1 Riemannian Hypersurfaces

Let us summarize some facts about Riemannian Geometry:

Tangent Vectors $\mathfrak{r}_j = \partial \mathfrak{r}/\partial x^j$, $\mathfrak{a} = a^i \mathfrak{r}_i$, second derivatives: $\mathfrak{r}_{ij} = \frac{\partial^2 \mathfrak{r}}{\partial x^i \partial x^j}$
Metric: $g_{ij} = \mathfrak{r}_i \cdot \mathfrak{r}_j = g_{ji}$ (symmetric).
Dot product: $\mathfrak{a} \cdot \mathfrak{b} = g_{ij} a^i b^j$.

Kronecker delta:

$$\delta_r^s = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } r \neq s \end{cases}$$

Permutation Symbol: Let σ be a permutation of the numbers $1, 2, ..., n$

$$\epsilon_{\sigma_1 \sigma_2 \ldots \sigma_n} = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \\ 0 & \text{if } \sigma \text{ has any duplicates} \end{cases}$$

We adopt the Range and Summation Conventions:

Definition Range Convention. *When a small Latin suffix (superscript or subscript) occurs unrepeated in a term, it is understood to take all the values $1, 2, ..., n$, where n is the number of dimensions of the space.*

Definition Summation Convention. *When a small Latin suffix is repeated in a term, summation with respect to that suffix is understood, the range of summation being $1, 2, ..., n$.*

Consider the determinant

$$g = \det |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{i1} & g_{i2} & \cdots & g_{in} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix} \tag{1}$$

We suppose, here and throughout, that g is not zero. Let Δ^{ij} be the cofactor of g_{ij} in this determinant, so that

$$g_{mr} \Delta^{ms} = g_{rm} \Delta^{sm} = \delta_r^s \, g,$$

which follows from the ordinary rules for developing a determinant.

Let us construct new quantities g^{ij}. The values of the components of g^{ij} are equal to the cofactor of the g_{ij}, divided by the full determinant g,

$$g^{kl} = g^{-1} \Delta^{kl}. \tag{2}$$

Alternatively the cofactor of g_{ij} can be expressed in terms of g^{ij} as folllows

$$\Delta^{ij} = g \, g^{ij}. \tag{3}$$

g^{ij} satisfies the following equations:

$$g_{ik} g^{kl} = \delta_i^l \quad \text{inverse} \tag{4}$$

Now g_{ij} is symmetric and we can also show that g^{ij} is symmetric. Multiply Eq. (4) by $g_{ls}g^{ir}$. The left hand side becomes

$$g_{ik}g^{kl}g_{ls}g^{ir} = g_{ki}g^{ir}g^{kl}g_{sl} = \delta^r_k g^{kl}g_{sl} = g_{sl}g^{rl},$$

while the right hand side becomes

$$\delta^l_i g_{ls}g^{ir} = g_{is}g^{ir} = g_{si}g^{ir} = \delta^r_s.$$

Therefore we obtain

$$g_{sl}g^{rl} = \delta^r_s \quad \text{inverse} \tag{5}$$

Comparing Eqs (4) and (5), we find that

$$g^{kl} = g^{lk}$$

Since g_{ij} is symmetric, it is obvious that g^{ij} should be symmetric also.

Let us examine the implications of the above applied to surfaces embedded in $\mathbb{R}_n$. A *curve* is defined as the totality of points given by the equations

$$x^r = f^r(u) \quad (r = 1, 2, ..., n). \tag{6}$$

Here u is a parameter and $f^r(u)$ are n functions.

Next consider the totality of points given by

$$x^r = f^r(t^1, t^2, ..., t^m) \quad (r = 1, 2, ..., n), \tag{7}$$

where the t's are parameters and $m < n$. This set of points may be called V_m, a subspace of V_n. In the case of V_{n-1}, it divides the neighboring portions of space into two parts. To see this, eliminate the parameters from (7). Since $m = n - 1$, the number or parameters is one less than the number of equations, and so, elimination gives just one equation

$$\phi(x^1, x^2, ..., x^n) = 0. \tag{8}$$

The adjacent portion of V_n is divided into two parts for which respectivel6 ϕ is positive and negative. V_{n-1} is commonly referred to as a *hypersurface* in V_n.

