Differential Equations Class Notes

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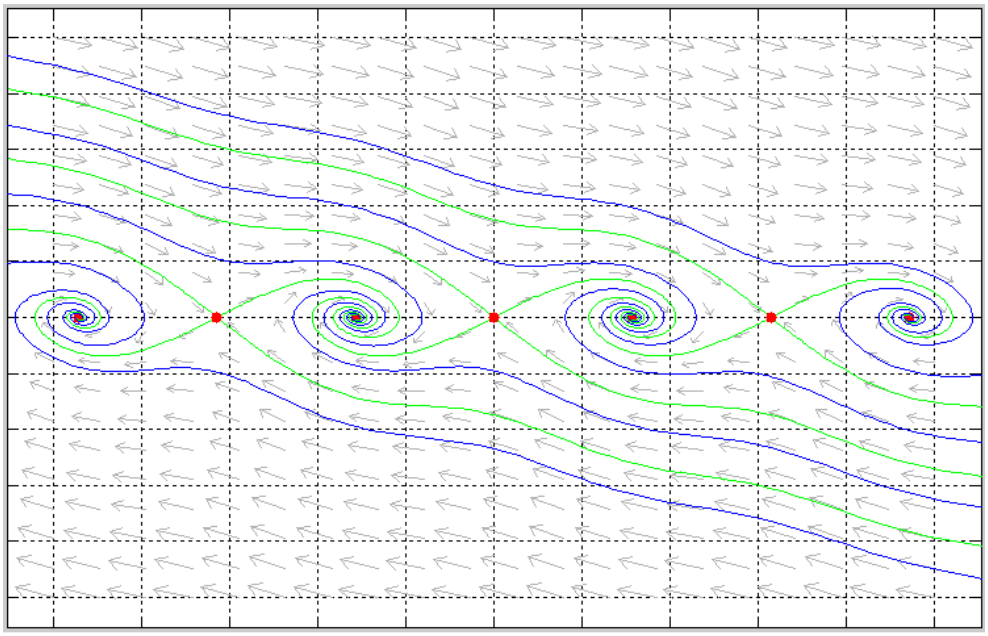
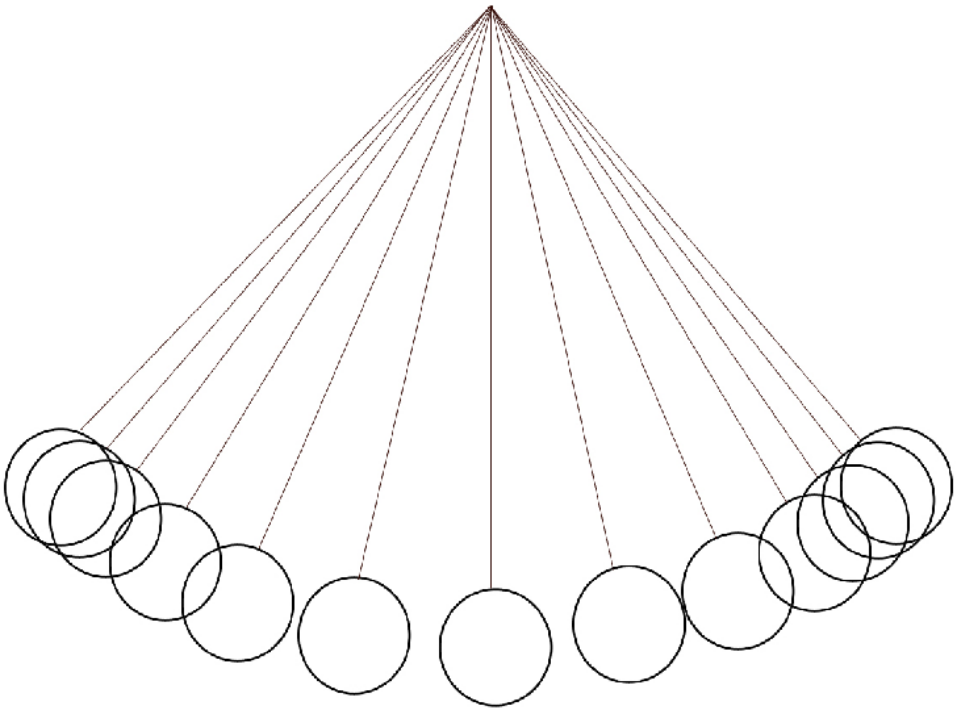
December 19, 2018





Contents

A Note About Supplementing the Textbook

The treatment of undergraduate ODE has barely changed over the past century. Therefore, there are myriad textbooks trying to explain the same material in different ways with contemporary methods and perspectives. I find that the tenth edition of Boyce and di Prima's *Elementary Differential Equations* is a very thorough treatment in our time. These notes are meant to be used in conjunction with this textbook. The hypocrisy and redundancy of these notes might seem laughable at this point, but I'd like to clarify why I found these necessary to write.

First, the formalism of a textbook can be daunting to students, especially on a first pass and in such bulk that these types of textbooks tend to amass. These notes stand to condense the text into a less formal structure and highlight important concepts concisely.

Second, everyone thinks to themselves at some point "This part is confusing - I could write this better." I am in no way claiming superiority over the textbook's authors; rather I am offering alternative explanations and perspectives that I found useful over the years of teaching and learning this subject. I hope that students and instructors alike can find my take on this course refreshing and novel enough to merit consideration.

Third, these notes use the same outline of topics as Boyce and di Prima's text, but they were written early enough in my career where the perspective is aligned more with the learner than with the instructor. It is my hope that these notes are more relatable to students than are the standard, verbose (and rather large!) textbooks that have become common (and sometimes unreadable) references in the United States.

A Note About Sequencing

Although one could conceivably study each section in numerical order as indicated by the numbering of these notes, it may be more fluid to take a slightly different ordering. In teaching this class, I take the following ordering of the chapters: 1, 2, 3, 7, 6, 5. I find that this accomplishes several goals:

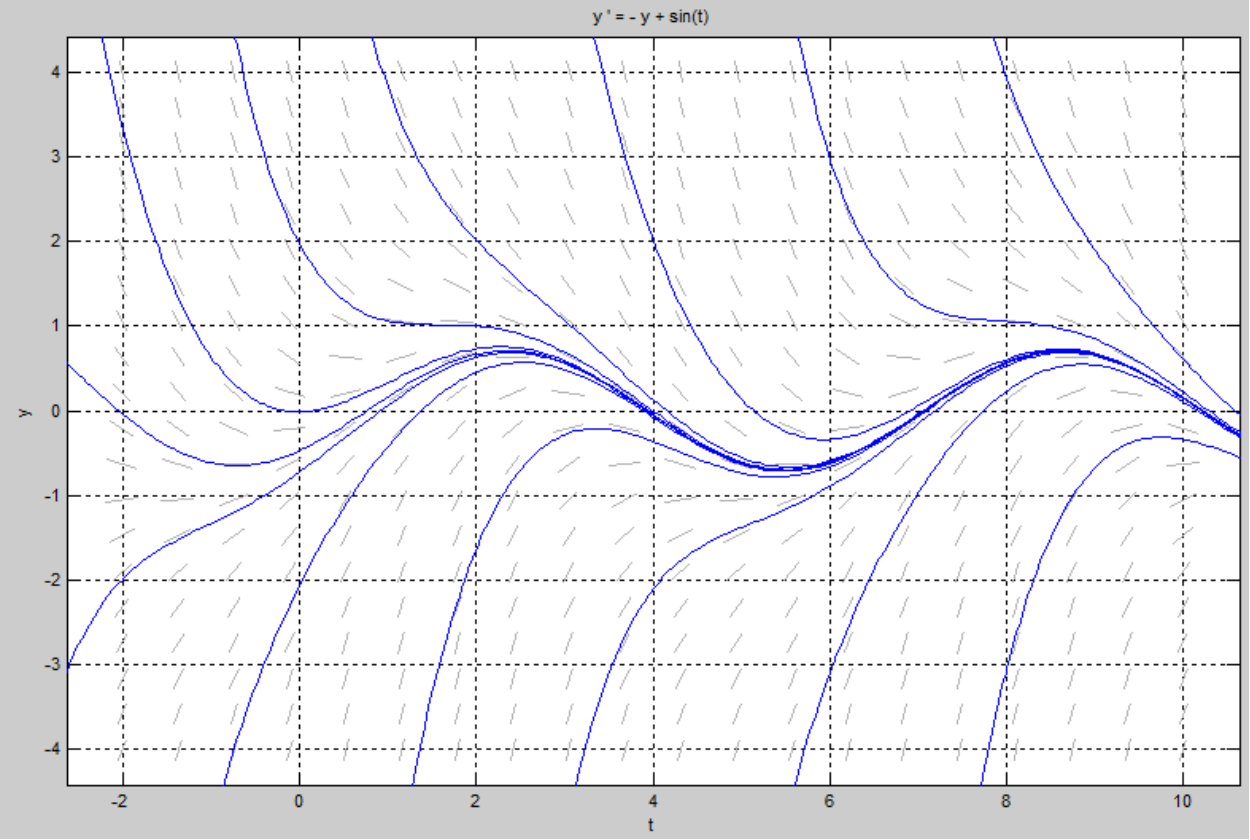
1. This ordering covers all the core material on the curriculum as outline by many universities' course descriptions. This material seems to be typical of other institutions as well and will adapt well to their curricula as well. Even within a constricted time frame (as in a summer class), these core topics will be covered based on this ordering.

2. More time can be given to explaining the core concepts in the beginning few chapters. Because there is so much padding at the end for extra topics, more attention can be given to some of the subtleties associated with this subject's key aspects.

3. The topics are presented in order of increasing sophistication and applicability in this way. In particular, the applications sections are generally relegated to the end of this course so that these are the freshest topics on the students' (mostly engineers) minds as they exit. This also bolsters student enthusiasm as the term winds down.

Of course, there are also optional topics that one may omit or include as time permits. These sections are indicated by asterisks and can typically be presented in whatever order the instructor finds most suitable. This is with the exception of the final section 7.8 which requires the coverage of sections 6.1 and 6.2 first.

1 Introduction and Definitions



1.1 Why study differential equations?

My job is to first try to convince you how great and useful differential equations are. So why **should** you study differential equations?

- They describe natural phenomena
- They simplify work that would otherwise be very tedious
- They allow us to predict future events or patterns
- They are ubiquitous in the physical sciences
- They're just plain AWESOME!

Definition 1.1. A *differential equation* is simply an equation that contains derivatives.

Example 1.1. Let me try to convince you how awesome differential equations really are:

1. In Calculus, we learned many different ways to solve the equation

$$\frac{dy}{dx} = f(x).$$

   The methods learned in foundational Calculus classes will be extremely useful in solving more general differential equations.

2. $$\frac{dB}{dt} = rB$$

   This equation models the growth of a colony of bacteria. Since the growth only depends directly on the number of individuals present in the population, this form of growth is called a *Malthusian growth model*.

3. $$y_{n+1} = ry_n(1 - y_n)$$

   Not all mathematical models are continuous by nature. In fact, there are many models that are only measured in discrete time intervals. This cousin of the continuous differential equation is called a *difference equation*. The equation above is called the *logistic equation*, and it is one of the simplest examples of a system that can exhibit chaotic behavior.

4. $$\begin{cases} \frac{dR}{dt} &= aR - bRF \\ \frac{dF}{dt} &= -cF + dRF. \end{cases}$$

   We can also keep track of multiple equations simultaneously. This system of equations models the interaction between a population of rabbits (R) and foxes (F). This form of predator-prey interaction is called the *Lotka-Volterra predator-prey model*.

5.
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$
This equation models a mass on a spring sliding back and forth in one direction. This is sometimes called a *harmonic oscillator*. When $b = 0$ and $F(t) = 0$, we call this a *simple harmonic oscillator*. This original equation also models certain kinds of electrical circuits. We will see this equation later in the course.

6.
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$
This equation models the swing of a pendulum under the force of gravity. Even though a pendulum demonstrates harmonic oscillations, this is not a simple harmonic oscillator as before. In fact, it only approximates simple harmonic motion for small angles θ (where $\sin\theta \cong \theta$ via Taylor's theorem).

7.
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$
This equation models many different mathematical and physical phenomena including complex-valued function, electrostatic potential, and fluid flow. It is called *Laplace's equation.*(Note the change from ordinary derivatives to partial derivatives.)

8.
$$\frac{\partial u}{\partial t} = k\Delta u$$
This equation models diffusion through a medium in any number of dimensions (usually one, two, or three). It is known as the *heat equation* because of its ability to model diffusion of heat through a medium.

These examples are indicative of the modeling capacity of differential equations. For this reason, we can call differential equations *mathematical models*. These terms will now be synonymous in this class.

Example 1.2. How do we construct a mathematical model?

Consider a falling object with mass m near sea level (perhaps a skydiver, if you want a concrete example). Newton's second law gives us the net force on the object as $F = ma$, where a is the acceleration of the object. We have learned that $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$, where $v(t)$ is the object's velocity and $x(t)$ is its position at a given time t. For this example, consider the effect of air resistance on the body. This can be modeled as $F_r(v) = \gamma v$ for some proportionality constant γ. Also notice that there is a constant gravitational acceleration $F_g = mg$. Combining these in a free-body diagram, we can easily see that
$$F = F_g - F_r = mg - \gamma v,$$
Newton's second law then tells us that
$$m\frac{dv}{dt} = mg - \gamma v. \tag{1}$$
We have now constructed a differential equation that describes the velocity of the falling object!

Definition 1.2. A *solution* of a differential equation is an open set called a *solution space* and a **function** (not just a variable) whose derivatives satisfy the differential equation on the solution space.

Example 1.3. Let's check a pair of examples - one where the solution space isn't terribly relevant, and one where it can't be ignored:

1. We claim that $B(t) = 7e^{3t}$ over the solution space $t \in \mathbb{R}$ solves the Malthusian growth equation given by
$$\frac{dB}{dt} = 3B.$$
To check this, we compute the derivative of the proposed solution:
$$\frac{dB}{dt} = 7(3e^{3t}) = 21e^{3t}.$$
Compare this against $3B$, which is
$$3B = 3(7e^{3t}) = 21e^{3t}.$$
Therefore, we have verified that indeed
$$\frac{dB}{dt} = 21e^{3t} = 3B.$$

2. We now claim that
$$y(t) = \begin{cases} \frac{-2}{t}, & t < 0 \\ \frac{1}{t^2}, & t > 0 \end{cases}$$
solves the equation
$$t^2y'' + 4ty' + 2y = 0.$$
To see this, we need to compute the first and second derivatives of the proposed solution $y(t)$. On the interval $(-\infty, 0)$, these are
$$y'(t) = \frac{2}{t^2} \quad \text{and} \quad y''(t) = \frac{-4}{t^3};$$
and on the interval $(0, \infty)$, these are
$$y'(t) = \frac{-2}{t^3} \quad \text{and} \quad y''(t) = \frac{6}{t^4}.$$
We need to consider the differential equation on each interval of the solution space individually. On the interval $(-\infty, 0)$, we see that
$$t^2y'' + 4ty' + 2y = t^2 \cdot \frac{-4}{t^3} + 4t \cdot \frac{2}{t^2} + 2 \cdot \frac{-2}{t} = \frac{1}{t}(-4 + 8 - 4) = 0;$$
and on the interval $(0, \infty)$, we see that
$$t^2y'' + 4ty' + 2y = t^2 \cdot \frac{6}{t^4} + 4t \cdot \frac{-2}{t^3} + 2 \cdot \frac{1}{t^2} = \frac{1}{t^2}(6 - 8 + 2) = 0.$$
Therefore, $y(t)$ is indeed a solution to the differential equation.

Remark 1.1. It is important as part of the definition that we include the solution space as a part of a solution to a differential equation. However, most of the time we only use the solution space as part of the rigor of the subject and can usually infer what it is when computing solutions. Most often we see these solution spaces as one contiguous interval (as in 1 above, but not as in 2).

Remark 1.2. It is also important to define an appropriate solution space especially when it comes to dealing with differential equations that appear as a model for a physical system. For example, one does not want to include solutions that extend to negative values for a population of organisms, so we can restrict time intervals to not extend beyond certain points.

As it turns out, we don't need to have an explicit solution in order to determine the behaviors of certain kinds of differential equations. In particular, if we can write the equation as

$$\frac{dy}{dt} = f(t, y),$$

we can graphically depict how the solution will behave at certain points in the domain of f.

Example 1.4. Let's use equation (1) calculated above with the solution space $v \in \mathbb{R}$ and $t \in (0, \infty)$ to demonstrate what we mean here. This equation may be put in the form

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v$$

so that $f(t, v) = f(v) = g - \frac{\gamma}{m}v$ is really just a function of the velocity. Recall that a derivative is really just a slope by its definition. In this context, it is the slope of a solution to the differential equation. So if we plug in various coordinates to $f(t, v)$, we will be able to determine the slope of a solution at that point in the solution space.

Let's take $g = 9.8\frac{m}{s^2}$, $\gamma = 2\frac{kg}{s}$, and $m = 10kg$ for definiteness. These are average values of a typical skydiver. Notice that for a fixed velocity v_0, the slope is the same for points (t, v_0) for all $t \in \mathbb{R}$. So let's check a few different velocities at time $t = 0$:

$$f(0, 0) = 9.8 \qquad f(0, 24.5) = 4.9 \qquad f(0, 49) = 0 \qquad f(0, 73.5) = -4.9.$$

With these few trial slopes in tow, we can slowly piece together what solutions of the differential equation will look like (see Figure 1). At each point, we place a small line segment of the given slope to indicate the direction that the solution will travel as we vary time. **Solutions** to this differential equation **will be tangent to the slope lines** that we have inserted. One can draw a few samples of solutions passing through the points we've chosen. Figure 1 has solutions drawn for the initial conditions $v(0) = 39$, $v(0) = 49$, and $v(0) = 60$.

One should remark that there is one solution in particular that is constant for all time t. This is a stationary solution $v(t) = 49$ of the equation and denotes the *terminal*

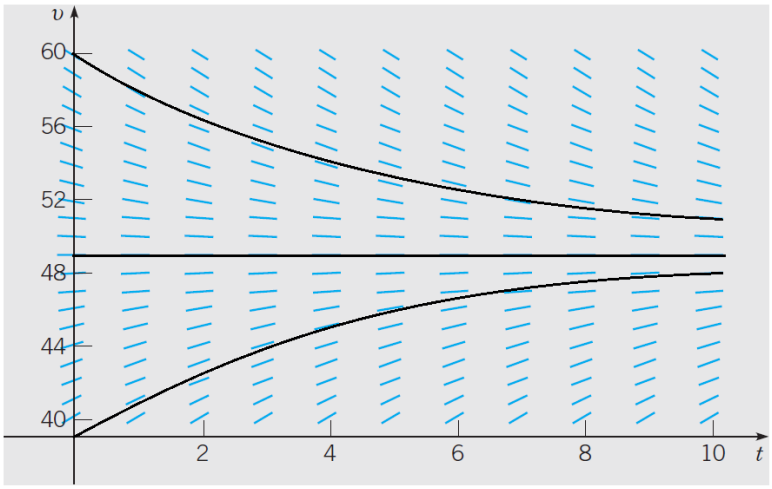


Figure 1: A slope field representing the solutions of the differential equation $\frac{dv}{dt} = 9.8 - \frac{2}{10}v$.

velocity of the skydiver. This is physically interpreted as the point at which the force of gravity is exactly canceled by the air resistance.

One more remark is that these **solutions will never intersect one another**. We will see why this is the case and what it would imply later in the course.

Definition 1.3. Let

$$\frac{dy}{dt} = f(t, y).$$

The function f that assigns to each point of a solution space a slope is called the *slope field*. This is what we constructed in the example above, and it is synonymous with the terms *direction field* and *rate function*. A solution $Y(t)$ where $f(t, Y(t)) = 0$ for all t is called an *equilibrium solution* of the differential equation.

Remark 1.3. All equilibrium solutions are in fact constants! Why is this the case? Because equilibrium solutions automatically set

$$\frac{dY}{dt} = 0,$$

whose solutions can only be constants.

Remark 1.4. Notice that we haven't actually solved the differential equation by creating a slope field and drawing in some solution curves. However, these are useful if we merely want some qualitative information about the differential equation like where equilibrium solutions will occur. Construction of a slope field is a perfect job for a computer to do. In fact, there are many applications online which will allow you to do exactly this.

1.2 Some first solutions to differential equations

If we can put a differential equation into the form

$$\frac{dy}{dt} = f(t, y),$$

then we stand a chance at solving it without too much trouble. Let's see an example of what is meant here.

Example 1.5. Let's solve two seemingly different differential equations. First, let's take a look at trying to solve

$$\frac{dy}{dt} = ay + b$$

for non-equilibrium solutions like $y(t) = -\frac{b}{a}$ (taking $a \neq 0$ and b to be constant). Our first step is to separate the equation into a 'y' side and a 't' side, treating the derivative term as a fraction, like so:

$$\frac{dy}{ay + b} = dt. \tag{2}$$

Now we can integrate the left side in y and the right side in t as follows:

$$\frac{1}{a} \ln |ay + b| = t + C.$$

Then we can solve this equation for y as follows:

$$\begin{aligned} \frac{1}{a} \ln |ay + b| &= t + C \\ |ay + b| &= e^{at + C_1} \\ ay + b &= \pm e^{C_1} e^{at} \\ y &= Ke^{at} - \frac{b}{a} \end{aligned}$$

where $K = \pm \frac{1}{a} e^{C_1}$ is just a constant. This is a **family of solutions** to the differential equation - notice that the equilibrium solution mentioned above is also in this family. We can check that each member of this family is indeed a solution to the differential equation by substituting their general representative back into equation (2):

$$\begin{aligned} \frac{d}{dt}\left(Ke^{at} - \frac{b}{a}\right) &= a\left(Ke^{at} - \frac{b}{a}\right) + b \\ aKe^{at} &= aKe^{t} - b + b \\ 0 &= 0, \end{aligned}$$

which is an identity know to be true. Hence,

$$y(t) = Ke^{at} - \frac{b}{a}$$

is a family of solutions solving the differential equation (2).

Observe that if $a > 0$, then our solutions will generally tend away from the equilibrium solution and off to $\pm\infty$. However, if $a < 0$, then solutions will generally tend toward the equilibrium solution.

Example 1.6. For our second example, let's try to solve the equation

$$\frac{dy}{dx} = \frac{1}{y(1+x^2)} \tag{3}$$

for solutions $y(x) \neq 0$ for any point x. We separate the equation as before:

$$y\,dy = \frac{1}{1+x^2}\,dx.$$

We can integrate both sides and solve for $y(t)$:

$$\begin{aligned} y\,dy &= \frac{1}{1+x^2}\,dx \\ \frac{1}{2}y^2 &= \tan^{-1}x + C \\ y(x) &= \pm\sqrt{2\tan^{-1}x + K}, \end{aligned}$$

where $K = 2C$ is an arbitrary constant. We can again check that these solutions solve the differential equation by substituting them into equation (3). We leave this verification to the student (go on, challenge yourself).

Definition 1.4. A set of *temporal (time) conditions* are merely values y_i that we wish a solution $y(t)$ to pass through at specific times t_i. An *initial condition* is one where we take only one temporal condition $y_0 = y(0)$ at $t = 0$. A differential equation equipped with an initial condition is called an *initial value problem.*

Definition 1.5. A family of equations that solves a differential equation is called a *general solution.* A solution solving a differential equation and passing through specified points is called a *particular solution* of the equation.

Remark 1.5. The astute reader will notice that the solution space of the first example solution is all real numbers but that the second example solution has a domain that depends on the coefficient K and on whether one takes the positive or negative square root. This brings to question what made the first equation different from the second equation so that the solution spaces varied so greatly from one another. This is a question we intend to answer as the course progresses.

We will study separable differential equations in more depth soon. There is still a smidgeon more terminology that we need to get through before we study techniques to solve differential equations.

1.3 Classification of Differential Equations

Differential equations come in a variety of shapes, sizes, and flavors. This section is a lot of terminology, but it is important to know these terms and why they are important to the study of differential equations. All terms will be given first, and then there will be a number of examples at the end of the section clarifying their meanings.

Differential equations come in two different flavors, and their solutions and solution techniques contrast starkly.

Definition 1.6. A differential equation whose dependent variables depend on a single independent variable is called an *ordinary differential equation (ODE)*. A differential equation whose dependent variables depend on more than one independent variable is called a *partial differential equation (PDE)*. If there is more than more than one dependent variable, we say that we are working with a *system* of ODE or PDE.

We will only study ODE in this class. In particular, we will first study ODE of one dependent variable and then move on at the end of the class to systems of ODE. It is of great use to us to learn how to solve ODE because they often appear in the implementation of solution techniques of PDE. If you want to study PDE, take MAT 118.

To some degree, the number of derivatives in a differential equation indicates its difficulty to solve and the richness of its solutions.

Definition 1.7. The *order* of a differential equation is the highest-power derivative that appears in the equation for which the equation cannot be written with lower-power derivatives.

All of the examples we have studied so far have been first-order ordinary differential equations (with the exception of one second-order equation).

Definition 1.8. A first-order ODE that can be written independently of the independent variable t, that is, in the form

$$\frac{dy}{dt} = f(y)$$

for some function f, is called *autonomous*. First-order ODE that fail to satisfy this condition are called *non-autonomous*.

The ODE can **always** be put into the general form

$$G(t, y, y', y'', \ldots, y^{(n)}) = 0,$$

where G is some function; however, this form is not very useful to us from a theoretical point of view (as there is no general solution technique for all differential equations). As such, we put the ODE into a standard form that will allow us to classify it. This classification will allow us to determine the best or most useful solution method. This standard form for an arbitrary n^{th}-order ODE is given by

$$F(y, y', y'', \ldots, y^{(n)}) = g(t). \tag{4}$$

Classification of the ODE is based on the different properties found in equation (4).

Definition 1.9. An ODE is called *linear* if F is a linear function of y and all of its derivatives. In particular, we must have

$$F(u+v,(u+v)',\ldots,(u+v)^{(n)}) = F(u,u',\ldots,u^{(n)}) + F(t,v,v',\ldots,v^{(n)})$$

and

$$F(\lambda u,(\lambda u)',\ldots,(\lambda u)^{(n)}) = \lambda F(u,u',\ldots,u^{(n)})$$

for any functions u and v and any nonzero constant λ. An ODE that is not linear is called *nonlinear*.

Remark 1.6. Because of the strictness of the linearity condition, we know exactly the form that a linear equation can take. They must all be of the form

$$F(y,y',\ldots,y^{(n)}) = a_n(t)y^{(n)} + \cdots + a_1(t)y' + a_0 y = g(t)$$

for some functions $a_i(t), g(t)$.

Definition 1.10. If a **linear** differential equation admits the constant function $y(t) = 0$ for any solution space, then we call the equation *homogeneous*. If this is not the case, then the equation is called *non-homogeneous*.

Remark 1.7.

1. The definition of linearity may be extended to PDE as well. The definition is a bit more cumbersome in this case, as we need to extend our function F to include all partial derivatives of $u = u(t,x_1,\ldots,x_k)$ as well. The definition will then say that a PDE $F(u,Du,\ldots,D^n u) = g(t,x_1,\ldots,x_k)$, where $D^i u$ is the vector of all i^{th}-order partial derivatives of u, is linear if F is a linear function.

2. Linear, homogeneous ODE are **nice**! Like... **really** nice! These equations allow for two remarkable properties of solutions. For one, if there are two solutions u and v to the differential equation, then the function $u+v$ is also a solution. Secondly, for any solution u and nonzero constant λ, the function λu is also a solution (sometimes **this** is what is meant by a "homogeneous" equation: one where this property is satisfied). This is not always the case for nonlinear equations. In fact, linear algebra plays a rather large role in studying solutions to differential equations, and it is the theory of linear algebra that allows for such a complete theory of linear ODE.

Example 1.7. These examples are exactly those from the beginning of the chapter and then some. The reader should try to classify each equation before reading its explanation.

1. $$\frac{dB}{dt} = rB$$

   This is a homogeneous, linear, first-order ODE.

2.
$$y_{n+1} = ry_n(1 - y_n)$$
We won't worry about classifying difference equations. It is useful, though, to know that the logistic equation above is a homogeneous, nonlinear, first-order (ordinary) difference equation.

3.
$$\begin{cases} \frac{dR}{dt} &= aR - bRF \\ \frac{dF}{dt} &= -cF + dRF. \end{cases}$$
This is a homogeneous, non-linear, first-order system of ODE.

4.
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$
This equation is a linear second-order ODE. It is homogeneous exactly when $F(t) = 0$.

5.
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$
This is a homogeneous, nonlinear second-order ODE.

6.
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$
This is a homogeneous, linear, second-order PDE.

7.
$$\frac{\partial u}{\partial t} = k\Delta u$$
This is a homogeneous, linear, second-order PDE.

8.
$$(\sin t)y + y''' = \ln t$$
This is a non-homogeneous, linear, third-order ODE.

9.
$$yy' + e^t = y''$$
This is a nonlinear, second-order ODE.

10.
$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \otimes \nabla \vec{v}\right) = -\nabla p + \nabla \cdot \vec{T} + \vec{f}$$
where ρ is a constant, p is a real-valued function of three spatial variables, and $\vec{v}$, $\vec{T}$, and $\vec{f}$ are three-dimensional, vector-valued functions of three spatial variables and one temporal variable.

This is a nonlinear, second-order system of PDE.

This is actually the famous *Navier-Stokes equation*. In 2000, the Clay Institute announced a one-million-dollar prize for anyone who can prove the existence of smooth solutions to this equation for general initial and boundary conditions (this is only one of seven *Millenium Problems* - one of which was solved in 2003, verified in 2006, and the winner, Grigori Perelman, was finally awarded the prize in 2010... which he then didn't accept).

There are a few final remarks to make about differential equations in general. So we ask a few questions:

1. *Existence* - Is a solution to an arbitrary ODE $F(t, y, y', y'', \ldots, y^{(n)}) = 0$ guaranteed to exist? If it does, what is the solution space? How can we tell if a solution to this ODE exists and over which solution spaces it can exist?

2. *Uniqueness* - If a solution exists, is it the only one? If there are more solutions, how many are there? Which conditions do we need in order to guarantee a unique solution?

3. *Computability* - Can we determine an explicit solution? How? Is it possible to determine this solution theoretically, numerically, or with nonelementary function?

Remark 1.8. In light of number 3 above, computers are an ever-present tool which may be used to determine **approximations** of solutions. However, this class deals with being able to solve ODE by hand and being able to recognize such ODE when they appear in practice. We will not worry about numerical solutions in this class; however, we will have a brief introduction to their implementation and use. If students want to see more about numerical methods in mathematics, I suggest taking a course typically called "Numerical Analysis" or "Numerical Methods".

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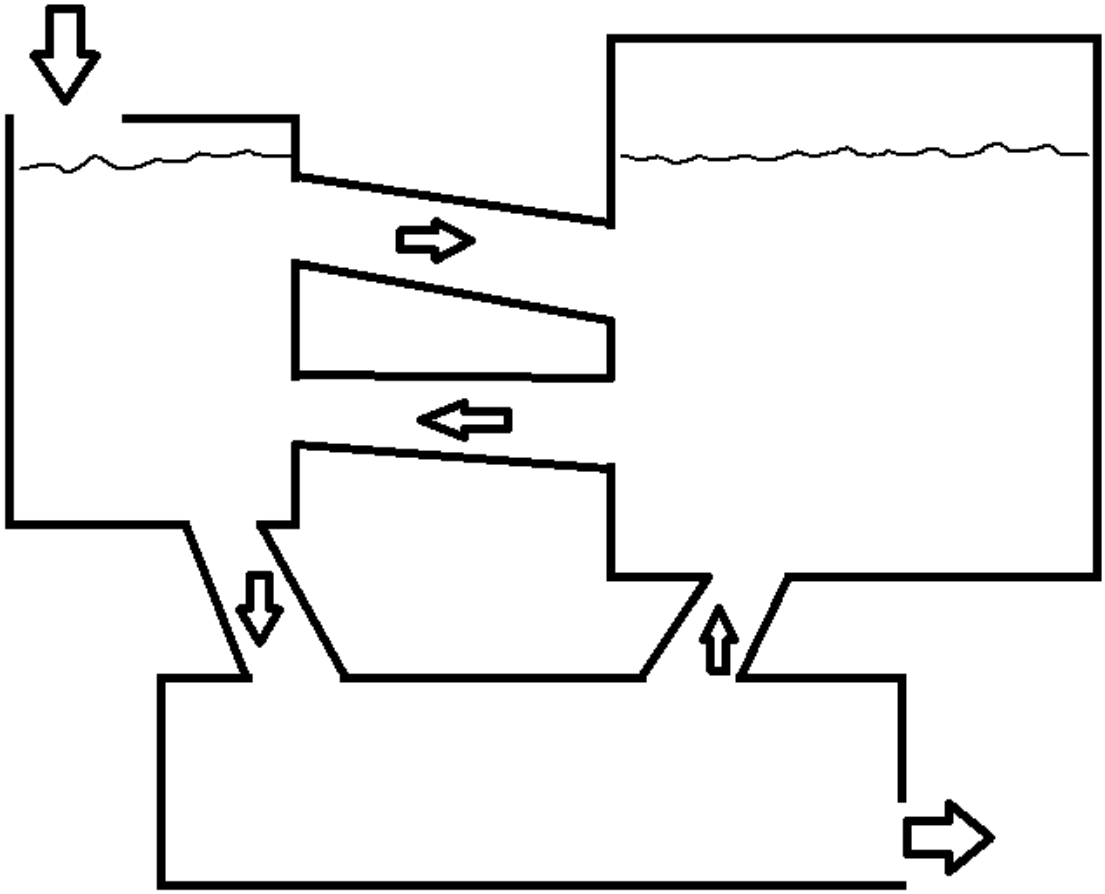
2 First-Order Differential Equations



2.1 Linear Equations and Integrating Factors

What form does a linear first-order ODE take? Recall that the standard form must look like

$$F(t, y, y') = a(t)y'(t) + b(t)y(t) + c(t) = 0.$$

We can then rewrite the equation as

$$\frac{dy}{dt} + p(t)y = q(t). \tag{5}$$

We take this to be our definition:

Definition 2.1. An equation of the kind in (5) is called a *linear, first-order ODE.*

Our goal for this section is to learn how to solve these. As we did in the first chapter, we aim to separate variables in order to solve this kind of equation. However, there is a problem with this approach. There may not be a way for us to separate variables in this kind of equation even in the simplest examples (e.g. $\frac{dy}{dt} = t + y$). We require a more general approach to solve these equations.

Our first step to solving such ODE is merely an observation. The left-hand side of (5) looks remarkably similar to a product rule. There seems to be something missing from it, though. If we were to multiply some function $\mu(t)$ into this equation, then perhaps we can represent the left-hand side as a derivative and then integrate. That is, we wish to find a function $\mu(t)$ where

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \frac{d}{dt}(\mu(t)y).$$

Then we can (relatively) easily solve the equation for a general solution

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)q(t)\, dt.$$

The question now is how to find such a miracle function $\mu(t)$. Notice here that if we had such an equality that

$$\left(\frac{d}{dt}(\mu(t)y) = \right) \mu(t)\frac{dy}{dt} + \mu'(t)y = \mu\frac{dy}{dt} + \mu(t)p(t)y.$$

Cancellation shows us that our function μ needs to satisfy, ironically, yet another ODE:

$$\mu'(t) = \mu(t)p(t).$$

This one, however, is separable! In fact, it is easily seen that

$$\mu(t) = Ce^{\int p(t)\, dt} \tag{6}$$

for any constant C. Since we only need one of these functions for our solution, we choose the solution for $C = 1$ so that the general solution to our general first-order linear ODE (5) is

$$y(t) = e^{-\int p(t)\, dt} \int [e^{\int p(t)\, dt}]q(t)\, dt. \tag{7}$$

Definition 2.2. Any function $\mu(t)$ (for $C \neq 0$) in equation (6) is called an *integrating factor* for equation (5).

If the work above was a bit cryptic in its exhibition, perhaps it would be prudent to see a few examples of this method in action.

Example 2.1. 1. "Solve the following first-order linear ODE:

$$\frac{dy}{dx} = x + y." \tag{8}$$

We first put this equation into the standard form

$$\frac{dy}{dx} - y = x.$$

Then we can take $p(x) = -1$ and $q(x) = x$. Then our formula for $\mu(x)$ is

$$\mu(x) = e^{\int -1\,dx} = e^{-x}.$$

We now multiply this factor into both sides of (8) to get

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}.$$

Then we rewrite the left-hand side of this new equation to yield

$$\frac{d}{dx}(e^{-x}y) = xe^{-x}.$$

Integrating both sides of this equation in x (integrating by parts on the right-hand side) gives us

$$e^{-x}y = -e^{-x} - xe^{-x} + C.$$

Diving both sides by e^{-x} gives us the final result

$$y(x) = -1 - x + Ce^{x}.$$

2. "Solve the following initial value problem:

$$\begin{cases} (1+x^2)\frac{dy}{dx} + 2y = 1 \\ y(0) = 2 \end{cases}."$$

Again, we put the equation in standard form so that

$$\frac{dy}{dx} + \frac{2}{1+x^2}y = \frac{1}{1+x^2}.$$

The we compute

$$\mu(x) = e^{\int \frac{1}{1+x^2}\,dx} = e^{2\tan^{-1}x}.$$

Multiplying by μ on both sides of the equation gives us

$$e^{2\tan^{-1}x}\frac{dy}{dx} + \frac{2}{1+x^2}e^{2\tan^{-1}x}y = \frac{1}{1+x^2}e^{2\tan^{-1}x}.$$

Rewriting the left-hand side of this equation and integrating (right side by substitution), we get

$$e^{2\tan^{-1}x}y = \frac{1}{2}e^{2\tan^{-1}x} + C.$$

Then dividing by $e^{2\tan^{-1}x}$ on both sides gives us

$$y(x) = \frac{1}{2} + Ce^{-2\tan^{-1}x}.$$

Since we were given an intial condition, we can use it to determine the constant C. That is, we use the coordinate $(x, y) = (0, 2)$ to find

$$2 = \frac{1}{2} + C$$

so that our final solution to the initial value problem is the particular solution

$$y(x) = \frac{1}{2} + \frac{3}{2}e^{-2\tan^{-1}x}.$$

Notice as well that this example can also be done by separating variables. The reader should, of course, show that the solutions are the same and independent of the method used.

The previous two examples both had solutions that were defined over all real values x. The following example shows us an instance where we cannot compute a solution that is valid for all values of the real line.

Example 2.2. "Solve the following ODE and explain what happens for solutions at $x = 0$:

$$x\frac{dy}{dx} + 2y = 4x^2.''$$

Putting this equation in standard form gives us

$$\frac{dy}{dx} + \frac{2}{x}y = 4x.$$

Then our integrating factor is

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2.$$

Multiplying this on both sides of the equation and rewriting it yields

$$\frac{d}{dx}(x^2y) = 4x^3.$$

Integrating this equation gives us

$$x^2y = x^4 + C$$

so that we have the general solution

$$y(x) = x^2 + \frac{C}{x^2}.$$

Notice that all solutions are discontinuous across $x = 0$. This means that the constant C can be different across this gap. Hence, if we specific some temporal condition at some point $x \neq 0$, we can only know the solution on one side of the singularity at $x = 0$. The only solution that does not "blow up" at $x = 0$ is when $C = 0$ - that is, when $y(x) = x^2$.

Remark 2.1. Notice that there are two pieces to every solution of a linear first-order ODE. The piece of the solution that has the coefficient in it is called the *homogeneous solution*, and the remaining piece is called an *inhomogeneous* or *particular solution*. The student of linear algebra will recognize this as a general solution to an underdetermined linear system. That is, the homogeneous solution defines the null space of the differential operator $D = \frac{d}{dt} + p(t) \cdot \mathrm{Id}$ (Id represents the identity operator), and the inhomogeneous solution is an "offset" of that homogeneous solution that solves the inhomogeneous differential equation.

2.2 Separable Equations

Our discussion of ODE originally began with separable equations. Here we discuss them in more detail. We need to see what kinds of consequences result from attempting to solve these kinds of ODE.

In solving separable ODE, recall that we are given a first-order ODE

$$\frac{dy}{dx} = f(x, y).$$

The goal was to separate the equation into functions only of x and only of y. That is, we wanted the form

$$g(y)\, dy = h(x)\, dx.$$

Definition 2.3. An ODE that can be written in the form

$$g(y)\, dy = h(x)\, dx$$

is called *separable*.

Let's do one example to remember how these ODE are solved.

Example 2.3. 1. "Solve the following ODE by separating variables:

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}."$$

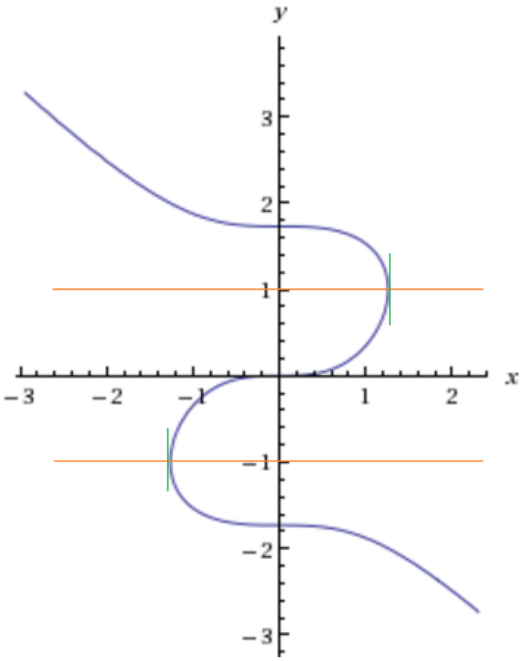


Figure 2: The graph of $y - \frac{1}{3}y^3 = \frac{1}{3}x^3$. To make this graph into a function $y = y(x)$, we must divide the range into branches where the graph is one-to-one. This occurs for the y intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

We first rewrite this equation as

$$(1 - y^2)\, dy = x^2\, dx.$$

Integrating both sides gives us

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C.$$

Unfortunately, this does not define an explicit function $y(x)$; however, this does give us an **implicit solution**. In fact, this equation prescribes different solutions $y = y(x)$ for different intervals of time! See Figure 2 for an illustration of the solution types for $C = 0$. However, if we solve for x as a function of y, then we avoid these complications altogether. That is,

$$x = \sqrt[3]{3y - y^3 - 3C};$$

however, this solution still has undefined derivatives at certain values of y.

