The Borel-Cantelli Lemma and its Applications

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Abstract

We state and prove the Borel-Cantelli lemma and use the result to prove another proposition.

1 Definitions and Identities

Definition 1 *Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets. The limit supremum of $\{E_k\}$ is the set*

$$\limsup_{k\to\infty}(E_k) := \{x \in \mathbb{R}^d \ : \ x \in E_k \textit{ for infinitely many } k\}$$

Proposition 1 *The following identity holds:*

$$\limsup_{k\to\infty}(E_k) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$$

Proof. Assume that $x \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$. So,

$$x \in \left(\bigcup_{k=1}^{\infty} E_k\right) \cap \left(\bigcup_{k=2}^{\infty} E_k\right) \cap \left(\bigcup_{k=3}^{\infty} E_k\right) \cap \ldots$$

Suppose that $x \notin \limsup_{k\to\infty}(E_k)$. By definition, this means that there is a positive integer k_0 such that for all $k \geq k_0$, $x \notin E_k$. Hence, $x \notin \bigcup_{k=k_0}^{\infty} E_k$. $\rightarrow\leftarrow$ Therefore, $x \in \limsup_{k\to\infty}(E_k)$. This means that

$$\limsup_{k\to\infty}(E_k) \supset \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$$

Conversely, assume that $x \in \limsup_{k\to\infty}(E_k)$. This means that x belongs to E_k for infinitely many k. That is to say, x continuously reappears as an element in a set of the sequence E_k. Then it is evident that $x \in \bigcup_{k=1}^{\infty} E_k$. It is equally evident that $x \in \bigcup_{k=2}^{\infty} E_k$, $x \in \bigcup_{k=3}^{\infty} E_k$, and so on. Thus,

$$\begin{aligned} x &\in \left(\bigcup_{k=1}^{\infty} E_k\right) \cap \left(\bigcup_{k=2}^{\infty} E_k\right) \cap \left(\bigcup_{k=3}^{\infty} E_k\right) \cap \ldots \\ &\in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k \end{aligned}$$

Therefore,

$$\limsup_{k\to\infty}(E_k) \subset \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$$

Thus,

$$\limsup_{k\to\infty}(E_k) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$$

$\square$

2 The Borel-Cantelli lemma and applications

Lemma 1 (Borel-Cantelli) *Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets of $\mathbb{R}^d$ such that*

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Then $\limsup_{k\to\infty}(E_k)$ is measurable and has measure zero.

Proof. Given the identity,

$$E = \limsup_{k\to\infty}(E_k) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$$

Since each E_k is a measurable subset of $\mathbb{R}^d$, $\bigcup_{k=n}^{\infty} E_k$ is measurable for each $n \in \mathbb{N}$, and so $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{n} E_k$ is measurable as well, Stein [1]. Therefore, E is measurable.

Suppose that $m(E) = \epsilon > 0$. Then

$$0 < \epsilon < m(E) = m\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right)$$

Since for all $n \in \mathbb{N}$, $E \subset \bigcup_{k=n}^{\infty} E_k$, by the monotonicity property, Stein [1],

$$m(E) \leq m\left(\bigcup_{k=n}^{\infty} E_k\right)$$

By the countable sub-additivity property, Stein [1], for all $n \in \mathbb{N}$,

$$m(E) = m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k)$$

By assumption, $\sum_{k=1}^{\infty} E_k < \infty$. It follows that the tail of the series can be made arbitrarily small. In other words, for any $\delta > 0$, there is an $N' \in \mathbb{N}$ such that

$$\sum_{k=N'}^{\infty} m(E_k) < \delta$$

However, if we choose $\delta = \epsilon/2$, we have

$$0 < \epsilon < m(E) \leq \sum_{k=N'}^{\infty} m(E_k) \leq \frac{\epsilon}{2}$$

$$\rightarrow\leftarrow$$

Therefore, $m(E) = 0$.

