Melting ice

In this section we investigate the following question:

> How long does it take for an ice cube of volume two to melt under steady conditions if an ice cube of unit volume melts in exactly one hour?

As we will find out, the simple guess "two hours" is incorrect or at least highly unlikely. More importantly, you will learn how to

- Construct a mathematical model for a physical process.
- Solve simple differential equations using calculus.
- Validate mathematical models using physical intuition.

The following discussion will further highlight the importance of related rates problems from Calculus I. Those were really practice problems for setting up *initial value problems* such as the one introduced below.

Setting up the model

Ice cubes melt because they receive heat from the surrounding environment. Henceforth we assume that the latter has constant temperature, as does the cube: otherwise the conditions are unsteady. We further stipulate that the cube melts evenly on all sides and retains its cubical shape. [5] Now, the heat enters the ice cube through its surface. Therefore *the rate at which the cube loses its volume must be related to its instantaneous surface area.* Notice how rates naturally enter the discussion of a dynamical process. We cannot really say very much about the volume of the cube, except what it was at time zero and that it eventually shrinks to zero. On the other hand, we can make a constructive statement about the time derivative of the volume: namely, that it is related to surface area.

At this point, it is useful to introduce some notation. Let us denote the volume of the cube by V and the surface area by S, bearing in mind that

[5] In real life ice cubes get rounded at the edges as they melt. This, however, makes the mathematics very difficult.

both quantities are functions of time. Then our modeling assumption can be written compactly as:

$$\frac{dV}{dt} = f(S), \tag{38}$$

where the nature of the function f remains to be specified. Equation (38) is a differential equation on account of the derivative on the left-hand side; it is an ordinary differential equation because differentiation is done in one variable[6], and it is of first order because the derivative is of first order. The unknowns in (38) are the two functions V and S. Clearly, we need to add at least one more equation in order to complete the model. In this particular case, the simple geometry of the cube suggests that we adjoin two more equations relating V and S to the side length x:

$$V = x^3 \tag{39}$$

$$S = 6\,x^2 \tag{40}$$

Now we have three equations in three unknowns and all that remains is the specification of the function f.

How should the derivative of the volume depend on the surface area? Here we can make three quick observations which will narrow down the possible choices. Firstly, since the cube is *melting* the values of f must be negative. Secondly, if the surface area is very small so is the rate of melting: hence, $f(0) = 0$. Thirdly, increasing S increases the magnitude of dV/dt. Thus the graph of $f(S)$ starts at the origin and goes to negative infinity as S goes to positive infinity. The *simplest* function with such a graph is $f = -k\,S$, where k is a positive[7] number. Therefore, we shall start with this choice of f and write Equation (38) as:

$$\frac{dV}{dt} = -k\,S.$$

Our model is now complete. In the next section we will study its output and see if it makes physical sense: if it does not, we will need to re-examine our choice of f and, perhaps, take additional physical factors into account.

[6]Differential equations relating partial derivatives are called, naturally, partial differential equations.

[7]By convention, all constant parameters in mathematical model are nonnegative.

Solving the model

Combining equations (38)–(40) (with $f(S) = -k\,S$), we obtain:

$$\frac{d(x^3)}{dt} = -6\,k\,x^2. \tag{41}$$

Now, the side length x depends on time. Therefore we must use the Chain Rule on the left-hand side of Equation (41) as follows:

$$\frac{d(x^3)}{dt} = 3\,x^2\,\frac{dx}{dt}.$$

Conveniently, we can now divide out the factor $3\,x^2$ which results in a very simple differential equation for the side length:

$$\frac{dx}{dt} = -2\,k. \tag{42}$$

Equation (42) says that the derivative of the side length is the constant $(-2\,k)$, so the sides shrink at a constant rate. The immediate conclusion is that x is a linear function of t with slope $(-2\,k)$. Denoting the initial value $x(0)$ by the symbol x_0, we can write:

$$x(t) = x_0 - 2\,k\,t.$$

It is now a simple matter to find the melting time T by setting the side length to zero. This leads to the simple formula below:

$$T = \frac{x_0}{2\,k}. \tag{43}$$

Returning to the original question, let V_0 denote the initial volume. Since $x_0 = \sqrt[3]{V_0}$, as follows from Equation (39), we can rewrite Equation (43) as

$$T = T(V_0) = \frac{\sqrt[3]{V_0}}{2\,k}.$$

We know that if $V_0 = 1$ then $T = 1$. Now let us double the initial volume:

$$T(2\,V_0) = \frac{\sqrt[3]{2\,V_0}}{2\,k} = \sqrt[3]{2}\,\frac{\sqrt[3]{V_0}}{2\,k} = \sqrt[3]{2}\,T(V_0).$$

