

1 Riemannian Hypersurfaces

Let us summarize some facts about Riemannian Geometry:

Tangent Vectors $\mathbf{r}_j = \partial \mathbf{r} / \partial x^j$, $\mathbf{a} = a^i \mathbf{r}_i$, second derivatives: $\mathbf{r}_{ij} = \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j}$

Metric: $g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j = g_{ji}$ (symmetric).

Dot product: $\mathbf{a} \cdot \mathbf{b} = g_{ij} a^i b^j$.

Kronecker delta:

$$\delta_r^s = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } r \neq s \end{cases}$$

Permutation Symbol: Let σ be a permutation of the numbers $1, 2, \dots, n$

$$\epsilon_{\sigma_1 \sigma_2 \dots \sigma_n} = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \\ 0 & \text{if } \sigma \text{ has any duplicates} \end{cases}$$

We adopt the Range and Summation Conventions:

Definition Range Convention. When a small Latin suffix (superscript or subscript) occurs unrepeated in a term, it is understood to take all the values $1, 2, \dots, n$, where n is the number of dimensions of the space.

Definition Summation Convention. When a small Latin suffix is repeated in a term, summation with respect to that suffix is understood, the range of summation being $1, 2, \dots, n$.

Consider the determinant

$$g = \det |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ g_{21} & g_{22} & \dots & g_{2n} \\ \dots & \dots & \dots & \dots \\ g_{i1} & g_{i2} & \dots & g_{in} \\ \dots & \dots & \dots & \dots \\ g_{n1} & g_{n2} & \dots & g_{nn} \end{vmatrix} \quad (1)$$

We suppose, here and throughout, that g is not zero. Let Δ^{ij} be the cofactor of g_{ij} in this determinant, so that

$$g_{mr} \Delta^{ms} = g_{rm} \Delta^{sm} = \delta_r^s g,$$

which follows from the ordinary rules for developing a determinant.

Let us construct new quantities g^{ij} . The values of the components of g^{ij} are equal to the cofactor of the g_{ij} , divided by the full determinant g ,

$$g^{kl} = g^{-1} \Delta^{kl}. \quad (2)$$

Alternatively the cofactor of g_{ij} can be expressed in terms of g^{ij} as follows

$$\Delta^{ij} = g g^{ij}. \quad (3)$$

g^{ij} satisfies the following equations:

$$g_{ik} g^{kl} = \delta_i^l \text{ inverse} \quad (4)$$

Now g_{ij} is symmetric and we can also show that g^{ij} is symmetric. Multiply Eq. (4) by $g_{ls}g^{ir}$. The left hand side becomes

$$g_{ik}g^{kl}g_{ls}g^{ir} = g_{ki}g^{ir}g^{kl}g_{sl} = \delta_k^r g^{kl}g_{sl} = g_{sl}g^{rl},$$

while the right hand side becomes

$$\delta_i^l g_{ls}g^{ir} = g_{is}g^{ir} = g_{si}g^{ir} = \delta_s^r.$$

Therefore we obtain

$$g_{sl}g^{rl} = \delta_s^r \quad \text{inverse} \quad (5)$$

Comparing Eqs (4) and (5), we find that

$$g^{kl} = g^{lk}$$

Since g_{ij} is symmetric, it is obvious that g^{ij} should be symmetric also.

Let us examine the implications of the above applied to surfaces embedded in \mathbb{R}_n . A *curve* is defined as the totality of points given by the equations

$$x^r = f^r(u) \quad (r = 1, 2, \dots, n). \quad (6)$$

Here u is a parameter and $f^r(u)$ are n functions.

Next consider the totality of points given by

$$x^r = f^r(t^1, t^2, \dots, t^m) \quad (r = 1, 2, \dots, n), \quad (7)$$

where the t 's are parameters and $m < n$. This set of points may be called V_m , a subspace of V_n . In the case of V_{n-1} , it divides the neighboring portions of space into two parts. To see this, eliminate the parameters from (7). Since $m = n - 1$, the number of parameters is one less than the number of equations, and so, elimination gives just one equation

$$\phi(x^1, x^2, \dots, x^n) = 0. \quad (8)$$

The adjacent portion of V_n is divided into two parts for which respectively ϕ is positive and negative. V_{n-1} is commonly referred to as a *hypersurface* in V_n .

