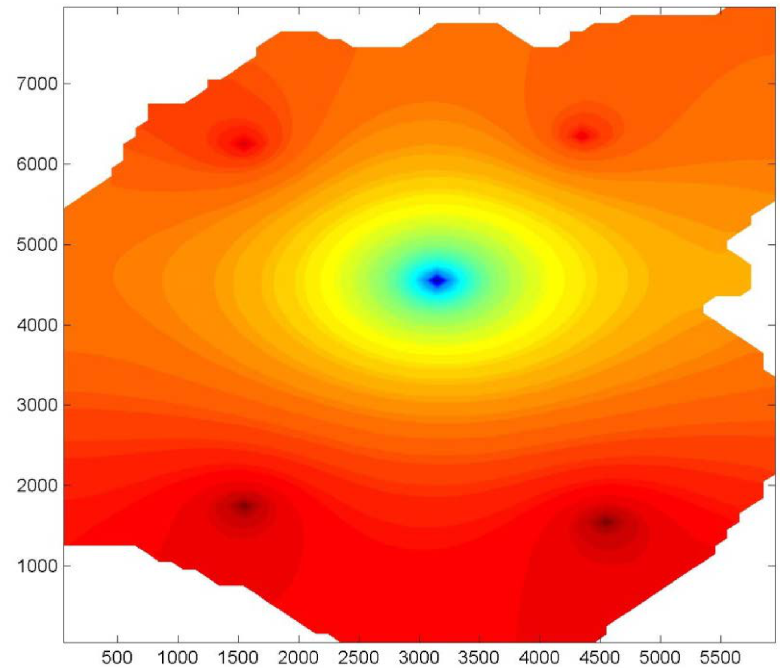
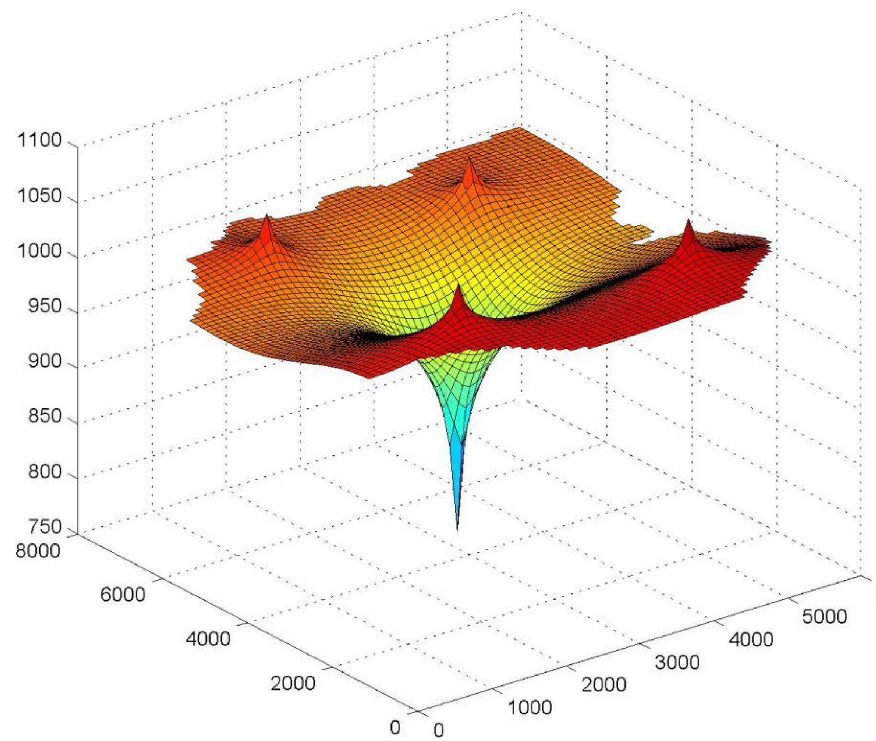
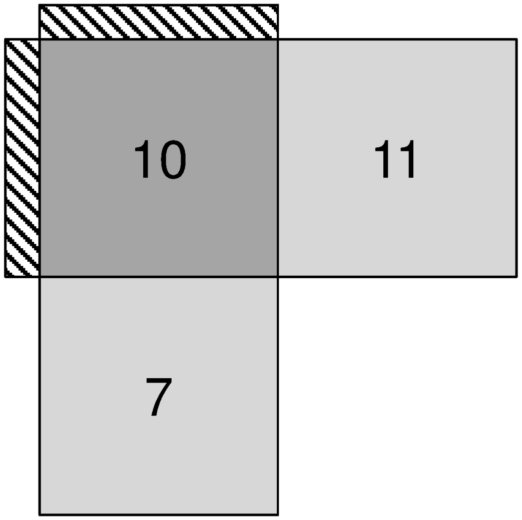
CHAPTER 6.
2D AND 3D FLOW

