Kep Schemes (Euler Equations) Non-relativistic

- $\partial_t \rho + \nabla\cdot(\rho u) = 0$ ①
- $\partial_t(\rho u) + \nabla\cdot(\rho u u) + \nabla P = 0$ ②
- $\partial_t \epsilon + \nabla\cdot([E+P]u) = 0$ ③

$(\epsilon = \rho E)$

- Integral formulation ① → ④ Using $\varphi = \varphi(\vec{x}, t)$

- $$\int_\Omega \varphi\, \partial_t \rho\, d^3x + \underbrace{\oint_{\partial\Omega} \varphi\rho\vec{u}\cdot d\vec{s} - \int_\Omega \nabla\varphi\cdot\rho\vec{u}\, d^3x}_{} = 0 \quad ④$$

Which comes from:

- $$\int_\Omega \varphi\,\nabla\cdot(\rho u)\, d^3x = \underbrace{\int_\Omega \nabla\cdot(\rho u\varphi)\, d^3x}_{\text{DT}} - \int_\Omega \underbrace{\rho u\cdot\nabla\varphi}_{\text{sym}}\, d^3x$$

$$= \oint_{\partial\Omega} \varphi\rho\vec{u}\cdot d\vec{s} - \int_\Omega \nabla\varphi\cdot(\rho u)\, d^3x$$

② → ⑤, $\psi = \psi(\vec{x}, t)$

$$\psi\nabla P = \nabla\psi P + P\nabla\psi$$

- $$\int_\Omega \psi\,\partial_t(\rho\vec{u})\, d^3x + \underbrace{\int_\Omega \psi\,\vec{\nabla}\cdot(\rho\vec{u}\vec{u})\, d^3x} + \int_\Omega \overbrace{\psi\nabla P}\, d^3x = 0$$

$$\psi\vec{\nabla}\cdot(\rho\vec{u}\vec{u}) = \nabla\cdot(\psi\rho\vec{u}\vec{u}) - \rho\vec{u}\vec{u}\cdot\nabla\psi$$

- $$\int_\Omega \psi\,\partial_t(\rho\vec{u})\, d^3x + \oint_{\partial\Omega} \psi\rho\vec{u}\vec{u}\cdot d\vec{s} - \int_\Omega \rho\vec{u}\vec{u}\cdot\nabla\psi\, d^3x + \oint_{\partial\Omega} \psi P\, d\vec{s}$$

$$- \int_\Omega P\nabla\psi\, d^3x = 0 \quad ⑤$$

- We note: $\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = u\cdot\partial_t(\rho u) - \frac{1}{2}u^2\partial_t\rho\right]$ ⊛

(See A1)

- We choose: (based on the prior note):

$\varphi = -\frac{1}{2}u^2$ $\quad \psi = \vec{u}\cdot(\;)$ $\quad$ and insert them into ⊛

- $$\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}\rho u^2 d^3x = \underbrace{\int_\Omega u\cdot\partial_t \rho u d\overset{3}{x}}_{⑤} + \underbrace{\int_\Omega\left(-\frac{1}{2}u^2\,\partial_t\rho\right)d^3x}_{④}$$

- $$\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}\rho u^2 d^3x + \Bigg[\oint_{\partial\Omega}(\tfrac{1}{2}+\cancel{\tfrac{1}{2}})\rho u^2\cdot\vec{u}\cdot d\vec{s} - \int_\Omega \rho(\vec{u}\vec{u}):\vec{\nabla}\vec{u}\,d\overset{3}{x}$$

originally

$\to$ or $\vec{u}\cdot(\rho\vec{u}\vec{u}\cdot d\vec{s})$ $\quad$ A2 $\quad \overset{\leftrightarrow}{A3}\ \cancel{\int_\Omega \vec{\nabla}(\tfrac{1}{2}u^2)\cdot\rho\vec{u}\,dx^3}$

$$+\oint_{\partial\Omega} P\vec{u}\cdot d\vec{s} - \int_\Omega P(\vec{\nabla}\cdot\vec{u})\,d^3x\Bigg]_{⑤}$$

$$+\left[\underbrace{\cancel{\oint_{\partial\Omega} -\frac{1}{2}u^2\rho\vec{u}\cdot d\vec{s}}}_{\text{cancels}} - \cancel{\int_\Omega -\nabla(\tfrac{1}{2}u^2)\cdot\rho\vec{u}\,d\overset{3}{x}}\right]_{④} = 0$$

