# Problem Set 8 Solutions

**Problem 1.** *Two-dimensional analogs of one-dimensional bifurcations*
Recall the three elementary bifurcations in one dimension:

- Saddle-node bifurcation:
$$\dot{x} = r + x^2$$
- Transcritical bifurcation:
$$\dot{x} = rx - x^2$$
- (Supercritical) Pitchfork bifurcation:
$$\dot{x} = rx - x^3$$

We can experiment with these bifurcations in two dimensions by replacing the '$\dot{x}$' terms with "$\ddot{x}$" instead. That is, for each $\dot{x} = f(x; r)$ above, we can set $\ddot{x} = f(x; r)$ instead. For example, we can write extend the saddle-node bifurcation as

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= r + x^2 \end{cases}$$

For each of the bifurcation types above:

(a) Find all fixed points $\vec{\mathbf{x}}^*$ of the two-dimensional system as functions of $r$. Count how many fixed points there are for each case of $r < 0$, $r = 0$, and $r > 0$.

(b) Construct the Jacobian matrix $J(x, y)$ and compute $J(\vec{\mathbf{x}}^*)$ for each fixed point found above.

(c) Classify the equilibrium type of each fixed point in each case of $r < 0$, $r = 0$, and $r > 0$.

(d) Plot a phase portrait for the system in each case of $r < 0$, $r = 0$, and $r > 0$.

(e) Describe what happens to the fixed points of the system and how their stabilities change as we increase $r$ from negative to positive.

## 0.1 Saddle-node bifurcation

We study the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = r + x^2. \end{cases}$$

(a) To find the fixed points of this system, we set $\dot{x} = 0$ and $\dot{y} = 0$ so that

$$\begin{cases} y &= 0 \\ r + x^2 &= 0. \end{cases}$$

Then we find the fixed points $\vec{\mathbf{x}}^* = (\pm\sqrt{-r}, 0)$.

When $r < 0$, we have two fixed points (since $-r$ is positive); when $r = 0$, the two points coalesce into one fixed point at the origin; and when $r > 0$, there are no fixed points (since $\pm\sqrt{-r}$ is now complex).

(b) We compute the Jacobian matrix as

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ 2x & 0 \end{bmatrix} \quad \text{so that} \quad J(\pm\sqrt{-r}, 0) = \begin{bmatrix} 0 & 1 \\ \pm 2\sqrt{-r} & 0 \end{bmatrix}.$$

(c) When $r < 0$, there are two fixed points at $\vec{\mathbf{x}}^* = (\pm\sqrt{-r}, 0)$ with Jacobian matrices

$$J(\pm\sqrt{-r}, 0) = \begin{bmatrix} 0 & 1 \\ \pm 2\sqrt{-r} & 0 \end{bmatrix}.$$

The matrix $J(\sqrt{-r}, 0)$ has trace $\tau = 0$ and determinant $\Delta = -\sqrt{-r} < 0$. Because $\Delta < 0$, we see that this fixed point is a saddle.

The matrix $J(-\sqrt{-r}, 0)$ has trace $\tau = 0$ and determinant $\Delta = \sqrt{-r}$. We see that $\tau^2 - 4\Delta < 0$ so that the eigenvalues of $J(-\sqrt{-r}, 0)$ are pure imaginary numbers (real part zero). Therefore, we classify this fixed point as a center.

When $r = 0$, there is one fixed point at the origin. The Jacobian $J(0, 0)$ has determinant $\Delta = 0$. This shows that there is at least one eigenvalue equal to zero. Therefore, we cannot conclude anything about the nature of this fixed point. Hence, we classify it as degenerate.

When $r > 0$, there are no fixed points.

(d) Figure 1 shows some representative phase portraits for this system when $r = -1, 0, 1$.

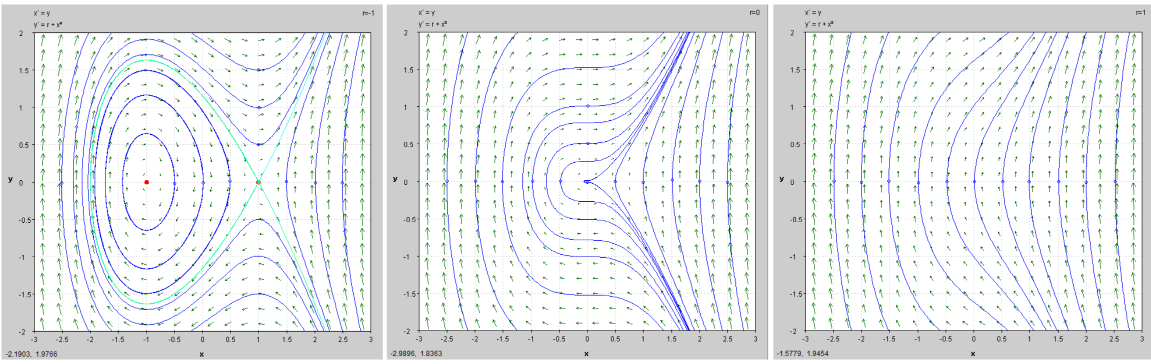

Figure 1: Saddle-node bifurcation

(e) The two fixed points that exist for $r < 0$ merge together at the origin when $r = 0$ and then cancel each other away so that there are no fixed points for $r > 0$. The bounded trajectories found for $r < 0$ disappear when $r \geq 0$, so there is a complete loss of stability in this type of bifurcation.

