1 The Operator Approach to the Harmonic Oscillator

The Hamiltonian for the one-dimensional harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{1}$$

The quantum mechanical commutation relations for the operators p and q is

$$[q, p] = i\hbar \tag{2}$$

Define two operators a and $a^\dagger$ as follows:

$$\begin{aligned} a &= \sqrt{m\omega/2\hbar}q + ip/\sqrt{2m\hbar\omega} \\ a^\dagger &= \sqrt{m\omega/2\hbar}q - ip/\sqrt{2m\hbar\omega} \end{aligned}$$

where $\omega = \sqrt{k/m}$ is the classical angular frequency of the harmonic oscillator.

Form the product

$$\begin{aligned} \hbar\omega a^\dagger a &= \hbar\omega\Big(\sqrt{\frac{m\omega}{2\hbar}}q - \frac{ip}{\sqrt{2m\hbar\omega}}\Big)\Big(\sqrt{\frac{m\omega}{2\hbar}}q + \frac{ip}{\sqrt{2m\hbar\omega}}\Big) \\ &= \frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{i\omega}{2}(qp - pq) \\ &= H - \frac{1}{2}\hbar\omega \end{aligned}$$

Using the above we can rewrite the Hamiltonian for the one dimensional harmonic oscillator as

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$