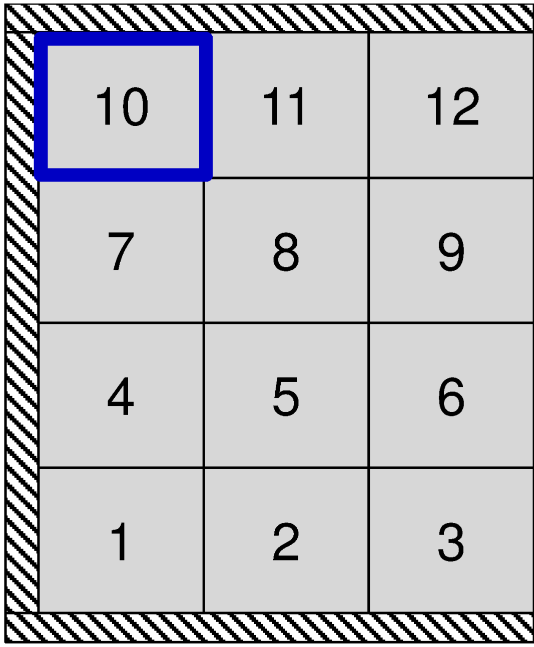




What about corner grids?



- Nothing really special about corner grids
- Just treat each boundary separate



$$T(\cancel{P_{\text{left}} - P_{10}}) + T(P_{11} - P_{10}) + T(P_7 - P_{10}) + T(\cancel{P_{\text{top}} - P_{10}}) = \frac{V_{10} c_t \phi_{10}}{B_w \Delta t}(P_{10}^{n+1} - P_{10}^{n}) - Q_{10}$$

Equation for block #10

$$\boxed{-TP_7^{n+1} + \left(2T + \frac{V_{10} c_t \phi_{10}}{B_w \Delta t}\right) P_{10}^{n+1} - TP_{11}^{n+1} = \frac{V_{10} c_t \phi_{10}}{B_w \Delta t} P_{10}^{n} + Q_{10}}$$

$T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$ | $P \equiv Pressure \left[\frac{M}{LT^2}\right]$ | $Q \equiv Source \left[\frac{L^3}{T}\right]$

Final T Matrix is Pentadiagonal

$$\mathbf{T} = \begin{bmatrix} 2T & -T & & -T & & & & & & & & \\ -T & 3T & -T & & -T & & & & & & & \\ & -T & 4T & & & -T & & & & & & \\ -T & & & 3T & -T & & -T & & & & & \\ & -T & & -T & 4T & -T & & -T & & & & \\ & & -T & & -T & 5T & & & -T & & & \\ & & & -T & & & 3T & -T & & -T & & \\ & & & & -T & & -T & 4T & -T & & -T & \\ & & & & & -T & & -T & 5T & & & -T \\ & & & & & & -T & & & 2T & -T & \\ & & & & & & & -T & & -T & 3T & -T \\ & & & & & & & & -T & & -T & 4T \end{bmatrix}$$

- Pentadiagonal, banded matrix
- Diagonally dominant
- Symmetric

<u>Simple rules for homogeneous, isotropic 2D systems</u>

- Matrix is square and the # rows= total # grid blocks
- Off diagonals are –T if that row (block) is connected to that column (block)
- Main diagonal = (# connected blocks + 2*dirichlet boundaries)T
  - Blocks 5 and 8 are interior blocks and have 4 neighbors, so “4T”
  - Blocks 2,4,7, and 11 have 3 neighbors and no flow boundaries, so “3T”
  - Blocks 1 and 10 have 2 neighbors and no flow boundaries, so “2T”
  - Blocks 6 and 9 have 3 neighbors and one dirichlet boundary, so “5T”
  - Blocks 3 and 12 have 2 neighbors and one dirichlet boundary, so “4T”

System of Equations in Matrix Notation

$$\left(\mathbf{T}+\frac{1}{\Delta t}\mathbf{B}\right)\vec{P}^{n+1}=\left(\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n}+\vec{Q}\right)$$

The B and Q arrays are listed as follows (assumes no wells)

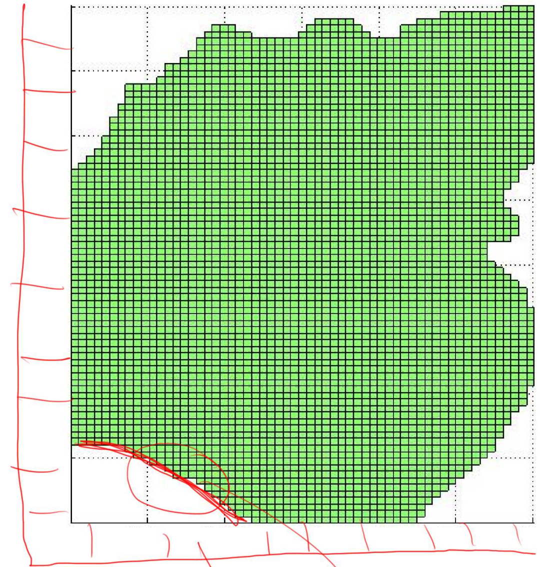
$$\mathbf{B}=\begin{bmatrix} \frac{V_1c_t\phi_1}{B_w} &&&&&&&&&&& \\ & \frac{V_2c_t\phi_2}{B_w} &&&&&&&&&& \\ && \frac{V_3c_t\phi_3}{B_w} &&&&&&&&& \\ &&& \frac{V_4c_t\phi_4}{B_w} &&&&&&&& \\ &&&& \frac{V_5c_t\phi_5}{B_w} &&&&&&& \\ &&&&& \frac{V_6c_t\phi_6}{B_w} &&&&&& \\ &&&&&& \frac{V_7c_t\phi_7}{B_w} &&&&& \\ &&&&&&& \frac{V_8c_t\phi_8}{B_w} &&&& \\ &&&&&&&& \frac{V_9c_t\phi_9}{B_w} &&& \\ &&&&&&&&& \frac{V_{10}c_t\phi_{10}}{B_w} && \\ &&&&&&&&&& \frac{V_{11}c_t\phi_{11}}{B_w} & \\ &&&&&&&&&&& \frac{V_{12}c_t\phi_{12}}{B_w} \end{bmatrix}$$