2. In the same vein as the previous example, let's consider solving the initial value problem

$$\begin{cases} \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1. \end{cases}$$

We can again separate it and integrate to find

$$y^2 - 2y = x^3 + 2x^2 + 2x + C.$$

Using the initial condition, we find that $C = 3$ so that

$$(y)^2 + (-2)y + (-x^3 - 2x^2 - 2x - 3) = 0$$

implicitly defines a solution to the ODE. Notice that we have written the solution so as to imply that we can solve for y using the quadratic formula. That is,

$$y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

So which solution should be take for the initial value problem? Of course, we should take the solution that actually passes through the initial value prescribed - namely, the negative branch of the function. Notice again that the domain of this function is bounded by a value of y that makes the derivative undefined.

Remark 2.2. In studying these separable equations a little further, we have found a new trend in the types of solutions we can obtain from them. There are a few other things we can observe while we're here.

1. Not all first-order equations can be put into the form

   $$\frac{dy}{dx} = f(x, y);$$

   but when they can, there is some simplicity to finding long-term behaviors of the solutions. The equilibrium solutions are readily found by setting $f(x, y) = 0$. For example,

   $$\frac{dy}{dx} = \frac{(y-3)\cos x}{1 + 2y^2}$$

   has an equilibrium solution at $y(x) = 3$.

2. Recall from parametric representations of functions that we can write a solution as the coordinate pair

   $$\vec{x}(t) = (x(t), y(t))$$

   for some parameter t in some interval I. Then the derivative of this parametrized curve is given by

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

   So we can sometimes view a general first-order equation

   $$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}$$

   as being the derivative of some parametrized curve with

   $$\frac{dy}{dt} = F(x(t), y(t)) \qquad \frac{dx}{dt} = G(x(t), y(t)).$$

As non-intuitive as this sounds, we can sometimes reduce the complexity of a differential equation by making an appropriate substitution of a parametrized curve. That is, we can turn a single ODE into a system of ODEs

$$\begin{cases} \frac{dx}{dt} &= G(x,y) \\ \frac{dy}{dt} &= F(x,y) \end{cases}$$

that will (hopefully) be easier to solve. We will see what to do with this method much later in this class.

3. In the above examples, we had a case where we could create an explicit solution from an implicit solution. This is almost always impossible to accomplish in general. Since this is the case, it is often easier to just leave an implicit solution the way it is after integration rather than to solve it for the explicit function $y(x)$.

2.3 Modeling with First-Order Equations

It was highlighted in the first chapter that differential equations are a natural way to model natural phenomena. After learning a few methods for solving relatively simple ODE, we give an exposition of how this sort of modeling is done.

The process of developing a mathematical model is usually broken into three distinct steps:

1. **Constructing the Model** - One first translates the physical scenario into easily grasped mathematical terms. This involves an attempt at estimating the factors that affect the rate of change of a physical quantity like heat, position, or population size.

   It is of the utmost importance that we step back and remember that all we can do with a mathematical model is *approximate*. Any assumptions that we make about our system are no more than guesses about what governs its change. There may be factors missing from our model or (less often) unnecessary factors present in it. By imposing these assumptions, we are also forced to recognize that our model is limited in its ability to predict the behavior of the system. Such an assumption may be that, in large populations, the birth rate of a species is in fact a discretely changing number (there are no "half-babies") but it can be approximated as a continuous quantity (which mathematically allows for such oddities to occur).

   Nevertheless, it is our goal in this step to create what would be a reasonable model governing how a physical system changes.

2. **Analyzing the Model** - This step can really be summarized by saying, "This is why we are taking a class in differential equations." It is this step that requires a strong aptitude for solving differential equations in many different qualitative and quantitative ways. This is also the step that causes most of the ambiguities because of its openness to interpretation. Sometimes one requires a fair bit of knowledge in a particular field of study (e.g. Physics, Chemistry, Ecology, etc.) in order to give a reasonable interpretation of what a differential equation says about a given system.

In practice, there is often a computer aspect to this step. Computer modeling is sometimes our only option when analyzing mathematical models simply because the differential equation cannot be solved - either explicitly or implicitly. This is not the focus for our class, but you must understand that this is a pervasive aspect of the study of differential equations.

Again, the goal in this step is interpretation of the model and its potential solutions.

3. **Comparison with Observations** - Sometimes we smoothly transition from the previous step to this one because they are so similar. This step is different in so much as this is our "reality check." We want to make sure that what our equation says is consistent with what we find from experiments or observations. If our equation agrees with our experiment, then we can move on to use the model and accurately predict future behavior and patterns. If our equation doesn't agree with observation, then we need to go back to our first step to determine which assumptions may have been wrong or unnecessary and potentially which features we may have forgotten to include. That is, we revise the model and repeat the whole process until we deem our equation "good enough" to model the physical process.

We can illustrate this entire process by looking at an example.

Example 2.4. Suppose that we work for a cheese factory. Such factories use brine (saltwater) solution to flavor and chill their cheese blocks during production. Suppose further that we have a 1,000-gallon brine tank filled with 900 gallons of pure water (no salt added), and we are going to circulate this tank to help mix the brine evenly. We are going to facilitate this mixing with a pump at the top of the tank and a pump at the bottom of the tank. The top pump will pump in brine solution at a fixed concentration of $\frac{1}{10}$ pounds per gallon, and the bottom pump will drain brine from the tank. For now, let's assume that these two flow rates r gallons per minute are exactly the same so as to keep the volume in the tank constant. We want to be able to carefully control how much salt is in our brine solution. Let's do this by creating a mathematical model.

(a) First, we need to define our parameters and then fix some assumptions. Let $w(t)$ be the weight of the salt in the tank at time $t \geq 0$ (the problem won't make sense for any $t < 0$). We have a lot of information given to us in this problem already. One piece of information we don't have is the concentration of the solution leaving the tank.

This is where we need to make our first assumption. We need to guarantee an "instantaneous mixing" of whatever salt is pumped into the tank. This way, we can guarantee that the concentration of the solution leaving the tank is exactly that of the current concentration of brine in the tank. That is, the concentration leaving the tank is $\frac{w(t)}{900}$ pounds per gallon.

Let's also assume a trivial (sort of) fact that solution is not lost through evaporation or leaks. This guarantees that whatever salt that goes into the tank either stays in the tank or leaves through the drain. This way all of the salt is accounted for.

We now need a physical law to describe the change in weight of salt $\frac{dw}{dt}$. We can make a very simply observation that there is a net flow of salt through the tank governed by the equation

$$\frac{dw}{dt} = \text{rate in} - \text{rate out}.$$

This part of the problem is very tricky to deal with without appealing to dimensional analysis. That is, we need to keep track of the units of the different quantities floating throughout the equation.

Since the units of $\frac{dw}{dt}$ are measure in pounds per minute, we need to construct rates in units of pounds per minute as well. The rate in can be given by

$$\text{rate in} = r \text{ gal/min} \cdot \frac{1}{10} \text{ lb/gal} = \frac{r}{10} \text{ lb/min}.$$

Given our assumption above, we can compute the rate out as

$$\text{rate out} = r \text{ gal/min} \cdot \frac{w(t)}{900} \text{ lb/gal} = \frac{rw(t)}{900} \text{ lb/min}.$$

Hence, we can now drop the units and pay strict attention to the mathematical aspects of our model, which is given now by the linear first-order ODE

$$\frac{dw}{dt} = \frac{r}{10} - \frac{rw}{900}.$$

(b) Now that we have constructed our model, we need to analyze (i.e. solve) it. To do this, we employ the methods developed in the first two sections of this chapter to yield

$$w(t) = 90 - Ce^{-rt/900}$$

(the reader should verify this calculation). Since we started with $w(0) = 0$ (pure water with no salt), we apply our initial condition to get $C = 90$ and our particular solution is

$$w(t) = 90(1 - e^{-rt/900}).$$

Hooray! We have solved our equation! Now what..? This leaves us the task of interpretation. Let's observe some long-term behavior of the solution. If we leave this process to run over a long period of time, we can take $t >> 0$. This would make the exponential term tend to zero so that our long-term (steady-state) solution is $w(t) = 90 = \frac{1}{10} \cdot 900$ pounds of salt in the tank. That is, the concentration in the tank will eventually equalize to that of the concentration being pumped in!

(c) We now seamlessly transition into our "reality check" step to see if this interpretation makes sense with what we should expect. Intuitively, this makes sense; but this is the point at which we would have to check our solution against a physical experiment.

Now we have to be humble about our almighty model of all things brine. We made a few assumptions at the outset of this problem. Let's acknowledge how their absence may affect our solution.

First, we assumed that there was an instantaneous mixing of the incoming brine with the brine in the tank. This is an entirely outrageous (and yet not too farfetched) assumption. This assumption describes a process that cannot be accomplished naturally. This effectively states that the incoming salt is at the top of the tank when it enters and that it is at the bottom of the tank an instant later. Even though this situation is preposterous, it allowed us to build a relatively simple model of the overall behavior of the system - and ultimately, this is exactly the purpose of the model. Something that may help make this condition more realistic is the addition of a mixer

to help homogenize the mixture in the tank. This makes the idea of instantaneous transport a little less abhorrent.

Our second assumption is not so appalling. Recall that we assumed that salt was not lost out of the tank. In reality, there are forces out of our control that make this not entirely realistic. In fact, evaporation at the sides of the tank will surely lead to crystallization of the solution at rim of the tank (kind of like a giant margarita glass filled with salt water - bleh!) especially if we oversaturate the water with salt.

This last comment also lends itself to hidden aspects that may or may not present themselves in the math of the problem. Physically there is a limit to how much salt can be dissolved into a certain amount of water. For NaCl, this is about 35 grams of salt for every 100 grams of water at sea level with a temperature of 25°C. A quick calculation shows that the maximum weight of salt that we can dissolve into the tank of 900 gallons of water is roughly 265 pounds of salt. So our model, which has a maximum value of 90 pounds of salt, is well within that range.

So our final assessment of our model is that it is decent but not a perfect model for short-term predictions, but it models long-term behavior very well.

We can easily see from the example that there is a lot more to mathematical modeling than just solving a differential equation. Granted: Solving the equation is a major battle, but the war is won in the interpretation of results. We will have a lot more practice with mathematical models in some worksheets; but for now, let it suffice to know that there is a formal process for devising these models.

2.4 Differences Between Linear and Nonlinear Equations

We mentioned three questions at the outset of this class:

1. For a given differential equation, does a solution **exist**?
2. For a differential equation that we know one solution, is it the **unique** solution?
3. For a solvable differential equation, is the solution **computable** by some means?

We plan on answering the first two of these in one fell swoop in this section.

In general, it is often impossible to exhibit an explicit solution (as we've seen in the previous sections). With this consideration in mind, we would still like to know whether or not a solution exists and to predict over which solution spaces a solution can exist. In particular, it has been the existence of particular solutions of initial value problems that have been the most cumbersome.

It would be nice to know whether or not a solution actually exists before putting in the time, effort, and resources to find it. Furthermore, we would like to know if there is only one solution or if there are several with the same initial condition (which is entirely possible!).

To answer these important questions, we actually have a theorem that will address both of these concerns simultaneously.

Theorem 2.1. (Existence and Uniqueness) If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

[Theorem statement from Boyce and DiPrima]

We have already seen a sort of "proof" for this theorem in the solutions we worked out with integrating factors. We can explicitly write the general solution as

$$y(t) = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} g(t)\,dt$$

and then use the temporal condition to find the particular solution.

For more general (perhaps nonlinear) first-order ODE, we need something a little more powerful.

Theorem 2.2. Let the function f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0. \end{cases}$$

[Theorem statement from Boyce and DiPrima]

Theorem 2.2 gives us the same result as the linear version, but it works for a much larger class of differential equations. Moreover, the only condition we need in order to guarantee existence and uniqueness of solutions is that f and $\frac{\partial f}{\partial y}$ are continuous in some open rectangle containing the initial condition! This is a powerful result, and we will see more about it in section 2.8.

Notice how theorem 2.2 includes theorem 2.1 as a special case, taking $f(t, y) = g(t) - p(t)y$ (theorem 2.2 would then require that $g(t) - p(t)y$ and $-p(t)$ be continuous, which they are if both $p(t)$ and $g(t)$ are continuous). In fact, we refer to theorem 2.2 as "the" existence and uniqueness theorem for this reason. For now, we should look at some examples of how we use both of these theorems.

Example 2.5. Consider the initial value problem

$$(t^2 - 1)y' - ty = \sin t; \quad y(2) = -5.$$

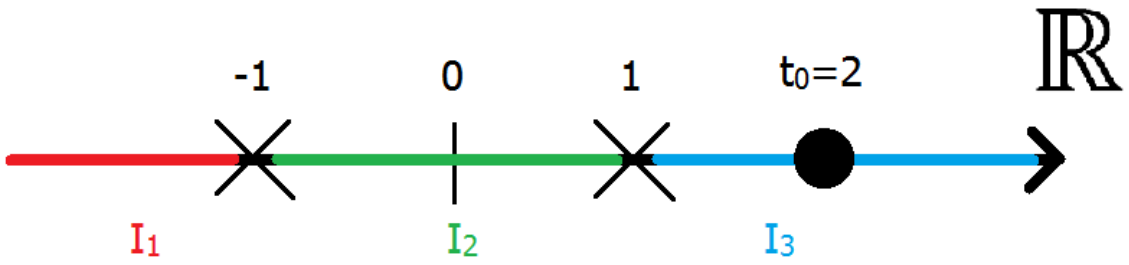


Figure 3: The candidate intervals for which theorem 2.1 is valid, along with the value of t_0 given in the example problem.

We can divide through this equation by $t^2 - 1$ so that we have the form indicated in theorem 2.1. That is, we have

$$y' - \frac{t}{t^2-1}y = \frac{\sin t}{t^2-1},$$

where $p(t) = \frac{-t}{t^2-1}$ and $q(t) = \frac{\sin t}{t^2-1}$. We cannot use our methods from before to solve this equation explicitly in terms of elementary functions (since an integral we would need to compute would be intractible - see for yourself!).

We need to first establish the intervals on which both p **and** q are continuous. Since p and q are both quotients of continuous functions, we need only consider where the denominators vanish. That is, we are unable to apply theorem 2.1 when $t_0 = \pm 1$, as these are the locations where the denominators of p and q are zero. Hence, both p and q are continuous away from these two points.

This decomposes the possible intervals of solutions into three candidates: $I_1 = (-\infty, -1)$, $I_2 = (-1, 1)$, and $I_3 = (1, \infty)$ (see Figure 3). Since we take $t_0 = 2$, we can apply theorem 2.1 in order to assert that the initial value problem indeed has a solution on $I_3 = (1, \infty)$ and that this solution is unique. In this case, I_3 denotes the maximal interval over which the solution is guaranteed to exist - and **we didn't have to explicitly solve the equation** to know this!

Example 2.6. Now consider the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}; \quad y(0) = -1.$$

We cannot apply theorem 2.1 since we cannot put the equation into the required form (a non-linear $y\frac{dy}{dx}$ term would appear). Hence, we must appeal to theorem 2.2 for existence. Observe that

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{3x^2 + 4x + 2}{2(y-1)^2}.$$

The only points of discontinuity lie on the line $y = 1$. Since our initial condition is away from these discontinuities, we know that both f and $\frac{\partial f}{\partial y}$ are continuous at $(0, -1)$. Even though the functions are both continuous in the semi-infinite rectangle $\mathbb{R} \times (-\infty, 1)$, the theorem only guarantees that the solution exists for some rectangle contained in this larger one. We don't know how big this solution space is, but we at least know that the solution exists in it and is unique there.

If our initial condition had been on the point $(y,t) = (1,0)$, then the theorem cannot tell us anything about the existence of solutions to this differential equation. Recall that we solved this very initial value problem in section 2.2, and our solution was

$$y(x) = 1 \pm \sqrt{x^3 + 2x^2 + 2x}.$$

That is, there are **two** solutions of the differential equation satisfying the same initial condition. Because f and $\frac{\partial f}{\partial y}$ are discontinuous, we don't get the luxury of uniqueness anymore.

Observe that just because the theorem's conditions fail doesn't mean that there is no solution to the initial value problem. Namely, even if f or $\frac{\partial f}{\partial y}$ or both are discontinuous, **this is not enough to dictate whether a solution does or does not exist**! On the worksheet you will see that the same can be said for the uniqueness of a solution - that just having one of the conditions of the theorem fail is not enough to say that a solution, if it even exists, is unique.

It is worth noting that the interval I in theorem 2.1 can be chosen to be as large as possible. That is, we take it to be the largest interval in which both p and q are defined **and** which contains t_0; and this turns out to be the solution space for the solution function. This is in stark contrast with theorem 2.2, which only guarantees that there is **some** interval in the maximal rectangle about $(y(t_0), t_0)$ in which the solution may be defined. This is illustrated in the following example:

Example 2.7. Consider the initial value problem

$$y' = y^2; \quad y(0) = 1.$$

This is non-linear, so our only choice is to use theorem 2.2. So we check that $f(y,t) = y^2$ is continuous over $\mathbb{R}^2$ and so is $\frac{\partial f}{\partial y} = 2y$. However, upon solving this equation, we find that separation of variables yields

$$y(t) = -\frac{1}{t-1}.$$

This is indeed a solution to the initial value problem, but **its domain is not the entire real line**! It is discontinuous at $t = 1$ and as such can only be defined for the interval $(-\infty, 1)$. There was nothing in the differential equation indicating that $t = 1$ was anything to watch out for, and yet there is a singularity in the solution all the same. Notice also that our general solution for $y(0) = y_0$ gives us

$$y(t) = \frac{y_0}{1 - y_0 t}.$$

Hence, this solution cannot be defined on the entire real line for **any** initial condition! This shows us that solutions of nonlinear equations may yield unexpected features, which again contrasts with linear equation in that all the information is encoded in the differential equation right from the beginning.

Remark 2.3. 1. *General solution.* Notice that our general solution for the previous example is given by

$$y = -\frac{1}{t+c},$$

wherein we take $c = -\frac{1}{y_0}$ for the initial condition $y(0) = y_0$. Notice that $y(t) = 0$ is clearly a solution to the differential equation, and yet there is no value of c that yields this solution! This is yet another feature of nonlinear equations: **General solutions depending on a constant of integration may not encompass all possible solutions**!

2. *Implicit solution.* Recall that we have an explicit solution for a general first-order linear equation, even if it is in terms of an integral. Unfortunately, the best we can hope for in nonlinear equations is a solutions of the form

$$F(t, y) = 0.$$

That is, we can't construct $y = y(t)$ in general. The solution F is then called a *first integral* for the system. Usually this pertains to a conserved quantity of the system (i.e. a value that stays constant along solutions). A common example of such a phenomenon is total mechanical energy in a closed kinematic system.

Even if we can show that a solution exists for an initial value problem, it is often impossible to find an implicit solution - even in linear equations! This is why we are mostly interested in just determining if a solution exists, since in general we can almost never find an equation detailing the solution of the initial value problem.

3. *Graphical or numerical construction of integral curves.* In light of the points above, we often appeal to computers to simulate solutions. We've seen that we can sometimes find slope fields to determine some qualitative behavior of the system. More such qualitative methods are discussed in the next section. Numerical solutions can be constructed using many different methods. Unsurprisingly, most of them involve Taylor series to approximate solutions. We will briefly introduce basic numerical methods a little later to illustrate how we treat differential equations in practice.

Before we end this section, there is an important fact for us to recognize. If our differential equation satisfies the existence and uniqueness theorem's conditions, we can conclude that no two solutions beginning at different initial values for the same time can ever cross paths. We quickly summarize this as **no two distinct solutions can ever intersect**. This allows us to create what is called a *foliation* of the solution space. We've seen foliations before in the sense that the "antiderivatives partition the solution space" (i.e. solutions never intersect, and they cover the solution space completely). This is a natural way for us to partition a surface, and it is actually used in many different applications of dynamical systems.

It should be noted that this is an artifact of restricting the freedom of solutions to one dimension. Such phenomena are not so easily observed in higher-order differential equations (but are still present). For example, $y(t) = \pm \sin t$ both solve the second-order IVP $\{y'' + y = 0,\ y(0) = 0\}$; however, these two solutions intersect one another at infinitely many points!

2.5 Autonomous Equations and Population Dynamics

Let's first recall a definition from section 1.3.

Definition 2.4. An ODE that is independent of its independent variable is called *autonomous*.

We've seen a few examples of this type of equation. In particular, we have seen first-order equations in the form

$$\frac{dy}{dt} = f(y).$$

They arise in a variety of applications, and are particularly useful in the study of population dynamics. These equations are always separable, and they allow us to very easily evaluate the qualitative behavior of a system. Let's see this in a few examples.

Example 2.8. *Exponential growth.* One of the simplest kinds of autonomous equations was studied by Malthus in the late 18^{th} century. Consider the initial value problem

$$\frac{dy}{dt} = ry; \quad y(0) = y_0,$$

where r is a real number. This equation models a population whose growth is only determined by how many member of the population there are at any given moment. Such a model is named after Malthus, and a population growth rate exhibiting this behavior is said to be *Malthusian*. We call the parameter r the *growth* or *decay parameter*. A common occurrence of such a model is a colony of bacteria cultured in a labaoratory. Such a model has been shown to be reasonably accurate for short periods of time. We can separate variables and solve this equation handily as

$$y(t) = y_0 e^{rt}.$$

Hence, if $r > 0$, the population grows without bound; and if $r < 0$, the population decays to extinction. For the $r > 0$ case, we know that a population obviously cannot grow unbounded. There are limiting factors like living space, a food source, or various other resources that will impede the growth of the population. We consider such limitations now.

Example 2.9. *Logistic growth.* With the obvious physical limitations of Malthusian growth, we try to mathematically force certain conditions in the model. For instance, there can be no growth if there is no population. We model this with

$$\frac{dy}{dt} = yf(y).$$

Hence, $\frac{dy}{dt} = 0$ if $y = 0$.

We also know from observation that there is a *carrying capacity* K of the population. This is the maximum number of organisms the population can support with the

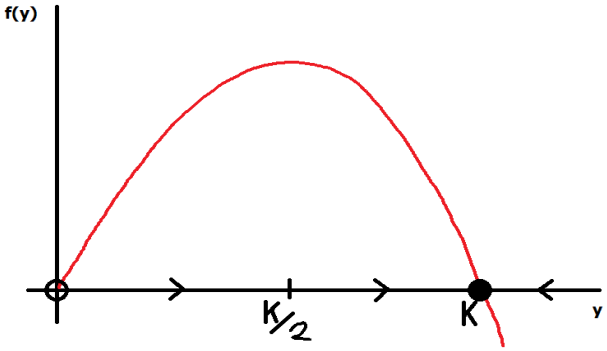


Figure 4: This shows the graphical method for the logistic growth model. Stable equilibria are usually solid dots, and unstable equilibria are usually open dots.

limited space and resources on hand. The population should, hence, not grow when the population is at this size. We model this by

$$\frac{dy}{dt} = \left(1 - \frac{y}{K}\right) g(y).$$

Combining these two results, we get the *logistic growth model*

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$

We call r the *intrinsic growth rate* of the population. Observe that this is exactly Malthusian growth for small values of y. The astute reader will also notice that we could have just as easily used the factor $\frac{y}{K} - 1$ in the logistic model. We will see shortly why we chose the other factor.

We mentioned earlier that there are certain "easy" ways to work with autonomous equations. In particular, we are going to use the logistic growth model to illustrate the *graphical method*.

Example 2.10. Consider the logistic model

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$

We can treat the right-hand-side of this equation as the function $f(y)$ (as we do with all autonomous equations). We can then graph this function against y as in Figure 4.

We aim to find all *equilibrium solutions* of this model. These are exactly the constant functions y^* such that

$$\frac{dy}{dt} = f(y^*) = 0.$$

These equilibria don't change as time progresses - they are constant. We can easily see in the figure that we have two equilibrium solutions $y_1^*(t) = 0$ and $y_2^*(t) = K$. What we want to do now is determine if the system wants to tend toward these solutions (*stable*) or away from them (*unstable*).

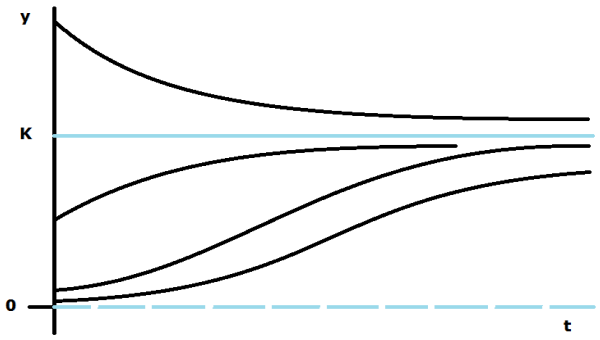


Figure 5: Solutions for varying values of $y(0) = y_0$. Notice that $y(t) = K$ is a stable equilibrium and $y(t) = 0$ is an unstable equilibrium.

To determine the stability of an equilibrium, we merely look at the figure and determine the direction that y "wants" to travel at any given value of y. The arrow in the figure are determined by evaluating whether the function f is above the y-axis (right/increasing arrows) or below the y-axis (left/decreasing arrows). We look at the arrow directions to either side of the equilibrium. If the arrows are both pointing toward the equilibrium, then it is stable and denoted by a solid dot. If the arrows are both pointing away, it is unstable and denoted by an open dot. There is another case that isn't on this figure. There are such equilibria that are stable on one side but unstable on the other. Such equilibria are called *semi-stable*. They are still considered to be unstable equilibria. Their representations are half-filled dots with the filled side associated with the stable side of the equilibrium.

The figure does not, however, graph the solution of the equation. These are given by a completely different graph shown in Figure 5.

Notice in the figure that solutions not beginning on the equilibria tend away from the unstable equilibrium $y(t) = 0$ and toward the stable equilibrium $y(t) = K$. Also observe that the inflection points for solutions beginning at $0 < y(0) < K$ all occur for the same value of $y = \frac{K}{2}$. There is much more analysis we can do for this equation, but it is not too enlightening at this point in time.

Remark 2.4. There is a final, quick remark about stable equilibria that we should make. Notice that solutions that tend toward the stable equilibrium **never reach the equilibrium**! The equilibrium acts as an asymptote since, by the existence and uniqueness theorem, no two disctinct solutions can ever cross paths. For this reason, we sometimes call the equilibria *asymptotically stable* or *unstable*.

We now return to the equation for the logistic model. We mentioned that we could have chosen either of the factors $\left(1 - \frac{y}{K}\right)$ or $\left(\frac{y}{K} - 1\right)$. The logistic model uses the former so as to force the equilibrium $y(t) = K$ to be a stable equilibrium. It is easily seen that the use of the other factor amounts to flipping over the parabola in Figure 4. This new figure actually switches the stabilities of the two equilibria. Now we see that if $0 < y_0 < K$, then our population will decay to extinction. However, past the threshold $y_0 \geq K$, we see

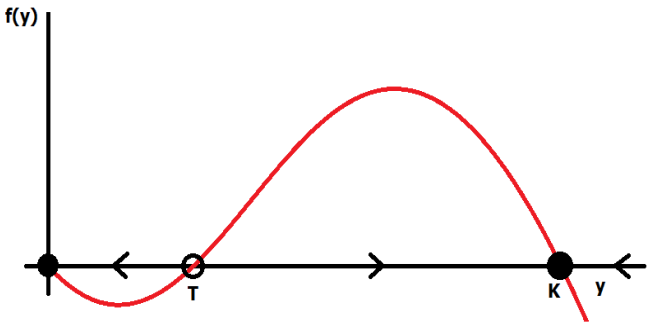


Figure 6: This is the graphical method for the logistic model with a critical threshold.

that the population either stabilizes at K or actually begins to thrive. We now call K a *critical threshold* for the population. Such a phenomenon is usually attributed to there not being enough individuals to propagate the population. Once past this threshold, the population will then be able to grow in the Malthusian sense. Again, we know that Malthusian growth is atypical of certain populations. With this in mind we implement yet another model.

We wish to construct a model that exhibits Malthusian decay near extinction, has an environmental carrying capacity K, and has a critical threshold T. So we begin simply by multiplying all of these factors together:

$$\frac{dy}{dt} = -ry\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right).$$

We can qualitatively see what this autonomous equation behaves like if we use the graphical method illustrated in Figure 6.

We challenge the reader to construct the solution graphs as we did in the logistic model above.

We end this section by remarking that there are many different models for population dynamics. We have only scratched the surface of their application. For more on population dynamics, I suggest the following two texts as a starting point:

Essential Mathematical Biology, N. Britton
Mathematical Methods in Biology, L. Edelstein-Keshet
Mathematical Biology, Volume I, J. D. Murray.

All of these texts are a wonderful exposition of mathematical methods in the biological sciences, and none of them require much more background than completion of this course.

2.7 Numerical Approximations: Euler's Method

[Note: We are skipping section 2.6 of the textbook. It uses methods of MAT 21D, and we have technically already seen in that class how to deal with such equations. The curious student can certainly read this section, but it will not be tested in this class.]

We now take our first look at numerical methods in differential equations. Recall that for any function, we may find a tangent line approximation at any point where

the function is differentiable. That is, we are given a function $y = f(x)$ and a point $(x_0, f(x_0))$, and we can find a linear approximation to the function by using the equation

$$y(x) \sim y'(x_0)(x - x_0) + y(x_0).$$

This is the key for Euler's method.

We apply this concept to the context of the initial value problem

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0.$$

That is, we wish to construct an approximation of the solution $y(t)$ to the initial value problem using a linear scheme. Let's see how to do this first by using an example system.

Example 2.11. Consider the initial value problem

$$\frac{dy}{dt} = 3 - 2t - \frac{1}{2}y; \quad y(0) = 1.$$

We know how to solve this equation, and we can find it to be

$$y(t) = 14 - 4t - 13e^{-\frac{t}{2}}.$$

But let's see how we can approximate this solution with only the differential equation and the starting point $(0, 1)$.

1. We can create the tanget line to the solution at the point $(0, 1)$ by using the formula

   $$y(t) \approx y'(0)(t - 0) + y(0).$$

   But we know from the differential equation that $y'(0) = f(0, 1)$. Hence, we can "take a step" forward in time to see where this line says the solution should be at a later time. Let's take $\Delta t = h = \frac{1}{5}$. Define $t_1 = t_0 + h$. Then we observe that the tangent line approximates

   $$y\left(0 + \frac{1}{5}\right) \approx \left(3 - 2 * \frac{1}{5} - \frac{1}{2} * 0\right)\left(\frac{1}{5} - 0\right) + 1 = \frac{3}{2}.$$

   So we define $y_1 = \frac{3}{2}$.

2. We create the tangent line to the solution now at the point $\left(0 + h, \frac{3}{2}\right)$. The tangent line approximation now looks like

   $$y(t) \approx y'(h)(t - h) + y_1.$$

   The differential equation tells us that $y'(h) = f(0+h, y_1)$ so that we can approximate

   $$y(0 + 2h) \approx 1.13 + 1.85 * 0.4 = 1.87.$$

   We define $y_2 = 1.87$ and take $t_2 = 0 + 2 * h = \frac{2}{5}$.

3. We iterate this process by using the equations

   $$t_n = t_0 + nh \quad \text{and} \quad y_n = y_{n-1} + f(t_{n-1}, y_{n-1})h$$

   for all $n \geq 1$. This scheme is called *Euler's method*. We can string together as many approximation points as we'd like in order to construct an approximation solution to the differential equation. We see how the approximation compares with the actual solution in Figure 7.

t	Exact	Euler with $h = 0.2$	Tangent line
0.0	1.00000	1.00000	$y = 1 + 2.5t$
0.2	1.43711	1.50000	$y = 1.13 + 1.85t$
0.4	1.75650	1.87000	$y = 1.364 + 1.265t$
0.6	1.96936	2.12300	$y = 1.6799 + 0.7385t$
0.8	2.08584	2.27070	$y = 2.05898 + 0.26465t$
1.0	2.11510	2.32363	

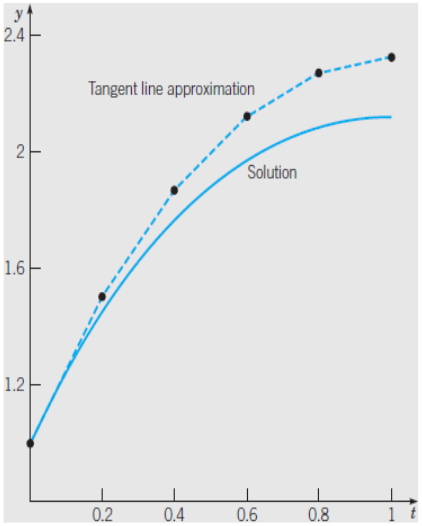


Figure 7: This illustrates the first five iterations of Euler's method for the differential equation in the example. Photo credit to the textbook.

This approximation seems to work pretty well at first, but then it begins to veer away. To remedy this, we can make our time step $\Delta t = h$ smaller. If we use the same initial value problem as before but change the step size, we can see how much better this makes the approximation in the table below:

t	Exact	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.0	1.0	1.0	1.0	1.0	1.0
1.0	2.1151	2.2164	2.1651	2.1399	2.1250
2.0	1.2176	1.3397	1.2780	1.2476	1.2295
3.0	-0.9007	-0.7903	-0.8459	-0.8734	-0.8898
4.0	-3.7954	-3.6707	-3.7152	-3.7373	-3.7506
5.0	-7.0671	-7.0003	-7.0337	-7.0504	-7.0604

Remark 2.5. There are a few noteworthy remarks about Euler's method:

1. The approximate solutions are really only reasonable when the time step $\Delta t = h$ is small. This means that we need to compute **a lot** of approximations in order to know how the solution is behaving. Usually, if we are just looking for qualitative behavior of the system, we assess the slope field. However, when we need to know the exact numbers and can't compute the solution explicitly, we need to implement a numerical method such as this to get them. This comes at the price of having to do a massive amount of calculations.

2. The smaller our time step $\Delta t = h$ is, the better our approximations will become. However, this comes with a bit of a trade-off. If we start at the same initial point with a time step of h_1, and we wish to use a smaller time step $h_2 < h_1$, we will need more steps using h_2 than h_1 to get to the same value of t. In practice, this means orders of magnitude greater computations that are necessary for not a lot more accuracy. In numerical analysis in general, there is a sort of balance between accuracy and computation time. In fact, there are ways for us to compute the optimal time step that will get us accurate measurements in a reasonable amount of time.

3. The approximate solution will inherently get worse the farther away from the initial point we wish to approximate. This is exactly the local behavior of the derivative shining through. This tells us that we will get very accurate approximations the closer we are to the initial value, and our approximations will only get worse the farther from it we want to go. This can be helped along by using a smaller h, but this comes at the price of having to expend even more computing power to deal with the massive number of approximations.

4. There are cases where Euler's method (and lots of other numerical methods) will break down. There is a worksheet problem that addresses exactly this, and we will hence explore it in a tiny bit of detail. We won't ellaborate on this fact, but we wish to at least address the fact that Euler's method may break down and give erroneous results for certain functions f and evaluations of their approximations.

For further study in numerical methods and their applications to differential equations and mathematical modeling, I suggest the student take MAT 128A.

2.8 The Existence and Uniqueness Theorem

We discussed this theorem at length in section 2.4 for both the linear and general cases. We shall restate it once more here:

Theorem 2.3. If f and $\frac{\partial f}{\partial y}$ are both continuous in a rectangle R: $|t| \leq a, |y| \leq b$, then there is some interval $|t| < h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem

$$\frac{dy}{dt} = f(t, y); \quad y(0) = 0.$$

Notice that we can always shift our initial value problem so that $t_0 = 0$ and $y_0 = 0$ by using a suitable change of variables. Our goal for this section is to illustrate a proof of this theorem. This is not the only proof of this theorem, but it works well enough and is accessible to students in this class. We break this process down into several steps.

1. We first write the differential equation a little differently. Just as it is possible to write down an equation in terms of derivatives, we can just as easily pose the equation in terms of integrals. In fact, we can write

$$\phi(t) = \int_0^t f(s, \phi(s))\, ds)$$

where we have made use of the initial condition $(t_0, y_0) = (0, 0)$ to take care of the stray $\phi(0)$. Such an equation is called an *integral equation*. In no way have we solved the differential equation; but rather we have given ourselves a new framing of the initial value problem, and we can now apply other methods of analysis to solve this problem. In this new context, we are looking for a function $\phi(t)$ that solves equation 1.

2. We are now going to implement the *method of successive approximations*. That is, we are going to construct a sequence (ϕ_n) of functions (each of which satisfies the initial condition!) whose limit is ideally going to solve the initial value problem.

   We need an initial "seed" ϕ_0 that is going to begin our sequence. Typically, we take $\phi_0(t) = 0$ because it is smooth and quite possibly the simplest function we could use for ϕ_0. The way we are going to define the rest of the elements of the sequence is with a recursion:

   $$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds.$$

   Notice that we indeed get $\phi_n(0) = 0$ for every element of the sequence. Our claim is that the limit of this sequence is going to solve the initial value problem in theorem 2.3.

There are four questions whose answers need to be established in the proof of theorem 2.3:

1. Do each of the ϕ_n equations actually exist, or does the iteration breakdown somewhere?
2. Does the sequence (ϕ_n) converge at all?
3. What sorts of properties does the limit function $\phi(t)$ have? In particular, does it solve equation 1?
4. Is this the only solution to equation 1?

We show how this iteration process works with an example:

Example 2.12. We are given the initial value problem

$$y' = 2t(y + 1); \quad y(0) = 0.$$

We set up the integral equation as

$$\phi(t) = \int_0^t 2s(1 + \phi(s))\, ds.$$

Define $\phi_0(t) = 0$ and proceed to calculate

$$\phi_1(t) = \int_0^t 2s(1 + 0)\, ds = \int_0^t 2s\, ds = t^2.$$

Using $\phi_1(t)$, we construct

$$\phi_2(t) = \int_0^t 2s(1 + s^2)\, ds = \int_0^t 2s + 2s^3\, ds = t^2 + \frac{1}{2}t^4.$$

We do this once more and then look for a pattern in the functions $\phi_n(t)$:

$$\phi_3(t) = \int_0^t 2s(1 + s^2 + \frac{1}{2}s^4)\, ds = \int_0^t 2s + 2s^3 + s^5\, ds = t^2 + \frac{1}{2}t^4 + \frac{1}{2 \cdot 3}t^6.$$

We notice that there seems to be a pattern emerging in the solutions. In fact, we can suggest that

$$\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!}.$$

We can prove this by induction, but this class doesn't mind letting the smaller details fly under the radar. So we just assert that the limit $\phi(t)$ of the sequence $(\phi_n(t))$ is given by

$$\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = -1 + \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} = -1 + e^{t^2}.$$

Yes, the reader should have recognized that this was indeed the Taylor expansion for e^{t^2} at $t = 0$. Notice that $\phi(t)$ is exactly the solution we would have gotten if we had used separation of variables from the beginning. This is validation (but not proof!) that the method is successful at **some** level.

To show that some other solution cannot exist for this equation, suppose that we had some other $\psi(t)$ derived from the sequence $(\psi_n(t))$ whose initial function $\psi_0(t) \neq 0$. We can look at their difference as

$$\phi(t) - \psi(t) = \int_0^t 2s(\phi(s) - \psi(s))\, ds.$$

Our goal is to show that $|\phi(t) - \psi(t)| = 0$. This would show that $\phi(t) = \psi(t)$ for all t so that they are the same function. So consider bounding this difference by

$$|\phi(t) - \psi(t)| = \left| \int_0^t 2s(\phi(s) - \psi(s))\, ds \right| \leq \int_0^t 2s|\phi(s) - \psi(s)|\, ds.$$

If we consider only values of t such that $0 \leq |t| \leq \frac{A}{2}$ (so that $|2t| \leq A$) for some arbitrary value $A > 0$, then we can further bound the difference by

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)\, ds.$$

Define the function

$$U(t) = \int_0^t |\phi(s) - \psi(s)|\, ds$$

for convenience. Then we can see that U satisfies the properties

$$U(0) = 0 \quad \text{and} \quad U(t) \geq 0$$

for all $t \geq 0$. In fact, we know that U is differentiable so that

$$U'(t) = |\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)|\, ds = AU(t)$$

for all $0 \leq t \leq \frac{A}{2}$.

We can substract the $AU(t)$ to the left side and multiply both sides of this inequality by e^{-At} so that

$$\frac{d}{dt}\left(e^{-At}U(t)\right) \leq 0$$

for $0 \leq t \leq \frac{A}{2}$. We can integrate this inequality and use the initial value $U(0) = 0$ so that

$$e^{-At}U(t) \leq 0 \Leftrightarrow U(t) \leq 0.$$

But we have $U(t) \geq 0$ for all $t \geq 0$. Because A was arbitrarily chosen, we know that we must have the relation $0 \leq U(t) \leq 0$ for all $t \geq 0$. Repeating this style of argument for $-t$, we get the same relation $0 \leq U(t) \leq 0$ for all $t \leq 0$. Hence, we know that $U(t) = 0$. Then the derivative is also zero so that $U'(t) = |\phi(t) - \psi(t)| = 0$, and we are done! Phew!!

