Group Testing

During World War II, the U.S. Army tested for syphilis by giving each soldier an individual blood test that was analyzed separately. One researcher suggested mixing pairs of blood samples. After the mixed pairs were tested, those with syphilis could be identified by retesting the few blood samples that were in the pairs that tested positive. Since the total number of analyses was reduced by pairing blood specimens, why not combine them in groups of three or four or more? This technique of combining samples in groups and retesting only those groups that test positive is known as *group testing* or *pooled testing*, or *composite testing*. University of Nebraska statistician Christopher Bilder wrote an article about this topic in *Chance* magazine, and he cited some real applications. He noted that the American Red Cross uses group testing to screen for specific diseases, such as hepatitis, and group testing is used by veterinarians when cattle are tested for the bovine viral diarrhea virus.

Mnemonics When trying to remember which of the two preceding terms involves order, think of **p**ermutations **p**osition, where the alliteration reminds us that with permutations, the positions of the items makes a difference. You might also use the alliteration in **c**ombinations **c**ommittee, where those words remind us that with members of a committee, rearrangements of the same members result in the same committee, so order does not count.

This section includes the following notation and counting rules. Illustrative examples follow.

Notation

The **factorial symbol (!)** denotes the product of decreasing positive whole numbers. For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. By special definition, $0! = 1$.

Counting Rules

1. **Fundamental Counting Rule**
$m \cdot n =$ Number of ways that two events can occur, given that the first event can occur m ways and the second event can occur n ways. (This rule extends easily to situations with more than two events.) *Example:* For a two-character code consisting of a letter followed by a digit, the number of different possible codes is $26 \cdot 10 = 260$.

2. **Factorial Rule**
$n! =$ Number of different *permutations* (order counts) of n different items when all n of them are selected. (This rule reflects the fact that the first item may be selected n different ways, the second item may be selected $n - 1$ ways, and so on.) *Example:* The number of ways that the five letters {a, b, c, d, e} can be arranged is as follows:
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

3. **Permutations Rule (When All of the Items Are Different)**
$_nP_r = \dfrac{n!}{(n-r)!} =$ Number of different *permutations* (order counts) when n different items are available, but only r of them are selected *without replacement*. (Rearrangements of the same items are counted as being different.) *Example:* If the five letters {a, b, c, d, e} are available and three of them are to be selected without replacement, the number of different permutations is as follows:
$$_nP_r = \frac{n!}{(n-r)!} = \frac{5!}{(5-3)!} = 60$$

4. **Permutations Rule (When Some Items Are Identical to Others)**
$\dfrac{n!}{n_1!n_2! \cdots n_k!} =$ Number of different *permutations* (order counts) when n items are available and all n are selected *without replacement*, but some of the items are identical to others: n_1 are alike, n_2 are alike, . . . , and n_k are alike. *Example:* If the 10 letters {a, a, a, a, b, b, c, c, d, e} are available and all 10 of them are to be selected without replacement, the number of different permutations is as follows:
$$\frac{n!}{n_1!n_2! \cdots n_k!} = \frac{10!}{4!2!2!} = \frac{3{,}628{,}800}{24 \cdot 2 \cdot 2} = 37{,}800$$

5. **Combinations Rule**
$_nC_r = \dfrac{n!}{(n-r)!r!} =$ Number of different combinations (order does not count) when n different items are available, but only r of them are selected *without replacement*. (Note that rearrangements of the same items are counted as being the same.) *Example:* If the five letters {a, b, c, d, e} are available and three of them are to be selected without replacement, the number of different combinations is as follows:
$$_nC_r = \frac{n!}{(n-r)!r!} = \frac{5!}{(5-3)!3!} = \frac{120}{2 \cdot 6} = 10$$