1 Notes on Riemannian Hypersurfaces

Let us summarize some facts about Riemannian Geometry:

Tangent Vectors $\mathfrak{r}_j = \partial\mathfrak{r}/\partial x^j$, $\mathfrak{a} = a^i\mathfrak{r}_i$, second derivatives: $\mathfrak{r}_{ij} = \frac{\partial^2\mathfrak{r}}{\partial x^i \partial x^j}$
Metric: $g_{ij} = \mathfrak{r}_i \cdot \mathfrak{r}_j = g_{ji}$ (symmetric).
Dot product: $\mathfrak{a} \cdot \mathfrak{b} = g_{ij}a^i b^j$.

Kronecker delta:

$$\delta_r^s = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } r \neq s \end{cases}$$

Also, the another form of the Kronecker delta is δ_{rs} which satisfies the same cases as above.

Permutation Symbol: Let σ be a permutation in the set, S_n, of all permutations of the numbers $1, 2, ..., n$

$$\epsilon_{\sigma_1\sigma_2\ldots\sigma_n} = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \\ 0 & \text{if } \sigma \text{ has any duplicates} \end{cases}$$

Application of permutation symbols to determinant of an n-by-n matrix A

$$\det|A| = \sum_{\sigma \in S_n} \epsilon_{\sigma_1\sigma_2\ldots\sigma_n} a_{1\sigma_1} a_{2\sigma_2} \ldots a_{n\sigma_n} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{1}$$

If the rows are permuted by $\rho \in S_n$, we find

$$\epsilon_{\rho_1\rho_2\ldots\rho_n} \det|A| = \sum_{\sigma \in S_n} \epsilon_{\sigma_1\sigma_2\ldots\sigma_n} a_{\rho_1\sigma_1} a_{\rho_2\sigma_2} \ldots a_{\rho_n\sigma_n} \tag{2}$$

Cofactor of an element a_{rs},

$$M_{rs} = \text{cofactor}(a_{rs}) = \sum_{\sigma \in S_n,\ \sigma_r = s} \epsilon_{\sigma_1\sigma_2\ldots\sigma_n} a_{1\sigma_1} a_{2\sigma_2} \ldots a_{r-1\sigma_{r-1}} a_{r+1\sigma_{r+1}} \ldots . a_{n\sigma_n} \tag{3}$$

Relation between cofactors and minors obtained by taking the determinant obtained from A by striking out row r and column s from A. We denote such minor determinants by A_{rs} :

$$M_{rs} = (-1)^{r+s} A_{rs}$$

For every r, $1 \leq r \leq n$ we can write $\det|A|$ in terms of a cofactor expansion

$$\det|A| = \sum_{s=1}^{n} a_{rs} M_{rs} = \sum_{s=1}^{n} (-1)^{r+s} a_{rs} A_{rs} \tag{4}$$

Also, if there is a mismatch between the row indices between the matrix element and the cofactor or minor, i.e.,

$$\sum_{s=1}^{n} a_{ks}\, M_{rs} = \sum_{s=1}^{n} (-1)^{r+s}\, a_{ks}\, A_{rs} = 0, \text{ for } \quad k \neq r \tag{5}$$

The case $k \neq r$ corresponds to a determinant with two identical rows and hence vanishes because the rows are linearly dependent. Putting both of the above equations together we find

$$\sum_{s=1}^{n} a_{ks}\, M_{rs} = \sum_{s=1}^{n} (-1)^{r+s}\, a_{ks}\, A_{rs} = \delta_{kr} \det |A| \tag{6}$$

Schur's complement: Suppose p, q are nonnegative integers, and suppose A, B, C, D are respectively p-by-p, p-by-q, q-by-p, and q-by-q matrices of complex numbers. Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{7}$$

so that M is a $(p+q)$-by-$(p+q)$ matrix.

