Consider a cart of mass m which is constrained to move along the x-axis. Imagine that the y-axis is a wall to which the cart is attached with a spring of stiffness k, as illustrated in Figure 8.

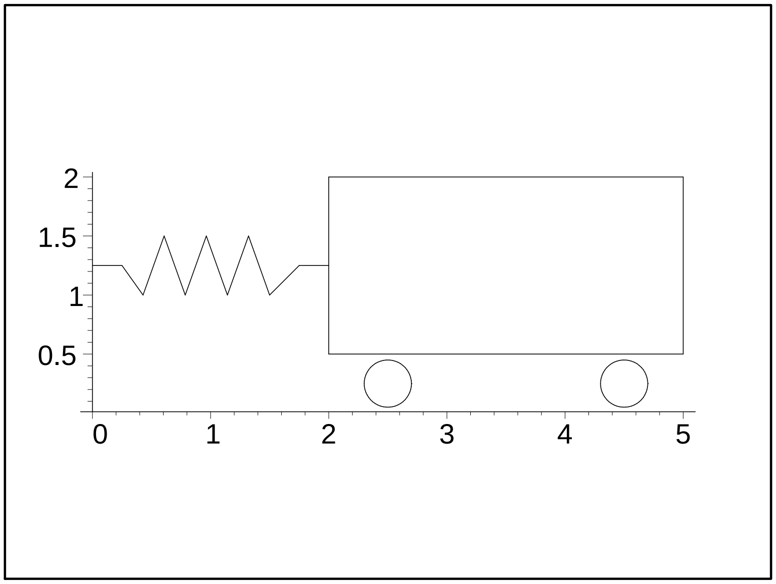

Figure 14. Mass-spring system

Let l be the natural length of the spring. If the cart is gently placed l units away from the wall with no initial velocity, it will remain motionless. However, if the cart is placed at any other location and released, the spring will either pull it to the wall or push it away from the wall thereby causing it to move. The motion can also be started from equilibrium by supplying the cart with initial velocity v_0. In fact, let us assume that that's the case as it will simplify calculations. The main question of this section is how to describe the motion of the cart. As we shall see, the answer involves a second order linear homogeneous ODE with constant coefficients.

Newton's second law. Instead of the actual position of the cart it is better to work with its displacement from equilibrium. Call the displacement $x(t)$. According to Newton's second law:

$$m\,\frac{d^2x}{dt^2} = F_1 + F_2,$$

where F_1 and F_2 denote the spring force and the frictional force, respectively. To proceed further, we require explicit formulas for F_1 and F_2 which we now derive.

Let us start with the spring force. According to Hooke's law, the force in a linear spring is proportional to displacement: therefore,

$$F_1 = -k\,x.$$

Notice the minus sign: if displacement is positive the spring is stretched and F_1 points in the negative direction, towards the wall; if displacement is negative, the spring is compressed and F_1 is directed away from the wall, in the positive direction.

Unlike the spring force, which depends on displacement, the force of resistance F_2 depends on the cart's velocity. For simplicity, we stipulate a linear relation, otherwise the ODE will be nonlinear and impossible to solve analytically:

$$F_2 = -r\,\frac{dx}{dt}.$$

Here r is a positive coefficient of friction measured in the units of "force divided by velocity" which is to say $\mathrm{kg} \cdot \mathrm{sec}^{-1}$. The minus sign in F_2 appears because frictional force opposes motion. Indeed, if $\frac{dx}{dt} > 0$, the cart is moving to the right and F_2 is pointing to the left, in the negative direction; similarly, if $\frac{dx}{dt} < 0$, the cart is moving to the left and F_2 is pointing to the right, in the positive direction.

We are now ready to form the IVP for the motion of the cart. Recall that we start the motion from the position of equilibrium $x(0) = 0$ with initial velocity $\frac{dx}{dt}(0) = v_0$. As our discussion of Newton's second law shows, the displacement $x(t)$ must satisfy:

$$m\,\frac{d^2x}{dt^2} = -k\,x - r\,\frac{dx}{dt}, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = v_0. \tag{30}$$

Equation (30) is of *second* order and therefore has *two* initial conditions. To see that it is linear, introduce

$$L(x) = m\,\frac{d^2x}{dt^2} + r\,\frac{dx}{dt} + k\,x,$$

so the ODE reads $L(x) = 0$. It is easy to check that $L(x_1 + x_2) = L(x_1) + L(x_2)$ and $L(ax) = a\,L(x)$. Therefore L is a *linear* operation and so is the ODE. Notice that after we collected all terms involving x on the left, the right-hand side became zero: the ODE is *homogeneous*. Last but not least, the coefficients m, r, and k on the left-hand side are positive constants. Therefore the full description of Equation (30) is "second order, linear, homogeneous, with constant coefficients."

