Thursday, January 31, 2019 9:05 PM

PS 2 Solutions

① (a) Note that method 1 yields more fish for higher populations.

Scenario 1: Business is booming and we want to increase profits. The fish have done well enough to sustain a population of thousands of fish. Method 2 is safe but only brings in so much money. Method 1 would allow us to sell many more fish than Method 2.

Scenario 2: It was a particularly rough winter, and we lost a lot of fish to the icy weather. Method 1 will not help us sell fish to keep up with demand because our population is so small. Method 2 would ensure we meet our quotas.

(b) Each equation has a logistic growth term in it, and the fish will grow until they find their carrying capacity if we don't harvest them.

The first equation has a $-EN$ term to indicate that some amount of fish proportional to the population size is being harvested.

The second equation has a constant $-h$ term instead to indicate a constant "death" rate.

(c) Set $\frac{dN}{dt} = rN(1-\frac{N}{K}) - EN = 0$

$$-\frac{r}{K}N^2 + (r-E)N = 0$$

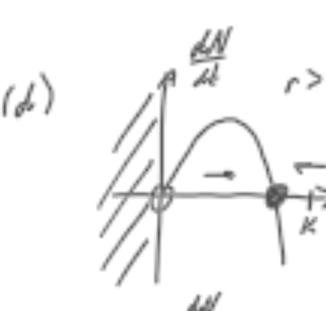
$$N\left(-\frac{r}{K}N + (r-E)\right) = 0$$

$$\Rightarrow N=0 \text{ or } -\frac{r}{K}N + (r-E) = 0$$

$$N - \frac{(r-E)K}{r} = 0$$

$$\boxed{N = \frac{K(r-E)}{r}}$$

(d)



The fish stabilize at $N = \frac{K(r-E)}{r}$ members. This is the new "carrying capacity."

The fish are doomed to extinction.

(e) Set $\frac{dN}{dt} = rN(1-\frac{N}{K}) - h = 0$

$$-\frac{r}{K}N^2 + rN - h = 0$$

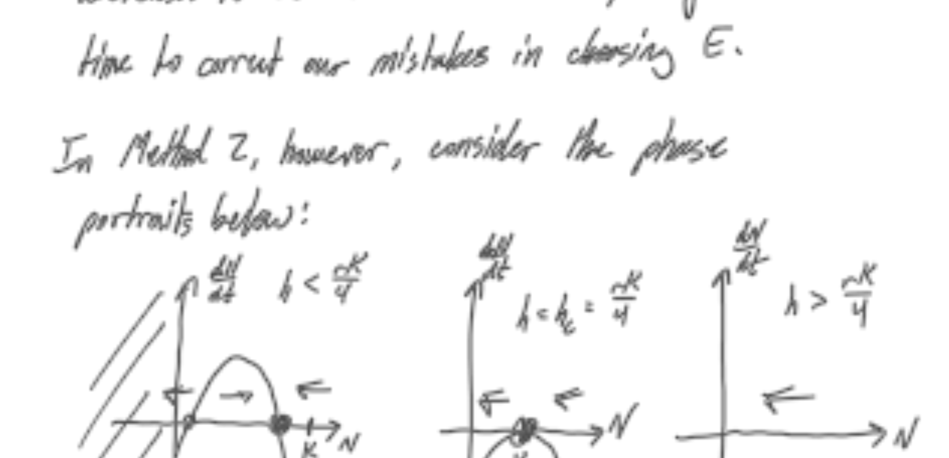
$$\boxed{N = \frac{-r \pm \sqrt{r^2 - \frac{4rh}{K}}}{-2r/K}}$$

(f) Quadratic polynomials only ever have 0 real solutions, 1 repeated real solution, or 2 distinct real solutions.

The value above will give real solutions only if $r^2 - \frac{4rh}{K} \geq 0$. This simplifies to $h \leq \frac{rK}{4}$. Set $\boxed{h_c = \frac{rK}{4}}$

(g) In Method 1, we can monitor our mistakes since the non-trivial equilibrium N^* gradually decreases to zero. There would be plenty of time to correct our mistakes in choosing E.

In Method 2, however, consider the phase portraits below:



If we are close to the $h = h_c$ case and accidentally put $h > \frac{rK}{4}$, the stable equilibrium vanishes! Instead of a gradual change, we have an instantaneous loss of stability. In this sense, Method 2 is much more dangerous to use. Method 1 is considered "safer".

