Relativistic KEP Scheme:

- $\partial_t \rho + \nabla \cdot (\rho v) = 0$ ①
- $\partial_t (\rho u) + \nabla \cdot (\rho u v) = 0$ ②
- $v = \dfrac{u}{\sqrt{1+u^2}}$ ③

- Integrating ①: Using $\varphi = \varphi(x,t)$

- $$\int_\Omega \varphi \partial_t \rho d^3x + \underbrace{\int_\Omega \varphi \nabla \cdot (\rho v)}_{} = 0$$

$$\underbrace{\int_\Omega [\nabla \cdot (\varphi \rho v) - \rho v \cdot \nabla(\varphi)] \, d^3x}_{}$$

$$\oint_{\partial\Omega} \varphi \rho v \cdot d\vec{s}$$

- $$\left[\int_\Omega \varphi \partial_t \rho d^3x + \oint_{\partial\Omega} \varphi \rho v \cdot d\vec{s} - \int_\Omega \rho v \cdot \nabla \varphi \, d^3x = 0 \right] \quad ④$$

- Integrating 2:

- $$\int_\Omega \psi \partial_t (\rho u) + \underbrace{\int_\Omega \psi \nabla \cdot (\rho u v) d^3x}_{\text{Same as above}}$$

- $$\left[\int_\Omega \psi \partial_t (\rho u) d^3x + \oint_{\partial\Omega} \psi \rho v u \cdot d\vec{s} - \int_\Omega \rho u v \cdot \nabla \psi \, d^3\vec{x} = 0 \right] \quad ⑤$$

or for $\vec{\Psi} \cdot (\cdots)$

$$v_j u_i = \frac{u_j u_i}{\gamma} = u_j v_i$$

Assume: v_i

$$\int_\Omega \vec{\Psi} \cdot [\nabla \cdot (\rho u v)] d^3x = \int_\Omega \psi_i \partial_j (\rho v_j u_i) \, d^3x$$

$$= \int_\Omega \partial_j (\psi_i \rho v_j u_i) d^3x - \int_\Omega \rho v_j u_i \underbrace{\partial_j \psi_i}_{} \, d^3x$$

$$= \int_\Omega \nabla \cdot (\rho \vec{v} (\vec{\Psi} \cdot \vec{u})) d^3x - \int_\Omega \rho u_j \underbrace{v_i \partial_j v_i}_{}$$

Dyadic Prod otherw.

$$\tfrac{1}{2} \partial_j v_i v_i = v_i \partial_j v_i$$

$$= \oint_{\partial\Omega} \rho \vec{v} (\vec{\Psi} \cdot \vec{u}) \cdot d\vec{s} - \int_\Omega \rho \tfrac{1}{2} \vec{u} \cdot \nabla v^2 d^3x$$

· $KE_{Rel} = (\gamma-1)\rho c^2 \qquad c=1, \quad$ Density KE

$$\cdot\ \partial_t((\gamma-1)\rho) = \underbrace{(\partial_t\gamma)\rho} + (\gamma-1)\partial_t\rho$$

$$\rightarrow \frac{\partial}{\partial t}\left(\sqrt{1+u^2}\right) = \frac{1}{2}\frac{1}{\sqrt{1+u^2}}\frac{\partial}{\partial t}(\vec{u}\cdot\vec{u})$$

$$= \frac{\vec{u}}{\sqrt{1+u^2}}\cdot\frac{\partial\vec{u}}{\partial t}$$

$$= \vec{v}\cdot\frac{\partial\vec{u}}{\partial t}$$

Also $\vec{v}\cdot\partial_t(\rho\vec{u}) = \vec{v}\cdot\vec{u}\,\partial_t\rho + \underbrace{\rho\vec{v}\cdot\partial_t\vec{u}}_{1}$

$$= \overbrace{\rho\vec{v}\cdot\partial_t\vec{u}}^{1} + (\gamma-1)\partial_t\rho$$

$$\cdot\ \boxed{\partial_t((\gamma-1)\rho) = \underbrace{\vec{v}}_{\psi}\cdot\underbrace{\partial_t(\rho\vec{u})}_{5} + \underbrace{(\gamma-1-\vec{v}\cdot\vec{u})}_{\varphi}\underbrace{\partial_t\rho}_{4}} \quad \circledast$$

· Integrate $\circledast$

$$\cdot\ \partial_t\int_\Omega(\gamma-1)\rho\, d^3x = \int_\Omega \vec{v}\cdot\partial_t\rho\vec{u}\, d^3x + \int_\Omega(\gamma-1-\vec{v}\cdot\vec{u})\partial_t\rho\, d^3x$$

$$= \cancel{\oint_{\partial\Omega}\rho\vec{v}(\vec{v}\cdot\vec{u})\cdot d\vec{s}} - \cancel{\int_\Omega \rho\,\tfrac{1}{2}\vec{u}\cdot\nabla v^2\, d^3x}$$

↕ cancel see pg 3

$$+ \underbrace{\oint_{\partial\Omega}(\gamma-1-\cancel{\vec{v}\cdot\vec{u}})\rho\vec{v}\cdot d\vec{s}}_{KE-flux} - \cancel{\int_\Omega \rho\vec{v}\cdot\nabla(\gamma-1-\vec{v}\cdot\vec{u})\, d^3x}$$

$$\cdot\ \boxed{\underbrace{\partial_t\int_\Omega(\gamma-1)\rho\, d^3x}_{KE-Volume} = \underbrace{\oint_{\partial\Omega}(\gamma-1)\rho\vec{v}\cdot d\vec{s}}_{KE-flux}}$$

• Volume terms:

$$\cdots = -\int_\Omega \rho \frac{1}{2} \vec{u} \cdot \vec{\nabla} v^2 d^3x - \int_\Omega \rho \vec{v} \cdot \underbrace{\vec{\nabla}(\gamma - 1 - \vec{v} \cdot \vec{u})}\, d^3x$$

$$\vec{\nabla}(\gamma - 1 - \gamma v^2)$$

$$\vec{\nabla}(\gamma \underbrace{(1 - v^2)}_{1/\gamma^2} - 1)$$

$$\vec{\nabla}\left(\frac{1}{\gamma} - 1\right)$$

$$\vec{\nabla} \frac{1}{\gamma} = \vec{\nabla}\sqrt{1 - v^2}$$

$$= \frac{1/2(-\vec{\nabla} v^2)}{\sqrt{1 - v^2}}$$

$$= -\frac{1}{2} \gamma \vec{\nabla} v^2$$

$$\cdots = -\int_\Omega \rho \frac{1}{2} \vec{u} \cdot \vec{\nabla} v^2 d^3x - \int_\Omega -\rho \frac{1}{2} \underbrace{\gamma \vec{v}}_{= \vec{u}} \cdot \nabla v^2 d^3x$$

$$= -\int_\Omega \rho \frac{1}{2} \vec{u} \cdot \vec{\nabla} v^2 d^3x + \int_\Omega \rho \frac{1}{2} \vec{u} \cdot \nabla v^2 d^3x$$

$$= 0 \quad \checkmark$$