (7.3) \end{aligned}$$

The approach is generally known as 'separation of variables'.

Since the Heat Equation is linear and has constant coefficients, we can construct (infinitely many) special solutions of the form $u = f(t)\,g(x)$.[17] Furthermore, we can insure that the special solutions satisfy the Neumann boundary conditions by imposing these conditions on $g(x)$. The actual separation of variables is accomplished by setting $u = f(t)\,g(x)$ in the Heat Equation

$$g(x)\,\frac{df}{dt} = f(t)\,\frac{d^2 g}{dx^2}$$

and dividing the result by $f(t)\,g(x)$:

$$\frac{1}{f}\,\frac{df}{dt} = \frac{1}{g}\,\frac{d^2 g}{dx^2}.$$

Now, since the left-hand side depends only on time while the right-hand side depends only on x, both sides must, in fact, be constant:

$$\frac{1}{f}\,\frac{df}{dt} = \frac{1}{g}\,\frac{d^2 g}{dx^2} = \lambda.$$

The triple equality is equivalent to a system of linear ODE:

$$\begin{aligned} &\frac{df}{dt} = \lambda\, f \\ &\frac{d^2 g}{dt^2} = \lambda\, g, \quad \frac{dg}{dx}(0) = \frac{dg}{dx}(L) = 0. \end{aligned}$$

Notice that we incorporated the Neumann boundary conditions into the ODE for $g(x)$. *In the future, we will progress from PDE directly to the system of ODE: that is, we will write "Separation of variables in product form leads to the following system of ODE" and list the ODE.*

At this point we can conclude that f must be a simple exponential. However, it is more expedient to start with the boundary value problem for $g(x)$. Indeed, the solution of the BVP for $g(x)$ produces not just

[17] True for any linear PDE with constant coefficients.

the functions $g(x)$ but also the corresponding eigenvalues λ; the latter should form a discrete sequence $\{\lambda_n\}$ since the interval $[0, L]$ is finite. If for whatever reason we fail to find the eigenvalues, the general formula $f(t) = A\, e^{\lambda\, t}$ is not going to be very helpful. We therefore turn our attention to $g(x)$ which has the general form:

$$g(x) = C_1\, e^{\sqrt{\lambda}\, x} + C_2\, e^{-\sqrt{\lambda}\, x}.$$

Imposing the boundary conditions and writing the resulting system in matrix-vector form, we get, after some algebra [18]:

$$\begin{bmatrix} 1 & -1 \\ e^{\sqrt{\lambda}\, L} & -e^{-\sqrt{\lambda}\, L} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A homogeneous linear system $A\,\mathbf{C} = \mathbf{0}$ has either only the zero solution, if $\det(A) \neq 0$, or infinitely many solutions, if $\det(A) = 0$. We want the latter option, hence we must have:

$$\det \begin{bmatrix} 1 & -1 \\ e^{\sqrt{\lambda}\, L} & -e^{-\sqrt{\lambda}\, L} \end{bmatrix} = e^{\sqrt{\lambda}\, L} - e^{-\sqrt{\lambda}\, L} = 0.$$

Rewriting the characteristic equation as

$$e^{2\,\sqrt{\lambda}\, L} = 1,$$

and using Euler's formula $e^{2\,\pi\, n\, i} = 1$, we conclude that

$$\lambda = -\frac{\pi^2\, n^2}{L^2}, \quad n = 1, 2, 3, \ldots$$

With this choice of λ and $C_1 = C_2 = \frac{1}{2}$ [19] we get infinitely many nontrivial solutions

$$g(x) = \frac{1}{2}\, e^{\frac{\pi\, n\, i}{L}\, x} + \frac{1}{2}\, e^{\frac{-\pi\, n\, i}{L}\, x} = \cos\left(\frac{\pi\, n}{L}\, x\right)$$

The corresponding $f(t)$ are given by

$$f(t) = A\, e^{-\frac{\pi^2\, n^2}{L^2}\, t}.$$

[18] In order to safely cancel $\sqrt{\lambda}$ we treat $\lambda = 0$ and the corresponding constant solutions $u = \text{const}$ as a special case.

