Lab 3: Projectile motion under the action of air resistance

Introduction:

Most introductory physics courses spend a considerable amount of time studying the motion of projectiles but almost always ignore the air resistance that inevitably impacts the motion of these objects. In many problems this is an excellent approximation; in others, air resistance is obviously very important and we need to know how to account for it. In this computational experiment, you will investigate air-resistance effects in the motion of falling objects and projectiles. By solving 2^{nd} Newton's law numerically, you will be able to test how good or how crude an approximation it is to neglect the influence of air resistance.

Air resistance is clearly dependent on the velocity with which the object moves, i.e., the higher the velocity, the higher the resistance. Mathematically, the resistive force represented by the vector **F** can be written as **F**=-f(v) **u**, where **u**=**V**/**|V|** is the unit vector along the direction of the velocity **V**. The minus sign guarantees that the air resistance acting on the projectile is always opposite to the direction of its velocity. The function f(V) is a positive quantity that describes how the magnitude of the air-resistance force depends on the magnitude of the velocity **V**. It is often a good approximation to write f(V) as $f(V) = b\,V + c\,V^2$. The coefficients b and c depend on the size and shape of the object. In the case of spherical objects, $b = B\,D$ and $c = C\,D^2$, where D denotes the diameter of the spherical object and the coefficients B and C depend on the nature of the medium. For a spherical projectile in air $B = 1.6 \times 10^{-4}$ N s/m^2 and $C = 0.25$ N s^2/m^4.

Exercise 1: How does the air resistance scale with the velocity?

Note that depending on the diameter and velocity of the projectile we may simplify the function f(V) by neglecting the linear or the quadratic term. In order to establish whether the linear or the quadratic terms can be neglected, you should:

(a) *Write a simple code that plots the function f(V) as a function of the velocity magnitude. Note that the function f(V) actually scales with the product D x V, so it is instructive to plot the separate contributions to f(V) as a function of D x V.*
In other words, plot the quantities bV and cV^2, both as a function of D x V.
By comparing their relative magnitudes, establish the range of values of D x V for which the linear term can be neglected, the range for which the quadratic term becomes negligible and the range for which both terms must be included.

(b) *Identify the ideal form for f(V) in the case of a baseball of diameter D=7cm traveling at a speed of V=5m/s; of a tiny drop of oil ($D=1.5 \times 10^{-6}$m) moving very slowly ($V= 5 \times 10^{-5}$ m/s); and of a raindrop of diameter D=1mm traveling at a speed of V=1 m/s*.

Exercise 2: Vertical motion under the action of air resistance

In the case of a spherical grain of dust of mass density of 2×10^3 kg/m^3 and diameter D = 10^{-4} m that is released from rest, how do we decide which approximation to take for the air resistance?
The maximum velocity reached by the particle is given by the terminal velocity V_T, which is the velocity for which the magnitude of the air resistance equals the weight force.

(a) *Compare the value of the product D x V_T with the ranges you obtained in problem 1 to convince yourself that in this case it is a good approximation to neglect the quadratic contribution to the air resistance, i.e., to assume that c=0*.

In this case, Newton's law can be expressed as

$$\frac{dV_y}{dt} = -g - \frac{b}{m} V_y$$

where V_y represents the vertical velocity of the particle and g is the acceleration due to gravity. The derivative on the left is the acceleration whereas on the right-hand side of the equation both the weight and the air-resistance forces are divided by the mass m.

(b) Write a simple code that obtains how the vertical velocity V_y of a spherical object varies with time t as it is released from rest.

The key to write codes of this type is to divide your time into small intervals Δt and assume that in the limit when Δt approaches zero all the relevant quantities are constant. In other words, you replace the differential equation above with

$$\Delta V_y = -g \Delta t - \frac{b}{m} V_y \Delta t$$

and assume that all quantities on the right-hand side of the equation are constant within the time interval Δt. It is as if the interval Δt is so small that there is not much time for the quantities on the right to vary. This will give you the change in velocity ΔV_y, which you will then use to update the velocity V_y. This has to be done repeatedly, always increasing the time t in steps of Δt and the velocity in steps of ΔV_y.

The figure below shows the graphical representation of an algorithm that might help you with the writing of your code.

