4 Fundamental Orbit Determination

The general non-linear problem of Orbit Determination involves estimating parameters associated with a state motion model of the form:

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, t), \qquad \mathbf{X}_k = \mathbf{X}(t_k) \tag{7}$$

$$\mathbf{Y}_i = \mathbf{G}(\mathbf{X}_i, t_i) + \epsilon_i; \;\; i \quad = 1, 2, \dots l \tag{8}$$

where $\mathbf{X}_k$ is an unknown n-dimensional State Vector at time t_k. Also $\mathbf{Y}_i$ is a p-dimensional set of measurements for $i = 1, 2, \dots l$, that are to be used to estimate the unknown value $\mathbf{X}_k$ which we denote by $\hat{\mathbf{X}}_k$. Finally ϵ_i represents the random measurement uncertainty which we usually assume to be given by a multivariate Gaussian distribution with $\mathbf{Cov} = \mathrm{E}[\epsilon\epsilon^T]$. In general $p < n$ and $m = p \times l \gg n$. Let $\mathbf{X}^*(t)$ represent the numerical integration of Eq.(**??**) with initial conditions $\mathbf{X}^*(t_0)$. $\mathbf{X}^*(t)$ will be referred to as the reference State Vector – we will implement an iterative method that numerically moves the reference State Vector so as to minimize an objective function so as to obtain the best estimate of the State Vector at time t_0.

Now let us look at small changes in the State Vector about the reference State Vector and consider their effect on the prediction of the RHS of the Eq.(**??**). Expanding to first order in a Taylor series about small differences in the reference State Vector

$$\begin{aligned} \mathbf{Y}_i &\approx \mathbf{G}(\mathbf{X}^*_; t_i) + \left[\frac{\partial \mathbf{G}}{\partial \mathbf{X}}\right]^*_i [\mathbf{X}(t_i) - \mathbf{X}^*(t_i)] + \epsilon_i, \\ &= \mathbf{G}(\mathbf{X}^*_; t_i) + \left[\frac{\partial \mathbf{G}}{\partial \mathbf{X}}\right]^*_i \mathbf{x}_i + \epsilon_i \end{aligned}$$

where $\mathbf{x}_i = \mathbf{X}(t_i) - \mathbf{X}^*(t_i)$.If we also use lower case for the difference $\mathbf{y}_i = \mathbf{Y}_i - \mathbf{G}(\mathbf{X}^*_; t_i)$ we can rewrite the above expression as

$$\begin{aligned} \mathbf{x}_i &= \mathbf{X}(t_i) - \mathbf{X}^*(t_i), \\ \mathbf{y}_i &\approx \left[\frac{\partial \mathbf{G}}{\partial \mathbf{X}}\right]^*_i \mathbf{x}_i + \epsilon_i \end{aligned}$$

If we use numerical integration to determine the time dependence of $\mathbf{X}^*(t)$, then the above can be recast to use the differences in the reference State Vector at t_0 and a nearby State Vector $\mathbf{x}_0 = \mathbf{X}(t_0) - \mathbf{X}^*(t_0)$

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{X}(t_0) - \mathbf{X}^*(t_0), \\ \mathbf{y}_i &\approx \left[\frac{\mathbf{G}(\mathbf{X}^{+\delta}, t_i) - \mathbf{G}(\mathbf{X}^{-\delta}, t_i)}{2\vec{\delta}}\right] \mathbf{x}_0 + \epsilon_i \end{aligned} \tag{9}$$

where $\mathbf{X}^{\pm\delta}$ is the numerical integration of

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, t)$$

with the initial conditions $\mathbf{X}^{\pm\delta}(t_0) = \mathbf{X}^*(t_0) \pm \vec{\delta}$. Let $\mathbf{H}_i$ be defined as

$$\mathbf{H}_i = \left[\frac{\mathbf{G}(\mathbf{X}^{+\delta}, t_i) - \mathbf{G}(\mathbf{X}^{-\delta}, t_i)}{2\vec{\delta}}\right] \tag{10}$$

and we can summarize the model in the form

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{X}(t_0) - \mathbf{X}^*(t_0), \\ \mathbf{y}_i &\approx \mathbf{H}_i \mathbf{x}_0 + \epsilon_i. \end{aligned} \tag{11}$$

5 Application of the first order method

Let us denote the prediction vector as

$$\alpha(t) = \begin{bmatrix} \theta(t) \\ \phi(t) \end{bmatrix} \text{ and the measurement data vectors as } \tilde{\alpha} = \begin{bmatrix} \tilde{\theta} \\ \tilde{\phi} \end{bmatrix}$$