$$Q=\begin{bmatrix} 0 \\ 0 \\ 2TP_B \\ 0 \\ 0 \\ 2TP_B \\ 0 \\ 0 \\ 2TP_B \\ 0 \\ 0 \\ 2TP_B \end{bmatrix}$$

Constant P boundary condition

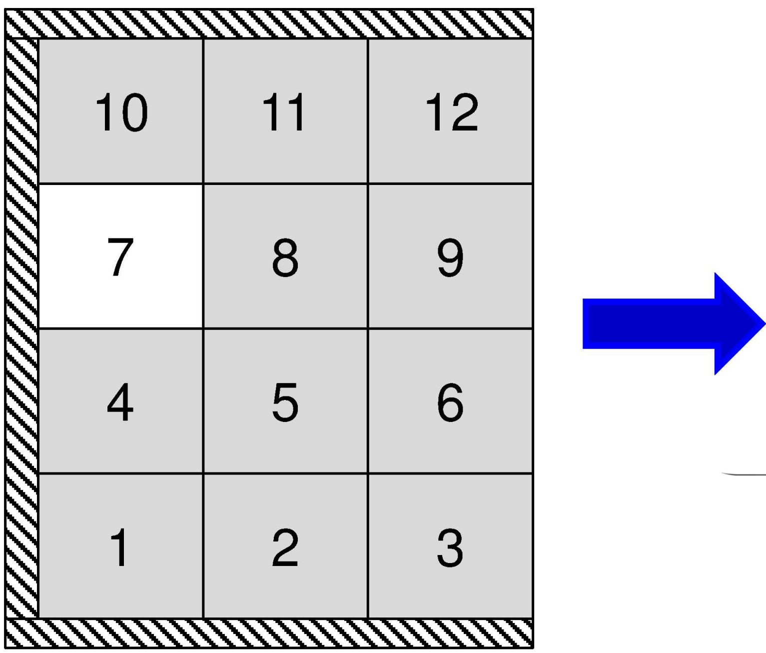
Irregular Geometries and Inactive Grids

- Inactive grids allow for irregular geometries
- Mathematically, assume the grid still exists but assign a permeability of zero
- This automatically gives harmonic mean of interblock transmissibilities equal to zero
- Result is the T matrix is different in the inactive block and its neighbors



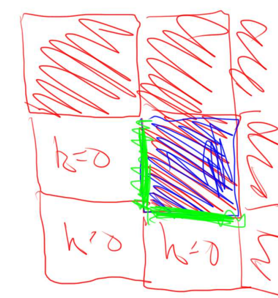
"fake"

"ghost"



$$
\mathbf{T} = \begin{bmatrix}
2T & -T & & -T & & & & & & & & \\
-T & 3T & -T & & -T & & & & & & & \\
& -T & 4T & & & -T & & & & & & \\
-T & & & 2T & -T & & 0 & & & & & \\
& -T & & -T & 4T & -T & & -T & & & & \\
& & -T & & -T & 5T & & & -T & & & \\
& & & 0 & & & 0 & 0 & & 0 & & \\
& & & & -T & & 0 & 3T & -T & & -T & \\
& & & & & -T & & -T & 5T & & & -T \\
& & & & & & 0 & & & T & -T & \\
& & & & & & & -T & & -T & 3T & -T \\
& & & & & & & & -T & & -T & 4T
\end{bmatrix}
$$

