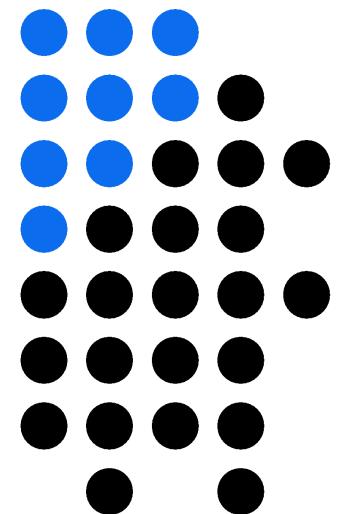
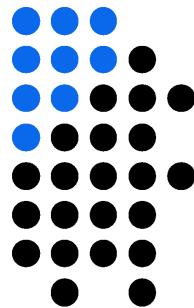


Ch 8. Multiphase Flow in Porous Media

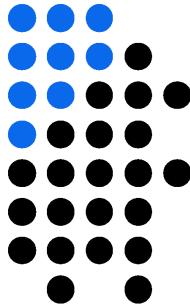
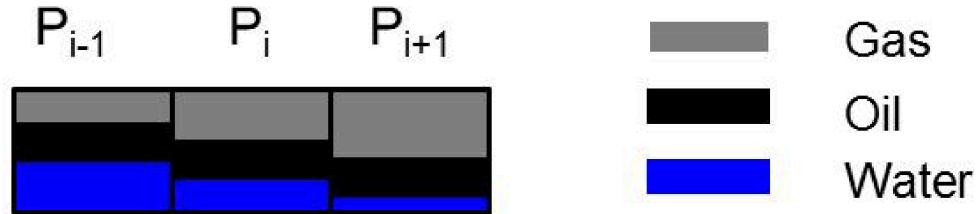


Review of Numerical Simulation of Single Phase Flow



1. Derived continuity and diffusivity equations
2. Derived finite difference approximations for derivatives
3. Discretized PDE to obtain algebraic equations
4. Included reservoir heterogeneities in model
5. Extended to multidimensions
6. Added wells (constant rate and BHP) into model

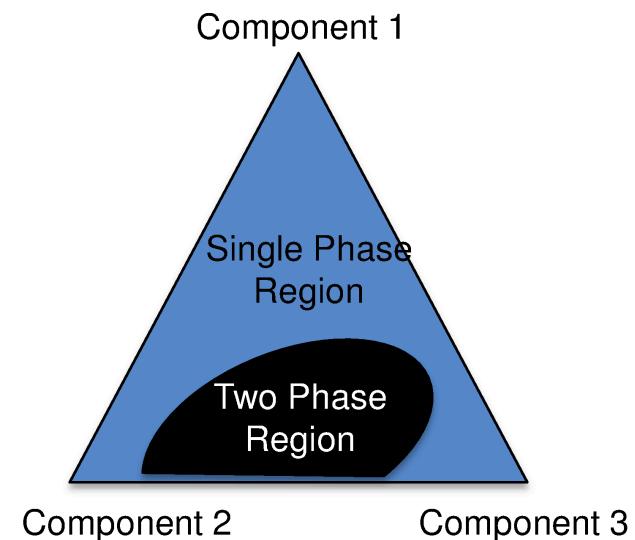
Multiphase Flow



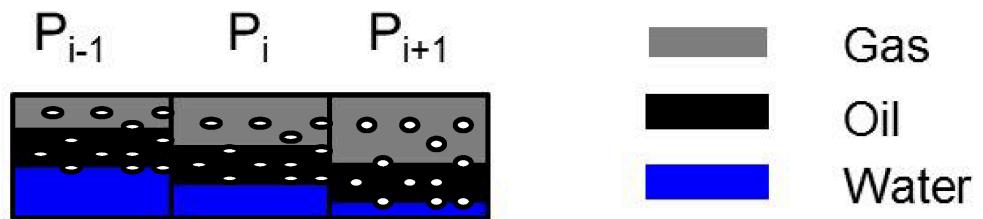
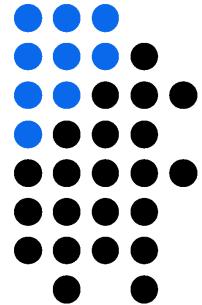
Applications: water flooding, aquifer support, saturated reservoirs, gas flooding, and more!

Modeling Approaches:

1. **Compositional:** Models several individual components. Accurate but very slow and computationally demanding
 - Component : any identifiable chemical entity
 - Phase : a homogeneous region of matter
 - Can have gas component in the oil *phase*!
 - Balance on the component, not the phase
2. **Black Oil:** Models 3 components (oil, gas, and water) in 3 *phases*: Oil (oleic), Water (aqueous), and Gas (gaseous)

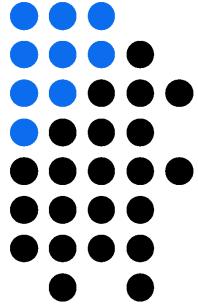


Black Oil Models (or β -model)

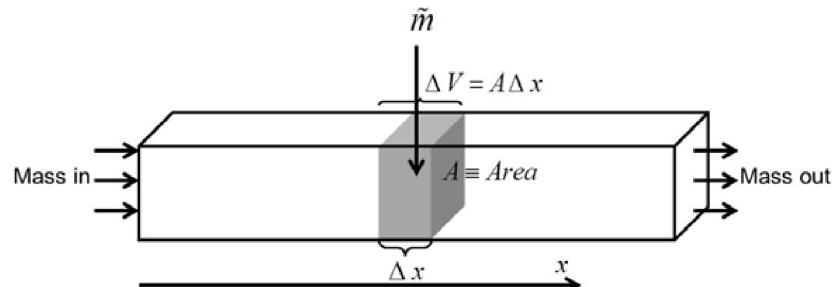


- Gas component in oil or flowing separately
- Gas does not dissolve in water
- Water and oil do not mix

Mass Balance of Water



Balance of water component:



$$\underbrace{\left(\dot{m}_w \Big|_x - \dot{m}_w \Big|_{x+\Delta x} \right) A \Delta t}_{\text{mass in-out water in aqueous phase}} + \underbrace{\tilde{m}_{w,\text{well}} \Delta V \Delta t}_{\substack{\text{mass in from well} \\ \text{water in aqueous phase}}} = \underbrace{\left(m_w \Big|_{t+\Delta t} - m_w \Big|_t \right)}_{\substack{\text{mass accumulated} \\ \text{water in aqueous phase}}} \quad S_w + S_o + S_g = 1$$

$S_w \rightarrow \% \text{ water in mixture}$

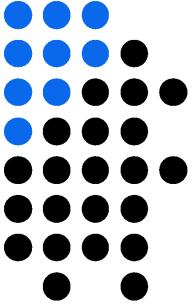
Recognizing: $\dot{m}_w = \rho_w u_w$ $m_w = \cancel{\rho_w} S_w \phi \Delta V$ $\rho_w \equiv \frac{\text{mass of water component}}{\text{volume aqueous phase}}$

$$\underbrace{\left(\rho_w u_w \Big|_x - \rho_w u_w \Big|_{x+\Delta x} \right) A \Delta t}_{\text{mass in-out water in aqueous phase}} + \underbrace{\tilde{m}_{w,\text{well}} \Delta V \Delta t}_{\substack{\text{mass in from well} \\ \text{water in aqueous phase}}} = \underbrace{\left(\rho_w S_w \phi \Delta V \Big|_{t+\Delta t} - \rho_w S_w \phi \Delta V \Big|_t \right)}_{\substack{\text{mass accumulated} \\ \text{water in aqueous phase}}}$$

Divide by $A \Delta x \Delta t$ and taking limits gives differential mass balance:

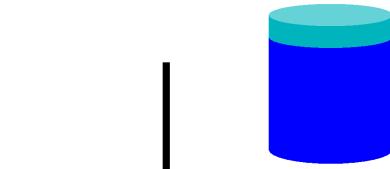
$$-\frac{\partial(\rho_w u_w)}{\partial x} + \tilde{m}_{w,\text{well}} = \frac{\partial(\rho_w \phi S_w)}{\partial t}$$

$\dot{m} \equiv \text{Mass flux} \left[\frac{M}{L^2 T} \right]$ $A \equiv \text{Area} \left[L^2 \right]$ $\tilde{m} \equiv \text{Sink or source rate} \left[\frac{M}{L^3 T} \right]$ $\Delta V \equiv \text{Volume} \left[L^3 \right]$ $\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$ $\phi \equiv \text{Porosity} \left[\frac{L^3}{L^3} \right]$



Water Formation Volume Factor (B_w)

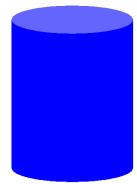
We measure volume at surface but do the mass balance at the reservoir



Surface/Standard Conditions

$$P = 14.7 \text{ psi}; T = 60^\circ\text{F}; V = V_{sc}$$

depth



Reservoir Conditions

$$P = P_R; T = T_R; V = V_R$$

$$B_w = \frac{\text{reservoir volume of aqueous phase}}{\text{std. volume aqueous phase}} = \frac{[V_w]^{RC}}{[V_w]^{SC}} = \frac{\rho_w^{SC}}{\rho_w^{RC}}$$

Note: B_w (formation volume factor for water) ~ 1.0

$$\Delta V \equiv \text{Volume} [L^3]$$

$$P \equiv \text{Pressure} \left[\frac{M}{LT^2} \right]$$

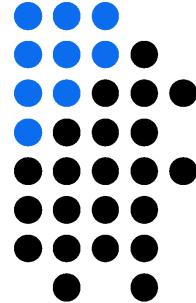
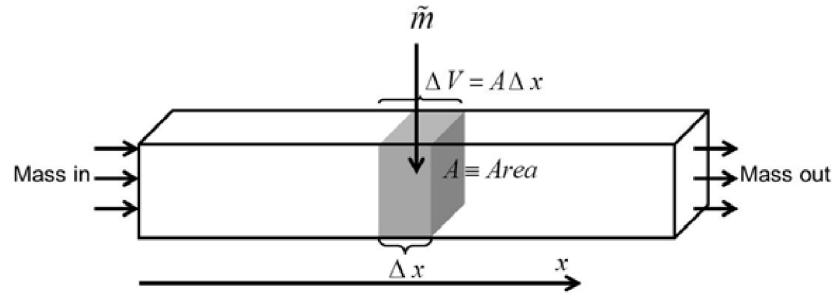
$$T \equiv \text{Temperature} [t]$$

$$B_w \equiv \text{Volumetric Factor (Water)} \left[\frac{L^3}{L^3} \right]$$

$$\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$$

Mass Balance of Water

$$-\frac{\partial(\rho_w u_w)}{\partial x} + \tilde{m}_{w,well} = \frac{\partial(\rho_w \phi S_w)}{\partial t}$$



