5 Subspaces

A subspace of a vector space V is any collection of vectors in V that is itself a vector space. As a simple example, consider the following collection of vectors in $\mathbb{R}^2$:

$$S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

Since all vectors in S are scalar multiples of a single vector, any linear combination of vectors in S is a scalar multiple of that same vector, and is therefore in S. Thus S is closed under vector operations and hence is a vector space residing in $\mathbb{R}^2$. In fact, as you may have already surmised, S is an example of a one-dimensional subspace of $\mathbb{R}^2$.

It should be clear that $\mathbb{R}^2$ has infinitely many one-dimensional subspaces. Indeed, we can pick any nonzero vector $v \in \mathbb{R}^2$ and set $S = \{t\,v \mid t \in \mathbb{R}\}$; of course, if $v = 0$, we get a trivial subspace consisting of just the zero vector. More generally, given any vector space V, we can pick any number of vectors, say, $v_1, \ldots, v_k$ and set

$$S = \operatorname{span}\{v_1, \ldots, v_k\}.$$

As explained in Section 3.2, the result is a subspace of V. Furthermore, if the vectors v_j are linearly independent, they form a basis of S. In this case we say that S is a k-dimensional subspace of V. Every subspace can be obtained in this manner, using the span operation.

5.1 Operations on subspaces

Let S_1 and S_2 be two subspaces of a vector space V. The intersection $S_1 \bigcap S_2$ is the set of all vectors in V which simultaneously belong to both S_1 and S_2: it is readily seen to be another subspace of V. The intersection of any two (or more) subspaces always has the zero vector (why?) and is therefore nonempty. However, if the intersection is just the zero vector, we will call the subspaces *disjoint* even though they are not disjoint as sets.

Subspaces can be added. By definition, the sum $S_1 + S_2$ is the set of all vector sums $v_1 + v_2$ where v_1 ranges over all vectors in S_1 and v_2 ranges over all vectors in S_2; showing that the sum of two or more subspaces is another subspace is left as an easy exercise. If S_1 and S_2 happen to be disjoint subspaces, we will use special notation $S_1 \oplus S_2$ for their sum and call it a *direct sum*.

Of particular importance to us will be situations where a vector space can be written as a direct sum of its subspaces. If $V = S_1 \oplus S_2$, we will call the subspaces

S_1 and S_2 *complimentary.* As a simple example, consider the following subspaces of $\mathbb{R}^2$:

$$S_1 = \left\{ \left[\begin{array}{c} x \\ 0 \end{array} \right] \middle| \, x \in \mathbb{R} \right\}, \quad S_2 = \left\{ \left[\begin{array}{c} 0 \\ y \end{array} \right] \middle| \, y \in \mathbb{R} \right\},$$

It is clear that $S_1 \bigcap S_2 = 0$, so the subspaces are disjoint. It should also be obvious that $\mathbb{R}^2 = S_1 \oplus S_2$.

5.2 Subspaces associated with linear transformations

Let $T : V \mapsto W$ be a linear transformation. There are two important subspaces associated with T.

Definition 9 (Null space)**.** *The null space of T is the set of all vectors in V which are mapped into the zero vector (in W):*

$$\text{null}(T) = \{v \in V \mid T(v) = 0\}.$$

The null space is readily seen to be a vector subspace of V. Indeed, let v_1 and v_2 be two arbitrary elements of null(T). Then $T(v_1) = T(v_2) = 0$. But then, by linearity, $T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2) = 0$. This shows that null$(T)$ is closed under vector operations and is therefore a subspace. The dimension of the null space is called *nullity.*

Definition 10 (Range)**.** *The range of T is the set of all vectors in W which are output by T:*

$$\text{Range}(T) = \{w \in W \mid w = T(v) \textit{ for some } v \in V\}.$$

Showing that Range(T) is a subspace of W is left as an easy exercise. The dimension of the range is called *rank.*

5.3 Subspaces associated with linear operators

If T is a linear operator on V, its null space and range are, clearly, subspaces of V. In addition, in light of Section 4, we may define *eigenspaces* of T as follows.

Definition 11 (Eigenspace)**.** *The eigenspace of T corresponding to eigenvalue λ is the set of all eigenvectors of T having eigenvalue λ augmented with the zero vector:*

$$E_\lambda(T) = \{v \in V \mid T(v) = \lambda \, v\}.$$

As an example, consider the operator on $\mathbb{R}^2$ defined by Equation (3) in Section 4:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -2\,a + b \\ a - 2\,b \end{bmatrix}.$$

We found that this operator has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. The corresponding eigenspaces are one-dimensional and given by:

$$E_{\lambda_1} = \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}, \quad E_{\lambda_2} = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\},$$

These eigenspaces are disjoint and, furthermore, it can be shown that $\mathbb{R}^2 = E_{\lambda_1} \oplus E_{\lambda_2}$. It is the last formula that should really be called eigendecomposition; the matrix factorization $A = V\,D\,V^{-1}$ is its direct consequence.