h=0



−T −T 4

−T T

Wells

(1) Constant Rate $\rightarrow$ P_{wf} vs. time

(2) Constant BHP $\rightarrow$ Q^{sc} vs. time

$$q_w = -J_\ell^{w} \left(P_\ell - P_{wf} \right)$$

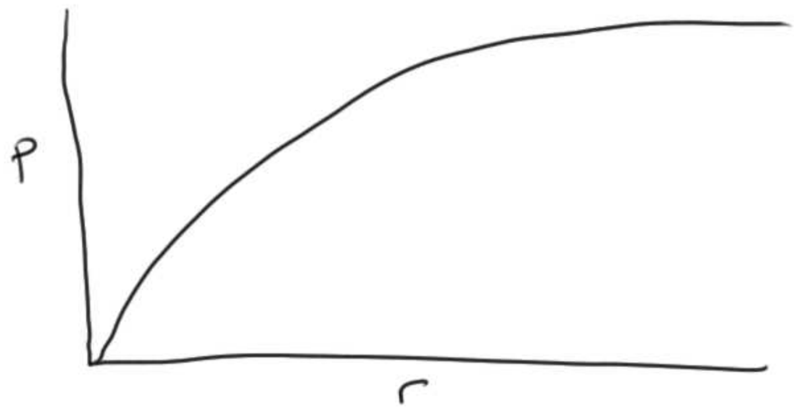
Flow around a wellbore

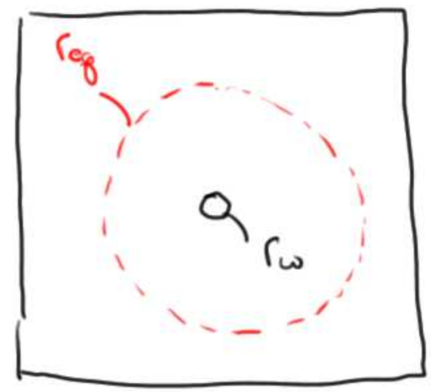
$$\frac{\phi \mu c_t}{k} \cancelto{0}{\frac{\partial p}{\partial t}} = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p}{\partial r} \right)$$ radial diffusivity equation

B.C.'s #1 $\lim_{r \to 0} \left(r \frac{\partial p}{\partial r} \right) = \frac{-q \mu B_w}{2 \pi k h}$

B.C.'s #2 $p = p_{ref}$ @ $r = r_{ref}$

$$\Rightarrow P(r) = P_{ref} - \frac{q_w \mu B_w}{2\pi k h} \ln\left(\frac{r}{r_{ref}}\right)$$



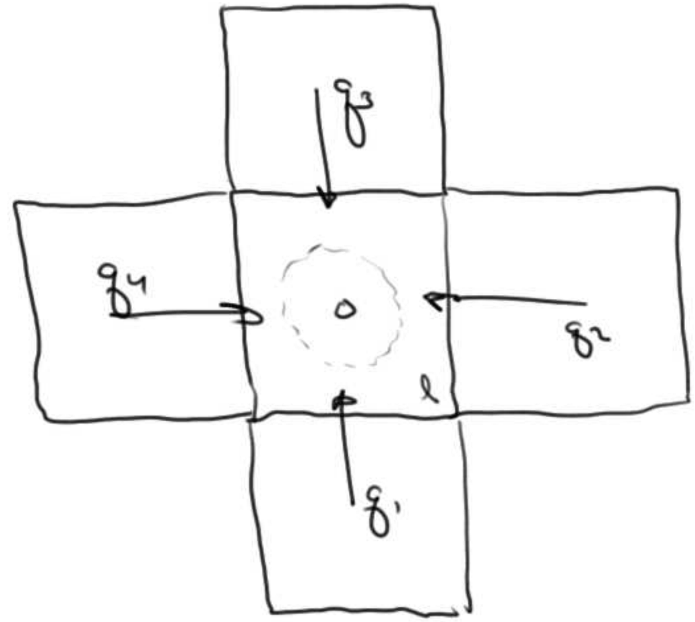


Let $P_{ref} = P_{wf}$ $\quad r = r_{ref} = r_w$

$r = r_{eq}$

$$P_e = P_{wf} - \frac{q_w \mu B_w}{2\pi k h} \ln\left(\frac{r_{eq}}{r_w}\right)$$

$$q_w + (q_1 + q_2 + q_3 + q_4) = 0$$



$$q_1 = \frac{kh\Delta x}{\mu B_w \Delta y}(P_1 - P_\ell)$$

$$q_2 = \frac{kh\Delta y}{\mu B_w \Delta x}(P_2 - P_\ell)$$

$$q_3 = \frac{kh\Delta x}{\mu B_w \Delta y}(P_3 - P_\ell)$$

$$q_4 = \frac{kh\Delta y}{\mu B_w \Delta x}(P_4 - P_\ell)$$

Block is square $\Delta x = \Delta y$

$$q_w = \frac{-kh}{\mu B_w}(P_1 + P_2 + P_3 + P_4 - 4P_\ell)$$

2D Flow problems

General diffusivity equation for 2D flow with anisotropic, heterogeneous permeability (without gravity) is given as:

$$\frac{\partial}{\partial x}\left(\frac{k_x}{\mu B_w}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k_y}{\mu B_w}\frac{\partial p}{\partial x}\right)=\frac{\phi c_t}{B_w}\frac{\partial p}{\partial t}+\tilde{q}_{sc}$$

Two types of 2D problems:

1. Areal view. Reservoir is flat so no gravity acting on system (we'll focus on these problems)
2. Side view. Gravity adds to potential gradient

Could discretize the PDE using finite differences, but here we'll use the "control volume" approach. Either way, we get a system of equations of the form:

$$\left(\mathbf{T}+\frac{1}{\Delta t}\mathbf{B}\right)\vec{P}^{n+1}=\left(\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n}+\vec{Q}\right)$$