We have by no means proven the theorem! This example merely illustrates the way in which one would go about implementing the proof in practice. We explain briefly how to answer the general questions posed before the example.

1. Regarding the existence of each ϕ_n, we are essentially looking to see if each previous iterate is continuous (this would give us integrability). The problem in general is that the functions f and $\frac{\partial f}{\partial y}$ are only assumed to be continuous in a small rectangle. Some iterate may contain points that lie outside of this rectangle. Hence, we are not guaranteed the existence of the next iterate.

   To remedy this, we may restrict t to a smaller interval (hence, the $h \leq a$ in the theorem). Since a continuous function on a closed interval is bounded, we can then assert the existence of a bound M so that

   $$|f(t, y)| \leq M$$

   in the original rectangle (and hence in the smaller one). This effectively limits how large the slope of the solution can be (since f is equal to y' after all). This bound (after we restrict t enough) will always guarantee us the existence of our ϕ_n iterates.

2. Convergence is yet another complicated matter. We actually employ a trick to ensure the convergence of our sequence. We observe that

   $$\phi_n(t) = \phi_1(t) + (\phi_2(t) - \phi_1(t)) + \cdots + (\phi_n(t) - \phi_{n-1}(t))$$

   so that we can write $\phi(t)$ as the series

   $$\phi(t) = \phi_1(t) + \sum_{n=1}^{\infty}(\phi_{n+1}(t) - \phi_n(t)).$$

   The trick to bounding this lies in something called the Monotone Convergence Theorem. This is something that is learned in a first course in real analysis and can be understood simply as "if the series is bounded above and all the partial sums are increasing, then the series must converge." We won't wrestle with these details in this class.

3. We next address what sorts of properties we get to conclude about the limit function $\phi(t)$. There are some problems in the worksheet for this section that bursts ones intuitive notions of what to expect from limit functions. For example, even if every function in the sequence is continuous, this is not enough to ensure that the limit function is continuous.

   To ensure that we get certain properties of $\phi(t)$, we need to establish a certain kind of convergence of the sequence. It is called *uniform convergence*, and it is more powerful that the usual notion of pointwise convergence. We don't discuss the details here, but there are incredibly important to solidify in order to complete the proof.

4. The question of uniqueness is handled very similarly to how we showed it in the example. Again, we need to be careful about the rectangle where f and $\frac{\partial f}{\partial y}$ are continuous; but this is easily dealt with in the same manner as in part (a) where we restrict the interval of t.

If the student is interested in the basic methods of real analysis discussed above, I suggest taking a course in "Real Analysis" or "Advanced Calculus".

2.9 First-Order Difference Equations*

We take a brief interlude in continuous differential equations and talk about their discrete counterparts: Difference equations. In practice, it is actually quite common for us to only have a discrete measurement of time. For example, instead of making a continuous measurement like setting up an instrument somewhere and having it transmit a live stream of data, we may only have the time or resources to make one measurement every week, month, or year. The latter example is what we mean by a discrete time variable. As is frequently the case, we can describe each measurement at a certain time by a recursion formula. That is, we create the following definition:

Definition 2.5. A *difference equation* is a recursion relation

$$y_{n+1} = f(n, y_0, y_1, \ldots, y_n)$$

for $n = 0, 1, 2, \ldots$. A difference equation is of *first order* if the next iterate only depends on the previous one. A difference equation is *linear* if f is linear in each y_k.

We are going to work only with first-order difference equations. These take on the general form

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \ldots.$$

We can easily add an initial condition to these as y_0, just as one would expect. In particular, we are going to concern ourselves with the autonomous version of these:

$$y_{n+1} = f(y_n), \quad n = 0, 1, 2, \ldots$$

A *solution* to a difference equation is a sequence y_n that satisfies the difference equation. Solving difference equations in general is often very difficult. We describe a few relatively simply examples below and observe how difficult the nonlinear case becomes even for first-order difference equations. Let's take a look at a few examples.

Example 2.13. *Growth models*

1. Just as the continuous case, we have an exponential growth model. Let's consider

$$y_{n+1} = \rho_n y_n, \quad n = 0, 1, 2, \ldots,$$

where $\rho_n = \rho(n)$ defines the growth parameter. Observe that we have

$$\begin{aligned} y_1 &= \rho_0 y_0 \\ y_2 &= \rho_1 y_1 = \rho_1 \rho_0 y_0 \\ &\vdots \\ y_n &= \rho_{n-1} \cdots \rho_1 \rho_0 y_0. \end{aligned}$$

This sequence solves the difference equation for any initial value y_0. Notice that if $\rho_n = \rho$ is a constant, then we have

$$y_n = \rho^n y_0.$$

Observe the similarity to the continuous case.

We are concerned with the limiting behavior of this solution. There are a few cases depending on ρ. If $|\rho| < 1$ or y_0, then the sequence certainly converges to 0 in the limit. If $\rho = 1$, then we have $y_n = y_0$ at each iteration so that the limit is y_0. However, the limit doesn't exist for any other case. That is, the limit is asymptotically stable for $|\rho| < 1$ and unstable for $|\rho| > 1$.

2. Consider modifying the previous example so that we have

$$y_{n+1} = \rho y_n + b_n, \quad n = 0, 1, 2, \ldots,$$

where ρ is constant and $b_n = b(n)$. This can be thought to model an intrinsic growth with an immigration or emmigration factor represented by the b_n. In a similar fashion as before, we write out the first few iterates and look for a pattern:

$$\begin{aligned} y_1 &= \rho y_0 + b_0 \\ y_2 &= \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1 \\ &\vdots \\ y_n &= \rho^n y_0 + \sum_{k=0}^{n-1} \rho^{n-1-k} b_k \end{aligned}$$

and this sequence solves the difference equation for a given y_0.

We can again consider what happens when $b_n = b$ is just a constant. The solution then becomes

$$y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \cdots + \rho^{n-1})b.$$

Notice that we can write the sum of powers of ρ as

$$1 + \rho + \rho^2 + \cdots + \rho^{n-1} = \frac{1 - \rho^n}{1 - \rho}.$$

Hence, we have

$$y_n = \rho^n y_0 + \frac{1-\rho^n}{1-\rho} b.$$

The limiting behavior of this solution is again found my elementary Calculus methods. Namely, if $|\rho| < 1$, then the ρ^n term tends to zero so that we are left with $\frac{b}{1-\rho}$. Again, there is no limit for $\rho = -1$ or $|\rho| > 1$. However, we look at the case for $\rho = 1$. Unfortunately, our shorthand for the sum of powers of ρ is no longer valid. So we just go back to what our sequence originally looked like. That is, we look at

$$y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \cdots + \rho^{n-1})b.$$

Since $\rho = 1$, we can simplify this to

$$y_n = y_0 + nb,$$

which also diverges in the limit. So the only asymptotically stable solutions occur for $|\rho| < 1$.

Let's take a look at one more example from the financial sciences.

Example 2.14. Suppose we had to take out a loan of \$10,000 from the bank at an annual interest rate of 12%. We would like to pay off this loan relatively quickly, so we set the goal of paying it off in four years. We now need to know how much we have to pay each month in order to reach this goal.

We can use exactly the equation from the previous example to model this scenario. How? We take $r = .01$ to be our monthly interest rate so that we should make $\rho = 1 + r$, and we should take b to be the amount of money we pay off every month. Observe that we should make $b < 0$ so as to account for decreasing the amount due on the loan. Applying the formula

$$y_n = \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}$$

taking $b < 0$ and $\rho = 1 + r$, we can see that we want to solve for b at $n = 4 * 12 = 48$ in the equation

$$y_n = (1.01)^n (10,000 + 100b) - 100b$$

and setting $y_{48} = 0$. Our work in the previous example shows us that

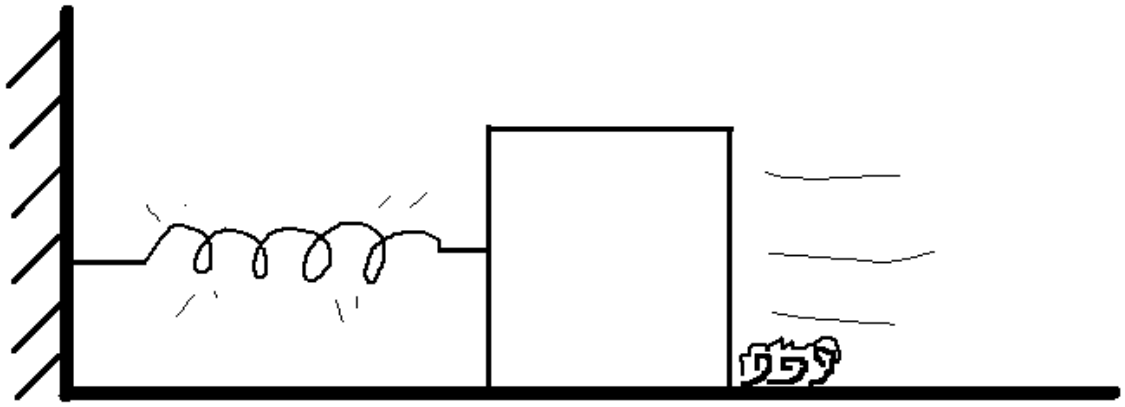
$$b = -100 \frac{(1.01)^{48}}{(1.01)^{48} - 1} = -263.34.$$

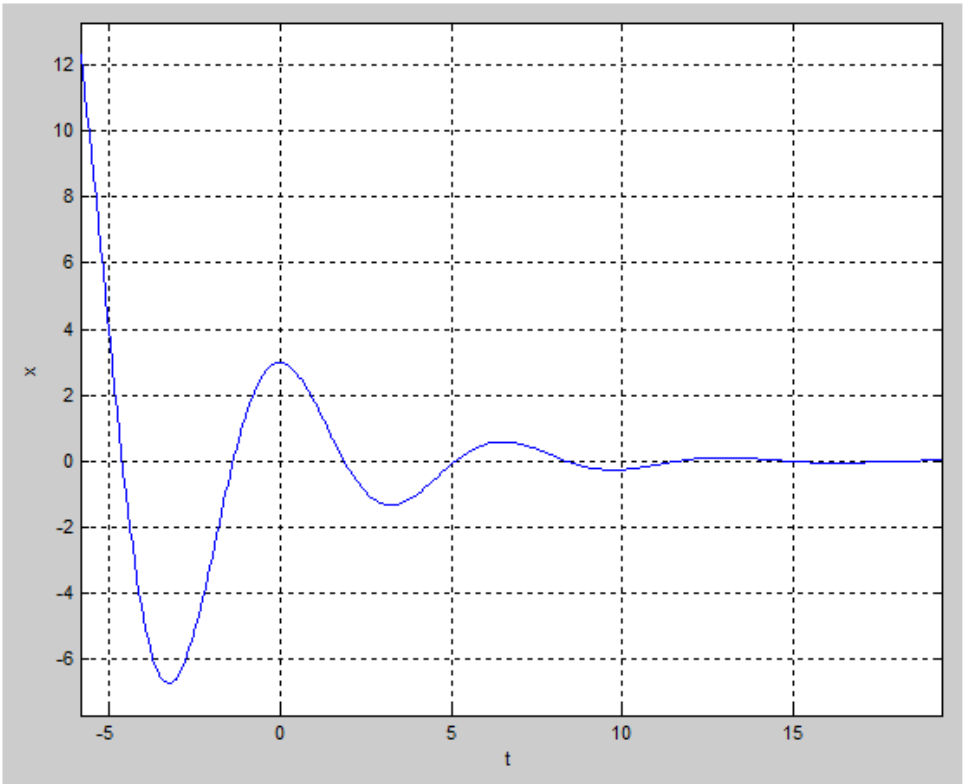
Hence, we would need to pay off \$263.34 every month for the next four years in order to fully pay off the loan by then. If this is the case, then we can see that $\$263.34 * 48 = \$12,640.32$ is the total amount that we would have to pay back to the bank. This is \$2,640.32 in interest.

We end this section without covering any nonlinear difference equations. They will show up on a worksheet in the form of the *logistic equation*. This is a simple nonlinear difference equation which has an incredibly complicated dynamics. The most notable characteristic of this system is that there are certain values of the "growth" parameter that evoke chaotic behavior. Chaos has only seriously been studied mathematically for the past 40 to 50 years. In those years there have been many interesting and surprising discoveries. Among these discoveries are the creatures called *strange attractors*. A strange attractor is essentially a stable equilibrium, but it doesn't have an integral dimension. Equilibria can lie on points (zero dimension), lines (one dimension), and hypersurfaces (two dimensions and higher); but strange attractors have dimension sitting between integer values. In particular, the equilibrium sets are actually *fractals*! That is, fractional dimensional entities.

This is an intensely fascinating subject that has boundless applications. If the student is interested in pursuing such phenomena, then they should look into a course in "Dynamical Systems" or "Nonlinear ODE".

3 Second-Order Linear Equations





3.1 Homogeneous Equations with Constant Coefficients

We have finally beaten first-order equations to death and are ready to move on to second-order equations! (Cue thunderous applause) Second-order differential equations are even more common in the physical sciences than the first-order equations. They are fundamental to the development of many principles in fluid mechanics, heat conduction, wave motion, and Maxwell's equations of electromagnetism. We will investigate some of these phenomena later in the chapter, but for now we need to develop our basic tools.

The equations we are turning to now are of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$

Granted: This is not the general form of a second-order equation, but it will suffice for our studies. Again, the differential equation is *linear* if the function f is linear in both y **and** $\frac{\partial f}{\partial y}$, and it is *nonlinear* otherwise. Our concerns in this class are with linear equations. We can put linear equations in the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

As with first-order systems, if the lone function $g(t)$ is identically zero, then we call the differential equation *homogeneous*. If $g(t) \neq 0$, then it is called *nonhomogeneous*. We will quickly see that homogeneous equations are vastly easier to solve than their nonhomogeneous counterparts. Moreover, the homogeneous solutions are a critical component of nonhomogeneous solutions.

Something that does change from first-order to second-order equations is how we define our initial conditions. We still need to know where the system starts, so we keep the condition $y(t_0) = y_0$. However, we will shortly see that this is not enough information to solve the ODE uniquely. Hence, we require a second initial condition, but this one is tied to the derivative $y'(t_0) = y_0'$. This can be thought of not as an initial displacement but rather an initial velocity.

For our beginning purposes, we wish to consider second-order linear homogeneous ODE with constant coefficients. That is, we should first look at systems of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are real constants.

Just by inspection of the above equation, we try to think of a function who cancels out its own derivatives by having its derivatives only be multiples of itself. Our thoughts immediately turn to the exponential function $y(t) = e^{rt}$ for any value r. Grabbing a solution (seemingly from nowhere) like this is called *ansatz* - this is German for "approach" or "starting point". In fact, let's look at an example to get a feel for our solution technique here.

Example 3.1. Let's find the general solution for the initial value problem

$$y'' - 4y = 0; \quad y(0) - 3, \quad y'(0) = 4.$$

We try our ansatz solution $y(t) = e^{rt}$. Then we differentiate twice to see that

$$0 = y'' - 4y = r^2e^{rt} - 4e^{rt} = (r^2 - 4)e^{rt}.$$

Because $e^{rt} \neq 0$ for any t, we know that $r^2 - 4 = 0$. The equation

$$r^2 - 4 = 0$$

is called the *characteristic equation* of the system. If we solve this equation for r, we easily see that the only values allowed are $r = \pm 2$. That is, we have two possible solutions for the differential equation:

$$y_1(t) = e^{2t} \quad \text{and} \quad y_2(t) = e^{-2t}.$$

It will be shown soon that because we are dealing with a linear ODE, any linear combination of these two solutions is also a solution. That is, our general solution looks like

$$y(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

To solve the IVP, we need to know this function's derivative. So we compute

$$y'(t) = 2c_1 e^{2t} - 2c_2 e^{-2t}.$$

Then we substitute in our initial conditions so that we have

$$\begin{cases} c_1 + c_2 & = -3 \\ 2c_1 - 2c_2 & = 4. \end{cases}$$

We recall from linear algebra how to solve such a system of two equations in two unknowns. In fact, we see that adding twice the first equation to the second gives us

$$4c_1 = -2 \Leftrightarrow c_1 = -\frac{1}{2}.$$

Substituting this into the first equation gives us

$$-\frac{1}{2} + c_2 = 2 \Leftrightarrow c_2 = -\frac{5}{2}.$$

Hence, the solution to the initial value problem is

$$y(t) = -\frac{1}{2}e^{2t} - \frac{5}{2}e^{-2t}.$$

In general, this is essentially how we solve second-order linear homogeneous ODE with constant coefficients. Let's see the same method above applied to the general equation

$$ay'' + by' + cy = 0.$$

Again, we use our ansatz solution $y(t) = e^{rt}$ so that we get

$$0 = ar^2 e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c).$$

Because $e^{rt} \neq 0$ for any t, we know that we must have

$$ar^2 + br + c = 0$$

for our characteristic equation.

All quadratic equations have exactly two complex-valued solutions (counting multiplicity - and yes, real-valued solutions count as complex-valued with no imaginary part). Call them r_1 and r_2. If we assume that $r_1 \neq r_2$, then we can again assume that our general solution takes the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

We leave it to the reader that if we are given the initial values of $y(t_0) = y_0$ and $y'(t_0) = y_0'$, then we get the coefficients

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0} \quad \text{and} \quad c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}.$$

It is easy to see here why we need r_1 and r_2 to be distinct. We will address the case when $r_1 = r_2$ a bit later. For now, let's try one more example.

Example 3.2. Let's consider the initial value problem

$$9y'' + 18y' - 16y = 0; \quad y(0) = 2, \quad y'(0) = -2.$$

Using our ansatz solution, we get the characteristic equation

$$9r^2 + 18r - 16 = 0.$$

It may or may not be evident that this polynomial is factorable, so we can apply the quadratic formula to solve for r_1 and r_2:

$$r = \frac{-18 \pm \sqrt{324 + 576}}{18} = \frac{-18 \pm 30}{18} = \frac{-3 \pm 5}{3}.$$

Then our general solution looks like

$$y(t) = c_1 e^{\frac{2t}{3}} + c_2 e^{-\frac{8t}{3}}.$$

Using the initial conditions, we solve for c_1 and c_2 as

$$\begin{cases} c_1 + c_2 & = 2 \\ \frac{2}{3}c_1 - \frac{8}{3}c_2 & = -2. \end{cases}$$

We compute here that

$$c_1 = 1 \quad \text{and} \quad c_2 = 1$$

so that the solution to the initial value problem is

$$y(t) = e^{\frac{2t}{3}} + e^{-\frac{8t}{3}}.$$

Remark 3.1. It should be noted here that it is entirely possible for $r = 0$. In this case, an ansatz solution will be constant. It isn't hard to see that a value of $r = 0$ comes from the equation

$$ar^2 + br = 0;$$

and, hence, would have come from the differential equation

$$ay'' + by' = 0,$$

for which it is obvious that any constant function is a solution.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

Before continuing our study of second-order linear homogeneous ODE, we need to establish a few properties about them. We've hinted at a few of these in the previous section, but we should rigorously demonstrate them before pressing forward.

Recall that our general form of such equations is

$$y'' + p(t)y' + q(t)y = 0.$$

We can write this as

$$L[y] = 0,$$

where we employ the use of the *differential operator*

$$L = \frac{d^2}{dx^2} + p(t)\frac{d}{dx} + q(t) = D^2 + p(t)D + q(t).$$

We can think of L as a function the eats a twice-differentiable function and spits out another function. This notation simply serves to compactify our notation (since it is getting rather lengthy).

We would first like to establish our existence and uniqueness theorem since it played such a large and monumental role in our study of first-order equations.

Theorem 3.1. Consider the initial value problem

$$y'' + p(t)y' + q(t) = g(t); \quad \text{and} \quad y(t_0) = y_0, \quad y'(t_0) = y_0',$$

where p, q, and g are continous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

To summarize, this theorem tells us that for linear second-order ODE where the coefficient equations are continuous on an interval containing the initial time t_0:

1. A solution exists for the IVP.
2. There is only one solution to the IVP.
3. The solution exists throughout the **entire** interval I.

These are exactly the same consequences we got for first-order linear ODE in section 2.4. Unfortunately, we aren't so lucky as in first-order linear equations. We don't have a nice, explicit formula for solutions as we did with the integrating factors in section 2.1. Since this formula was essentially the proof for first-order linear equations, we need to find a more general method to prove this result for second-order linear equations. We won't prove this result, but it is still important to recognize its consequences.

Example 3.3. 1. Consider the initial value problem

$$ty'' - \frac{5t^2}{t^2 - 4}y' + (\cos t)y = 1; \quad y(3) = -5, \quad y'(3) = 4.$$

We put this in the standard form

$$y'' - \frac{5t}{t^2-4}y' + \frac{\cos t}{t}y = \frac{1}{t}.$$

Here we take $p(t) = -\frac{5t}{t^2-4}$, $q(t) = \frac{\cos t}{t}$, and $g(t) = \frac{1}{t}$. Observe that p is discontinuous at $t = \pm 2$ and that q and g are discontinuous at $t = 0$. Hence, the intervals I that we can choose are

$$I_1 = (-\infty, -2), \quad I_2 = (-2, 0), \quad I_3 = (0, 2), \quad \text{and } I_4 = (2, \infty).$$

The only one of these that contains the initial $t_0 = 3$ is $I_4 = (2, \infty)$. Hence, the existence and uniqueness theorem guarantees us a unique solution only on the interval $I_3 = (2, \infty)$. The solution may extend beyond this, but we are only guaranteed its existence in I_4.

2. Suppose we are given the general initial value problem

$$y'' + p(t)y' + q(t)y = 0; \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where p and q are both continous on some open interval I that contains t_0.

We quickly notice that $y(t) = 0$ solves this IVP. The existence and uniqueness theorem instantly tells us that this is the **only** solution to this IVP. It is not often that this scenario occurs; but if one can find a solution by inspection and the IVP satisfies the conditions of the existence and uniqueness theorem, then we know that this is the only solution to the equation.

We now turn to the linearity of the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Let's talk about something we've been hinting at since day one:

Theorem 3.2. If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

We quickly notice that if one of the coefficients is zero, then we have the corrollary that scalar multiples of solutions are also solutions:

If y is a solution to $L[y] = 0$, then cy is also a solution for any coefficient c.

We can handily prove theorem 3.2. Let $y = c_1y_1 + c_2y_2$ for solutions y_1 and y_2 of $L[y] = 0$. Consider the calculation

$$\begin{aligned} L[y] &= L[c_1y_1 + c_2y_2] \\ &= (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\ &= (c_1y_1'' + c_1p(t)y_1' + c_1q(t)y_1) + (c_2y_2'' + c_2p(t)y_2' + c_2q(t)y_2) \\ &= c_1L[y_1] + c_2L[y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0. \end{aligned}$$

Therefore, y also solves the differential equation and the theorem is proven.

We would like to address the initial value problem again. Even though we have solved this for linear second-order ODE, we would like to see what kinds of consequences we get from theorem 3.2. Who knows? Maybe there's something else that can happen now that we've increased the freedom of solutions.

Consider that if we have two solutions y_1 and y_2 to system $L[y]$, then the initial conditions would require our general solution to solve

$$\begin{cases} c_1y_1(t_0) + c_2y_2(t_0) & y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) & y_0' \end{cases}$$

for the variables c_1 and c_2. Our prerequisite knowledge of linear algebra tells us that this system has a unique solution for any choice of y_0 and y_0' if and only if the determinant of the matrix

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$$

is nonzero. We can write the determinant of this matrix as

$$W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0).$$

This function is called the *Wronskian*. We have just proven:

Theorem 3.3. Suppose that y_1 and y_2 are two solutions to $L[y] = 0$ and that the initial conditions have been chosen to be $y(t_0) = y_0$ and $y'(t_0) = y_0'$. Then it is always possible to choose the constants c_1 and c_2 so that

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

satisfies the initial value problem **if and only if** the Wronskian $W(y_1, y_2)(t_0) \neq 0$.

This tells us exactly when two **functions** y_1 and y_2 are **linearly independent** in the **vector space** of twice-differentiable functions. We knew exactly how to do this when we had vectors of $\mathbb{R}^n$, but we only have the single equation

$$c_1y_1 + c_2y_2 = 0$$

to use in our calculation of linear independence. We have to extrapolate this equation in order to extract another equation from this equality. In fact, we can differentiate both sides to get the system

$$\begin{cases} c_1 y_1 + c_2 y_2 &= 0 \\ c_1 y_1' + c_2 y_2' &= 0, \end{cases}$$

which is exactly the system we solved earlier. In general, this is indeed how one determines the linear independence of differentiable functions.

We can illustrate this theorem with a quick example.

Example 3.4. Consider the differential equation

$$y'' - (r_1 + r_2)y' + r_1 r_2 y = 0,$$

where $r_1 \neq r_2$ are constants. Using our methods from section 3.1, we know that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are both solutions. By theorem 3.3, we can check where we may not be guaranteed a solution (or a unique solution) by evaluating the Wronskian. So consider that

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1 + r_2)t}.$$

We see that the Wronskian is never zero since $r_1 - r_2 \neq 0$ so that we can always find a unique solution for **any** initial conditions for the initial value problem. Moreover, because the differential equation had continuous coefficients, the solution to any initial value problem is guaranteed to exist **for all** t. Powerful, powerful stuff.

Theorem 3.3 justifies the following theorem:

Theorem 3.4. Suppose that y_1 and y_2 are two solutions of the differential equation $L[y]$. Then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of $L[y] = 0$ if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero.

What the combination of the last three theorem implies is that our **fundamental solutions** that we found in section 3.1 **form a basis for the space of solutions of the differential equation**. Hence, **the solutions to a homogeneous differential equation forms a vector subspace of twice-differentiable functions**. We can clearly see the ties that differential equations have to linear algebra and why we require students to take a linear algebra class before (or concurrently with) differential equations classes.

Let's check two more examples before moving on to our final theorem of the section.

Example 3.5. 1. Recall from section 3.1 that we found the fundamental solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. We assumed that we always had $r_1 \neq r_2$. Let's see why we did that by computing

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_1 - r_2)e^{(r_1 + r_2)t}.$$

Observe that if $r_1 \neq r_2$, then the Wronskian is always nonzero. If $r_1 = r_2 = r$, then we have the Wronskian begin the zero function. Hence, there are solutions that cannot be represented by the single solution $y = ce^{rt}$.

2. We show here that the functions $y_1 = e^{rt}$ and $y_2 = te^{rt}$ are linearly independent. So we calculate the Wronskian of these two functions as

$$W(y_1, y_2) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{vmatrix} = e^{2rt},$$

which is again never zero. This realization will be of great importance to us in later sections.

3. We can also use the Wronskian for functions outside the context of solutions to differential equations. Let's find out when $\{t^3, \sin 2t\}$ is a linearly independent subset of twice-differentiable functions:

$$W(t^3, \sin 2t) = \begin{vmatrix} t^3 & \sin 2t \\ 3t^2 & 2\sin 2t \end{vmatrix} = 2t^3 \sin 2t - 3t^2 \sin 2t = t^2 \sin 2t(2t - 3).$$

This function has zeroes at every integer multiple of $t = \frac{\pi}{2}$ as well as, strangely, at the point $t = \frac{3}{2}$. Hence, because the Wronskian is not the zero function, $\{t^3, \sin 2t\}$ is a linearly independent subset of twice-differentiable functions.

We have been dealing with real-valued functions so far. It is time to observe that there are such things as complex-valued functions - that is, functions whose images lie in $\mathbb{C}$, the field of complex numbers. These functions can always be decomposed into a sum of real and imaginary parts. We would like to point out one key feature of these kinds of functions before we continue:

Theorem 3.5. Suppose that

$$y(t) = u(t) + iv(t),$$

where u and v are real-valued functions (the real and imaginary components of y, respectively), is a complex-valued solution of the differential equation $L[y] = 0$ (recall that i is the positive square root of -1). Then each of the parts u and v also solve the differential equation.

This theorem follows immediately from theorem 3.2 by taking $y_1 = u$, $y_2 = v$, $c_1 = 1$, and $c_2 = i$. In view of the title of the next section, one might think this theorem plays a vital role in our study of second-order differential equations - and he or she would indeed be correct.

3.3 Complex Roots of the Characteristic Equations

We continue our work with second-order linear differential equations of the form

$$ay'' + by' + cy = 0$$

by consider the case of complex roots of the characteristic equation

$$ar^2 + br + c = 0.$$

It is well-known (from the quadratic formula) that if a, b, and c are real numbers, then complex roots will occur only in conjugate pairs r and $\overline{r}$. So we let $r = \lambda + i\mu$ and $\overline{r} = \lambda - i\mu$, where $i = \sqrt{-1}$ is the positive imaginary root of -1 and λ and μ are real numbers. This would give us ansatz solutions of the form

$$y(t) = e^{(\lambda \pm i\mu)t} = e^{\lambda t}e^{\pm i\mu t}.$$

We are already familiar with the $e^{\lambda t}$ part of this solution, but what do with do with the complex exponential? This is where we need to recall some Calculus. Remember that the Taylor series for e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Let's consider what happens if we let $x = it$. We simply substitute this into the series to see that

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots.$$

Since $i^2 = -1$, we know that each even term of this series alternates sign. This also implies that odd powers are imaginary and also alternate signs. That is, we reduce the above to

$$\begin{aligned} e^{it} &= 1 + it - \tfrac{t^2}{2!} - i\tfrac{t^3}{3!} + \tfrac{t^4}{4!} + i\tfrac{t^5}{5!} - \cdots \\ &= \left[1 - \tfrac{t^2}{2!} + \tfrac{t^4}{4!} - \cdots\right] + i\left[t - \tfrac{t^3}{3!} + \tfrac{t^5}{5!} - \cdots\right]. \end{aligned}$$

Staring long enough at the real and imaginary parts of this last expression, we recognize these as the Taylor series of cosine and sine, respectively. We finally conclude that

$$\boxed{e^{it} = \cos t + i \sin t.}$$

This equation is called *Euler's identity*, and it gives us a way to represent complex exponential functions as a sum of real and imaginary parts. Something similar occurs for $e^{i\mu t}$. Substituting $x = i\mu t$ gives us

$$e^{i\mu t} = \cos(\mu t) + i \sin(\mu t).$$

However, we are only concerned with real-valued solutions when it comes to our differential equations in this class. This is why we mentioned theorem 3.5 at the end of the last section. It allows us to find two linearly independent *real-valued* solutions. Certainly, the real and imaginary parts are independent since the Wronskian (which we leave to the reader to check) is

$$W(e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)) = \mu e^{2\lambda t},$$

which is always nonzero (since the case for $\mu = 0$ implies r is, in fact, a repeated real root of the characteristic equation). We consolidate our results in the following theorem:

Theorem 3.6. If $r = \lambda \pm i\mu$ as given above are roots of the characteristic equation of

$$ay'' + by' + cy = 0,$$

then the general solution of this differential equation is

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t).$$

Let's see what kinds of consequences we can expect from this by doing some examples.

Example 3.6. "Find the general solution to the differential equation

$$y'' + \omega^2 y = 0,$$

where ω is a real number."

This is the equation for an undriven *simple harmonic oscillator*, a model for periodic motion that appears quite often in practice. Its characteristic equation is

$$r^2 + \omega^2 = 0,$$

which has roots of $r = \pm\omega i$. Hence, the general solution looks like

$$y(t) = c_1 e^{0t}\cos(\omega t) + c_2 e^{0t}\sin(\omega t) = c_1\cos(\omega t) + c_2\sin(\omega t).$$

The ω represents the frequency of oscillation of whatever this equation models. We will return to the modeling aspect of this equation later.

This is a standard solution that any student of differential equations should know. That being said: Know it, love it.

Example 3.7. Consider the differential equation

$$y'' - 2y' + 5y = 0.$$

The characteristic equation for this equation is

$$r^2 - 2r + 5 = 0.$$

We use the quadratic formula to find that

$$r = \frac{2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

These complex solutions give us $\lambda = 1$ and $\mu = 2$ so that our general solution is

$$y(t) = c_1 e^{t}\cos(2t) + c_2 e^{t}\sin(2t).$$

The limit of this **function** as $t \to \infty$ **does not exist** because of the oscillations of sine and cosine. However, notice that the **amplitude** grows without bound. This is typically characteristic of systems that are in phase with a resonating frequency. We will see more of this in a later section.

Example 3.8. "Solve the initial value problem

$$y'' + y' + \frac{37}{4}y = 0; \quad y(0) = 2, \quad y'(0) = 8."$$

This has characteristic equation

$$r^2 + r + \frac{37}{4} = 0,$$

which has roots $r = -\frac{1}{2} \pm 3i$. Then our general solution takes on the form

$$y(t) = c_1 e^{\frac{-t}{2}} \cos(3t) + c_2 e^{\frac{-t}{2}} \sin(3t).$$

Substituting in $y(0) = 2$, we see that $c_1 = 2$. Then we differentiate the solution to get

$$y'(t) = c_1(-\frac{1}{2}e^{-t/2}\cos(3t) - 3e^{-t/2}\sin(3t)) + c_2(-\frac{1}{2}e^{-t/2}\sin(3t) + 3e^{-t/2}\cos(3t)).$$

Using $c_1 = 2$ and $y'(0) = 8$, we get

$$8 = -1 + 3c_2$$

so that $c_2 = 3$. So our solution to the initial value problem is

$$y(t) = e^{\frac{-t}{2}}(2\cos(3t) + 3\sin(3t)).$$

Notice that as $t \to \infty$, y tends to zero because of the decaying amplitudes of the sine and cosine functions. This is characteristic of a damped oscillation where friction *gently* pulls energy away from the system so that the total mechanical energy is depleted.

One final remark of this section is to notice that the long-term tendencies of solutions with complex characteristic roots is *entirely* dependent on the real part of these roots. A real part that is less than zero indicates an exponentially decaying amplitude. A real part greater than zero indicates a exponentially growing amplitude. The final case here is when the real part of the characteristic roots is identically zero. The solutions are neither blowing up nor decaying. They are happily in what is called *simple harmonic motion*. Energy is neither entering nor leaving whatever system the differential equation represents. We will indeed come back to all of these cases when we talk about mechanical and electrical vibrations.

3.4 Repeated Roots; Reduction of Order

There is one final case that we need to consider before we can claim to have completely solved the homogeneous case of these second-order linear equations. This is the case of repeated roots. At this point, we technically already have a solution of this case. If the differential equation

$$ay'' + by' + cy = 0$$

with characteristic equation

$$ar^2 + br + c = 0$$

has a repeated root R, then we know that the function

$$y(t) = ce^{Rt}$$

satisfies the equation - it was designed to do exactly this! However, the only other solution we can construct from what we know is the other e^{Rt}. We cannot use this solution because it is not linearly independent from the e^{Rt} we already have in the solution. So our goal here is to find another linearly independent solution to this equation. This is generally done by simply multiplying our existing solution by some other function. Hence, our method is going to be a simple case of the method of *variation of parameters*. We'll explore this method soon in a more general setting.

Let's begin by assuming that our new target solution has the form

$$y(t) = u(t)e^{Rt}.$$

Then we plug this trial solution into the original differential equation; so we need to calculate a few derivatives:

$$y'(t) = u'e^{Rt} + Rue^{Rt} = [u' + Ru]e^{Rt}$$

$$y''(t) = u''e^{Rt} + Ru'e^{Rt} + Ru'e^{Rt} + R^2ue^{Rt} = [u'' + 2Ru' + R^2u]e^{Rt}.$$

Substituting these into the differential equation and recalling that R is a root of the characteristic equation, we have

$$\begin{aligned} a[u'' + 2Ru' + R^2u]e^{Rt} + b[u' + Ru]e^{Rt} + cue^{Rt} &= 0 \\ au'' + (2aR + b)u' + (aR^2 + c)u &= 0. \end{aligned}$$

In order to continue, we use two pieces of information we get from assuming that R is a repeated root of the characteristic equation - both coming from the quadratic formula. That is, we know that $R = -\frac{b}{2a}$ so that the u' term above vanishes since

$$2a\left(-\frac{b}{2a}\right) + b = 0;$$

and we know that the discriminant $b^2 - 4ac$ must be zero in this case:

$$\begin{aligned} b^2 - 4ac &= 0 \\ 4ac &= b^2 \\ c &= \tfrac{b^2}{4a}. \end{aligned}$$

Therefore, the coefficient for the u term above is

$$a\left(-\frac{b}{2a}\right)^2 + \frac{b^2}{4a} = 0,$$

causing this term to vanish as well.

Hence, the only functions $u(t)$ that we can use must satisfy the equation

$$u''(t) = 0.$$

Integrating this twice gives us

$$u(t) = At + B.$$

We may choose any coefficients of A and B here, but only those where $A \neq 0$ will yield a new solution to the original differential equation. So for simplicity, we take $A = 1$ and $B = 0$. We check the Wronskian of these solutions to guarantee their linear independence:

$$W(e^{Rt}, te^{Rt}) = \begin{vmatrix} e^{Rt} & te^{Rt} \\ Re^{Rt} & e^{Rt} + Rte^{Rt} \end{vmatrix} = e^{2Rt} + Rte^{2Rt} - Rte^{Rt} = e^{2Rt}.$$

This quantity is never zero, so these functions are indeed linearly independent from one another. Therefore, our new solution to the differential equation is given by

$$y(t) = c_1 e^{Rt} + c_2 te^{Rt}.$$

Let's quickly look at a few examples.

Example 3.9. Find the general solution to the differential equation

$$y'' - 2y' + y = 0.$$

The characteristic equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0$$

so that $r = 1$ is the single repeated root of this equation. So our general solution must take the form

$$c_1 e^t + c_2 te^t.$$

Example 3.10. Solve the initial value problem

$$y'' + 6y' + 9y = 0; \quad y(0) = 3, \quad y'(0) = -5$$

One can easily determine that $r = -3$ is the single repeated root of the characteristic equation here. Hence, the general solution is

$$y(t) = c_1 e^{-3t} + c_2 te^{-3t}.$$

Then we find

$$y'(t) = -3c_1 e^{-3t} + c_2(e^{-3t} - 3te^{-3t}).$$

Using the initial conditions, we find that

$$\begin{aligned} c_1 &= 3 \\ -3c_1 + c_2 &= -5 \end{aligned}$$

so that $c_1 = 3$ and $c_2 = 4$. Therefore, the solution to the initial value problem is

$$y(t) = 3e^{-3t} + 4te^{-3t}.$$

We can again look for long-term behavior of the solution. Once again it is determined by whether or not this repeated root of the characteristic equation is positive or negative. Notice that if the root is zero (i.e. a solution of $y'' = 0$), then our general solution does in fact predict the general linear solution.

We have now completed our study of homogeneous second-order linear constant-coefficient ODE. Let us quickly summarize what solutions we can expect from this kind of ODE:

1. *Distinct real roots.* For distinct real roots r_1 and r_2 of the characteristic equation, we have a general solution of
$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

2. *Complex conjugate roots.* For complex roots $r = \lambda + i\mu$ and $\overline{r}$ of the characteristic equation, we have a general solution of
$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).$$

3. *Repeated roots.* For a repeated root r of the characteristic equation, we have a general solution of
$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

We now take a moment to remark that our method for constructing the second linearly independent solution in the case of repeated roots can actually be generalized. We mentioned that the method we used was a simpler case of *variation of parameters*. This simpler method is called *reduction of order*. **This method absolutely requires that we have already found one solution $y_1(t)$ to our given differential equation**. Then our aim is to find another solution to the differential equation by implementing a trial function of $y_2(t) = u(t)y_1(t)$ for some function $u(t)$. We then substitute this function into our equation to determine a differential equation in $u(t)$ and its derivatives. Then we can choose a general solution of this new equation that makes y_2 linearly independent from y_1. Let's see this process in action with an example.

Example 3.11. Let's consider the differential equation
$$2t^2 y'' + 3ty' - y = 0$$
on the interval $t > 0$. Suppose that we have somehow determined that $y_1(t) = \frac{1}{t}$ is a solution to this equation (go on - verify that y_1 is indeed a solution). To find another linearly independent solution to this equation, we employ our method of reduction of order. So let's assume that our solution takes the form
$$y_2(t) = v(t)t^{-1}.$$
We then differentiate this twice so that we may substitute it into the differential equation:
$$y_2'(t) = v't^{-1} - vt^{-2} \qquad y_2''(t) = v''t^{-1} - 2v't^{-2} + 2vt^{-1}.$$
Using this in the differential equation gives us
$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-1}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1}$$

$$= 2tv'' - v' = 0.$$

Even though this new equation for v is of second-order, it can be reformulated as a first-order equation taking $w = v'$ so that

$$2tw' - w = 0.$$

This is exactly the reason for the name "reduction of order." We have reduced the order of the equation in order to determine another linearly independent solution. We can easily see that this is a separable equation with solution

$$w(t) = C\sqrt{t}.$$

Since $w = v'$, we must have

$$v(t) = C_1 t^{3/2} + C_2.$$

Taking $C_2 = 0$ (since it corresponds to the solution y_1), we notice that

$$y_2(t) = v(t)y_1(t) = C_1\sqrt{t}.$$

We can check the Wronskian of y_1 and y_2 to be

$$W(y_1, y_2) = \frac{3}{2}t^{-3/2},$$

which is defined and nonzero for all $t > 0$. Hence, y_1 and y_2 are linearly independent. Then our general solution to the differential equation is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 t^{-1} + c_2\sqrt{t}, \quad t > 0.$$

3.5 Nonhomogeneous Equations: Method of Undetermined Coefficients

Now that we have completely solved our second-order linear constant-coefficient ODE for the homogeneous case, we must now tackle the nonhomogeneous case. We begin with our general equation

$$L[\phi] = \phi'' + p(t)\phi' + q(t)\phi = g(t).$$

Suppose that we **were** able to find some solution $Y(t)$ such that $L[Y] = g(t)$. Then I pose to you one question: What would happen if I were to add to Y some other function ϕ such that $L[\phi] = 0$? Observe now that $y(t) = Y(t) + \phi(t)$ must also solve the equation since the linearity of the operator allows us to distribute it across both terms:

$$L[Y + \phi] = L[Y] + L[\phi] = g(t) + 0 = g(t).$$

That is, by adding a **homogeneous** solution to any one solution $Y(t)$ of the nonhomogeneous equation provides us with yet another one of its solutions! Hence, we know that if the general solution of the homogeneous equation is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t),$$

then our solution to the nonhomogeneous equation is given by

$$y(t) = y_h(t) + y_p(t) = [c_1 y_1(t) + c_2 y_2(t)] + y_p(t),$$

where y_p denotes some solution of the nonhomogeneous equation. We call y_p the *particular solution* and y_h the *homogeneous solution*. Together, they give the general solution to the nonhomogeneous ODE. Observe here two things:

1. We saw the same phenomenon in the solution of the general first-order, linear equation. The only difference is that there is no closed form solution of the general second-order, linear equation.