Let V_n be a region of $\mathbb{R}_n$, bounded by a smooth (i.e. sufficiently differentiable) surface B, which is an $n-1$ dimensional closed manifold. We take B as an example of a hypersurface V_{n-1}. We has suppose that B is parameterized by $n-1$ independent parameters $t^\alpha (\alpha = 1, 2, ..., n-1)$. Let the hypersurface B be defined implicitly by the equation (8).

On the surface of B, $x^i = x^i(t^\alpha)$, and therefore

$$\frac{\partial \phi}{\partial x^j}\frac{\partial x^j}{\partial t^\alpha} = 0. \tag{9}$$

Equation (9) may be regarded as $n-1$ linear conditions upon the n quantities $\partial\phi/\partial x^j$. Since the matrix

$$\left[\frac{\partial x^j}{\partial t^a}\right]$$

is assumed to have its full rank value $n-1$ on B, the mutual ratios of the partials $\partial\phi/\partial x^j$ are fully determined by (9).

For future reference let us define a set of determinants with respect to the change of variables between the x^i and the t^α

$$D_k = (-1)^{k-1} \det \left[\frac{\partial(x^1, ..., x^{k-1}, x^{k+1}, ..., x^n)}{\partial(t^1, ..., t^{n-1})} \right], \quad k = 1, 2, ..., n \tag{10}$$

The n determinants D_k satisfy the $n-1$ linear conditions

$$D_k \frac{\partial x^k}{\partial t^\alpha} = 0, \tag{11}$$

which can be established by noting that the left hand side can be written as an n-by-n determinant which has two rows the same and therefore vanishes. In view of (9, conditions (11) imply that the $\partial\phi/\partial x^k$ and D_k are proportional

$$\frac{\partial\phi}{\partial x^k} = \alpha D_k,$$

for some scalar factor of proportionality α. Hence the $\partial\phi/\partial x^k$ determine the unit normal to B, namely

$$n_k = \frac{1}{\sqrt{(\nabla\phi)^2}} \frac{\partial\phi}{\partial x^k} \tag{12}$$

$$n_k = \frac{D_k}{\sqrt{g^{ij} D_i D_j}} = \frac{D_k}{D}, \tag{13}$$

where

$$D^2 = g^{ij} D_i D_j = g^{ij} \frac{\partial\phi}{\partial x^i} \frac{\partial\phi}{\partial x^j} \alpha^{-2}.$$

We consider an $n-1$ dimensional hypersurface, H, embedded in a Riemannian manifold M_n (which we assume to be sufficiently differentiable). Let $x^1, ..., x^n$ be local coordinates in M_n and suppose that H is represented locally in the form

$$x^j = x^j(t^1, ..., t^{n-1}), \quad j = 1, ..., n$$

The metric tensor of M_n is denoted by g_{ij}. and the metric tensor of H by $\overline{g}_{\alpha\beta}$. These two tensors are related by

$$\overline{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial t^\alpha} \frac{\partial x^j}{\partial t^\beta} = \frac{\partial x^i}{\partial t^\alpha} g_{ij} \frac{\partial x^j}{\partial t^\beta} \tag{14}$$

The element of area (or volume) in M_n is

$$dV = \sqrt{g}\, dx^1 ... dx^n, \tag{15}$$

where g is the determinant of the metric tensor in M_n. Similarly the element of area (volume) in the hypersurface H is given by

$$dS = \sqrt{\overline{g}}\, dt^1 ... dt^{n-1}. \tag{16}$$

We will use (14) to compute the element of area in H based on expansion of the determinant of $\overline{g}$ in terms of products of the various minors in a manner similar to the Cauchy-Binet formula or the Laplace expansion of a determinant. Let S_n stand for the set of permutations of the numbers $\{1, 2...., n\}$ and let P represent one of the members of S_n.
$\overline{g}_{\alpha\beta}$ is an (n-1)-by-(n-1) matrix and we can consider its determinant.