If D is invertible, then the Schur complement of the block D of the matrix M is the p-by-p matrix defined by

$$M/D := A - BD^{-1}C. \tag{8}$$

and

$$\det M = \det D \det(A - BD^{-1}C).$$

If A is invertible, the Schur complement of the block A of the matrix M is the q-by-q matrix defined by

$$M/A := D - CA^{-1}B. \tag{9}$$

and

$$\det M = \det A \det(D - CA^{-1}B).$$

In the case that A or D is singular, substituting a generalized inverse for the inverses on M/A and M/D yields the generalized Schur complement.

Laplace's General Expansion Theorem: Divide the set of integers $1, 2, ..., n$ into two complementary sets $\{\alpha_1, \alpha_2, ..., \alpha_p\}$ and $\{\beta_1, \beta_2, ..., \beta_q\}$ where $n = p + q$ and

$$\alpha_1 < \alpha_2 < ... < \alpha_p \text{ and } \beta_1 < \beta_2 < ... < \beta_q.$$

These two sets remained fixed throughout the following:

$$\det A = \sum_{\substack{\rho_1,\rho_2,...,\rho_p \\ \rho_1<\rho_2<...<\rho_p}} (-1)^{(\alpha_1+\alpha_2+...+\alpha_p+\rho_1+\rho_2...+\rho_p)} \begin{vmatrix} a_{\alpha_1\rho_1} & a_{\alpha_1\rho_2} & \cdots & a_{\alpha_1\rho_p} \\ a_{\alpha_2\rho_1} & a_{\alpha_2\rho_2} & \cdots & a_{\alpha_2\rho_p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\alpha_i\rho_1} & a_{\alpha_i\rho_2} & \cdots & a_{\alpha_i\rho_p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\alpha_p\rho_1} & a_{\alpha_p\rho_2} & \cdots & a_{\alpha_p\rho_p} \end{vmatrix} \begin{vmatrix} a_{\beta_1\sigma_1} & a_{\beta_1\sigma_2} & \cdots & a_{\beta_1\sigma_q} \\ a_{\beta_2\sigma_1} & a_{\beta_2\sigma_2} & \cdots & a_{\beta_2\sigma_q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\beta_i\sigma_1} & a_{\beta_i\sigma_2} & \cdots & a_{\beta_i\sigma_q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\beta_q\sigma_1} & a_{\beta_q\sigma_2} & \cdots & a_{\beta_q\sigma_q} \end{vmatrix} \tag{10}$$

The sets $\{\alpha_1, \alpha_2, ..., \alpha_p\}$ and $\{\beta_1, \beta_2, ..., \beta_q\}$ were defined to be fixed. The summation is taken over all p-tuples $(\rho_1, \rho_2, ..., \rho_p)$ from among the numbers $1, 2, ..., n$ for which $\rho_1 < \rho_2 < ... < \rho_p$ and where

$$\sigma_1 < \sigma_2 < ... < \sigma_q$$

are the remaining q integers.

Application 1: **Example Partitioned Matrix and the Cauchy-Binet Formula**

$$M = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1n} \\ 0 & 0 & \dots & 0 & b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & b_{m1} & b_{m2} & \dots & b_{mn} \\ \hline c_{11} & c_{21} & \dots & c_{m1} & 1 & 0 & \dots & 0 \\ c_{12} & c_{22} & \dots & c_{m2} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ c_{1n} & c_{2n} & \dots & c_{mn} & 0 & 0 & \dots & 1 \end{array}\right] \tag{11}$$

