Quadratic variation property of Brownian motion

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1 Unbounded variation of a Brownian motion

Any sequence of values $0 < t_0 < t_1 < \cdots < t_n < T$ is called a partition $\Pi = \Pi(t_0, \ldots, t_n)$ of an interval $[0, T]$. Given a continuous function $f : [0, T] \to \mathbb{R}$ its *total variation* is defined to be

$$LV(f) \triangleq \sup_{\Pi} \sum_{1 \le k \le n} |f(t_k) - f(t_{k-1})|,$$

where the supremum is taken over all possible partitions Π of the interval $[0, T]$ for all n. A function f is defined to have bounded variation if its total variation is finite.

Theorem 1. *Almost surely no path of a Brownian motion has bounded variation for every $T \ge 0$. Namely, for every T*

$$\mathbb{P}(\omega : LV(B(\omega)) < \infty) = 0.$$

The main tool is to use the following result from real analysis, which we do not prove: if a function f has bounded variation on $[0, T]$ then it is differentiable almost everywhere on $[0, T]$. We will now show that quite the opposite is true.

Proposition 1. *Brownian motion is almost surely nowhere differentiable. Specifically,*

$$\mathbb{P}(\forall\, t \ge 0 : \limsup_{h \to 0} |\frac{B(t+h) - B(t)}{h}| = \infty) = 1.$$

Proof. Fix $T > 0, M > 0$ and consider $A(M,T) \subset C[0,\infty)$ – the set of all paths $\omega \in C[0,\infty)$ such that there exists at least one point $t \in [0,T]$ such that

$$\limsup_{h \to 0} |\frac{B(t+h) - B(t)}{h}| \le M.$$

We claim that $\mathbb{P}(A(M,T)) = 0$. This implies $\mathbb{P}(\cup_{M \ge 1} A(M,T)) = 0$ which is what we need. Then we take a union of the sets $A(M,T)$ with increasing T and conclude that B is almost surely nowhere differentiable on $[0,\infty)$. If $\omega \in A(M,T)$, then there exists $t \in [0,T]$ and n such that $|B(s) - B(t)| \le 2M|s-t|$ for all $s \in (t - \frac{2}{n}, t + \frac{2}{n})$. Now define $A_n \subset C[0,\infty)$ to be the set of all paths ω such that for some $t \in [0,T]$

$$|B(s) - B(t)| \le 2M|s-t|$$

for all $s \in (t - \frac{2}{n}, t + \frac{2}{n})$. Then

$$A_n \subset A_{n+1} \tag{1}$$

and

$$A(M,T) \subset \cup_n A_n. \tag{2}$$

Find $k = \max\{j : \frac{j}{n} \le t\}$. Define

$$Y_k = \max\{|B(\frac{k+2}{n}) - B(\frac{k+1}{n})|, |B(\frac{k+1}{n}) - B(\frac{k}{n})|, |B(\frac{k}{n}) - B(\frac{k-1}{n})|\}.$$

In other words, consider the maximum increment of the Brownian motion over these three short intervals. We claim that $Y_k \le 6M/n$ for every path $\omega \in A_n$.

To prove the bound required bound on Y_k we first consider

$$\begin{aligned} |B(\frac{k+2}{n}) - B(\frac{k+1}{n})| &\le |B(\frac{k+2}{n}) - B(t)| + |B(t) - B(\frac{k+1}{n})| \\ &\le 2M\frac{2}{n} + 2M\frac{1}{n} \\ &\le \frac{6M}{n}. \end{aligned}$$

The other two differences are analyzed similarly.

Now consider event B_n which is the set of all paths ω such that $Y_k(\omega) \le 6M/n$ for *some* $0 \le k \le Tn$. We showed that $A_n \subset B_n$. We claim that $\lim_n \mathbb{P}(B_n) =$

0. Combining this with (1), we conclude $\mathbb{P}(A_n) = 0$. Combining with (2), this will imply that $\mathbb{P}(A(M,T)) = 0$ and we will be done.

Now to obtain the required bound on $\mathbb{P}(B_n)$ we note that, since the increments of a Brownian motion are independent and identically distributed, then

$$\begin{aligned}\mathbb{P}(B_n) &\leq \sum_{0\leq k\leq Tn} \mathbb{P}(Y_k \leq 6M/n) \\ &\leq Tn\mathbb{P}(\max\{|B(\frac{3}{n}) - B(\frac{2}{n})|, |B(\frac{2}{n}) - B(\frac{1}{n})|, |B(\frac{1}{n}) - B(0)|\} \leq 6M/n) \\ &= Tn[\mathbb{P}(|B(\frac{1}{n})| \leq 6M/n)]^3. \qquad (3)\end{aligned}$$

Finally, we just analyze this probability. We have

$$\mathbb{P}(|B(\frac{1}{n})| \leq 6M/n) = \mathbb{P}(|B(1)| \leq 6M/\sqrt{n}).$$

Since $B(1)$ which has the standard normal distribution, its density at any point is at most $1/\sqrt{2\pi}$, then we have that this probability is at a most $(2(6M)/\sqrt{2\pi n})$. We conclude that the expression in (3) is, ignoring constants, $O(n(1/\sqrt{n})^3) = O(1/\sqrt{n})$ and thus converges to zero as $n \to \infty$. We proved $\lim_n \mathbb{P}(B_n) = 0$. □

2 Bounded *quadratic* variation of a Brownian motion

Even though Brownian motion is nowhere differentiable and has unbounded total variation, it turns out that it has bounded *quadratic* variation. This observation is the cornerstone of Ito calculus, which we will study later in this course.

