CMSC351: Prerequisite

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1 Symbols

1. For all ($\forall$)
2. There exists ($\exists$)
3. Element in set ($\in$)
4. Greater than ($>$)
5. Less than ($<$)
6. Greater than equal ($\geq$)
7. Less than equal ($\leq$)

2 Proofs

2.1 Weak Induction

First, we need to prove $\forall n \geq n_0\ P(n)$ we first prove $P(n_0)$ (which is the base case) and then we prove $\forall k \geq n_0\ P(k) \rightarrow P(k+1)$ (which is the inductive step) (you can also prove $P(k-1)$). The assumption of $P(k)$ in the inductive step is the inductive hypothesis.

Ultimately, for the inductive step we are trying to find that for any $k \geq n_0$ if $P(k)$ is true, then $P(k+1)$ is also true.

Example: Suppose we have a set of nested Russian dolls (Matryoshka dolls). Each doll is contained with another doll, each doll is labeled 1, 2, 3 and so on.

In this hypothetical, there are two things that are true. Let's say $M(n)$ is true iff (if and only if) doll n has another doll contained.

(a) The first doll has another doll contained. That is, $M(n)$ is true.

(b) For every doll k, if doll k has another doll inside then doll $k+1$ has another doll inside.

$$\forall k \geq 1, M(k) \rightarrow M(k+1)$$

We can now conclude that $\forall n \geq 1, M(n)$.

2.2 Strong Induction

The goal of strong induction is we need to find some property, we can say $P(n)$ and we need to find some n greater than equal to a ($n \geq a$).

How can we accomplish this?

Step 1 (Basis step): We are going to prove for $P(a)$ (we would just prove $P(a)$ for weak induction), $P(a+1)$, ..., for some finite number say $P(b)$.

$(P(a),\ P(a+b),\ ...,\ P(b)) \leftarrow$ we prove each of these.

Step 2 (Induction): We can now assume $P(i)$ where $a \leq i \leq k$. (Assume $P(i)$). Then, we prove $P(k+1)$.

Example: Fibonacci Sequence

Proof. Claim: The Fibonacci Sequence is defined as the following: $F(0) = 0, F(1) = 1$ and for $n \geq 2, F(n) = F(n-1) + F(n-2)$. We want to prove that for all $n \geq 0, F(n) \leq 2^n$

Base cases:

$$For\ n = 0 : F(0) = 0 \leq 2^0 = 1$$
$$For\ n = 1 : F(1) = 1 \leq 2^1 = 1$$

For both these base case, they hold true.

Inductive Step: Assume that for all i, $0 \leq i \leq k$, we have $F(k) \leq 2^k$.

(This could also be known as the inductive hypothesis)

We now need to prove $F(k+1) \leq 2^{k+1}$.

By the inductive hypothesis, we have $F(n) \leq 2^n$ and $F(n-1) \leq 2^{n-1}$.

We can prove by which:

$$\begin{aligned}
F(k+1) &= F(k) + F(k-1) \\
&\leq 2^{k-1} + 2^{k-2} && \text{(IH)} \\
&\leq 2^{k-1} + 2^{k-1} && (2^{k-2} \text{ is less than } 2^{k-1}) \\
&= 2^k && \text{(Simplify)}
\end{aligned}$$

□

2.3 Constructive Induction

When we are solving recurrences and we have guessed the general form, and we do not know the constants, we usepackage constructive induction.

Example: We know that $\sum_{i=1}^{n} i = \frac{1}{2}n^2 - \frac{1}{2}n$. But how do we determine $\sum_{i=1}^{n} i^2$? Since we know the solution for the first sum is a quadratic, we can guess that for the second sum that is cubic ($an^3 + bn^2 + cn + d$). We start with assumptions. $\sum_{i=1}^{n-1} i = a(n-1)^3 + b(n-1)^2 + c(n-1) + d$ and $n > 0$ We need to prove that $\sum i = 1^n i^2 = an^3 + bn^2 + cn + d$ So, we need:

$$\begin{aligned}
\sum_{i=1}^{n} i &= an^3 + bn^2 + cn + d \\
\sum_{i=1}^{n-1} i + n^2 &= an^3 + bn^2 + cn + d \\
a(n-1)^3 + b(n-1)^2 + c(n-1) + d + n^2 &= an^3 + bn^2 + cn + d \\
a(n^3 - 3n^2 + 3n - 1) + b(n^2 - 2n + 1) + c(n-1) + d &= an^3 + bn^2 + cn + d \\
an^3 + (b - 3a)n^2 + (3a - 2b + c)n + (d - a + b - c) &= an^3 + bn^2 + cn + d \\
an^3 + (b - 3a + 1)n^2 + (3a - 2b + c)n + (d - a + b - c) &= an^3 + bn^2 + cn + d
\end{aligned}$$

We arrive at a systems of equations:

$$\begin{aligned} b - 3a + 1 &= b \\ 3a - 2b + c &= c \\ d - a + b - d &= d \end{aligned}$$

Thus, $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$ and $d = 0$

Therefore,

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}$$

2.4 Structural Induction

To prove some property, $P(x)$ holds for all x in recursively defined set R.

- $P(b)$ for each base case $b \in R$
- $P(c(x))$ for each constructor, c, assuming inductive hypothesis $P(x)$.