In the above M is an $(m+n)$-by-$(m+n)$ matrix which is partitioned into: A an m-by-m matrix of 0's, B an m-by-n matrix, C an n-by-m matrix and D is the n-by-n Identity matrix. Then using the determinant in the Schur's Complement form we have

$$\det M = \det D \det(A - BD^{-1}C) = \det(-BC) = (-1)^m \det(BC).$$

Another representation of $\det M$ can be obtained by applying Laplace's Generalized Theorem successively. First swap the bottom n rows with the top m rows (by moving each of the bottom n rows up m positions and changing the sign of $\det M$ accordingly

$$\det M = (-1)^{m^2} \det \left[\begin{array}{cccc|cccc} c_{11} & c_{21} & \dots & c_{m1} & 1 & 0 & \dots & 0 \\ c_{12} & c_{22} & \dots & c_{m2} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ c_{1n} & c_{2n} & \dots & c_{mn} & 0 & 0 & \dots & 1 \\ \hline 0 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1n} \\ 0 & 0 & \dots & 0 & b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & b_{m1} & b_{m2} & \dots & b_{mn} \end{array}\right] \tag{12}$$

Using Laplace's Generalized Theorem the first time across the bottom m rows and taking advantage of the m-by-m block of zeros in the lower left hand corner to drop some of the minors in the expansion we have only a sum over the m-by-m minors in the b_{ij} elements starting in columns $m+1$ through $m+n$ of M. After forming any one of these minors we have to select m columns to remove to form the algebraic complement to use in the Laplace Theorem. The n-by-n complementary minor consists of the n-by-m c_{ij} matrix elements with an additional $n-m$ columns which remain in from the former Identity matrix in the upper right hand side of the working matrix.

We now apply Laplace's Generalized Theorem a second time, now expanding over the first m columns of the complementary n-by$n$ minor. We must again sum over the choice of m rows from the first n rows. The complementary minor to this second set of minors is obtained from the original n-by-n Identity matrix, I_n, with m columns removed by forming the first set of minors in the b_{ij} and removing the m rows to form the second set of minors. We are left with an $n-m$-by-$n-m$ matrix from I_n. If the rows that were struck out do not match exactly the columns that were struck out then there will appear in the the reduced matrix rows and columns with all zero entries and the corresponding determinant vanishes. So in the second application of Laplace's Generalized theorem there is only one term surviving in the sum, for any given choice of m numbers $1 \le \rho_1 \le \rho_2 \le ... \le \rho_m \le n$, the only term that survive is

$$\begin{vmatrix} c_{1\rho_1} & c_{1\rho_2} & \dots & c_{1\rho_m} \\ c_{2\rho_1} & c_{2\rho_2} & \dots & c_{2\rho_m} \\ \dots & \dots & \dots & \dots \\ c_{m\rho_1} & c_{m\rho_2} & \dots & c_{m\rho_m} \end{vmatrix} \begin{vmatrix} b_{1\rho_1} & b_{1\rho_2} & \dots & b_{1\rho_m} \\ b_{2\rho_1} & b_{2\rho_2} & \dots & b_{2\rho_m} \\ \dots & \dots & \dots & \dots \\ b_{m\rho_1} & b_{m\rho_2} & \dots & b_{m\rho_m} \end{vmatrix} \tag{13}$$

Putting both results together and using the fact that both methods give an expression for $\det M$ we obtain the Cauchy-Binet Formula for matrices B (n-by-m) and C (m-by-n) with $m \le n$:

$$\det(BC) = \sum_{\substack{\rho_1,\rho_2,\dots,\rho_m=1 \\ \rho_1<\rho_2<\dots<\rho_m}}^{n} \begin{vmatrix} c_{1\rho_1} & c_{1\rho_2} & \dots & c_{1\rho_m} \\ c_{2\rho_1} & c_{2\rho_2} & \dots & c_{2\rho_m} \\ \dots & \dots & \dots & \dots \\ c_{m\rho_1} & c_{m\rho_2} & \dots & c_{m\rho_m} \end{vmatrix} \begin{vmatrix} b_{1\rho_1} & b_{1\rho_2} & \dots & b_{1\rho_m} \\ b_{2\rho_1} & b_{2\rho_2} & \dots & b_{2\rho_m} \\ \dots & \dots & \dots & \dots \\ b_{m\rho_1} & b_{m\rho_2} & \dots & b_{m\rho_m} \end{vmatrix} \tag{14}$$

