CMSC351: Prerequisite

Johning To

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1 Symbols

1. For all ($\forall$)
2. There exists ($\exists$)

2 Proofs

2.1 Weak Induction

First, we need to prove $\forall n \geq n_0 \ P(n)$ we first prove $P(n_0)$ (which is the base case) and then we prove $\forall k \geq n_0 \ P(k) \rightarrow P(k+1)$ (which is the inductive step) (you can also prove $P(k-1)$). The assumption of $P(k)$ in the inductive step is the inductive hypothesis.

Ultimately, for the inductive step we are trying to find that for any $k \geq n_0$ if $P(k)$ is true, then $P(k+1)$ is also true.

Example: Suppose we have a set of nested Russian dolls (Matryoshka dolls). Each doll is contained with another doll, each doll is labeled 1, 2, 3 and so on.

In this hypothetical, there are two things that are true. Let's say $M(n)$ is true iff (if and only if) doll n has another doll contained.

(a) The first doll has another doll contained. That is, $M(n)$ is true.

(b) For every doll k, if doll k has another doll inside then doll $k+1$ has another doll inside.

$$\forall k \geq 1, M(k) \rightarrow M(k+1)$$

We can now conclude that $\forall n \geq 1, M(n)$.

2.2 Strong Induction

The goal of strong induction is we need to find some property, we can say $P(n)$ and we need to find some n greater than equal to a ($n \geq a$).

How can we accomplish this?

Step 1 (Basis step): We are going to prove for $P(a)$ (we would just prove $P(a)$ for weak induction), $P(a+1)$, ..., for some finite number say $P(b)$.

$(P(a),\ P(a+b),\ ...,\ P(b)) \leftarrow$ we prove each of these.

Step 2 (Induction): We can now assume $P(i)$ where $a \leq i \leq k$. (Assume $P(i)$). Then, we prove $P(k+1)$.

Example: Fibonacci Sequence

Proof. Claim: The Fibonacci Sequence is defined as the following: $F(0) = 0, F(1) = 1$ and for $n \geq 2, F(n) = F(n-1) + F(n-2)$. We want to prove that for all $n \geq 0, F(n) \leq 2^n$

Base cases:

$$For\ n = 0 : F(0) = 0 \leq 2^0 = 1$$
$$For\ n = 1 : F(1) = 1 \leq 2^1 = 1$$

For both these base case, they hold true.

Inductive Step: Assume that for all i, $0 \leq i \leq k$, we have $F(k) \leq 2^k$.

(This could also be known as the inductive hypothesis)

We now need to prove $F(k+1) \leq 2^{k+1}$.

By the inductive hypothesis, we have $F(n) \leq 2^n$ and $F(n-1) \leq 2^{n-1}$.

We can prove by which:

$$\begin{aligned}
F(k+1) &= F(k) + F(k-1) \\
&\leq 2^{k-1} + 2^{k-2} && \text{(IH)} \\
&\leq 2^{k-1} + 2^{k-1} && (2^{k-2} \text{ is less than } 2^{k-1}) \\
&= 2^k && \text{(Simplify)}
\end{aligned}$$

□

2.3 Constructive Induction

When we are solving recurrences and we have guessed the general form, and we do not know the constants, we usepackage constructive induction.

Example: We know that $\sum_{i=1}^{n} i = \frac{1}{2}n^2 - \frac{1}{2}n$. But how do we determine $\sum_{i=1}^{n} i^2$? Since we know the solution for the first sum is a quadratic, we can guess that for the second sum that is cubic $(an^3 + bn^2 + cn + d)$. We start with assumptions. $\sum_{i=1}^{n-1} i = a(n-1)^3 + b(n-1)^2 + c(n-1) + d$ and $n > 0$ We need to prove that $\sum i = 1^n i^2 = an^3 + bn^2 + cn + d$ So, we need:

$$\begin{aligned}
\sum_{i=1}^{n} i &= an^3 + bn^2 + cn + d \\
\sum_{i=1}^{n-1} i + n^2 &= an^3 + bn^2 + cn + d \\
a(n-1)^3 + b(n-1)^2 + c(n-1) + d + n^2 &= an^3 + bn^2 + cn + d \\
a(n^3 - 3n^2 + 3n - 1) + b(n^2 - 2n + 1) + c(n-1) + d &= an^3 + bn^2 + cn + d \\
an^3 + (b - 3a)n^2 + (3a - 2b + c)n + (d - a + b - c) &= an^3 + bn^2 + cn + d \\
an^3 + (b - 3a + 1)n^2 + (3a - 2b + c)n + (d - a + b - c) &= an^3 + bn^2 + cn + d
\end{aligned}$$

We arrive at a systems of equations

$$\begin{aligned} b - 3a + 1 &= b \\ 3a - 2b + c &= c \\ d - a + b - d &= d \end{aligned}$$

Thus, $a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$ and $d = 0$

Therefore,

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}$$

2.4 Structural Induction