Derivatives and Commutators

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1 Derivatives and Commutators

Using the Chain Rule of Calculus we define the derivative of X^n as

$$\frac{dX^n}{dt} = nX^{n-1}\frac{dX}{dt}$$

We ask: Can the derivative operation be defined in terms of discrete operations, or operators which appear to be discrete?

Let use imagine two operators A and B. The commutator of A and B is written as $[A, B]$ and is defined as

$$[A, B] = AB - BA$$

For example the operators may be defined by their action on a set of functions. Such is the case in Quantum Mechanics for the operators p and q which play the role of kinematic variables corresponding to "observables" of momentum and position.

Suppose the operators A and B commute with their commutaors, i.e.,

$$[B, [A, B]] = [A, [A, B]] = 0$$

Consider the following procedure:

$$\begin{aligned}
[A, B^n] &= AB^n - B^n A \\
&= ABB^{n-1} - BAB^{n-1} + B(AB)B^{n-2} - B(BA)B^{n-3} + ... + B^{n-1}AB - B^{n-1}BA \\
&= [A, B]B^{n-1} + B[A, B]B^{n-2} + ... + B^{n-1}[A, B]
\end{aligned}$$

Using the fact that B commutes with $[A, B]$ we obtain

$$[A, B^n] = B^{n-1}[A, B] + B^{n-1}[A, B] + ... + B^{n-1}[A, B] = nB^{n-1}[A, B]$$

Also, using $[A^n, B] = -[B, A^n]$ we can also obtain

$$[A^n, B] = -nA^{n-1}[B, A] = nA^{n-1}[A, B]$$

It seems therefore that we can get similar properties as differentiation using operators A and B by identifying the derivative operation by this correspondence:

$$\frac{dA}{dt} = [A, B]$$

and therefore

$$\frac{dA^n}{dt} = [A^n, B] = nA^{n-1}\frac{dA}{dt} = nA^{n-1}[A, B]$$

The above correspondence appears to be fundamental to Quantum Mechanics using the dynamical variables $q = A$ and $p = B$ which lead to the Schrödinger picture with position operator $q = x$ and momentum operator $p = -i\hbar\frac{\partial}{\partial_x}$ with the famous commutation rule

$$[q, p] = i\hbar$$

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Perhaps such commutation algebra can be used to eliminate the derivative at a more fundamental level and free mechanics from continuous coordinates?

Also, observe that the Right Hand Side (RHS) of the commutation rule can be written in terms of the identity operator I (in matrix language I is the $n \times n$ unit matrix for some finite vector space of dimension n) as follows:

$$[q, p] = i\hbar\, I$$

and so the commutator of q and p is a multiple of the unit matrix I. This presents an interesting problem because there is a matrix representation associated with our operators A and B (or p and q) – this is similar to the correspondence between the Heisenberg pictures and the Schrödinger pictures. Therefore we can think of A and B as"square" matrices corresponding to a transformation on a linear vector space. But this is exactly where a complication develops. Because, for any square matrices A and B we have that the trace operation is cyclic, viz. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and hence

$$\mathrm{Tr}([A, B]) = \mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0.$$

This fact contradicts the commutation relation because $\mathrm{Tr}(I) = n$ for any finite dimensional vector space in n dimensions. This contraction appears to go away if we use infinite-dimensional vector spaces! This is one way to understand why infinite dimensional vector spaces (etc. with scalar products, etc. Hilbert Spaces) are fundamental to Quantum Mechanics.

Conclusion – Commutators have similar properties as derivative operators. Perhaps commutators are more useful in situations were there is no way to define a derivative due to fundamental limits on discreteness of the coorindates in space-time.