Application 2: **Alternate Derivation of the Cauchy-Binet Formula**

Let A be an m-by-n matrix and let B be an n-by-m matrix. Let $1 \le j_1, j_2, ..., j_m \le n$. Let $A_{j_1 j_2 \dots j_m}$ denote the m-by-m matrix consisting of the m columns $j_1, j_2, ..., j_m$ of A. Let $B_{j_1 j_2 \dots j_m}$ denote the m-by-m matrix consisting of the m rows $j_1, j_2, ..., j_m$ of B. Let $\{k_1, k_2, ..., k_m\}$ be an ordered m-tuple of integers and let $\{j_1, j_2, ..., j_m\}$ be the same set of integers but arranged into non-decreasing order:

$$j_1 \le j_2 \le ... \le j_m.$$

Then we have the relationships that

$$\begin{aligned} \det\left(B_{k_1 k_2 \dots k_m}\right) &= \epsilon_{k_1 k_2 \dots k_m} \det\left(B_{j_1 j_2 \dots j_m}\right), \\ \det\left(B_{j_1 j_2 \dots j_m}\right) &= \epsilon_{k_1 k_2 \dots k_m} \det\left(B_{k_1 k_2 \dots k_m}\right) \text{ no implied sum on } k_i's. \end{aligned}$$

Now from the definition of determinant for the m-by-m matrix AB we have

$$\begin{aligned} \det(AB) &= \sum_{1\le l_1,\dots l_m \le m} \epsilon_{l_1 l_2 \dots l_m} \left(\sum_{k=1}^{n} a_{1k} b_{kl_1}\right) ... \left(\sum_{k=1}^{n} a_{mk} b_{kl_m}\right) \\ &= \sum_{1\le k_1,\dots k_m \le n} \left(a_{1k_1} ... a_{mk_m}\right) \sum_{1\le l_1,\dots l_m \le m} \epsilon_{l_1 l_2 \dots l_m} b_{k_1 l_1} ... b_{k_m l_m} \\ &= \sum_{1\le k_1,\dots k_m \le n} \left(a_{1k_1} ... a_{mk_m}\right) \det\left(B_{k_1 \dots k_m}\right) \\ &= \sum_{1\le k_1,\dots k_m \le n} \left(a_{1k_1} ... a_{mk_m} \epsilon_{k_1 \dots k_m}\right) \det\left(B_{j_1 \dots j_m}\right) \\ &= \sum_{1\le k_1,\dots k_m \le n} \left(\epsilon_{k_1 \dots k_m}\, a_{1k_1} ... a_{mk_m}\right) \det\left(B_{j_1 \dots j_m}\right) \\ &= \sum_{1\le j_1 \le j_2 \le \dots \le j_m \le n} \det\left(A_{j_1 \dots j_m}\right) \det\left(B_{j_1 \dots j_m}\right) \end{aligned}$$

If any two j's are equal then $\det(A_{j_1\cdots j_m}) = 0$

In the case of A and $B = A^T$ we immediately have

$$\det(AA^T) = \sum_{1\le j_1\le j_2\le\cdots\le j_m\le n} [\det(A_{j_1\cdots j_m})]^2$$

Conventions: We adopt the Range and Summation Conventions:

Definition Range Convention. *When a small Latin suffix (superscript or subscript) occurs unrepeated in a term, it is understood to take all the values $1, 2, ..., n$, where n is the number of dimensions of the space.*

Definition Summation Convention. *When a small Latin suffix is repeated in a term, summation with respect to that suffix is understood, the range of summation being $1, 2, ..., n$.*

Consider the determinant

$$g = \det|g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{i1} & g_{i2} & \cdots & g_{in} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix} \tag{15}$$

We suppose, here and throughout, that g is not zero. Let Δ^{ij} be the cofactor of g_{ij} in this determinant, so that

$$g_{mr}\Delta^{ms} = g_{rm}\Delta^{sm} = \delta^s_r\, g,$$

which follows from the ordinary rules for developing a determinant.