2. Any two particular solutions Y_1 and Y_2 of the nonhomogeneous equation must solve the homogeneous equation. Again, the linearity of the operator is what allows us to distribute itself across both terms and, additionally, extract the negative sign:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0.$$

Our observations above tell us exactly how to approach solving the nonhomogeneous ODE:

1. First determine the homogeneous solution y_h of the differential equation $L[\phi] = 0$.

2. Then find a solution y_p of the nonhomogeneous equation.

3. The general solution is given by $y(t) = y_p(t) + y_h(t)$.

We have just spent the past four sections working on step one of this method. Our goal for this section is to introduce a method for particular functions $g(t)$ in the nonhomogeneous equation $L[\phi] = g(t)$. To understand this method, it may actually be best to learn it by example.

Example 3.12. Find a particular solution $y_p(t)$ of the following equation:

$$y'' - 5y' + 6y = e^{-t}.$$

It stands to reason that we may wish to try to find solutions of the form $w(t) = Ae^{-t}$. This is that same idea of *ansatz solution* that we were seemingly divinely inspired by in the first section. So we substitute this into the differential equation as

$$\begin{aligned} Ae^{-t} + 5Ae^{-t} + 6Ae^{-t} &= e^{-t} \\ 12Ae^{-t} &= \end{aligned}$$

so that we must have $12A = 1$. Hence, we need $A = \frac{1}{12}$. Therefore, our particular solution is given by

$$y_p(t) = \frac{1}{12}e^{-t}.$$

Then the general solution of the differential equation is

$$y(t) = \frac{1}{12}e^{-t} + c_1 e^{2t} + c_2 e^{3t}.$$

Example 3.13. Find the particular solution of the following equation:

$$y'' - y = t^3.$$

Once again, it would stand to reason that we want solutions of the form $y(t) = At^3$. So we naïvely attempt to use this solution:

$$6At - At^3 = t^3.$$

Equating coefficients on the left and right sides gives us $6A = 0$ and $-A = 1$. So we would need $A = -1$ and $A = 0$. This apparent contradiction tells us that **we must not have the correct form of the solution**!

We try a more general approach. Instead of just taking $y(t) = At^3$, we may need all terms of lower degrees in order to find to correct solution - a general third-degree polynomial. So we try a solution of the form $At^3 + Bt^2 + Ct + D$. This gives us

$$\begin{aligned}(6At + 2B) - (At^3 + Bt^2 + Ct + D) &= t^3\\ -At^3 - Bt^2 + (6A - C)t + (2B - D) &= t^3.\end{aligned}$$

Equating coefficients again shows us that we need

$$\begin{cases} -A & = 1\\ -B & = 0\\ 6A - C & = 0\\ 2B - D & = 0.\end{cases}$$

We can easily solve this system as $A = -1$, $B = 0$, $C = -6$, and $D = 0$. That is, our general solution takes the form

$$y_p(t) = -t^3 - 6t.$$

The general solution is then

$$y(t) = -t^3 - 6t + c_1e^t + c_2e^{-t}.$$

In general, if the nonhomogeneous part $g(t)$ of the equation is a polynomial, then we need to use trial solutions that are general polynomials of the degree of the polynomial in g.

Example 3.14. Find a particular solution of the following equation:

$$y'' - 3y' + 2y = \cos(2t).$$

We would think that we should try to find solutions of the form $A\cos(2t)$. So we try this first:

$$\begin{aligned}(-4A\cos(2t)) - 3(-2A\sin(2t)) + 2(A\cos(2t)) &= \cos(2t)\\ -2A\cos(2t) + 6A\sin(2t) &= \cos(2t).\end{aligned}$$

Equation coefficients shows us that $-2A = 1$ as well as $6A = 0$. This again leads to a contradiction showing us that we must have the wrong general form of the solution. So we stare at the work we've done up to this point and think that maybe we need a sine

term in our general equation - a general *trigonometric polynomial* of frequency 2. We then try the function $A\cos(2t) + B\sin(2t)$:

$$\begin{aligned}(-4A\cos(2t) - 4B\sin(2t)) - 3(-2A\sin(2t) + 2B\cos(2t)) & \\ +2(A\cos(2t) + B\sin(2t)) &= \cos(2t)\\ (-2A - 6B)\cos(2t) + (6A - 2B)\sin(2t) &= \cos(2t).\end{aligned}$$

Equation the coefficients again gives us

$$\begin{cases} -2A + 6B &= 1\\ 6A - 2B &= 0.\end{cases}$$

We can again solve this simple linear system to find $A = -\frac{1}{20}$ and $B = -\frac{3}{20}$ so that our particular solution is

$$y_p(t) = -\frac{1}{20}\cos(2t) - \frac{3}{20}\sin(2t).$$

Then the general solution is

$$y(t) = -\frac{1}{20}\cos(2t) - \frac{3}{20}\sin(2t) + c_1e^t + c_2e^{2t}.$$

We can also combine trial solutions when g is a product of the types considered above:

Example 3.15. Let's consider the equation

$$y'' - 5y' + 6y = (12t^2 + 10t - 24)e^{-t}.$$

The kind of trial solution we need to consider should be a combination of a general second-degree polynomial and the exponential function e^{-t}. An easy choice is to use the trial function

$$e^{-t}(At^2 + Bt + C).$$

Using this in the above equation, we find

$$e^{-t}[At^2 + (-4A + B)t + (2A - 2B + C)] - 5e^{-t}[-At^2 + (2A - B)t + (B - C)]$$

$$+6e^{-t}[At^2 + Bt + C] = (12t^2 + 10t - 24)e^{-t}$$

$$e^{-t}[(12A)t^2 + (-14A + 12B)t + (2A - 7B + 12C)] = (12t^2 + 10t - 24)e^{-t}.$$

Matching coefficients of corresponding terms allows us to see that we must solve the system

$$\begin{cases} 12A &= 12\\ -14A + 12B &= 10\\ 2A - 7B + 12C &= -24.\end{cases}$$

The solution here is $A = 1$, $B = 2$, and $C = -1$. Therefore, the particular solution we seek is given by

$$y_p(t) = e^{-t}(t^2 + 2t - 1).$$

A similar method is employed for other products. We find our general solution to be

$$y(t) = e^{-t}(t^2 + 2t - 1) + c_1e^{2t} + c_2e^{3t}.$$

We have now seen a few different examples of how to find particular solutions when the function g is exponential, polynomial, trigonometric, and products thereof; but there are a few special cases that we need to worry about. In particular, we need to determine what general form of solutions we can take when the nonhomogeneous function $g(t)$ itself satisfies the homogeneous equation. This was never the case in the examples above, so our solutions are certainly valid. In general, though, we need to check our trial solutions against the homogeneous solutions as in the following examples.

Example 3.16. 1. We begin with

$$y'' - y = e^t.$$

The homogeneous solution is given as $c_1e^t + c_2e^{-t}$. Notice that the nonhomogeneous function $g(t) = e^t$ is actually of this form. In other words, if we used a trial solution of Ae^t, it would evaluate on the left-hand-side as zero (check this!) and couldn't possibly solve the nonhomogeneous equation. So we need to find another trial solution. Our next guess is to use the same technique as in section 3.4. We just need another linearly independent solution that will allow us to try another general form. So let's try a trial solution of $y_p(t) = t(Ae^t)$. This yields

$$y_p'(t) = Ae^t + Ate^t \qquad y_p''(t) = 2Ae^t + Ate^t$$

so that substituting these into the equation gives us

$$\begin{aligned}(2Ae^t + Ate^t) - (Ate^t) &= e^t \\ 2Ae^t &= e^t.\end{aligned}$$

Equating coefficients gives us $A = \frac{1}{2}$ so that the particular solution is

$$y_p(t) = \frac{1}{2}te^t.$$

The general solution is then

$$y(t) = \frac{1}{2}te^t + c_1e^2 + c_2e^{-t}.$$

2. Let's consider the equation

$$y'' - 2y' + y = e^t(4 + 6t).$$

We now need to account for both summands of the nonhomogeneous function simultaneously. We would initially search for solutions of the form $Ae^t + Bte^t$. However, this is identically the homogeneous solution of this differential equation! So we must try another solution. Again, our solution is merely to multiply in another factor of t to our trial solution. Hence, we should try solutions of the form $y_p(t) = t^2(At+B)e^t$. This gives us

$$\begin{aligned}y_p'(t) &= (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t \\ &= e^t[At^3 + (3A + B)t^2 + 2Bt]\end{aligned}$$

$$y_p''(t) = e^t[At^3 + (3A+B)t^2 + 2Bt] + e^t[3At^2 + (12A+2B)t + 2B]$$
$$= e^t[At^3 + (6A+B)t^2 + (12A+4B)t + 2B]$$

so that we have

$$\begin{aligned} y_p''(t) - 2y_p'(t) + y_p(t) &= 4e^t + 6te^t \\ (12At + 2B)e^t &= 4e^t + 6te^t. \end{aligned}$$

We equate coefficients so that

$$\begin{cases} 12A &= 6 \\ 2B &= 4. \end{cases}$$

Hence, we have $A = \frac{1}{2}$ and $B = 2$ so that the particular solution is given by

$$y_p(t) = \left(\frac{1}{2}t^3 + 2t^2\right)e^t.$$

The general solution is then

$$y(t) = \left(\frac{1}{2}t^3 + 2t^2\right)e^t + c_1te^t + c_2e^t = e^t\left(\frac{1}{2}t^3 + 2t^2 + c_1t + c_2\right).$$

3. We end with one more example:

$$y'' + 4y = 3\sin(2t) - \cos(2t).$$

To find this equation's particular solution, we observe first that the homogeneous solution takes the form $A\sin(2t) + B\cos(2t)$. Since the nonhomogeneous function is of this form, we cannot use this solution as our particular solution. We then try solutions of the form $y_p(t) = t(A\sin(2t) + B\cos(2t))$. We find that

$$y_p'(t) = (A\sin(2t) + B\cos(2t)) + t(2A\cos(2t) - 2B\sin(2t))$$
$$= \cos(2t)(2At + B) + \sin(2t)(A - 2Bt)$$
$$y_p''(t) = [-2\sin(2t)(2At + B) + \cos(2t)(2A)] + [2\cos(2t)(A - 2Bt) + \sin(2t)(-2B)]$$
$$= \cos(2t)(4A - 4Bt) + \sin(2t)(-2B - 4At)$$

so that we have

$$\begin{aligned} y_p'' + 4y_p &= 3\sin(2t) - \cos(2t) \\ 4A\cos(2t) - 4B\sin(2t) &= 3\sin(2t) - \cos(2t). \end{aligned}$$

Equating coefficients gives us $4A = 3$ and $-4B = -1$ so that $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Then the particular solution of the equation is

$$y_p(t) = t\left(\frac{3}{4}\sin(2t) + \frac{1}{4}\cos(2t)\right).$$

The general solution is then

$$y(t) = t\left(\frac{3}{4}\sin(2t) + \frac{1}{4}\cos(2t)\right) + c_1\cos(2t) + c_2\sin(2t)$$
$$= \cos(2t)\left(\frac{3}{4}t + c_1\right) + \sin(2t)\left(\frac{1}{4}t + c_2\right).$$

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin \beta t \\ \cos \beta t\end{cases}$	$t^s[(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos \beta t + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin \beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

Figure 8: This figure illustrates the general form of particular solutions for various non-homogeneous functions in the differential equation $ay'' + by' + cy = g(t)$. Photo credit: Boyce and DiPrima.

There is a general method for what to take our general form of the particular solution to be. It is given in Figure 8. This table details exactly what to do for very specific cases. It by no means solves the general nonhomogeneous case. In order to solve this case entirely, we will need to appeal to a much stronger method than a glorified "guess and check."

It is normal for students to see this method and think it a bit arbitrary. Worry not! We will soon be able to justify everything done in this section with very precise methods using the general variation of parameters and Laplace transforms. We will see these in the next section and Chapter 6, respectively. Mastery of this method of undetermined coefficients comes from many, many hours of practice. Students should work through not just the worksheet problems but extra problems from the textbook or the text reference on the class webpage.

3.6 Variation of Parameters

We saw from the previous section that if we are given a homogeneous solution

$$y_h(t) = c_1y_1(t) + c_2y_2(t)$$

to a differential equation, then we can actually find the particular solution by assuming that the general solution took on the form

$$y_h(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where we have **replaced the constants** c_1 and c_2 by the **functions** $u_1(t)$ and $u_2(t)$. This method is attributed to Lagrange, and we can actually use it to prove that the general

form of the particular solutions we came up with in the previous section are natural choices for u_1 and u_2. We should probably illustrate this method using an example or two before tackling the general problem.

Example 3.17. Let's find a particular solution of the differential equation

$$y'' + 4y = 3\csc t.$$

We unfortunately (or fortunately, depending on how one looks at it) cannot use the method of undetermined coefficients since $3\csc t$ is a quotient of the prescribed functions rather than a product. So we know by now that the homogeneous solution is

$$y_h(t) = c_1\cos(2t) + c_2\sin(2t).$$

We now apply variation of parameters to replace c_1 and c_2 by functions so that we look for solutions of the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t).$$

We now try to plug this function into our differential equation, but we quickly realize that there is only going to be one final equation in two unknowns u_1 and u_2. Thus, we anticipate there existing **many** solutions of this form! We need to impose another equation in these variables so that we can narrow our choices to one unique solution. Let's differentiate our trial solution once to see what sort of condition we should impose:

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t).$$

Even this first expression is quite messy. Let's use our extra condition to clean it up a little bit. So let's try to get rid of the derivatives of u_1 and u_2 that are already present - this way we are still aiming to reduce the order of the new differential equation in u_1 and u_2. That is, take

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0.$$

Now we are left with

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t).$$

We can now differentiate this once more to find

$$y''(t) = -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t).$$

We now substitute these expression into the original differential equation to find

$$y'' + 4y = -2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 3\csc t.$$

We can now rewrite our extra condition 3.17 to find an expression of u_2' in terms of u_1':

$$u_2'(t) = -u_1'(t)\frac{\cos(2t)}{\sin(2t)}.$$

Using this in our substituted differential equation and simplifying, we get

$$u_1'(t) = -\frac{3}{2}\csc(t)\sin(2t) = -\frac{3}{2}\frac{1}{\sin t}2\sin t\cos t = -3\cos t.$$

We can then put this solution back into our rewritten extra condition for

$$u_2'(t) = \frac{3}{2}\frac{\cos t \cos(2t)}{\sin(2t)} = \frac{3}{2}\frac{1 - 2\sin^2 t}{2\sin t} = \frac{3}{2}\csc t - 3\sin t.$$

To quickly summarize, we have just found

$$\begin{cases} u_1'(t) &= -3\cos t \\ u_2'(t) &= \frac{3}{2}\csc t - 3\sin t. \end{cases}$$

We can easily integrate these to find

$$u_1(t) = -3\sin t + c_1$$

and

$$u_2(t) = \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2.$$

We then know that our general solution to the differential equation takes the form

$$\begin{aligned} y(t) &= -3\sin t\cos(2t) + \frac{3}{2}\ln|\csc t - \cot t|\sin(2t) + 3\cos t\sin(2t) + \\ &\quad c_1\cos(2t) + c_2\sin(2t) \\ &= 3\sin t + \frac{3}{2}\ln|\csc t - \cot t|\sin(2t) + c_1\cos(2t) + c_2\sin(2t). \end{aligned}$$

The reader should check that this does indeed still constitute a general solution to the original nonhomogeneous equation.

After seeing how easily we can construct a particular solution by replacing the constant coefficients of the homogeneous solution by functions, we want to be able to extend this method to the general case of the equation

$$y'' + p(t)y' + q(t)y = g(t),$$

where we usually take p, q, and g to be continuous functions. We take y_1 and y_2 to be a basis of solutions of the homogeneous equation (where $g(t) = 0$). Already, we are asking for a lot. We technically only know how to solve the case where p and q are constant functions. There are indeed methods to deal with the more general case, but let's just assume for now that we can find such functions y_1 and y_2. Then we start with a trial function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Again we realize that we are going to have infinitely many potential solutions for u_1 and u_2 (not just limited to the values of the integration constants). With this foresight, we look to find a way to reduce the order of the ensuing differential equations for u_1 and u_2. Differentiating the trial solution once gives us

$$y' = (u_1'y_1 + u_1y_1') + (u_2'y_2 + u_2y_2').$$

We take our extra condition in general to get rid of the initial u_1' and u_2' derivatives so that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0.$$

We are then left with

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t).$$

Differentiating once more gives us

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Substituting these expressions into the original differential equation will give us

$$u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)]$$

$$+u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)]$$

$$+u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t).$$

Since each y_1 and y_2 were assumed to solve the homogensous equation, we are merely left with

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t).$$

We can combine this with our general extra condition where we know that

$$u_2'(t) = -u_1'(t)\frac{y_1(t)}{y_2(t)}.$$

Substituting this into the substituted differential equation gives us the pair of equations

$$u_1'(t) = -\frac{y_2(t)g(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}$$

and

$$u_2'(t) = \frac{y_1(t)g(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}.$$

Notice that the denominators are exactly the Wronskian $W(y_1, y_2)(t)$! Then we can integrate these functions so that

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_1$$

and

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_2.$$

We summarize our results in the following theorem:

Theorem 3.7. If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are a linearly independent set of solutions to the homogeneous equation corresponding to

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution of the nonhomogeneous equation above is given by

$$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)}\,ds,$$

where t_0 is a point in I. The general solution to the above differential equation is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t).$$

Notice now that we have a definitive answer for what a particular solution to a nonhomgeneous differential equation needs to be. This solidifies our method of undetermined coefficients. In fact, we can double check that the general forms of solutions that we used in the previous section are what are prescribed by theorem 3.7.

Example 3.18. Let's reconsider the equation

$$y'' - y = e^t,$$

which we solved in the previous section via undetermined coefficients. The homogeneous solution is given by

$$y_h(t) = c_1 e^t + c_2 e^{-t}$$

so that $y_1 = e^t$ and $y_2 = e^{-t}$. The Wronskian of these functions can be shown to be $W(y_1, y_2)(t) = -2$. Then we can use theorem 3.7 to show that

$$y_p(t) = -e^t \int \frac{e^{-t} \cdot e^t}{-2}\, dt + e^{-t} \int \frac{e^t \cdot e^t}{-2}\, dt.$$

Then we can evaluate these integrals as

$$y_p(t) = \frac{1}{2} t e^t + \frac{1}{4} e^t.$$

Notice that we can omit the integration constants so as to just get **some** particular solution to this equation. Then we now see that our general solution is given by

$$y(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2} t e^t + \frac{1}{4} e^t.$$

This is exactly what we would have found had we used the method of undetermined coefficients - the extra e^t term is acceptable since it may be absorbed into the $c_1 e^t$ term. Hence, variation of parameters agrees with our previous method as we had hoped.

Remark 3.2. We now see how powerful variation of parameters is. It not only solves the general nonhomogeneous second-order linear differential equations, but it completely replaces the method of undetermined coefficients! However, one should note that this method does still have its limitations.

1. Recall that we still only know how to find the homogeneous solutions to **constant-coefficient** differential equations. We may explore this topic later in Chapter 5, but the student should know that this is a difficult problem to approach from elementary beginnings. In fact, even if we were able to find just one homogeneous solution for a variable-coefficient equation, we would need to find yet another linearly independent one in order to even begin to use variation of parameters!

2. The final solution given in variation of parameters is in terms of integrals. There are still some integrals that we cannot compute explicitly, and hence in general **we may not be able to compute solutions explicitly**! But the conditions of the theorem guarantee that they exist by theorem 3.1.

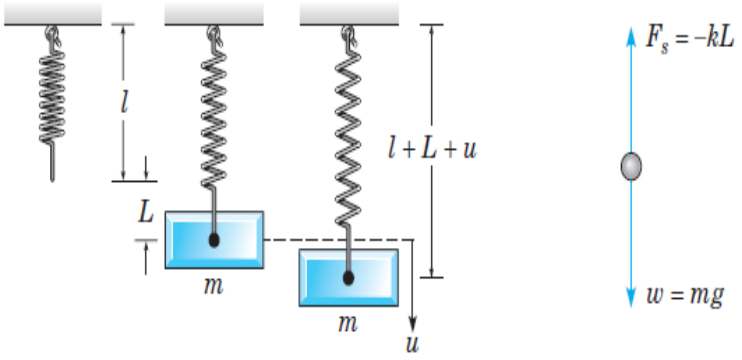


Figure 9: A free-body diagram of a hanging mass-spring system with rest length $\ell + L$. Photo credit: Boyce and DiPrima.

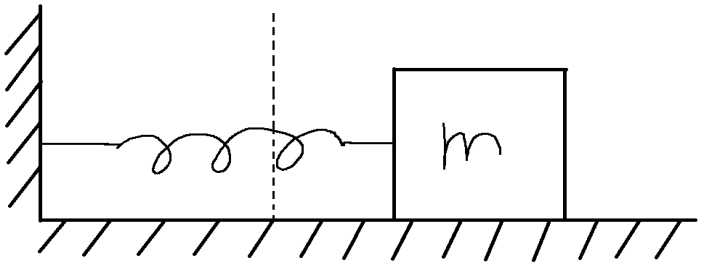


Figure 10: A diagram of a potential damped mass-spring system. The dashed line indicated the equilibrium or rest length.

3.7 Mechanical and Electrical Vibrations

Now that we have enough background in second-order equations, we can effectively put our knowledge to good use. The last two sections of this chapter have to do with applications to oscillating systems. The examples we can use are exactly those in the previous sections, but now the coefficients of the differential equations have physical meaning corresponding to masses, damping effects, and spring constants. As such, we will not emphasize the examples of these systems so much as the physical theory behind them. There are two main mechanical systems that we can study with the knowledge we have from sections 3.1 through 3.6: Hanging mass-spring systems and damped, sliding mass-spring systems. In general, we can model both of these systems using the same differential equation:

$$ay'' + by' + cy = g(t).$$

We can physically interpret what each of these terms means if we merely apply our (limited) knowledge of physics and intuition. In a hanging mass-spring system, there is assumed to be no damping effects. Figure 9 depicts exactly what is meant by the hanging system, and figure 10 shows a kind of damped, sliding system that one might encounter in practice.

Let ℓ be the rest length in the system without the mass, L be the rest length with the mass ($L = \ell$ in the damped case in figure 10, k be the spring constant of the spring, g be the gravitational acceleration, and u be the displacement of the block away from the rest length L. We can derive the differential equation that describes both of these systems in the following analysis:

1. We always have a downward force of gravity $F_g = mg$ acting on the mass.

2. We can go ahead and assume Hooke's law for the force imparted by the spring. That is, we can take the spring force $F_s = -k(L + u(t))$. Notice that the negative sign accounts both for compression (where the spring wants to **push** the mass toward the equilibrium) and relaxation (where the spring want to **pull** the mass back to equilibrium).

3. We can account for a damping force in the damped system by assuming the same model we did for air resistance back in chapter 1. That is, we assume that drag force is directly proportional and in the opposite direction of the velocity u' of the mass. So we define $F_d = -\gamma u'(t)$ for some $\gamma > 0$. Note that we may be able to change this drag force depending on which model we wish to use. For now, we assume a simple model so as to be able to apply our known techniques to the ensuing differential equation.

4. We can also have an applied external force $F(t)$ that can influence the motion of the mass. We can use Newton's second law of motion to show that the net force is equal to the mass times the net acceleration of the mass. That is,

$$mu''(t) = F_g + F_s + F_d + F(t).$$

   Because we assume that $mg - kL = 0$ by the definition of the rest length, we can see that the resulting equation is

$$mu''(t) + \gamma u'(t) + ku(t) = F(t).$$

   This is the model for a general mass-spring system! To guarantee that we are talking about a single system, we tie up the differential equation with in initial condition $u(0) = u_0$ and $u'(0) = v_0$ corresponding to the initial displacement and velocity of the mass.

To elaborate on the differences between hanging and damped mass-spring systems, let's consider that the hanging system has no appreciable damping. That is, in the hanging system we take $\gamma = 0$. Then our equation becomes

$$mu'' + ku = 0.$$

We can easily solve this differential equation since m and k are both constants. That is, we know that

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

where $\omega_0^2 = \frac{k}{m}$. Using some trigonometric identities, it is possible to rewrite this equation simply in terms of either sine or cosine and not both. Hence, we can write the solution as

$$u(t) = R \cos(\omega_0 t - \delta),$$

where $R^2 = c_1^2 + c_2^2$ and $\tan \delta = \frac{c_2}{c_1}$. The resulting function is shifted by the *phase* δ and exhibits *simple harmonic motion*. Simple harmonic motion is simply periodic motion with constant maximum amplitude. The periodic motion described here has a *frequency* of ω_0. The *period* T of this motion is determined by the relation $2\pi = T\omega_0$ so that $T = 2\pi\sqrt{\frac{m}{k}}$. A larger period corresponds to a smaller frequency and vice versa. Sometimes ω_0 is called

the *resonant frequency* of the system. We will see that this value is very important in the next section, and it should usually to be avoided in practice.

In the presence of damping, simple harmonic motion is impossible. There is, however, a similar feature called *quasi-harmonic motion*. This is essentially simple harmonic motion, but energy is being added to or dissipated from the system that either increases or decreases the amplitude as a function of time. Let's begin with the general unforced damped system

$$mu'' + \gamma u' + ku = 0.$$

This equation has characteristic equation $mr^2 + \gamma r + k = 0$ so that solutions will be of the form

$$u(t) = c_1 e^{r_+ t} + c_2 e^{r_- t},$$

where r_+ and r_- are the positive and negative, respectively, conjugates of the roots

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

The behavior of the solutions heavily depends on the value of the discriminant

$$D = \gamma^2 - 4k.$$

If $D > 0$, then there are two distinct real **negative** values of r so that solutions are exponential in nature and decay to zero. Such a system is called *overdamped*. If $D < 0$, then the values of r are complex conjugates and solutions of the equation are oscillatory - but there is always a negative real part of these roots so that the solutions not only oscillate but decay over time. Such a system is called *underdamped*. If $D = 0$, then there is a repeated root $r = -\frac{\gamma}{2m}$. Then our solutions either immediately decay, or our solutions given one quick flurry of motion to a peak amplitude and then decay without oscillations. Such a system is called *critically damped*.

Even though these are not simple harmonic oscillators, we can still carry over some of the terminology and machinery from these oscillators to their damped counterparts. The frequency term has been altered slightly so that we have a *quasi-frequency* of

$$\omega_q = \frac{\sqrt{4km - \gamma^2}/2m}{\sqrt{k/m}}.$$

This is exactly the ratio of the imaginary part of the r value of the characteristic equation and the resonant frequency of the undamped sytem. It can be simplified so that

$$\omega_q = \sqrt{1 - \frac{\gamma^2}{4km}} \approx 1 - \frac{\gamma^2}{8km},$$

where the approximation is found by using the Taylor expansion of the square root function. We can play the same game to find the *quasi-period* as

$$\frac{T_d}{T} = \frac{1}{\sqrt{1 - \frac{\gamma^2}{4km}}} \approx 1 + \frac{\gamma^2}{8km}.$$

The funny thing about physics is that many different phenomena are modeled by exactly the same equations. We can see this now by recalling Kirchoff's law that the

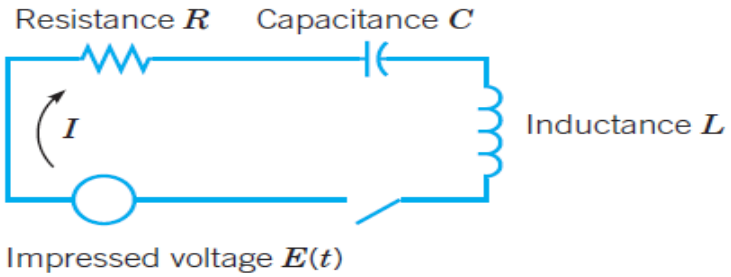


Figure 11: This is a circuit diagram for one possible configuration of a resistor-capacitor-inductor (RLC) circuit. Photo credit: Boyce and DiPrima.

imposed voltage on a closed circuit is equal to the sum of the voltage drops across each component in the rest of the circuit. Consider now the closed resistor-capacitor-inductor (RCL) circuit in Figure 11.

We find that the voltage drop across the resistor is given by Ohm's law $V = IR$, where I is the current passing through the resistor of resistance R. The voltage drop across the capacitor of capacitance C is by definition given as $V = \frac{Q}{C}$, where Q is the charge on the plates of the capacitor. The voltage drop across the inductor is given by a combination of Ampère's and Faraday's laws as $V = L\frac{dI}{dt}$. Then with the imposed voltage of $E(t)$, Kirchoff's law ($\frac{dV}{dt} = 0$) tells us that

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

If we recall that $I = \frac{dQ}{dt}$, then we can rewrite this equation as a second-order linear equation (solely in terms of the charge Q and its derivatives) with constant coefficients. That is, we can rewrite this as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Hence, solving one type of differential equation actually leads us to be able to analyze models of many different kinds of physical situations.

3.8 Forced Vibrations

Now that we've analyzed the unforced case of oscillations (modeled by homogeneous equations), we need to consider what happens if we apply an external driving force to the system. This corresponds to our study of nonhomogeneous equations from section 3.5 and 3.6. We see what sorts of consequences arise in this case.

First we consider if we have a damped mass-spring system. Recall that earlier in our study of first-order equations we found that our solutions took the form of

$$y(t) = y_p(t) + y_h(t).$$

That is, we had an asymptotic part of our solution as well as a homogeneous part. There is a similar phenomenon for second-order systems. We can still separate our general solutions to look like the above, but we rename the particular solution the *steady-state solution* since it is the eventual behavior of the system; and we rename the homogeneous solution the *transient solution* to account for behavior that deviates from the steady-state solution. We can see this sort of behavior in the following example.

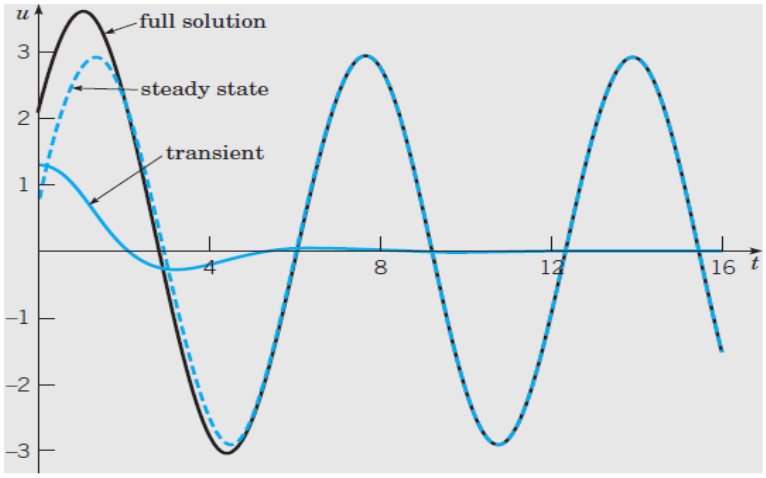


Figure 12: The two different terms of the solution to example 3.19 and their sum, the full solution. Photo credit: Boyce and DiPrima.

Example 3.19. Suppose we have modeled our system by the initial value problem

$$u'' + u' + \frac{5}{4}u = 3\cos t; \quad u(0) = 2, \quad u'(0) = 3.$$

We can think of this as a damped mass-spring system where we have pulled the mass away from the anchor and proceeded to throw it further away from the anchor. We can think of the nonhomogeneous function as perhaps an alternating electromagnetic field that is switched on when we throw the mass. We can easily solve this equation using the methods of the previous sections as

$$u(t) = (\frac{12}{17}\cos t + \frac{48}{17}\sin t) + (\frac{22}{17}e^{-t/2}\cos t + \frac{14}{17}e^{-t/2}\sin t).$$

Figure 12 shows exactly how the steady-state and transient solutions overlap in order to create the full solution. The steady-state solution is exactly just the particular solution to the original equation. This represents the natural tendency of the system under damping effects if we had simply started the system in equilibrium at rest (with a potential phase shift). That is to say that **every solution tends toward the steady-state solution as** $t \to \infty$. Another name for the steady-state solution is the *forced response* of the system. This isn't quite the same as an equilibrium solution since the steady-state isn't a constant function. However, the two concepts are complementary and describe the same type of phenomenon. The transient solution tends to zero as time moves on. As such, it is of little consequence in practice and is often ignored.

It is often the case that the nonhomoheneous, or *forcing*, function is periodic. As such, we can write any forcing function of this kind as

$$F(t) = F_0\cos(\omega t).$$

It isn't too hard to convince oneself that the forced response under such a forcing function is also periodic of the same frequency. We can then write our steady-state solution will take on the form

$$U(t) = R\cos(\omega t - \delta).$$

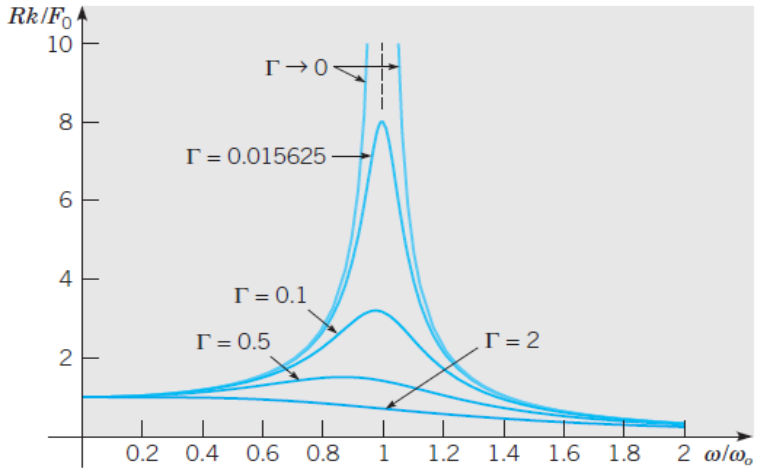


Figure 13: Amplitude of steady-state solutions versus the ratio of forcing and resonant frequencies. Photo credit: Boyce and DiPrima.

Just as in the case of the unforced systems in the previous section, we can determine several properties of the system from this steady-state solution. In particular, after a lot of dense algebra, we find that

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta},$$

where we define

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2) + \gamma^2\omega^2} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}.$$

Recall that ω_0 is the resonant frequency of the system. Let's compare the amplitudes of the forcing function to the steady-state solution. We use the ratio of resulting forces

$$\frac{Rk}{F_0} = \left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}\right]^{-1/2},$$

where we simply let $\Gamma = \frac{\gamma^2}{mk}$. In order to understand a particular consequence of a forced system, we need to consider not the individual frequencies ω and ω_0, but rather we need to compare how closely the forced frequency matches that of the resonant frequency. That is, we study what happens when we vary the ratio $\frac{\omega}{\omega_0}$. We notice first that $\frac{Rk}{F_0}$ reaches a maximum. This maximum will occur when

$$\omega = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right).$$

We denote this value by $\omega_{\max}$. Then the maximum amplitude of the response can be found to be

$$R_{\max} = \frac{F_0}{\gamma\omega_0}\sqrt{1 - (\gamma^2/4mk)} \approx \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right).$$

If $\gamma^2/mk > 2$, then $\omega_{\max}$ is imaginary; and in this case, $R_{\max}$ occurs for $\omega = 0$.

Figure 13 depicts different plots of $\frac{Rk}{F_0}$ for varying values of $\frac{\omega}{\omega_0}$. Notice the dramatic spike as the forcing frequency reaches the resonant frequency. This is a major problem in practice. What happens in this case is that mechanical energy is being added to the

system in a way that it is not being dissipated at a greater or equal rate. That is, **we are adding an unbounded amount of mechanical energy to the system**! Engineering students are no doubt familiar with the famous Tacoma Narrows bridge failure of 1940. This is a classic example of what happens when care is not taken in determining the resonant frequency of a structure and comparing it with naturally occuring environmental oscillations (in the case of the Tacoma Narrows bridge, this environmental osciallation was driven by high winds passing through the valley). The main point to take away from this analysis is that there are three dimensionless parameters ($\frac{Rk}{F_0}$, $\frac{\omega}{\omega_0}$, and Γ) that determine the resonant behavior of the system, and special care must be taken to ensure that the forcing frequency does not match the resonant frequency so that we don't cause a catastrophic failure of our system.

We can also consider the undamped mass-spring system just as before. In the case of including a forcing function without a damping effect, we don't lose energy like we did before. In fact, there is a very similar analysis of the system

$$mu'' + ku = F_0 \cos(\omega t)$$

that tells us what kind of behavior this system has under varying forcing frequencies. If $\omega \neq \omega_0$, we can write the general solution as

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) + c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

An interesting case of this type of system is when we take it to initially be at equilibrium and at rest (i.e. $u(0) = 0$ and $u'(0) = 0$). In this case, we must have a particular solution of

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos(\omega t) - \cos(\omega_0 t)).$$

We can use a trigonometric identity to rewrite the difference in cosines so that we have

$$u(t) = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)\right] \sin\left(\frac{(\omega + \omega_0)t}{2}\right).$$

If $|\omega_0 - \omega|$ is small, then $\omega + \omega_0$ is much greater in comparison. Then the second sine term above oscillates faster than the first. If we consider this system modeling a sound wave, we would physically hear a "wah-wah" effect. This phenomenon is called a *beat*, and it is often used in the practice of tuning musical instruments (like pianos and various string instruments). We can determine the amplitude of the waves as

$$A = \frac{2F_0}{m|\omega_0^2 - \omega^2|} \left|\sin\left(\frac{(\omega_0 - \omega)t}{2}\right)\right|.$$

This phenomenon of oscillating amplitude is seen in electronics as *amplitude modulation*. This is exactly what we listen to on the radio as "AM frequency." This can be seen in Figure 14.

