Borel-Cantelli Lemmas

Suppose that $\{A_n : n \geq 1\}$ is a sequence of events in a probability space. Then the event $A(i.o.) = \{A_n$ ocurrs for infinitely many n $\}$ is given by

$$A(i.o.) = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n,$$

Lemma 1 *Suppose that $\{A_n : n \geq 1\}$ is a sequence of events in a probability space. If*

$$\sum_{n=1}^{\infty} P(A_n) < \infty, \tag{1}$$

then $P(A(i.o.)) = 0$; only a finite number of the events occur, wp1.

Proof :

Let $I_n = I\{A_n\}$ denote the indicator rv for the event A_n, and let

$$N = \sum_{n=1}^{\infty} I_n,$$

denote the total number of the events to occur. Then $P(A(i.o.)) = 0$ if and only if $P(N < \infty) = 1$. But if $E(N) < \infty$, then $P(N < \infty) = 1$ (as is the case with any rv N), and by Tonelli's (Fubini's) theorem,

$$E(N) = \sum_{n=1}^{\infty} P(A_n), \tag{2}$$

which is assumed finite, thus completing the proof. ■

In general, the converse is not true. Essentially, this is because there exists rvs N such that $P(N < \infty) = 1$ but $E(N) = \infty$. (For example choose an N such that $P(N = n) = c/n^2$, $n \geq 1$, and define $A_n = \{N = n\}$.) But if the events are *independent*, then the converse holds:

Lemma 2 *Suppose that $\{A_n : n \geq 1\}$ is a sequence of* **independent** *events in a probability space. If*

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \tag{3}$$

then $P(A(i.o.)) = 1$.

Proof : Suppose that (3) holds, and note that if it holds then

$$\sum_{n=k}^{\infty} P(A_n) = \infty, \ k \geq 1. \tag{4}$$

Let $\overline{A}_n$ denote the complement of the set A_n.

$$P(A(i.o.)) = \lim_{k\to\infty} P(\cup_{n=k}^{\infty} A_n) = 1 - \lim_{k\to\infty} P(\cap_{n=k}^{\infty} \overline{A}_n).$$

To complete the proof we will show that

$$P(\cap_{n=k}^{\infty} \overline{A}_n) = 0, \ k \geq 1.$$

By independence, and the basic fact that $1 - x \leq e^{-x}, \ x \geq 0$,

$$\begin{aligned}
P(\cap_{n=k}^{\infty} \overline{A}_n) &= \prod_{n=k}^{\infty} P(\overline{A}_n) && (5)\\
&= \prod_{n=k}^{\infty} (1 - P(A_n)) && (6)\\
&\leq \prod_{n=k}^{\infty} e^{-P(A_n)} && (7)\\
&= e^{-\sum_{n=k}^{\infty} P(A_n)} && (8)\\
&= e^{-\infty} = 0, && (9)
\end{aligned}$$

where the last equality is from (4).

■

As an immediate corollary to the two Lemmas, we have a special case of a "0-1" law:

Proposition 0.1 *If $\{A_n : n \geq 1\}$ is a sequence of* **independent** *events in a probability space, then either $P(A(i.o.)) = 0$ ($E(N) < \infty$ case) or $P(A(i.o.)) = 1$ ($E(N) = \infty$ case), where N denotes the total number of A_n to occur;*

$$N = \sum_{n=1}^{\infty} I_n,$$

where $I_n = I\{A_n\}$ denote the indicator rv for the event A_n.

0.1 Applications

1. Suppose $\{X_n : n \geq 1\}$ are rvs such that $P(X_n = 1) = p_n, \ P(X_n = 0) = 1 - p_n$. Thus these are Bernoulli rvs in which the success probability depends on n and we

do not assume independence. Using $A_n = \{X_n = 1\}$ we deduce from Lemma 1 that if

$$\sum_{n=1}^{\infty} p_n < \infty,$$

then X_n only visits 1 a finite number of times, hence eventually must take on value 0, wp1. Thus there exists a random time N such that $X_n = 0, \ n \geq N$ and so $\lim_{n\to\infty} X_n = 0$, wp1. To make this happen, we need that $p_n \to 0$ fast enough; we could, for example, choose $p_n = 1/n^2$ or choose $\{p_n\}$ to be *any* probability distribution, such as a geometric distribution, $p_n = (1-p)^{n-1}p, \ n \geq 1$ (some $0 < p < 1$). (This result remains valid even if the rvs are independent.)

Now assume that the rvs are independent so that by Lemma 2 if

$$\sum_{n=1}^{\infty} p_n = \infty,$$

then $X_n = 1$ for infinitely many n, wp1. To achieve this, we could, of course take $p_n = p$ for some fixed p, but more interesting would be to take, for example, $p_n = 1/n$ in which case $q_n = 1 - p_n = P(X_n = 0)$ also sums to ∞ since $q_n \to 1$; $X_n = 0$ for infinitely many n, wp1 too. Thus in this case X_n continues to visit both 1 and 0 over and over infinitely often.