Where T is "pentadiagonal" in 2D

Block Numbering for 2D Grids

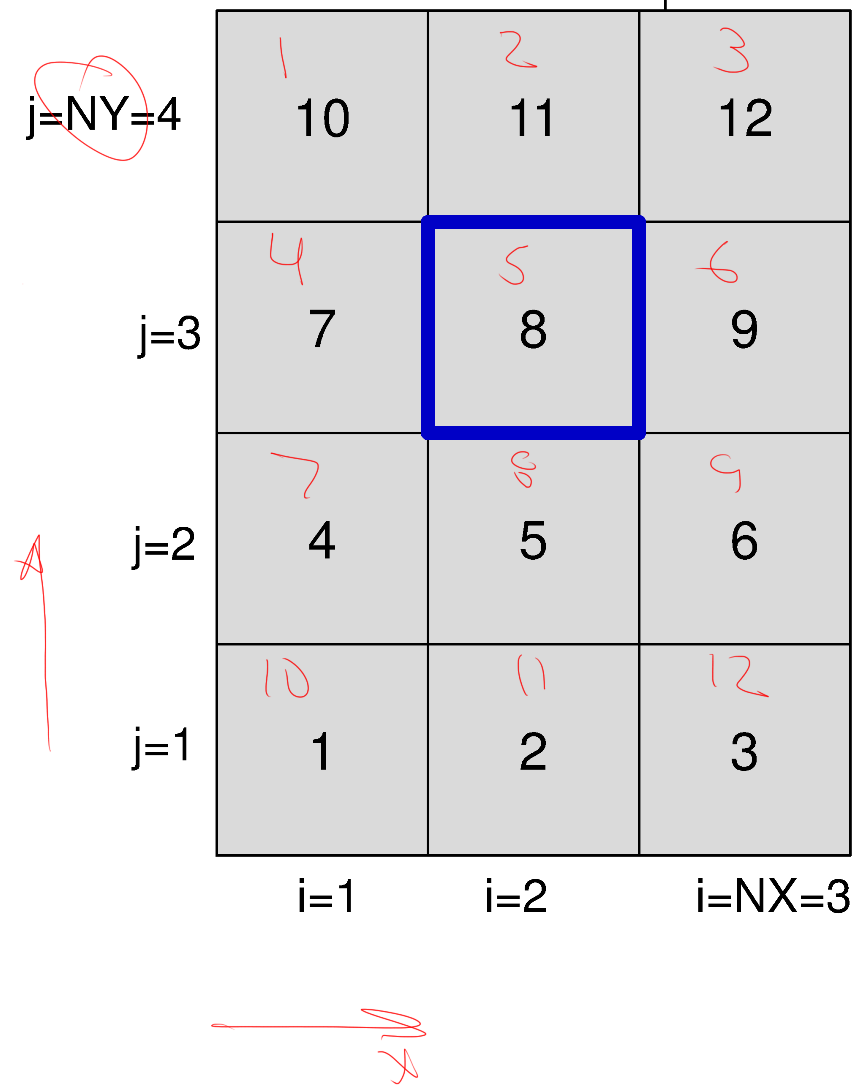
- 2D grid systems have NX grids in the x-direction and NY grids in the y-direction. (Example problem: NX=3 and NY=4)

- Label grids by i,j numbering. Block highlighted in red is i = 2, j = 3

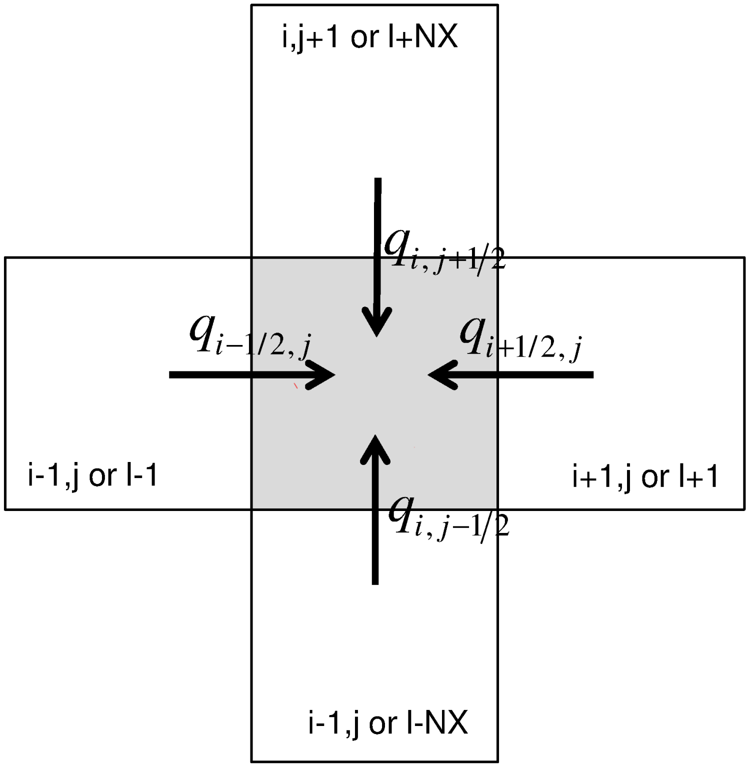
- Can also number grids using "l" numbering, which is related to i and j. Block highlighted in red is also block l = 8.

$$l = (j-1)N_x + i$$

- We'll use both block numbering systems interchangeably in slides that follow
  - "i,j" numbering useful when writing mass balance and interblock T's
  - "l" numbering useful for creating matrix