(h) The maximum yield is proportional to the harvest rates EN & h. The largest we can make these are when $E=r$ & $h = \frac{rK}{4}$; but these are both hard to maintain since this puts the fish near extinction in Method 1 and near the "scary" value h_c in Method 2.

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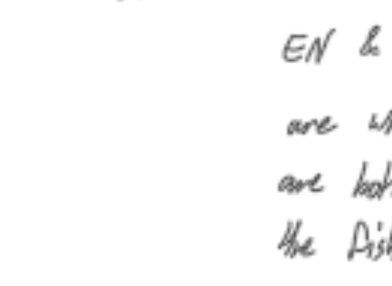
② (a)



$$\frac{dw}{dt} = 30\frac{\text{gal}}{\text{min}} \cdot 0.2\frac{\text{lbs}}{\text{gal}} - \frac{w \text{ lbs}}{V(t) \text{ gal}} \cdot 30\frac{\text{gal}}{\text{min}}$$

$$\boxed{\begin{cases} \frac{dw}{dt} = 6 - \frac{1}{2}w \\ w(0) = 0 \end{cases}}$$

Because $\begin{cases} \frac{dV}{dt} = 30\frac{\text{gal}}{\text{min}} - 30\frac{\text{gal}}{\text{min}} = 0 \\ V(0) = 60 \text{ gal} \end{cases}$

$$\Rightarrow V(t) = 60, \ t \geq 0$$

(b) Observe that $\frac{dw}{dt} + \frac{1}{2}w = 6$ is in standard form. Then $p(t) = \frac{1}{2}$ and $g(t) = 6$ so the LINEAR existence & uniqueness theorem states that there is a unique solution to our IVP that exists for all time t.

(c) $\frac{dw}{dt} + \frac{1}{2}w = 6 \Rightarrow \mu(t) = e^{\int \frac{1}{2}dt} = e^{t/2}$

So $e^{t/2}\left(\frac{dw}{dt} + \frac{1}{2}w\right) = 6e^{t/2}$

$$\left(e^{t/2}w\right)' =$$

$$e^{t/2}w = \int 6e^{t/2}dt$$

$$= 12e^{t/2} + C$$

$$w(t) = 12 + Ce^{-t/2}$$

Since $w(0) = 0$, we have $0 = 12 + C$ so that

$$\boxed{w(t) = 12 - 12e^{-t/2}.}$$

Although this solution exists for all t, it only matters to us when $t \geq 0$.

(d) Note that $w_\infty = \lim_{t\to\infty} w(t) = \lim_{t\to\infty}(12 - 12e^{-t/2}) = \boxed{12 \text{ lbs}}$.

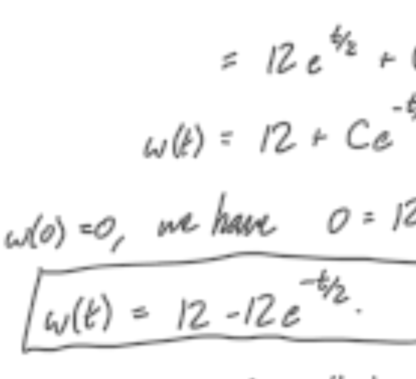
(e) The differential equation is autonomous, so set

$$\frac{dw}{dt} = 6 - \frac{1}{2}w = 0$$

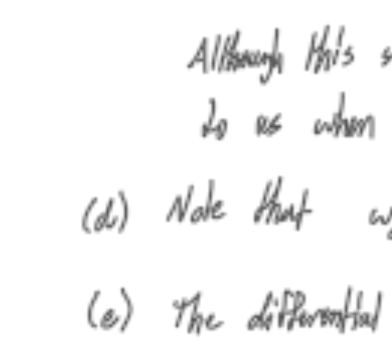
$$6 = \frac{1}{2}w$$

$$\boxed{w = 12 \text{ lbs}}$$

Consider that this is a stable equilibrium by viewing its phase portrait:



(f)



The first change occurs in $V(t)$:

$$\begin{cases} \frac{dV}{dt} = 35\frac{\text{gal}}{\text{min}} - 30\frac{\text{gal}}{\text{min}} \\ V(0) = 60 \text{ gal} \end{cases}$$

So $V(t) = 5t + C$.