## 0.2 Transcritical bifurcation

We study the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = rx - x^2. \end{cases}$$

(a) To find the fixed points of this system, we set $\dot{x} = 0$ and $\dot{y} = 0$ so that

$$\begin{cases} y & = 0 \\ x(r - x) & = 0. \end{cases}$$

Then we find the fixed points $\vec{\mathbf{x}}^* = (0, 0), (r, 0)$.

When $r < 0$, we have two fixed points - one in the left half-plane; when $r = 0$, the two points coalesce into one fixed point at the origin; and when $r > 0$, there are again two fixed points - one in the right half-plane.

(b) We compute the Jacobian matrix as

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ r - 2x & 0 \end{bmatrix}$$

so that we have

$$J(r, 0) = \begin{bmatrix} 0 & 1 \\ -r & 0 \end{bmatrix} \quad \text{and} \quad J(0, 0) = \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix}.$$

(c) When $r < 0$, there are two fixed points at $\vec{\mathbf{x}}^* = (r, 0), (0, 0)$ with Jacobian matrices

$$J(r, 0) = \begin{bmatrix} 0 & 1 \\ -r & 0 \end{bmatrix} \quad \text{and} \quad J(0, 0) = \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix}.$$

The matrix $J(r, 0)$ has trace $\tau = 0$ and determinant $\Delta = r < 0$. Because $\Delta < 0$, we see that this fixed point is a saddle.

The matrix $J(0, 0)$ has trace $\tau = 0$ and determinant $\Delta = -r > 0$. We see that $\tau^2 - 4\Delta < 0$ so that the eigenvalues of $J(0, 0)$ are pure imaginary numbers (real part zero). Therefore, we classify this fixed point as a center.

When $r = 0$, there is one fixed point at the origin. The Jacobian $J(0, 0)$ has determinant $\Delta = 0$. This shows that there is at least one eigenvalue equal to zero. Therefore, we cannot conclude anything about the nature of this fixed point. Hence, we classify it as degenerate.

When $r > 0$, there are two fixed points at $\vec{\mathbf{x}}^* = (r, 0), (0, 0)$ with Jacobian matrices

$$J(r, 0) = \begin{bmatrix} 0 & 1 \\ -r & 0 \end{bmatrix} \quad \text{and} \quad J(0, 0) = \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix}.$$

The matrix $J(r, 0)$ has trace $\tau = 0$ and determinant $\Delta = r > 0$. We see that $\tau^2 - 4\Delta < 0$ so that the eigenvalues of $J(r, 0)$ are pure imaginary numbers (real part zero). Therefore, we classify this fixed point as a center.

The matrix $J(0, 0)$ has trace $\tau = 0$ and determinant $\Delta = -r < 0$. Because $\Delta < 0$, we see that this fixed point is a saddle.

(d) Figure 2 shows some representative phase portraits for this system when $r = -1, 0, 1$.

(e) The fixed point at $(r, 0)$ is saddle for $r < 0$, and the fixed point at $(0, 0)$ is a center. When we increase $r$ through zero, we notice that the two fixed points meet at the origin; and then it appears that the saddle point "kicks away" the center at the origin to its new location at $(r, 0)$. In effect, these two fixed points "bounce" off of one another like billiards balls. The origin is always a fixed point in this system, so it would also be valid to say that the origin has transferred its stability to the $(r, 0)$ fixed point.

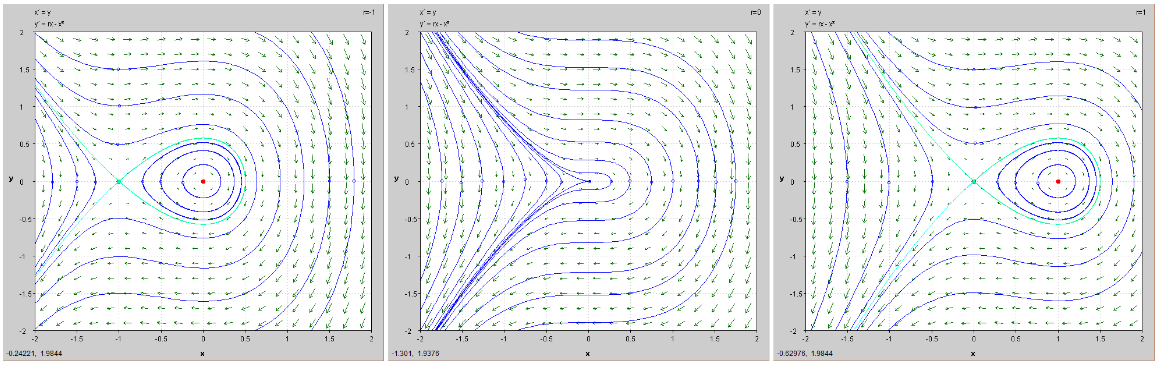

Figure 2: Transcritical bifurcation

## 0.3 Pitchfork bifurcation

We study the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = rx - x^3. \end{cases}$$