Equations for 2D Grids: Control Volume Approach



Flux in/out of grid "l"

$$q_{i-1/2,j} = TX_{i-1/2,j}(P_{i-1,j} - P_{i,j}) = TX_{i-1/2,j}(P_{l-1} - P_l)$$

$$q_{i+1/2,j} = TX_{i+1/2,j}(P_{i+1,j} - P_{i,j}) = TX_{i+1/2,j}(P_{l+1} - P_l)$$

$$q_{i,j-1/2} = TY_{i,j-1/2}(P_{i,j-1} - P_{i,j}) = TY_{i,j-1/2}(P_{l-NX} - P_l)$$

$$q_{i,j+1/2} = TY_{i,j+1/2}(P_{i,j+1} - P_{i,j}) = TY_{i,j+1/2}(P_{l+NX} - P_l)$$

Accumulation

$$\frac{V_{i,j}c_t\phi_{i,j}}{B_w\Delta t}(P_{i,j}^{n+1} - P_{i,j}^n) = \frac{V_l c_t\phi_l}{B_w\Delta t}(P_l^{n+1} - P_l^n)$$

Mass Balance on block "i,j" or "l"

$$\underbrace{TX_{i-1/2,j}(P_{l-1} - P_l)}_{\text{left}} + \underbrace{TX_{i+1/2,j}(P_{l+1} - P_l)}_{\text{right}} + \underbrace{TY_{i,j-1/2}(P_{l-NX} - P_l)}_{\text{bottom}} + \underbrace{TY_{i,j+1/2}(P_{l+NX} - P_l)}_{\text{top}} = \underbrace{\frac{V_l c_t\phi_l}{B_w\Delta t}(P_l^{n+1} - P_l^n)}_{\text{accumulation}} - \underbrace{Q_l}_{\text{sources}}$$

$T \equiv Transmissibility\left[\frac{L^4T}{M}\right]$	$P \equiv Pressure\left[\frac{M}{LT^2}\right]$	$Q \equiv Source\left[\frac{L^3}{T}\right]$

Heterogeneities and Anisotropy

- Three complications can occur in multidimensions regarding the interblock transmissibility:
  1. Heterogeneities in permeability and grid size
  2. Anisotropy (kx ≠ ky)
  3. Grid sizes may be different in each direction ($\Delta x \neq \Delta y$)

- Therefore, transmissibility in all 4 connecting blocks may be different, e.g.

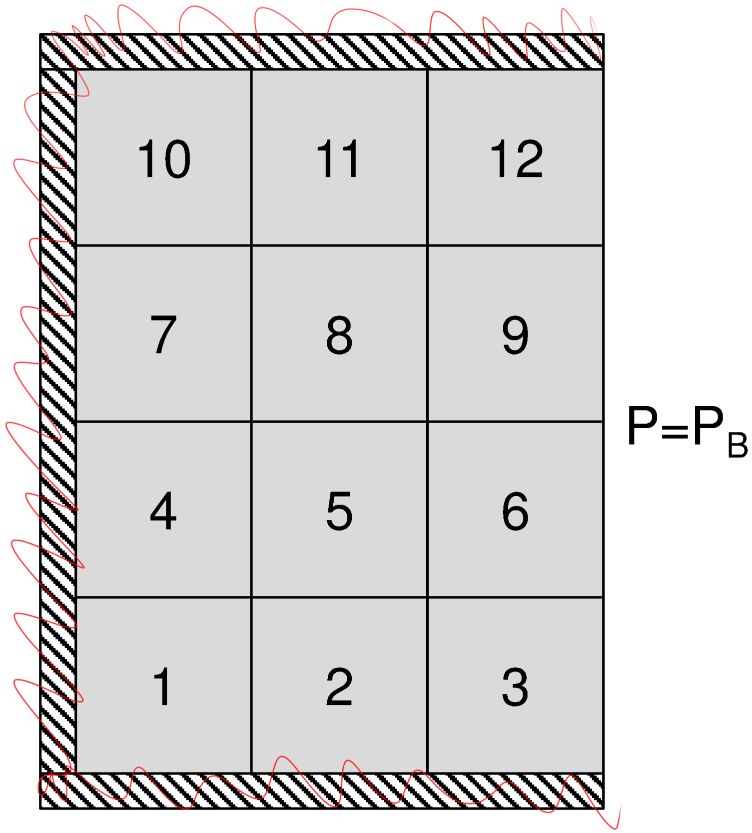
$$kx_{i+1/2,j} = \frac{\Delta x_{i,j} + \Delta x_{i+1,j}}{\dfrac{\Delta x_{i,j}}{kx_{i,j}} + \dfrac{\Delta x_{i+1,j}}{kx_{i+1,j}}}; \quad TX_{i+1/2,j} = \frac{kx_{i+1/2,j}\left(\Delta y_{i,j} h\right)}{\mu B_w \Delta x_{i+1/2,j}}$$

$$ky_{i,j+1/2} = \frac{\Delta y_{i,j} + \Delta y_{i,j+1}}{\dfrac{\Delta y_{i,j}}{ky_{i,j}} + \dfrac{\Delta y_{i,j+1}}{ky_{i,j+1}}}; \quad TY_{i,j+1/2} = \frac{ky_{i+1/2,j}\left(\Delta x_{i,j} h\right)}{\mu B_w \Delta y_{i,j+1/2}}$$

- If reservoir is homogeneous, isotropic and uniform, square ($\Delta x = \Delta y$) grids employed transmissibility same in all directions