$$\det|\overline{g}| = \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} \overline{g}_{1p_1} \overline{g}_{2p_2} \ldots \overline{g}_{n-1 p_{n-1}} \tag{17}$$

We can think of the right hand side of (14) as representing the matrix product of three matrices.

$$\begin{aligned} J &= \left[\frac{\partial x^i}{\partial t^\alpha}\right] \text{ is } n\text{-by-}(n-1), \\ G &= \left[g\right] \text{ is } n\text{-by-}n \end{aligned}$$

and (14) can be written in the form:

$$\begin{aligned} \overline{g}_{\alpha\beta} &= \sum_{i,j=1}^{n} \left[\frac{\partial x^i}{\partial t^\alpha}\right]^T_{\alpha i} [g]_{ij} \left[\frac{\partial x^i}{\partial t^\beta}\right]_{j\beta} = \frac{\partial x^i}{\partial t^\alpha} g_{ij} \frac{\partial x^j}{\partial t^\beta} \text{ summation convention on } i \text{ and } j \\ \left[\overline{g}_{\alpha\beta}\right] &= J^T G J \end{aligned}$$

Inserting this computation into (17) we have $\overline{g}_{\alpha\beta}$ is an (n-1)-by-(n-1) matrix and we can consider its determinant. Let $\overline{g} = \det|\overline{g}_{\alpha\beta}|$.

$$\begin{aligned} \overline{g} &= \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,...,k_{n-1}=1}}^{n} (J^T_{1i_1} G_{i_1 k_1} J_{k_1 p_1})(J^T_{2i_2} G_{i_2 k_2} J_{k_2 p_2})...(J^T_{n-1 i_{n-1}} G_{i_{n-1} k_{n-1}} J_{k_{n-1} p_{n-1}}) \\ &= \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,...,k_{n-1}=1}}^{n} (J^T_{1i_1} ... J^T_{n-1 i_{n-1}})(G_{i_1 k_1} ... G_{i_{n-1} k_{n-1}})(J_{k1 p_1} ... J_{k_{n-1} p_{n-1}}) \\ &= \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,...,k_{n-1}=1}}^{n} (J^T_{1i_1} ... J^T_{n-1 i_{n-1}})(G_{i_1 k_1} ... G_{i_{n-1} k_{n-1}}) \sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} (J_{k1 p_1} ... J_{k_{n-1} p_{n-1}}) \end{aligned}$$

At this point notice that the sum over the $n-1$ variables $k1, k2, ..., k_{n-1}$ involves summing over all values from 1 to n for each k_i, but that these values must be distinct. If $k_i = k_j$ for some pair i, j, then the sum is zero. This means that there are $n-1$ distinct k values chosen from the full set of n values, therefore one of the k_i values is to be omitted in this sum. Let k stand for this missing value and J_k stand for the matrix J with the k-th row deleted.

$$\sum_{P \in S_{n-1}} \epsilon_{p_1 p_2 \ldots p_{n-1}} (J_{k1 p_1} ... J_{k_{n-1} p_{n-1}}) = \epsilon_{k_1 k_2 \ldots k_{n-1}} |J_k| = \epsilon_{k_1 k_2 \ldots k_{n-1}} (-1)^{1+k} D_k \tag{18}$$

There will be n such cases and therefore we insert a sum over $k = 1, 2, ..., n$ to account for each case in the following. The sum over the $i_1, ..., i_{n-1}$ indices follows that same way.

$$
\begin{aligned}
\overline{g} &= \sum_{k=1}^{n} \sum_{\substack{i1,..,i_{n-1}=1 \\ k_1,...,k_{n-1}=1,\neq k}}^{n} (J^T_{1i_1} ... J^T_{n-1 i_{n-1}})(G_{i_1 k_1} ... G_{i_{n-1} k_{n-1}}) \epsilon_{k_1 k_2 ... k_{n-1}} (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^{n} \sum_{i1,..,i_{n-1}=1,\neq i}^{n} (J^T_{1i_1} ... J^T_{n-1 i_{n-1}}) \epsilon_{i_1 i_2 ... i_{n-1}} (-1)^{i+k} \text{cofactor}(g_{ik}) (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^{n} \sum_{i1,..,i_{n-1}=1,\neq i}^{n} \epsilon_{i_1 i_2 ... i_{n-1}} (J^T_{1i_1} ... J^T_{n-1 i_{n-1}}) (-1)^{i+k} \Delta^{ik} (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^{n} (-1)^{1+i} D_i (-1)^{i+k} \Delta^{ik} (-1)^{1+k} D_k \\
&= \sum_{i,k=1}^{n} D_i \, g \, g^{ik} \, D_k = g \sum_{i=1}^{n} D^k \, D_k = g \, D^2,
\end{aligned}
$$