Divide by density at standard conditions, ρ_w^{sc} , and use our definition of formation volume factor of water

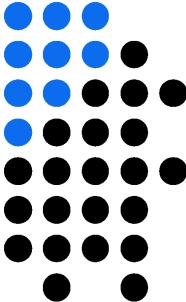
$$B_w = \frac{\rho_w^{sc}}{\rho_w^{rc}}$$

We get balance equation in terms of formation volume factor

$$-\frac{\partial}{\partial x} \left(\frac{u_w}{B_w} \right) + \tilde{q}_w = \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right)$$

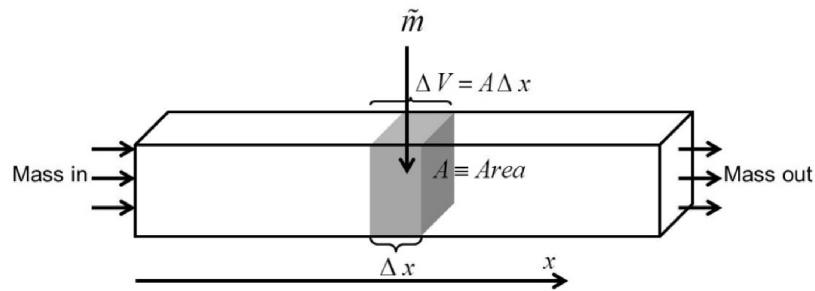
$\dot{m} \equiv \text{Mass flux} \left[\frac{M}{L^2 T} \right]$ $A \equiv \text{Area} \left[L^2 \right]$ $\tilde{m} \equiv \text{Sink or source rate} \left[\frac{M}{L^3 T} \right]$ $\Delta V \equiv \text{Volume} \left[L^3 \right]$ $\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$ $\phi \equiv \text{Porosity} \left[\frac{L^3}{L^3} \right]$

Mass Balance of Oil



Balance of oil component:

$$\underbrace{\left(\dot{m}_o \Big|_x - \dot{m}_o \Big|_{x+\Delta x} \right) A \Delta t}_{\text{mass in-out oil in oleic phase}} + \underbrace{\tilde{m}_{o,\text{well}} \Delta V \Delta t}_{\text{mass in from well oil in oleic phase}} = \underbrace{\left(m_o \Big|_{t+\Delta t} - m_o \Big|_t \right)}_{\text{mass accumulated oil in oleic phase}}$$



Recognizing: $\dot{m}_o = \rho_o u_o$ $m_o = \rho_o S_o \phi \Delta V$ $\rho_o \equiv \frac{\text{mass of oil component}}{\text{volume oleic phase}}$

$$\underbrace{\left(\rho_o u_o \Big|_x - \rho_o u_o \Big|_{x+\Delta x} \right) A \Delta t}_{\text{mass in-out oil in oleic phase}} + \underbrace{\tilde{m}_{o,\text{well}} \Delta V \Delta t}_{\text{mass in from well oil in oleic phase}} = \underbrace{\left(\rho_o S_o \phi \Delta V \Big|_{t+\Delta t} - \rho_o S_o \phi \Delta V \Big|_t \right)}_{\text{mass accumulated oil in oleic phase}}$$

Divide by $A \Delta x \Delta t$ and taking limits:

$$-\frac{\partial(\rho_o u_o)}{\partial x} + \tilde{m}_{o,\text{well}} = \frac{\partial(\rho_o \phi S_o)}{\partial t}$$

$\omega \rightarrow \sigma$

| | | | | | |
|--|---|--|--|---|--|
| $\dot{m} \equiv \text{Mass flux} \left[\frac{M}{L^2 T} \right]$ | $A \equiv \text{Area} \left[L^2 \right]$ | $\tilde{m} \equiv \text{Sink or source rate} \left[\frac{M}{L^3 T} \right]$ | $\Delta V \equiv \text{Volume} \left[L^3 \right]$ | $\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$ | $\phi \equiv \text{Porosity} \left[\frac{L^3}{L^3} \right]$ |
|--|---|--|--|---|--|

Formation Volume Factors and Solubility Ratios

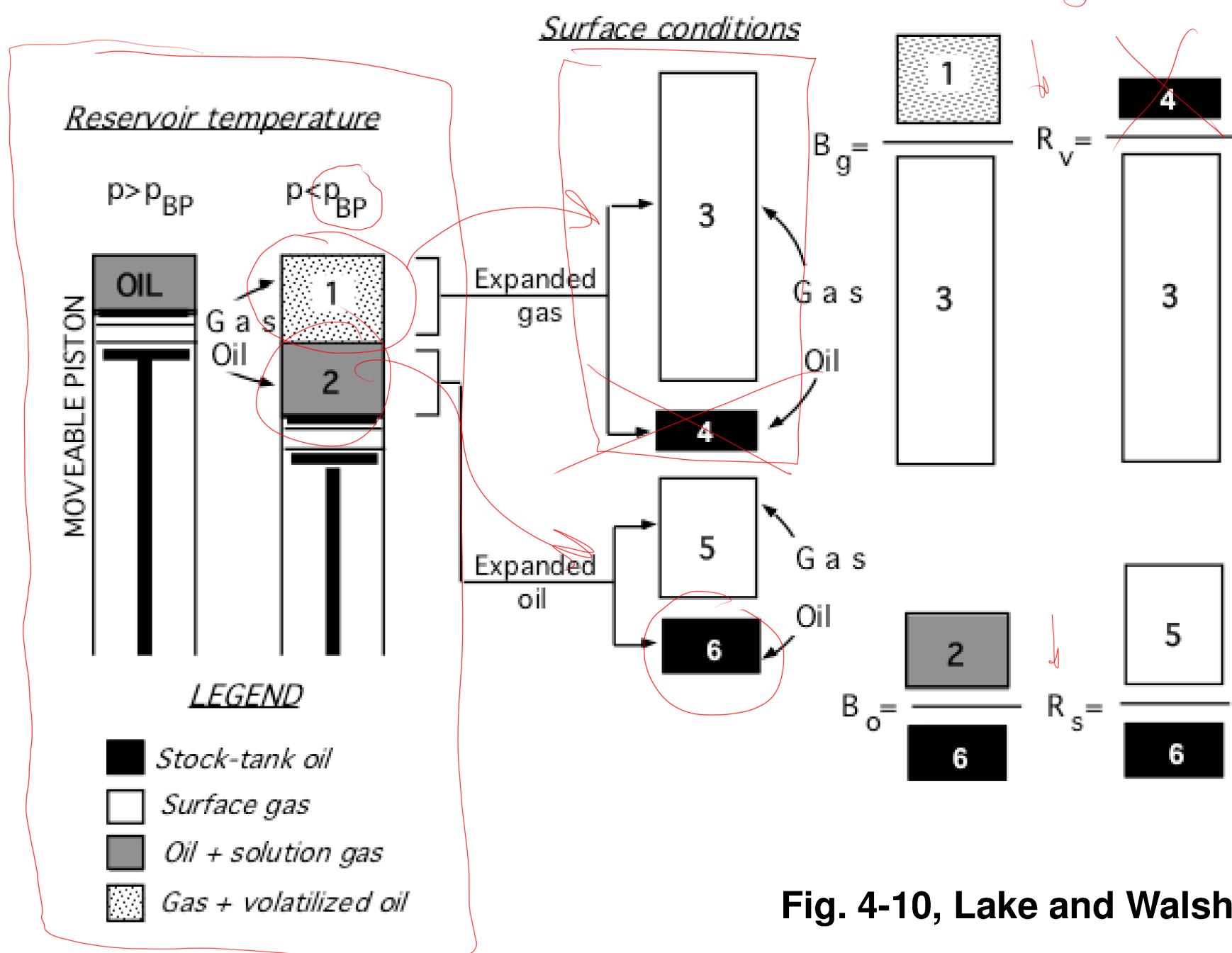
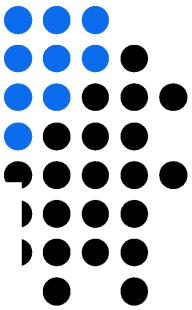
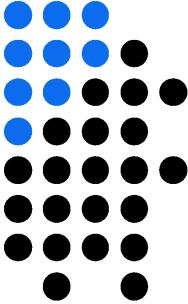


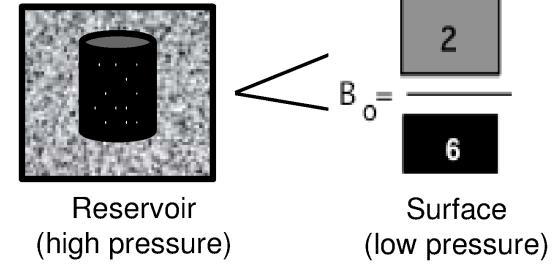
Fig. 4-10, Lake and Walsh (2003)



Oil Formation Volume Factor (B_o)

Oil formation volume factor is defined as:

$$B_o = \frac{\text{res. vol. of oleic (oil + dissolved gas) phase}}{\text{std. volume oleic phase (oil only)}} = \frac{\left[V_o^{RC} \right]}{\left[V_o^{SC} \right]} [=] \frac{\text{RB oleic}}{\text{STB oil}}$$



Mass of oleic phase at RC is mass of liquid + mass of dissolved gas at SC

$$\text{mass oleic}^{RC} = \text{mass oil}^{SC} + \text{mass gas}^{SC}$$

$$\rho_{oleic}^{RC} V_{oleic}^{RC} = \rho_o^{SC} V_o^{SC} + \rho_g^{SC} V_{dg}^{SC}$$