Let us construct new quantities g^{ij}. The values of the components of g^{ij} are equal to the cofactor of the g_{ij}, divided by the full determinant g,

$$g^{kl} = g^{-1}\Delta^{kl}. \tag{16}$$

Alternatively the cofactor of g_{ij} can be expressed in terms of g^{ij} as folllows

$$\Delta^{ij} = g\, g^{ij}. \tag{17}$$

g^{ij} satisfies the following equations:

$$g_{ik}g^{kl} = \delta^l_i \;\; \text{inverse} \tag{18}$$

Now g_{ij} is symmetric and we can also show that g^{ij} is symmetric. Multiply Eq. (18) by $g_{ls}g^{ir}$. The left hand side becomes

$$g_{ik}g^{kl}g_{ls}g^{ir} = g_{ki}g^{ir}g^{kl}g_{sl} = \delta^r_k g^{kl}g_{sl} = g_{sl}g^{rl},$$

while the right hand side becomes

$$\delta^l_i g_{ls}g^{ir} = g_{is}g^{ir} = g_{si}g^{ir} = \delta^r_s.$$

Therefore we obtain

$$g_{sl}g^{rl} = \delta^r_s \;\; \text{inverse} \tag{19}$$

Comparing Eqs (18) and (19), we find that

$$g^{kl} = g^{lk}$$

Since g_{ij} is symmetric, it is obvious that g^{ij} should be symmetric also.

Let us examine the implications of the above applied to surfaces embedded in $\mathbb{R}_n$. A *curve* is defined as the totality of points given by the equations

$$x^r = f^r(u) \quad (r = 1, 2, ..., n). \tag{20}$$

Here u is a parameter and $f^r(u)$ are n functions.

Next consider the totality of points given by

$$x^r = f^r(t^1, t^2, ..., t^m) \quad (r = 1, 2, ..., n), \tag{21}$$

where the t's are parameters and $m < n$. This set of points may be called V_m, a subspace of V_n. In the case of $m = n - 1$, it divides the neighboring portions of space into two parts. To see this, eliminate the parameters from (21). The number or parameters is one less than the number of equations, and so, elimination gives just one equation

$$\phi(x^1, x^2, ..., x^n) = 0. \tag{22}$$

The adjacent portion of V_n is divided into two parts for which respectivel ϕ is positive and negative. V_{n-1} is commonly referred to as a *hypersurface* in V_n.

Let V_n be a region of $\mathbb{R}_n$, bounded by a smooth (i.e. sufficiently differentiable) surface B, which is an $n-1$ dimensional closed manifold. We take B as an example of a hypersurface V_{n-1}. We has suppose that B is parameterized by $n - 1$ independent parameters $t^\alpha(\alpha = 1, 2, ..., n - 1)$. Let the hypersurface B be defined implicitly by the equation (22).

On the surface of B, $x^i = x^i(t^\alpha)$, and therefore

$$\frac{\partial \phi}{\partial x^j} \frac{\partial x^j}{\partial t^\alpha} = 0. \tag{23}$$

Equation (23) may be regarded as $n - 1$ linear conditions upon the n quantities $\partial\phi/\partial x^j$. Since the matrix

$$\left[\frac{\partial x^j}{\partial t^a}\right]$$

is assumed to have its full rank value $n - 1$ on B, the mutual ratios of the partials $\partial\phi/\partial x^j$ are fully determined by (23).