In Section 4 we mentioned *geometric multiplicity* of eigenvalues. We now have the language to define it. The geometric multiplicity of an eigenvalue is the dimension of the corresponding eigenspace. In the above example the geometric multiplicity of both eigenvalues is one—the same as algebraic multiplicity. The geometric and algebraic multiplicities are not always the same. An operator can have a simple eigenvalue (algebraic multiplicity is one) whose eigenspace is k-dimensional (geometric multiplicity is k).

5.4 Computational aspects

In order to find any of the subspaces defined above, one works with a matrix of the transformation. Let us look at a simple example.

Suppose that the matrix of $T : V \mapsto W$ is given by:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

So, V is two dimensional and can be identified with $\mathbb{R}^2$ while W can be identified with $\mathbb{R}^3$. To find the null space, we need to find all $x \in \mathbb{R}^2$ such that $A\,x = 0$. This leads us to consider the following system:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Although there are three equations for the unknowns x_1 and x_2, they are all readily seen to be linearly dependent. Therefore we can write:

$$\text{null}(T) = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \middle| \, x_1 + 2\,x_2 = 0 \right\}$$

Better yet, let $x_2 = t$ be the free variable. From $x_1 + 2\,x_2 = 0$ follows that $x_1 = -2\,t$. Hence

$$\text{null}(T) = \left\{ t \left[\begin{array}{r} -2 \\ 1 \end{array} \right] \middle| \, t \in \mathbb{R} \right\}$$

The null space is one-dimensional. If the basis of V is $\{v_1, v_2\}$, the basis of the null space of T is the single vector $-2\,v_1 + v_2$.

In `Matlab` the null space can be found using `null` command as follows:

```
A = [1 2; 2 4; 3  6]

A =

     1     2
     2     4
     3     6

null(A)

ans =

   0.894427190999916
  -0.447213595499958
```

The output of `null` is a matrix whose columns form the basis of the null space. Above we got only one column, so the null space is one-dimensional. Notice that the answer is a scalar multiple of the vector

$$\left[\begin{array}{r} -2 \\ 1 \end{array} \right]$$

The components are not integer because, just like `eig`, the `null` command scales the vectors to have unit length.

Finding the range of a matrix is equally easy. In `Matlab` it is done using `orth` command:

```
orth(A)

ans =

  -0.267261241912424
  -0.534522483824849
  -0.801783725737273
```

Again, the output is a matrix whose columns are the basis of the range; the columns are scaled to have unit length.

In order to find the range by hand, we will use the fact that the result of matrix-vector multiplication is a linear combination of columns of the matrix. This implies that:

> The range of a matrix is the span of its columns—the column space.

In our case, the columns are linearly dependent. Therefore:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = (x_1 + 2\,x_2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Since x_1 and x_2 can be arbitrary, the range

$$\text{Range}(T) = \left\{ t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

Thus the range is one-dimensional. If $\{w_1, w_2, w_3\}$ is the basis of W, the basis of the range of T is the single vector $w_1 + 2\,w_2 + 3\,w_3$. Convince yourself that our characterization of the range agrees with the output of `orth`.

Homework

1. Suppose that T can be represented with a matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   Find the null space and the range of T. Confirm your results in `Matlab`.

2. Repeat the previous exercise with the matrices:

   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

3. The matrix of a linear operator is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   (a) Find the null space and nullity.

   (b) Find the range and rank.

   (c) Find the eigenvalues and the corresponding eigenspaces. Describe the algebraic and geometric multiplicities of the eigenvalues. Can this matrix be diagonalized? Why?

4. Find examples of three-by-three matrices whose null spaces have dimension zero (trivial null space), one, two, and three (the whole space). Give two examples for each dimension. In each case compute rank and nullity. What observation can you make about rank and nullity? Are they related? What could be the general statement?

5. Let S_1 and S_2 be subspaces of $\mathbb{R}^5$ having dimensions 3 and 4, respectively. Is it possible for $S_1 \bigcap S_2$ to have dimension 2? If yes, give a concrete example. If no, explain why.

6. The two-by-two zero matrix (`zeros(2)`) and the identity matrix (`eye(2)`) satisfy the matrix equation: $A^2 = A$. Find three other two-by-two matrices with that property and examine their null spaces, ranges, and eigenspaces. Write down your observations.

7. Consider

$$S_1 = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$$

   as a subspace of $\mathbb{R}^3$. Find another subspace S_2 in $\mathbb{R}^3$ such that $\mathbb{R}^3 = S_1 \oplus S_2$.