UNIVERSITY OF MARY
SCHOOL OF ENGINEERING
**ENR 338: Assignment 06**
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# Problem

A certain engineering problem has been modeled by the following differential equation

$$\frac{d^2y}{dx^2} + y = x^3$$

where it has been found that $y(0) = 1$ and $dy/dx(0) = 0$ and $x$ ranges between $x = 0$ and $x = 2$. We will solve this problem using three different methods: two of them mathematical and one of them numerical. We will then plot the solutions and compare them.

# Power Series Solution

We assume the solution can be written as a power series in $x$ as follows

$$y = \sum_{n=0}^{\infty} a_n x^n$$

The derivatives are therefore

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

where we have dropped the zero terms by starting the sum at $n = 1$ and $n = 2$ respectively. If we put these expressions into the original differential equation we have

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = x^3$$

Now we shift the index of the first sum by letting $n \to n + 2$ and starting the sum at $n = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = x^3$$

Now that both sums have the same index and they start at the same index value we can combine them into a single sum

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = x^3$$

Next we note that different powers of $x$ are linearly independent which means that this equation must be true separately for every term in the sum. For the $n = 0$ term we have

$$[n=0] \quad 2 \cdot 1 a_2 + a_0 = 0$$
$$\implies \boxed{a_2 = -\frac{a_0}{2!}}$$

Continuing from here we have for $n = 1$

$$[n=1] \quad 3 \cdot 2 a_3 + a_1 = 0$$
$$\implies \boxed{a_3 = -\frac{a_1}{3!}}$$

Then for more steps up to 7:

$$[n=2]: \quad a_4 = \frac{-a_2}{4 \cdot 3} = \frac{(-1)^2}{4!}$$
$$[n=3]: \quad a_5 = \frac{1 - a_3}{5 \cdot 4} = \frac{1}{20}$$
$$[n=4]: \quad a_6 = \frac{-a_4}{6 \cdot 5} = \frac{(-1)^3}{6!}$$
$$[n=5]: \quad a_7 = \frac{-a_5}{7 \cdot 6} = \frac{(-1) \cdot 3!}{7!}$$
$$[n=6]: \quad a_8 = \frac{-a6}{8 \cdot 7} = \frac{(-1)^4}{8!}$$
$$[n=7]: \quad a_9 = \frac{-a_7}{9 \cdot 8} = \frac{(-1) \cdot 3!}{9!}$$

From this we can recognize a pattern emerging for the even and oddd terms:

Even:

$$a_{2k} = -\frac{1^k}{(2k)!}$$

Odd:

$$a_{2k+1} = \frac{-1^k 3!}{(2k+1)!}, \quad k \geq 2$$

We can combine these along with k¡2 terms we excluded. This gives us our final summation.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^k}{(2k)!} + \sum_{n=0}^{\infty} \frac{(-1)^k 3!}{(2k+1)!} - 6x + x^3$$

In this summation we can recognize $\cos(x)$ and $\sin(x)$. Simplify, and we're left with a final answer:

$$\boxed{y = \cos(x) + 6\sin(x) - 6x + x^3}$$

## Laplace Transform Solution

We solve the same equation using Laplace Transforms by first transforming the whole equation:

$$\mathcal{L}\left(\frac{d^2y}{dx^2} + y = x^3\right)$$
$$\mathcal{L}\left(\frac{d^2y}{dx^2}\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(x^3\right)$$

and then using the boundary conditions along with the known transforms of the individual terms we can solve for $\mathcal{L}(y)$ and then perform and inverse transform (along with tables of known transforms) to find our function $y(x)$. We have

$$s^2L(s) - sy(0) - y'(0) + L(s) = \frac{6}{s^4} \tag{1}$$

Simplifying with initial conditions gives us

$$s^2\mathcal{L}(s) - s + \mathcal{L}(s) = \frac{6}{s^4} \tag{2}$$

Then we find $\mathcal{L}(s)$

$$\mathcal{L}(s)[s^2+1] = \frac{6}{s^4} + s \tag{3}$$

$$\mathcal{L}(s) = \frac{\frac{6}{s^4}+s}{s^2+1} = \frac{6+s^5}{s^4(s^2+1)} \tag{4}$$

Now we must use partial fraction decomposition to determine the coefficients:

$$\frac{6+s^5}{s^4(s^2+1)} = \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} + \frac{Es+F}{s^2+1}$$

After some algebra happens, we have

$$\begin{aligned} D+E &= 1 \\ C+F &= 0 \\ B+D &= 0 \\ A+C &= 0 \\ B &= 0 \\ A &= 6 \\ C &= -6 \\ D &= 0 \\ E &= 1 \\ F &= 6 \end{aligned}$$

$$\frac{6}{s^4} - \frac{6}{s^2} + \left(\frac{s}{s^2+1} + \frac{6}{s^2+1}\right)$$

Taking $\mathcal{L}^{-1}$ of this yields our final answer:

$$\boxed{y = x^3 - 6x + \cos(x) + 6\sin(x)}$$

## Numerical Solution

Finally, we can use Eulers method to solve the equation using a C program and compare against the other methods. First we apply Eulers method:

$$\begin{aligned} y'' + y &= x^3 \\ \text{Let } \alpha &= y' \\ \alpha' + y &= x^3 \\ \alpha' &= x^3 - y \\ \frac{dy}{dx} &= (x^3 - y)dx \end{aligned}$$

Now we write a C program which solves the differential equation numerically.

