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Problem 1:

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Homework - 5

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CS688 - Machine Learning

1. (a) For T time steps, both the coin that I flip ($z_t \in \{1,2\}$ at time step t) and the outcome variable x_t which is 1 if the coin comes up heads and 0 otherwise.

To write the log-likelihood function, we need to consider the probability for parameters $\theta = (p_1, p_2, \pi_1)$

The ~~log~~ likelihood function can be expressed as:

$$\log L(\theta) = \sum_t \Big[I(z_t=1)\log\left(\pi_1 \cdot p_1^{x_t} \cdot (1-p_1)^{(1-x_t)}\right) + I(z_t=2)\log\left((1-\pi_1)\cdot p_2^{x_t} \cdot (1-p_2)^{(1-x_t)}\right)\Big] \rightarrow eq\ (1)$$

where $I[z_t=1]$ is indicator function, that equals 1 if coin 1 is selected at time 't', and 0 otherwise.

To find the maximum likelihood estimates (MLEs) of $\theta = (p_1, p_2, \pi_1)$, we take the partial derivative with respect to each parameter and set them equal to zero.

For p_1

$$\frac{\partial \log L(\theta)}{\partial p_1} = 0$$

$$\sum_t \Big[I(z_t=1) \underbrace{\frac{\partial}{\partial p_1}\left(\log\left(\pi_1 \cdot p_1^{x_t} \cdot (1-p_1)^{(1-x_t)}\right)\right)}_{\text{Let this derivate be } M} + 0\Big] = 0$$

solve M

$$M = \frac{\partial}{\partial p_1}\left(\log \pi_1 \cdot p_1^{x_t} \cdot (1-p_1)^{(1-x_t)}\right) = \log \pi_1 \frac{\partial}{\partial p_1}\left[x_t \log p_1 + (1-x_t)\log(1-p_1)\right]$$

$$= x_t \cdot \frac{1}{P_1} + (1-x_t)\cdot\left(\frac{1(-1)}{1-P_1}\right)$$

$$M = \frac{x_t}{P_1} - \frac{(1-x_t)}{1-P_1} \longrightarrow eq\ 2$$

substitute eq 2 in eq 1

$$\Sigma_t \left[I(z_t=1)\cdot\left(\left(\frac{x_t}{P_1}\right) - \frac{(1-x_t)}{1-P_1}\right)\right] = 0$$

$$\Sigma_t \left[I(z_t=1)\cdot\left(\frac{x_t}{P_1}\right)\right] - \Sigma_t \left[I(z_t=1) * \frac{(1-x_t)}{(1-P_1)}\right] = 0$$

multiply by $P_1 * (1-P_1)$

$$\Sigma_t \left[I(z_t=1) * x_t \right] - P1 / (1-P_1) * \Sigma_t \left[I(z_t=1) * (1-x_t) \right] = 0$$

$$\frac{P1}{(1-P1)} = \frac{\Sigma_t \left[I(z_t=1) * x_t \right]}{\Sigma_t \left[I(z_t=1) * (1-x_t) \right]}$$

$$\boxed{P_1 = \frac{\Sigma_t \left[I(z_t=1) * x_t \right]}{\Sigma_t \left[I(z_t=1) \right]}}$$

similarly for P2:

$$\boxed{P_2 = \frac{\Sigma_t \left[I(z_t=2) * x_t \right]}{\Sigma_t \left[I(z_t=2) \right]}}$$

(2)

For π_1:

$$\frac{\partial(\log L(\theta))}{\partial \pi_1} = \sum_t \left[I(z_t = 1)/\pi_1 - I(z_t=2)/(1-\pi_1) \right] = 0$$

~~solving~~ $\sum_t \left[\frac{I(z_t = 1)}{\pi_1} \right] = \sum_t \left[\frac{I(z_t=2)}{(1-\pi_1)} \right]$

solving for π_1,

we get

$$\boxed{\pi_1 = \sum_{t=1}^{T} \frac{I(z_t = 1)}{T}}$$

(b) In this case, since we don't observe which coin was flipped, we need to consider the probability of observing the outcome x_t under either coin, weighted by the probability of selecting each coin.