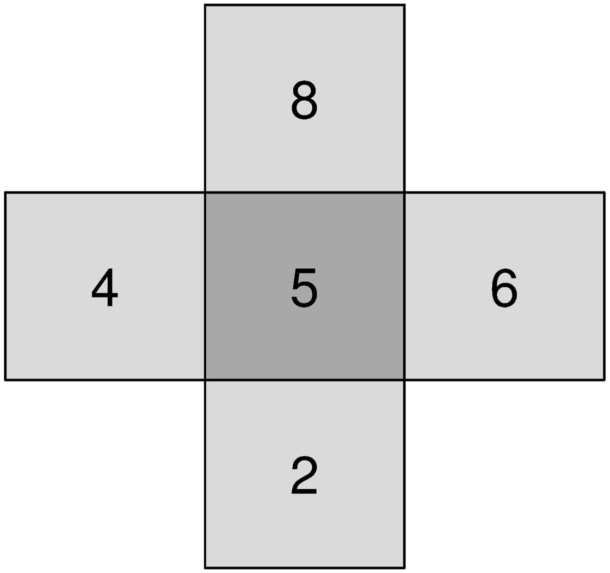
$$TX_{i+1/2,j} = TX_{i-1/2,j} = TY_{i,j+1/2} = TY_{i,j-1/2} = \boxed{T = \frac{kA}{\mu B_w \Delta x}}$$

An Example Problem:

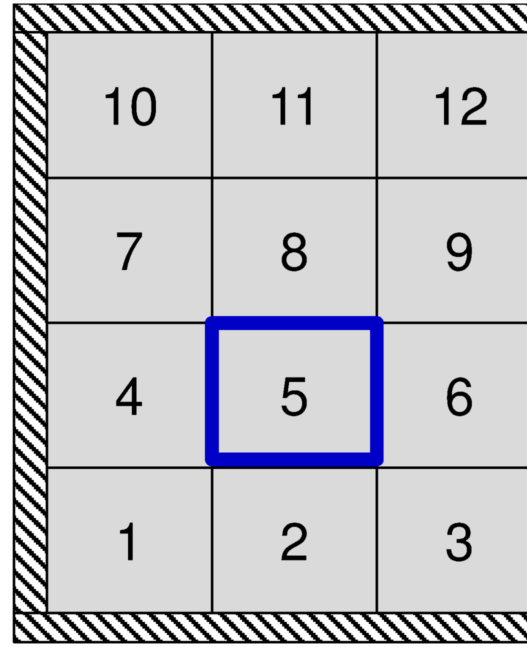


- 2D grid with NX=3 and NY=4
- “No flow” boundary conditions on 3 edges (sealed boundaries)
- “Constant pressure” on right edge (open boundary)
- Assume homogeneous and isotropic permeability along with uniform, square grids

"Interior" Blocks Have 4 Neighbors



Note: Block l=5 is also block i=2, j=2



Mass balance on block #5

$$T(P_4 - P_5) + T(P_6 - P_5) + T(P_2 - P_5) + T(P_8 - P_5) = \frac{V_5 c_t \phi_5}{B_w \Delta t}(P_5^{n+1} - P_5^n) - Q_5$$

Grouping terms & multiplying through by "-1" gives eqn for block #5

$$\boxed{TP_2^{n+1} + TP_4^{n+1} - \left(\frac{V_5 c_t \phi_5}{B_w \Delta t} + 4T\right) P_5^{n+1} + TP_6^{n+1} + TP_8^{n+1} = \frac{V_5 c_t \phi_5}{B_w \Delta t} P_5^n + Q_5}$$

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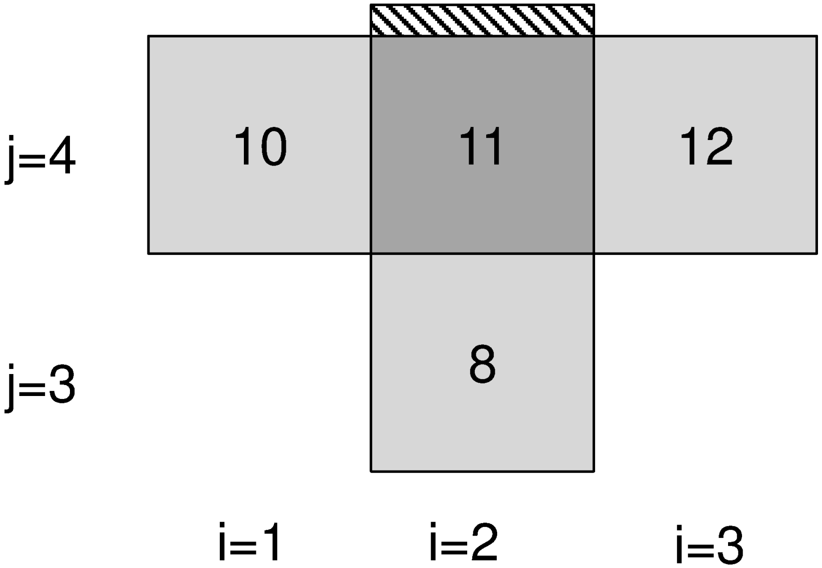
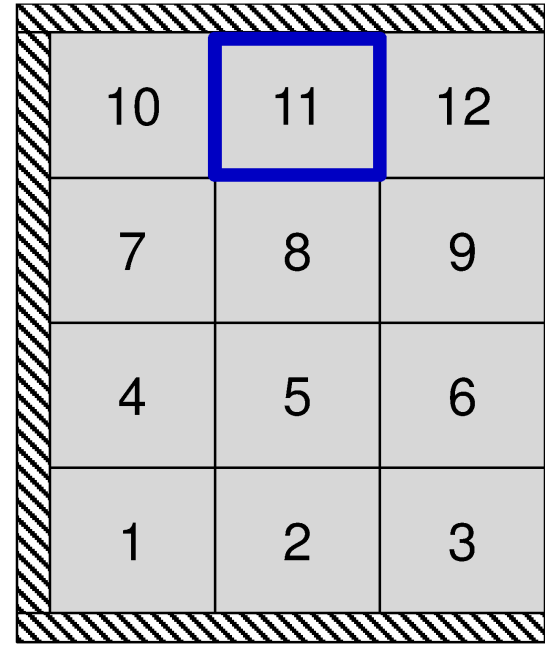
"No Flow" Boundary Blocks