$V(0) = 60 = C$.

Then $V(t) = 5t + 60$

So $\frac{dw}{dt} = 35\frac{\text{gal}}{\text{min}} \cdot 0.2\frac{\text{lbs}}{\text{gal}} - \frac{w(t) \text{ lbs}}{V(t) \text{ gal}} \cdot 30\frac{\text{gal}}{\text{min}}$

$$= 7\frac{\text{lbs}}{\text{gal}} - \frac{30}{5t+60}w\frac{\text{lbs}}{\text{gal}}.$$

This makes the IVP

$$\boxed{\begin{cases} \frac{dw}{dt} = 7 - \frac{30}{5t+60}w \\ w(0) = 0 \end{cases}}$$

Then $\frac{dw}{dt} + \frac{30}{5t+60}w = 7$ is in standard form. Set $p(t) = \frac{30}{5t+60}$ and $g(t) = 7$ so that the linear existence & uniqueness theorem tells us that there is still a unique solution to the IVP, but it only exists for $t > -12$ (since $p(t)$ is discontinuous at $t = -12$).

The model is solved with $\mu(t) = e^{\int \frac{30}{5t+60}dt}$

$$= e^{6\ln|5t+60|}$$

$$= e^{\ln(5t+60)^6}$$

$$= (5t+60)^6.$$

We compute

$$(5t+60)^6\left(\frac{dw}{dt} + \frac{30}{5t+60}w\right) = 7(5t+60)^6$$

$$\left((5t+60)^6 w\right)' =$$

$$(5t+60)^6 w = \frac{1}{5}(5t+60)^7 + C$$

$$w(t) = \frac{1}{5}(5t+60) + \frac{C}{(5t+60)^6}$$

The initial condition gives $w(0) = 12 + \frac{C}{60^6} = 0$

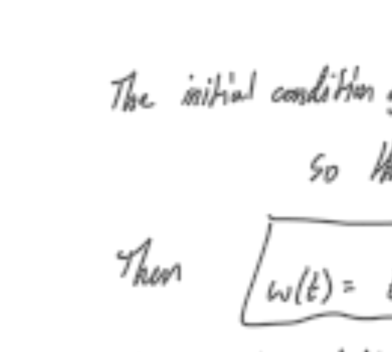
so that $C = -\frac{60^6}{12}$

Then $\boxed{w(t) = t + 12 - \frac{1}{12}\left(\frac{60}{5t+60}\right)^6.}$

Again, this solution exists for all $t > -12$, but we will only use it for $t \geq 0$.

(g) Remember that the tank is only able hold 100 gallons of water. The volume $V(t) = 5t + 60$ will exceed this capacity at time $t = 8$ minutes. Then $T = 8$ so that the function $w(t)$ in (f) is valid only for $\boxed{0 \leq t \leq 8.}$

After time $t = 8$, the model looks like:



Then $\begin{cases} \frac{dV}{dt} = 35 - (5+30) = 0 \\ V(0) = 100 \end{cases}$

so that $V(t) = 100$ gal.

Therefore, $\frac{dw}{dt} = 35\frac{\text{gal}}{\text{min}} \cdot 0.2\frac{\text{lbs}}{\text{gal}} - \frac{w \text{ lbs}}{100 \text{ gal}} \cdot 35\frac{\text{gal}}{\text{min}}$

$$= 7 - \frac{7}{20}w.$$

The initial condition NOW is where the previous function failed at $t = 8$. This was $w(8) = 20 - \frac{1}{12}\left(\frac{60}{100}\right)^6$

$$= 20 - \frac{1}{12}\left(\frac{3}{5}\right)^6 \text{ lbs}$$

$$(\approx 19.996 \text{ lbs})$$

Therefore, the new model should be

$$\boxed{\begin{cases} \frac{dw}{dt} = 7 - \frac{7}{20}w \\ w(8) = 20 - \frac{1}{12}\left(\frac{3}{5}\right)^6 \end{cases}}$$

(h) The equilibrium occurs for $\frac{dw}{dt} = 7 - \frac{7}{20}w = 0$

$$7 = \frac{7}{20}w$$

$$\boxed{20 = w}$$

This is stable for the same reason as before: