Peridynamic Beams, Plates, and Shells

a non-ordinary state-based model

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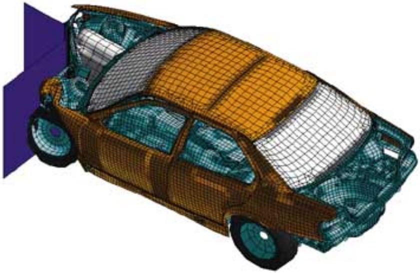
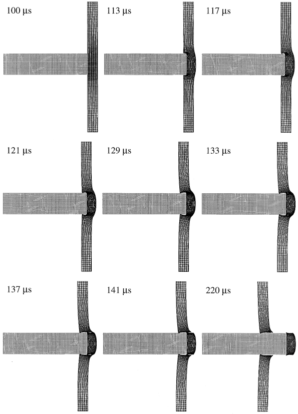
1. Motivation
2. Background
   - Classical Modeling Approaches
   - Peridynamics
   - Thin Features
3. Objectives
4. Methodology
   - Bond-pair Models
5. Preliminary Results
6. Conclusion
   - Summary
   - Ongoing Work

Material Failure Drives Design

Sometimes,
we *need* to model failure



Car crash



Ballistic plate penetration[1]

[1] T Børvik et al. "Ballistic penetration of steel plates". In: *International Journal of Impact Engineering* 22.9 (1999), pp. 855–886

Classical Dynamics

PDE based methods like XFEM and RKPM solve the classical dynamics equation for continuum momentum conservation to find the relationship:

Displacement $\leftrightarrow$ Strain $\leftrightarrow$ Stress $\leftrightarrow$ Force $\leftrightarrow$ Accelleration

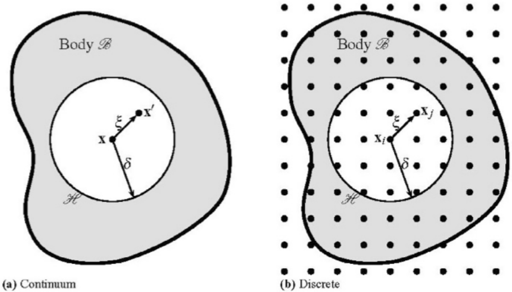
Classical Dynamics has some inconvenient features

- Strain is the spatial derivative of displacement: $\epsilon := \frac{\partial u}{\partial \mathbf{X}}$
    - not defined if displacements are discontinuous
- Stress and Strain are *local* functions of displacement: $\sigma(\mathbf{x}) = f(\epsilon(\mathbf{x}))$
    - some materials exhibit nonlocal dependence

So solving a PDE is begging the question: What is the spatial derivative at a crack?

What is Peridynamics?

- Greek: *peri* - near or around, *dyna* - force
- Force at a point is a function of the condition of the surrounding area (Strongly Nonlocal)
- Naturally handles discontinuous displacements



Two peridynamic bodies[2]

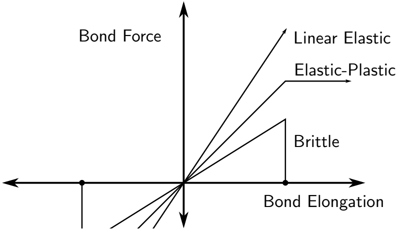
[2] J.T. Foster, S.A. Silling, and W.W. Chen. "State based peridynamic modeling of dynamic fracture". In: *SEM Annual Conf and Exposition on Experimental and Applied Mechanics, Albuquerque, USA*. 2009, pp. 2312–2317.

Bond-based Models

Bond-based peridynamic Equation of Motion[3] :

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}) = \int_{\mathcal{H}} \mathbf{f}\left(\mathbf{u}\left(\mathbf{x}', t\right) - \mathbf{u}\left(\mathbf{x}, t\right), \mathbf{x}' - \mathbf{x}\right) dV_{\mathbf{x}'} + \mathbf{b}\left(\mathbf{x}, t\right)$$

The force at a point is an integral of contributions from all the relative displacements $(\mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t))$ inside a horizon $\mathcal{H}$. (no strain term ϵ)



With a linear elastic bond force, the peridynamic model can reduce to a classical linear elastic solid with $\nu = \frac{1}{4}$ or a 2D plate with $\nu = \frac{1}{3}$.

[3] S.A. Silling. "Reformulation of elasticity theory for discontinuities and long-range forces". In: *Journal of the Mechanics and Physics of Solids* 48.1 (2000), pp. 175–209.