In order to see how the attributes of IVP (30) can be constructively used, let us examine a few simple cases.

Undamped motion. As a highly idealized case, consider the motion without resistance. If we set $r = 0$ in Equation (30), we obtain:

$$m\,\frac{d^2x}{dt^2} = -k\,x, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = v_0.$$

Dividing both sides by the mass m leads to:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}\,x. \tag{31}$$

It is easy to guess that the solution of Equation (31) should be a simple harmonic. Indeed, if $k = m$ then Equation (31) reads $\dfrac{d^2x}{dt^2} = -x$ and is solved by $\sin(t)$, $\cos(t)$, and any linear combination thereof. Inspired by this observation, let us look for the solution of (31) in the form

$$x = A\,\cos(\omega_0 t) + B\,\sin(\omega_0 t),$$

where ω_0 is to be determined. Substituting our harmonic guess into (31), we get

$$\frac{d^2x}{dt^2} = -\omega_0^2\,(A\,\cos(\omega_0 t) + B\,\sin(\omega_0 t)) = -\omega_0^2\,x = -\frac{k}{m}\,x,$$

which shows that

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

We conclude that

> In the absence of resistance the cart oscillates harmonically with *natural frequency* ω_0.

In particular, if the cart starts from equilibrium with velocity v_0 then

$$x(t) = v_0\,\frac{\sin(\omega_0 t)}{\omega_0}, \tag{32}$$

whose graph is shown in Figure 15 below.

Figure 15 conforms with our intuition: a mass on a spring is supposed to oscillate. Of course, if $r > 0$ the oscillations will eventually die out. To see at which rate the oscillations are dampened, let us consider another extreme case.

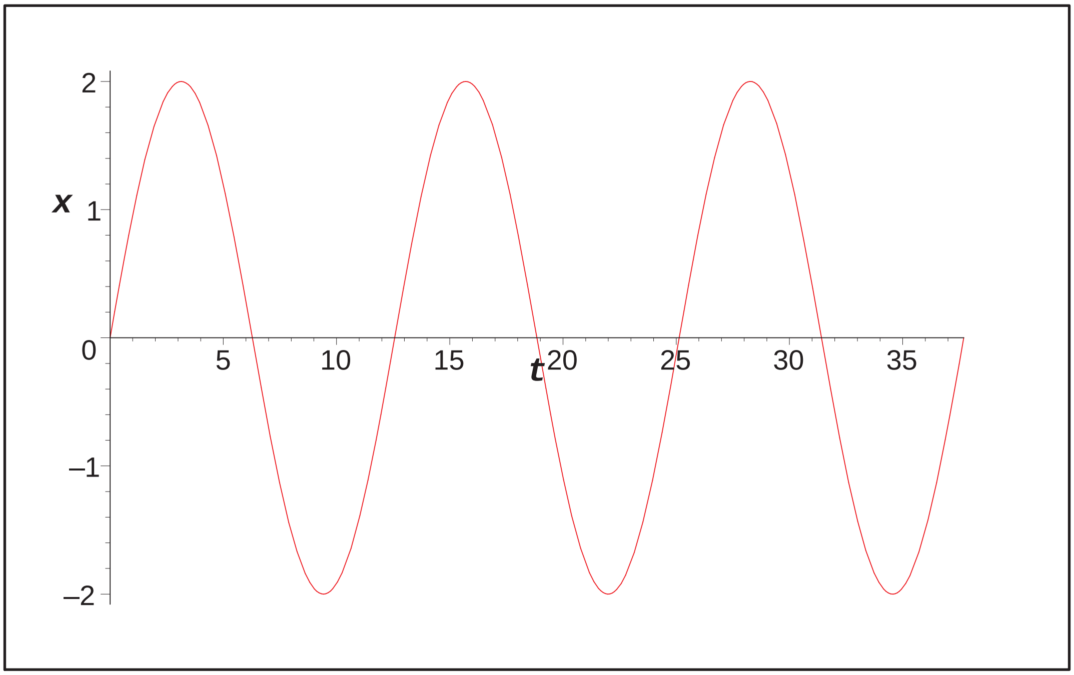

FIGURE 15. Solution for the frictionless case

Extremely damped motion. Suppose that the spring is very weak and the motion of the cart is dominated by friction.