Evidently, doubling the initial volume increases the melting time by the factor $\sqrt[3]{2}$: so, the cube of volume two will melt in about one hour and sixteen minutes.

A different computational pathway

Our solution above commenced with the replacement of V and S with their expressions in terms of x. However, this is not the only path to the answer. For instance, since we are interested in relating the melting time to the initial volume, we may prefer to actually keep V in the differential equation. Then we must replace S with its expression in terms of V

$$S = 6\,x^2 = 6\left(\sqrt[3]{V}\right)^2 = 6\,V^{\frac{2}{3}},$$

which leads to the following equation:

$$\frac{dV}{dt} = -6\,k\,V^{\frac{2}{3}}. \tag{44}$$

Solving Equation (44) leads to the same answer, as we will demonstrate below. First, however, we must discuss some typical mistakes which often discourage students from such experimentation.

Typical pitfalls

In order to solve Equation (44), students often integrate

$$\int dV = -6\,k\int V^{\frac{2}{3}}\,dt$$

incorrectly, either as

$$V = -6\,k\,V^{\frac{2}{3}}t + C$$

or

$$V = -\frac{12}{5}\,k\,V^{\frac{5}{2}} + C.$$

Let us briefly discuss the source of these errors in order to see how they should be corrected. Firstly, the transition to

$$\int dV = -6\,k\int V^{\frac{2}{3}}\,dt,$$

is usually caused by the often subconscious dislike of denominators. Think back to the very early days of algebra: how many fractions did you have to vanquish or simplify away? Now, there is absolutely nothing wrong with multiplying both sides of a differential equation by dt. In fact, it is true that

$$V = -6\,k \int V^{\frac{2}{3}}\,dt + C.$$

However, the multiplication by dt is a set up for mistakes in integration. The volume V is a function of time $V(t)$: this means that $V^{\frac{2}{3}}$ cannot be integrated as a constant. Nor can it be integrated as a power since

$$\int g^n(t)\,dt \neq \frac{g^{n+1}(t)}{n+1} + C,$$

as can be seen by differentiating both sides or trying a few simple choices for $g(t)$.

Separation of variables

The problem with the integral

$$\int V^{\frac{2}{3}}\,dt$$

lies in the simultaneous presence of two *distinct* variables: V and t. If we knew $V(t)$ or, conversely, $t(V)$ we would be able to integrate. Unfortunately, $V(t)$ is what we need to find in the first place. There is, however, a different scenario which allows integration: we would have no difficulty integrating $V^{\frac{2}{3}}$ if we had dV in place of dt in the integral. This suggests that, after multiplying by dt, we should divide by $V^{\frac{2}{3}}$ in order to *separate the variables* as follows:

$$V^{-\frac{2}{3}}\,dV = -6k\,dt$$

Now integration presents no difficulties, and we get

$$\int V^{-\frac{2}{3}}\,dV = 3\,V^{\frac{1}{3}} = \int (-6k)\,dt = -6k\,t + C, \tag{45}$$

which is called the *general solution* of Equation (44). The term *general* means that any *particular* solution of Equation (45) has the *general form* (45). Notice that we added only one indeterminate constant C on the right. Adding different constants to both sides does not produce anything new because the two constants can be combined on the right, say, into a single constant. We can now deduce, through simple algebra, that

$$V = (-2k\,t + C)^3\,.$$

At time $t = 0$ the volume is $V(0) = V_0$ which means

$$V_0 = C^3$$

and, consequently,

$$C = \sqrt[3]{V_0}.$$

Hence,

$$V = \left(\sqrt[3]{V_0} - 2k\,t\right)^3,$$

which leads to the same formula for the melting time

$$T = \frac{\sqrt[3]{V_0}}{2k}.$$

Validating the results

The best way to validate a model is to compare its predictions with experimental data. In our case, if we make two ice cubes with volumes one and two and melt them, the ratio of the melting times should be close to $\sqrt[3]{2}$. Try it in the kitchen!