Let V_n be a region of \mathbb{R}_n , bounded by a smooth (i.e. sufficiently differentiable) surface B , which is an $n - 1$ dimensional closed manifold. We take B as an example of a hypersurface V_{n-1} . We have supposed that B is parameterized by $n - 1$ independent parameters t^α ($\alpha = 1, 2, \dots, n - 1$). Let the hypersurface B be defined implicitly by the equation (8).

On the surface of B , $x^i = x^i(t^\alpha)$, and therefore

$$\frac{\partial \phi}{\partial x^j} \frac{\partial x^j}{\partial t^\alpha} = 0. \quad (9)$$

Equation (9) may be regarded as $n - 1$ linear conditions upon the n quantities $\partial \phi / \partial x^j$. Since the matrix

$$\left[\frac{\partial x^j}{\partial t^\alpha} \right]$$

is assumed to have its full rank value $n - 1$ on B , the mutual ratios of the partials $\partial\phi/\partial x^j$ are fully determined by (9).

For future reference let us define a set of determinants with respect to the change of variables between the x^i and the t^α

$$D_k = (-1)^{k-1} \det \left[\frac{\partial(x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^n)}{\partial(t^1, \dots, t^{n-1})} \right], \quad k = 1, 2, \dots, n \quad (10)$$

The n determinants D_k satisfy the $n - 1$ linear conditions

$$D_k \frac{\partial x^k}{\partial t^\alpha} = 0, \quad (11)$$

which can be established by noting that the left hand side can be written as an n -by- n determinant which has two rows the same and therefore vanishes. In view of (9, conditions (11) imply that the $\partial\phi/\partial x^k$ and D_k are proportional

$$\frac{\partial\phi}{\partial x^k} = \alpha D_k,$$

for some scalar factor of proportionality α . Hence the $\partial\phi/\partial x^k$ determine the unit normal to B , namely

$$n_k = \frac{1}{\sqrt{(\nabla\phi)^2}} \frac{\partial\phi}{\partial x^k} \quad (12)$$

$$n_k = \frac{D_k}{\sqrt{g^{ij} D_i D_j}} = \frac{D_k}{D}, \quad (13)$$

where

$$D^2 = g^{ij} D_i D_j = g^{ij} \frac{\partial\phi}{\partial x^i} \frac{\partial\phi}{\partial x^j} \alpha^{-2}.$$

We consider an $n - 1$ dimensional hypersurface, H , embedded in a Riemannian manifold M_n (which we assume to be sufficiently differentiable). Let x^1, \dots, x^n be local coordinates in M_n and suppose that H is represented locally in the form

$$x^j = x^j(t^1, \dots, t^{n-1}), \quad j = 1, \dots, n$$

The metric tensor of M_n is denoted by g_{ij} . and the metric tensor of H by $\bar{g}_{\alpha\beta}$. These two tensors are related by

$$\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial t^\alpha} \frac{\partial x^j}{\partial t^\beta} = \frac{\partial x^i}{\partial t^\alpha} g_{ij} \frac{\partial x^j}{\partial t^\beta} \quad (14)$$

The element of area (or volume) in M_n is

$$dV = \sqrt{g} dx^1 \dots dx^n, \quad (15)$$

where g is the determinant of the metric tensor in M_n . Similarly the element of area (volume) in the hypersurface H is given by

$$dS = \sqrt{\bar{g}} dt^1 \dots dt^{n-1}. \quad (16)$$

We will use (14) to compute the element of area in H based on expansion of the determinant of \bar{g} in terms of products of the various minors in a manner similar to the Cauchy-Binet formula or the Laplace expansion of a determinant. Let S_n stand for the set of permutations of the numbers $\{1, 2, \dots, n\}$ and let P represent one of the members of S_n .