The liklihood of observing x_t given by:

$$P\left(\frac{x_t}{\theta}\right) = P_r(x_t \mid z_t = 1, \theta) * P_r(z_t = 1 \mid \theta) + P_r(x_t \mid z_t = 2, \theta) * P_r(z_t = 2 \mid \theta)$$

From eq (2) and (3)

$$= P_1^{(x_t)} \cdot (1-P_1)^{(1-x_t)} * \pi_1 + P_2^{(x_t)} \cdot (1-P_2)^{(1-x_t)} * (1-\pi_1)$$

$\longrightarrow$ eq (1)

where $\theta = (P_1, P_2, \pi_1)$

① $P(x_t \mid z_t, \theta)$:

↳ if $z_t = 1$, the prob. of observing x_t is P_1 if $x_t = 1$ (heads) and $1-P_1$ if $x_t = 0$ (tails).

↳ if $z_t = 2$, the prob. of observing $x_t = 1$ is P_2 if $x_t = 1$ (heads) and $1-P_1$ if $x_t = 0$ (tails).

we can express this as:

$$P(x_t \mid z_t, \theta) = \begin{cases} P_1^{x_t}(1-P_1)^{1-x_t} & \text{if } z_t = 1 \\ P_2^{x_t}(1-P_2)^{1-x_t} & \text{if } z_t = 2 \end{cases} \longrightarrow \text{eq ②}$$

② $P(z_t \mid \theta)$:

↳ The prob. of choosing coin $z_t = 1$ is π_1 and $z_t = 2$ is $1-\pi_1$.

↳ we can express this as:

$$P(z_t \mid \theta) = \begin{cases} \pi_1 & \text{if } z_t = 1 \\ 1-\pi_1 & \text{if } z_t = 2 \end{cases} \longrightarrow \text{eq ③}$$

Taking log of the likelihood for the eq ①

$$\log L(\theta) = \sum_{t=1}^{T} \log \left(P_1^{x_t} * (1-P_1)^{(1-x_t)} * \pi_1 + P_2^{x_t} * (1-P_2)^{(1-x_t)} * (1-\pi_1) \right)$$

Hence proved.

(3)

(c) Using the above, In the E-step of the EM algorithm, we need to compute the expected value of the complete data log likelihood function, given the observed data and the current parameter estimates.

To do this we need to calculate $Pr(Z_t = 1 | x_t, \theta)$, which is the probability that coin 1 was flipped, given the observed outcome x_t and the current parameter estimates θ.

Using Bayes's rule, we can derive:

$$\hookrightarrow Pr(Z_t = 1 | x_t, \theta) = \frac{Pr(x_t | Z_t = 1, \theta) * Pr(Z_t = 1 | \theta)}{Pr(x_t | \theta)}$$

$$= \frac{P_1^{x_t} * (1-P_1)^{(1-x_t)} * \pi_1}{P_1^{x_t} (1-P_1)^{(1-x_t)} * \pi_1 + P_2^{x_t} * (1-P_2)^{(1-x_t)} * (1-\pi_1)}$$

Similarly

$$Pr(Z_t = 2 | x_t, \theta) = \frac{Pr(x_t | Z_t = 2, \theta) * Pr(Z_t = 2 | \theta)}{Pr(x_t | \theta)}$$

(or)

$$\hookrightarrow Pr(Z_t = 2 | x_t, \theta) = 1 - Pr(Z_t = 1 | x_t, \theta)$$

(d) In the M-step, we maximize the expected complete log likelihood function to update p1, p2, π1

The expected complete log likelihood function can be written as

$$Q\left(\frac{\theta}{\theta^t}\right) = \sum_{t=1}^{T}\Big[P_r\left(z_t=1 \mid x_t, \theta^t\right) * \log\left(\pi_1 * p1^{x_t} * (1-p1)^{(1-x_t)}\right)$$
$$+ P_r\left(z_t=2 \mid x_t, \theta^t\right) * \log\left((1-\pi_1) * P_2^{x_t} * (1-P_2)^{(1-x_t)}\right)\Big]$$

where θ^t represents the current parameter estimates at iteration 't' of the EM algorithm.

To update the parameters in the M-step, we take the partial derivatives of $Q(\theta \mid \theta^t)$ with respect to each parameter and set them to zero.