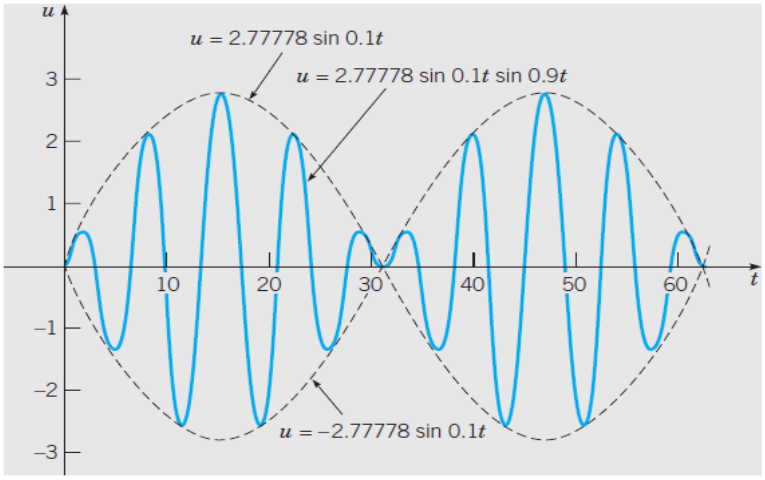
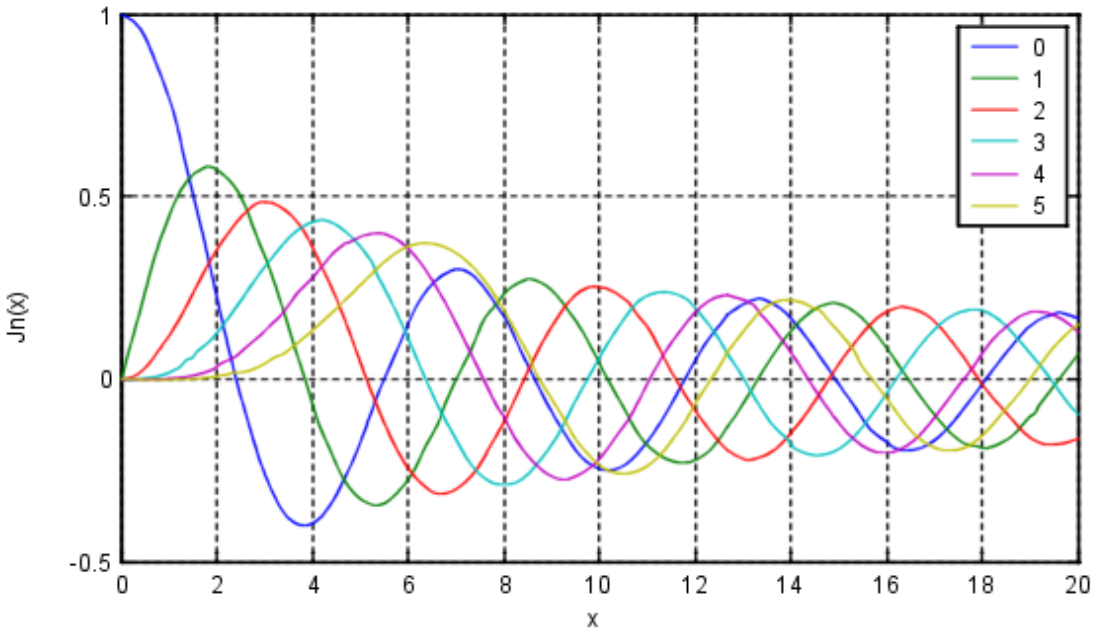
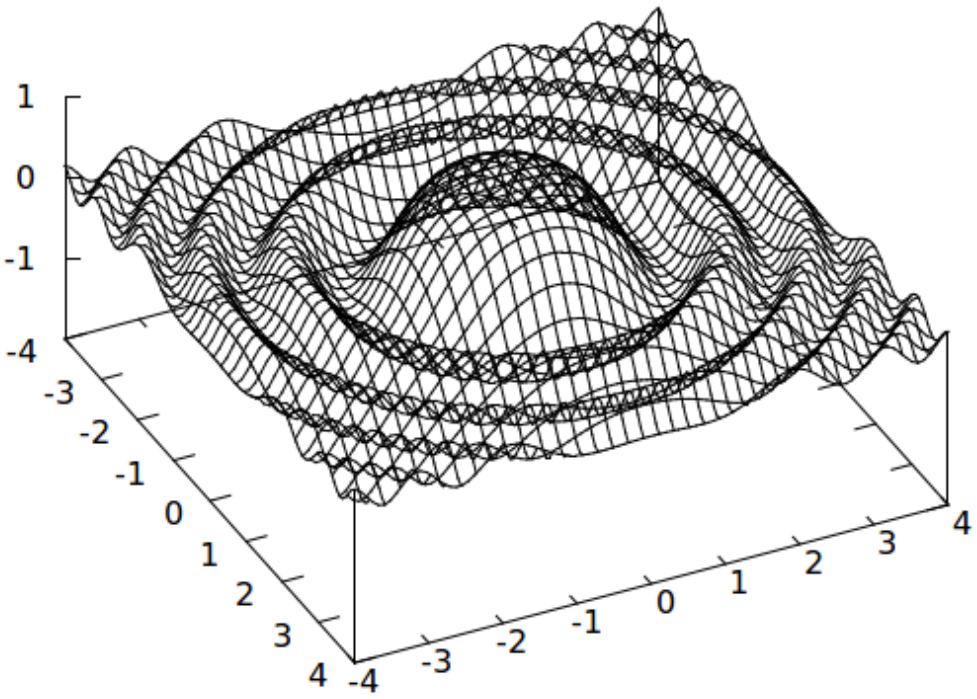


Figure 14: An example showing what happens when forcing the forcing frequency doesn't match the resonant frequency in an undamped forced system. Photo credit: Boyce and DiPrima.

If we consider what happens when the frequency of the forcing function matches the resonant frequency, then we expect similar finding as in the damped systems. Namely, we will easily be able to see that the particular solution to the differential equation must have the form $At\cos(\omega_0 t)$ so that the amplitude increases without bound once more. This again leads us to a catastrophic failure of the system. The analysis here is infinitely simpler than that we did for the damped systems, and we don't need to elaborate on it anymore here.

5 Series Solutions of Second-Order Linear Equations*





5.2 Series Solutions Near an Ordinary Point, Part I

Our scope for this course so far has been for **linear** differential equations with **constant** coefficients. Linearity is indispensible, so we don't want to remove this condition from the types of equations we study; however, we will remove the restriction to constant-coefficient equations and study those of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

That is, the goal of this section is to try to solve the general, linear, homogeneous, second-order ODE.

It would seem that our study is not without motivation, as many questions in physics lead to just such equations. For example, there is the *Bessel equation* of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

that describes rotationally symmetric vibrations. There is the *Legendre equation* of order α

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

which arises as part of the separation-of-variables solution to the Schrödinger equation for the hydrogen atom. Because of these models, we typically take our coefficient functions to be polynomials; however, the method we are going to study (due to Frobenius) can account for *analytic* functions.

To begin our study of these equations, let's assume that P, Q, and R have no common factors. Our goal here is to find a **power series solution** to the differential equation around some point x_0. Initially, to make life easier, we assume that $P(x_0) \neq 0$. This makes x_0 what we call an *ordinary point* of the equation. Because P is continuous (as it's a polynomial), we know there must exist an interval I about x_0 where P is nonzero on all of I. Hence, within this interval, we can rewrite the differential equation as

$$y'' + p(x)y' + q(x)y = 0$$

by simply dividing away $P(x)$. Moreover, both p and q are still continuous on I. Therefore, our second-order existence and uniqueness theorem tells us that (when we specify initial conditions) there must exist a unique solution to this equation on I. Notice that we may not apply this theorem when $P(x_0) = 0$. If this occurs, we say that x_0 is instead a *singular point* of the equation. Our methods will need to change to accommodate such points, but we study ordinary points for now.

Again, we aim to solve the equation near x_0, so it makes sense to use a local tool to analyze such a solution. Power series prove to be a good first thought here. Hence, we are looking for a solution of the form

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n.$$

In order to make this study interesting, we need to guarantee that we have a nonzero radius of convergence ρ for this series. That is, $y(x)$ is a solution that converges for all x such that $|x - x_0| < \rho$ for some radius $\rho > 0$. Our goal is then translated as **find the coefficients a_n necessary to satisfy the differential equation**.

We need to have a sort of "proof of concept" to make sure that this method is even viable as a solution technique (maybe it's just always going to be intractably difficult?).

Example 5.1. Let's begin with a familiar equation, whose solution is already known to exist for all x:

$$y'' + \omega^2 y = 0, \quad \omega > 0.$$

We identify our coefficient functions as $P(x) = 1$, $Q(x) = 0$, and $R(x) = \omega^2$. We choose an *expansion point* that will make computation relatively manageable - $x_0 = 0$ seems a good choice. We check that $x_0 = 0$ is indeed ordinary, and we proceed to find a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

centered at $x_0 = 0$ (where we assume that this solution converges over a nonzero radius ρ of $x_0 = 0$).

We now need to see what the differential equation looks like after we assume this is a solution, so we must differentiate the power series (term by term, of course) and substitute the results into the equation:

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Hence, upon substituting back into the differential equation, we find

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \omega^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

This is all well and good... but how do we work with this mess?

Recall that two (formal) power series are considered the same if they have the same coefficients. Namely, because the right-hand side is also a (trivially zero) power series, we just need to find a way to write the left-hand side as a **single** power series and equate the coefficients. To do this, let's align the indices of the two series on the left-hand side:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \omega^2 a_n x^n = \sum_{n=0} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0} \omega^2 a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + \omega^2 a_n] x^n.$$

Great! Now we can equate this to the zero series (from the original differential equation) so that we require

$$(n+2)(n+1) a_{n+2} + \omega^2 a_n = 0$$

for $n = 0, 1, 2, \ldots$ This is called a *recurrence relation* between the coefficients. Therefore, once we know a_0 and a_1, we can determine all other coefficients. Our goal now becomes **find a formula for a_n in terms of n, a_0, and a_1.**

Starting with a general a_0, we find

$$a_2 = -\frac{\omega^2}{2} a_0, \quad a_4 = -\frac{\omega^2}{4 \cdot 3} a_2 = \frac{(\omega^2)^2}{4!} a_0, \quad a_6 = -\frac{\omega^2}{6 \cdot 5} a_4 = -\frac{(\omega^2)^3}{6!} a_0, \ldots$$

We can see the pattern for the even terms is

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0.$$

In a similar fashion, the pattern for the odd terms is

$$a_{2n+1} = (-1)^n \frac{\omega^{2n+1}}{(2n+1)!} a_1.$$

This shows us that our power series solution takes on the form

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n}}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!} = a_0 \cos(\omega x) + a_1 \sin(\omega x),$$

which converges for an infinite radius of $x_0 = 0$ and is **exactly** what our usual methods tells us the solution should be - arbitrary coefficients and all!

We were lucky in this case: We **recognized** the power series as some other function we already knew! We will not always be so fortunate in general. Hence, we need a way to deal with these power series in general. We can actually find out quite a bit of information about

$$C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n}}{(2n)!} \quad \text{and} \quad S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!}.$$

For instance, we know that our solution

$$y(x) = a_0 C(x) + a_1 S(x).$$

We can determine that $C(0) = 1$ and $S(0) = 0$, that $C'(x) = -S(x)$ and $S'(x) = C(x)$, that C is even and S is odd, and (particularly usefully) that their Wronskian in nonzero everywhere - making $C(x)$ and $S(x)$ a fundamental set of solutions.

Let's try a new equation on for size:

Example 5.2. Consider *Airy's equation*

$$y'' - xy = 0.$$

Here $P(x) = 1$, $Q(x) = 0$, and $R(x) = -x$ so that we are looking at our first non-constant coefficient example (how exciting!). Because P is a nonzero constant, every point is ordinary for this equation - so we choose the easiest such point to work with: $x_0 = 0$.

We assume that we have a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

representing the solution for all points $|x| < \rho$ for some nonzero radius ρ. Then, using our computations for y' and y'' from before, we see that Airy's equation becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}a_n x^n = 0.$$

We now try to consolidate the two series on the left-hand side into one again:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}a_n x^{n+1} = \left(2\cdot 1a_2 + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n\right) - \sum_{n=1}a_{n-1}x^n$$

$$2a_2 + \sum n = 1^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1}]x^n.$$

Hence, we merely equation the coefficients (all of them) to zero so that

$$2a_2 = 0 \quad \text{and} \quad (n+2)(n+1)a_{n+2} = a_{n-1}$$

for $n = 1, 2, 3, \ldots$

Because $a_2 = 0$, we can immediately conclude that $0 = a_5 = a_8 = a_{11} = \cdots = a_{3n+2} = \cdots$. Our attention immediately falls to separating out the recurrence relation above into manageable pieces. The pattern for the remaining coefficients can be divided up into cases: (1) the index is divisible by three and (2) the index has remainder one after division by three. We find the pattern for a_{3n} to be

$$a_{3n} = \frac{a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}, \quad n = 1, 2, 3, \ldots$$

and the pattern for a_{3n+1} to be

$$a_{3n+1} = \frac{a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}, \quad n = 1, 2, 3, \ldots$$

Therefore, we conclude that the solution to Airy's equation about $x_0 = 0$ can be seen as

$$y(x) = a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{(2\cdot 3)\cdot(5\cdot 6)} + \cdots + \frac{x^{3n}}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}\right]$$

$$+a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{(3\cdot 4)\cdot(6\cdot 7)} + \cdots + \frac{x^{3n+1}}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}\right]$$

which we can write more compactly as

$$y(x) = a_0 Ai(x) + a_1 Bi(x).$$

These two *Airy functions* are not elementary functions; however, the ratio test will confirm that both $Ai(x)$ and $Bi(x)$ will converge for an infinite radius.

A rather curious observation is that these functions are oscillatory for $x < 0$ and exponential for $x > 0$. This is easily explained by the coefficient function $R(x) = x$. For $x < 0$ we see that Airy's equation mimics simple harmonic motion, while for $x > 0$ it mimics an exponential growth equation. Airy's functions $Ai(x)$ and $Bi(x)$ are, in some sense, a combination of these two types of phenomena - *bounding* and *scattering*.

Example 5.3. We actually continue the Airy equation example from above in this example. We noted that every point was ordinary for this equation, so why not expand at a different point? Maybe it will make our work easier?

Let's now consider a power series solution about $x_0 = 1$ for some nonzero radius ρ. That is, take

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n.$$

Similar computations for the derivatives hold so that

$$y' = \sum_{n=0} (n+1) a_{n+1} (x-1)^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n.$$

Then the Airy equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0.$$

Now we are unable to merely absorb the x term into the second series as we have been before. Instead, we write $x = (x-1) + 1$ so that we can say

$$\sum_{n=0} (n+2)(n+1) a_{n+2} (x-1)^n - (1 + (x-1)) \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = 0.$$

Pulling out the first term of the second series and shifting the index of the third series above shows that

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - a_0 - \sum_{n=1}^{\infty} [a_n + a_{n-1}] (x-1)^n = 0.$$

That is to say that our recurrence relation looks a little like

$$2a_2 = a_0 \quad \text{and} \quad (n+2)(n+1) a_{n+2} = a_n + a_{n-1}, \quad n = 1, 2, 3, \ldots.$$

Surprisingly enough, this actually **simplifies** what the resulting power series looks like (check the coefficients for yourself):

$$y(x) = a_0 \left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots \right]$$

$$+ a_1 \left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots \right].$$

There was **no way** for us to predict such a nicer form of a solution by shifting the expansion point, nor did we have any reason to suspect it would do anything better. Yet we have a simpler expression. In fact, calling $y(x) = a_0 U(x) + a_1 V(x)$, we see that these

two functions again form a fundamental set of solutions for the Airy equation. What will really get you thinking about it is that there is a way to equate the two solutions we've found. That is, we can find coefficients so that

$$aAi(x) + bBi(x) = a_0U(x) + a_1V(x).$$

This is just a matter of linear algebra.

Remark 5.1. In light of how rough it was initially to deal with a nonzero expansion point x_0, we can see one way to fix this up before we dive into some nasty algebra. We can **shift the equation** to the new coordinates $t = x - x_0$ and then use a power series in the variable t centered at $t_0 = 0$. That is, we change the equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$
$$\Leftrightarrow P(t + x_0)y''(t + x_0) + Q(t + x_0)y'(t + x_0) + R(t + x_0)y(t + x_0) = 0$$

to match the power series

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \Leftrightarrow y(t + x_0) = \sum_{n=0}^{\infty} a_n t^n.$$

We then proceed as we did in the first example of the section without having to play this algebraic game of "shift the coefficient functions" while we're dealing with infinitely many terms.

In the case of the previous example, we use the change of variables $t = x - 1$ so that the problem can been presented equivalently as

$$y'' - (t + 1)y = 0.$$

We would then solve it using the power series

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Without diving too far into the next section, we can outline the main result for ordinary points of more general equations whose coefficient functions aren't only mere polynomials. The type of functions that are allowable in order for us to apply our power series solution technique are called analytic functions. That is, a function $f(x)$ is *analytic at* $x = x_0$ if it may be expanded as a power series

$$f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

that converges within a **nonzero radius** of x_0 (that is, $|x - x_0| < \rho$ with $\rho > 0$). For the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

we say that x_0 is an **ordinary point of the differential equation** provided $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are both analytic at x_0. Otherwise, we say x_0 is a **singular point** for the equation.

With this terminology in mind, we can state the general theorem for ordinary points as follows:

Theorem 5.1. If x_0 is an ordinary point of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \Leftrightarrow \quad y'' + p(x)y' + r(x)y = 0,$$

and the functions p and q are analytic at x_0, then the general solution of this differential equation is

$$y(x) = \sum_{n=0}^{\infty} a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary constants, and y_1 and y_2 are two power series solutions to the equation that are analytic at x_0. Moreover, y_1 and y_2 are a fundamental set of solutions for the equation, and the radius of convergence for y is the minimum radius of convergence between the solutions y_1 and y_2.

This theorem allows us to solve equations like

$$y'' + \sin xy - \frac{y}{1+x^2} = 0$$

at any point and

$$xy'' + \sin xy = 0$$

near $x_0 = 0$... at least in theory.

5.4 Euler Equations; Regular Singular Points

We now move on to solving our differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

around a singular point x_0; that is, if $P(x_0) = 0$. A good, prototypical problem of this scenario is the second-order *Euler equation*:

$$L[y] = x^2y'' + \alpha xy' + \beta y = 0,$$

where α and β are real constants. We can easily see that $x = 0$ is the only singular point here. Because of this fact, our existence and uniqueness theorem for linear equations can only assume solutions exist for either $x < 0$ or $x > 0$.

At first, because our usual power series solution won't apply here, we can try to guess an ansatz solution. That is, we need a function whose derivatives lose a power of x only to have it filled in again by the coefficient functions. A reasonable candidate function here is

$$y(x) = x^r$$

for some real r, since $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Using this candidate solution, we obtain

$$L[x^r] = x^2[r(r-1)x^{r-2}] + \alpha x[rx^{r-1}] + \beta[x^r] = x^r[r(r-1) + \alpha r + \beta] = 0$$

so that either $x^r = 0$ for all x (which is impossible) or

$$r(r-1) + \alpha r + \beta = r^2 + (\alpha - 1)r + \beta = 0.$$

This eqation is called the *indicial equation* for the differential equation.

We can solve the indicial equation for r to find

$$r_{\pm} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}.$$

Hence, $y(x) = x^r$ will solve the equation only for these particular values of r. As with the characteristic polynomial for second-order equations with constant coefficients, we can separate our approach based on the types of r values we can expect to find: real and distinct, real and repeated, or complex conjugates (conjugates arise since we again assumed real α and β coefficients).

For the real-and-distinct case, we can easily see that the functions x^{r_+} and x^{r_-} form a fundamental set of solutions (check that their Wronskian is $\pm(r_+ - r_-)x^{r_+ + r_- - 1}$) so that

$$x^2y'' + \alpha xy' + \beta y = 0$$

is completely solved by

$$y(x) = c_+x^{r_+} + c_-x^{r_-},$$

where $x > 0$ or $x < 0$ (but not necessarily both!).

Example 5.4. Suppose we wish to solve the initial value problem

$$6x^2y'' - 7xy' - 3y = 0; \quad y(1) = -4,\ y'(1) = 5.$$

We can quickly rewrite this as

$$x^2y'' - \frac{7}{6}xy' - \frac{1}{2}y = 0$$

so that $\alpha = -\frac{7}{6}$ and $\beta = -\frac{1}{2}$ from our above discussion. Hence, solving the indicial equation for $r_{\pm}$ yields

$$r_+ = \frac{3}{2} \quad r_- = -\frac{1}{3}.$$

Then our general solution is

$$y(x) = c_+x^{\frac{3}{2}} + c_-x^{-\frac{1}{3}}.$$

We see that this solution will only be valid for $x > 0$ (therefore, our problem is well posed and can be solved uniquely).

Using our initial conditions gives us

$$\begin{aligned} c_+ + c_- &= -4 \\ \tfrac{3}{2}c_+ - \tfrac{1}{3}c_- &= 5 \end{aligned}$$

so that $c_+ = 2$ and $c_- = -6$. Then we finally conclude that

$$y(x) = 2x^{\frac{3}{2}} - 6x^{-\frac{1}{3}}, \quad x > 0.$$

Now we move to the repeated-root case. The previous method will only give us one solution: $y(x) = x^{\frac{1-\alpha}{2}}x$. As before, we can use reduction of order to find out candidate second solution in our fundamental set. For the sake of generality, however, we are going to employ another method.

Consider the function $F(r) = (r - r_+)(r - r_-)$. The equation $F(r) = 0$ is the indicial equation from above. Notice that our function in the case where $r_+ = r_- = R$ makes $F(r) = (r - R)^2$. Then $F(R) = 0$ and, more interestingly, $F'(R) = 0$ as well.

Why is this so much more interesting? Consider differentiating the differential equation $L[x^r] = 0$ with respect to r:

$$0 = \frac{\partial}{\partial r}L[x^r] = \frac{\partial}{\partial r}[x^r F(r)] = x^r \ln x \cdot F(r) + x^r \cdot F'(r).$$

At $r = R$, we see that

$$0 = x^R \ln x \cdot F(R) + x^R \cdot F'(R) = x^R \ln x \cdot F(R) = L[x^R]$$

so that $y(x) = x^R \ln x$ should also solve the equation (note that this was only possible because $F'(R) = 0$).

We can see that x^R and $x^R \ln x$ are indeed a fundamental set of solutions (check that their Wronskian is $\pm x^{2R-1}$) so that the general solution to the repeated-root case is

$$y(x) = c_1 x^R + c_2 x^R \ln x = x^R(c_1 + c_2 \ln x), \quad x > 0.$$

Example 5.5. Let's solve the equation

$$x^2 y'' + 3xy' + y = 0.$$

We can see that $R = -1$ solves the indicial equation so that

$$y(x) = x^{-1}(c_1 + c_2 \ln x), \quad x > 0$$

generally solves the differential equation.

This leads us to the complex-conjugate-root case. Let's call our solutions $r_\pm = \lambda \pm i\mu$, where λ and $\mu \neq 0$ are real numbers. Just a quick reality check here: What is x^i..?

We've never seen such a function before, so we need to re-frame it in such a way that we can make it more familiar. An easy way to do this is to observe that, in the case where r is real,

$$x^r = e^{\ln(x^r)} = e^{r \ln x}.$$

Then we can **define** x^r in the same way for the complex numbers as well (in general, such an extension is called *analytic continuation* of the real function to a complex domain). We can then continue with this definition and the fact that $r_\pm = \lambda \pm i\mu$ to find

$$x^{r_\pm} = x^\lambda x^{\pm i\mu} = x^\lambda e^{\pm i\mu \ln x} = x^\lambda[\cos(\mu \ln x) \pm i \sin(\mu \ln x)], \quad x > 0.$$

We can once again use the real and imaginary parts of this solution to obtain a pair of real-valued solutions:

$$y_1(x) = x^\lambda \cos(\mu \ln x) \quad \text{and} \quad y_2(x) = x^\lambda \sin(\mu \ln x).$$

These indeed form a fundamental set of solutions (check that their Wronskian is $\pm\mu x^{2\lambda-1}$). Then our general solution looks like

$$y(x) = x^\lambda[c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)].$$

Example 5.6. Let's solve

$$x^2y'' - 3xy' + 10y = 0.$$

We find that the indicial equation gives us $r_\pm = -2 \pm 4i$. Our general solution above then tells us that

$$y(x) = x^{-2}[c_1 \cos(4 \ln x) + c_2 \sin(4 \ln x)], \quad x > 0$$

generally solves the differential equation.

The astute reader will observe that all of our solutions have been found for the domain with $x > 0$. Does that mean it's impossible to make these solutions work for $x < 0$? Of course not! Our theorem **guarantees** existence of solutions on this interval as well! So how do we adapt our method so that we can account for these solutions? The reader can provide the details, but suffice it to say that we can merely substitute $x \mapsto -x$ to get exactly those solutions that will work on the interval $x < 0$. We can then combine our results in general to find that we have solutions of the form

$$y(x) = \begin{cases} c_1|x|^{r_+} + c_2|x|^{r_-}, & \text{real } r_+ \neq r_- \\ |x|^R(c_1 + c_2 \ln|x|), & \text{repeated } R \\ |x|^\lambda[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)], & r = \lambda \pm i\mu, \text{ with } \lambda, \mu \text{ real } (\mu \neq 0). \end{cases}$$

We again turn our attention to the general second-order equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \Leftrightarrow \quad y'' + p(x)y' + q(x)y = 0.$$

Because we are only interested in solutions near certain points, we can use this to our advantage to change our view of the differential equation itself. That is, **if we zoom in close enough, we only have to pay attention to the constant terms of the coefficient functions' Taylor series**! This is easy enough for zooming in around ordinary points: Their Taylor series tells us everything we need to know. For singular points, however, there is no such Taylor series; so zooming in is not quite so easy.

As it turns out, if we brazenly charge in and make it so that Taylor series are "close enough" to solutions of the equation, we can indeed solve the equation using an expansion about a singular point. Namely, we want to say that a series solution is off by a factor of x^r from a true solution. That is, in the next section, we will try to find solutions of the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

for the appropriate values of r.

We then ask ourselves what is a good enough condition to make Taylor series only off by a factor of x^r? This leads us to the following definition:

Definition 5.1. For a differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

a singular point x_0 (that is, a point for which $P(x_0) = 0$) is called a *regular singular point* of the equation if the limits

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x))}{P(x)} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

exist and are finite. If one or both of these limits does not exist or are infinite, then we say that x_0 is an *irregular singular point* of the equation.

Remark 5.2. The main takeaway from this definition is this: We have a chance of solving equations about singular points **only if the coefficient functions don't blow up too quickly**.

Example 5.7. The Euler equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

has a regular singular point at $x = 0$. (They're the "prototype", remember?)

Example 5.8. Suppose we wanted to solve the equation

$$x^2(x-4)^3y'' + 16\sin x\, y' - y = 0.$$

We can easily see that the only singular points of this equation are at $x = 0$ and $x = 4$. Checking the limits

$$\lim_{x\to 0} x\frac{16\sin x}{x^2(x-4)^3} = -\frac{1}{4} \quad \text{and} \quad \lim_{x\to 0} x^2\frac{-1}{x^2(x-4)^3} = -\frac{1}{64},$$

we see that $x = 0$ is a regular singular point. However, we see that the limit

$$\lim_{x\to 4}(x-4)^2\frac{-1}{x^2(x-4)^3}$$

does not exist. Hence, $x = 4$ must be an irregular singular point (we don't even need to check the other limit).

5.5 Series Solutions Near a Regular Singular Point, Part I

We now need to see why we studied the Euler equation. Again, we're turning to our general second-order linear equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

whose solution we would like to know near a regular singular point which, for simplicity, we assume to be $x_0 = 0$.

The real trick here is the fact that even though $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ aren't analytic at x_0, the functions $xp(x)$ and $x^2q(x)$ are (since their limits exist as $x \to x_0$). Therefore, we may write

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^2,$$

where **both** series converge on the interval $|x| < \rho$ for some $\rho > 0$. Once we've made this observation, we can multiply the original differential equation by x^2 and regroup those x's on the appropriate terms. That is, we write

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0.$$

Observe that if $xp(x)$ and $x^2q(x)$ are constant, we reduce ourselves to the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0.$$

What if the series $xp(x)$ and $x^2q(x)$ are not constant (which is more generally the case)? It turns out we really don't have too much to go on here. Our best guess it to use our solutions to the Euler equation to try to "fill in" the rest of the power series solution. That is, we try solutions of the form

$$y(X) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n},$$

where x^r satisfies the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0$$

and $a_0 \neq 0$.

Example 5.9. Let's solve an interesting problem, one that appears frequently in mechanical systems. The *Bessel function of order* ν is given as

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

We immediately see that $x = 0$ is the only singular point of the equation. Looking a little more closely, we can easily see that $x = 0$ is in fact a regular singular point ($p_0 = 1$ and $q_0 = -\nu^2$). This shows us that our indicial equation is

$$r(r-1) + r - \nu^2 = 0 \quad \Leftrightarrow \quad r^2 - \nu^2 = 0$$

so that $r = \pm\nu$.

We then use our ansatz solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

and find

$$y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}.$$

Substituting these into the Bessel equation shows us

$$\sum_{n=0}^{\infty} a_n x^{r+n+2} + \sum_{n=0}^{\infty} a_n [(r+n)^2 - \nu^2] x^{r+n} = 0.$$

In order to bring these series under a single summation, we pull out the first two terms of the second series and shift the index of the first:

$$a_0[(r+0)^2 - \nu^2]x^r + a_1[(r+1)^2 - \nu^2]x^{r+1} + \sum_{n=2}^{\infty} [a_{n-2} + a_n((r+n)^2 - \nu^2)]x^{r+n} = 0.$$

The a_0-term vanishes since the indicial equation kills it. For the other terms, we recall that we are working in the interval $x > 0$ so that the only way for this equality to hold is by making all coefficients in this series zero. We then find that

$$a_1[(r+1)^2 - \nu^2] = 0 \quad \Rightarrow a_1 = 0$$

and

$$a_{n-2} + a_n[(r+n)^2 - \nu^2] = 0, \quad n \geq 2.$$

Combining the recurrence equation with the finding that $a_1 = 0$ shows that $a_{2k+1} = 0$ for all $k = 0, 1, 2, \ldots$. Solving the recurrence equation for a_n gives us

$$a_n = -\frac{a_{n-2}}{n(n+2\nu)}, \quad n = 2, 4, 6, \ldots,$$

where a_0 is arbitrary and we have used the indicial equation again to simplify the denominator on the right-hand side.

We could go on to solve the Bessel equation in general, but this is tricky for the general order-ν equation. Hence, we simplify matters and solve the order-$\frac{1}{2}$ equation:

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0.$$

Why do we do this? It turns out that the recurrence equation (with the substitution $n = 2m$ becomes much more manageable in this case:

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)}, \quad n = 1, 2, 3, \ldots$$

Term by term, we can solve for a pattern (closed form) in the a_{2m} terms:

$$a_2 = -\frac{a_0}{3!}, \quad a_4 = \frac{a_0}{5!}, \quad \cdots, \quad a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad m = 1, 2, 3, \ldots$$

Then our ansatz solution looks like

$$y(x) = x^{\frac{1}{2}} \left[a_0 + \sum_{n=0}^{\infty} \frac{(-1)^m a_0 x^{2m}}{(2m+1)!} \right] = x^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m a_0 x^{2m+1}}{(2m+1)!}, \quad x > 0.$$

It should be noted here that the series can be shown (using the ratio test) to have an infinite radius of convergence. In fact, the second series above should look quite familiar: It's the **sine function**! Therefore, we find the solution associated with $x^{\frac{1}{2}}$ is

$$y_1(x) = a_0 \frac{\sin x}{\sqrt{x}}, \quad x > 0.$$

A similar argument shows that the solution associated with $x^{-\frac{1}{2}}$ is

$$y_2(x) = b_0 \frac{\cos x}{\sqrt{x}} + b_1 \frac{\sin x}{\sqrt{x}}, \quad x > 0.$$

Since we already have the second term as a solution associated with $x^{\frac{1}{2}}$, we see that our complete general solution is

$$y(x) = a_0 \frac{\sin x}{\sqrt{x}} + a_1 \frac{\cos x}{\sqrt{x}}, \quad x > 0.$$

In general, we use a sort of "normalized" function to describe this general solution:

$$y(x) = a_0 J_{\frac{1}{2}}(x) + a_1 J_{-\frac{1}{2}}(x), \quad x > 0,$$

where

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \quad x > 0.$$

These J functions are called *Bessel functions of the first kind of order* $\frac{1}{2}$. Observe that $J_{\frac{1}{2}}(x)$ approaches zero as $x \to 0$ but that $J_{-\frac{1}{2}}(x)$ approaches ∞ as $x \to 0$. Hence, we usually use the former in physical scenarios.

Remark 5.3. This is just one example of how we can solve this type of equation in general. The important things to note are:

1. The values r that solve the indicial equation tell us exactly how the solution behaves near the singularity. Namely, they tell us "how badly" the solution blows up there. As such, the r values that solve the indicial equation are sometimes called the *exponents of the singularity.*

2. Notice that if the exponents of the singularity differ by an integer, then the two solutions found by this *method of Frobenius* differ only by a multiplication by x^k for some integer k. That means that we may not have found a second independent solution to the equation. Typically (but not always), as in the repeated-root case of the Euler equation, we need to add a logarithmic term to the first solution in order to get our second independent solution. That is, **at least one solution accounts for the nasty behavior near the singular point**.

3. Polynomial coefficients are vastly easier to work with compared to general analytic functions. If we find ourselves having to multiply two power series together, it's awful but possible to do so with the closed formula

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

This coefficient is called the n^{th} *convolution of* a_n *and* b_n.

We could go on into the next section to see the full theory behind these equations. Frankly, it's not much more enlightening than what we've already seen from the theory of the Euler equation. Hence, it will suffice to just tell you that the derivation of the general solutions look very similar to the previous section. The following theorem summarizes the results:

Theorem 5.2. Consider the differential equation

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0,$$

where $x_0 = 0$ is a regular singular point. Then $xp(x)$ and $x^2q(x)$ are analytic at $x_0 = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2q(x) = \sum_{n=0}^{\infty}$$

for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2p(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right],$$

where the $a_n(r_1)$ are given by the recurrence relation

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0, \quad n \geq 1$$

using $a_0 = 1$ and $r = r_1$.

If $r_1 - r_2$ is not zero or a positive integer, then in either interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a second solution of the form

$$y_2(x) = |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right].$$

The $a_n(r_2)$ are determined by the same recurrence relation above with $a_0 = 1$ but with $r = r_2$ instead. The power series above converge at least for $|x| < \rho$.

If $r_1 = r_2$, then the second solution is instead

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n.$$

If $r_1 - r_2 = N$, a positive integer, then

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right].$$

The coefficients $a_n(r_1)$, $b_n(r_1)$, $x_n(r_2)$, and the constant a can be determined by substituting the form

of the series solution into the differential equation. The constant a may turn out to be zero, in which case there is no logarithmic term in the solution. Each of the series in the previous two cases converge at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x_0 = 0$.

In all three cases, the two solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the given differential equation.

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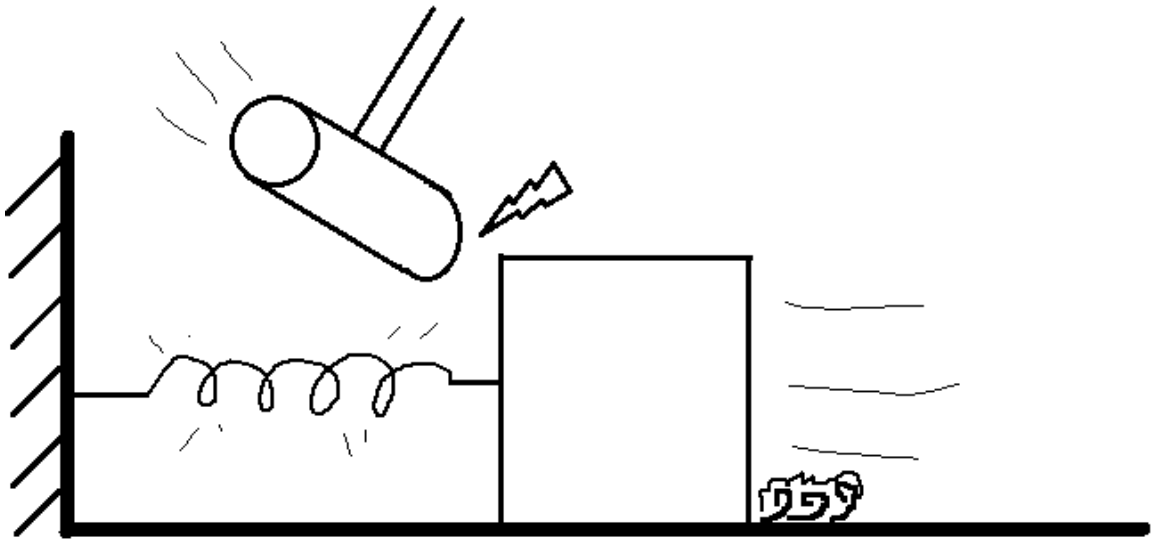
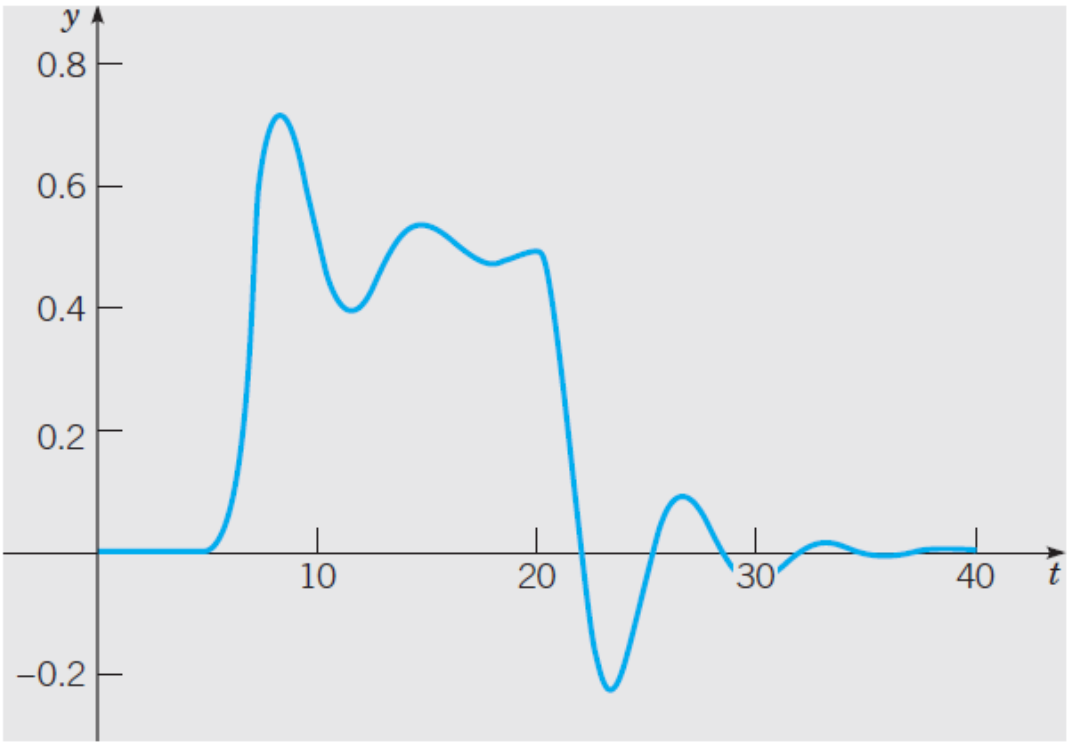
6 The Laplace Transform





6.1 Definition of the Laplace Transform

Now that we have established how to solve for general solutions of some simple first- and second-order linear differential equations, we can move on to more common solution techniques which may be extended to higher-ordered equations. In order to do this, we need to remind ourselves about improper integrals.

Definition 6.1. Let f be continuous on the interval (a, ∞). We define

$$I = \int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt$$

as the *improper integral* of f on the interval (a, ∞). If the limit exists and is finite, we say that I *converges*. If the limit exists and is infinite or if the limit doesn't exist, we say that I *diverges*.

We remember how to calculate these sorts of definite integrals from integral Calculus.

Example 6.1. We can compute a few examples to warm up to the tool that we wish to use later.

1. Let's compute
$$I = \int_0^\infty e^{ct}\,dt,$$
where c is just some real number. We easily find
$$I = \lim_{A\to\infty} \int_0^A e^{ct}\,dt = \lim_{A\to\infty} \left[\frac{1}{c}e^{ct}\right]_0^A = \lim_{A\to\infty} \left[\frac{1}{c}e^{Ac} - \frac{1}{c}e^0\right].$$
The limit does not exist if $c > 0$ since the exponential term would tend to infinity. The limit again doesn't exist if $c = 0$ since we would be reduced to evaluating the integral of $\int_0^\infty dt$, which is divergent. The limit only exists when $c < 0$ so that the exponential term tends to zero and
$$I = -\frac{1}{c}, \quad c < 0.$$

2. Consider the improper integral
$$I = \int_1^\infty \frac{1}{t}\,dt.$$
We compute
$$I = \lim_{A\to\infty} \int_1^A \frac{1}{t}\,dt = \lim_{A\to\infty} \left[\ln|t|\right]_1^A = \lim_{A\to\infty} \ln A = \infty.$$
Hence, this improper integral diverges.

3. Let $f(t) = t^{-p}$ for $t \geq 1$, where p is real and $p \neq 1$. Then the improper integral

$$I = \int_1^\infty t^{-p}\,dt = \lim_{A\to\infty}\int_1^A t^{-p}\,dt = \lim_{A\to\infty}\frac{1}{1-p}(A^{1-p}-1).$$

If $p > 1$, then this integral converges to $\frac{1}{p-1}$. Otherwise, the term $A^{1-p} \to \infty$ as $A \to \infty$ so that the integral diverges. These integrals are comparable to the series $\sum_{n=1}^\infty \frac{1}{n^p}$.

To continue on our path to our main tool, the Laplace transform, we need to build one more piece of terminology. So we have the following definition:

Definition 6.2. A function f is said to be *piecewise continuous* on an interval $\alpha \leq t \leq \beta$ if the interval may be broken up into a **finite** number of subintervals $\alpha = t_0 < t_1 < t_2 < \cdots < t_n = \beta$ where f is continuous on each of these subintervals and approaches a finite limit at the endpoints of each subinterval (considered from the appropriate sides).

This definition just says that these types of functions are allowed to be discontinuous at a finite number of points, but they must still be bounded. This allows us to break up integrals of these types of functions into their continuous components as

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \cdots + \int_{t_{n-1}}^\beta f(t)\,dt.$$

We can also assign whatever finite value of f at each point of discontinuity without changing the value of the above integral. This has to do with the fact that the points of discontinuity form what is called a *set of Lebesgue measure zero*. It is a subtlety that can't be ignored in general, but it works well for us now to be able to split integrals without consequence.

If we can't compute an improper integral explicitly, we often need to appeal to the methods of Calculus to compare this integral to some other known improper integral that either converges or diverges. We summarize this as the following theorem:

Theorem 6.1. If f is piecewise continuous for $t \geq a$, if $|f(t)| \geq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

The reader should recall this theorem as the Comparison Test from Calculus. Now that we have this theorem accounted for, we have enough of a foundation to define the Laplace transform.

The Laplace transform is among a class of tools called *integral transforms*. A general integral transform takes on the form

$$F(s) = \int_\alpha^\beta K(s,t)f(t)\,dt,$$

where α and β are fixed limits and K is a fixed function called the *kernel* of the transform. It can be so that $\alpha = -\infty$ or $\beta = \infty$ or both. The resulting function $F(s)$ is called the *transform* of f. There are many integral transforms, but we are only going to be interested in the Laplace transform since it has its most useful applications in differential equations.

Definition 6.3. Let $f(t)$ be a function defined on $t > 0$. Define

$$\mathcal{L}[f(t)](s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

If F converges for some interval in the s-domain, then we call F the *Laplace transform* of f.

It is not always the case that these integrals converge (as we saw in the examples). Hence, we establish the following theorem:

Theorem 6.2. Suppose that

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality, $K > 0$, a, and $M > 0$ are real constants. (Another way of saying this is in "big-oh" notation - that f is $O(e^{at})$).