For future reference let us define a set of determinants with respect to the change of variables between the x^i and the t^α

$$D_k = (-1)^{k-1} \det \left[\frac{\partial(x^1, ..., x^{k-1}, x^{k+1}, ..., x^n)}{\partial(t^1, ..., t^{n-1})}\right], \quad k = 1, 2, ..., n \tag{24}$$

The n determinants D_k satisfy the $n-1$ linear conditions

$$D_k \frac{\partial x^k}{\partial t^\alpha} = 0, \tag{25}$$

which can be established by noting that the left hand side can be written as an n-by-n determinant which has two rows the same and therefore vanishes. In view of (23, conditions (25) imply that the $\partial\phi/\partial x^k$ and D_k are proportional

$$\frac{\partial \phi}{\partial x^k} = \alpha D_k,$$

for some scalar factor of proportionality α. Hence the $\partial\phi/\partial x^k$ determine the unit normal to B, namely

$$n_k = \frac{1}{\sqrt{(\nabla\phi)^2}} \frac{\partial \phi}{\partial x^k} \tag{26}$$

$$n_k = \frac{D_k}{\sqrt{g^{ij} D_i D_j}} = \frac{D_k}{D}, \tag{27}$$

where

$$D^2 = g^{ij} D_i D_j = g^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \alpha^{-2}.$$

2 Hypersurfaces and Concepts of Area

Consider an $n-1$ dimensional hypersurface, B_{n-1}, embedded in a Riemannian manifold M_n (which we assume to be sufficiently differentiable). Let $x^1, ..., x^n$ be local coordinates in M_n and suppose that B_{n-1} is represented locally in the form

$$x^j = x^j(t^1, ..., t^{n-1}), \quad j = 1, ..., n$$

The metric tensor of M_n is denoted by g_{ij}. and the metric tensor on B_{n-1} by $\overline{g}_{\alpha\beta}$. These two tensors are related by

$$\overline{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial t^\alpha} \frac{\partial x^j}{\partial t^\beta} = \frac{\partial x^i}{\partial t^\alpha} g_{ij} \frac{\partial x^j}{\partial t^\beta} \tag{28}$$

The element of area (or volume) in M_n is

$$dV = \sqrt{g}\, dx^1 ... dx^n, \tag{29}$$

where g is the determinant of the metric tensor in M_n. Similarly the element of area (volume) in the hypersurface B_{n-1} is given by

$$dS = \sqrt{\overline{g}}\, dt^1 ... dt^{n-1}. \tag{30}$$

We will use (28) to define the element of area in B_{n-1} based on expansion of the determinant of $\overline{g}$ in terms of products of the various minors in a manner similar to the Cauchy-Binet formula or the Laplace expansion of a determinant. Let S_n stand for the set of permutations of the numbers $\{1, 2...., n\}$ and let P represent one of the members of S_n.

$\overline{g}_{\alpha\beta}$ is an (n-1)-by-(n-1) matrix and we can consider its determinant.

$$\overline{g} = \det|\overline{g}| = \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} \overline{g}_{1p_1} \overline{g}_{2p_2} \ldots \overline{g}_{n-1 p_{n-1}} \tag{31}$$

We can think of the right hand side of (28) as representing the matrix product of three matrices.

$$\begin{aligned} J &= \left[\frac{\partial x^i}{\partial t^\alpha}\right] \text{ is } n\text{-by-}(n-1), \\ G &= \Big[g\Big] \text{ is } n\text{-by-}n \end{aligned}$$

and (28) can be written in the form:

$$\begin{aligned} \overline{g}_{\alpha\beta} &= \sum_{i,j=1}^{n} \left[\frac{\partial x^i}{\partial t^\alpha}\right]^T_{\alpha i} [g]_{ij} \left[\frac{\partial x^i}{\partial t^\beta}\right]_{j\beta} = \frac{\partial x^i}{\partial t^\alpha} g_{ij} \frac{\partial x^j}{\partial t^\beta} \text{ summation convention on } i \text{ and } j \\ \Big[\overline{g}_{\alpha\beta}\Big] &= J^T G J \end{aligned}$$

Inserting this computation into (31) we have $\overline{g}_{\alpha\beta}$ is an (n-1)-by-(n-1) matrix and we can consider its determinant. Let $\overline{g} = \det|\overline{g}_{\alpha\beta}|$.