[19] The values λ_n reduce the rank of the linear system $A(\lambda)\,\mathbf{C} = \mathbf{0}$ from 2 to 1. In other words, the two linear equations become the same condition $C_1 - C_2 = 0$ and, consequently, we get infinitely many solutions. These solutions form a one-dimensional linear vector space: that is, they differ by a scalar multiple. We are thus free to choose any $C_1 = C_2 \neq 0$ to get a basis vector. The choice made in the text is motivated by Euler's formula and leads to a particularly simple real-valued form of $g(x)$.

We can set the constant A to 1 [20] and conclude that the sequence of functions

$$u_n(x,t) = e^{-\frac{\pi^2 n^2}{L^2} t} \cos\left(\frac{\pi n}{L} x\right),$$

satisfies the Heat Equation and the Neumann boundary conditions for all integer n including $n = 0$.

7.1. **Fourier series.** By linearity, any series of the form

$$u(x,t) = \sum_{n=0}^{\infty} C_n \, e^{-\frac{\pi^2 n^2}{L^2} t} \cos\left(\frac{\pi n}{L} x\right)$$

satisfies the Heat Equation and the Neumann boundary conditions. It remains to satisfy the initial condition $u(x,0) = u_0(x)$. This leads us to the problem of expanding $u_0(x)$ into the Fourier series of cosines:

$$u_0(x) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{\pi n}{L} x\right).$$

Every Fourier expansion is an orthogonal projection. To speak of orthogonality one needs to define the dot product. In general, the appropriate definition of the dot product depends on both the PDE and the boundary conditions and may not be obvious. In our case, however, the usual L^2-dot product does the job:

$$\phi(x) \cdot \psi(x) = \int_0^L \phi(x)\, \psi(x)\, dx.$$

Indeed, recall from Calculus II that

$$\begin{aligned}\int_0^L \cos\left(\frac{\pi n}{L} x\right) \cos\left(\frac{\pi m}{L} x\right) dx &= \frac{L}{\pi} \int_0^{\pi} \cos(n\, s)\, \cos(m\, s)\, ds \\ &= \begin{cases} 0, & n \neq m, \\ L, & n = m = 0 \\ \frac{L}{2}, & n = m \neq 0. \end{cases}\end{aligned}$$

Therefore the cosines

$$\cos\left(\frac{\pi n}{L} x\right)$$

are mutually orthogonal with respect to the standard dot product. Using orthogonality, we can compute

$$C_0 = \frac{1}{L} \int_0^L u_0(y)\, dy$$

[20]This corresponds of assigning initial value $f(0) = 1$ which we will use by default in future discussion.

and

$$C_n = \frac{2}{L} \int_0^L \cos\left(\frac{\pi\, n}{L}\, y\right)\, u_0(y)\, dy, \quad n = 1, 2, 3, \ldots$$

Hence the general solution is given by the following Fourier series:

$$\begin{aligned} u(x,t) = & \frac{1}{L} \int_0^L u_0(y)\, dy \\ & + \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L \cos\left(\frac{\pi\, n}{L}\, y\right)\, u_0(y)\, dy \right] e^{-\frac{\pi^2\, n^2}{L^2}\, t} \cos\left(\frac{\pi\, n}{L}\, x\right) \end{aligned}$$

HOMEWORK

7.2. **Exercises.**

(1) Use separation of variables to derive a general formula for the solution of the following problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\pi < x < \pi, \quad t > 0, \\ u(\pi, t) &= u(-\pi, t) = 0, \\ u(x, 0) &= f(x). \end{aligned}$$

Test your solution using the following initial conditions:

(a) $f = \sin(3\, x)$

(b) $f = |x|$

(c) $f = \begin{cases} -1, & x < 0 \\ +1, & x \geq 0 \end{cases}$

By testing, I mean: compare the solution obtained numerically, through semi-discretization, with the analytic formula. If you do things correctly, the agreement will be noticeable. Think about the best way of comparing the numerical solution and the analytic solution.

(2) Repeat the previous exercise with:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad -\pi < x < \pi, \quad t > 0, \\ u(-\pi, t) &= u(\pi, t) = 0, \\ u(x, 0) &= f(x), \\ \frac{\partial u}{\partial t}(x, 0) &= 0. \end{aligned}$$

(3) Hypothesize about what happens to the solution of the Heat Equation derived in this handout as $L \to \infty$; try to make a strong case.