- $$\underbrace{\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}\rho u^2 dx}_{\text{KE - in volume}} + \underbrace{\oint_{\partial\Omega}\left[\frac{1}{2}\rho u^2 + P\right]\vec{u}\cdot d\vec{s}}_{\text{KE - flux}} - \underbrace{\int_\Omega P(\vec{\nabla}\cdot\vec{u})d^3x}_{\text{Work Done}} = 0$$

1-D Euler Equations:

$$\partial_t \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + P \\ (\mathcal{E} + P) u \end{bmatrix} = 0$$

FV Scheme:

- $$\frac{\partial \rho_j}{\partial t} + \frac{(\rho u)_{j+1/2} - (\rho u)_{j-1/2}}{\Delta x} = 0 \qquad (7)$$

- $$\frac{\partial}{\partial t}(\rho_j u_j) + \frac{(\rho u^2)_{j+1/2} - (\rho u^2)_{j-1/2}}{\Delta x} + \frac{P_{j+1/2} - P_{j-1/2}}{\Delta x} = 0 \qquad (8)$$

- $$\frac{\partial}{\partial t}(\mathcal{E}_j) + \frac{[(\mathcal{E}+P)u]_{j+1/2} - [(\mathcal{E}+P)u]_{j-1/2}}{\Delta x} = 0 \qquad (9)$$

- multiply (7) $\times -\frac{1}{2}u_j^2$ & (8) by $\vec{u}_j$ (...) no dot (10)

From (A):
- $$\frac{\partial}{\partial t}\sum_j \frac{1}{2}(\rho u^2)_j = \sum_j u_j \cdot \underbrace{\partial_t(\rho u)_j}_{(8)} - \sum_j -\frac{1}{2}u_j^2 \underbrace{\partial_t \rho_j}_{(7)}$$

- $$\Delta x \frac{\partial}{\partial t}\sum_j \frac{1}{2}(\rho u^2)_j + \sum_j u_j \left[(\rho u^2)_{j+1/2} - (\rho u^2)_{j-1/2}\right]$$
$$+ \sum_j u_j \left[P_{j+1/2} - P_{j-1/2}\right]$$
$$- \sum_j \frac{1}{2}u_j^2\left[(\rho u)_{j+1/2} - (\rho u)_{j-1/2}\right] = 0 \qquad (10)$$

$\rightarrow$ Sum by parts (A4) Eq (10):

- $$\Delta x \frac{\partial}{\partial t}\sum_j \left(\frac{1}{2}\rho u^2\right)_j + \sum_j (u_j - u_{j+1})(\rho u^2)_{j+1/2} + BT_1$$
$$- \sum_j \frac{1}{2}(u_j^2 - u_{j+1}^2)(\rho u)_{j+1/2} + BT_2$$
$$+ \sum_j u_j (P_{j+1/2} - P_{j-1/2}) = 0$$

· Combining terms:

$$\Delta x \frac{\partial}{\partial t} \sum_j \frac{1}{2} \rho_j u_j^2 + \sum_j (u_j - u_{j+1}) \left[(\rho u)^2_{j+1/2} - \frac{1}{2}(u_j + u_{j+1})(\rho u)_{j+1/2} \right]$$

$$+ \sum_j u_j (P_{j+1/2} - P_{j-1/2}) = 0$$

Now we have (if $P \to 0$) conservation of energy if:

$$\text{I} \quad \left[(\rho u)^2_{j+1/2} = \frac{1}{2}(u_j + u_{j+1})(\rho u)_{j+1/2} \right]$$

· Discrete local balance

Using Ⓘ we can re-write our Eqs: (For FV 7-8), $\rho \to 0$

- $(\partial_t \rho_j)\Delta x + (\rho u)_{j+1/2} - (\rho u)_{j-1/2} = 0$
- $(\partial_t(\rho_j u_j))\Delta x + \frac{1}{2}(u_j + \underbrace{u_{j+1}}_{!!})(\rho u)_{j+1/2} - \frac{1}{2}(u_{j-1} + \underbrace{u_j}_{!!})(\rho u)_{j-1/2} = 0$