For p1 :

$$\frac{\partial Q}{\partial P_1} = \sum_t P_r\left(z_t=1 \mid x_t, \theta^t\right) * \left[(x_t/P_1) - (1-x_t)/(1-p1)\right] = 0$$

$$\sum_t P_r\left(z_t=1 \mid x_t, \theta^t\right) * \left(\frac{x_t}{P_1}\right) = \sum_t P_r\left(z_t=1 \mid x_t, \theta^t\right) \cdot \frac{(1-x_t)}{(1-p1)}$$

$$\Rightarrow P_1 = \frac{\sum P_r\left(z_t=1 / x_t, \theta^t\right) * x_t}{\sum P_r\left(z_t=1 \mid x_t, \theta^t\right)}$$

Therefore, the solution for P1 is

$$P_1^{(t+1)} = \frac{\sum_t Pr(z_t=1|x_t,\theta^t) \cdot x_t}{\sum_t Pr(z_t=1|x_t,\theta^t)}$$

Similarly for P_2

$$\frac{\partial Q}{\partial P_2} = 0 \Rightarrow P_2^{(t+1)} = \sum_t \frac{Pr(z_t=2|x_t,\theta^t) * x_t}{\sum_t Pr(z_t=2|x_t,\theta^t)}$$

For π_1

$$\frac{\partial Q}{\partial \pi_1} = \sum_t \left[\frac{Pr(z_t=1|x_t,\theta^t)}{\pi_1} - \frac{Pr(z_t=2|x_t,\theta^t)}{(1-\pi_1)} \right] = 0$$

solving for π_1, we get:

$$\pi_1^{(t+1)} = \frac{\sum Pr(z_t=1|x_t,\theta^t)}{T}$$

$\therefore$ These updates are based on the expectations computed in the E-step.