Setting $k = 0$ in Equation (30), we arrive at:

$$m\,\frac{d^2x}{dt^2} = -r\,\frac{dx}{dt}, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = v_0. \tag{33}$$

The special form of Equation (33) suggests that we substitute

$$\frac{dx}{dt} = v.$$

The IVP for the cart's velocity is

$$m\,\frac{dv}{dt} = -r\,v, \quad v(0) = v_0,$$

with the familiar exponential solution:

$$v = v_0\,e^{-\frac{r}{m}\,t}.$$

Integrating velocity, and using the initial condition $x(0) = 0$ leads to

$$x(t) = \frac{mv_0}{r}\,\left(1 - e^{-\frac{r}{m}\,t}\right). \tag{34}$$

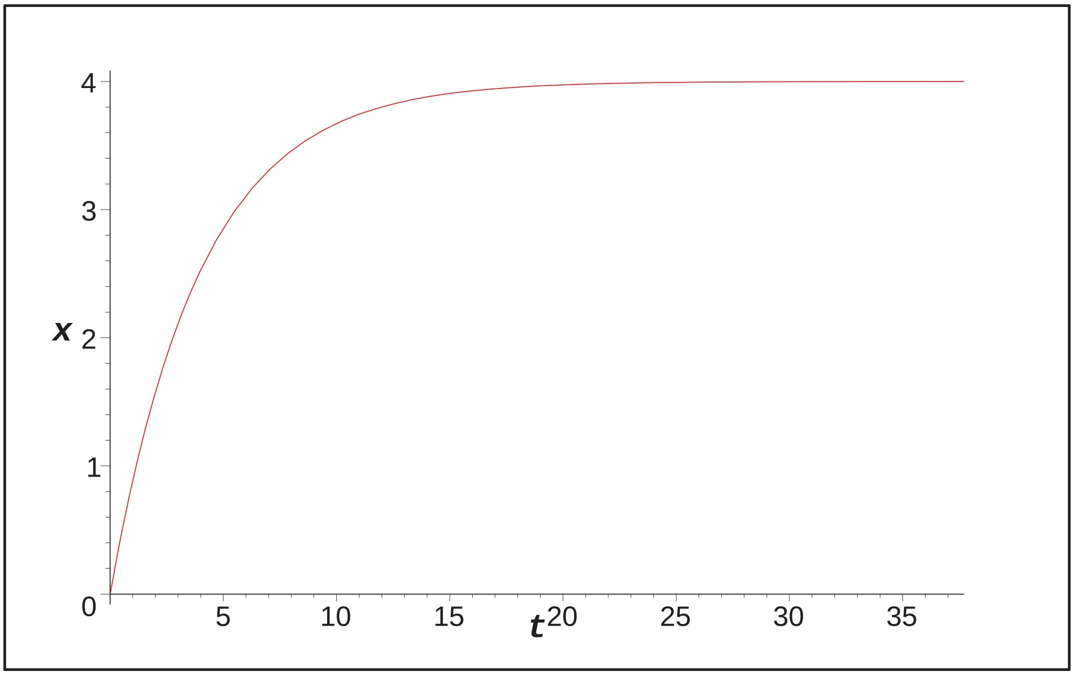

FIGURE 16. Solution for the springless case

As Figure 16 shows, the cart is moving to the right tending asymptotically to

$$x_\infty = \lim_{t\to\infty} x(t) = \frac{m v_0}{r}.$$

Obviously, since there is no spring, the cart never reverses direction. More importantly,

> the cart's velocity is a decaying exponential as is its total energy.

Underdamped motion. We now know how the cart moves in two extreme cases:

No friction: The motion is harmonic with natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$.

Lots of friction: The cart decelerates exponentially fast.

It stands to reason that in the intermediate case the motion may be governed by an exponentially decaying harmonic:

$$x = A\, e^{a\, t} \sin(b\, t), \quad a < 0, \quad b > 0. \tag{35}$$

A typical graph of the function (35) is shown in Figure 17.