$\bar{g}_{\alpha\beta}$ is an $(n-1)$ -by- $(n-1)$ matrix and we can consider its determinant.

$$\det |\bar{g}| = \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \dots p_{n-1}} \bar{g}_{1p_1} \bar{g}_{2p_2} \dots \bar{g}_{n-1 p_{n-1}} \quad (17)$$

We can think of the right hand side of (14) as representing the matrix product of three matrices.

$$\begin{aligned} J &= \left[\frac{\partial x^i}{\partial t^\alpha} \right] \text{ is } n\text{-by-}(n-1), \\ G &= [g] \text{ is } n\text{-by-}n \end{aligned}$$

and (14) can be written in the form:

$$\begin{aligned} \bar{g}_{\alpha\beta} &= \sum_{i,j=1}^n \left[\frac{\partial x^i}{\partial t^\alpha} \right]_{\alpha i}^T [g]_{ij} \left[\frac{\partial x^j}{\partial t^\beta} \right]_{j\beta} = \frac{\partial x^i}{\partial t^\alpha} g_{ij} \frac{\partial x^j}{\partial t^\beta} \text{ summation convention on } i \text{ and } j \\ [\bar{g}_{\alpha\beta}] &= J^T G J \end{aligned}$$

Inserting this computation into (17) we have $\bar{g}_{\alpha\beta}$ is an $(n-1)$ -by- $(n-1)$ matrix and we can consider its determinant. Let $\bar{g} = \det |\bar{g}_{\alpha\beta}|$.

$$\begin{aligned} \bar{g} &= \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \dots p_{n-1}} \sum_{\substack{i_1, \dots, i_{n-1}=1 \\ k_1, \dots, k_{n-1}=1}}^n (J_{1i_1}^T G_{i_1 k_1} J_{k_1 p_1}) (J_{2i_2}^T G_{i_2 k_2} J_{k_2 p_2}) \dots (J_{n-1 i_{n-1}}^T G_{i_{n-1} k_{n-1}} J_{k_{n-1} p_{n-1}}) \\ &= \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \dots p_{n-1}} \sum_{\substack{i_1, \dots, i_{n-1}=1 \\ k_1, \dots, k_{n-1}=1}}^n (J_{1i_1}^T \dots J_{n-1 i_{n-1}}^T) (G_{i_1 k_1} \dots G_{i_{n-1} k_{n-1}}) (J_{k_1 p_1} \dots J_{k_{n-1} p_{n-1}}) \\ &= \sum_{\substack{i_1, \dots, i_{n-1}=1 \\ k_1, \dots, k_{n-1}=1}}^n (J_{1i_1}^T \dots J_{n-1 i_{n-1}}^T) (G_{i_1 k_1} \dots G_{i_{n-1} k_{n-1}}) \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \dots p_{n-1}} (J_{k_1 p_1} \dots J_{k_{n-1} p_{n-1}}) \end{aligned}$$

Examining the last term in the sum on the right hand side we find a sum over the $n-1$ set $\{p_1, p_2, \dots, p_{n-1}\}$ for a particular choice of index values: $\{k_1, \dots, k_{n-1}\}$

$$\sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \dots p_{n-1}} (J_{k_1 p_1} \dots J_{k_{n-1} p_{n-1}}) = \epsilon_{k_1 k_2 \dots k_{n-1}} |J_k| = \epsilon_{k_1 k_2 \dots k_{n-1}} (-1)^{1+k} D_k \quad (18)$$

where the index k is not included in $\{k_1, \dots, k_{n-1}\}$ and final result involves the respective determinant D_k .

At this point notice that the sum over the $n - 1$ variables k_1, k_2, \dots, k_{n-1} involves summing over all values from 1 to n for each k_i , but that these values must be distinct. If $k_i = k_j$ for some pair i, j , then the sum is zero because of the presence of the permutation symbol $\epsilon_{k_1 k_2 \dots k_{n-1}}$. This means that there are $n - 1$ distinct k values chosen from the full set of n values, therefore one of the k_i values is to be omitted in this sum. Let k stand for this missing value and J_k stand for the matrix J with the k -th row deleted. The set of k_i values in any set is also propagated on to the G_{ik} matrices and the corresponding k -th column of G will not contribute this particular set of choices. There will be n such cases and therefore we insert a sum over $k = 1, 2, \dots, n$ to account for each case in the following. The sum over the i_1, \dots, i_{n-1} indices follows that same way.