Problem 2:

```
Testing on Eleven-armed bandit:
Algorithm: greedy_bandit
Iterations: 1000, Mean reward: 0.0, Mean regret: 1.0
Iterations: 10000, Mean reward: 0.0, Mean regret: 1.0
Iterations: 100000, Mean reward: 0.0, Mean regret: 1.0
Algorithm: ucb1_bandit
Iterations: 1000, Mean reward: 846.12, Mean regret: 0.15328999999999926
Iterations: 10000, Mean reward: 9634.36, Mean regret: 0.03638260000000045
Iterations: 100000, Mean reward: 99418.14, Mean regret: 0.005800630000000247
Algorithm: thompson_sampling_bandit
Iterations: 1000, Mean reward: 986.08, Mean regret: 0.014254999999999999
Iterations: 10000, Mean reward: 9986.1, Mean regret: 0.0013903
Iterations: 100000, Mean reward: 99986.17, Mean regret: 0.0001377999999999999
Testing on Five-armed bandit:
Algorithm: greedy_bandit
Iterations: 1000, Mean reward: 299.48, Mean regret: 0.5500000000000044
Iterations: 10000, Mean reward: 3007.03, Mean regret: 0.5500000000001022
Iterations: 100000, Mean reward: 29998.07, Mean regret: 0.5500000000008705
Algorithm: ucb1_bandit
Iterations: 1000, Mean reward: 790.84, Mean regret: 0.05796530000000053
Iterations: 10000, Mean reward: 8298.44, Mean regret: 0.01960542000000157
Iterations: 100000, Mean reward: 84448.09, Mean regret: 0.005631584999998115
Algorithm: thompson_sampling_bandit
Iterations: 1000, Mean reward: 830.4, Mean regret: 0.019289899999999863
Iterations: 10000, Mean reward: 8453.91, Mean regret: 0.004518010000000033
Iterations: 100000, Mean reward: 84918.16, Mean regret: 0.0008003910000000199
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Testing on Eleven-armed bandit:
Algorithm: greedy_bandit
Algorithm: ucb1_bandit
Algorithm: thompson_sampling_bandit
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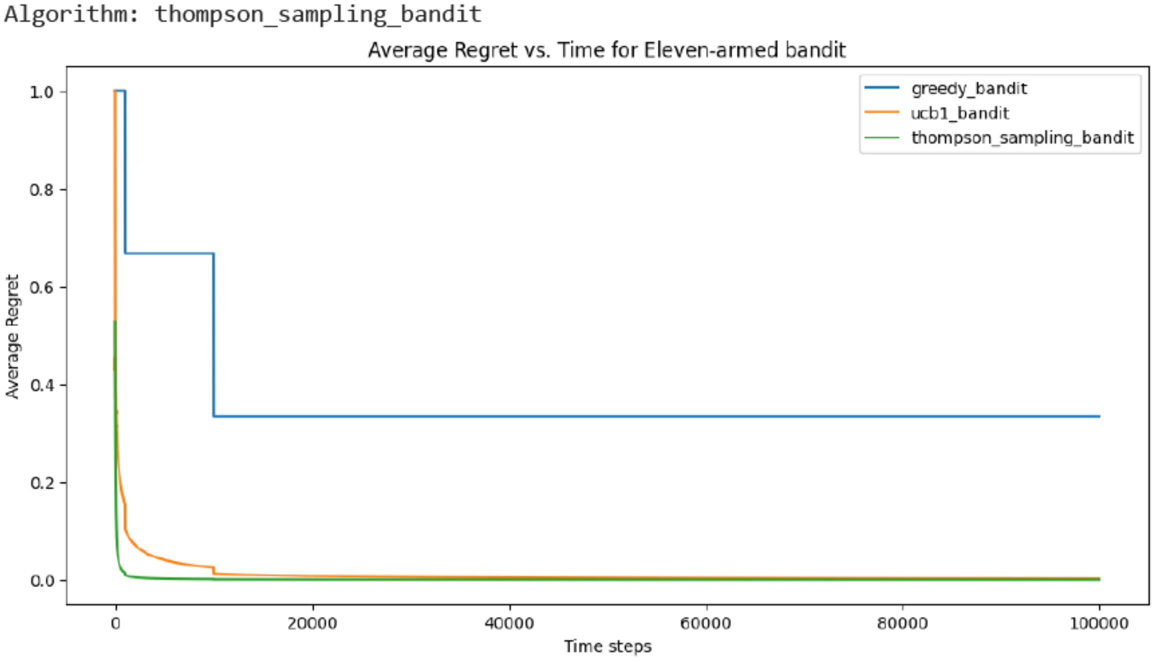

Based on the graphs and results, we can analyze the properties of the three different algorithms: Greedy, UCB1, and Thompson Sampling.

1. Average Regret vs. Time:

    - For the eleven-armed bandit setting, the Greedy algorithm performs poorly, with a constant high regret throughout the time steps. UCB1 and Thompson Sampling algorithms have significantly lower regret, with Thompson Sampling having the lowest regret overall.
    - For the five-armed bandit setting, the Greedy algorithm still performs poorly compared to UCB1 and Thompson Sampling. However, the difference in regret between UCB1 and Thompson Sampling is smaller compared to the eleven-armed bandit case.
2. Action Selection Over Time:
    - For the eleven-armed bandit setting, the Greedy algorithm quickly converges to selecting the arm with the highest probability (arm 10), but it takes a long time to explore and identify the optimal arm.
    - UCB1 and Thompson Sampling explore more efficiently and converge faster to the optimal arm (arm 10) compared to the Greedy algorithm.
    - For the five-armed bandit setting, all three algorithms converge to the optimal arm (arm 4) relatively quickly, but UCB1 and Thompson Sampling still explore more efficiently and converge faster than the Greedy algorithm.

Interesting Insights:

1. The Greedy algorithm performs poorly in both settings, as it lacks an exploration mechanism and can get stuck on sub-optimal arms, leading to high regret.
2. UCB1 and Thompson Sampling outperform the Greedy algorithm by balancing exploration and exploitation effectively. They have lower regret and converge faster to the optimal arm.
3. Thompson Sampling generally performs better than UCB1, especially in the eleven-armed bandit setting, where the number of arms is larger. This suggests that Thompson Sampling is more efficient in exploring and identifying the optimal arm in complex environments with more choices.
4. The difference in performance between UCB1 and Thompson Sampling is more pronounced in the eleven-armed bandit setting compared to the five-armed bandit setting. This indicates that as the number of arms increases, the advantage of Thompson Sampling over UCB1 becomes more significant.
5. The shape of the average regret curves for UCB1 and Thompson Sampling suggests that they have a logarithmic regret bound, which is a desirable property for bandit algorithms.

Overall, the results demonstrate the superiority of UCB1 and Thompson Sampling over the Greedy algorithm in multi-armed bandit problems, with Thompson Sampling having a slight edge, especially in more complex environments with a larger number of arms.