(a) To find the fixed points of this system, we set $\dot{x} = 0$ and $\dot{y} = 0$ so that

$$\begin{cases} y & = 0 \\ x(r - x^2) & = 0. \end{cases}$$

Then we find the fixed points $\vec{\mathbf{x}}^* = (0,0), (\pm\sqrt{r}, 0)$.

When $r < 0$, we have exactly one fixed point at the origin; when $r = 0$, we still have that same fixed point; but when $r > 0$, the fixed point at the origin persists and there are now two additional fixed points to either side of it.

(b) We compute the Jacobian matrix as

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ r - 3x^2 & 0 \end{bmatrix}$$

so that we have

$$J(\pm\sqrt{r}, 0) = \begin{bmatrix} 0 & 1 \\ -2r & 0 \end{bmatrix} \quad \text{and} \quad J(0,0) = \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix}.$$

(c) When $r \leq 0$, there is exactly one fixed point at $\vec{\mathbf{x}}^* = (0,0)$ with Jacobian matrix

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix}.$$

The matrix $J(0,0)$ has trace $\tau = 0$ and determinant $\Delta = -r \leq 0$. When $r < 0$, $\tau^2 - 4\Delta = r < 0$ so that $(0,0)$ is a center; but $(0,0)$ is degenerate for $r = 0$ since $\Delta = 0$ implies the existence of at least one zero-eigenvalue of $J(0,0)$.

When $r > 0$, there are two additional fixed points at $\vec{\mathbf{x}}^* = (\pm\sqrt{r}, 0)$ with Jacobian matrices

$$J(\pm\sqrt{r}, 0) = \begin{bmatrix} 0 & 1 \\ -2r & 0 \end{bmatrix}$$

The matrices $J(\pm\sqrt{r}, 0)$ have trace $\tau = 0$ and determinant $\Delta = 2r > 0$. We see that $\tau^2 - 4\Delta < 0$ so that the eigenvalues of $J(\pm\sqrt{r}, 0)$ are pure imaginary numbers (real part zero). Therefore, we classify these fixed points as centers.

The matrix $J(0, 0)$ has trace $\tau = 0$ and determinant $\Delta = -r < 0$. Because $\Delta < 0$, we see that this fixed point is a saddle.

(d) Figure 3 shows some representative phase portraits for this system when $r = -1, 0, 1$.

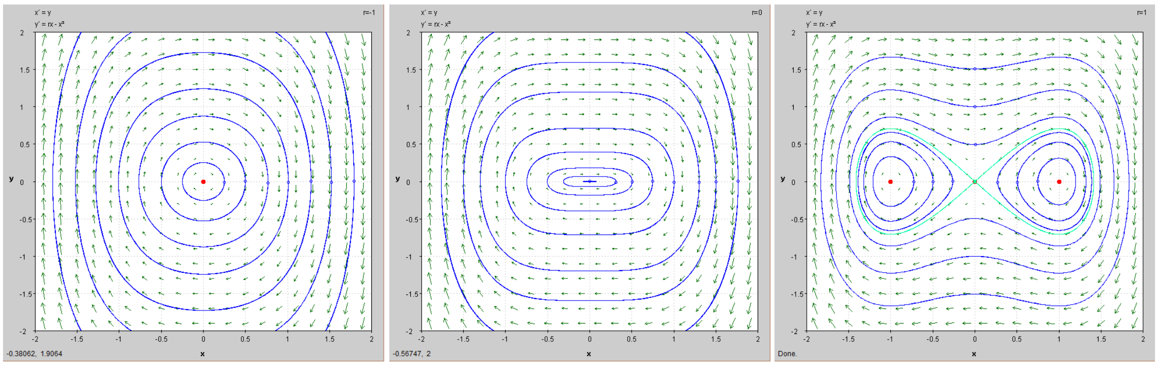

Figure 3: Pitchfork bifurcation

(e) For $r \leq 0$ the fixed point at the origin is a center. As $r$ is increased, this center begins to pinch flatter and flatter in the vertical direction. Once $r > 0$, this pinch creates a lemniscate that separates the plane into three regions. The closed orbits from $r \leq 0$ persist outside of this lemniscate, but they are now split into pairs of closed orbits within the bounded regions. That is, the center at the origin as now become a saddle, and there are now two bounded regions containing one center each surrounded by closed trajectories inside the lemniscate. Of particular note is the fact that the origin has changed its stability from center to saddle.