$$\begin{aligned}
\bar{g} &= \sum_{k=1}^n \sum_{\substack{i_1, \dots, i_{n-1}=1 \\ k_1, \dots, k_{n-1}=1, \neq k}}^n (J_{1i_1}^T \dots J_{n-1i_{n-1}}^T) (G_{i_1 k_1} \dots G_{i_{n-1} k_{n-1}}) \epsilon_{k_1 k_2 \dots k_{n-1}} (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^n \sum_{i_1, \dots, i_{n-1}=1, \neq i}^n (J_{1i_1}^T \dots J_{n-1i_{n-1}}^T) \epsilon_{i_1 i_2 \dots i_{n-1}} (-1)^{i+k} \text{cofactor}(g_{ik}) (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^n \sum_{i_1, \dots, i_{n-1}=1, \neq i}^n \epsilon_{i_1 i_2 \dots i_{n-1}} (J_{1i_1}^T \dots J_{n-1i_{n-1}}^T) (-1)^{i+k} \Delta^{ik} (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^n (-1)^{1+i} D_i (-1)^{i+k} \Delta^{ik} (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^n D_i g^{ik} D_k = g \sum_{i=1}^n D^i D_i = g D^2,
\end{aligned}$$

where $g = |g_{ij}|$ as before and the sum is over the n values of i and k which correspond to the rows or columns which are crossed out to form the n minors each of dimension $(n-1)$ -by- $(n-1)$. Hence we find that

$$\bar{g} = g D^2.$$

We finally derive the relationship between the element of surface are on the hyperplane H and the metric of the n -dimensional Riemann M_n space that H is embedded inside.

$$dS = \sqrt{\bar{g}} dt^1 \dots dt^{n-1} = \sqrt{g D^2} dt^1 \dots dt^{n-1} \quad (19)$$

If the $n-1$ dimensional hypersurface H which is embedded in Cartesian \mathbb{R}_n is given in parametric form as follows:

$$\mathbf{r}(x) = \{x, f(x)\}$$

Then $g_{ij} = I_n$ (n -by- n identity matrix) and J is an n -by- $n - 1$ matrix:

$$J = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \dots & \frac{\partial f}{\partial x^{n-1}} \end{bmatrix} \quad (20)$$

$$J^T = \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{\partial f}{\partial x^1} \\ 0 & 1 & \dots & 0 & \frac{\partial f}{\partial x^2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \frac{\partial f}{\partial x^n} \end{bmatrix} \quad (21)$$

Working out the minors, we find $D_n = 1$ (cross out the last row of J), and crossing out any of the other rows $i < n$ inserts a 0 in the i -th diagonal and shifts all the other rows $i + 1, \dots, n$ up one row. The resulting determinant is equal to $D_k = (-1)^{1+k} \frac{\partial f}{\partial x_k}$. Therefore we obtain

$$\bar{g} = \sum_{k=1}^n D_k = \left(\frac{\partial f}{\partial x^1}\right)^2 + \left(\frac{\partial f}{\partial x^2}\right)^2 + \dots + \left(\frac{\partial f}{\partial x^n}\right)^2 + 1 \quad (22)$$

And the resulting magnitude, given that $g_{ij} = I_n$ is $\sqrt{D^2} = \sqrt{1 + |\nabla f|^2}$ and

$$dS = \sqrt{1 + |\nabla f|^2} dt^1 \dots dt^{n-1}. \quad (23)$$

2 Application to the Green Theorem

In Riemannian Geometry the divergence of a vector \mathbf{b} is given by

$$\nabla \cdot \mathbf{b} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} b^i) \text{ summation convention}$$

Let us form the integral of the divergence of \mathbf{b} over a region R in V_n bounded by the closed surface B

$$\int_R \nabla \cdot \mathbf{b} dV = \int_R \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} b^i) dV = \int_R \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} b^i) \sqrt{g} dx^1 \dots dx^n \quad (24)$$

Now let us integrate the differentiated terms over the range of the appropriate variable for each one. We obtain an integral over the boundary surface B : for the k term this is (k not summed)

$$\int_B \sqrt{g} b^k dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^n = \int_B \sqrt{g} b^k D_k dt^1 \dots dt^{n-1}, \quad (25)$$

where D_k is given as in (10) represents the transformation of the integration variables from the x^i to the coordinates as given on B .

From the above set of equations we see that

$$\begin{aligned} \int_R \nabla \cdot \mathbf{b} dV &= \int_B b^k D_k \sqrt{g} dt^1 \dots dt^{n-1} \\ &= \int_B b^k n_k \sqrt{g} D dt^1 \dots dt^{n-1} \\ &= \int_B \mathbf{b} \cdot \mathbf{n} dS, \end{aligned}$$

where dS is given by (19).