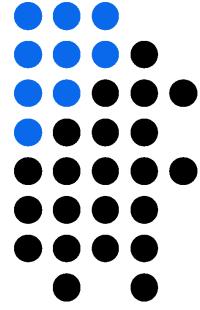
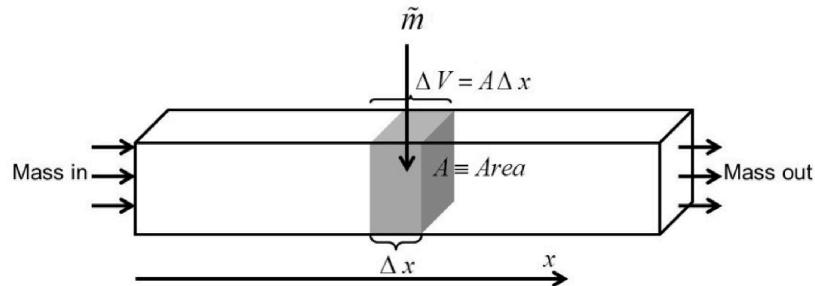
Divide through by volume of oleic phase we get:

$$\begin{aligned} \rho_{oleic}^{RC} &= \rho_o^{SC} \frac{V_o^{SC}}{V_{oleic}^{RC}} + \rho_g^{SC} \frac{V_{dg}^{SC}}{V_{oleic}^{RC}} \cdot \frac{V_o^{SC}}{V_o^{SC}} = \frac{\rho_o^{SC} + R_s \rho_g^{SC}}{B_o} = \underbrace{\frac{\rho_o^{SC}}{B_o}}_{\rho_o^{RC}} + \underbrace{\frac{R_s \rho_g^{SC}}{B_o}}_{\rho_{dg}^{RC}} \\ P_{oleic}^{RC} &= P_o^{RC} + P_{dg}^{RC} \end{aligned}$$

Note: $B_o > 1$ (usually, $1.0 < B_o < 2.0$) because dissolved gas comes out of solution

Mass Balance of Oil

$$-\frac{\partial(\rho_o u_o)}{\partial x} + \tilde{m}_{o,well} = \frac{\partial(\rho_o \phi S_o)}{\partial t}$$



Divide by density at standard conditions, ρ_o^{sc} , and use our definition of formation volume factor of oil

$$B_o = \frac{\rho_o^{sc}}{\rho_o^{rc}}$$

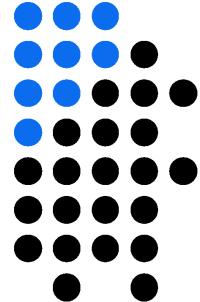
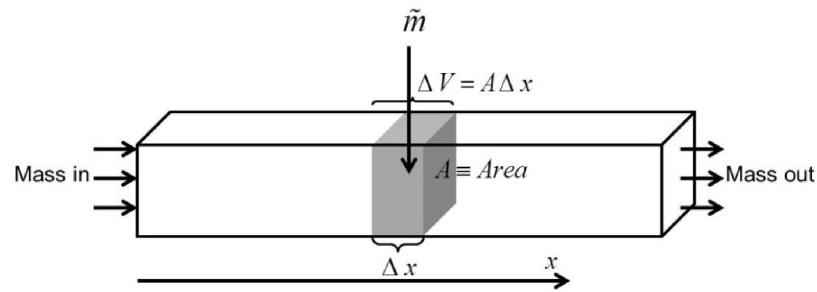
We get balance equation in terms of formation volume factor

$$-\frac{\partial}{\partial x} \left(\frac{u_o}{B_o} \right) + \tilde{q}_o = \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right)$$

$\dot{m} \equiv \text{Mass flux} \left[\frac{M}{L^2 T} \right]$ $A \equiv \text{Area} \left[L^2 \right]$ $\tilde{m} \equiv \text{Sink or source rate} \left[\frac{M}{L^3 T} \right]$ $\Delta V \equiv \text{Volume} \left[L^3 \right]$ $\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$ $\phi \equiv \text{Porosity} \left[\frac{L^3}{L^3} \right]$

Mass Balance of Gas

Balance of gas component:



$$\begin{aligned}
 & \underbrace{\left(\dot{m}_g \Big|_x - \dot{m}_g \Big|_{x+\Delta x} \right)}_{\text{mass in-out gas in gaseous phase}} A \Delta t + \underbrace{\left(\dot{m}_{dg} \Big|_x - \dot{m}_{dg} \Big|_{x+\Delta x} \right)}_{\text{mass in-out dissolved gas in oleic phase}} A \Delta t + \underbrace{\tilde{m}_{g,\text{well}} \Delta V \Delta t}_{\substack{\text{mass in from well} \\ \text{gas in gaseous phase}}} + \underbrace{\tilde{m}_{dg,\text{well}} \Delta V \Delta t}_{\substack{\text{mass in from well} \\ \text{dissolved gas in oil phase}}} \\
 &= \underbrace{\left(m_g \Big|_{t+\Delta t} - m_g \Big|_t \right)}_{\substack{\text{mass accumulated} \\ \text{gas in gaseous phase}}} + \underbrace{\left(m_{dg} \Big|_{t+\Delta t} - m_{dg} \Big|_t \right)}_{\substack{\text{mass accumulated} \\ \text{dissolved gas in oil phase}}}
 \end{aligned}$$

Recognizing:

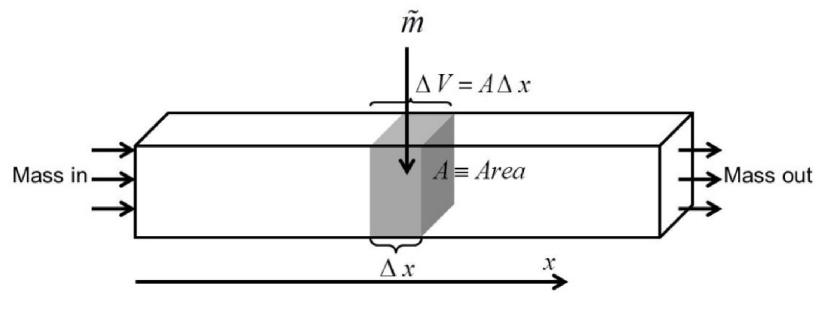
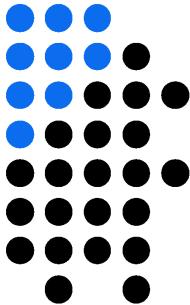
$$\begin{aligned}
 \dot{m}_g &= \rho_g u_g & m_g &= \rho_g S_g \phi \Delta V \\
 \dot{m}_{dg} &= \rho_{dg} u_o & m_{dg} &= \rho_{dg} S_o \phi \Delta V
 \end{aligned}$$

$$\begin{aligned}
 \rho_g &\equiv \frac{\text{mass of gas component}}{\text{volume gaseous phase}} \\
 \rho_{dg} &\equiv \frac{\text{mass of dissolved gas}}{\text{volume oleic phase}}
 \end{aligned}$$

$$\begin{aligned}
 & \underbrace{\left(\rho_g u_g \Big|_x - \rho_g u_g \Big|_{x+\Delta x} \right)}_{\text{mass in-out gas in gaseous phase}} A \Delta t + \underbrace{\left(\rho_{dg} u_o \Big|_x - \rho_{dg} u_o \Big|_{x+\Delta x} \right)}_{\text{mass in-out dissolved gas in oleic phase}} A \Delta t + \underbrace{\tilde{m}_{g,\text{well}} \Delta V \Delta t}_{\substack{\text{mass in from well} \\ \text{gas in gaseous phase}}} + \underbrace{\tilde{m}_{dg,\text{well}} \Delta V \Delta t}_{\substack{\text{mass in from well} \\ \text{dissolved gas in oil phase}}} = \\
 & \underbrace{\left(\rho_g S_g \phi \Delta V \Big|_{t+\Delta t} - \rho_g S_g \phi \Delta V \Big|_t \right)}_{\substack{\text{mass accumulated} \\ \text{gas in gaseous phase}}} + \underbrace{\left(\rho_{dg} S_o \phi \Delta V \Big|_{t+\Delta t} - \rho_{dg} S_o \phi \Delta V \Big|_t \right)}_{\substack{\text{mass accumulated} \\ \text{dissolved gas in oil phase}}}
 \end{aligned}$$

| | | | | | |
|--|---|--|--|---|--|
| $\dot{m} \equiv \text{Mass flux} \left[\frac{M}{L^2 T} \right]$ | $A \equiv \text{Area} \left[L^2 \right]$ | $\tilde{m} \equiv \text{Sink or source rate} \left[\frac{M}{L^3 T} \right]$ | $\Delta V \equiv \text{Volume} \left[L^3 \right]$ | $\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$ | $\phi \equiv \text{Porosity} \left[\frac{L^3}{L^3} \right]$ |
|--|---|--|--|---|--|

Mass Balance of Gas

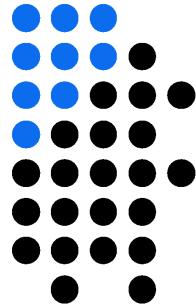


$$\underbrace{\left(\rho_g u_g \Big|_x - \rho_g u_g \Big|_{x+\Delta x} \right) A \Delta t}_{\text{mass in-out gas in gaseous phase}} + \underbrace{\left(\rho_{dg} u_o \Big|_x - \rho_{dg} u_o \Big|_{x+\Delta x} \right) A \Delta t}_{\text{mass in-out dissolved gas in oilic phase}} + \underbrace{\tilde{m}_{g,well} \Delta V \Delta t}_{\text{mass in from well gas in gaseous phase}} + \underbrace{\tilde{m}_{dg,well} \Delta V \Delta t}_{\text{mass in from well dissolved gas in oil phase}} = \\
 \underbrace{\left(\rho_g S_g \phi \Delta V \Big|_{t+\Delta t} - \rho_g S_g \phi \Delta V \Big|_t \right)}_{\text{mass accumulated gas in gaseos phase}} + \underbrace{\left(\rho_{dg} S_o \phi \Delta V \Big|_{t+\Delta t} - \rho_{dg} S_o \phi \Delta V \Big|_t \right)}_{\text{mass accumulated dissolved gas in oil phase}}$$

Divide by $A \Delta x \Delta t$ and taking limits:

$$- \frac{\partial (\rho_g u_g + \rho_{dg} u_o)}{\partial x} + \tilde{m}_{g,well} + \tilde{m}_{dg,well} = \frac{\partial}{\partial t} (\rho_g \phi S_g + \rho_{dg} \phi S_o)$$

$\dot{m} \equiv \text{Mass flux } \left[\frac{M}{L^2 T} \right]$
 $A \equiv \text{Area } [L^2]$
 $\tilde{m} \equiv \text{Sink or source rate } \left[\frac{M}{L^3 T} \right]$
 $\Delta V \equiv \text{Volume } [L^3]$
 $\rho \equiv \text{Density } \left[\frac{M}{L^3} \right]$
 $\phi \equiv \text{Porosity } \left[\frac{L^3}{L^3} \right]$



Solution Gas-Oil Ratio (R_s)

Solution Gas-Oil Ratio is volume of dissolved gas **at surface conditions** divided by the volume of oil **at surface conditions**

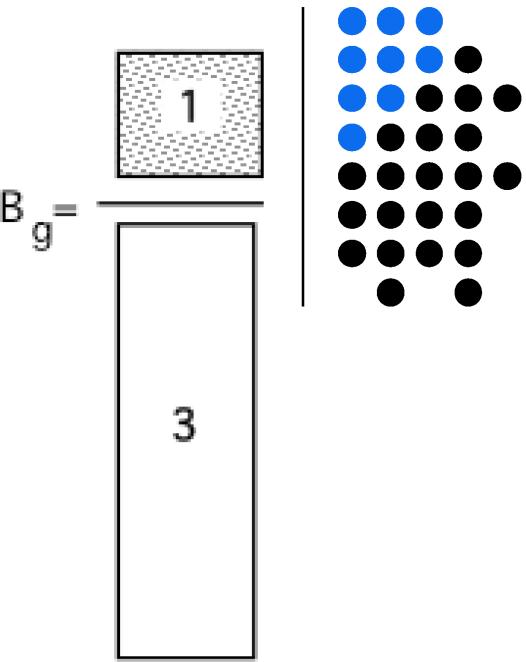
$$R_s = \frac{5}{6}$$

$$R_s [=] \frac{\text{std. vol. gas}}{\text{std. vol. oil}} = \frac{[V_{dg}]^{\text{SC}}}{[V_o]^{\text{SC}}} [=] \frac{\text{scf}}{\text{STB}}$$

Gas Formation Volume Factor (B_g)

Definition of gas formation volume factor:

$$B_g = \frac{\text{reservoir volume of gaseous (gas + volatile oil) phase}}{\text{std. volume gaseous phase (gas only)}} = \frac{[V_g]^{RC}}{[V_g]^{SC}} [=] \frac{\text{ft}^3}{\text{scf}}$$



Mass of gaseous phase is mass of gas + liquids at SC:

$$\text{mass gaseous}^{RC} = \text{mass gas}^{SC} + \text{mass oil}^{SC}$$

$$\rho_g^{RC} V_g^{RC} = \rho_g^{SC} V_g^{SC} + \rho_o^{SC} V_{vo}^{SC}$$

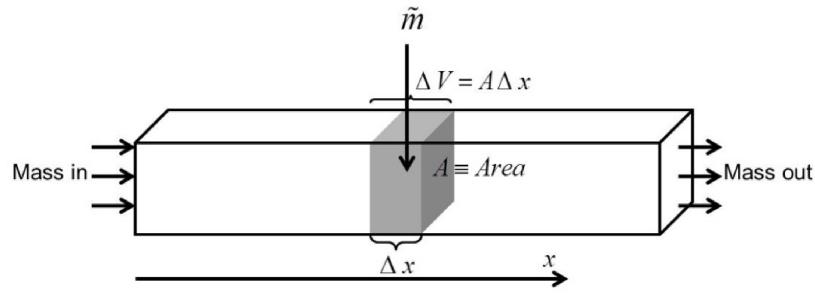
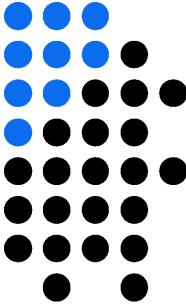
Divide through by volume of gaseous phase at reservoir conditions

$$\rho_g^{RC} = \rho_g^{SC} \frac{V_g^{SC}}{V_g^{RC}} + \rho_o^{SC} \frac{V_{vo}^{SC}}{V_g^{RC}} \cdot \frac{V_g^{SC}}{V_g^{SC}} = \frac{\rho_g^{SC} + R_v \rho_o^{SC}}{B_g}$$

If no liquid comes out of the gaseous phase, $R_v = 0$ (Black Oil Model)

$$\boxed{\rho_g^{RC} = \frac{\rho_g^{SC}}{B_g}}$$

Mass Balance of Gas



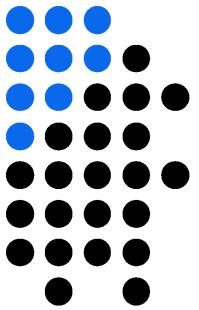
$$-\frac{\partial(\rho_g u_g + \rho_{dg} u_o)}{\partial x} + \tilde{m}_{g,well} + \tilde{m}_{dg,well} = \frac{\partial}{\partial t}(\rho_g \phi S_g + \rho_{dg} \phi S_o)$$

Divide by density (ρ_g^{sc}) at standard conditions and recall:

$$\rho_g = \frac{\rho_g^{sc}}{B_g} \quad \rho_{dg} = \frac{R_s \rho_g^{sc}}{B_o}$$

We get:

$$\boxed{-\frac{\partial}{\partial x}\left(\frac{u_g}{B_g} + R_s \frac{u_o}{B_o}\right) + \tilde{q}_{g,well} + R_s \tilde{q}_{o,well} = \frac{\partial}{\partial t}\left[\phi\left(\frac{1}{B_g} S_g + \frac{R_s}{B_o} S_o\right)\right]}$$



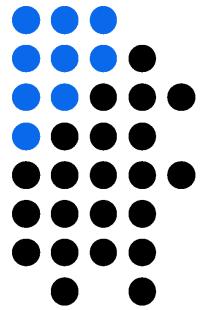
Summary of 3 Phase Balance Equations

water:
$$-\frac{\partial}{\partial x} \left(\frac{u_w}{B_w} \right) + \tilde{q}_w = \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right)$$

oil:
$$-\frac{\partial}{\partial x} \left(\frac{u_o}{B_o} \right) + \tilde{q}_o = \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right)$$

gas:
$$-\frac{\partial}{\partial x} \left(\frac{u_g}{B_g} + R_s \frac{u_o}{B_o} \right) + \tilde{q}_g + R_s \tilde{q}_o = \frac{\partial}{\partial t} \left[\phi \left(\frac{1}{B_g} S_g + \frac{R_s}{B_o} S_o \right) \right]$$

| | | | | | |
|--|---|--|--|---|--|
| $\dot{m} \equiv \text{Mass flux} \left[\frac{M}{L^2 T} \right]$ | $A \equiv \text{Area} \left[L^2 \right]$ | $\tilde{m} \equiv \text{Sink or source rate} \left[\frac{M}{L^3 T} \right]$ | $\Delta V \equiv \text{Volume} \left[L^3 \right]$ | $\rho \equiv \text{Density} \left[\frac{M}{L^3} \right]$ | $\phi \equiv \text{Porosity} \left[\frac{L^3}{L^3} \right]$ |
|--|---|--|--|---|--|



Darcy's Law for Multiphase Flow

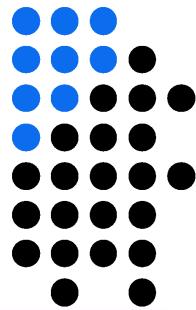
$$u_w = \frac{-kk_{rw}}{\mu_w} \left[\frac{\partial p_w}{\partial x} - \rho_w g \frac{\partial z}{\partial x} \right]$$

$$u_o = \frac{-kk_{ro}}{\mu_o} \left[\frac{\partial p_o}{\partial x} - \rho_o g \frac{\partial z}{\partial x} \right]$$

$$u_g = \frac{-kk_{rg}}{\mu_g} \left[\frac{\partial p_g}{\partial x} - \rho_g g \frac{\partial z}{\partial x} \right]$$

$$\rho_w g \approx 0.433 \text{ psi/ft (water)}$$

| | | | | |
|-----------------------|----------------------------|---------------|------------------------------|------------------------------------|
| u = velocity [L/time] | k = perm [L ²] | D = depth [L] | μ = viscosity [M/L/time] | g = gravity [L/time ²] |
|-----------------------|----------------------------|---------------|------------------------------|------------------------------------|



Pressure Dependent Fluid Properties

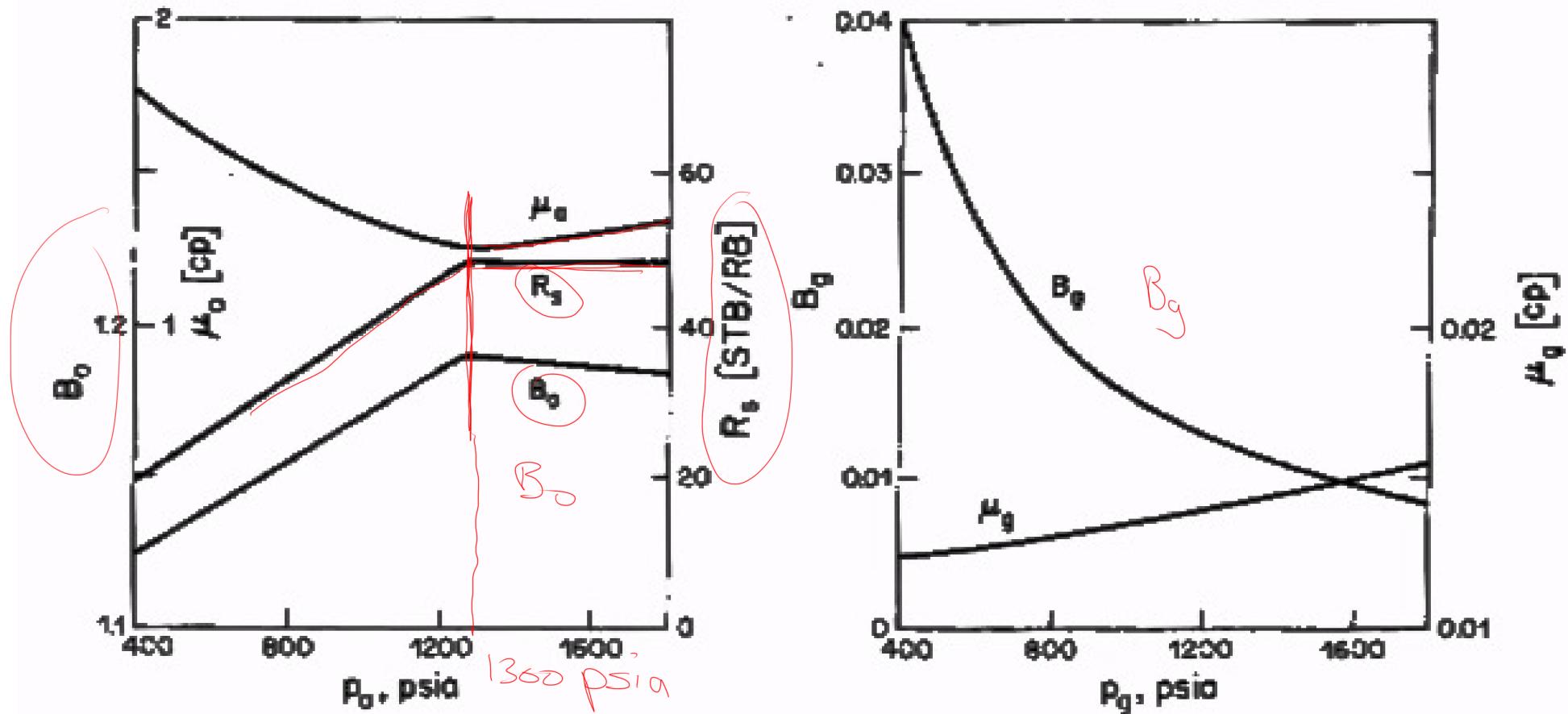
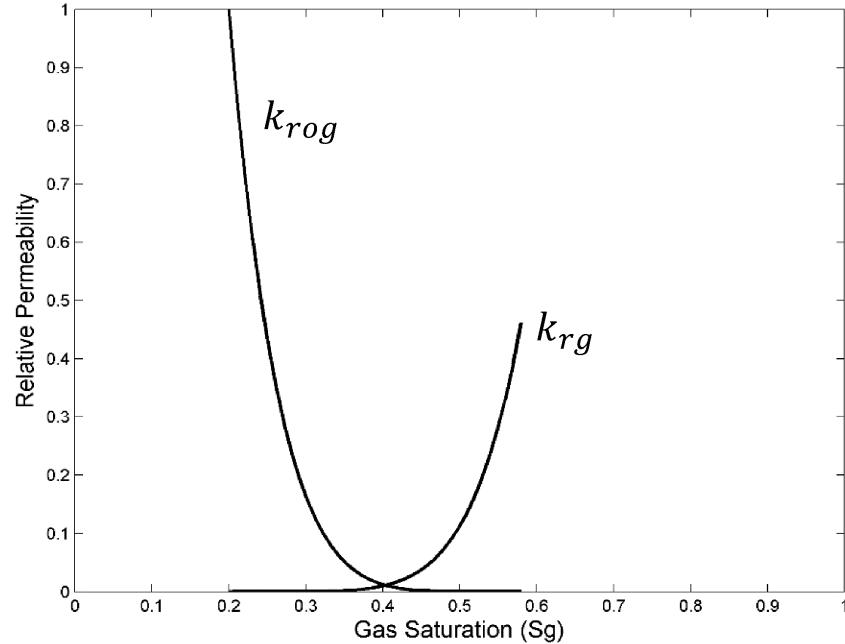
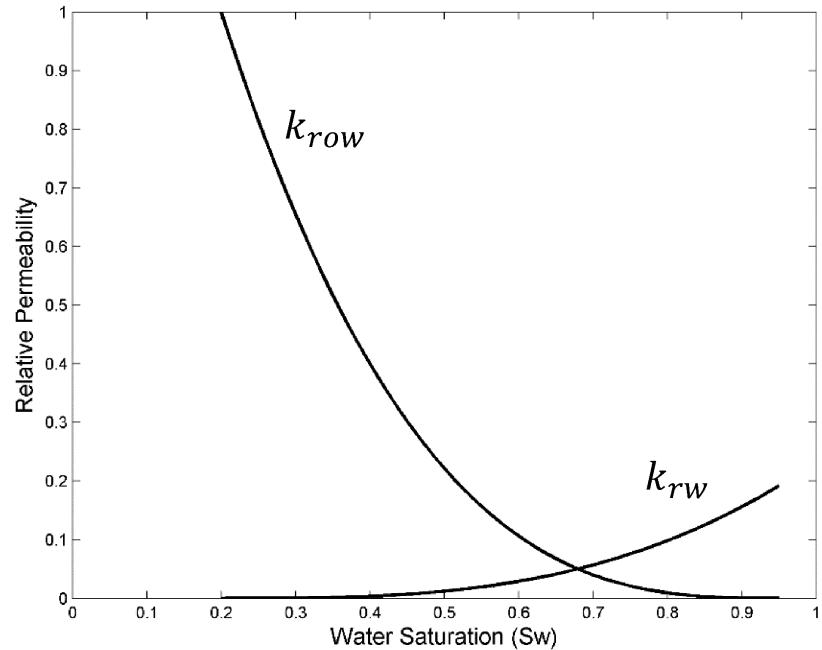
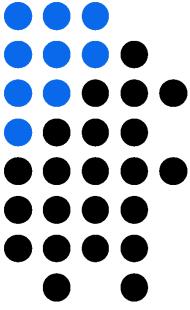


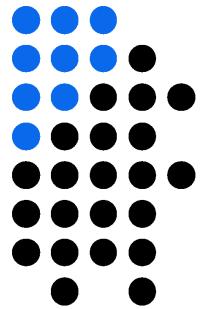
FIG. 2.4. Typical pressure-dependent functions (from Settari and Aziz, 1975).

Brooks-Corey Relative Permeability Model



Brooks-Corey empirical models

$$k_{rw} = k_{rw}^0 \left(\frac{S_w - S_{wr}}{1 - S_{or} - S_{wr} - S_{gc}} \right)^{n_w}; \quad k_{ro} = k_{row} k_{rog} = k_{ro}^0 \left(\frac{1 - S_o - S_{or}}{1 - S_{or} - S_{wr} - S_{gc}} \right)^{n_o}; \quad k_{rg} = k_{rg}^0 \left(\frac{S_g - S_{gc}}{1 - S_{or} - S_{wr} - S_{gc}} \right)^{n_g}$$

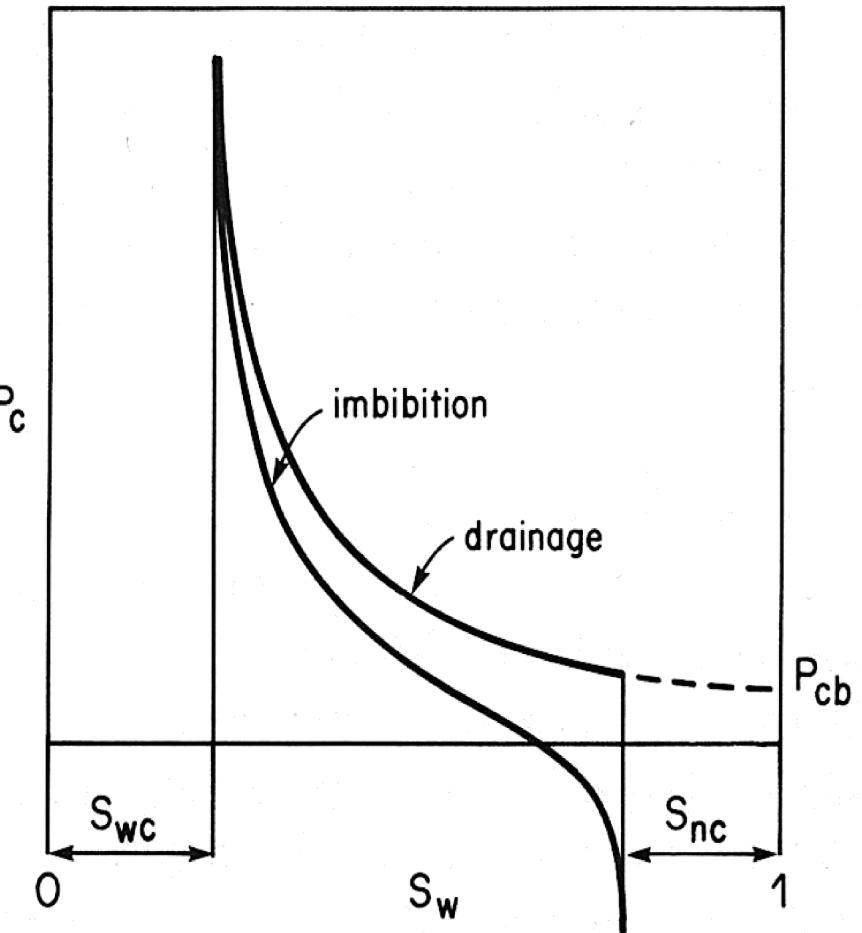


Capillary Pressure Curves

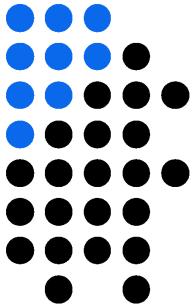
- $P_c(S_w) = P_o - P_w$
- Most simulators include P_c
 - Small (at the reservoir scale) compared to the reservoir pressure
 - We will assume $P_c = 0$
- Van Genuchten empirical model

$$P_c = \frac{1}{\alpha} \left[S^{-1/m} - 1 \right]^{1/n}$$

Note: "n" and "m" parameters are different than in Corey-Brooks relative permeability model



(Aziz and Settari, 1979)



Substituting Darcy's Law into Mass Balance:

PDEs (without gravity) describing 3-phase flow in porous media

$$\frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) = \frac{\partial}{\partial x} \left(\frac{1}{B_w} \frac{k k_{rw}}{\mu_w} \frac{\partial P_w}{\partial x} \right) + \tilde{q}_w$$

$$\frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) = \frac{\partial}{\partial x} \left(\frac{1}{B_o} \frac{k k_{ro}}{\mu_o} \frac{\partial P_o}{\partial x} \right) + \tilde{q}_o$$

$$\frac{\partial}{\partial t} \left[\phi \left(\frac{1}{B_g} S_g + \frac{R_s}{B_o} S_o \right) \right] = \frac{\partial}{\partial x} \left(\frac{1}{B_g} \frac{k k_{rg}}{\mu_g} \frac{\partial P_g}{\partial x} + R_s \frac{1}{B_o} \frac{k k_{ro}}{\mu_o} \frac{\partial P_o}{\partial x} \right) + \tilde{q}_g + R_s \tilde{q}_o$$

Auxiliary Equations Required

$$S_w + S_o + S_g = 1$$

$$P_{c,ow}(S_w) = P_o - P_w$$

$$P_{c,og}(S_w) = P_g - P_o$$

$$k_{rw} = f(S_w)$$

$$k_{ro} = f(S_w, S_g)$$

$$k_{rg} = f(S_g)$$

Rarely can these coupled PDEs be solved analytically. An exception is the Buckley-Leverett problem