- we can recover $(\rho u)_{j+1/2} \to$ Upwind values (for now)
  Recovery

A1

$$\partial_t \tfrac{1}{2}\rho u^2 = \tfrac{1}{2} u\cdot u \frac{\partial \rho}{\partial t} + \tfrac{1}{2}\rho\, 2u\cdot\partial_t u$$

$$\partial_t\left(\tfrac{1}{2}\rho\vec{u}\cdot\vec{u}\right) = \underbrace{\tfrac{1}{2}\vec{u}\cdot\partial_t(\rho\vec{u})} + \tfrac{1}{2}\rho\vec{u}\cdot\partial_t u$$

$$= \tfrac{1}{2}\vec{u}\cdot[\rho\partial_t\vec{u} + \vec{u}\partial_t\rho]$$

$$= \tfrac{1}{2}u^2\partial_t\rho + \tfrac{1}{2}\rho\vec{u}\cdot\partial_t u + \tfrac{1}{2}\rho\vec{u}\cdot\partial_t\vec{u}$$

$$= \tfrac{1}{2}u^2\partial_t\rho + \underbrace{\rho\vec{u}\cdot\partial_t\vec{u}}$$

$$\partial_t\left(\tfrac{1}{2}\rho\vec{u}\cdot\vec{u}\right) = \tfrac{1}{2}u^2\partial_t\rho + \partial_t(\rho\vec{u}\cdot\vec{u}) - u\cdot\partial_t\rho u$$

$$-\partial_t\left(\tfrac{1}{2}\rho\vec{u}\cdot\vec{u}\right) = \tfrac{1}{2}u^2\partial_t\rho - u\cdot\partial_t(\rho u)$$

$$\rightarrow \partial_t\left(\tfrac{1}{2}\rho\vec{u}\cdot\vec{u}\right) = u\cdot\partial_t(\rho u) - \tfrac{1}{2}u^2\partial_t\rho$$

A2

$$\int_\Omega \vec{\varphi}\cdot[\vec{\nabla}\cdot(\rho\vec{u}\vec{u})]\,d^3x = \int_\Omega \varphi_j[\partial_i(\rho u_i u_j)]\,d^3x$$

$$= \int_\Omega \partial_i(\rho u_i \varphi_j u_j)\,d^3x - \int_\Omega \rho u_i u_j \partial_i\varphi_j\,d^3x$$

$$= \int_\Omega \vec{\nabla}\cdot(\rho\vec{u}(\vec{\varphi}\cdot\vec{u}))\,d^3x - \int_\Omega \rho(\vec{u}\vec{u}) : \vec{\nabla}\vec{\varphi}\,d^3x$$

A3

$$\int_\Omega \rho u_i \underbrace{u_j\partial_i u_j}\,d^3x = \int_\Omega \rho u_i \tfrac{1}{2}\partial_i u_j u_j = \int_\Omega \tfrac{1}{2}\rho\vec{u}\cdot\vec{\nabla}u^2\,d\upsilon$$

$$u_j\partial_i u_j = \partial_i u_j u_j - u_j\partial_i u_j \rightarrow \tfrac{1}{2}\partial_i u_j u_j = u_j\partial u_j$$

Summation by Parts

- $$\sum_{j=1}^{N} \phi_j (U_{j+1/2} - U_{j-1/2}) = \sum_{j=1}^{N} \phi_j U_{j+1/2} - \underbrace{\sum_{j=1}^{N} \phi_j U_{j-1/2}}_{(\text{II})}$$

- set $m = j-1$ in (II): $\displaystyle\sum_{m=0}^{N-1} \phi_{m+1} U_{m+1/2}$

- Extract Boundary terms: $\displaystyle\sum_{m=1}^{N} \phi_{m+1} U_{m+1/2} + \underbrace{\phi_1 U_{1/2} - \phi_{N+1} U_{N+1/2}}_{\text{Boundary}}$

- Hence,

$$\sum_{j=1}^{N} \phi_j (U_{j+1/2} - U_{j-1/2}) = \underbrace{\sum_{j=1}^{N} (\phi_j - \phi_{j+1}) U_{j+1/2}}_{-\sum_{j=1}^{N} (\phi_{j+1} - \phi_j) U_{j+1/2}} + \underbrace{\phi_1 U_{1/2} - \phi_{N+1} U_{N+1/2}}_{\text{Boundary}}$$