Suppose S is a recursively defined set

To prove a statement of the form:

$$\forall s \in S.\ P(s)$$

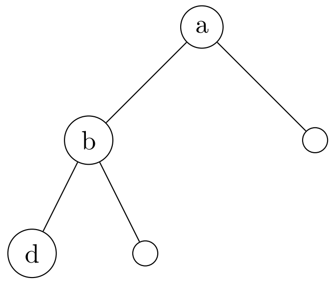
- Base Case: Show P holds for each element defined to be in S as a base case.
- Inductive Step: For each recursive rule used to build S, show that if P holds for things already in S, then P also holds for things already in S, then P also holds for the new elemnts constructed by the rules.

Example: Let $t \in B$ be a binary tree.

We define a binary tree as the following:

- A single node is a binary tree.
- If B_1 and B_2 are binary trees then a single node as parent B_1 and B_2 is also a binary tree.

Let $e(t) =$ the number of empty trees contained in t.



From this graph, $e(t) = 4$ and $b(t) = 3$

Theroem: For all $t \in B$, $e(t) = 1 + b(t)$

We can prove this by structural induction!

Proof. By structural induction.

Base Case:

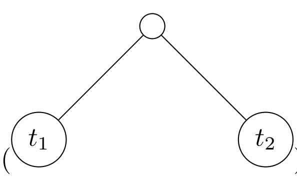
The number of empty trees in an empty tree is 1. ✓

The number of binary trees in an empty tree is 0. ✓

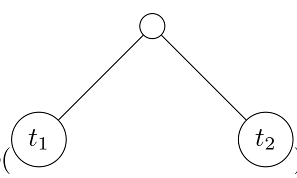
Inductive Step:

Let t_1, $t_2 \in B$ and suppose $e(t_1) = 1 + b(t_1)$ and $e(t_2) = 1 + b(t_2)$.

Then,

$e($ 

$) = e(t_1) + e(t_2)$

$1 + b($ 

$) = 1 + (1 + b(t_1) + b(t_2))$

$$= 2 + (b(t_1) + b(t_2))$$

$$e(t_1) + e(t_2) = (1 + b(t_1)) + (1 + b(t_2))$$

$\square$

3 Combinatorics

3.1 Permutations and Combinations

Formulas:

$$n \text{ objects, permutate } k = \frac{n!}{(n-k)!}$$
$$n \text{ objects, choose } k = \frac{n!}{k!(n-k)!}$$
$$n \text{ categories, permutate } k = n^k$$

3.2 Probability and Expected Value

Suppose X is a random variable that represents the different outcomes $x_1, ..., x_n$ with respective probabilities $p_1, ..., p_n$ then the expected value of X is:

$$E(X) = p_1 x_1 + ... + p_n x_n$$

Example. Suppose an algorithm sorts the values in a list and returns the alternating sum/difference of the result. For example, if you give it $[5, 8, 4, 1]$ and then sorts to $[1, 4, 5, 8]$ and then returns $1 - 4 + 5 - 8 = -6$.

If we are given the following inputs: $[5, 8, 4, 1], [10, 20, 0], [2, 1]$ *and* $[0, 5, 2, -3]$. The following outcomes of each:

$$[5, 8, 4, 1] \to -6$$
$$[10, 20, 0] \to -10$$
$$[2, 1] \to -1$$
$$[0, 5, 2, -3] \to -1$$

Since all inputs are equally likely, the probability of each input set is $\frac{1}{4}$.

$$0.25(-6) + 0.25(10) + 0.25(-1) + 0.25(-6) = 0.75$$

4 Calculus

4.1 Sequences and Sums

Some must know sums:

$$\sum_{i=1}^{n} 1 = n$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$
$$\sum_{i=0}^{n} 2^i = 2^{n+1}-1$$
$$\sum_{i=1}^{n} i * 2^i = (n-1)2^{n+1}+2$$

Example: Let's solve the following sum:

$$\sum_{i=2}^{n} 2i + 2^{-i} + i^2$$

First, we split can split the summation up (because of the addition).

$$\sum_{i=2}^{n} 2i + \sum_{i=2}^{n} 2^{-i} + \sum_{i=2}^{n} i^2$$

We can solve each one of these separately now.

$$\sum_{i=2}^{n} 2i = 2\sum_{i=2}^{n} i = 2\left[\frac{n(n+1)}{2} - 1\right]$$
$$\sum_{i=2}^{n} 2^{-i} = \sum_{i=2}^{n} \frac{1}{2}^{n} = \left[\sum_{i=0}^{n} \frac{1}{2}^{n}\right] - 1 - \frac{1}{2} = \left[\frac{\frac{1}{2}^{n+1}-1}{\frac{1}{2}-1}\right] - \frac{1}{2}$$
$$\sum_{i=2}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} - 1$$

The results of the sum of those:

$$2\left[\frac{n(n+1)}{2} - 1\right] + \left[\frac{\frac{1}{2}^{n+1}-1}{\frac{1}{2}-1}\right] - \frac{1}{2} + \left[\frac{n(n+1)(2n+1)}{6}\right] - 1$$

4.2 L'hopital Rule

Things to remember:

Theorem. Suppose we are trying to evaluate:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

- If $\lim_{x\to\infty} f(x) \ = \lim_{x\to\infty} g(x) = 0$ then:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

- If $\lim_{x\to\infty} f(x) \ = \lim_{x\to\infty} g(x) = \infty$ then:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

4.3 Manipulation of Logs

$$\begin{aligned}
log_b a &= \frac{log_c a}{log_c b} \text{ (base changes)} \\
log_b(xy) &= log_b x + log_b y \\
log_b(\frac{x}{y}) &= log_b x - log_b y \\
log_b(x^p) &= p log_b x
\end{aligned}$$