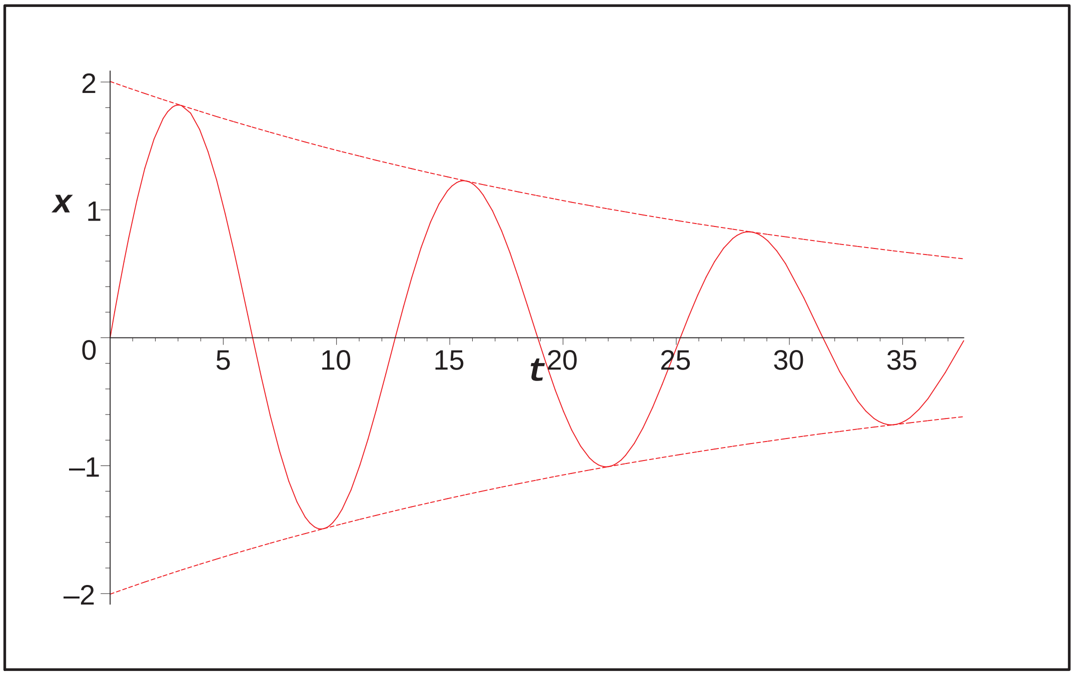

FIGURE 17. Underdamped motion

The dashed lines in Figure 17 are exponentials $\pm A\, e^{at}$; these define what is called an *envelope* of the harmonic.

In order to find a and b exactly, substitute (35) into Equation (30) and combine the like terms:

$$e^{a\,t}\left\{(r\,b + 2\,m\,a\,b)\cos(b\,t) + \left(m\,a^2 + k - m\,b^2 + r\,a\right)\sin(b\,t)\right\} = 0.$$

The exponential term cannot be zero. Hence we must choose a and b so that the coefficients in front of the sine and cosine are zero:

$$\begin{aligned} r\,b + 2\,m\,a\,b &= 0 \\ m\,a^2 + k - m\,b^2 + r\,a &= 0 \end{aligned}$$

Since b cannot be zero, we must have:

$$a = -\frac{r}{2m}.$$

Consequently,

$$\begin{aligned} b &= \sqrt{a^2 + \frac{r}{m}\,a + \frac{k}{m}} = \sqrt{\frac{r^2}{4m^2} - \frac{r^2}{2m^2} + \frac{k}{m}} \\ &= \frac{\sqrt{4k\,m - r^2}}{2\,m} \end{aligned}$$

and the solution of Equation (30) for $k > 0$ and *small* $r > 0$ is given by:

$$x = v_0\, e^{-\frac{r}{2m}\, t}\, \frac{\sin(\omega t)}{\omega}, \quad \omega = \frac{\sqrt{4k\, m - r^2}}{2\, m}. \tag{36}$$

Clearly, ω makes physical sense only if $r^2 < 4km$: the coefficient of friction is relatively small and the motion is *underdamped.* Notice that the frequency ω of damped oscillations is smaller than the natural frequency ω_0:

$$\omega = \frac{\sqrt{4k\, m - r^2}}{2\, m} < \sqrt{\frac{k}{m}} = \omega_0.$$

In fact, if r is increased the frequency of the oscillations tends to zero. That means that the period goes to infinity: highly damped motion is not oscillatory.

Critically damped motion. What happens to Equation (36) when $r^2 = 4km$? The frequency ω approaches zero. By L'Hospital's Rule:

$$\lim_{\omega \to 0} \frac{\sin(\omega t)}{\omega} = t.$$

Therefore, the solution of (30) must approach:

$$x = v_0\, e^{-\frac{r}{2m}\, t}\, t. \tag{37}$$

This is critically damped motion with the graph shown in Figure 18.

Overdamped motion. It remains to investigate the case where $r^2 > 4\, k\, m$ but the damping is not so extreme that we can disregard the presence of the spring as we did earlier. We know that in the extreme case the solution is given by an exponentially decreasing function. Let us therefore look for exponential solutions. To this end, we substitute

$$x(t) = e^{\lambda\, t}$$

in Equation (30). This leads to

$$\left(m\, \lambda^2 + r\, \lambda + k\right)\, e^{\lambda t} = 0.$$

Since exponentials are never zero we must choose λ so that

$$m\, \lambda^2 + r\, \lambda + k = 0. \tag{38}$$

Equation (38) is called the *characteristic equation* of IVP (30). Solving the quadratic equation we get two *characteristic roots*:

$$\lambda_1 = \frac{-r + \sqrt{r^2 - 4\, k\, m}}{2\, m}, \quad \lambda_2 = \frac{-r - \sqrt{r^2 - 4\, k\, m}}{2\, m}.$$