where $g = |g_{ij}|$ as before and the sum is over the n values of i and k which correspond to the rows or columns which are crossed out to form the n minors each of dimension (n-1)-by-(n-1). Hence we find that

$$\overline{g} = g \, D^2.$$

We finally derive the relationship between the element of surface are on the hyperplane H and the metric of the n-dimensional Riemann M_n space that H is embedded inside.

$$dS = \sqrt{\overline{g}} \, dt^1 ... dt^{n-1} = \sqrt{g \, D^2} \, dt^1 ... dt^{n-1} \tag{19}$$

If the n-1 dimensional hypersurface H which is embedded in Cartesian $\mathbb{R}_n$ is given in parametric form as follows:

$$\mathfrak{r}(x) = \{x, f(x)\}$$

Then $g_{ij} = I_n$ (n-by-n identity matrix) and J is an n-by-$n-1$ matrix:

$$J = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \dots & \frac{\partial f}{\partial x^{n-1}} \end{bmatrix} \tag{20}$$

$$J^T = \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{\partial f}{\partial x^1} \\ 0 & 1 & \dots & 0 & \frac{\partial f}{\partial x^2} \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & 1 & \frac{\partial f}{\partial x^n} \end{bmatrix} \tag{21}$$

Working out the minors, we find $D_n = 1$ (cross out the last row of J), and crossing out any of the other rows $i < n$ inserts a 0 in the i-th diagonal and shifts all the other rows $i+1, ..., n$ up one row. The resulting determinant is equal to $D_k = (-1)^k \frac{\partial f}{\partial x_k}$

Therefore we obtain

$$\overline{g} = \sum_{k=1}^{n} D_k = (\frac{\partial f}{\partial x^1})^2 + (\frac{\partial f}{\partial x^2})^2 + ... + (\frac{\partial f}{\partial x^n})^2 + 1 \tag{22}$$

And the resulting magnitude, given that $g_{ij} = I_n$ is $\sqrt{D^2} = \sqrt{1 + |\nabla f|^2}$ and

$$dS = \sqrt{1 + |\nabla f|^2}\ dt^1...dt^{n-1}. \tag{23}$$

2 Application to the Green Theorem

In Riemannian Geometry the divergence of a vector $\mathfrak{b}$ is given by

$$\nabla \cdot \mathfrak{b} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, b^i) \text{ summation convention}$$

Let us form the integral of the divergence of $\mathfrak{b}$ over a region R in V_n bounded by the closed surface B

$$\int_R \nabla \cdot \mathfrak{b}\, dV = \int_R \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, b^i)\, dV = \int_R \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\, b^i) \sqrt{g}\, dx^1...dx^n \tag{24}$$

Now let us integrate the differentiated terms over the range of the appropriate variable for each one. We obtain an integral over the boundary surface B: for the k term this is (k not summed)

$$\int_B \sqrt{g}\, b^k dx^1...dx^{k-1} dx^{k+1}...dx^n = \int_B \sqrt{g}\, b^k D_k\, dt^1...dt^{n-1}, \tag{25}$$

where D_k is given as in (10) represents the transformation of the integration variables from the x^i to the coordinates as given on B.

From the above set of equations we see that

$$\begin{aligned} \int_R \nabla \cdot \mathfrak{b}\, dV &= \int_B b^k D_k \sqrt{g}\, dt^1...dt^{n-1} \\ &= \int_B b^k n_k \sqrt{g}\, D\ dt^1...dt^{n-1} \\ &= \int_B \mathfrak{b} \cdot \mathfrak{n}\, dS, \end{aligned}$$

where dS is given by (19).