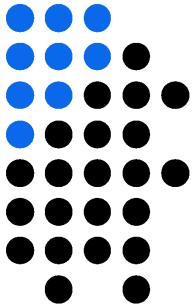
S = saturation

k = perm [L²]

B = FVF

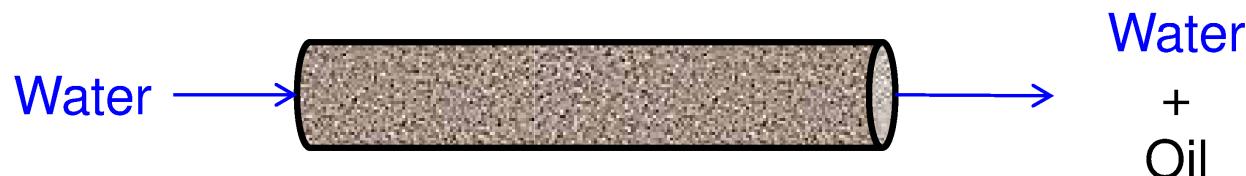
μ = viscosity [M/L/time]

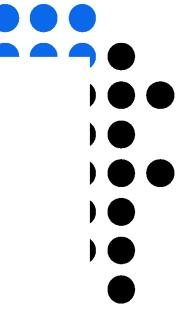
ρ = density [M/V]



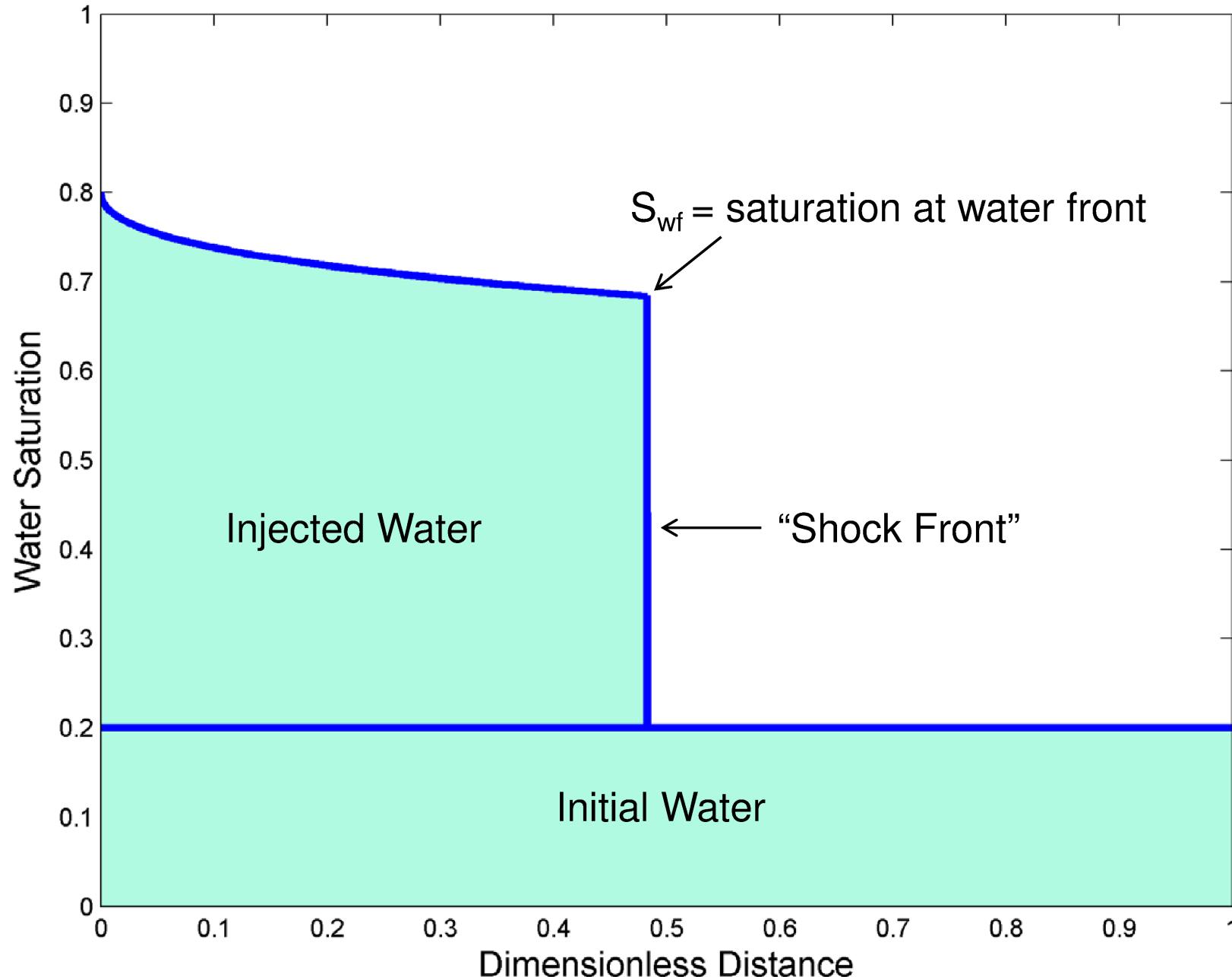
Buckley-Leverett Theory for 1D Displacement

- Semi-analytical solution saturation equation in 1D
- Useful for benchmarking reservoir simulation (numerical) solutions
- Assumptions
 1. 2-phase flow only (oil and water)
 2. 1D flow
 3. Incompressible fluids
 4. no capillary pressure/gravity,
 5. no sources or sinks
- Initial and Boundary conditions
 - Core initially saturated with water at residual water saturation ($S_{wi}=S_{wr}$)
 - Constant injection rate (q) at $x=0$
 - Constant production rate (q) at $x=L$



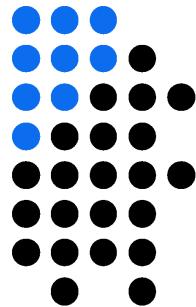


Buckley-Leverett Theory for 1D Displacement



Water front displaces as a shock wave

Buckley-Leverett Equations



Recall mass balance equation for water:

$$\frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) = - \frac{\partial}{\partial x} \left(\frac{u_w}{B_w} \right) + \tilde{q}_w$$

No sources/sinks

Since rock and fluids are incompressible, porosity and FVF are constant

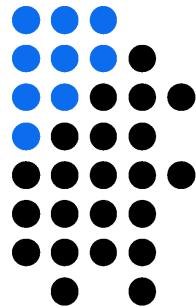
$$\phi \frac{\partial S_w}{\partial t} = - \frac{\partial u_w}{\partial x}$$

Let f_w be fractional flow, q_w be water rate, and q the total rate (which is constant):

$$u_w = \frac{q_w}{A} = \frac{q f_w}{A}$$

Mass balance equation for water can be simplified:

$$\phi \frac{\partial S_w}{\partial t} = - \frac{q}{A} \frac{\partial f_w}{\partial x}$$



Buckley-Leverett Equations

Using chain rule, the mass balance equation becomes:

$$\frac{\partial S_w}{\partial t} = - \frac{q}{A\phi} \frac{\partial f_w}{\partial S_w} \frac{\partial S_w}{\partial x}$$

Note that $S_w = f(x,t)$, so the partial (dS_w) can be expanded as:

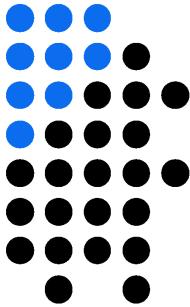
$$dS_w = \frac{\partial S_w}{\partial x} dx + \frac{\partial S_w}{\partial t} dt$$

Search for velocity front of constant saturation (i.e. $dS_w = 0$). Setting the above expression to zero we get:

$$u = \left. \frac{\partial x}{\partial t} \right|_{S_w} = - \frac{\left. (\partial S_w / \partial t) \right|_{S_w}}{\left. (\partial S_w / \partial x) \right|_{S_w}} = \frac{q}{A\phi} \frac{\partial f_w}{\partial S_w}$$

Velocity of constant-saturation front is proportional to the derivative of the fractional flow curve:

$$\left. \frac{\partial x}{\partial t} \right|_{S_w} = \frac{q}{A\phi} \frac{\partial f_w}{\partial S_w}$$



Dimensionless Variables

It is convenient to introduce some dimensionless variables .

$$x_D = x / L \equiv \text{dimensionless distance}$$

$$t_D = \frac{qt}{\phi AL} \equiv \frac{\text{pore volumes injected}}{\text{pore volume of core}}$$

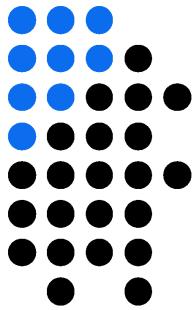
Dimensionless velocity front can then be written as:

$$\left. \frac{\partial x}{\partial t} \right|_{S_w} = \frac{q}{A\phi} \frac{\partial f_w}{\partial S_w} \quad \rightarrow \quad \left. \frac{\partial x_D}{\partial t_D} \right|_{S_w} = \frac{\partial f_w}{\partial S_w}$$