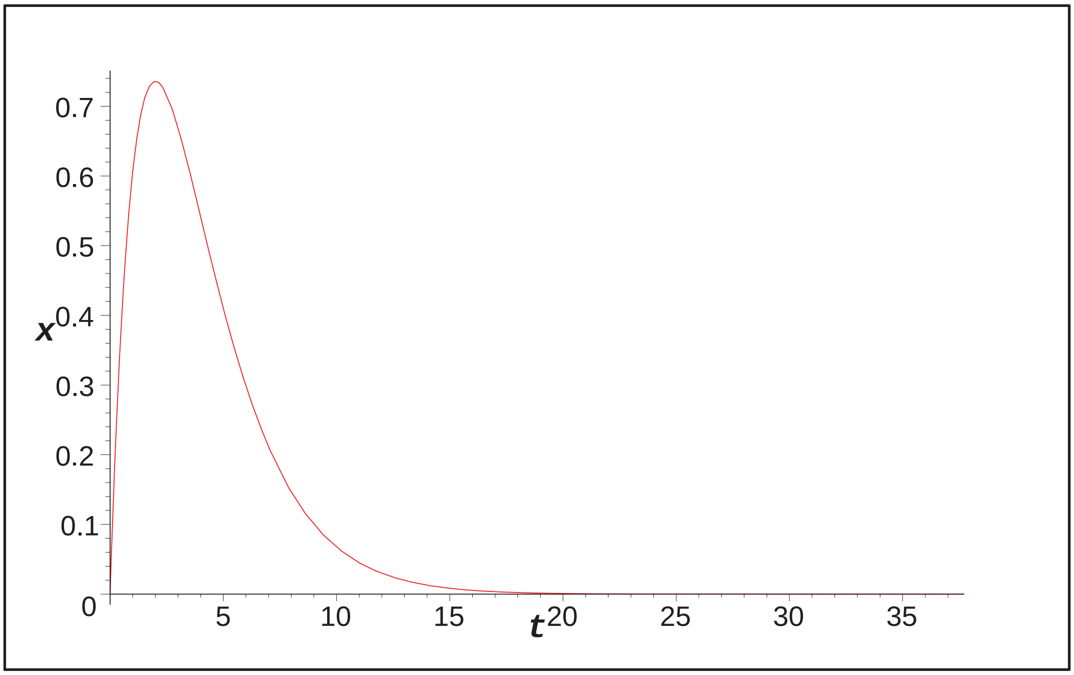

FIGURE 18. Critically damped motion

This time $r^2 > 4\,k\,m$ so both λ's are real-valued. Evidently, Equation (30) has two exponential solutions:

$$x_1 = e^{\lambda_1\,t}, x_2 = e^{\lambda_2\,t}.$$

By linearity, the general solution of Equation (30) is given by the linear combination:

$$x = C_1\,x_1 + C_2\,x_2 = C_1\,e^{\lambda_1\,t} + C_2\,e^{\lambda_2\,t}.$$

Therefore, the solution of the IVP for the overdamped case is given by the difference of two exponentials:

$$x = v_0\,\frac{e^{\lambda_1\,t} - e^{\lambda_2\,t}}{\lambda_1 - \lambda_2}, \quad \lambda_{1,2} = \frac{-r \pm \sqrt{r^2 - 4\,k\,m}}{2\,m}. \tag{39}$$

At first blush Equation (39) seems to have nothing in common with the underdamped solution given by Equation (36). However, if we set

$$\omega = \frac{\sqrt{4\,k\,m - r^2}}{2m}$$

then

$$\lambda_1 - \lambda_2 = \frac{\sqrt{r^2 - 4\,k\,m}}{m} = 2\omega\,\sqrt{-1}$$

and

$$e^{\lambda_1 t} - e^{\lambda_2 t} = e^{-\frac{r}{2m}t} \left(e^{\frac{\sqrt{r^2-4\,k\,m}}{2m}t} - e^{-\frac{-\sqrt{r^2-4\,k\,m}}{2m}t} \right) = 2\, e^{-\frac{r}{2m}t} \sinh(\omega\, \sqrt{-1}\, t),$$

where $\sinh(s) = \frac{e^s - e^{-s}}{2}$ is the *hyperbolic sine* function. Hence for the overdamped case the solution can be written in terms of the hyperbolic sine as

$$x = v_0\, e^{-\frac{r}{2m}t} \frac{\sinh(\omega\sqrt{-1}\, t)}{\omega\, \sqrt{-1}}.$$

This is exactly Equation (36) with ω replaced by $\omega\sqrt{-1}$.