- Read g, b,m, Δt, t_{max}…
- Initialize t=0,V_y=0
- Repeat until t reaches maximum value t_{max}
  - $\Delta V_y = -g\Delta t - \frac{b}{m} V_y \Delta t$
  - $V_y = V_y + \Delta V_y$
  - $t = t + \Delta t$
  - print t and V_y
- End repeat-until
- Plot V_y vs t

The code should ask for the values of *m, g, b and* Δt. Once these values are defined, the code should provide a series of values of V_y for each time t.

(c) *You should then plot graphs showing V_y as a function of time t. Repeat this procedure for grains of different masses. What happens when the mass gets very large*? *Furthermore, convince yourself that results for larger masses are similar to the cases of smaller resistances. In other words, increases in m are similar to reductions in the coefficient b.*

It turns out that there is an analytical solution to this problem and it is given by

$$V_y = \frac{mg}{b}\left(e^{-bt/m} - 1\right)$$

(d) C*ompare the results obtained with your code with those obtained by the analytical expression above*. *Plot a graph of the error and how it evolves with time.*
What can you do to improve the accuracy of your computer-generated results?

Now that you have investigated how the velocity increases with time, you should describe how the position of your projectile varies with the same quantity. Knowing that V_y=dY/dt, simply manipulate the V_y x t result obtained earlier to find how the position Y varies with time. Imagine that the grain is released from a height H=5m and calculate the time it takes the grain to reach the ground. Show that this time depends on the mass of the grain.

e) *Plot a graph of the time to reach the ground as a function of the mass of the object. What can you say about the often-quoted statement that all objects fall together with the same acceleration regardless of their masses? When is this a good approximation*?

Exercise 3: **Projectile motion under the action of air resistance - Part 1**

Consider now a spherical object launched with a velocity V forming an angle theta with the horizontal ground. In the absence of air resistance, the trajectory followed by this projectile is known to be a parabola. This follows from writing Newton's law separately for the horizontal and vertical coordinates. The former scales linearly with time whereas the latter varies quadratically. Therefore, when time is eliminated, we are left with a quadratic equation that gives rise to a parabolic trajectory. Let's see how the trajectory changes when air resistance is no longer neglected. In the case of a resistive force that grows linearly with velocity (c=0), we can still separate the motion between horizontal and vertical coordinates. 2nd Newton's law for both the horizontal and vertical coordinates become

$$\frac{dV_x}{dt} = -\frac{b}{m} V_x$$

$$\frac{dV_y}{dt} = -g - \frac{b}{m} V_y$$

The code written earlier can be applied to both directions separately, the difference being that gravity acts on the vertical direction (Y-axis) but not on the horizontal one (X-axis). Once again, you will have results relating the coordinates X and Y with the time t.

(a) Eliminate the time and plot the relationship between X and Y, which will give you the trajectory followed by the object under the action of air resistance. Superimpose this trajectory with the one which you would obtain in vacuum to see how different the two cases are.

Another well-known fact, often derived in introductory Physics courses, is that the launching angle of 45^o leads to the maximum horizontal displacement in a projectile motion. This is the case in the absence of air resistance. The question we now pose is whether this is also the case when air resistance is not neglected.

(b) You can now use your code to determine what the optimum launching angle is. How does that depend on the mass m? Plot theta_optimum as a function of m.

Exercise 4: Projectile motion under the action of air resistance - Part 2

Imagine now that you are in a situation where the air resistance depends quadratically on the velocity. In this case, b=0. Obtain the trajectory of a spherical object of diameter D launched with a speed V_0 that forms an angle theta with the horizontal ground. Pay attention to the fact that now the differential equations describing the time evolution of the velocity components are no longer decoupled. In other words, the differential equation for the horizontal component V_x depends on the vertical component V_y, and vice versa. Mathematically, we have

$$\frac{dV_x}{dt} = -\frac{c}{m}\sqrt{V_x^2+V_y^2}\,V_x \qquad \frac{dV_y}{dt} = -g-\frac{c}{m}\sqrt{V_x^2+V_y^2}\,V_y$$

a) Adapt your code to account for the quadratic dependence of the air resistance.
Once again, plot the trajectory of the projectile, this time superimposing it with the trajectory you would obtain for the linearly-dependent air resistance and with the case of no air resistance at all. This should give you 3 different trajectories.
Do that for a few different values of masses, launching angles and initial velocities.