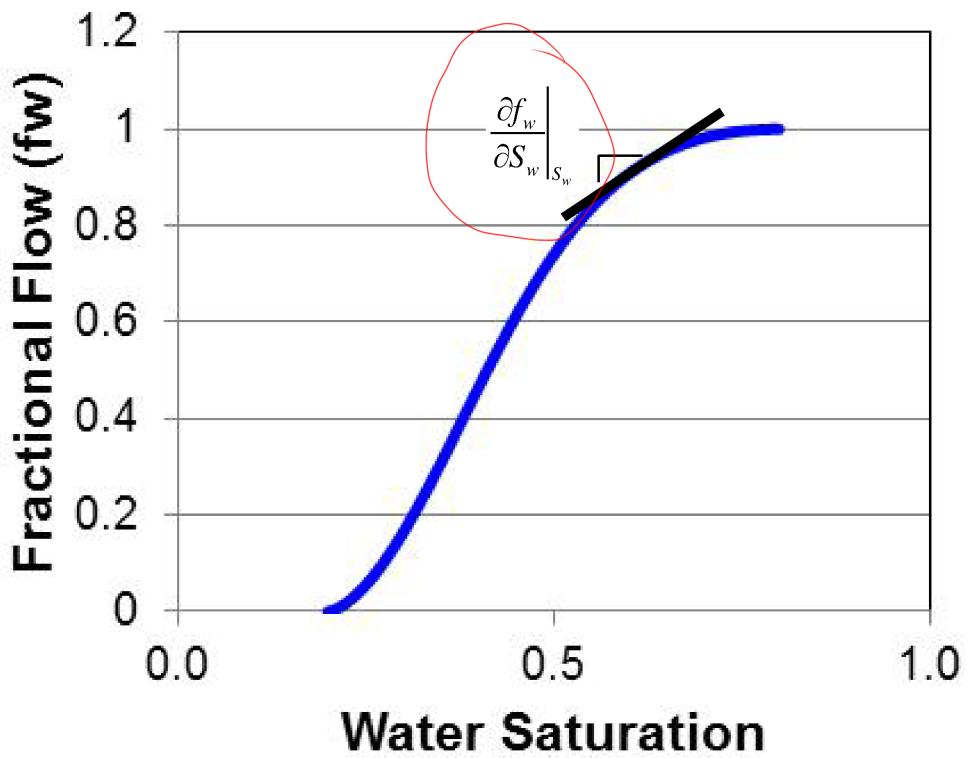
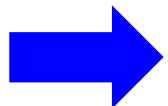
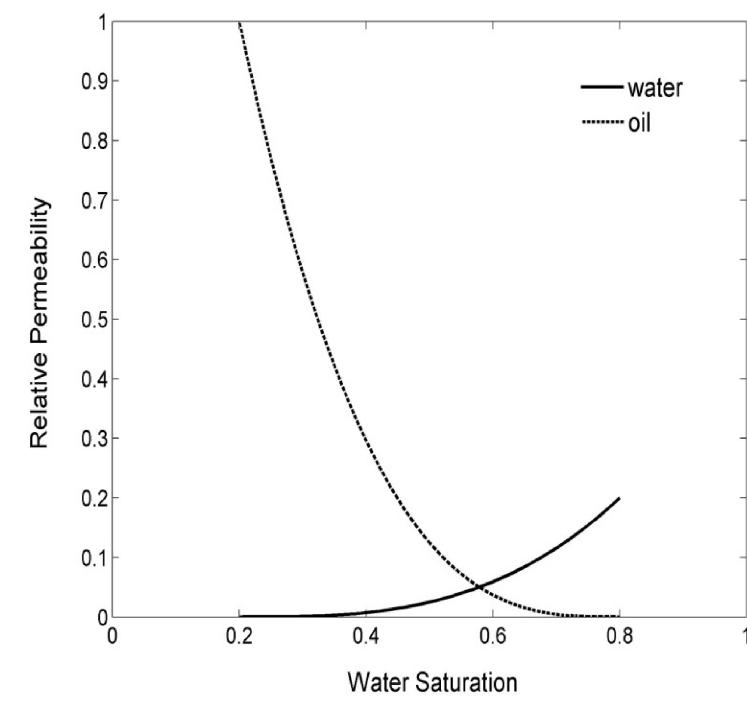
Integrating and recognizing that $x_D = 0$ at $t_D = 0$

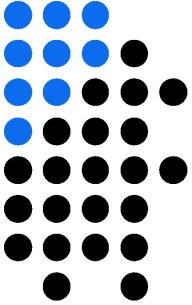
$$\rightarrow \quad x_D(S_w) = t_D \left. \frac{\partial f_w}{\partial S_w} \right|_{S_w}$$

Fractional flow curve is function of S_w through relative permeability



$$f_w(S_w) = \frac{q_w}{q_o + q_w} = \frac{k_{rw}}{k_{rw} + k_{ro}} \frac{\mu_w}{\mu_w + \mu_o}$$





Steps for Generating Saturation Profile at Snapshot in time (t_D)

1. Choose a time/dimensionless time to make the plot $t_D = \frac{qt}{\phi AL}$
2. Create fractional flow curve (f_w vs. S_w) using relative permeability curves

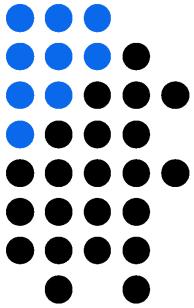
$$f_w(S_w) = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{ro}/\mu_o}$$

3. At a given S_w (e.g. $1-S_{or}$), compute $\frac{\partial f_w}{\partial S_w} \Big|_{S_w}$, numerically (slope of tangent to the fractional flow curve)
4. Compute x_D at that S_w and t_D . Plot point on S_w vs. x_D curve

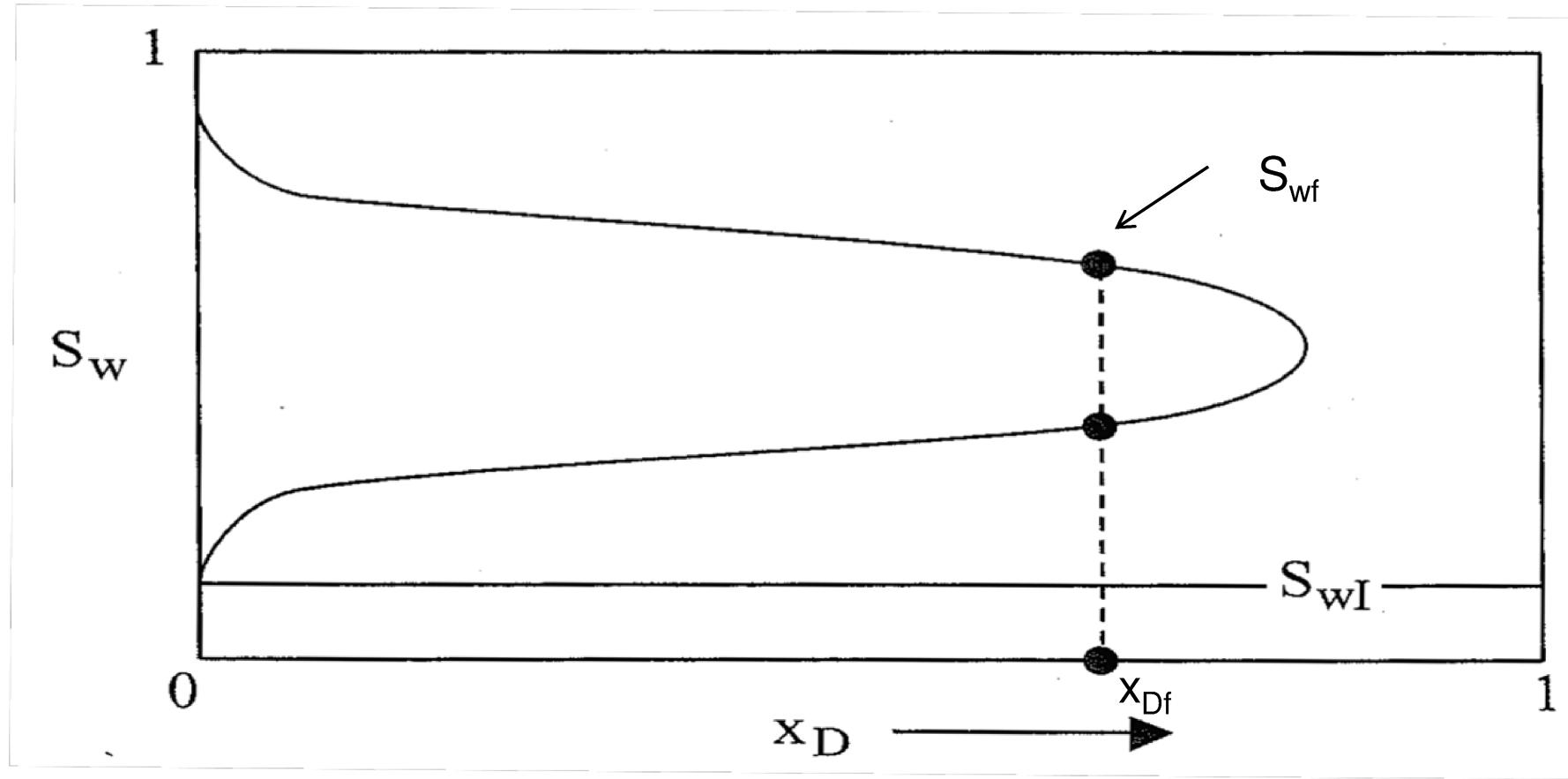
↗

$$x_D(S_w) = t_D \frac{\partial f_w}{\partial S_w} \Big|_{S_w}$$

5. Repeat for all S_w ($S_{wr} < S_w < 1-S_{or}$) to make an “ S_w versus x_D ” curve

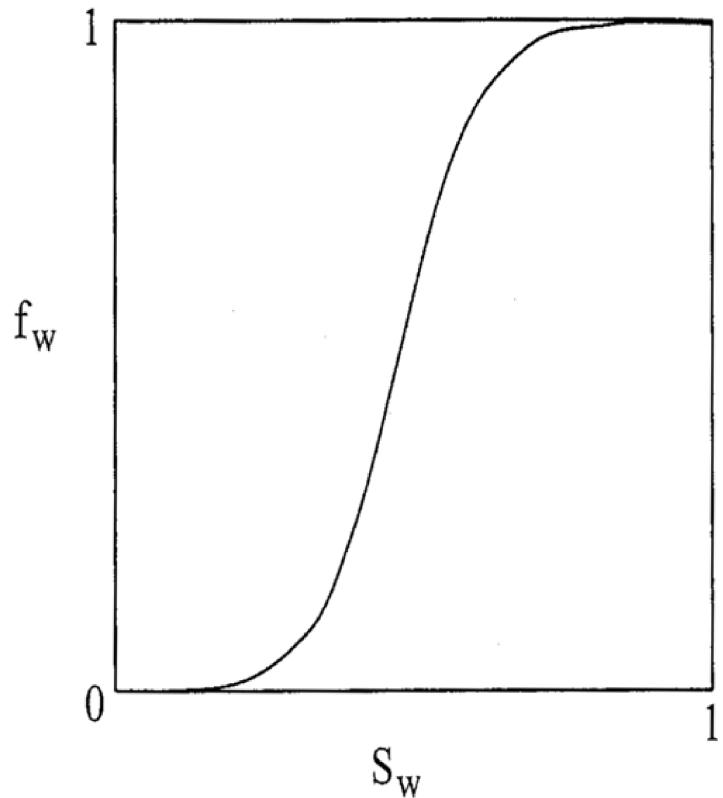
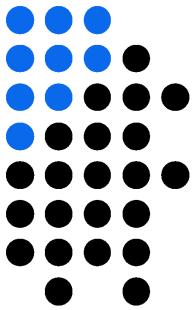


Resulting saturation plot is non-physical; at a given distance in the core, more than one saturation exists