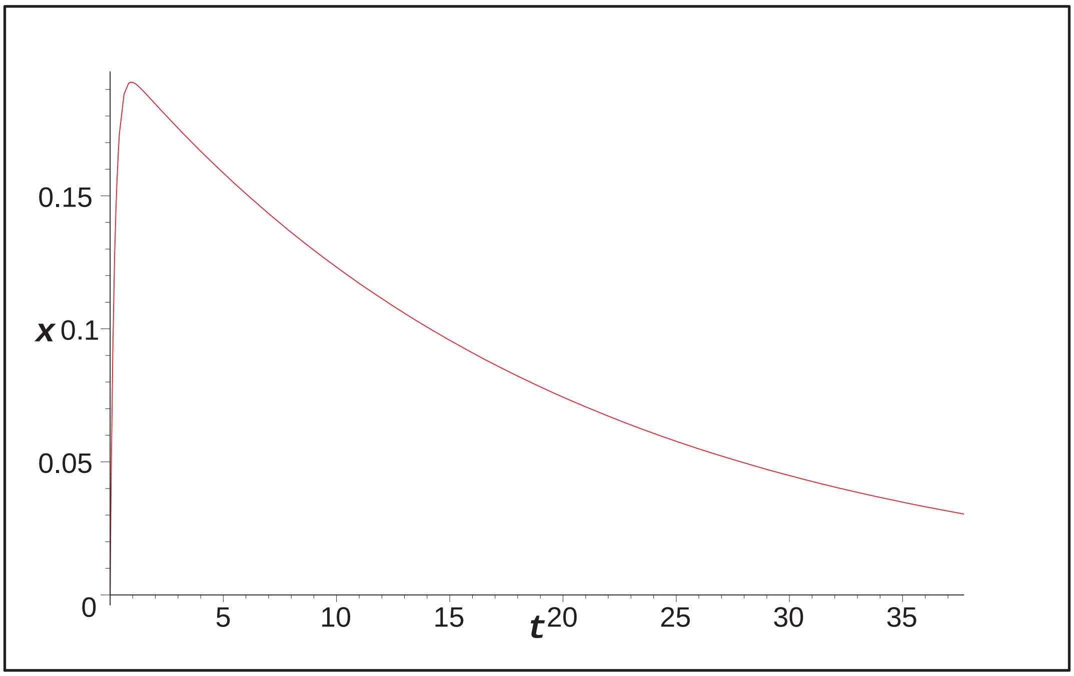

FIGURE 19. Overdamped motion

Complex exponentials. In algebra one uses complex numbers $a + b\,\sqrt{-1} = a + b\,i$ to make sense out of polynomial equations. If one works only with real numbers, an equation of fourth degree may have four roots, two roots, or no roots at all. On the other hand, if one allows complex roots then every quartic has exactly four roots; more generally, every polynomial of degree n has exactly n complex roots counted with multiplicities.

In Differential Equations one needs complex numbers in order to unify the treatment of linear ODE with constant coefficients. We have shown that if $r^2 > 4\,m\,k$ then solving

$$m\,\frac{d^2x}{dt^2} + r\,\frac{dx}{dt} + k\,x = 0$$

amounts to finding the roots of the characteristic equation:

$$m\,\lambda^2 + r\,\lambda + k = 0.$$

Indeed, let

$$\lambda_1 = \frac{-r + \sqrt{r^2 - 4\,k\,m}}{2\,m}, \quad \lambda_2 = \frac{-r - \sqrt{r^2 - 4\,k\,m}}{2\,m}$$

be the characteristic roots. It is straightforward to verify that

$$x_1 = e^{\lambda_1\,t}, \quad x_2 = e^{\lambda_2\,t}$$

are particular solutions whose superposition is the general solution:

$$x = C_1\,x_1 + C_2\,x_2 = C_1\,e^{\lambda_1\,t} + C_2\,e^{\lambda_2\,t}.$$

The trouble is that if $r^2 - 4\,k\,m < 0$ the exponentials do not seem to be well-defined.

If we wish to treat the underdamped motion in the same manner as the overdamped motion then we must find a suitable interpretation for complex exponentials. This interpretation is provided by the Taylor series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

Let us examine e^{bit} where b is a real number. Using the Taylor expansion of the exponential function we can write:

$$e^{b\,i\,t} = \sum_{n=0}^{\infty} \frac{(b\,i\,t)^n}{n!}.$$

Now split the summation into "even" and "odd" parts as follows:

$$\sum_{n=0}^{\infty} \frac{(b\,i\,t)^n}{n!} = \sum_{n=0}^{\infty} \frac{(b\,i\,t)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(b\,i\,t)^{(2n+1)}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{i^{2n}\,(b\,t)^{2\,n}}{(2\,n)!} + i\,\sum_{n=0}^{\infty} \frac{i^{2\,n}\,(b\,t)^{2n+1}}{(2n+1)!}.$$

Since $i^{2n} = (-1)^n$, we can write

$$e^{b\,i\,t} = \sum_{n=0}^{\infty} \frac{(-1)^n\,(b\,t)^{2\,n}}{(2\,n)!} + i\,\sum_{n=0}^{\infty} \frac{(-1)^n\,(b\,t)^{2n+1}}{(2n+1)!} = \cos(b\,t) + i\,\sin(b\,t).$$

Now, let us look at the more general case of $e^{(a+b\,i)t}$. Since $e^{A+B} = e^A\,e^B$ we can write:

$$e^{(a+b\,i)\,t} = e^{a\,t}\,\left(\cos(b\,t) + i\,\sin(b\,t)\right). \tag{40}$$

The remarkable formula (40) was discovered by Euler and is named in his honor.