Then the Laplace transform $\mathcal{L}[f(t)](s) = F(s)$ exists for $s > a$.

Remark 6.1. It is conventional to denote the Laplace transform of a function with the corresponding capital letter. Here, $\mathcal{L}[f(t)](s) = F(s)$. As another example, convention would dictate that $\mathcal{L}[q(t)](s)$ be denoted $Q(s)$.

We can convince ourselves that this theorem does indeed guarantee the existence of the Laplace transform. First, we notice that we can split the integral into two parts

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t)\, dt + \int_M^\infty e^{-st} f(t)\, dt.$$

The first integral converges since the function is piecewise continuous on $0 \leq t \leq M$. So our attention turns to the second integral. We know that the theorem tells us that f is of *exponential order*. Hence, we know that

$$|e^{-st} f(t)| \leq Ke^{-st} e^{at} = Ke^{(a-s)t}.$$

Then theorem 6.1 tells us that if $a - s < 0$, then the exponential integral converges so that the second integral above also converges. Hence, the theorem is indeed correct.

Remark 6.2. In general, the variable s should be taken to be complex. The issue of convergence can usually be relegated to studying the real part $\mathrm{Re}(s)$ of s, since this is the only component whose absolute value can grow or diminish. The imaginary part becomes a unit-length complex number, as seen by using Euler's formula.

Remark 6.3. It should be said that there are certainly functions that will not satisfy the second criterion of the theorem. For example, the function e^{t^2} is not of exponential order. This function increases faster than any function e^{at}.

Let's see a few examples of Laplace transforms:

Example 6.2. 1. Let $f(t) = 1$ for $t \geq 0$. Then, we can compute

$$F(s) = \int_0^\infty e^{-st} 1 \, dt = \frac{1}{s},$$

provided $s > 0$. This integral was computed in a previous example where $s = -c$.

2. Consider $f(t) = e^{at}$ for $t \geq 0$. Then we can compute

$$F(s) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{(a-s)t} \, dt = \frac{1}{s-a},$$

provided $s > a$.

3. Let's define

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1, \end{cases}$$

where k is any finite constant. This function is indeed piecewise continuous, so we move on to compute its Laplace transform:

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt = \int_0^1 e^{-st} \, dt = \left[-\frac{e^{-st}}{s} \right]_0^1 = \frac{1 - e^{-s}}{s}, \quad s > 0.$$

Notice here that it certainly did not matter which value of k we chose since this definition occurred on the discontinuity at $t = 1$. This shows us that there are **many** functions that differ only at a single point but still have the same Laplace transform.

[Note: It is often convenient (for a vast number of reasons) to view these functions as equivalent. The general theory was considered by Lebesgue, and the equivalence classes of functions (that differ only at a relatively few number of points) are now called *Lebesgue functions*.]

4. Consider now the function $f(t) = \sin(at)$ for $t \geq 0$. We compute

$$F(s) = \int_0^\infty e^{st} \sin(at) \, dt = \lim_{A \to \infty} \int_0^A e^{-st} \sin(at) \, dt.$$

We leave it to the reader to confirm that this can be evaluated as

$$F(s) = \lim_{A \to \infty} \left[-e^{-st} \frac{s \sin(at) + a \cos(at)}{a^2 + s^2} \right]_0^A.$$

We then evaluate this limit so that

$$F(s) = \frac{a}{a^2 + s^2}, \quad s > 0.$$

We take a moment at the end of the section here to remark that the Laplace transform at its basic foundation is simply an integral. This means that if the integrand is "nice" enough, then we can extend the linearity of the integral to the Laplace transform. That is, for any constants c_1 and c_2 and any admissible functions f_1 and f_2, we can see that

$$\begin{aligned}\mathcal{L}[c_1 f_2(t) + c_2 f_2(t)](s) &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt \\ &= c_1 \mathcal{L}[f_1(t)](s) + c_2 \mathcal{L}[f_2(t)](s).\end{aligned}$$

Hence, the Laplace transform is a *linear operator*. This will be of great importance in our study of linear differential equations in the next section.

6.2 Solution of Initial Value Problems

Now we come to the reason why we have defined the Laplace transform: **They help us solve initial value problems**! We begin this section by building up some machinery of the Laplace transform to see where we can use them.

Theorem 6.3. Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}[f'(t)]$ exists for $s > a$. Moreover, we have

$$\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)] - f(0).$$

We can easily prove this theorem by considering the integral

$$I_A = \int_0^A e^{-st} f'(t)\,dt.$$

If the limit of this integral as $A \to \infty$ exists, then it is the Laplace transform of f'. Since f' is piecewise continuous on $[0, A]$, we can write the above integral as

$$I_A = \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \cdots + \int_{t_k}^{A} e^{-st} f'(t)\,dt.$$

The real key to this problem is an application of integration by parts on each of these subintegrals. Then we have

$$\begin{aligned}I_A = \left([e^{-st} f(t)]_0^{t_1} + [e^{-st} f(t)]_{t_2}^{t_3} + \cdots + [e^{-st} f(t)]_{t_k}^{A}\right) \\ +s\left(\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \cdots + \int_{t_k}^{A} e^{-st} f(t)\,dt\right).\end{aligned}$$

Because f is continuous, the integral contributions from each of the $t_1, t_2, \ldots, t_k$ limits all cancel. As such, we have

$$I_A = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\,dt.$$

As we take $A \to \infty$, the integral becomes $\mathcal{L}[f(t)]$, and the first term tends to zero because we assume $s > a$ and f is of exponential order beyond $t \geq M$. Therefore,

$$\mathcal{L}[f'(t)](s) = \lim_{A\to\infty} I_A = s\mathcal{L}[f(t)](s) - f(0).$$

Having proven this fact, we can use mathematical induction to establish a powerful corollary:

Corollary 6.1. Suppose that the functions $f, f', f'', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants K, a, and M such that $|f(t)|, |f'(t)|, \ldots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}[f^{(n)}(t)]$ exists for $s > a$ and is given by

$$\mathcal{L}[f^{(n)}(t)](s) = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Let's see how this is applied in a few examples.

Example 6.3. Consider the initial value problem

$$y'' - y' - 2y = 0; \quad y(0) = 1, \quad y'(0) = 0.$$

We can apply the Laplace transform to both sides of the equation and make use of its linearity to find

$$(s^2Y(s) - sy(0) - y'(0)) - (sY(s) - y(0)) - 2Y(s) = 0,$$

where $Y(s)$ denotes the Laplace transform of $y(t)$. Using the assumed initial conditions, we can simplify this to

$$(s^2 - s - 2)Y(s) - s + 1 = 0.$$

We can now solve this **algebraic** expression for $Y(s)$ so that

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s+1)(s-2)}.$$

We have now reached a pivotal point in solving the initial value problem. We have found out what the solution looks like **in s-space**! This is great!.. except now we need to know what this looks like as a function of t instead of s. The general way to do this is to use the *inverse Laplace transform*. However, it requires that we know complex analysis to apply it. This is not in the scope of this class, so we will resort to a commonly used practical method.

We can split our expression for $Y(s)$ into two summands using partial fractions. The reader can verify that this is given by

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

We then have a moment of clarity and realize that these look very similar to the Laplace transforms of e^{-t} and e^{2t}. In this realization, we apply the inverse operation to both sides of the above equation so that

$$\mathcal{L}^{-1}[Y(s)] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right].$$

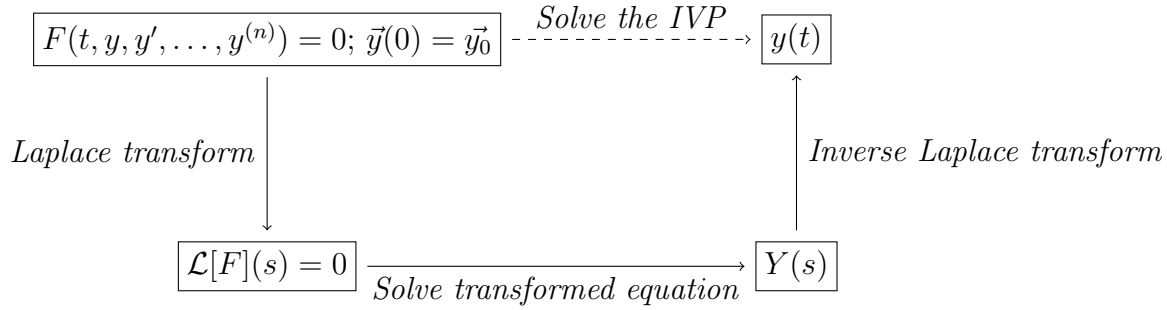
We finish our analysis by recalling that the Laplace transform will naturally cancel with its inverse so that we have

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Let's break down our method from above.

1. We apply the Laplace transform to the differential equation in the intial value problem so that we have an algebraic equation involving $Y(s)$.
2. We solve this equation algebraically for $Y(s)$.
3. Beat this expression for $Y(s)$ into a form where we can recognize the individual terms as Laplace transforms of functions we know. This is usually accomplished by appealing to a table to recognize the forms of the terms of the expression for $Y(s)$.

To summarize what is happening here, consider the following diagram:



.

This diagram emphasizes that we are taking an alternate route to solving the IVP which is (supposedly) easier work. Because we need to appeal to a table of Laplace transforms, it might be reasonable to expect that we are going to present a table. So Figure 15 gets this job done. Step 3 above is often the most difficult part of solving initial value problems in this fashion. The trick is to find a way to write $Y(s)$ as a linear combination of familiar Laplace transforms. In should be noted that the inverse Laplace transform is also linear so that we can invert each summand of the function $Y(s)$ individually and then add the results together to find $y(t)$.

A natural question to ask at the end of this method is if there are multiple possible outcomes for $y(t)$. We did see in the previous section that there can be an infinite number of functions that have the same Laplace transform. However, this only occurred when we didn't assume that the solution function wasn't continuous. We eliminate any ambiguity by imposing this continuity condition on our solution $y(t)$. This now establishes a near one-to-one correspondence between functions and Laplace transforms, and it makes sense to have created our table. [Note: To make it a true one-to-one correspondence, we need to use Lebesgue's equivalence mentioned in the previous section.]

We should do a few more examples to illustrate this process more lucidly.

Example 6.4. 1. Let's solve the initial value problem

$$y'' + y = \sin(2t); \quad y(0) = 2, \quad y'(0) = 1$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. t^n, $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$

Figure 15: Table of common Laplace transforms. Photo credit: Boyce and DiPrima.

using Laplace transforms. We follow the first step by applying the Laplace transform to both sides of the differential equation, using the table 15 for the sine term:

$$(s^2Y(s) - sy(0) - y'(0)) + (Y(s)) = \frac{2}{s^2 + 2^2}.$$

We can simplify this as

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}.$$

This doesn't match any of the forms on the right-hand side of table 15. However, we employ partial fractions to separate it. The reader can check that this decomposition is

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4} = 2\frac{s}{s^2 + 1} + \frac{5}{3}\frac{1}{s^2 + 1} - \frac{1}{3}\frac{2}{s^2 + 4}.$$

We see that each of these can be matched with one of the terms in the table, and hence we can invert $Y(s)$ so that

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t).$$

2. One particularly useful feature of Laplace transforms (which was alluded to in corollary 6.1) is that we can use them on higher-ordered differential equations. Let's use the same method as above to solve

$$y^{(4)} - y = 0; \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0.$$

Applying the Laplace transform to both sides of the differential equation gives us

$$(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) - (Y(s)) = 0,$$

which can be simplified to

$$Y(s) = \frac{s^2}{s^4 - 1} = \frac{s^2}{(s^2 - 1)(s^2 + 1)}.$$

Using partial fractions to separate this expression gives us

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}.$$

We can once again invert the transform so that (after consulting the table)

$$y(t) = \frac{1}{2}\sinh t + \frac{1}{2}\sin t = \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t.$$

The most useful feature of the Laplace transform is that of the method of variation of parameters: There is no "guess and check" methodology. This minimizes the chances of human-based error in the calculation of the solution. One could always implement an algorithm in a language that supports symbolic algebra, and the methods developed in sections 3.5 and 3.6 would work just as well as the methods developed here. However,

there are always going to be exceptions that "break" our algorithms. Often times it is **exactly** these problems that show up in practice. Hence, to fully understand the power of the Laplace transform, one should take a course or two in "Complex Analysis".

Perhaps the most important applications of the Laplace transform occurs in physics and engineering. In particular, it is used in exactly the systems considered in sections 3.7 and 3.8 that we have already studied. What we are going to do now is find a way to represent practical effects in mathematical terms. These are at the heart of how we come to studying step functions and the delta functional.

6.3 Step Functions*

Knowing how to solve an initial value problem with Laplace transforms is good and all... but we know how to solve these same problems **without** them. So why did we learn about them? Their principal use only reveals itself once we consider differential equations that are modeled with **discontinuous**(!) functions.

There are two main varieties of discontinuous functions that we usually consider. The first (which we save for the next section) models an *impulse*. When we want to consider systems in which a force is only briefly (almost instantaneously) present and then (just as quickly) is removed from the equation, we turn toward what is called a *delta functional* (yes, the "-al" makes this a different object than just a "function"). Think of delta functionals as modeling a blow from a hammer or maybe a static shock of electricity.

The second discontinuous function models *instantaneous presence or absence*. These *step functions* will be the focus of this section. Intuitively, you can think of them as "light switches" turning external forces on and off. Perhaps we're modeling an electromagnet turning on or an object moving from one medium to another (an object falling into water is a good example here).

We define the *unit step function* (or *Heaviside function*) as the piecewise function

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$

This is the function that models "turning $f(t) = 1$ on" at $t = c$. Naturally, we can ask if there is a specific function that we can use to model "turning $f(t) = 1$ off" at $t = c$. There's no need to define a new function for this because we can use u_c to do it. That is, "turning $f(t) = 1$ off" at $t = c$ is modeled by adding to $f(t)$ the function $-u_c(t)$.

It's possible to model piecewise functions using **only** the unit step function, and it's much easier to work with than defining multiple domains in the piecewise definition. Let's see some examples of how this is done.

Example 6.5. Suppose we wanted to turn on the function $f(t) = 5$ at $t = 3$. Not only do we wish to turn it on, we wish to leave it on only for ten units; and when we do, we only turn it down to $g(t) = 2$. That is, we should (a) turn f on at $t = 3$, (b) turn f off at $t = 13$, and (c) turn on g at $t = 13$.

For (a), we need to have an "on switch" at $t = 3$; and we need to make it turn on the function $f(t) = 5$. Here we can use $5u_3(t)$ - hat is, we merely scale the unit step function by 5.

For (b), we need to have an "off switch" for f at $t = 13$. We simply add $5(-u_{13}(t))$ at $t = 13$ to do this.

For (c), we need to have another "on switch" at $t = 13$, but this one needs to turn on $g(t) = 2$. We use the function $2u_{13}(t)$ to do so.

Putting all of these together is just as easy as adding all three models together:

$$F(t) = 5u_3(t) + 5(-u_{13}(t)) + 2u_{13}(t).$$

We can check this by using the piecewise definition of the step function:

$$F(t) = \begin{cases} 5 \cdot 0 + 5 \cdot (-0) + 2 \cdot 0, & t < 3 \\ 5 \cdot 1 + 5 \cdot (-0) + 2 \cdot 0, & 3 \le t < 13 \\ 5 \cdot 1 + 5 \cdot (-5) + 2 \cdot 1, & t \ge 13 \end{cases} = \begin{cases} 0, & t < 3 \\ 5, & 3 \le t < 13 \\ 2, & t \ge 13, \end{cases}$$

which is indeed the function that we wanted to model.

It is worth remarking that we could have combined steps (b) and (c) above into one step (b′): "take off f **while adding on** g". This would have looked like the function $f(t)(-u_{13}(t)) + g(t)u_{13}(t) = (g(t) - f(t))u_{13}(t)$, which is effectively the same process described above.

Example 6.6. Now suppose we didn't just want to turn on a constant. How would we turn on the function $f(t) = t^2$ at $t = 2$? Similarly to the previous example, we scale the unit step function and move it to the correct location.

An initial guess would be to use the function

$$F(t) = t^2 u_2(t).$$

However, as before, this model function is **discontinuous**. Solutions to differential equations typically must be **continuous**. So how do we still model a parabola that doesn't turn on until a certain point by a continuous function?

It is easiest to see what to do here by using the piecewise definition of $F(t)$ for various choices of the parabola. That is, we need to figure out which **translates** of the parabola will align with the zero function:

$$F_c(t) = (t - c)^2 u_2(t).$$

For which c is this a continuous function? Well, we merely "slide" the parabola over to the point of the discontinuity - at $c = 2$. That is, we use

$$F_2(t) = (t - 2)^2 u_2(t).$$

Now how does this fit in with Laplace transforms? The Laplace transform of the unit step function is actually remarkably easy to compute:

$$\mathcal{L}[u_c(t)] = \int_0^\infty e^{-st} u_c(t)\, dt = \int_0^c e^{-st} \cdot 0\, dt + \int_c^\infty 1 \cdot e^{-st}\, dt = \frac{e^{-cs}}{s}, \quad s > 0.$$

In a similar fashion, functions of the form

$$g(t) = \begin{cases} 0, & t < c, \\ f(t - c), & t \ge c \end{cases} = f(t - c)u_c(t)$$

also have simply computed Laplace transforms - provided $\mathcal{L}[f(t)]$ exists in the first place.

Theorem 6.4. If $F(s) = \mathcal{L}[f(t)]$ exists for $s > a \geq 0$, and if c is a positive constant, then

$$\mathcal{L}[f(t-c)u_c(t)] = e^{-cs}\mathcal{L}[f(t)] = e^{-cs}F(s), \quad s > a.$$

Conversely, if $f(t) = \mathcal{L}^{-1}[F(s)]$, then

$$f(t-c)u_c(t) = \mathcal{L}^{-1}[e^{-cs}F(s)].$$

The proof of this theorem is proven simply via a change of variables where $\xi = t - c$ (the reader should carry out this computation now - don't forget to change the limits of integration along the way).

Remark 6.4. As the reader verifies this theorem for his or herself, we can see now the importance of translating our functions to make them continuous upon connecting them to the unit step function. If we **don't** translate, then the change of variables prescribed above will not be able to shift the limits of integration to the appropriate coordinates. Otherwise, we would end up with

$$\mathcal{L}[f(t)u_c(t)] = \int_{-c}^{\infty} e^{-c\xi} f(\xi)\, d\xi,$$

where it is much more difficult to relate to $F(s)$.

So, really, we make the function a little "worse" in the t-space so that its Laplace transform is easier to describe in the s-space.

We'll see how to use this theorem in a moment, but for now it's important to consider another perspective. We can see it most easily in terms of another theorem:

Theorem 6.5. If $F(s) = \mathcal{L}[f(t)]$ exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}[e^{ct}f(t)] = F(s-c), \quad s > a + c.$$

Conversely, if $f(t) = \mathcal{L}^{-1}[F(s)]$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}[F(s-c)].$$

It is a quick matter to verify this theorem:

$$\mathcal{L}[e^{ct}f(t)] = \int_0^{\infty} e^{-st}e^{ct}f(t)\, dt = \int_0^{\infty} e^{-(s-c)t}f(t)\, dt = F(s-c).$$

Again there is the subtlety of considering the domain of convergence for this improper integral, but in this course we will not worry about this detail any further.

Remark 6.5. Theorem 6.5 shows us what a translation in s-space looks like in t-space. This amounts to a multiplication by an exponential function e^{ct}. It is then natural to ask what happens in t-space after we multiply a function in s-space (whose Laplace transform we already know) by the exponential function e^{-st}. By Theorem 6.4, we see that the result is a "translation" in t-space that involves the use of the unit step function. Remember that the odd form of the "translation" is due to two facts:

1. The resulting change of variables in the Laplace transform integral must be recognizable as an exponential multiple of a known Laplace transform.
2. The inverse Laplace transform must be continuous across the shift in domains of the step function.

Let's see these theorems applied to a physical problem:

Example 6.7. Suppose we had a simple mass-spring system modeled by the initial value problem

$$y'' + y = 0; \quad y(0) = 1,\ y'(0) = 0.$$

Assume now that the mass is actually made of a magnetic material so that when we turn on an electromagnet at time $t = 3\pi$ the mass feels a constant force of 1 Newton. We can model this **new** scenario by the initial value problem

$$y'' + y = u_{3\pi}(t), \quad y(0) = 1,\ y'(0) = 0.$$

The traditional way to solve this without Laplce transforms is to break up the original system into domains of continuity. That is, every time we see a "jump" in the model, we split it into another system. Namely, we solve

$$y'' + y = 0, \quad y(0) = 1,\ y'(0) = 0$$

first on the interval $0 \leq t < 3\pi$, and we recall that this is solved by $y(t) = \cos t$ for $0 \leq t < 3\pi$. The problem now **changes** as we have hit a discontinuity in our model. We set up the new equation as

$$y'' + y = 1,$$

but we appeal to the previous solution to gain our new initial conditions: $y(3\pi) = \cos(3\pi) = -1$ and $y'(3\pi) = -\sin(3\pi) = 0$ (this forces our final solution to be continuous across the discontinuity). That is, we now solve

$$y'' + y = 1, \quad y(3\pi) = -1,\ y'(3\pi) = 0$$

on the interval $3\pi \leq t$. We use whatever method we'd like here in order to solve this as $y(t) = 1 + 2\cos t$ on $3\pi \leq t$.

All we do now is "glue" these solutions together to find that

$$y(t) = \begin{cases} \cos t, & 0 \leq t < 3\pi \\ 1 + 2\cos t, & 3\pi \leq t. \end{cases}$$

There is indeed a simpler way to accomplish exactly the same thing using Laplace transforms - that is, via Theorems 6.4 and 6.5. We merely apply the Laplace operator to both sides of the equation

$$y'' + y = u_{3\pi}(t)$$

to see that

$$Y(s)(s^2+1) - s = \frac{e^{-3\pi s}}{s}.$$

Hence,

$$Y(s) = \frac{s}{s^2+1} + e^{-3\pi s}\frac{1}{s(s^2+1)} = \frac{1}{s^2+1} + e^{-3\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right).$$

Applying the inverse Laplace transform operation to each side yields (based on Theorem 6.4)

$$y(t) = \cos t + (1-\cos(t-3\pi))u_{3\pi}(t) = \begin{cases} \cos t, & 0 \le t < 3\pi \\ \cos t + (1-\cos(t-3\pi)), & t \ge 3\pi \end{cases}$$

$$= \begin{cases} \cos t, & 0 \le t < 3\pi \\ 1 + 2\cos t, & t \ge 3\pi, \end{cases}$$

since $\cos(t-3\pi) = -\cos t$. Hence, this method agrees with the "classical" approach.

Example 6.8. Consider an RCL (DC) circuit modeled by the equation

$$y'' + 5y' + 6y = 1 - 3e^{-(t-5)}u_5(t); \quad y(0) = 0,\ y'(0) = 1.$$

That is to say that we are given a circuit with no initial charge in it and intially has, say, a magnetically induced current in it. This circuit is hooked up to a constant electromotive force of magnitude 1; but then we flip a switch at time $t = 5$ that introduces a second, stronger battery to the circuit wired **backwards** (gasp!) - but this battery is old and shorts itself out very quickly (sure this is contrived, but we can still solve a model for this scenario).

Let's solve this via Laplace transforms. Applying the Laplace transform operator to this equation yields

$$(s^2Y(s) - sy(0) - y'(0)) + 5(sY(s) - y(0)) + 6(Y(s)) = \frac{1}{s} - \frac{3e^{-5s}}{s+1}.$$

This simplifies to

$$(s^2 + 5s + 6)Y(s) - 1 = \frac{1}{s} - \frac{3e^{-5s}}{s+1}$$

so that

$$Y(s) = \frac{s+1}{s(s+2)(s+3)} - \frac{3e^{-5s}}{(s+1)(s+2)(s+3)}.$$

We appeal to some algebra to split this transform into manageable pieces:

$$Y(s) = \left(\frac{1/6}{s} + \frac{1/2}{s+2} - \frac{2/3}{s+3}\right) + e^{-5s}\left(\frac{-3/2}{s+1} + \frac{3}{s+2} - \frac{3/2}{s+3}\right).$$

Applying the inverse Laplace transform operator now (invoking Theorems 6.4 and 6.5 where necessary) gives us

$$y(t) = \left(\frac{1}{6} + \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t}\right) + \left(-\frac{3}{2}e^{-(t-5)} + 3e^{-2(t-5)} - \frac{3}{2}e^{-3(t-5)}\right) u_5(t).$$

It's worth pointing out here that the faulty battery in this scenario is actually powerful enough to **reverse the flow of electricity**! Sometimes we need to stem the flow of electricity in a circuit so that it only flows in one direction. If we could control the initial "pulse" from this battery, we could determine what *critical voltage* will stop the flow of electricity in the circuit. This merely amounts to ensure that the solution to the model is a non-negative function. Incidentally, this critical voltage here is roughly 2.47 (this coefficient would replace the '3' in the forcing term of the model).

Remark 6.6. Notice that most of the "hard" work comes from the algebra in between applying the Laplace and inverse Laplace transform operators. If we are sure to group terms in an efficient manner, then the inverse Laplace transform is relatively easily computed. This saves a lot of headaches in "bookkeeping" when there are a lot of pieces in the forcing term of the model.

6.5 Impulse Functions*

An *impulse* in the strictest sense actually doesn't exist. It is an idealization of spreading a fixed amount of "stuff" across smaller and smaller intervals of time. That is, it's a **limit**!

As for the step functions in the previous section, we can construct a model function representing all impulses. To do this, consider the family of functions

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{otherwise,} \end{cases}$$

where we assume $\tau > 0$. One can check (do this, it's fast) that the total integral of this function

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\, dt = 1.$$

That is, each $d_\tau(t)$ bounds the same amount of "stuff" beneath it. We now need to spread this "stuff" over a very short interval. In rigorous terms, we need to know what happens in the limit as we take $\tau \to 0^+$.

It's easy to see that for each $t \neq 0$, the limit

$$\lim_{\tau \to 0} d_\tau(t) = 0.$$

In technical terms, we say $d_\tau(t)$ converges *pointwise* to zero away from $t = 0$. It then remains to see what happens at $t = 0$. This is where something strange occurs - something

we (probably) haven't seen before. The limiting "function" blows up at $t = 0$; however, **it still bounds the same amount of area**! Why? Because

$$\lim_{\tau \to 0} I(\tau) = \lim_{\tau \to 0} 1 = 1.$$

We use these features to define the **unit impulse function**, which we call δ. That is, δ satisfies

1. $\delta(t) = 0$ for all $t \neq 0$, and
2. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$.

We're hesitant to use the term "function" here. In fact, the actual limit $\lim_{\tau \to 0} d_\tau(t)$ doesn't exist as a function in the traditional sense. The kind of object that came from this process is known as a *generalized function* (or more colloquially as a *distribution*). It is all right to use the term "delta function" in practice; just bear in mind that this object is not actually a function at all!

Great! We now have δ as our model impulse. We can treat this like a function for most intents and purposes. In particular, we can translate it away from zero and scale it by other functions. It is then, perhaps, not so surprising that we can compute its Laplace transform as well (even though it's not a function!).

To compute a **formal** Laplace transform for δ, we need to define it as a limit of the Laplace transforms of the d_τ's that we constructed before. That is, given a fixed translation distance t_0, define

$$\mathcal{L}[\delta(t - t_0)] = \lim_{\tau \to 0^+} \mathcal{L}[d_\tau(t - t_0)] = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} e^{-st} d_\tau(t - t_0)\, dt.$$

We can then continue computing this by first evaluating each $d_\tau(t - t_0)$ transform:

$$\mathcal{L}[d_\tau(t - t_0)] = \int_{\infty}^{\infty} e^{-st} d_\tau(t - t_0)\, dt = \frac{1}{2\tau} \int_{t_0 - \tau}^{t_0 + \tau} e^{-st}\, dt$$

$$= \frac{-1}{2s\tau} [e^{-st}]_{t = t_0 - \tau}^{t_0 + \tau} = \frac{1}{2s\tau} e^{-st_0} (e^{s\tau} - e^{-s\tau}) = \frac{\sinh(s\tau)}{s\tau} e^{-st_0}.$$

(recall that sinh is the hyperbolic sine function). Our job now is to evaluate this limit as $\tau \to 0$. This can be done simply using L'Hôpital's rule:

$$\lim_{\tau \to 0} \frac{\sinh(s\tau)}{s\tau} \overset{\text{LH}}{=} \frac{s \cosh(s\tau)}{s\tau} = 1.$$

Therefore,

$$\mathcal{L}[\delta(t - t_0)] = e^{-st_0}.$$

In particular,

$$\mathcal{L}[\delta(t)] = 1.$$

Remark 6.7. We have the formula

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

for all positive integers. Since the "simplest" Laplace transform we've been able to compute thus far is the function $\frac{1}{s}$, students often ask at this point what the inverse Laplace transform of the constant function $F(s) = 1$. Well, it's not a **function**, per se, that accomplishes this... Indeed, it is the delta functional that does.

Now that we know what the Laplace transform of δ is, we can more generally determine the Laplace transform of the function $\hat{f}(t) = \delta(t-t_0)f(t)$. Again, we accomplish this by a limiting process. Noting that $f(t_0)\delta(t-t_0) = \lim_{\tau\to 0} f(t)d_\tau(t)$ we compute

$$\mathcal{L}[f(t)\delta(t-t_0)] = \lim_{\tau\to 0+}\int_{-\infty}^{\infty} f(t)d_\tau(t-t_0)\,dt = \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)\,dt = f(t_0).$$

If the reader isn't convinced that we are able to draw the limit inside the integral, they can (and should) compute the result above in a fashion that evaluates the integral first while there is a d_τ function there (be careful with how you treat the $f(t)$ term).

Remember the we outlined in the previous section a few physical examples that are modeled by an idealized impulse - things like a hammer blow or a static shock are great examples. To see how this function is used in practice, let's do an example or two.

Example 6.9. Let's use the same mass-spring system as before; but instead we replace the electromagnet by a mallet that hits the mass sharply at time $t = \frac{17\pi}{4}$. This hammer-blow scenario is modeled by a delta function at the time $t = \frac{17\pi}{4}$:

$$y'' + y = \delta\left(t - \frac{17\pi}{4}\right); \quad y(0) = 1,\ y'(0) = 0.$$

Applying the (formal) Laplace transform operator here yields

$$(s^2+1)Y(s) - s = e^{-\frac{17\pi s}{4}}.$$

Then we have

$$Y(s) = \frac{s}{s^2+1} + e^{-\frac{17\pi s}{4}}\frac{1}{s^2+1}.$$

We use Theorem 6.5 to evaluate the inverse Laplace transform of the exponential term and find

$$y(t) = \cos t + \sin\left(t - \frac{17\pi}{4}\right)u_{\frac{17\pi}{4}}(t).$$

We immediately notice that the effect of adding the delta function is that of adding on a step function term at the appropriate location. This is seen in the "jerky" behavior of the graph at $t = \frac{17\pi}{4}$. The function is indeed continuous there, but we sacrifice its differentiability there in order to idealize our scenario.

Example 6.10. Let's use the same RCL (DC) circuit as in the previous section. Now instead of attaching a secondary battery unit to the system, we merely shock the system (at the location where we're measuring it). This is modeled by

$$y'' + 5y' + 6y = 1 + 3\delta(t-5); \quad y(0) = 0,\ y'(0) = 1$$

(we're **adding** the delta function since it **adds** energy to the system rather than removing it). Proceeding as before, we apply the (formal) Laplace transform operator to the equation and get

$$(s^2+5s+6)Y(s) - 1 = \frac{1}{s} + 3e^{-5s}.$$

This yields

$$Y(s) = \frac{s+1}{s(s+2)(s+3)} + 3e^{-5s}\frac{1}{(s+2)(s+3)}.$$

In a similar fashion before, we use a partial fraction decomposition to separate this out and apply the inverse Laplace transform operator. This gives us

$$Y(s) = \left(\frac{1/2}{s+2} - \frac{2/3}{s+3} + \frac{1/6}{s}\right) + e^{-5s}\left(\frac{3}{s+2} - \frac{3}{s+3}\right),$$

whose inverse Laplace transform we calculate easily as

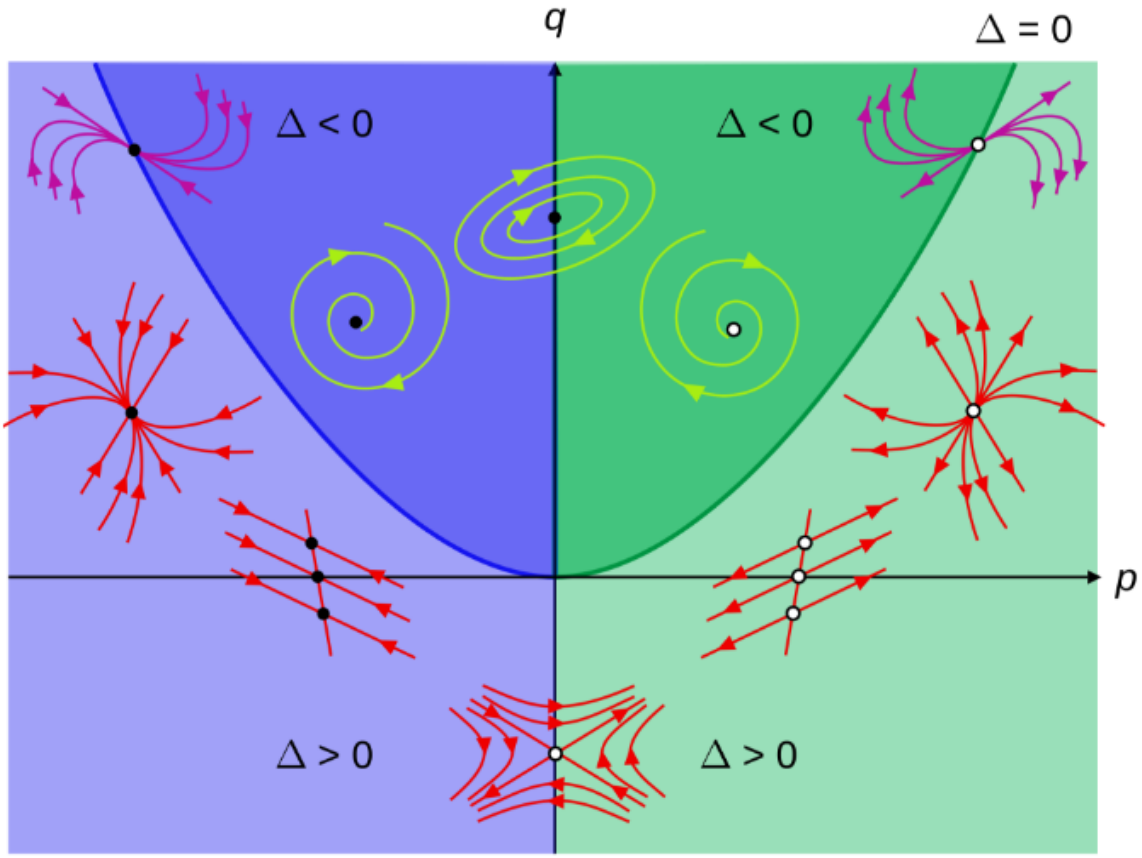
$$y(t) = \left(\frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t} + \frac{1}{6}\right) + \left(3e^{-2(t-5)} - 3e^{-3(t-5)}\right)u_5(t).$$

By plotting this solution, we see that there is a sharp spike in charge at $t = 5$. However, this is quickly dissipated, and the system returns to its equilibrium state of $\hat{y}(t) = \frac{1}{6}$.

Remark 6.8. Again we see how important organization of one's work becomes in this setting. It is very easy to lose oneself in the tedium of keeping track of all relevant terms. Practice keeping all exponential terms of the transform together.

We can see how closely related step functions and impulse functions are. They give us a formal mathematical framework within which we can model practical situations. There is vast literature on these functions and generalized functions in particular. Studying them is actually much more intensive than what we've discussed here. Suffice it to say that we will not proceed the theory of the delta functional further within this class. It is certainly enough just to have a functioning knowledge of how to apply the Laplace transform to it.

7 Systems of First-Order Linear Equations



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$
$$\Delta = p^2 - 4q$$

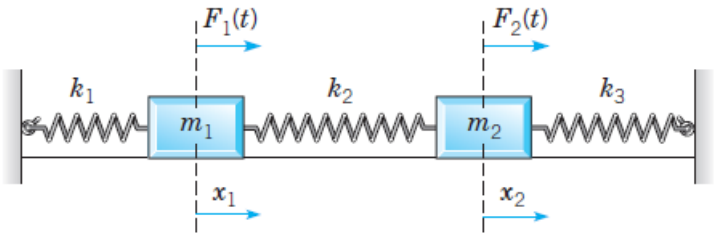


Figure 16: A damped and forced mass-spring system for two masses anchored at the endpoints. Photo credit: Boyce and DiPrima.

7.1 Introduction

Up until now, our studies have focused on differential equations of a single dependent variable of a single independent variable. We are still going to focus on a single independent variable, but we are now going to shift our attention to multiple dependent variables. Not only that, but we wish to solve many differential equations in these variables **simultaneously**. There is a plethora of reasons and motivation for considering such *systems of ordinary differential equations* as they naturally come about from physical and theoretical phenomena. For example, consider a mass-spring system consisting of two masses connected to one another in the configuration of Figure 16. We can easily analyze the forces acting on both of these masses and show that

$$\begin{cases} m_1 \frac{d^2x_1}{dt^2} &= k_2(x_2 - x_1) - k_1 x_1 + F_1(t) \\ &= -(k_1 + k_2)x_1 + k_2 x_2 + F_1(t) \\ m_2 \frac{d^2x_2}{dt^2} &= -k_3 x_2 - k_2(x_2 - x_1) + F_2(t) \\ &= k_2 x_1 - (k_2 + k_3)x_2 + F_2(t). \end{cases}$$

Hence, we can describe the locations of both masses using the same time variable. Since both force equations above depend on the locations one another, we say that such a system is *coupled*.

For another example, consider the RCL-circuits from before but wired instead in parallel rather than in series. This setup is given in Figure 17. We have the same laws of electromagnetism as before (thanks to Ohm, Kirchoff, Faraday, and Ampére), but we present them in a different way. We can write a system of equations in terms of current I and voltage V as

$$\begin{cases} \frac{dI}{dt} &= \frac{V}{L} \\ \frac{dV}{dt} &= -\frac{I}{C} - \frac{V}{RC}. \end{cases}$$

One rather peculiar reason for why we care so much about system of equations is because **every higher-order ODE can be written as such a system**. Let's take a look at the following example:

Example 7.1. Recall from the single mass-spring systems of Chapter 3 that the general equation governing the mass's motion is given by

$$mu'' + \gamma u' + ku = F(t).$$

Let's solve this for u'' so that

$$u'' = \frac{1}{m}F(t) - \frac{\gamma}{m}u' - \frac{k}{m}u.$$

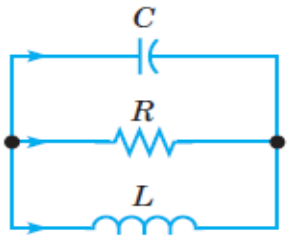


Figure 17: A partial RCL-circuit wired in parallel. Photo credit: Boyce and DiPrima.

If we let $u_1(t) = u(t)$ and $u_2(t) = u'(t)$, then we can write

$$\frac{du_1}{dt} = u' = u_2 \quad \text{and} \quad \frac{du_2}{dt} = u'' = \frac{1}{m}F(t) - \frac{\gamma}{m}u' - \frac{k}{m}u.$$

This translates into the system

$$\begin{cases} \frac{du_1}{dt} &= u_2 \\ \frac{du_2}{dt} &= \frac{1}{m}F(t) - \frac{\gamma}{m}u_2 - \frac{k}{m}u_1. \end{cases}$$

Hence, this second-order model may be written as a system of first-order ODE!

Example 7.2. We can employ the same trick for higher-order equations. Consider, for instance, the equation

$$y^{(v)} + 2y^{(iv)} + 3y''' + 4y'' + 5y' + 6y = f(t).$$

We can solve this equation for $y^{(5)}$ so that

$$y^{(v)} = f(t) - 2y^{(iv)} - 3y''' - 4y'' - 5y' - 6y.$$

We make the change of variables as $y_1 = y$ and $y_{i+1} = y_i'$ for $i \geq 1$. Hence, we get the system

$$\begin{cases} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= y_5 \\ y_5' &= f(t) - 2y_5 - 3y_4 - 4y_3 - 5y_2 - 6y_1. \end{cases}$$

Hence, this fifth-order ODE may be written as a first-order system!.

This trick may be used for any ODE which we can explicitly solve for its highest-order derivative. In fact, let's take the general case of

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}).$$

Again taking

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \ldots, x_n = y^{(n-1)},$$

we can rewrite the general ODE as a first-order system so that

$$\begin{cases} x_1' & = x_2 \\ x_2' & = x_3 \\ \vdots & \vdots \\ x_{n-1}' & = x_n \\ x_n' & = F(t, x_1, x_2, \ldots, x_n). \end{cases}$$

This should definitely be enough motivation to study the general first-order system

$$\begin{cases} x_1' & = F_1(t, x_1, \ldots, x_n) \\ x_2' & = F_2(t, x_1, \ldots, x_n) \\ \vdots & \vdots \\ x_n' & = F_n(t, x_1, \ldots, x_n). \end{cases} \tag{9}$$

We aim to find a *solution*, which is an interval $I\colon \alpha < t < \beta$ along with a vector $\vec{\phi}(t) = (\phi_1(t), \ldots, \phi_n(t))$ whose components are differentiable on I which satisfy each row of differential equations on I. We can additionally ask for an *initial condition* $\vec{\phi}(0) = (\phi_1(0), \ldots, \phi_n(0))$ to be satisfied, which is similar to the previous ODE we have studied. As such, a system of differential equations coupled with an initial condition is labeled an *initial value problem*. Solutions to systems of differential equations can be thought of as **parametric equations** by which a particle travels through n-dimensional space over a time variable t.