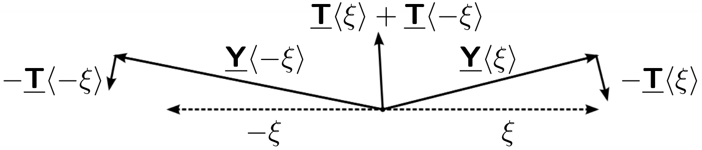
State-based Models

In a state-based model, the force between two points can depend on the behavior of the other bonds surrounding them.

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{H}} \left\{\underline{\mathbf{T}}[\mathbf{x},t]\langle \mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{q},t]\langle \mathbf{x}-\mathbf{q}\rangle\right\}\; dV_q + \hat{\mathbf{b}}(\mathbf{x},t),$$

where $\underline{\mathbf{T}}$ is the vector state force function.

$\underline{\mathbf{T}}$ need not be oriented along the bond, this *non-ordinary* state-based model could represent rotational springs between pairs of bonds.



Model Types

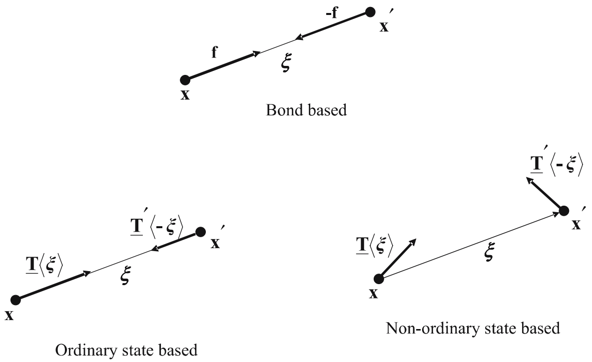


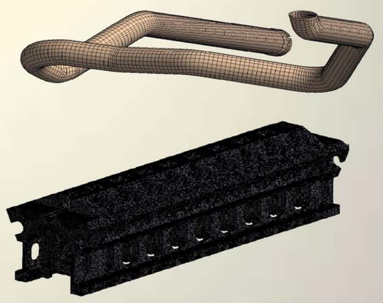
Illustration of the three types of peridynamic models, from specific to general[4]

[4]SA Silling et al. "Peridynamic states and constitutive modeling". In: *Journal of Elasticity* 88.2 (2007), pp. 151–184.

Thin Features

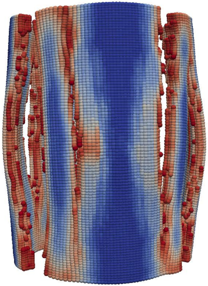
Thin features present challenges for 3D solid models. To accurately capture some material behavior, discretization must be very dense.

Finite Element thin features can be greatly simplified



ANSYS brochure

Peridynamics has no equivalent models



[5]

Thin models use the same failure modeling techniques as solid models

[5]Image from David John Littlewood. *Simulation of dynamic fracture using peridynamics, finite element modeling, and contact.* Tech. rep. Sandia National Laboratories, 2010

Objectives

Lay the foundation:

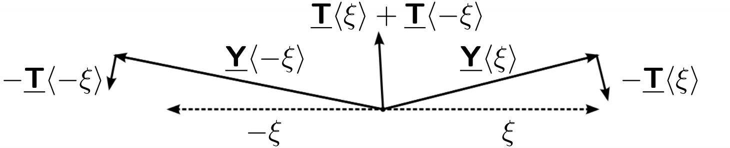
- Create peridynamic beam, plate, and shell models
- Verify correspondence to classical continuum models

Ultimate Goal:

- Simulate thin feature behavior
- Develop and validate failure models
- Simulate thin feature failure

Bond Pair Model

Starting with Silling's proposed non-ordinary State-based model[6]



in which

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = \frac{\alpha}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}\frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|} \times \left[\frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|} \times \frac{\underline{\mathbf{Y}}\langle-\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle-\boldsymbol{\xi}\rangle|}\right]$$

This force state is the Fréchet derivative of the energy state

$$\underline{w}[x]\langle\boldsymbol{\xi}\rangle = \alpha[1 + \cos(\underline{\theta}[x]\langle\boldsymbol{\xi}\rangle)]$$

[6] SA Silling et al. "Peridynamic states and constitutive modeling". In: *Journal of Elasticity* 88.2 (2007), pp. 151–184.