We again start with partitions $\Pi = \Pi(t_0, \ldots, t_n)$ of a fixed interval $[0, T]$, but now consider instead

$$Q(\Pi, B) \triangleq \sum_{1\leq k\leq n} (B(t_k) - B(t_{k-1}))^2.$$

where, we make (without loss of generality) $t_0 = 0$ and $t_n = T$. For every partition Π define

$$\Delta(\Pi) = \max_{1\leq k\leq n} |t_k - t_{k-1}|.$$

Theorem 2. *Consider an arbitrary sequence of partitions* $\Pi_i, i = 1, 2, \ldots$. *Suppose* $\lim_{i\to\infty} \Delta(\Pi_i) = 0$. *Then*

$$\lim_{i\to\infty} \mathbb{E}[(Q(\Pi_i, B) - T)^2] = 0. \qquad (4)$$

Suppose in addition $\lim_{i\to\infty} i^2\Delta(\Pi_i) = 0$ *(that is the resolution* $\Delta(\Pi_i)$ *converges to zero faster than* $1/i^2$*). Then almost surely*

$$Q(\Pi_i, B) \to T. \tag{5}$$

In words, the standard Brownian motion has almost surely finite quadratic variation which is equal to T.

Proof. We will use the following fact. Let Z be a standard Normal random variable. Then $\mathbb{E}[Z^4] = 3$ (cute, isn't it?). The proof can be obtained using Laplace transforms of Normal random variables or integration by parts, and we skip the details.

Let $\theta_i = (B(t_i)-B(t_{i-1}))^2-(t_i-t_{i-1})$. Then, using the independent Gaussian increments property of Brownian motion, θ_i is a sequence of independent zero mean random variables. We have

$$Q(\Pi_i) - T = \sum_{1\le i\le n} \theta_i.$$

Now consider the second moment of this difference

$$\begin{aligned}\mathbb{E}(Q(\Pi_i) - T)^2 = &\sum_{1\le i\le n} \mathbb{E}(B(t_i) - B(t_{i-1}))^4 \\ &- 2\sum_{1\le i\le n} \mathbb{E}(B(t_i) - B(t_{i-1}))^2(t_i - t_{i-1}) + \sum_{1\le i\le n}(t_i - t_{i-1})^2.\end{aligned}$$

Using the $\mathbb{E}[Z^4] = 3$ property, this expression becomes

$$\begin{aligned}&\sum_{1\le i\le n} 3(t_i - t_{i-1})^2 - 2\sum_{1\le i\le n}(t_i - t_{i-1})^2 + \sum_{1\le i\le n}(t_i - t_{i-1})^2 \\ &= 2\sum_{1\le i\le n}(t_i - t_{i-1})^2 \\ &\le 2\Delta(\Pi_i)\sum_{1\le i\le n}(t_i - t_{i-1}) \\ &= 2\Delta(\Pi_i)T.\end{aligned}$$

Now if $\lim_i \Delta(\Pi_i) = 0$, then the bound converges to zero as well. This establishes the first part of the theorem.

To prove the second part identify a sequence $\epsilon_i \to 0$ such that $\Delta(\Pi_i) = \epsilon_i/i^2$. By assumption, such a sequence exists. By Markov's inequality, this is bounded by

$$\mathbb{P}((Q(\Pi_i) - T)^2 > 2\epsilon_i) \leq \frac{\mathbb{E}(Q(\Pi_i) - T)^2}{2\epsilon_i} \leq \frac{2\Delta(\Pi_i)T}{2\epsilon_i} = \frac{T}{i^2} \qquad (6)$$

Since $\sum_i \frac{T}{i^2} < \infty$, then the sum of probabilities in (6) is finite. Then applying the Borel-Cantelli Lemma, the probability that $(Q(\Pi_i) - T)^2 > 2\epsilon_i$ for infinitely many i is zero. Since $\epsilon_i \to 0$, this exactly means that almost surely, $\lim_i Q(\Pi_i) = T$. $\square$

3 Additional reading materials

- Sections 6.11 and 6.12 of Resnick's [1] chapter 6 in the book.

References

[1] S. Resnick, *Adventures in stochastic processes*, Birkhuser Boston, Inc., 1992.

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