- Block #l=11 is also i=2, j=4
- Neighbors to left, right, bottom
- No flow BC on top



Mass Balance: There is no flow from the "top"!

$$T(P_{10}-P_{11})+T(P_{12}-P_{11})+\cancel{T(P_{top}-P_{11})}+T(P_{8}-P_{11})=\frac{V_{11}c_t\phi_{11}}{B_w\Delta t}(P_{11}^{n+1}-P_{11}^{n})-Q_{11}$$

After grouping terms:

$$\boxed{-TP_8^{n+1}-TP_{10}^{n+1}+\left(3T+\frac{V_{11}c_t\phi_{11}}{B_w\Delta t}\right)P_{11}^{n+1}-TP_{12}^{n+1}=\frac{V_{11}c_t\phi_{11}}{B_w\Delta t}P_{11}^{n}+Q_{11}}$$

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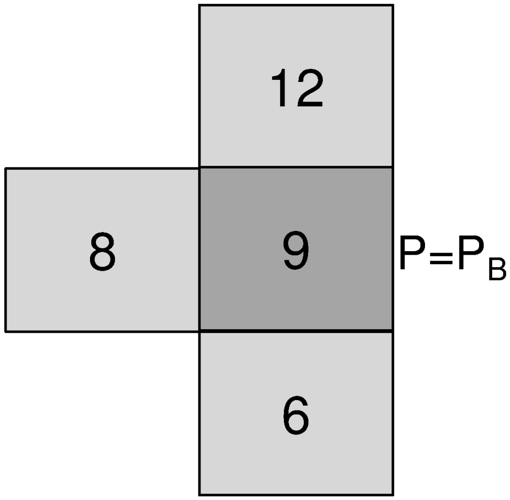
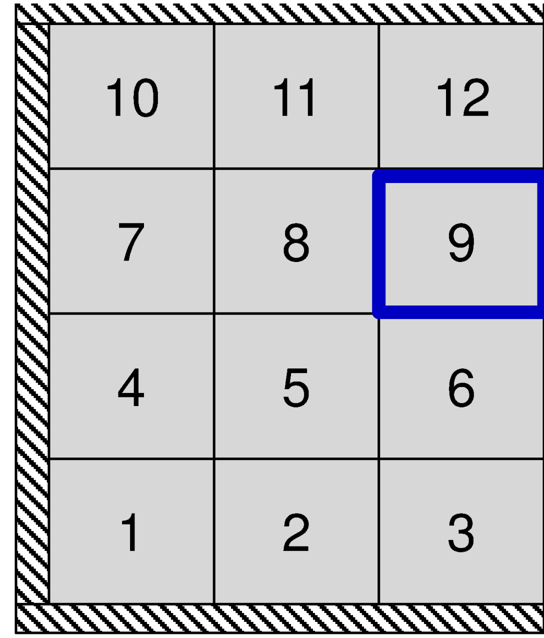
“Constant” Pressure Boundary Condition



- Block #9 has no neighbor to its right
- But, pressure at boundary is fixed (P_B)
- Q= 2T (P_B-P_9)
- “2T” appears because the boundary is only a half Δx away from block center



Mass Balance in “I” numbering:

$$T(P_8 - P_9) + 2T(P_B - P_9) + T(P_6 - P_9) + T(P_{12} - P_9) = \frac{V_9 c_t \phi_9}{B_w \Delta t}(P_9^{n+1} - P_9^n) - Q_9$$

Grouping Terms and moving known values to right hand side:

$$\boxed{-TP_6^{n+1} - TP_8^{n+1} + \left(5T + \frac{V_9 c_t \phi_9}{B_w \Delta t}\right) P_9^{n+1} - TP_{12}^{n+1} = \frac{V_9 c_t \phi_9}{B_w \Delta t} P_9^n + Q_9 + 2TP_B}$$

$T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$ | $P \equiv Pressure \left[\frac{M}{LT^2}\right]$ | $Q \equiv Source \left[\frac{L^3}{T}\right]$