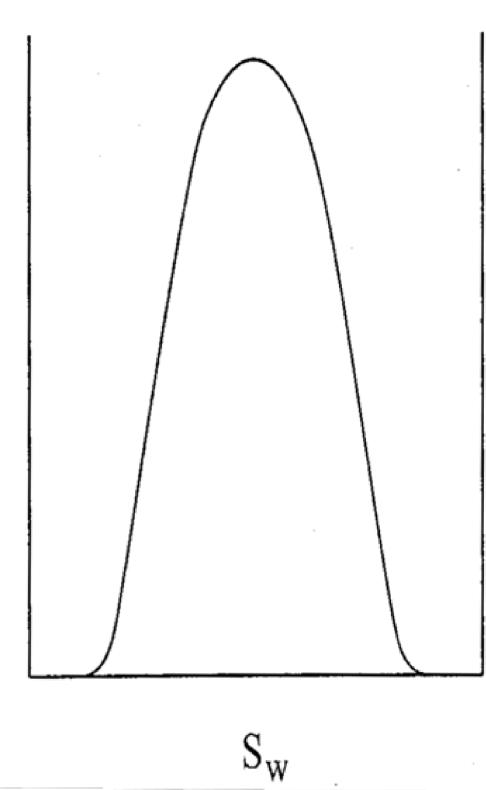


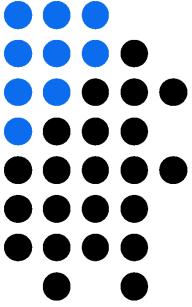
In reality, a shock front occurs at some x_{Df} at snapshot in time, t_D . Saturation has a “step change” from S_{wf} to S_{wi} at that distance x_{Df} . But how do we determine x_{Df} ?

Fractional Flow Curve has an inflection point. As a result, the derivative goes through a maximum



$$\left. \frac{\partial f_w}{\partial S_w} \right|_{S_w}$$

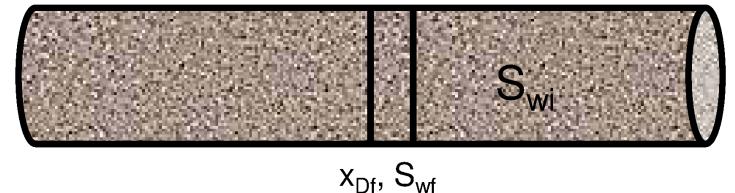




Obtaining a “Physical” Solution and determine S_{wf}

Mass balance on water in the core where $S_w = S_{wf}$

$$\underbrace{\rho_w q_w|_x \Delta t}_{\text{mass into CV}} - \underbrace{\rho_w q_w|_{x+\Delta x} \Delta t}_{\text{mass out of CV}} = \underbrace{\rho_w A\phi \Delta x S_w|_{t+\Delta t} - \rho_w A\phi \Delta x S_w|_t}_{\text{accumulation}}$$



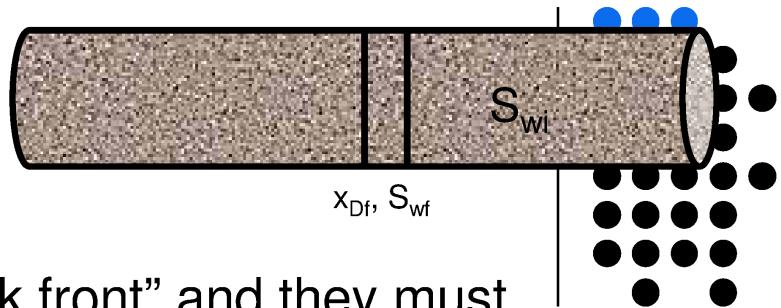
We know from shock front that S_w goes from S_{wi} to S_{wf} and therefore f_w goes from f_{wi} to f_{wf} . Mass balance becomes:

$$\rightarrow q(f_{wf} - f_{wi})\Delta t = A\phi(S_{wf} - S_{wi})\Delta x$$

Using some algebra and using dimensionless variables:

$$\frac{\Delta x}{\Delta t} = \frac{u}{\phi} \frac{(f_{wf} - f_{wi})}{(S_{wf} - S_{wi})} \Rightarrow \boxed{\frac{\Delta x_D}{\Delta t_D} = \frac{f_{wf}}{(S_{wf} - S_{wi})}}$$

Obtaining a “Physical” Solution



We have two equations describing velocity at “shock front” and they must be equal for mass conservation

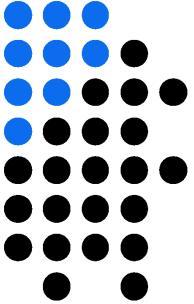
$$\frac{\Delta x_D}{\Delta t_D} = \frac{f_{wf}}{(S_{wf} - S_{wi})}$$

$$\left. \frac{\partial x_D}{\partial t_D} \right|_{S_{wf}} = \left. \frac{\partial f_w}{\partial S_w} \right|_{S_{wf}}$$

Equation suggests a secant line can be drawn connecting two points on a f_w versus S_w plot; (S_{wi}, f_{wi}) and (S_{wf}, f_{wf})

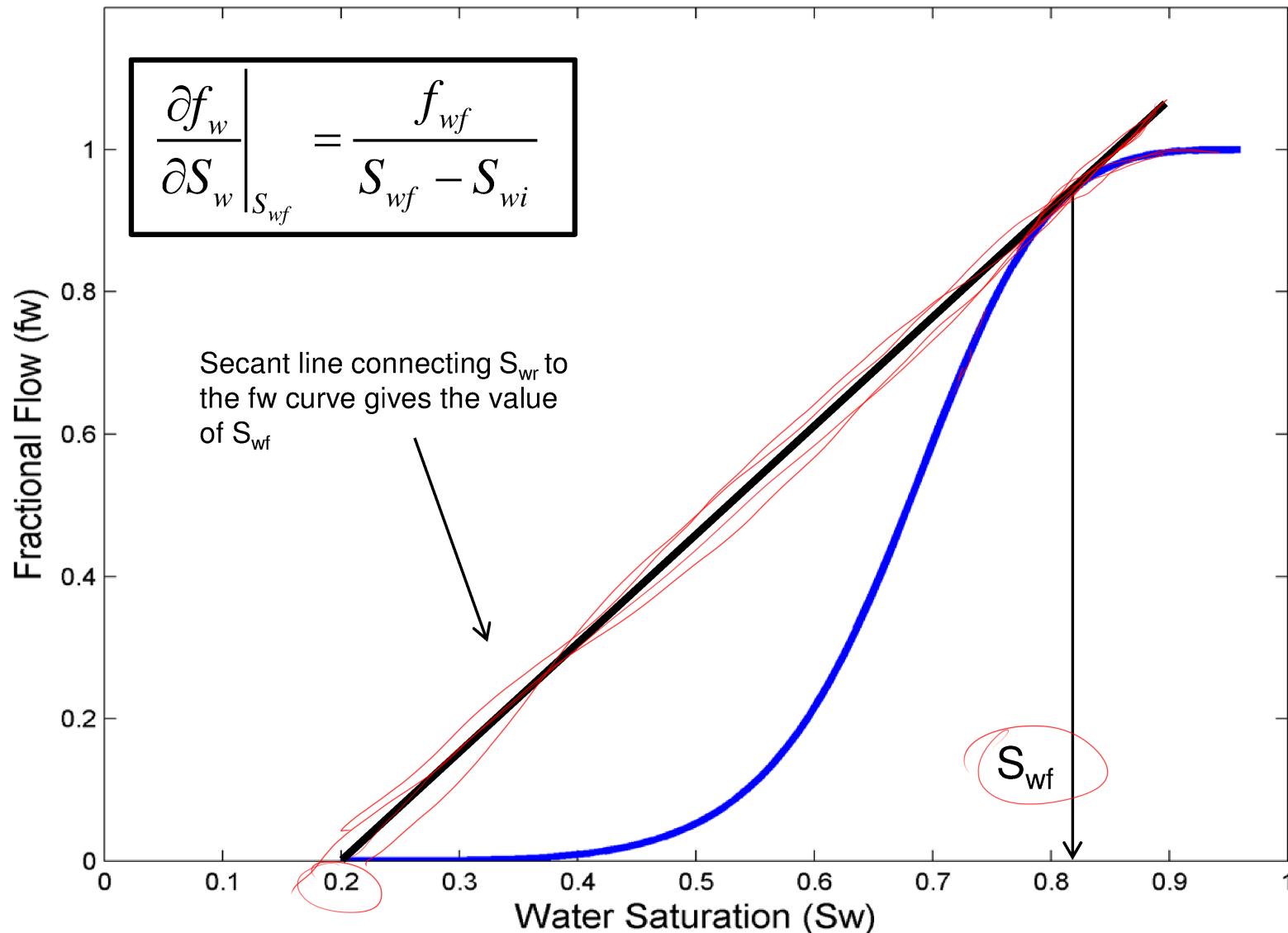
$$f_{wf} = \left. \frac{\partial f_w}{\partial S_w} \right|_{S_{wf}} (S_{wf} - S_{wi})$$

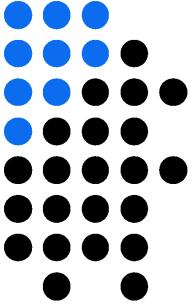
So right at the shock front ($x_D=x_{Df}$, $S_w=S_{wf}$), the slope of the line tangent to the fractional flow curve must equal the line intersecting (S_{wi}, f_{wi}) ; i.e. the point of interest (S_{wf}, f_{wf})



Fractional Flow Curve

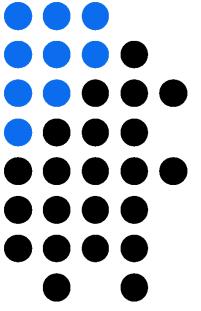
$$f_w(S_w) = \frac{q_w}{q_w + q_o} = \frac{\frac{\mu_w}{k_{rw}(S_w)} + \frac{\mu_o}{k_{ro}(S_w)}}{\mu_w + \mu_o}$$





Steps for Generating Saturation Profile

1. Compute S_{wf} by using secant method on fractional flow curve
2. For all $S_w > S_{wf}$, compute $\frac{\partial f_w}{\partial S_w} |_{S_w}$, numerically
3. Compute dimensionless distance, x_D at each $S_w > S_{wf}$
4. Plot S_w versus x_D using points obtained from #3 and use $S_w = S_{wi}$ for $x_D > x_{Df}$



Water Saturation Profile from BL Theory