**assignment06.c**

```
#include <stdio.h>
#include <stdlib.h>

double x = 0;
double y = 1;
double dy = 0;
double dx = 0.0001;

int main()
{
  for (x = 0; x <= 2; x += dx)
  {
    dy += (x*x*x - y) * dx;
    y += dy*dx;
    printf("%lf\t%lf\t%lf\n", x, y, dy);
  }
  return 0;
}
```

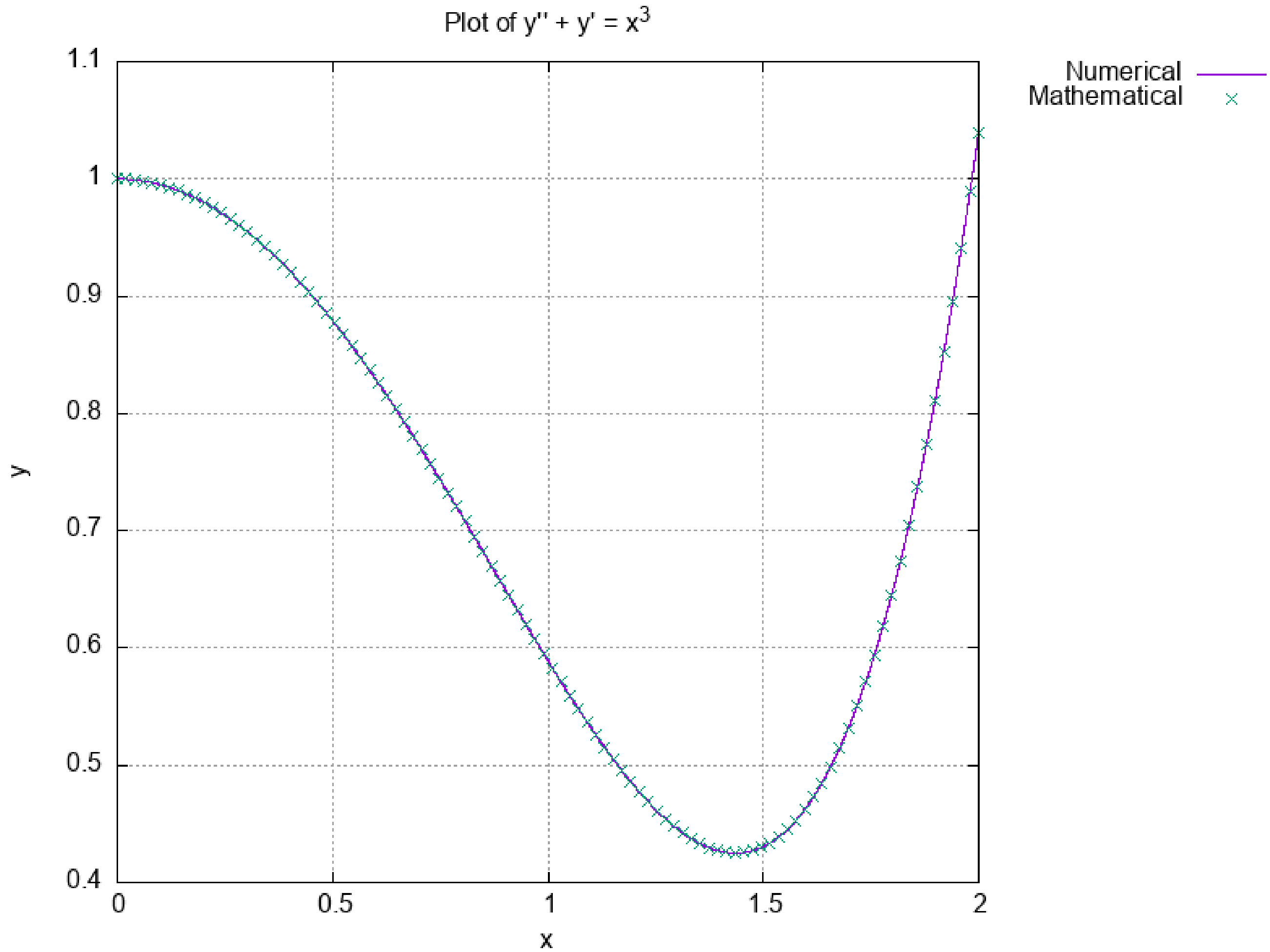
We use the following Gnuplot script to compare the numerical solution with the mathematical one.

**assignment06.gnuplot**

```
set term png size 800, 600
set key outside
set title "Plot of y'' + y' = x^{3}"
set ylabel "y"
set xlabel "x"
set grid


plot "data.dat" u 1:2 w l t "Numerical"
```

Here are both the mathematical solution and the numerical solution plotted on the same graph:




Since the solutions from power series and Laplace are the same, we define one function to represent them and plot as points. We can see the line tracks perfectly along the points, so the two results are equivalent.

## Conclusion

Solving $y'' + y = x^3$ with power series, Laplace transforms, and the numerical method with C programs all result in the same answer. The numerical method is short and sweet, and can easily solve differential equations, assuming you have a computer and can write C.