Continuous Bond Pair Beam

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Start by modeling a 1D beam in bending.
The deformed bond-pair angle is:

$$\theta(\underline{\mathbf{Y}}[x]\langle\xi\rangle, \underline{\mathbf{Y}}[x]\langle-\xi\rangle) \approx \pi - \frac{y(x+\xi) - 2y(x) + y(x-\xi)}{\xi}$$

which is very similar to the finite difference second derivative

$$\theta(\underline{\mathbf{Y}}\langle\xi\rangle, \underline{\mathbf{Y}}\langle-\xi\rangle) \approx \pi - \xi\frac{\partial^2 y}{\partial x^2} = \pi - \xi\kappa$$

The resulting total strain energy density

$$W(x) \approx \int_{-\delta}^{\delta} \omega(\xi)\alpha\frac{\xi^2}{2}\kappa^2 d\xi = \frac{\alpha}{2}\kappa^2\int_{-\delta}^{\delta}\omega(\xi)\xi^2 d\xi,$$

in which $\omega(\xi)$ is a weighting function

Bond Pair Beam Energy

Choosing α carefully results in the classical Euler beam strain energy:

$$\alpha = \frac{EI}{m};\ m = \int_{-\delta}^{\delta} \omega(\xi)\xi^2 d\xi \implies W = \frac{EI}{2}\kappa^2$$

or the discrete version:

$$\alpha = \frac{EI\ \Delta x}{m};\ m = \sum_{i=1}^{n} \omega(\xi_i)\xi_i^2 \implies$$

$$W = \Delta x \sum_{i=1}^{n} \frac{EI}{2}\left(\frac{y(x+\xi_i) - 2y(x) + y(x-\xi_i)}{\xi_i}\right)^2$$

Bond Pair Damage Models

Determine critical angle θ_c from δ, ϵ_c, and t.
Brittle material: bond pair ceases to exist
Nonlinear elastic material: same moment regardless of angle

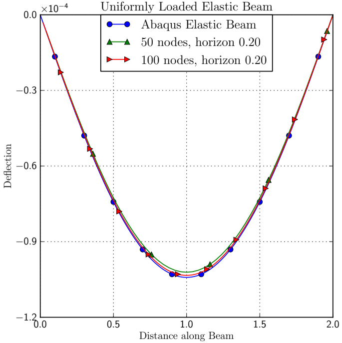
$$|\underline{\mathbf{T}}\langle\xi\rangle| = \begin{cases} \alpha \frac{\sin(\theta(\underline{\mathbf{Y}}\langle\xi\rangle,\underline{\mathbf{Y}}\langle-\xi\rangle))}{|\underline{\mathbf{Y}}\langle\xi\rangle|} & \text{if } \theta < \theta_c \\ \alpha \frac{\sin(\theta_c)}{|\underline{\mathbf{Y}}\langle\xi\rangle|} & \text{if } \theta \geq \theta_c \end{cases}$$

Elastic perfectly-plastic material: track the plastic deformation
$\theta^p(\xi) = \theta - \theta_c$ and let $\theta^e(\xi) = \theta - \theta^p$

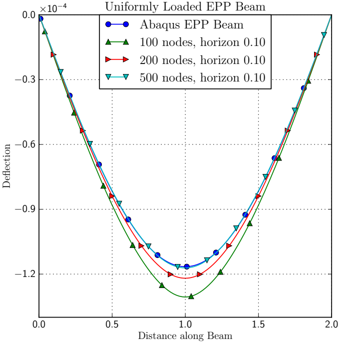
$$|\underline{\mathbf{T}}\langle\xi\rangle| = \begin{cases} \alpha \frac{\sin(\theta^e(\underline{\mathbf{Y}}\langle\xi\rangle,\underline{\mathbf{Y}}\langle-\xi\rangle))}{|\underline{\mathbf{Y}}\langle\xi\rangle|} & \text{if } \theta^e < \theta_c \\ \alpha \frac{\sin(\theta_c)}{|\underline{\mathbf{Y}}\langle\xi\rangle|} & \text{if } \theta^e \geq \theta_c \end{cases}$$

Beam Results

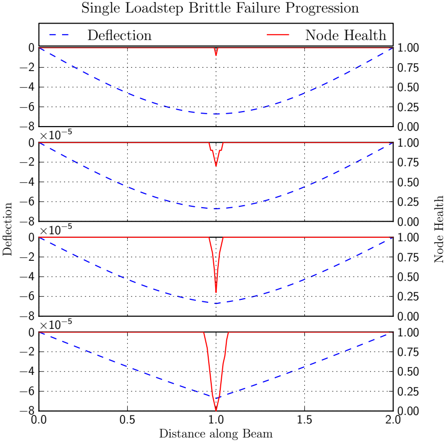
Elastic



Elastic-Plastic

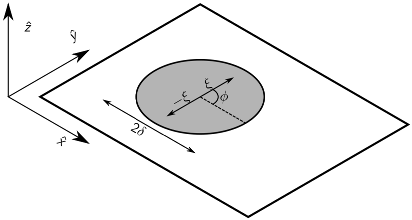


Brittle Beam:



Bond Pair Plate

The same equations can model a peridynamic plate in bending:



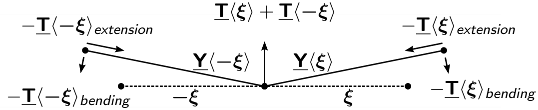
By choosing α as before, we find that this matches the classical strain energy for a plate with $\nu = 1/3$

$$\alpha = \frac{c}{m};\; c = \frac{Gt^3}{6\pi};\; m = \int_0^\delta \omega(r)\frac{r^3}{2}dr;\; \nu = \frac{1}{3} \implies$$

$$W = \frac{Gt^3}{12(1-\nu)}\left((\kappa_1)^2 + (\kappa_2)^2 + 2\nu\,(\kappa_1\kappa_2) + 2(1-\nu)\,(\kappa_3)^2\right)$$

In-Plane Deformation

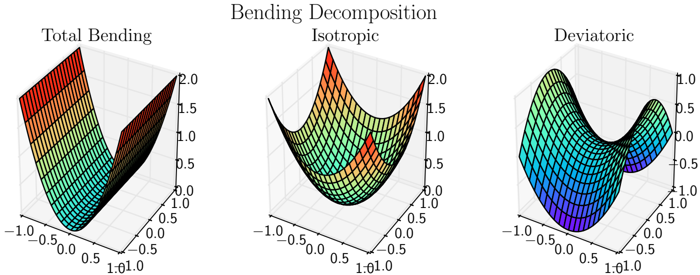
Pure bending model does not resist in-plane stretch or shear, but we can combine it with an extension-based model



Failure of bond-pairs and failure of extension bonds can be coupled so that either mode of damage reduces both modes of stiffness

Arbitrary Poisson's Ratio

Bending can be divided into 2 types:

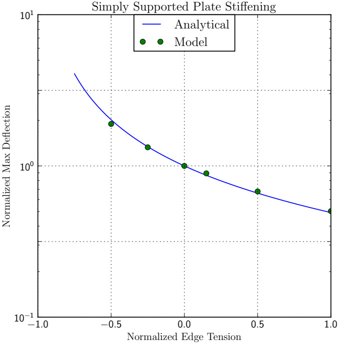


A bending "pressure" proportional to the isotropic curvature $\bar{\boldsymbol{\kappa}}$ allows simulation of arbitrary ν

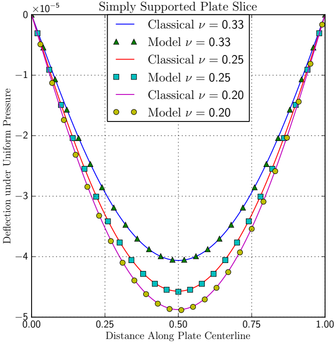
$$\underline{\mathbf{T}'}\langle\boldsymbol{\xi}\rangle = \frac{8G}{m}\frac{h^3}{12}\frac{3\nu-1}{1-\nu}\frac{\omega(\boldsymbol{\xi})}{\xi^2}\bar{\boldsymbol{\kappa}}$$

Plate Results

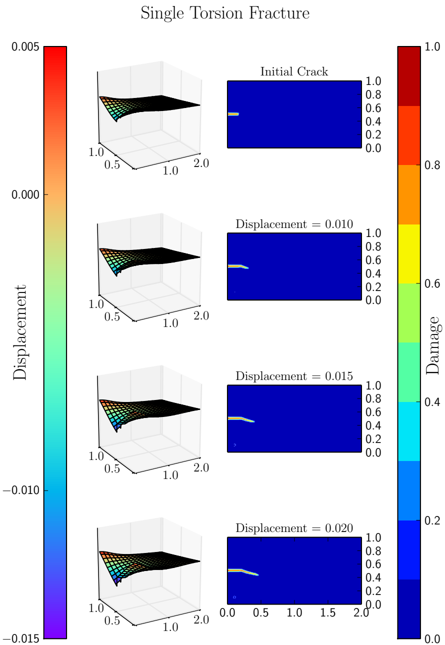
Tension-Stiffening



Arbitrary ν



Brittle Plate:



Accomplished

The first state-based peridynamic thin features

- Strain energy equivalence demonstrated
- Failure models proposed
- Numerical models coded and evaluated
- Good beam, plate, and flat shell results

In progress

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Virtual point pairing

- Irregular discretization
- Curved beams, plates, shells

Energy rate based failure models

Questions?