Another source of validation for our model comes from basic Physics. Fourier law, discovered experimentally, states that the heat flux across a planar boundary is proportional to the area and the difference of temperatures. Therefore, our choice of $f(S)$ is sensible and furthermore k is proportional to the difference of temperatures between the ice and its environment.

Why differential equations are inevitable

Differential equations govern almost every single branch of mathematical physics. Take Newton's Second Law, for example. Acceleration, in Calculus terms, is the second derivative of position. Therefore, "mass times acceleration equals force" is really a second order differential equation. Thus almost the entire subject of dynamics is reduced to solving second order differential equations.

As a typical example, let $y(t)$ be the height of a ball dropping under the influence of the force of gravity. In Calculus terms, Newton's Second Law is:

$$m\,\frac{d^2y}{dt^2} = -m\,g, \quad y(0) = y_0, \quad \frac{dy}{dt}(0) = v_0.$$

The solution of this equation is easily seen to be

$$y = -\frac{g\,t^2}{2} + v_0\,t + y_0,$$

which is familiar from general physics. If we want to account for the force of friction, we can simply add the appropriate term to the right-hand side. The simplest model where friction is proportional to velocity, would be:

$$m\,\frac{d^2y}{dt^2} = -m\,g - k\,\frac{dy}{dt}, \quad y(0) = y_0, \quad \frac{dy}{dt}(0) = v_0.$$

If we set $v = \frac{dy}{dt}$ the equation becomes

$$m\,\frac{dv}{dt} = -m\,g - k\,v, \quad v(0) = v_0.$$

This is a separable equation whose solution is one of the exercises. If the motion happens in the plane, we simply apply Newton's Second Law to another coordinate:

$$\begin{aligned} m\,\frac{d^2x}{dt^2} &= -k\,\frac{dx}{dt}, \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = u_0 \\ m\,\frac{d^2y}{dt^2} &= -m\,g - k\,\frac{dy}{dt}, \quad y(0) = y_0, \quad \frac{dy}{dt}(0) = v_0. \end{aligned}$$

The right-hand side for the first equation is simpler than for the second because the force of gravity does not act along the x-axis, so there is just the force of resistance. The motion of any number of particles under any kinds of forces can be expressed similarly with a system of second order ODE.

As another example let us consider chemical kinetics. This is a science about the rate of chemical reactions and much of it is governed by the Law of Mass Action (LMA). To illustrate the latter, let us apply it to a monomolecular reaction A $\to$ B. If use the chemistry bracket notation for concentrations of the species, the LMA model is:

$$\begin{aligned} \frac{d[A]}{dt} &= -k\,[A] \\ \frac{d[B]}{dt} &= k\,[A] \end{aligned}$$

Here k is a positive rate constant. Notice that the first equation says that the rate at which A is consumed is proportional to the current concentration of A. Meanwhile B is created with exactly the same rate, so the total mass is conserved. For a bi-molecular reaction $\mathrm{A} + \mathrm{B} \to \mathrm{C}$ the LMA model becomes:

$$\begin{aligned}\frac{d[A]}{dt} &= -k\,[A]\,[B]\\ \frac{d[B]}{dt} &= -k\,[A]\,[B]\\ \frac{d[C]}{dt} &= k\,[A]\,[B]\end{aligned}$$

So the interaction between two species is modeled with a product of concentrations.

We conclude the discussion of the Law of Mass Action with a mention that it is also used outside of chemistry. For instance, in population biology it may describe the dynamics of populations competing for resources and predating on each other. Generally, wherever there is a system evolving in time there is a good chance that the evolution is described by a set of differential equations.