Because each of these equations in the general system 9 above are first order equations, we have an existence and uniqueness theorem that is very similar to that of section 2.8.

Theorem 7.1. Let each of the functions $F_1, \ldots, F_n$ and the partial derivatives $\frac{\partial F_i}{\partial x_j}$ for $1 \leq i, j \leq n$ be continuous in a region R of $tx_1x_2\cdots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, ..., $\alpha_n < x_n < \beta_n$, and let the point $(t_0, x_1^0, \ldots, x_n^0)$ be in R. Then there is an interval $|t - t_0| < h$ in which there exists an unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system (9) of differential equations that also satisfies the initial condition $\vec{\phi}(t_0) = (x_1^0, \ldots, x_n^0)$.

In a similar fashion, the system (9) is *linear* if each of the $F_1, \ldots, F_n$ are linear in $x_1, \ldots, x_n$. Otherwise, the system is *nonlinear*. We will only consider linear systems in this class. Hence, we consider the general first-order linear system

$$\begin{cases} x_1' & = p_{1,1}(t)x_1 + \cdots + p_{1,n}(t)x_n + g_1(t) \\ x_2' & = p_{2,1}(t)x_1 + \cdots + p_{2,n}(t)x_n + g_2(t) \\ \vdots & \vdots \\ x_n' & = p_{n,1}(t)x_1 + \cdots + p_{n,n}(t)x_n + g_n(t). \end{cases} \tag{10}$$

If each of the functions $g_1, \ldots, g_n$ are the zero function, then the system is called *homogeneous*. Otherwise, it is called *nonhomogeneous*.

We end this section with a new (yet familiar) observation. Just like the first-order ODE case, the existence and uniqueness theorem 7.1 has a simpler statement in the linear case.

Theorem 7.2. If the functions $p_{i,j}$ and g_i in the system (10) are continuous on an open interval $I\colon \alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system (10) satisfying the initial condition $\vec{\phi}(0) = (x_1^0, \ldots, x_n^0)$, where t_0 is any point in I, and $x_1^0, \ldots, x_n^0$ are arbitrary. Furthermore, the solution exists throughout the entire interval I.

7.2 Review of Matrices

In order to study systems of equations, it is convenient to frame them as matrix equations. In doing so, we can apply much of the theory of linear algebra to these systems. It is this motivation to requires us to review some of the fundamentals of matrices and matrix equations.

Let us first define an $m \times n$ *matrix* as a rectangular array of numbers (called *elements*) arranged into m rows and n columns. We denote the element in the i^{th} row and j^{th} column by a_{ij}. We denote the matrix with a capital letter A or the collection (a_{ij}) of elements. We typically write such a matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

We can exchange the rows of any matrix (a_{ij}) with its columns. The resulting matrix (a_{ji}) is called the *transpose*, and it's denoted $(a_{ij})^T = (a_{ji}$. If the matrix's elements are complex numbers, we can also compute the *conjugate* matrix. Given a matrix (a_{ij}), we denote the conjugate matrix by $\overline{(a_{ij})} = (\overline{a_{ij}})$. That is, the conjugate of a matrix is simply the matrix containing the conjugate of each element. Combining the transposition and conjugation operations, we obtain a matrix $A^* = \overline{A}^T = \overline{A^T}$. This matrix is called the *conjugate transpose* or (more commonly) the *adjoint* of A.

To exemplify the above, let's consider the matrix

$$A = \begin{bmatrix} 2+i & 3 & 4-2i \\ -5 & 4 & 3i \end{bmatrix}.$$

Then we can compute

$$A^T = \begin{bmatrix} 2+i & -5 \\ 3 & 4 \\ 4-2i & 3i \end{bmatrix} \quad \text{and} \quad \overline{A} = \begin{bmatrix} 2-i & 3 & 4+2i \\ -5 & 4 & -3i \end{bmatrix}.$$

We can also easily see that

$$\overline{A^T} = \overline{\begin{bmatrix} 2+i & -5 \\ 3 & 4 \\ 4-2i & 3i \end{bmatrix}} = \begin{bmatrix} 2-i & -5 \\ 3 & 4 \\ 4+2i & -3i \end{bmatrix} = \begin{bmatrix} 2-i & 3 & 4+2i \\ -5 & 4 & -3i \end{bmatrix}^T = \overline{A}^T.$$

In the study of systems of differential equations, we often encounter *square* matrices - that is, matrices with the same number of columns as rows. We also come across *row vectors* (i.e. $m \times 1$ matrices) and *column vectors* (i.e. $1 \times n$ matrices). We typically denote vectors with an "over arrow." That is, we denote a row or column vector by $\vec{x}$. Usually, the vector notation is reserved for column vectors, and we then denote row vectors by transposes of column vectors. That is, for the column vectors $\vec{x}$, the corresponding row vector is $\vec{x}^T$.

We now list a few useful properties of matrices that we will encounter often in the ensuing sections.

1. **Equality**. Two $m \times n$ matrices (a_{ij}) and (b_{ij}) are equal if all of the elements occupying the same location in the array are equal. That is, $(a_{ij}) = (b_{ij})$ if and only if $a_{ij} = b_{ij}$ for each i and j.

2. **Zero**. The symbol $\vec{0}$ denotes the column vector whose entries are all zero. The symbol 0 will be used both for the zero scalar and the zero matrix (i.e. the matrix with zero for each entry). Whether we refer to the scalar or the matrix will be clear from the context.

3. **Addition**. The sum of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined by

   $$A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

   for each pair i and j. It is easily seen at this point that

   $$A + B = B + A \quad \text{and} \quad A + (B + C) = (A + B) + C$$

   so that matrix addition is both commutative and associative.

4. **Scalar Multiplication**. The product of a matrix (a_{ij}) by a complex number α is defined as

   $$\alpha(a_{ij}) = (\alpha a_{ij})$$

   for all pairs i and j. Then we see that

   $$\alpha(A + B) = \alpha A + \alpha B \quad \text{and} \quad (\alpha + \beta)A = \alpha A + \beta A,$$

   where β is another complex number. In particular, we define

   $$-A = (-1)A.$$

5. **Subtraction**. We can see subtraction of two matrices A and B as a combination of the above properties so that

   $$A - B = A + (-1)B.$$

6. **Multiplication of Vectors**. There are many legitimate ways to multiply to vectors together in a sensible way. We define one of these as the *dot product*. This is given as

   $$\vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i.$$

The reader can now verify that

$$\vec{x}^T\vec{y} = \vec{y}^T\vec{x}, \quad \vec{x}^T(\vec{y}+\vec{z}) = \vec{x}^T\vec{y} + \vec{x}^T\vec{z}, \quad \text{and} \quad (\alpha\vec{x})^T\vec{y} = \alpha(\vec{x}^T\vec{y}) = \vec{x}^T(\alpha\vec{y}).$$

There is one more product of vectors that we can define and should mention before continuing (especially since we are using complex entries for matrices). Let's define the *standard inner product* by

$$(\vec{x},\vec{y}) = \sum_{i=1}^{n} x_i\overline{y_i} = \vec{x}^T\overline{\vec{y}}.$$

We can now verify each of the following:

$$\begin{array}{rclcrcl} (\vec{x},\vec{y}) & = & \overline{(\vec{y},\vec{x})} & & (\vec{x},\vec{y}+\vec{z}) & = & (\vec{x},\vec{y})+(\vec{x},\vec{z}), \\ (\alpha\vec{x},\vec{y}) & = & \alpha(\vec{x},\vec{y}) & & (\vec{x},\alpha\vec{y}) & = & \overline{\alpha}(\vec{x},\vec{y}). \end{array}$$

Even if the vectors have complex components, we have a nonnegative quanitity

$$(\vec{x},\vec{x}) = \sum_{i=1}^{n} x_i\overline{x_i} = \sum_{i=1}^{n} |x_i|^2.$$

We then define the *magnitude* of a vector as

$$\|\vec{x}\| = \sqrt{(\vec{x},\vec{x})}.$$

If we have $(\vec{x},\vec{y}) = 0$, then we say that the two vectors are *orthogonal*. Geometrically, for the standard inner product, this means that the two vectors are perpendicular to one another.

7. **Multiplication of Matrices**. This operation is only defined for two matrices A and B where the number of columns of A matches the number of rows of B. That is, we can only multiply an $m \times n$ matrix into an $n \times r$ matrix. In order to define this product, let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times r$ matrix. We define the product

$$AB = (a_{ij})(b_{ij}) = \left(\sum_{k=1}^{m} a_{ik}b_{kj} \right).$$

One can think of matrix multiplication as a number of dot products between the rows of A with the columns of B. The reader can verify by direct computation that

$$A(BC) = (AB)C \quad \text{and} \quad A(B+C) = AB + AC.$$

In general, it should be noted that $AB \neq BA$. We leave it to the reader to come up with creative examples of this phenomenon.

8. **Identity**. With most multiplicative operations, there comes an element that leaves the others unchanged by multiplication. In matrices, this element is denoted

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}.$$

Thus, for any square matrix A, we have

$$AI_n = I_nA = A.$$

9. **Inverse**. A square matrix A is called *nonsingular* or *invertible* if there is another matrix B such that $AB = BA = I_n$. Such an inverse is unique if it exists. Hence, we denote "the" inverse of A as $B = A^{-1}$. Matrices that don't have inverses are called *singular* or *noninvertible*.

   There are many ways to compute the inverse of a square matrix should it exist, but we don't go into too much detail here about how to do so. However, we would like to emphasize that a matrix has an inverse if and only if its determinant $\det(A) \neq 0$. If this is the case, then one may compute this inverse by putting the *augmented matrix*

$$[A|I_n] = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nn} & & & 1 \end{array}\right]$$

   into reduced row-echelon form. That is to say that

$$[A|I_n] \sim [I_n|A^{-1}].$$

Instead of just putting numbers as the entries of matrices, we can insert functions in their places. We denote such *matrix functions* by

$$vecx(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \text{and} \quad A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t). \end{bmatrix}$$

All of the usual operations for functions may be done componentwise on matrix functions. For example,

$$\frac{d}{dt}(A(t)) = \left(\frac{da_{ij}}{dt}\right)$$

and

$$\int_a^b A(t)\,dt = \left(\int_a^b a_{ij}(d)\,dt\right).$$

The reader should verify that

$$\frac{d}{dt}(A+B) = \frac{dA}{dt} + \frac{dB}{dt},$$

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B,$$

and

$$\frac{d}{dt}(CA) = C\frac{dA}{dt},$$

where C is a constant matrix.

7.3 Systems of Linear Equations; Linear Independence, Eigenvalues/vectors

We take some time to recall a few results from linear algebra that will be of use to us later. To begin, we recall that we can write any linear system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n & = b_1 \\ \vdots & \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n & = b_n \end{cases}$$

of n equations in n unknowns as a matrix equation

$$A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \dots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

We don't consider the general $m \times n$ system here since we will not encounter them in our studies. If $\vec{b} = \vec{0}$, then we call the system *homogeneous*. Otherwise, it is called *nonhomogeneous*.

Recall that this system has a unique solution if and only if A is invertible. In this case, we solve the system by taking

$$\vec{x} = A^{-1}\vec{b}.$$

Notice here that the only solution of the homogeneous system in this case is the zero vector.

In the case of A being noninvertible, the system $A\vec{x} = \vec{0}$ will have infinitely many solutions including the trivial one.

For the nonhomogeneous system $A\vec{x} = \vec{b}$, solutions only exist if $(\vec{b}, \vec{y}) = 0$ where $A^*\vec{y} = \vec{0}$. The solutions to the nonhomogeneous equation are given by

$$\vec{x} = \vec{x}^{(0)} + \vec{\xi},$$

where $\vec{x}^{(0)}$ is a particular solution to $A\vec{x} = \vec{b}$ and $\vec{\xi}$ solves the homogeneous system $A\vec{x} = \vec{0}$. In general, solving linear systems is done typically by row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right].$$

One should recall all of these results from a prerequisite linear algebra course.

Example 7.3. 1. One should verify that the system

$$\begin{cases} x_1 - 2x_2 + 3x_3 & = 7 \\ -x_1 + x_2 - 2x_3 & = -5 \\ 2x_1 - x_2 - x_3 & = 4 \end{cases}$$

has $\vec{x} = (x_1, x_2, x_3)^T = (2, -1, 1)^T$ as a solution. The work is omitted here in light of the next example which is very similar.

2. Solve the general system

$$\begin{cases} x_1 - 2x_2 + 3x_3 & = b_1 \\ -x_1 + x_2 - 2x_3 & = b_2 \\ 2x_1 - x_2 + 3x_3 & = b_3. \end{cases}$$

One can row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right]$$

to the similar (equivalent) system

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{array}\right].$$

Hence, this system only has a solution when

$$b_1 + 3b_2 + b_3 = 0.$$

In fact, let's take the specific example of

$$\vec{b} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}.$$

Since we've already done the work in the general case, we insert these values for b_1, b_2, and b_3 so that we must make

$$\begin{cases} x_1 - 2x_2 + 3x_3 & = 2 \\ x_2 - x_3 & = -3 \end{cases}.$$

We can solve each of these equations for x_1 and x_2, respectively, so that

$$\begin{aligned} x_1 &= 2x_2 - 3x_3 + 2 \\ x_2 &= x_3 - 3. \end{aligned}$$

This leaves x_3 to be what is called a *free variable* or *parameter*. We take $x_3 = t$ for any real number t. By using this parameter t, we can express the general solution to the above system by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2(t-3) - 3t + 2 \\ t - 3 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}$$

We now need to recall what it means for vectors to be linearly independent. A set of vectors $\{\vec{x}^{(1)}, \ldots, \vec{x}^{(k)}\}$ is *linearly dependent* if there exist scalars $c_1, \ldots, c_k$, of which at least one is nonzero, such that the linear combination

$$c_1\vec{x}^{(1)} + \cdots + c_k\vec{x}^{(k)} = \vec{0}.$$

If the only such scalars for which this occurs is when $c_1 = \cdots = c_k = 0$, then the set is called *linearly independent*.

Determining whether or not a set of m-dimensional vectors is linearly independent is accomplished by row reducing a matrix whose columns are exactly the vectors in question.

We can extend this idea to a set of vector functions. That is, a set of vector functions $\{\vec{x}^{(1)}(t), \ldots, \vec{x}^{(k)}(t)\}$ is *linearly dependent on* $\alpha < t < \beta$ is there exist scalars $c_1, \ldots, c_k$, at least one of which is nonzero, such that the linear combination

$$c_1\vec{x}^{(1)}(t) + \cdots + c_k\vec{x}^{(k)}(t) = \vec{0}$$

on $\alpha < t < \beta$. If the only such scalars are $c_1 = \cdots = c_k = 0$, then the set is called *linearly independent* on $\alpha < t < \beta$. We have already seen this in detail for sets of two vector functions, and we can now see that it easily extends to sets of k many vector functions as well.

We now come to the real meat of this section. What follows is the reason we built up so much material before. Let's say we're given a matrix equation

$$A\vec{x} = \vec{y}.$$

This can easily be thought of as a linear transformation taking $\vec{x} \mapsto \vec{y}$. We wish to find all (nonzero) vectors such that

$$A\vec{x} = \lambda\vec{x}$$

for some $\lambda \in \mathbb{C}$. That is, we want to find all of the vectors (and their corresponding λ's) such that multiplication by a matrix A is really just multiplication by a scalar λ. We can reframe this equation as

$$A\vec{x} - \lambda\vec{x} = (A - \lambda I)\vec{x} = \vec{0}.$$

Hence, we are really trying to solve the homogeneous system for nonzero vectors. Clearly, the zero vector solves any homogeneous system; but we are particularly interested in those nonzero vectors that solve this. As we have seen before, **this system has infinitely many solutions if the matrix $A - \lambda I$ is noninvertible**. Solutions to the above homogeneous equation are called *eigenvectors* of A. The values of λ that yield these solutions are the corresponding *eigenvalues* of A.

Let's solidify these concepts (which, again, should be familiar from a prerequisite course of linear algebra) with an example or two.

Example 7.4. Let's find the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}.$$

We begin by considering the matrix

$$\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{bmatrix}.$$

This matrix is noninvertible if and only if its determinant is zero. So we compute the determinant to be

$$|A - \lambda I| = (3 - \lambda)(-2 - \lambda) - (-1)(4) = \lambda^2 - \lambda - 2 = 0.$$

This factors so that

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0,$$

and therefore $\lambda = 2$ or $\lambda = -1$ are the only values of λ for which this matrix is noninvertible. Hence, the eigenvalues of A are $\lambda \in \{-1, 2\}$.

We now need to find the eigenvectors. This is accomplished by using the eigenvalues found above. We begin by using $\lambda = 2$ so that

$$(A - 2I)\vec{x} = \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \vec{x} = \vec{0}.$$

This translates to the single equation

$$x_1 - x_2 = 0.$$

Then we can make $x_2 = t$ our parameter so that

$$\vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then we see that $\vec{v}_2 = (1, 1)^T$ is **an** eigenvector of A corresponding to the eigenvalue $\lambda = 2$.

We do the same sort of computation for $\lambda = -1$. That is, we find solutions to

$$(A - (-1)I)\vec{x} = \begin{bmatrix} 3 - (-1) & -1 \\ 4 & -2 - (-1) \end{bmatrix} \vec{x} = \vec{0}.$$

This translates into the equation

$$4x_1 - x_2 = 0.$$

We can solve this so that $x_2 = 4x_1$. We then take $x_1 = t$ to be our parameter so that

$$\vec{x} = t \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Hence, $\vec{v}_{-1} = (1, 4)^T$ is **an** eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

Remark 7.1. It should be noted here that the vectors we found in the preceeding example are just representatives for eigenvectors of the matrix A. We could have chosen any multiple of the vectors found above as eigenvectors. The point here is to find a **basis** for the space of eigenvectors corresponding to a given eigenvalue λ. Hence, $\vec{v}_2$ and $\vec{v}_{-1}$ are bases for the *eigenspaces* corresponding to $\lambda = 2$ and $\lambda = -1$, respectively.

In order to remedy this ambiguity, one often thinks it wise to *normalize* the vectors found. That is, a vector is *normal* if its length is 1. In this case, there are then only two vectors (rather than infinitely many of them) that can represent a given eigenspace.

In general, the polynomial in the equation that determines the eigenvalues for a matrix is called the *characteristic polynomial* of A. Being a polynomial, the characteristic polynomial has as many complex roots (counting multiplicity) as its degree. That is, a polynomial $p(\lambda) = \sum_{i=0}^{n} c_i \lambda^i$ has n complex roots. The number of times m that a root here is repeated is called the *algebraic multiplicity* of the root. The dimension q of an eigenspace corresponding to that root is called the *geometric multiplicity* of that same root. It should be noted that these numbers do not coincide in general. In fact, we have $1 \leq q \leq m$. A root/eigenvalue that has algebraic multiplicity 1 is said to be *simple*. Simple eigenvalues will inherently have identical algebraic and geometric multiplicities.

Example 7.5. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We again look to make

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

noninvertible. One can compute the determinant of this matrix as

$$|A - \lambda I| = -\lambda^3 + 3\lambda + 2 = 0.$$

The roots can easily be found as $\lambda = 2$ and $\lambda = -1$. We see that 2 is a simple eigenvalue, but -1 is of algebraic multiplicity 2. We compute the eigenvectors for $\lambda = 2$ by reducing the matrix

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right].$$

Then we find the solutions to be spanned by the vector $\vec{v}_2 = (1, 1, 1)^T$.

For $\lambda = -1$, we get the equation

$$x_1 + x_2 + x_3 = 0$$

where we can make both $x_2 = t$ and $x_3 = s$ parameters. This then yields the general solution

$$\vec{x} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, there is a two-dimensional eigenspace (so that $\lambda = -1$ has geometric multiplicity 2) for $\lambda = -1$. It is spanned by the vectors $\vec{v}_{-1,1} = (-1, 0, 1)^T$ and $\vec{v}_{-1,2} = (-1, 0, 1)^T$ which are also eigenvectors for $\lambda = -1$.

Example 7.6. We can even find eigenvalues for matrices with no real eigenvalues. We merely shift the field of our vector space from the real numbers $\mathbb{R}$ to the complex numbers $\mathbb{C}$. Take, for example, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This is the matrix that rotates vectors in the plane by a counter-clockwise, quarter turn. Clearly, there are no real eigenvalues or eigenvectors (since all nonzero vectors aren't scaled, but are instead rotated). The characteristic equation for this matrix is

$$\lambda^2 + 1 = 0,$$

whose solutions are given as $\lambda = \pm i$. Then $A - \lambda I$ is now

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}.$$

For the former, an eigenvector is

$$\vec{v_i} = \begin{bmatrix} i \\ 1 \end{bmatrix};$$

and for the latter, an eigenvector is

$$\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

The algebra is all the same as before - we just now use complex numbers for row reduction.

Remark 7.2. There is one more string of remarks we should make before proceeding to solving systems of differential equations. This involves the matrices who are equal to their own adjoints - i.e. $A = A^*$. These are called *self-adjoint* or *Hermitian* matrices. They satisfy the following properties:

1. All of their eigenvalues are real.
2. There are always a full set of linearly independent eigenvectors, regardless of algebraic multiplicities.
3. All eigenvectors that correspond to distinct eigenvalues are orthogonal to one another.
4. Given an eigenvalue of algebraic multiplicity m, it is possible to choose m eigenvectors that are mutually orthogonal. Hence, we can always choose all eigenvectors to be orthogonal as well as linearly independent.

Since the matrix in the previous example is self-adjoint, the reader should verify all of these conditions for it.

7.4 Basic Theory of Systems of First-Order Linear Equations

Recall the general system

$$\begin{cases} x_1'(t) &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ \vdots &= \vdots \\ x_n'(t) &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t). \end{cases}$$

We can model what we do here after what we have done for the constant-coefficient matrix equations of the previous sections. That is, we can write this system as

$$\vec{x}'(t) = A(t)\vec{x} + \vec{g}(t),$$

where $\vec{x}(t) = (x_1(t), \ldots, x_n(t))^T$, $A(t) = (p_{ij}(t))$, and $\vec{g}(t) = (g_1(t), \ldots, g_n(t))^T$. Our interests mainly lie in the study of the homogeneous case where $\vec{g}$ is the zero vector-function. Our goal in this case is to find a set $\{\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)\}$of linearly independent vector functions that solve the homogeneous matrix equation above. In this homogeneous case, we still have the following (familiar) theorem.

Theorem 7.3. If the vector functions $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ are solutions of the homogeneous system in equation 10, then the linear combination $c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$ is also a solution for any constants c_1 and c_2.

We have seen this principle in action many times before. We can illustrate it once more using the example that

$$\vec{x}^{(1)}(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} = e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{x}^{(2)}(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

are both solutions to the equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{x}(t)$$

(check this). So according to theorem 7.3, any linear combination

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

is also a solution. It should be noted that we can apply this theorem for any finite number of solutions $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(k)}(t)$. That is, if each of those vector functions is a solution to a given matrix equation, then any linear combination

$$c_1\vec{x}^{(1)}(t) + \cdots + c_k\vec{x}^{(k)}(t)$$

of them is also a solution.

It is in the case of systems of differential equations that we can revert back to thinking of vectors as elements of finite-dimensional vector spaces so that we can simply put these solutions $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ into the columns of a matrix

$$X(t) = \begin{bmatrix} \vec{x}^{(1)}(t) & \cdots & \vec{x}^{(n)}(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{bmatrix}$$

and find the determinant to find when the vectors are linearly independent. This is exactly what we have been accustomed to in the prerequisite linear algebra courses. We can summarize this as the following theorem.

Theorem 7.4. If the vector functions $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are a basis solutions of the (homogeneous) system $\vec{x}'(t) = A(t)\vec{x}(t)$ for each point t in the open interval (α, β) [i.e. they are linearly independent and span the solution space], then each solution $\vec{x} = \vec{\phi}(t)$ of the system can be expressed as a linear combination

$$\vec{\phi}(t) = c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t)$$

in exactly one way.

This tells us precisely that if we find a set of n linearly independent vector functions of the n-dimensional system $\vec{x}' = A\vec{x}$, then they constitute a basis of the solutions. Again, the linear combination in the above theorem is called a *general solution*, and the functions $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are called a *fundamental set of solutions* for this system. We omit the proof of this theorem as it is not terribly enlightening.

Recall our definition of the Wronskian for second-order equations as the determinant of a certain matrix of vector functions and their derivatives. This notion can be extended to higher dimensions by defining the Wronskian as

$$W[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}](t) = \begin{vmatrix} \vec{x}_1^{(1)}(t) & \cdots & \vec{x}_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ \vec{x}_n^{(1)}(t) & \cdots & \vec{x}_n^{(n)}(t) \end{vmatrix},$$

where $\vec{x}_j^{(i)}(t)$ is the j^{th} component of the i^{th} vector function (notice the lack of derivatives in this setting). We get another familiar theorem in this case:

Theorem 7.5. If $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are solutions to the system $\vec{x}'(t) = A(t)\vec{x}(t)$ on the interval $\alpha < t < \beta$, then in this interval the Wronskian $W[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}](t)$ is either identically zero or else never vanishes.

This theorem saves us the trouble of checking the Wronskian for all points in the interval (α, β). Namely, we just have to check one point to see if these functions are a spanning set of solutions. The reader should consult the textbook for a proof of this fact.

There is one more condition that we can check to determine if a set of solutions forms a fundamental set of solutions.

Theorem 7.6. Let

$$\vec{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \vec{e}^{(n)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Further, let $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ be solutions of the system $\vec{x}'(t) = A(t)\vec{x}(t)$ that satisfy the initial conditions

$$\vec{x}^{(1)}(t_0) = \vec{e}^{(1)}, \ldots, \vec{x}^{(n)}(t_0) = \vec{e}^{(n)},$$

respectively, where t_0 is any point in the interval (α, β). Then $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ form a fundamental set of solutions of the system.

Remark 7.3. Notice that $\{\vec{e}^{(1)}, \ldots, \vec{e}^{(n)}\}$ constitutes a basis for $\mathbb{R}^n$. Hence, any (constant) vector may be written uniquely as a linear combination of these vectors. That is, for any vector $\vec{v} = \sum_{i=1}^{n} c_i \vec{e}^{(i)}$, we easily see that the function $\vec{x}(t) = \sum_{i=1}^{n} c_i \vec{x}^{(i)}(t)$ solves our system with the prescribed initial conditions $\vec{v}$. Hence, these $\vec{x}^{(i)}(t)$ are a sort of "standard" set of solutions for our system (i.e. $\vec{x}(0) = \vec{v}$).

This only works because the matrix $P(t)$ in $\vec{x}'(t) = P(t)\vec{x}(t)$ is linear and because the system is homogeneous. (Check the assertion made in this remark.)

There is one more theorem that we should note now for when it shows up in section 7.6. It is an analog for what we did with complex-valued functions in the second-order equation case.

Theorem 7.7. Consider the system

$$\vec{x}'(t) = P(t)\vec{x}(t),$$

where each element of P is a real-valued continuous functions. If $\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$ is a complex-valued solution of this matrix equation, then its real and imaginary parts $\vec{u}$ and $\vec{v}$ are both solutions to this matrix equation as well.

This theorem again tells us what to do when we encounter complex-valued solutions: Just use the real and imaginary parts of these solutions!

With these theorems in our tool kits, we can now press onward to actually solving first-order systems!

7.5 Homogeneous Linear Systems with Constant Coefficients

In the same fashion as for first- and second-order equations with one dependent variable, we begin by solving some systems of homogeneous differential equations with the condition that the coefficients are constant (and real) - this is mostly for simplicity. Our concerns, then, will mainly lie with the homogeneous system

$$\vec{x}'(t) = A\vec{x}(t),$$

where A is a square matrix of constant, real coefficients. If $n = 1$, we are reduced to the autonomous equation

$$\frac{dx}{dt} = ax,$$

which has solution $x(t) = Ce^{at}$. We know the long-term behavior of this system is governed by the value of a. Near the equilibrium solution $x(t) = 0$, we know that there are three different possibilities of other solutions nearby:

1. $a > 0$: Solutions will deviate away from the equilibrium.
2. $a < 0$: Solutions will tend toward the equilibrium.
3. $a = 0$: Solutions are actually constant here. (In general systems, this is the case where the matrix A is not invertible. We won't worry about this case until later, if at all.)

This clues us in to how we should be determining long-term behavior for multiple differential equations. **Without computing any solutions directly, we can determine the behavior of solutions near equilibrium solutions**!

Notice that the matrix equation that we want to study is in fact an autonomous system (i.e. one that doesn't depend on t). Hence, we can search for equilibrium solutions by setting $\vec{x}'(t) = A\vec{x}(t) = \vec{0}$. We will assume (most of the time) that A is an invertible matrix so that $\vec{x}(t) = \vec{0}$ is the only equilibrium solution for the system. Our goal is then clear: **Find out what kind of equilibrium the solution $\vec{x} = \vec{0}$ is**.

Our first goal is to find out what kind of behavior we can expect from two-dimensional systems. We can work our way up from there. To study the stability of the origin, we often appeal to the use of the *phase plane*. This is very similar (if not identical) to the analysis we did in section 2.5 with autonomous equations in one dimension - back then, we worked on the phase *line*. That is, we are making a two-dimensional analog to the phase line used in section 2.5. To create the phase plane, we are essentially constructing a vector field using $A\vec{x}$. This, again, is almost identical to what we did in section 1.1 by creating slope/direction fields.

Consider why this works: We know that solutions $\vec{x}(t)$ have a velocity vector given by $\vec{x}'(t)$. In our matrix system, we have the relation $\vec{x}'(t) = A\vec{x}(t)$. Hence, at any point $\vec{x} = (x_1, x_2)^T$ of the plane, we can associate with it a velocity (tangent) vector given by $A\vec{x}$. After constructing this vector field, we can plot several representative solutions. The resulting plots together with the vector field are called a *phase portrait*.

Finding explicit solutions of this general n-dimensional system is done by using the theory from the previous sections 7.1-7.4. From our analysis of the one-dimensional system, we are motivated to believe that solutions should be exponential in nature. That is, we think our solutions might be exponential vector functions. Taking the hint from the one-dimensional case, let's take an ansatz solution of

$$\vec{x}(t) = e^{rt}\vec{\xi},$$

where r is a (complex) constant, and $\vec{\xi}$ is a fixed, non-zero vector - both are to be determined later. (Notice that it's all right to multiply a vector by a function, seeing how the function is just a real number for any particular value of t.) Let's see what consequences this kind of trial solution yields. We differentiate $\vec{x}(t)$ so that

$$re^{rt}\vec{\xi} = \vec{x}'(t) = A\vec{x} = Ae^{rt}\vec{\xi}.$$

We can then multiply both sides by e^{-rt} and be left with

$$r\vec{\xi} = A\vec{\xi},$$

which may be written as

$$(A - rI)\vec{\xi} = \vec{0}.$$

This equation is exactly the same one used to find the eigenvalues of A! Hence, **the only values of r that will yield our ansatz solution are exactly the eigenvalues of A!**

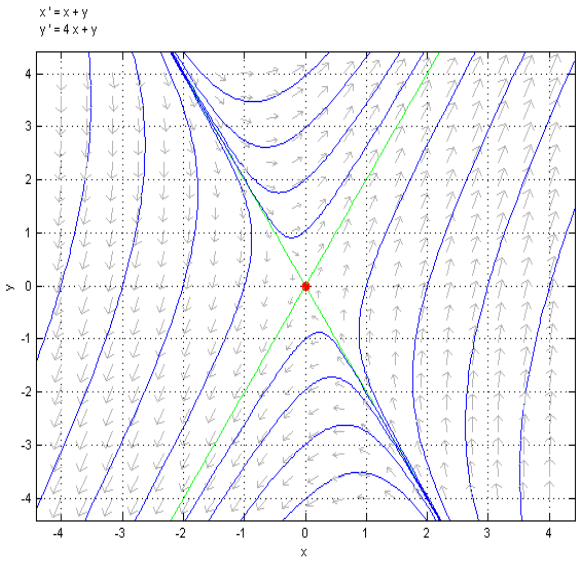


Figure 18: A phase portrait for the system in the first example. Green solutions are those along the eigenspaces of A.

Remark 7.4. Let's take one moment to understand the implications of the previous statement. Eigenvector solutions to $\vec{x}'(t) = A\vec{x}(t)$ starting in an eigenspace will stay in that eigenspace. In the case that the eigenspace is one-dimensional, solutions beginning in an eigenspace can only flow along the line (through the origin) in which they started.

We should now do a few examples to illustrate this method of solution.

Example 7.7. Let's solve the system

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}(t).$$

We can construct a vector field to show what the solution behavior looks like with arbitrary initial conditions. This is given in Figure 18.

For explicit solutions, we first need to find the eigenvalues of A. We can find these by our usual method, and we determine quickly that the characteristic polynomial of A is $r^2 - 2r - 3$. Hence, the roots to this equation are the eigenvalues of A. They are $r = 3$ and $r = -1$.

We can then determine the associated eigenvectors. One can find these to be

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v}_{-1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then our ansatz solutions are given by

$$\vec{x}^{(1)}(t) = e^{3t}\vec{v}_3 \quad \text{and} \quad \vec{x}^{(2)} = e^{-t}\vec{v}_{-1}.$$

Therefore, the general solution to the differential equation is given by

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

One can also check that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are linearly independent by checking that their Wronskian is

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Now that we have a general solution, we can try to make sense of its long-term behavior. The first term $\vec{x}^{(1)}$ wants to tend to infinity as $t \to \infty$. However, this is going to blow up in a very specific direction - it's given by $\vec{v}_3$. The second term $\vec{x}^{(2)}$ wants to tend to zero as $t \to \infty$. Again, this is happening in a very specific direction given by $\vec{v}_{-1}$. We can group these two results together to reason that solutions want to tend **away** from the origin along the line spanned by $\vec{v}_3$ and **toward** the origin along the line spanned by $\vec{v}_{-1}$. These two results together give us the phase portrait from Figure 18. Because solutions both tend away from and toward the origin, we call such an equilibrium a *saddle point*. (In fact, we could have concluded this just by observing that we began with two distinct, real eigenvalues with opposite signs.)

We can also separate the solutions of the system into $x_1(t)$ and $x_2(t)$ by reading of their respective components from

$$\vec{x}(t) = \begin{bmatrix} c_1e^{3t} + c_2e^{-t} \\ 2c_1e^{3t} - 2c_2e^{-t} \end{bmatrix}.$$

That is,

$$x_1(t) = c_1e^{3t} + c_2e^{-t} \quad \text{and} \quad x_2(t) = 2c_1e^{3t} - 2c_2e^{-t}.$$

Each of these may be graphed against t to see what each component of the solution does as $t \to \infty$.

Remark 7.5. It is important to note here that we can't absorb constants into our arbitrary "integration" coefficients anymore. This is chiefly due to the fact that **the basis of solutions are coupled and, as such, depend on the same collection of coefficients**. This is why we've left the above solution for $x_2(t)$ as we have.

There is one other kind of solution type that appears for distinct, real eigenvalues. This is the case of a *node*, which can be either *stable* or *unstable*.

Example 7.8. Let's take a look at the system

$$\vec{x}'(t) = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \vec{x}(t).$$

We can construct a direction field as before. This can be found in the textbook. The reader should try creating the direction field before consulting the textbook.

We can find the characteristic equation as before. For this matrix, it is $r^2 + 5r + 4$. Hence, the eigenvalues are $r = -1$ and $r = -4$. We can then compute the eigenvectors to be

$$\vec{v}_{-1} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad \vec{v}_{-4} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}.$$

So the ansatz solutions are

$$\vec{x}^{(1)}(t) = e^{-t}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad \vec{x}^{(2)}(t) = e^{-4t}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}.$$

Therefore, the general solution to the system is

$$\vec{x}(t) = c_1 e^{-t}\vec{v}_{-1} + c_2 e^{-4t}\vec{v}_{-4}.$$

The solutions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ both tend to zero as $t \to \infty$. This indicates that all solutions in this case will tend toward the origin. They will do so along the vectors $\vec{v}_{-1}$ and $\vec{v}_{-4}$, but they will all tend toward the origin. It should be noted, though, that they will tend to zero faster along $\vec{v}_{-4}$. Hence, most solutions will want to tend toward the line spanned by $\vec{v}_{-1}$ before they tend toward the origin (a similar phenomenon can also be seen in the saddle point example above). The reader should verify this by checking trial initial conditions on their direction fields created from before.

Because all solutions will tend toward the origin in this case, the origin equilibrium is called a *stable node*. One can reasonably assume that if both eigenvalues were distinct positive real numbers, then the origin would be an *unstable node* where all solutions would tend away from the origin as $t \to \infty$.

We can dissect the possible outcomes and long-term behaviors of solutions into the following three cases:

1. The eigenvalues are real and distinct.
2. The eigenvalues are complex conjugates.
3. The eigenvalues are repeated real roots.

This list is identical to that studied in second-order equations. This is no coincidence since these eigenvalues are, again, found to be roots of a real-coefficient polynomial. We have already seen what kinds of solutions we can expect from the first case. We should check one more example to see what other kinds of cases occur when we expand beyond two-dimensional systems.

Example 7.9. Let's extract the general solution of the system

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\vec{x}(t).$$

We have already found the eigenvalues and eigenvectors of this matrix in section 7.3. Recall that they were $r = 2$ and $r = -1$ so that

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_{-1,1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_{-1,2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

We use those results to show that the general solution is

$$\vec{x}(t) = c_1 e^{2t}\vec{v}_2 + c_2 e^{-t}\vec{v}_{-1,1} + c_3 e^{-t}\vec{v}_{-1,2}.$$

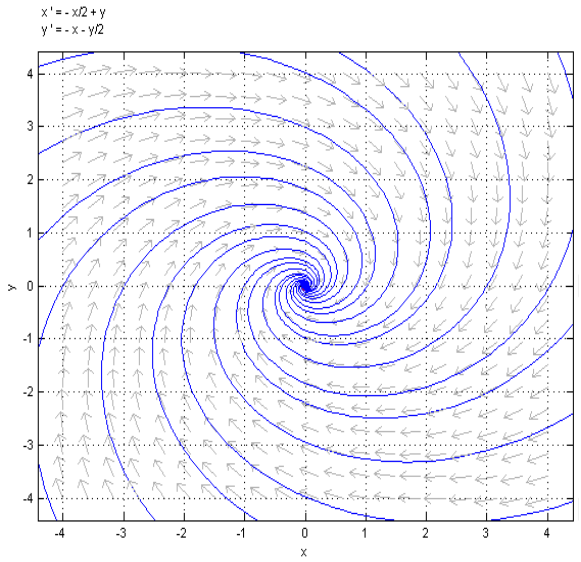


Figure 19: A phase portrait for the system in the first example. Notice how there are no separatrices as in real-eigenvalue systems.

We can think of this solution as a curve laying in three-dimensional space. If we let $t \to \infty$, we see that the two solutions corresponding to $r = -1$ will tend to zero. This tells us that solutions along the **plane** spanned by $\vec{v}_{-1,1}$ and $\vec{v}_{-1,2}$ will tend to zero. Hence, the origin **looks like** a stable node in this plane; however, there is the unstable axis of $\vec{v}_2$ to take into account. If any solution deviates from the plane (a *stable manifold*), then they will be "seen" by the unstable axis and be pulled away from the plane toward infinity in the direction of $\vec{v}_2$. Therefore, this is type of three-dimensional analog to a saddle point.

Even though we have considered a three-dimensional system, we are going to mainly keep to studying two-dimensional systems for the time being. We should now move on to other cases of the eigenvalues.

7.6 Complex Eigenvalues

Given the list of possibilities discussed in the previous section, we now wish to consider what happens when our characteristic polynomial has complex roots. Perhaps an example is the best way to introduce this new concept.

Example 7.10. "Find a fundamental set of real-valued vector functions that solve the system

$$\vec{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \vec{x}$$

, then plot a phase portrait of this system."

We can easily create the phase portrait for this system as in Figure 19.

This figure indicates that we should expect solutions that want to tend toward the origin; however, they will do so in a way that is distinct from the behavior of solutions

near a stable node. Let's use our ansatz solution to figure out what eigenvalues and eigenvectors we need to use.

The matrix has a characteristic polynomial of $r^2 + r + \frac{5}{4}$. One can check that the roots of this polynomial are $r_+ = -\frac{1}{2} + i$ and $r_- = \overline{r_+}$. We can then quickly find the eigenvectors to be

$$\vec{v}_{r_+} = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \vec{v}_{r_-} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

This is a new one for us: What do we do with complex-valued eigenvalues and eigenvectors? The answer is in theorem 7.7. Namely, if we can find a way to represent our solutions as a sum of real and imaginary parts, then those real and imaginary parts are the solutions we should use to make our solutions real-valued.