$$\begin{aligned} \overline{g} &= \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,\ldots,k_{n-1}=1}}^{n} (J^T_{1i_1} G_{i_1 k_1} J_{k_1 p_1})(J^T_{2i_2} G_{i_2 k_2} J_{k_2 p_2}) \ldots (J^T_{n-1 i_{n-1}} G_{i_{n-1} k_{n-1}} J_{k_{n-1} p_{n-1}}) \\ &= \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,\ldots,k_{n-1}=1}}^{n} (J^T_{1i_1} \ldots J^T_{n-1 i_{n-1}})(G_{i_1 k_1} \ldots G_{i_{n-1} k_{n-1}})(J_{k1 p_1} \ldots J_{k_{n-1} p_{n-1}}) \\ &= \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,\ldots,k_{n-1}=1}}^{n} (J^T_{1i_1} \ldots J^T_{n-1 i_{n-1}})(G_{i_1 k_1} \ldots G_{i_{n-1} k_{n-1}}) \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} (J_{k1 p_1} \ldots J_{k_{n-1} p_{n-1}}) \end{aligned}$$

Examining the last term in the sum on the right hand side we find a sum over the $n-1$ set $\{p1, p2, ..., p_{n-1}\}$ for a particular choice of index values: $\{k1, ...k_{n-1}\}$

$$\sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} (J_{k1 p_1} \ldots J_{k_{n-1} p_{n-1}}) = \epsilon_{k_1 k_2 \ldots k_{n-1}} |J_k| = \epsilon_{k_1 k_2 \ldots k_{n-1}} (-1)^{1+k} D_k \tag{32}$$

where the index k is not included in $\{k1, ...k_{n-1}\}$ and final result involves the respective determinant D_k.

At this point notice that the sum over the $n-1$ variables $k1, k2, ..., k_{n-1}$ involves summing over all values from 1 to n for each k_i, but that these values must be distinct. If $k_i = k_j$ for some pair i, j, then the sum is zero because of the presence of the permutation symbol $\epsilon_{k_1 k_2 \ldots k_{n-1}}$. This means that there are $n-1$ distinct k values chosen from the full set of n values, therefore one of the k_i values is to be omitted in this sum. Let k stand for this missing value and J_k stand for the matrix J with the

k-th row deleted. The set of k_i values in any set is also propagated on to the G_{ik} matrices and the corresponding k-th column of G will not contribute this particular set of choices. There will be n such cases and therefore we insert a sum over $k = 1, 2, ..., n$ to account for each case in the following. The sum over the $i_1, ..., i_{n-1}$ indices follows that same way.

$$\begin{aligned}
\overline{g} &= \sum_{k=1}^{n} \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,...,k_{n-1}=1,\neq k}}^{n} (J^T_{1i_1}...J^T_{n-1i_{n-1}})(G_{i_1k_1}...G_{i_{n-1}k_{n-1}})\epsilon_{k_1k_2...k_{n-1}}(-1)^{1+k}D_k \\
&= \sum_{i,k=1}^{n} \sum_{i1,..,i_{n-1}=1,\neq i}^{n} (J^T_{1i_1}...J^T_{n-1i_{n-1}})\epsilon_{i_1i_2...i_{n-1}}(-1)^{i+k}\text{cofactor}(g_{ik})(-1)^{1+k}D_k \\
&= \sum_{i,k=1}^{n} \sum_{i1,..,i_{n-1}=1,\neq i}^{n} \epsilon_{i_1i_2...i_{n-1}}(J^T_{1i_1}...J^T_{n-1i_{n-1}})(-1)^{i+k}\Delta^{ik}(-1)^{1+k}D_k \\
&= \sum_{i,k=1}^{n} (-1)^{1+i}D_i(-1)^{i+k}\Delta^{ik}(-1)^{1+k}D_k \\
&= \sum_{i,k=1}^{n} D_i\, g\, g^{ik}\, D_k = g\sum_{i=1}^{n} D^k\, D_k = g\, D^2,
\end{aligned}$$

where $g = |g_{ij}|$ as before and the sum is over the n values of i and k which correspond to the rows or columns which are crossed out to form the n minors each of dimension (n-1)-by-(n-1). Hence we find that

$$\overline{g} = g\, D^2.$$

We finally derive the relationship between the element of surface are on the hyperplane H and the metric of the n-dimensional Riemann M_n space that H is embedded inside.