Series solution

The easiest and, arguably, the best way to solve a differential equation is by separating variables. However, this strategy is very limited because most ODE's are actually nonseparable. Fortunately, there are many other techniques for solving ODE. Here we will consider the Taylor expansion method which we will illustrate using the following example:

$$\frac{dx}{dt} = t - x, \quad x(0) = 1. \tag{46}$$

IVP (46) may describe the cooling of an object in an environment whose temperature is rising at a constant rate. Convince yourself that (46) is not separable before reading on.

Set $x(t) = a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3 + \ldots$ and substitute this expansion into (46). We get

$$a_1 + 2\,a_2\,t + 3\,a_3\,t^2 + \ldots = t - (a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3 + \ldots)$$

We now have two *power series* that equal each other. This means that the like coefficients must be equal. In particular, the constant on the left is a_1 while the constant on the right is $(-a_0)$: therefore $a_1 = -a_0$. Likewise, comparing the linear terms leads to $2\,a_2 = 1 - a_1$, and so on. We thus have the following infinite system of equations:

$$\begin{aligned} a_1 &= -a_0 \\ 2\,a_2 &= 1 - a_1 \\ 3\,a_3 &= -a_2 \\ 4\,a_4 &= -a_3 \\ 5\,a_5 &= -a_4 \\ &\cdots \end{aligned}$$

Now $a_0 = 1$ because of the initial condition $x(0) = 1$. It follows that $a_1 = -a_0 = -1$, $a_2 = (1 - a_1)/2 = 1$, $a_3 = -a_2/3 = -1/3$, and so on. Continuing the *recursion* we find that the expansion of x is

$$x(t) = 1 - t + t^2 - \frac{t^3}{3} + \frac{t^4}{3 \cdot 4} - \frac{t^5}{3 \cdot 4 \cdot 5} + \frac{t^6}{3 \cdot 4 \cdot 5 \cdot 6} - \ldots$$

In principle the problem is solved—we got the expansion we sought. However in this case we can go one step further. Write the solution in the form:

$$x(t) = 1 - t + 2\left(\frac{t^2}{2} - \frac{t^3}{2 \cdot 3} + \frac{t^4}{2 \cdot 3 \cdot 4} - \frac{t^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{t^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \ldots\right)$$

Now note that the expression in parenthesis is the expansion of e^{-t} which is missing the first two terms. Therefore the solution in "closed form" is:

$$x = 1 - t + 2\,(e^{-t} - (1 - t)) = 2\,e^{-t} + t - 1.$$

As an exercise, check that this is indeed the solution of Equation (46).

Exercises

1. Solve the system of equations

$$\begin{aligned} \frac{dV}{dt} &= -k\,S \\ V &= x^3 \\ S &= 6\,x^2 \end{aligned}$$

by replacing V in the differential equation with its expression in terms of S. Having found $S(t)$, use that expression to find the melting time T.

2. Modify the ice cube model to find the melting time of an ice sphere. What melts faster, the cube or the sphere of the same volume? Provide full mathematical exposition.

3. Investigate the ice cube model of the form

$$\frac{dV}{dt} = -k\,S^{\alpha},$$

where $\alpha > 0$. Find the melting time T in terms of V_0 and k for three choices of alpha: $\alpha = \frac{1}{2}$, $\alpha = \frac{3}{2}$ and $\alpha = 2$.

4. The fall with resistance is modeled by the following IVP:

$$m\,\frac{d^2y}{dt^2} = -m\,g - k\,\frac{dy}{dt}, \quad y(0) = y_0, \quad \frac{dy}{dt}(0) = v_0.$$

Solve this equation by substituting $v = \frac{dy}{dt}$ as suggested in the text of the handout. Validate your solution and illustrate it with a plot of the position y against time t. For plotting, use the following values of the parameters: $m = 1$, $g = 10$, $k = .1$, $y_0 = 20$, $v_0 = 0$. Also, superimpose on the same axes the graph of the solution of the same model but without resistance.

5. Solve the following IVP using separation of variables:

$$\frac{dy}{dx} = \frac{y^2}{x^2}, \quad y(1) = 2.$$

Don't forget to verify the answer.

6. Solve the following modification of (46)

$$\frac{dx}{dt} = t^2 - x, \quad x(0) = 1.$$

First find the series expansion and then rearrange it so that it looks like a familiar series as we did above. Plot the solution on $[0, 1]$.