Underdamped motion revisited. As an illustration of Euler's formula, let us solve Equation (30) for the underdamped case using complex exponentials. We know that

$$x = v_0 \, \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2},$$

where the characteristic roots

$$\lambda_{1,2} = \frac{-r \pm \sqrt{r^2 - 4\,k\,m}}{2\,m}$$

are complex. We also know that the real solution is an exponentially damped sinusoid. Introduce the following notation:

$$a = -\frac{r}{2\,m}, \quad \omega = \frac{\sqrt{4\,k\,m - r^2}}{2\,m}.$$

Note that both a and ω are real numbers. Using the new notation, we can write

$$\lambda_1 = a + i\,\omega, \quad \lambda_2 = a - i\,\omega.$$

Clearly,

$$\lambda_1 - \lambda_2 = 2\,i\,\omega.$$

Now, according to Euler's formula:

$$\begin{aligned} e^{\lambda_1 t} - e^{\lambda_2 t} &= e^{a\,t}\,(\cos(\omega\,t) + i\,\sin(\omega\,t)) - e^{a\,t}\,(\cos(-\omega\,t) + i\,\sin(-\omega\,t)) \\ &= 2\,i\,e^{at}\,\sin(\omega\,t). \end{aligned}$$

Upon division the imaginary units cancel and we get

$$x = v_0\,\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = v_0\,e^{at}\,\frac{\sin(\omega t)}{\omega} = v_0\,e^{-\frac{r}{2m}\,t}\,\frac{\sin(\omega t)}{\omega}.$$

which is the same as Equation (36).

Summary. Consider an IVP of the form

$$m\,\frac{d^2x}{dt^2} + r\,\frac{dx}{dt} + k\,x = 0, \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0,$$

or, more generally, any linear homogeneous ODE with constant coefficients. The solution process consists of the following steps:

(1) Derive the characteristic equation by guessing $x = e^{\lambda t}$ and checking the guess. For a second order ODE the characteristic equation is a quadratic.
(2) Find the characteristic roots λ_1 and λ_2.

(3) If the roots are distinct, form the general solution by superposing exponentials:

$$x = C_1\, e^{\lambda_1 t} + C_2\, e^{\lambda_2 t}.$$

If the roots coincide, $\lambda_1 = \lambda_2 = \lambda$ then form

$$x = C_1\, e^{\lambda t} + C_2\, t\, e^{\lambda t}.$$

(4) Use initial conditions to find constants C_1 and C_2.

(5) If λ_1 and λ_2 are complex, use Euler's formula to interpret complex exponentials. If your math is right, after applying Euler's formula all imaginary numbers will drop out and you will be left with a real-valued expression.

Appendix: How to solve second order ODE using `ode45`. In order to solve *any* ODE using `ode45`, or another MATLAB ode solver, one must rewrite the ODE in the form: $y' = f(t, y)$. This may seem impossible, at first glance. However, a key thing to remember is that y may be a *vector*—this is what makes `ode45` universal.

Let us consider an example. Suppose we wish to solve:

$$\frac{d^2x}{dt^2} + x = 0, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 1;$$

Think of this IVP as an instance of Newton's second law applied to a frictionless cart on spring; the solution, as we now know, should be $x = \sin(t)$. Introducing the velocity $v = \frac{dx}{dt}$, we can rewrite the second order scalar ODE as a system of two first order ODE:

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = -x.$$

Now, in order to use `ode45`, we must rewrite the system of scalar ODE's as one single vector ODE. To this end, we introduce the vector y whose components are the displacement x and velocity v:

$$y = \begin{bmatrix} x \\ v \end{bmatrix}$$

Then

$$\frac{dy}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ -x \end{bmatrix} = f(t, y).$$

So, the right-hand side function $f(t, y)$ should simply rearrange the vector y: namely, interchange the two components and multiply the second component by (-1):

```
rhs = @(t,y) [y(2); -y(1)];
y0 = [0; 1];
[t,y] = ode45(rhs,[0 6*pi],y0);

figure
plot(t,sin(t),'bo')
hold on
plot(t,y)
legend('sin(t)','x','v')
```