$$dS = \sqrt{\overline{g}}\, dt^1...dt^{n-1} = \sqrt{g\, D^2}\, dt^1...dt^{n-1} \tag{33}$$

If the n-1 dimensional hypersurface H which is embedded in Cartesian $\mathbb{R}_n$ is given in parametric form as follows:

$$\mathfrak{r}(x) = \{x, f(x)\}$$

Then $g_{ij} = I_n$ (n-by-n identity matrix) and J is an n-by-$n-1$ matrix:

$$J = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \dots & \frac{\partial f}{\partial x^{n-1}} \end{bmatrix} \tag{34}$$

$$J^T = \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{\partial f}{\partial x^1} \\ 0 & 1 & \dots & 0 & \frac{\partial f}{\partial x^2} \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 1 & \frac{\partial f}{\partial x^n} \end{bmatrix} \tag{35}$$

Working out the minors, we find $D_n = 1$ (cross out the last row of J), and crossing out any of the other rows $i < n$ inserts a 0 in the i-th diagonal and shifts all the other rows $i+1, ..., n$ up one row. The resulting determinant is equal to $D_k = (-1)^{1+k}\frac{\partial f}{\partial x_k}$
Therefore we obtain

$$\overline{g} = \sum_{k=1}^{n} D_k = (\frac{\partial f}{\partial x^1})^2 + (\frac{\partial f}{\partial x^2})^2 + ... + (\frac{\partial f}{\partial x^n})^2 + 1 \tag{36}$$

And the resulting magnitude, given that $g_{ij} = I_n$ is $\sqrt{D^2} = \sqrt{1 + |\nabla f|^2}$ and

$$dS = \sqrt{1 + |\nabla f|^2}\; dt^1 ... dt^{n-1}. \tag{37}$$

3 Application to the Green Theorem

In Riemannian Geometry the divergence of a vector $\mathfrak{b}$ is given by

$$\nabla \cdot \mathfrak{b} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, b^i) \text{ summation convention}$$

Let us form the integral of the divergence of $\mathfrak{b}$ over a region R in V_n bounded by the closed surface B_{n-1}

$$\int_R \nabla \cdot \mathfrak{b}\, dV = \int_R \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, b^i)\, dV = \int_R \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, b^i)\, \sqrt{g}\, dx^1 ... dx^n \tag{38}$$

Now let us integrate the differentiated terms over the range of the appropriate variable for each one. We obtain an integral over the boundary surface B_{n-1}: for the k term this is (k not summed)

$$\int_B \sqrt{g}\, b^k dx^1 ... dx^{k-1} dx^{k+1} ... dx^n = \int_B \sqrt{g}\, b^k D_k\, dt^1 ... dt^{n-1}, \tag{39}$$

where D_k is given as in (24) represents the transformation of the integration variables from the x^i to the coordinates as given on B_{n-1}.

From the above set of equations we see that

$$\begin{aligned} \int_{R_n} \nabla \cdot \mathfrak{b}\, dV_n &= \int_{B_{n-1}} b^k D_k \sqrt{g}\, dt^1 ... dt^{n-1} \\ &= \int_{B_{n-1}} b^k n_k \sqrt{g}\, D\;\; dt^1 ... dt^{n-1} \\ &= \int_{B_{n-1}} \mathfrak{b} \cdot \mathfrak{n}\, dS, \end{aligned}$$

where dS is given by (33).