$\square$

Proposition 2 *Let $\{f_n(x)\}$ be a sequence of measurable functions on [0,1] with $|f_n(x)| < \infty$ for a.e. $x \in [0,1]$. Then there exists a sequence $\{c_n\}$ of positive real numbers such that*

$$\frac{f_n(x)}{c_n} \to 0 \quad a.e.\ x \in [0,1]$$

Proof. Given a sequence of positive numbers $\{c_n\}$, consider the set

$$E_n = \big\{x \in [0,1] \ : \ \frac{|f_n(x)|}{c_n} > \frac{1}{n}\big\}$$

Suppose that there is no sequence of positive numbers $\{c_n\}$ such that $m(E_n) \leq 2^{-n}$. Without loss of generally, we can assume that $\{c_n\}$ is a sequence of positive numbers. Fix $n \in \mathbb{N}$. Then it follows that for any $N \in \mathbb{N}$,

$$m(A_N) = m\big(\big\{x \in [0,1] \ : \ \frac{|f_n(x)|}{N} > \frac{1}{n}\big\}\big) > 2^{-n}$$

$$m(A_N) = m\big(\big\{x \in [0,1] \ : \ |f_n(x)| > \frac{N}{n}\big\}\big) > 2^{-n}$$

So,

$$\begin{aligned}
A_1 &= \big\{x \in [0,1] \ : \ |f_n(x)| > \frac{1}{n}\big\} \\
A_2 &= \big\{x \in [0,1] \ : \ |f_n(x)| > \frac{2}{n}\big\} \\
A_3 &= \big\{x \in [0,1] \ : \ |f_n(x)| > \frac{3}{n}\big\} \\
&\vdots \\
A_\infty &= \big\{x \in [0,1] \ : \ |f_n(x)| = \infty\big\}
\end{aligned}$$

It is easy to see that this is a decreasing sequence of sets: $A_1 \supset A_2 \supset A_3 \supset \ldots$. Since A_∞ is a subset of each A_N, $A_\infty = \bigcap_{N=1}^{\infty} A_N$. Hence, $m(\bigcap_{N=1}^{\infty} A_N) = m(A_\infty)$, and so

$$2^{-n} < m\left(\bigcap_{N=1}^{\infty} A_N\right) = m(A_\infty)$$

However, by assumption $m(A_\infty) = 0$. $\rightarrow\leftarrow$ Therefore, there is a sequence of positive numbers $\{c_n\}$ such that

$$m(E_n) = m(\big\{x \in [0,1] \ : \ \frac{|f_n(x)|}{c_n} > \frac{1}{n}\big\}) \leq 2^{-n}$$

Thus the series converges by comparison to a geometric series:

$$\begin{aligned}
\sum_{n=1}^{\infty} m(E_k) &\leq \sum_{n=1}^{\infty} 2^{-n} \\
\sum_{n=1}^{\infty} m(E_k) &\leq 1 \\
&< \infty
\end{aligned}$$

According to the Borel-Cantelli lemma then, $\limsup_{n\to\infty} E_n$ has measure zero. By definition,

$$\limsup_{n\to\infty}(E_n) = \{x \ : \ x \in E_n \text{ for infinitely many } n\}$$

So if $x \in \limsup_{n\to\infty}(E_n)$, then x is a number in [0,1] such that for infinitely many n,

$$\frac{|f_n(x)|}{c_n} > \frac{1}{n}$$

Negating this statement, if $x \notin \limsup_{n\to\infty}(E_n)$, then there is a $k_0 \in \mathbb{N}$ such that $|f_n(x)|/c_n \leq 1/n$ for all $n \geq k_0$. By comparison then, $\{|f_n(x)|/c_n\}$ would converge to 0 since $\{1/n\}$ converges to 0. Therefore, since $m(\limsup_{n\to\infty}(E_n)) = 0$, the conclusion is

$$\frac{f_n(x)}{c_n} \to 0 \quad a.e.\ x \in [0,1]$$

References

[1] E. M. Stein and R. Shakarchi. *Real Analysis.* Princeton University Press, Princeton and Oxford, 2005.