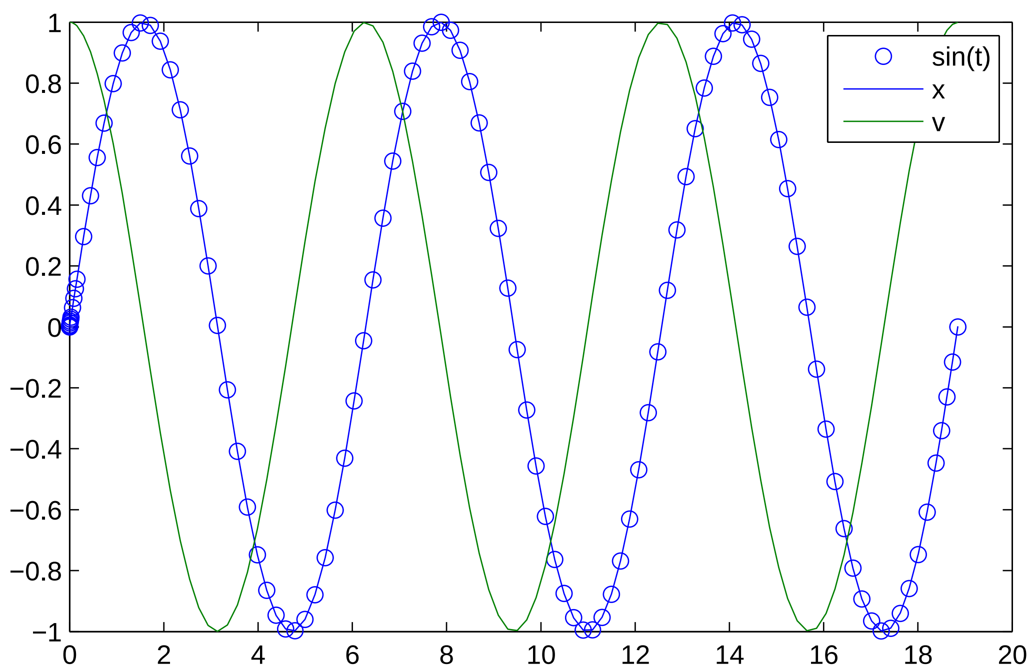

FIGURE 20. The output of `ode45` applied to $x'' + x = 0$ with $x(0) = 0$ and $x'(0) = 1$ as initial conditions. Plotted as circles is the analytic solution $x = \sin(t)$.

Figure 20 shows the output of `ode45` confirming the analytic solution $x = \sin(t)$. Notice that the plotting command `plot(t,y)` produces two plots at once. This happens because `y` is a matrix with two columns: the first column has the values of x while the second column has the values of v (corresponding to the times stored in `t`). Indeed, typing `y` into the command window results in the following display (which we truncated):

```
y

     0    1.0000
0.0001    1.0000
0.0001    1.0000
0.0002    1.0000
0.0002    1.0000
0.0005    1.0000
0.0007    1.0000
0.0010    1.0000
0.0012    1.0000
0.0025    1.0000
0.0037    1.0000
0.0050    1.0000
0.0062    1.0000
0.0125    0.9999
0.0188    0.9998
0.0251    0.9997
0.0313    0.9995
0.0627    0.9980
0.0940    0.9956
0.1252    0.9921

...
```

If only the displacement is desired, it can be plotted with the command:

```
plot(t,y(:,1))
```

Exercises.

(1) Solve each of the following homogeneous equations using $x(0) = 0$ and $\frac{dx}{dt}(0) = 1$ as initial conditions. Illustrate each solution with a computer generated plot similar to Figure 20. Suggestions for plotting: use $0 < t < 12$ as the plotting range; show the plot of the formula for $x(t)$ (which you will have derived analytically) as a solid curve and the output of `ode45` as circles. The circles, obviously, need to match the curves.

(a) $\dfrac{d^2x}{dt^2} + 4\,\dfrac{dx}{dt} + 3\,x = 0.$

(b) $\dfrac{d^2x}{dt^2} + 4\,\dfrac{dx}{dt} + 4\,x = 0.$

(c) $\dfrac{d^2x}{dt^2} + 4\,\dfrac{dx}{dt} + 10\,x = 0.$

(2) Consider a particle of unit mass under the influence of a 2π-periodic force. If initially the particle is resting at the origin, Newton's law leads to the following IVP:

$$\frac{d^2x}{dt^2} = a\,\cos(t) + b\,\sin(t), \quad x(0) = \frac{dx}{dt}(0) = 0.$$

For some choices of a and b the particle oscillates near the origin but for others it drifts away from the origin. Solve the IVP and answer the following question: Under what conditions on a and b does the particle drift away from the origin? Illustrate your answer with computer generated plots.

(3) Use the techniques discussed in this handout to find the general solution of

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} = 0.$$

Confirm your formula for the general solution using `ode45` with several sets of initial conditions of your choice.