## 2. Derivation of the Heat Equation

In this section we construct a model for the flow of heat in a metal rod using the following simplifying assumptions:

(1) Geometry: The rod is a circular cylinder of length $L$ with cross-sectional area $A$.
(2) Material properties: The rod has uniform density $\rho$ and heat capacitance $c$. We will use mass heat capacitance which is the amount of heat needed to raise the temperature of one unit of mass by one degree.
(3) Insulation: The entire surface of the rod is perfectly insulated so that the total amount of heat is conserved.
(4) Initial State: The initial temperature of the rod is constant over each cross-section and is thus a function of one variable. As a consequence of Fourier's Law, the temperature will remain constant over cross-sections at all times, varying only along the axis of the rod.

The derivation is based on the principle of conservation of energy. The Heat Equation is thus a conservation law.

### 2.1. **Energy balance.**

Since the flow of heat is one-dimensional, we can identify the rod with the interval $[0, L]$ on the $x$-axis. Consider an *arbitrary* subinterval $[a, b]$ that is properly contained[1] in $[0, L]$; the word 'arbitrary' is emphasized because it is key to the argument. The total amount of thermal energy $Q$ contained in that subinterval is given by

$$Q = \int_a^b c\, u(x,t)\, \rho\, A\, dx,$$

where $u(x, t)$ denotes temperature at $x$ at time $t$. Not much can be said about $Q$ itself other than that it is a function of time. However, we can make constructive statements about its rate of change:

$$\frac{dQ}{dt} = \frac{d}{dt} \int_a^b c\, u(x,t)\, \rho\, A\, dx. \tag{2.1}$$

Recall that we assume the surface of the rod to be perfectly insulated. This implies that the energy content of $[a, b]$ can change only due to *flux* across the boundary, that is the cross-sections at $x = a$ and $x = b$. In the present context, the heat flux $\phi(x, t)$ is the rate at which thermal energy crosses the cross-section at $x$ at time $t$ going from left to right;

[1] A subset is called 'proper' if it is smaller than the whole set.

negative flux corresponds to energy flow from right to left. Using the flux $\phi$, we can express $dQ/dt$ as:

$$\frac{dQ}{dt} = -A\,\phi(b,t) + A\,\phi(a,t). \tag{2.2}$$

Combining Equations (2.1) and (2.2), and canceling the cross-sectional area $A$, leads to the following *conservation law*:

$$\frac{d}{dt}\int_a^b c\,u(x,t)\,\rho\,dx = -\phi(b,t) + \phi(a,t). \tag{2.3}$$

Equation (2.3) is an example of a *global* conservation law. This means that it applies to a finite (as opposed to infinitesimal) portion of the rod and involves integration. We will now use the Fundamental Theorem of Calculus to convert Equation (2.3) into an equivalent, but much more useful, *local* conservation law.

2.2. **Equation of Continuity.** In order to pass to a local conservation law, we manipulate Equation (2.3) as follows. On the left-hand side, interchange integration and differentiation[2]. Note that the full derivative becomes a partial derivative once brought inside the integral sign:

$$\frac{d}{dt}\int_a^b c\,u(x,t)\,\rho\,dx = \int_a^b c\,\rho\,\frac{\partial u}{\partial t}(x,t)\,dx.$$

Note also that we moved some constants around for neatness. As the next step, rewrite the right-hand side of Equation (2.3) as an integral, using the Fundamental Theorem of Calculus:

$$-\phi(b,t) + \phi(a,t) = -\int_a^b \frac{\partial \phi}{\partial x}(x,t)\,dx.$$

Combining both integrals on the left-hand side brings Equation (2.3) to the following form:

$$\int_a^b \left( c\,\rho\,\frac{\partial u}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) \right)\,dx = 0.$$

We now claim that since the limits of integration are *arbitrary*, the *integrand* must vanish:

$$c\,\rho\,\frac{\partial u}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) = 0. \tag{2.4}$$

Equation (2.4) is the sought local conservation law. It is called *Equation of Continuity.*

[2]Intuitively, this is possible because $x$ and $t$ are independent variables. There do exist pathological functions for which such interchanges are not legal but we do not need to be concerned about that.

2.3. **Fourier's Law and the Heat Equation.** Equation of Continuity (2.4) contains two unknown functions: temperature $u(x,t)$ and flux $\phi(x,t)$. In order to close the model, we need another equation relating these quantities. Such an equation is furnished by Fourier's Law of heat conduction. In words, the law states that:

> The heat flux is proportional to the gradient of the temperature and is directed against it.

In one dimension, the gradient is just a single partial derivative. So, in symbols, the one-dimensional Fourier's Law of heat conduction is

$$\phi = -k\,\frac{\partial u}{\partial x},$$

where $k$ is a positive constant called *thermal conductivity.* Combining Equation (2.4) with Fourier's Law, we get:

$$\frac{\partial u}{\partial t} = K\,\frac{\partial^2 u}{\partial x^2}, \quad \left(K = \frac{k}{c\,\rho}\right). \tag{2.5}$$

Equation (2.5) is called the *Heat Equation*; the aggregated constant $K$ is *thermal diffusivity.* Here we also would like to remind the reader that the Heat Equation is often called the Diffusion Equation.

2.4. **Initial-Boundary Value Problem.** To make the solution of Equation (2.5) unambiguous, we need to specify *initial* and *boundary* conditions [3].

Since Equation (2.5) is of first order in the time variable, we need one initial condition:

$$u(x,0) = f(x).$$

Now let us discuss the boundary conditions. These can vary for one and the same rod depending on the experimental setup. In our case, since the rod is completely insulated, there should be no flux of heat at the end points $x = 0, L$. This corresponds to *Neumann* boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

[3] Here we can draw a parallel with modeling the motion of a particle. It is not enough to have Newton's Second Law—one also needs the initial position and velocity.

Together, the Heat Equation, the initial condition, and the Neumann boundary conditions are called Neumann Initial Boundary Value Problem (IBVP):

$$\begin{aligned}
&\frac{\partial u}{\partial t} = K\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\
&u(x,0) = f(x),\\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0.
\end{aligned} \tag{2.6}$$

In Equation (2.6) the positive constants $K$ and $L$ are parameters of the model and are assumed to be given along with the initial temperature $f(x)$.

## Homework

(1) Derive the Heat Equation for the rod whose heat capacitance $c$ and density $\rho$ vary along the axis and are thus functions of the position $x$.

(2) *Attempt* to derive the Heat Equation in two dimensions. You may not be able to finish the derivation unless you have taken Vector Calculus (Math 152) or advanced Physics. Still, it is instructive to try. Assume that you have an insulated metal plate. Consider the energy content of an arbitrary region and try to follow the logic of the handout. If you run into a difficulty, write a clear sentence explaining what that difficulty is so that we could discuss it in class at some point.

(3) Let $x_1 = 0 < x_2 < \ldots < x_{n-1} < x_n = 1$ be an even subdivision of the unit interval $[0,1]$. Such a subdivision can be generated with the command `linspace(0,1,n)` in Matlab. For any such subdivision one can construct a nonzero function $f$ with the property:

$$\int_{x_i}^{x_j} f(x)\,dx = 0, \quad \text{for all choices of } i \text{ and } j.$$

This is trivial when $n = 2$ and easy when $n = 3$. Do it for $n = 4$. Provide both the equation $y = f(x)$ and its plot as your solution. Extra points if you produce a scheme for constructing such functions for arbitrary $n$. In which case, what happens as $n \to \infty$?