

$P_1$ 
 $X_t \in \{A, B, C, D, E, F\}$ 
 $X_1 = A$ 
 $E_t \in \{\text{hot}, \text{cold}\}$

$X_{t+1} \backslash X_t$		A	B	C	D	E	F
$X_1$	A	0.2	0.8	0	0	0	0
$X_2$	B	0	0.2	0.8	0	0	0
$X_3$	C	0	0	0.2	0.8	0	0
$X_4$	D	0	0	0	0.2	0.8	0
$X_5$	E	0	0	0	0	0.2	0.8
$X_6$	F	0	0	0	0	0	0.2

$E_t \backslash X_t$		hot	cold
	A	1	0
	B	0	1
	C	0	1
	D	1	0
	E	0	1
	F	0	1

transition  $P(X_t | X_{t-1})$   
 observation  $P(E_t | X_t)$

(1) filtering

$$P(X_3 | \text{hot}_1, \text{cold}_2, \text{cold}_3) \propto P(\text{cold}_3 | X_3) \sum_{X_2} P(X_3 | X_2) \underbrace{P(X_2 | \text{hot}_1, \text{cold}_2)}$$

$$P(X_2 | \text{hot}_1, \text{cold}_2) \propto P(\text{cold}_2 | X_2) \sum_{X_1} P(X_2 | X_1) \underbrace{P(X_1 | \text{hot}_1)}$$

$$P(X_1 | \text{hot}_1) \propto P(\text{hot}_1 | X_1) P(X_1)$$

$$\propto [P(\text{hot}_1 | X_1=A), P(\text{hot}_1 | X_1=B), \dots, P(\text{hot}_1 | X_1=F)]$$

$$[P(X_1=A), P(X_1=B), \dots, P(X_1=F)]$$

$$\propto [1 \cdot 1, 0, 0, 0, 0, 0]$$

$$= [1, 0, 0, 0, 0, 0]$$

$$P(\text{cold}_2 | X_2) = \begin{matrix} & A & B & C & D & E & F \\ \begin{bmatrix} 0, 1, 1, 0, 1, 1 \end{bmatrix} \end{matrix}$$

We found out that only entry of B, C, E, F is not multiplied by zero.

$$P(\text{cold}_2 | X_2 = B) \cdot [P(X_2 = B | X_1 = A) P(X_1 = A | \text{hot}_1) + 0 + \dots + 0]$$

$$= 1 \cdot [0.8 \cdot 1]$$

$$= 0.8$$

$$P(\text{cold}_2 | X_2 = C) \cdot [P(X_2 = C | X_1 = A) P(X_1 = A | \text{hot}_1) + 0 + \dots + 0]$$

$$= 1 \cdot [0 \cdot 1]$$

$$= 0$$

Therefore, according to observation, only  $X_2 = B$  is not multiplied by zero.

$$P(X_2 | \text{hot}_1, \text{cold}_2) = \begin{bmatrix} A & B & C & D & E & F \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P(\text{cold}_3 | X_3) = \begin{bmatrix} A & B & C & D & E & F \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We found out that only entry of B, C, E, F is not multiplied by zero.

$$P(\text{cold}_3 | X_3 = B) [0 + P(X_3 = B | X_2 = B) P(X_2 = B | \text{hot}_1, \text{cold}_2) + 0 + \dots + 0]$$

$$= 1 \cdot [0.2 \cdot 1]$$

$$= 0.2$$

$$\begin{aligned}
 & P(\text{cold}_3 | X_3 = C) [0 + P(X_3 = C | X_2 = B) P(X_2 = B | \text{hot}_1, \text{hot}_2) + 0 + \dots + 0] \\
 &= 1 \cdot [0.8 \cdot 1] \\
 &= 0.8
 \end{aligned}$$

$$\begin{aligned}
 & P(\text{cold}_3 | X_3 = E) [0 + P(X_3 = E | X_2 = B) P(X_2 = B | \text{hot}_1, \text{hot}_2) + 0 + \dots + 0] \\
 &= 0
 \end{aligned}$$

$$P(X_3 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = [0, 0.2, 0.8, 0, 0, 0]$$