We begin by noting that our ansatz solutions are

$$\vec{X}^{(1)}(t) = e^{(-1/2+i)t}\vec{v}_{r_+} \quad \text{and} \quad \vec{X}^{(2)}(t) = e^{(-1/2-i)t}\vec{v}_{r_-}.$$

Let's look at $\vec{X}^{(1)}(t)$ first. (We can do the exact same analysis with $\vec{X}^{(2)}(t)$, and we will get very similar results). The exponential function can be broken into

$$e^{(-1/2+i)t} = e^{-t/2}(\cos t + i \sin t).$$

Hence, $\vec{X}^{(1)}$ can be rewritten as

$$\vec{X}^{(1)}(t) = \begin{bmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{bmatrix} + i \begin{bmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{bmatrix}.$$

By theorem 7.7, we can take the real and imaginary parts of this solution to be our fundamental set of real-valued vector solutions. That is,

$$\vec{x}^{(1)}(t) = e^{-t/2}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \vec{x}^{(2)}(t) = e^{-t/2}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

We can double-check the Wronskian between these two solutions to see that they are indeed linearly independent. (I suggest the reader do exactly this before continuing.) Therefore, our general solution of the system (in real-valued vector functions) is given by

$$\vec{x}(t) = c_1 e^{-t/2}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t/2}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

Notice here that the decaying exponential terms forces these solutions do indeed tend toward zero as $t \to \infty$. However, there is a second feature of these solutions that shouldn't at all be overlooked: **They oscillate**! When solutions near the origin have such an oscillating behavior, there are three classifications of the equilibrium at zero. First, the solutions can either decay to zero or blow up to infinity. In the decaying case, the origin is called a *stable spiral*. In the unbounded case, the origin is called an *unstable spiral*. The second case is if solutions neither decay nor blow up. If the solutions about the origin come back to where they started, the origin is then called a (linear) *center*.

We can summarize the work that we have done in the preceeding example in the general case of a pair of complex conjugate roots $r = \lambda \pm i\mu$. If they have corresponding eigenvectors $\vec{v}_+$ and $\vec{v}_-$, then we have ansatz solutions

$$\vec{X}^{(1)}(t) = e^{r+t}\vec{v}_+ \quad \text{and} \quad \vec{X}^{(2)}(t) = e^{r-t}\vec{v}_-.$$

We ignore the second of these in general so that we create our fundamental solutions from $\vec{X}^{(1)}$. (Using $\vec{X}^{(2)}$ instead will result in a difference in sign that can be later absorbed into the general coefficients - hence, the solution will be identical to the method outlined here.) This is done by using Euler's formula to rewrite the complex exponential function so that

$$\begin{aligned} \vec{X}^{(1)}(t) &= e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))\vec{v}_+ \\ &= e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))(\vec{a} + i\vec{b}), \end{aligned}$$

where we have broken $\vec{v}_+ = \vec{a} + i\vec{b}$ into real and imaginary parts as well. Then we finally have

$$\vec{x}^{(1)}(t) = e^{\lambda t}(\cos(\mu t)\vec{a} - \sin(\mu t)\vec{b}) + ie^{\lambda t}(\sin(\mu t)\vec{a} + \cos(\mu t)\vec{b}).$$

This can then be broken into real and imaginary parts so that $\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$, where

$$\vec{u}(t) = e^{\lambda t}(\cos(\mu t)\vec{a} - \sin(\mu t)\vec{b})$$

and

$$\vec{v}(t) = e^{\lambda t}(\sin(\mu t)\vec{a} + \cos(\mu t)\vec{b}.$$

Notice now that these solutions will always come in pairs. This aligns with the idea that complex eigenvalues will always come in complex conjugate pairs. Again, we should emphasize that this only worked when the functions/coefficients of A are real. Otherwise, the complex numbers need not occur in pairs.

We can now summarize the kinds of solutions we have found so far in two-dimensional systems as:

1. Eigenvalues are oppositely signed real values - origin is a saddle point.
2. Eigenvalues have same sign and are distinct - origin is a node.
3. Eigenvalues are complex conjugates with nonzero real part - origin is a spiral.
4. Eigenvalues are complex conjugates with zero real part - origin is a center.

Remark 7.6. In general, there is the possibility of having an eigenvalue equal to zero. One should recall, however, that this is equivalent to the fact that the matrix A is not invertible. It is then the case that there are nonzero solution vectors for our system. In particular, there is more than just the constant zero vector for an equlibrium - this is something we stated we were going to avoid by ensuring that A was invertible. Hence, we won't speak about the zero eigenvalue case any further.

There is also the case of repeated real eigenvalues. Namely, eigenvalues that aren't simple can show up, and we will consider this case in section 3.8. We can expect our solutions to look similar to that of the repeated roots case of second-order systems; but we will cross that bridge when we come to it.

We mentioned a motivational problem for studying systems of differential equations. We study now the unforced double mass-spring problem stated generally at the beginning of this chapter.

Example 7.11. Recall the anchored double mass-spring system

$$\begin{cases} m_1 \frac{d^2x_1}{dt^2} &= -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 \frac{d^2x_2}{dt^2} &= k_2 x_1 - (k_2+k_3)x_2. \end{cases}$$

We can easily transform this into a system of first-order equations. This is accomplished by using the (most convenient) change of variables

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = y_1' = x_1', \text{ and } \quad y_4 = y_2' = x_2'.$$

This consequently transforms the system of second-order equations into the first-order system $\vec{y}' = A\vec{y}$, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & 0 & 0 \\ k_2/m_2 & -(k_2+k_3)/m_2 & 0 & 0 \end{bmatrix}.$$

If we use the specific values of

$$m_1 = 2, \quad m_2 = \frac{9}{4}, \quad k_1 = 1, \quad k_2 = 3, \text{ and } \quad k_3 = \frac{15}{4},$$

then we get the A matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{bmatrix}.$$

One can find the characteristic polynomial by hand (remarkably easily) or by using a computer to get $r^4 + 5r^2 + 4 = (r^2+1)(r^2+4)$. Hence, the eigenvalues are $r_1 = i$, $r_2 = -i$, $r_3 = 2i$, and $r_4 = -2i$. We can just as easily find the eigenvectors to be

$$\vec{v}_i = \begin{bmatrix} 3 \\ 2 \\ 3i \\ 2i \end{bmatrix}, \vec{v}_{-i} = \begin{bmatrix} 3 \\ 2 \\ -3i \\ -2i \end{bmatrix}, \vec{v}_{2i} = \begin{bmatrix} 3 \\ -4 \\ 6i \\ -8i \end{bmatrix}, \text{ and } \quad \vec{v}_{-2i} = \begin{bmatrix} 3 \\ -4 \\ -6i \\ 8i \end{bmatrix}.$$

We can use $\vec{v}_i$ and $\vec{v}_{2i}$ to find the two real-valued vector functions for each pair of eigenvalues. Let's start with $\vec{v}_i$. The ansatz solution here is

$$\vec{Y}^{(1)}(t) = (\cos t + i \sin t)\vec{v}_i$$

so that

$$\vec{y}^{(1)}(t) = \begin{bmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{bmatrix} + i \begin{bmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{bmatrix} = \vec{u}^{(1)}(t) + i\vec{v}^{(1)}(t).$$

Similarly, we find that

$$\vec{Y}^{(2)}(t) = (\cos(2t) + i\sin(2t))\vec{v}_{2i},$$

and that

$$\vec{y}^{(2)}(t) = \begin{bmatrix} 3\cos(2t) \\ -4\cos(2t) \\ -6\sin(2t) \\ 8\sin(2t) \end{bmatrix} + i \begin{bmatrix} 3\sin(2t) \\ -4\sin(2t) \\ 6\cos(2t) \\ -8\cos(2t) \end{bmatrix} = \vec{u}^{(2)}(t) + i\vec{v}^{(2)}(t).$$

It is left to the reader to confirm the linear independence of all four of these solutions.

We have solved the first-order system in four-dimensional space. However, our original system was given as a two-dimensional system of second-order equations. We now need to procure from this a solution for the original system. This is done by recalling our change of variables that $x_1(t) = y_1(t)$ and $x_2(t) = y_2(t)$. This corresponds to our first and second components of the solution $\vec{y}(t) = c_1\vec{y}^{(1)}(t) + c_2\vec{y}^{(2)}(t)$. Then we have

$$x_1(t) = 3(c_1\cos t + c_2\sin t) + 3(c_3\cos(2t) + c_4\sin(2t))$$

and

$$x_2(t) = 2(c_1\cos t + c_2\sin t) - 4(c_3\cos(2t) + c_4\sin(2t)).$$

Hence, we have simultaneously solved for x_1 and x_2 so that we know the behavior of this system. Since there was assumed to be no damping forces involved in this model, we can get an idea for what each x_1 and x_2 looks like over time. They are given in Figure 20. The figure shows what the solutions look like superimposed over each other against a time t axis as well as parametric displacement-velocity plots.

The first two rows of plots in the figure correspond to the two *fundamental modes* of this model. These are the displacements resulting from specific initial conditions where the period intervals of the frequency-1 terms are in a sense evenly divided by the frequency-2 terms. The third row is the more common result for more general initial conditions. Notice that the motion in any of these cases is still periodic - that is, this is harmonic motion. It just doesn't fall under the case of simple harmonic motion where there is one underlying frequency involved.

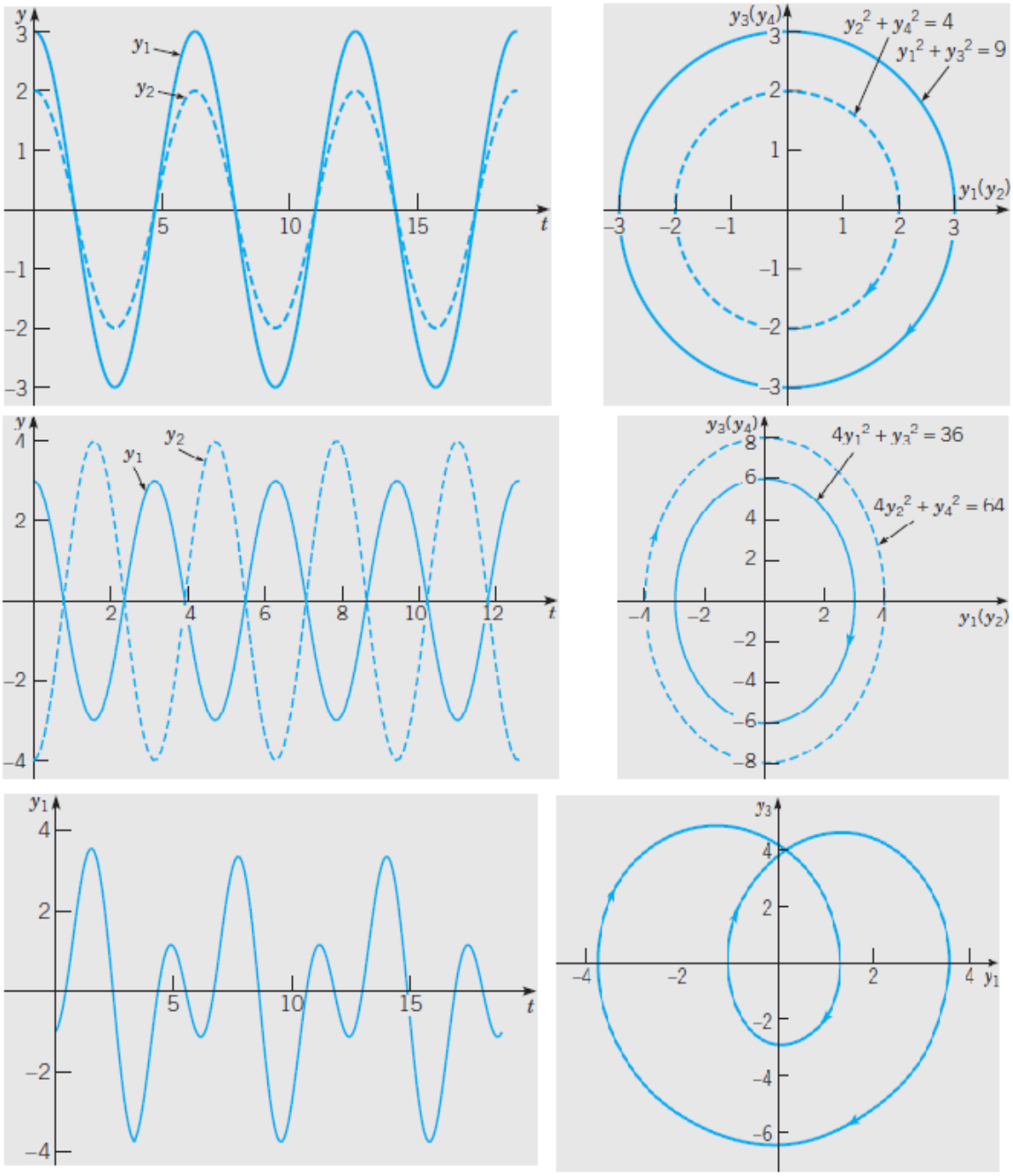


Figure 20: The solutions to the double mass-spring system with various initial conditions. These plots should be considered as the pairs in which the rows present them. Plots on the left are superimposed displacements of the two masses, whereas the plots on the right shows the mass's displacement along with its velocity.

7.7 Fundamental Matrices

We now have enough information to work with a general theory for first-order systems of differential equations. Our first order of business is to justify our ansatz solution, explaining why it should make sense to choose it as our trial solution.

So first, we suppose that we have already found a fundamental set of linearly independent solutions $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(p)}(t)$ (not necessarily as in Theorem 7.6) for the equation

$$\vec{x}'(t) = P(t)\vec{x}$$

on some interval $\alpha < t < \beta$. Consider the matrix

$$\Psi(t) = [\ \vec{x}^{(1)}(t)\ \cdots\ \vec{x}^{(n)}(t)\ ] = \begin{bmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & \ddots & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{bmatrix}.$$

We call this a *fundamental matrix* of the system. By its construction, Ψ is an invertible matrix on $\alpha < t < \beta$.

As an example, consider the system

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}.$$

We have found a fundamental set of solutions for this system to be

$$\vec{x}^{(1)}(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}, \quad \vec{x}^{(2)}(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}.$$

Hence, we can construct a fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

With this simple construction in mind, we can explicitly describe solutions $\vec{x}(t)$ of the system as a product of Ψ with a coefficient vector $\vec{c}$. That is,

$$\vec{x}(t) = \Psi(t)\vec{c},$$

where $\vec{c} = (c_1, \ldots, c_n)^T$, is the general solution of the system $\vec{x}'(t) = P(t)\vec{x}$. (Notice the similarity with solutions to the analogous first-order equation $x' = kx$ with solutions $x(t) = ce^{kt}$. This is what tells us that this Ψ might be a good construction.)

Consider then that we can ask for an initial condition $\vec{x}(t_0) = \vec{x}^0$ (where $\alpha < t_0 < \beta$ and try to find the coefficient vector $\vec{c}$ that yields the solution with this initial condition. So we know here that we must have

$$\vec{x}^0 = \vec{x}(t_0) = \Psi(t_0)\vec{c}$$

so that

$$\vec{c} = \Psi(t_0)^{-1}\vec{x}^0.$$

(This is a compact way to write out all that row reduction that we've been using in the two- and three-dimensional examples to find our coefficients for particular solutions.) Therefore, using this in place of $\vec{c}$ in the solution vector, we find

$$\vec{x}(t) = \Psi(t)\left(\Psi(t_0)^{-1}\vec{x}^0\right)$$

is the solution to the initial value problem $\vec{x}' = P(t)\vec{x}$ with $\vec{x}(t_0) = \vec{x}^0$.

It should also be noted before we continue that $\Psi(t)$ itself satisfies the original system in the following way:

$$\Psi'(t) = P(t)\Psi(t).$$

(That is, we are compactly writing that each column of $\Psi(t)$ is a solution to the original system.) This should not be confused with the original matrix equation seeing how the "solutions" here are matrices rather than column vectors. One can verify this equality by checking the terms column-by-column.

We can now ask for a **convenient** fundamental set of solutions. That is, if we take each of our fundamental solutions to have the initial conditions $\vec{x}^{(j)}(t_0) = \vec{e}_j$ (the j^{th} standard basis vector) and let their fundamental matrix be denoted by $\Phi(t)$, then $\Phi(t_0) = I_n$ (the $n \times n$ identity matrix). Hence, the calculation of the inverse matrix above is trivial is a trivial multiplication making the solution vector

$$\vec{x} = \Phi(t)\vec{x}^{(0)}.$$

That is, we would only need to multiply our fundamental matrix by the initial conditions to find our solution. Nifty! How does one find this convenient matrix $\Phi(t)$? Well, we appeal to the fact that solutions are unique and regroup the parentheses on the particular solution we found before:

$$\vec{x}(t) = \Psi(t)(\Psi(t_0)^{-1}\vec{x}^0) = (\Psi(t)\Psi(t_0)^{-1})\vec{x}^0 = \Phi(t)\vec{x}^0.$$

Hence, we would have

$$\Phi(t) = \Psi(t)\Psi^{-1}(t_0).$$

Let's put this to good use in an example.

Example 7.12. Recall the example from above:

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}.$$

We found one fundamental matrix $\Psi(t)$ to be

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}.$$

We then use this matrix to solve for the solution with initial conditions $\vec{x}^{(1)}(0) = \vec{e}_1$ and $\vec{x}^{(2)}(0) = \vec{e}_2$ so that we have

$$\Psi(0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix},$$

which has the inverse

$$\Psi^{-1}(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

We then apply this inverse to each initial condition separately as follows:

$$\vec{X}^{(1)}(t) = \Psi(t)\Psi^{-1}(0)\vec{e}_1 = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix} \quad \text{and} \quad \vec{X}^{(2)}(t) = \Psi(t)\Psi^{-1}(0)\vec{e}_2 = \begin{bmatrix} \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

Hence, the convenient fundamental matrix $\Phi(t)$ is given by the matrix whose columns are those vector just found above. Therefore, we have

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

This matrix could have also been found simply by the matrix multiplication

$$\Phi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

Sure enough, we can easily see that $\Phi(0) = I_2$.

In order to further elaborate on why our ansatz solutions were accurate, we should recall what the motivation was for their use in the first place. This was in the initial value problem

$$x' = ax; \quad x(0) = x_0,$$

where a was just some real constant. It has the solution

$$x(t) = x_0 e^{at}.$$

We just found an **extremely** similar result for systems of first-order equations. Namely, for the system

$$\vec{x}' = A\vec{x}; \quad \vec{x}(0) = \vec{x}^0,$$

we have the solution

$$\vec{x}(t) = \Phi(t)\vec{x}^0.$$

It would stand to reason that we would think that

$$\Phi(t) = e^{At},$$

but how exactly do we exponentiate a **matrix**?

This is accomplished by means of the Taylor series for e^x. Recall that we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots.$$

We can simply define our matrix exponential as what happens when we replace x by At. There is of course a little bit of work to do to ensure that A^n converges as $n \to \infty$, but we can take it on faith in this class that our definition

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots$$

does indeed converge for any **square** matrix we use. Notice here that we also had to replace the constant 1 term by the appropriately sized identity matrix I. However, there are some things that we **can** check (and really should) and don't have to take on faith. For example, we can check that the derivative of this function is what we expect it to be.

Consider the calculation

$$\begin{aligned}
\frac{d}{dt}(e^{At}) &= \frac{d}{dt}\left(\sum_{n=0}^{\infty}\frac{1}{n!}A^n t^n\right) && (11)\\
&= \sum_{n=0}^{\infty}\frac{d}{dt}\left(\frac{1}{n!}A^n t^n\right) && (12)\\
&= \sum_{n=0}^{\infty}\frac{n}{n!}A^n t^{n-1} && (13)\\
&= A\sum_{n=0}^{\infty}\frac{1}{n!}A^n t^n && (14)\\
&= Ae^{At}. && (15)
\end{aligned}$$

Hence, we see that $\vec{x}(t) = e^{At}$ satisfies the equation $\vec{x}' = A\vec{x}$. Therefore, by the uniqueness of solutions of initial value problems, we must have

$$\Phi(t) = e^{At}.$$

Therefore, we can write all solutions to the initial value problem

$$\vec{x}' = A\vec{x}; \quad \vec{x}(0) = \vec{x}^0$$

as

$$\vec{x}(t) = e^{At}\vec{x}^0.$$

This is most certainly consistent with our first-order theory of a single differential equation. So this answers why our ansatz solutions were correct to begin with, but how do eigenvalues and eigenvectors fit in with this brilliant new theory we are developing? This is where we take a detour through linear algebra for a few moments.

Every so often, we can write a first-order system in the form

$$\vec{x}' = D\vec{x},$$

where D is a diagonal matrix. In this case, we don't need to use all of the fancy machinery built over the past six sections. In fact, each row of equations in this system is a separable equation of the form

$$x_j' = d_j x_j.$$

Hence, the solution is easily found in this case by solving each row and then putting each component into the solution vector $\vec{x}(t)$. We want to extend this idea to the more general case of

$$\vec{x}' = A\vec{x}.$$

Unfortunately, we don't have a diagonal matrix A in general. However, there are certain matrices that can be made to **look** like they're diagonal. This is the idea of *diagonalization*. We want to find a way to transform a matrix A into a diagonal matrix D. We have seen eactly how to do this in our prerequisite linear algebra courses. We simply find the eigenvalues and eigenvectors for A and group the vectors into one matrix T. That is, for eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, we create the matrix

$$T = \left[\begin{array}{ccc} \vec{v}_1 & \cdots & \vec{v}_n \end{array}\right].$$

It can easily be shown that

$$AT = TD,$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Here, the eigenvalue λ_j is associated to the eigenvector $\vec{v}_j$. Since the set of eigenvectors is linearly independent, we can invert T so that

$$D = T^{-1}AT.$$

This is what is called a *similarity transformation*. If we can find such a similarity transformation of A into D, then we say that A is *diagonalizable*. It should be noted here that this similarity transformation effectively takes a basis of eigenvectors to the standard basis vectors $\vec{e}_1, \ldots, \vec{e}_n$.

Let's use our familiar first-order system as an example.

Example 7.13. Recall the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

from our previous examples. We have already found that the eigenvectors are

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v}_{-1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Hence, we should take

$$T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

so that

$$T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

We can then check the matrix multiplication

$$D = T^{-1}AT = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence, A is diagonalizable.

How does solving the D system help us solve the A system? It is finally time to come full circle and tie together the theory developed in this section with solutions of first-order systems of differential equations. Consider once more the A system

$$\vec{x}' = A\vec{x}.$$

Let's assume that A has a full set of eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with the associated eigenvalues $\lambda_1, \ldots, \lambda_n$. Let T denote the matrix whose columns are the eigenvectors of A, and let $\vec{x}$

denote a solution to the above system. Then we define a new vector $\vec{y}$ so that $\vec{x} = T\vec{y}$. From this we see that

$$T\vec{y}' = \vec{x}' = A\vec{x} = AT\vec{y}$$

so that

$$\vec{y}' = (T^{-1}AT)\vec{y} = D\vec{y},$$

where D is the diagonal matrix whose components are the eigenvectors $\lambda_1, \ldots, \lambda_n$ associated with $\vec{v}_1, \ldots, \vec{v}_n$.

Now, define the matrix

$$Q(t) = e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}.$$

Obervse that this is a fundamental matrix for the D system. We can construct a fundamental matrix $\Psi(t)$ for this system by taking

$$\Psi(t) = TQ(t).$$

Hence, the general solution of this system is given by

$$\vec{x}(t) = \Psi(t)\vec{c}$$

for some vectors $\vec{c}$ of coefficients as before.

It should be noted that there is no advantage of using this method over what has already been done in the previous sections. This is chiefly due to the fact that both methods require one to find the eigenvalues and eigenvectors of the matrix A - and as such, these methods will take roughly the same time to compute (the fundamental matrix method here is just a compact way to explain why everything before this has worked so well and that there is indeed a larger, overarching theory behind it). To bring our main example to a close, consider it one final time.

Example 7.14. Given the system of equations

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x},$$

we already know what T is so that we can take

$$\vec{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \vec{y}.$$

We then see that

$$Q(t) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix},$$

and we can compute

$$\Psi(t) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix},$$

as we found before.

7.8 Repeated Eigenvalues*

Now that we have considered all of the cases where we have distinct eigenvalues, we need to consider what happens when we have an eigenvalue that is repeated. Remember from our study of second-order differential equations that we only got one ansatz solution, so we needed to find a way to create a second linearly independent solution here. Let's see what this looks like in a series of examples.

Example 7.15. We begin with the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

We can construct the characteristic polynomial as

$$(1-r)(3-r)+1 = r^2 - 4r + 4 = (r-2)^2 = 0$$

so that $R = 2$ is a repeated eigenvalue. We can immediately find its associated eigenvectors as

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, there is only one eigenvector associated with $R = 2$.

The fact that we could only find one eigenvector for $R = 2$ in the above example poses a problem. We need to find solutions numbering the algebraic multiplicity of this eigenvalue in order to construct a fundamental set of solutions for the first-order system of equations. In the case where the geometric multiplicity matches the algebraic multiplicity m, there is no problem since we can find m linearly independent vector solutions to this system. The problem comes in as in the example where the geometric multiplicity was less than the algebraic multiplicity.

Following the cue from our study of second-order equations, we suppose that we have a solution of the form

$$\vec{x}(t) = te^{2t}\vec{\xi}.$$

We need to find the vector ξ, if it exists, that will make this $\vec{x}$ a solution to the differential equation. If we can find just one more solution here, then we are done and can take this to be our fundamental set of solutions. So we quickly compute

$$2te^{2t}\vec{\xi} + e^{2t}\vec{\xi} - te^{2t}A\vec{\xi} = \vec{0}.$$

Hence, we would need to have $\vec{\xi} = \vec{0}$. Since we are looking for nonzero vectors of this form, there are no solutions $\vec{x}$ of that particular form. However, just like with undetermined coefficients, we can include another vector function to this solution to see if things will work out. Namely, take

$$\vec{x}(t) = te^{2t}\vec{\xi} + e^{2t}\vec{\eta}.$$

We now compute

$$2te^{2t} + (\vec{\xi} + 2\vec{\eta})e^{2t} = A(te^{2t} + e^{2t}\vec{\eta}).$$

This time, we realize that we need to have

$$(A - 2I)\vec{\xi} = \vec{0}$$

and

$$(A - 2I)\vec{\eta} = \xi.$$

This can also be seen as

$$(A - 2I)^2\vec{\eta} = \vec{0}.$$

This makes $\vec{\eta}$ a *generalized eigenvector* of A. To see how we compute this, we can use any eigenvector of A to stand in for ξ. So we take

$$\vec{xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Now we solve the system

$$(A - 2I)\vec{\eta} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This is the augmented system

$$\left[\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right].$$

Hence, if $\vec{\eta} = (\eta_1, \eta_2)^T$, we need to have

$$\eta_1 + \eta_2 = -1,$$

where $\eta_2 = t$ is our free variable. Solving this then gives us

$$\vec{\eta} = \begin{bmatrix} -1 - \eta_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Notice that the last term above is yet another linear combination (multiple) of the solution $\vec{x}^{(1)}(t)$ that we have already found. So we can group this into that solution and ignore it here. Hence, we have a solution

$$\vec{x}^{(2)}(t) = te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

One should check that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ for a linearly independent set of solutions to this system. Therefore, the general solution of the system is given by

$$\vec{x}(t) = c_1e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\left(te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right).$$

Let's see what the phase portrait of this system looks like so that we can get a feeling for how these *improper node* equilibria treat solutions of the system. This is in Figure 21.

Notice how, at the root of it all, this is still an unstable equilibrium. However, it is not in the class of unstable nodes that we saw before. In fact, there is only the one eigenspace that the solutions will adhere to should the begin on it. This is in contrast to the nodes from before where we had two distinct eigenspaces that corresponded to different eigenvalues and dictated different rates at which the solutions would travel. Here there is only one rate, and we don't even see any geometric implications of the vector $(-1, 0)^T$ that we created in the example.

The way we treat repeated roots for systems is in stark contrast with how we treated them in the second-order equations before. The trial solutions we have tried here in the

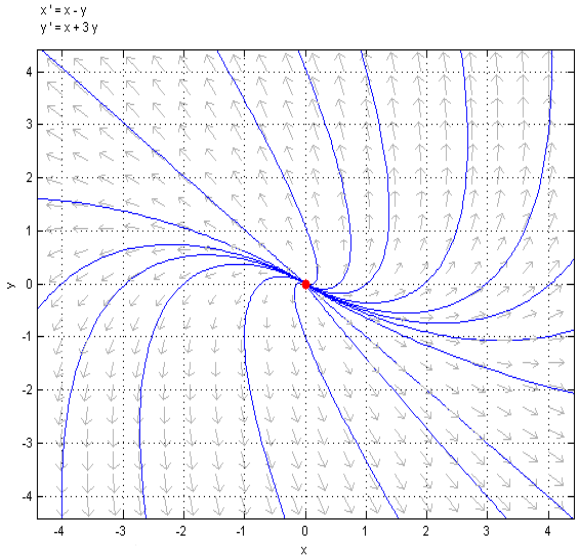


Figure 21: A phase portrait of the system in the example. This illustrates an improper node which is the result of having an eigenvalue whose geometric multiplicity is less than its algebraic multiplicity.

system case would have yielded an $\vec{\eta} = \vec{0}$. Hence, this method is vastly more general than what we saw in those earlier cases. We need to hang on to that extra $e^{2t}\vec{\eta}$ term in the general case of systems.

Again, we saw something new in solving this type of system. There was the appearance of *generalized eigenvectors*. These are incredibly useful in theory, and they are absolutely necessary here in solving systems of differential equations. The general way to solve this case where the geometric multiplicity was less that the algebraic multiplicity of R is to take a general solution

$$\vec{x}(t) = te^{Rt}\vec{\xi} + e^{Rt}\vec{\eta},$$

where we specified that

$$(A - RI)\vec{\xi} = \vec{0} \quad \text{and} \quad (A - RI)\vec{\eta} = \vec{\xi}.$$

The second of these equations can be rewritten as

$$(A - RI)^2\vec{\eta} = \vec{0}.$$

Hence, we need $\vec{\xi}$ to be an eigenvector for R and $\vec{\eta}$ to be a generalized eigenvector for R.

We should double-check that this is in agreement with our work for fundamental matrices. Using our solutions from above, we can take

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} - e^{2t} \\ -e^{2t} & -te^{2t} \end{bmatrix} = e^{2t}\begin{bmatrix} 1 & t-1 \\ -1 & -t \end{bmatrix}.$$

To get our convenient fundamental matrix, we compute as before so that

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = e^{2t}\begin{bmatrix} 1 & t-1 \\ -1 & -t \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} = e^{2t}\begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}.$$

One can verify that this is exactly $\Phi(t) = e^{At}$.

7.9 Nonhomogeneous Linear Systems*

Now that we have seen how to solve homogeneous linear systems of constant coefficients, we wish to expand our horizons a bit and extend our results to nonhomogeneous systems of the form

$$\vec{x}' = A(t)\vec{x} + \vec{g}(t).$$

As one might expect, solutions will be of the form

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t),$$

where $\vec{x}_h$ is the general homogeneous solution and $\vec{x}_p$ is a particular solution. We need to find a way to solve for this particular solution. We are going to take a single example and show several different ways to solve it. These methods are implemented by means of

1. Diagonalization of the matrix,
2. Undetermined coefficients,
3. Variation of parameters, and
4. Laplace transforms.

Let's begin with diagonalization of the matrix. The idea here is to take a coupled system of first-order equations and turn it into an uncoupled system. From there, we can solve each equation via the method of section 2.1. Let's outline this for the general system

$$\vec{x}' = A\vec{x} + \vec{g}(t).$$

Suppose that A is diagonalizable (this is a rather large assumption). Then we can find a full set of eigenvectors so that we can create the matrix T of eigenvectors. We can then write $\vec{x}$ in terms of the eigenvector basis. That is, we take

$$\vec{x} = T\vec{y}$$

for some vector $\vec{y}$ of weights of the eigenvectors that creates $\vec{x}$. Differentiating this expression gives us

$$\vec{x} = T\vec{y}'$$

because the T matrix is filled with constants (we also note that T is invertible here - why?). Hence, we must have

$$T\vec{y}' = A(T\vec{y}) + \vec{g}(t)$$

so that we now have a system in the transformed $\vec{y}$ coordinates

$$\vec{y}' = (T^{-1}AT)\vec{y} + T^{-1}\vec{g}(t).$$

Since A is diagonalizable, we recall that $T^{-1}AT = D$ for the diagonal matrix of eigenvalues (in the appropriate locations along the diagonal). So we have now turned our system into an uncoupled system in terms of the vector $\vec{y}$. Once here, we solve for each component $y_i(t)$ so that we have

$$\vec{x}(t) = T\vec{y}(t),$$

and we then have the general solution. Let's see this process in action in an example.

Example 7.16. Consider the nonhomogeneous system

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

We quickly determine the eigenvalues to be $\lambda = -3$ and $\lambda = -1$. The associated eigenvectors are

$$\vec{v}_{-3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v}_{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then we see that the homogeneous solution of the system is

$$\vec{x}_h(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To find the particular solution, we implement our diagonalization method. Recall that T is simply made by using the eigenvectors of A as the columns of the matrix. This is given by

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then we can create the uncoupled system

$$\vec{y}' = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \vec{y} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \begin{bmatrix} -3y_1 \\ -y_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{bmatrix}.$$

Therefore, we have the system

$$\begin{cases} y_1' + 3y_1 &= e^{-t} - \frac{3}{2}t \\ y_2' + y_2 &= e^{-t} + \frac{3}{2}t. \end{cases}$$

We can then solve each of these indivually (because they are only dependent on their own respective dependent variables) to get

$$y_1(t) = \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6} + c_1 e^{-3t}$$

and

$$y_2(t) = te^{-t} + \frac{3}{2}t - \frac{3}{2} + c_2 e^{-t}.$$

We then apply our original change of variables to find that

$$\vec{x} = T\vec{y} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{y}$$

$$= te^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -4 \\ -5 \end{bmatrix} + c_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

One can directly check that this is indeed the general solution to the nohomogeneous system.

There is a way to adjust this method for when the matrix A is not diagonalizable, but it is not within the scope of this course.

We now move on to the next method: Undetermined coefficients. This works very similarly to what we have already seen in second-order equations. The key difference between second-order equations and first-order systems is in the case where the function $\vec{g}(t)$ contains a part of the homogeneous solution of the system. Instead of just adding a factor of t to the homogeneous function, we need to include all terms of the general polynomial $At + B$ in the factor. Let's see what we mean in this version of the example.

Example 7.17. Recall the system from before:

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

First consider that the function $\vec{g}(t)$ can be written as

$$\vec{g}(t) = e^{-t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Recall that the homogeneous solution also contains a term of $e^{-t}\vec{v}_{-1}$. Hence, the trial particular solution should be

$$\vec{x}_p(t) = te^{-t}\vec{a} + e^{-t}\vec{b} + t\vec{c} + \vec{d}.$$

We substitute this trial solution into the differential equations and find that we need

$$A\vec{a} = -\vec{a},$$

$$A\vec{b} = \vec{a} - \vec{b} - \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$A\vec{c} = -\begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

and

$$A\vec{d} = \vec{c}.$$

The first equation tells us that $\vec{a}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$. So we take $\vec{a} = \alpha\vec{v}_{-1}$, where α is some constant. Looking to the second equation, this can only be solved if $\alpha = 1$. Hence, we need to have

$$\vec{b} = \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where β is any constant. We can simply take $\beta = 0$ in this case so that

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = [0 - 1].$$

The third equation is easily solved as

$$\vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

so that

$$\vec{d} = \begin{bmatrix} -4/3 \\ -5/3 \end{bmatrix}.$$

Therefore, our general solution is

$$\vec{x} = T\vec{y} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{y}$$

$$= te^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -4 \\ -5 \end{bmatrix} + c_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Notice that this particular solution doesn't quite match the particular solution found in the preceeding example. Nevertheless, these two solutions identify the same general solution (since we can extract the missing factor of $\frac{1}{2}$ from the homogeneous constant term).

We now move on to the more precise version of undetermined coefficients. Just as with second-order equations, this more precise method is found in variation of parameters. Again, our goal is to find particular solutions by using the homogeneous solutions that we already have. Once more, we can apply this method to the general nonhomogeneous version of our equation:

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t).$$

It is assumed that the components of $P(t)$ and $\vec{g}(t)$ are continuous for some interval $\alpha < t < \beta$. We also assume that we have a fundamental set of solutions to the homogeneous equation

$$\vec{x}' = P(t)\vec{x},$$

and we make them the columns of a fundamental matrix $\Psi(t)$. Recall that the general solution of the homogeneous system can be written as $\vec{x} = \Psi(t)\vec{c}$ for some constant vector $\vec{c}$. To use variation of parameters, we instead assume that the general solution takes the form $\vec{x} = \Psi(t)\vec{u}(t)$ for some vector $\vec{u}(t)$. We now substitute this into our nonhomogeneous system to find that we need

$$\Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t) = P(t)\Psi(t)\vec{u}(t) + \vec{g}(t).$$

Because $\Psi(t)$ is a fundamental matrix, we know that $\Psi' = P\Psi$. Hence, we can reduce the above equation to

$$\Psi(t)\vec{u}'(t) = \vec{g}(t).$$

Recall that $\Psi(t)$ is invertible on the interval (α, β). Therefore, to solve for $\vec{u}'(t)$, we just need to multiply both sides of this equation by $\Psi^{-1}(t)$. Hence,

$$\vec{u}(t) = \int \Psi^{-1}(t)\vec{g}(t)\,dt + \vec{c}$$

for some arbitrary constant vector $\vec{c}$. Therefore, the final general solution of the system is given by

$$\vec{x}(t) = \Psi(t)\vec{c} + \Psi(t)\int_{t_1}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds,$$

where t_1 is some point in the interval (α, β).

If we are given an initial value problem with $\vec{x}(t_0) = \vec{x}^{(0)}$, we can modify this solution to be based at the point t_0. This can be realized as

$$\vec{x}(t) = \Psi(t)\vec{c} + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds.$$

At this point, we can determine $\vec{c}$ explicitly as

$$\vec{c} = \Psi^{-1}(0)\vec{x}^{(0)}$$

so that

$$\vec{x}(t) = \Psi(t)\Psi^{-1}(0)\vec{x}^{(0)} + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\vec{g}(s)\,ds.$$

The student should recognize at this point that we can instead use our "convenient" fundamental matrix $\Phi(t)$ corresponding to $\Phi(t_0) = I$. Then we can rewrite our solution as

$$\vec{x}(t) = \Phi(t)\vec{x}^{(0)} + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\vec{g}(s)\,ds.$$

Let's see this put to use in the example.

Example 7.18. Recall once again our example system

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

We can put our homogeneous solution to use in the fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}.$$

We take our general solution to be of the form $\vec{x}(t) = \Psi(t)\vec{u}(t)$ so that we have

$$\begin{bmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{-t} 3t \end{bmatrix}.$$

We can multiply both sides by $\Psi^{-1}(t)$ and integrate to find

$$u_1(t) = \frac{1}{2}e^{2t} - \frac{1}{2}e^{3t} + \frac{1}{6}e^{3t} + c_1$$

and

$$u_2(t) = t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2.$$

Then we can conclude that

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + te^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix} + c_1e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

. Notice that this solution is now identical to the one found in the first example.

Notice that our examples have been progressively taking less and less work. Our final example will also take a small amount of work to implement. The use of Laplace transforms can be extended to vector functions since they are, after all, merely integrals. Hence, the workhorse for this kind of problem still lies in the formula for the Laplace transform of a function's derivative:

$$\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)] - f(0).$$

We extend this to vector functions in exactly the same way:

$$\mathcal{L}[\vec{x}'(t)](s) = s\mathcal{L}[\vec{x}(t)] - \vec{x}(0).$$

We again use the convention that the captial letter corresponds to the Laplace transform of the function denoted by the lowercase letter. That is, we can rewrite the above formula as

$$\mathcal{L}[\vec{x}'(t)](s) = s\vec{X}(s) - \vec{x}(0).$$

Let's see the example in this setting.

Example 7.19. For the last time, consider the system

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}.$$

If we take the Laplace transform of both sides of the equation, we find

$$s\vec{X}(s) - \vec{x}(0) = A\vec{X}(s) + \vec{G}(s).$$

Here, we take our initial condition to be $\vec{x}(0) = \vec{x}^{(0)} = \vec{0}$. We solve this equation for $\vec{X}(s)$ as before to find that

$$(sI - A)\vec{X}(s) = \vec{G}(s).$$

Our hope here is that the matrix $sI - A$ is invertible. If it exists, this matrix $(sI - A)^{-1}$ is called the *transfer matrix*.

We can compute

$$\vec{G}(s) = \begin{bmatrix} 2/(s+1) \\ 3/s^2 \end{bmatrix}$$

and

$$sI - A = \begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix}$$

so that

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}.$$

Then we compute the multiplication

$$\vec{X}(s) = (sI - A)^{-1}\vec{G}(s) = \begin{bmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{bmatrix}.$$

From here, all that is left to do is determine the inverse Laplace transform of $\vec{X}(s)$. This is accomplished in practice by appealing to computer software. Here, we use the table from before that find that

$$\vec{x}(t) = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{2}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + te^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

gives us the particular solution of the initial value problem.

Now that we have seen four (count 'em: FOUR!) different ways to solve nonhomogeneous systems, we should compare them against one another to determine which one works well in which situation. This can be summarized in the table below. In general, there is little advantage of choosing one method over another. Pick your poison.

Method	Pros	Cons
Diagonalization	Great for self-adjoint matrices (inverse can be written without computation)	(i) Compute an inverse matrix (ii) Solve several 1^{st}-order equations (iii) Matrix multiplication
Undetermined Coefficients	No integration	(i) Limited by which functions $\vec{g}(t)$ can be (ii) Solve several sets of algebraic equations
Variation of Parameters	Most general method	(i) Solution of general homogeneous system (ii) Integration (iii) Matrix multiplication (iv) Overkill for smaller systems
Laplace Transform	Useful in problems with forcing functions (especially with δ-functions)	(i) Compute an inverse matrix (ii) Matrix multiplication (iii) Compute inverse Laplace transforms