(2) Smoothing

$$P(X_2 | \text{hot}_1, \text{cold}_2, \text{cold}_3) \propto P(X_2 | \text{hot}_1, \text{cold}_2) P(\text{cold}_3 | X_2)$$

$$\text{According to (1), } P(X_2 | \text{hot}_1, \text{cold}_2) = \begin{bmatrix} A & B & C & D & E & F \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P(\text{cold}_3 | X_2) = \sum_{X_3} P(\text{cold}_3 | X_3) P(X_3 | X_2)$$

Through observation, we can see only entry "B" would not be multiplied by zero.

$$\begin{aligned}
 P(\text{cold}_3 | X_2 = B) &= P(\text{cold}_3 | X_3 = A) P(X_3 = A | X_2 = B) + P(\text{cold}_3 | X_3 = B) P(X_3 = B | X_2 = B) \\
 &\quad + P(\text{cold}_3 | X_3 = C) P(X_3 = C | X_2 = B) + P(\text{cold}_3 | X_3 = D) P(X_3 = D | X_2 = B) \\
 &\quad + 0 + \dots + 0 \\
 &= 0 + 1 \cdot 0.2 + 1 \cdot 0.8 + 0 \\
 &= 1
 \end{aligned}$$

$$P(X_2 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = [0, 1, 0, 0, 0, 0, 0]$$

(3) Prediction

$$P(\text{hot}_4 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = \sum_{X_3} P(X_3 | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(\text{hot}_4 | X_3)$$

According to (1),  $P(X_3 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = \begin{bmatrix} A & B & C & D & E & F \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \end{bmatrix}$

$$P(\text{hot}_4 | X_3) = \sum_{X_4} P(\text{hot}_4 | X_4) P(X_4 | X_3)$$

Through observation, we observe that only entry "B" "C" have non-zero value.

$$\begin{aligned} P(\text{hot}_4 | X_3 = B) &= P(\text{hot}_4 | X_4 = A) P(X_4 = A | X_3 = B) + P(\text{hot}_4 | X_4 = B) P(X_4 = B | X_3 = B) \\ &\quad + P(\text{hot}_4 | X_4 = C) P(X_4 = C | X_3 = B) + P(\text{hot}_4 | X_4 = D) P(X_4 = D | X_3 = B) \\ &\quad + 0 + \dots + 0 \\ &= 0 + 0 + 0 + \dots + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} P(\text{hot}_4 | X_3 = C) &= 0 + 0 + P(\text{hot}_4 | X_4 = C) P(X_4 = C | X_3 = C) \\ &\quad + P(\text{hot}_4 | X_4 = D) P(X_4 = D | X_3 = C) + P(\text{hot}_4 | X_4 = E) P(X_4 = E | X_3 = C) \\ &\quad + 0 \\ &= 0 + 1 \cdot 0.8 + 0 \\ &= 0.8 \end{aligned}$$

$$P(\text{hot}_4 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = 0.64$$

(4) Prediction

$$P(X_4 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = \sum_{X_3} P(X_3 | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4 | X_3)$$

According to (1),  $P(X_3 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = \begin{bmatrix} A & B & C & D & E & F \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \end{bmatrix}$

Through observation, we observe that only entry "B" "C" have non-zero value.

$$P(X_3=B | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=A | X_3=B) + P(X_3=C | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=A | X_3=C)$$
$$= 0 + 0$$

$$= 0$$

$$P(X_3=B | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=B | X_3=B) + P(X_3=C | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=B | X_3=C)$$
$$= 0.2 \cdot 0.2 + 0$$

$$= 0.04$$

$$P(X_3=B | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=C | X_3=B) + P(X_3=C | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=C | X_3=C)$$
$$= 0.2 \cdot 0.8 + 0.8 \cdot 0.2$$

$$= 0.16 + 0.16$$

$$= 0.32$$

$$P(X_3=B | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=D | X_3=B) + P(X_3=C | \text{hot}_1, \text{cold}_2, \text{cold}_3) P(X_4=D | X_3=C)$$

$$= 0 + 0.8 \cdot 0.8$$

$$= 0.64$$

$$P(X_4 | \text{hot}_1, \text{cold}_2, \text{cold}_3) = [0, 0.04, 0.32, 0.64, 0, 0]$$