Transmissibility Form of Problem

Imp

$$-\eta P_{i-1}^{n+1} + (1+2\eta)P_i^{n+1} - \eta P_{i+1}^{n+1} = P_i^n \qquad [\equiv \text{psi}]$$

Recall definition of η and α (diffusivity constant)

$$\eta = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{k}{\mu \phi c_t}\frac{\Delta t}{(\Delta x)^2}$$

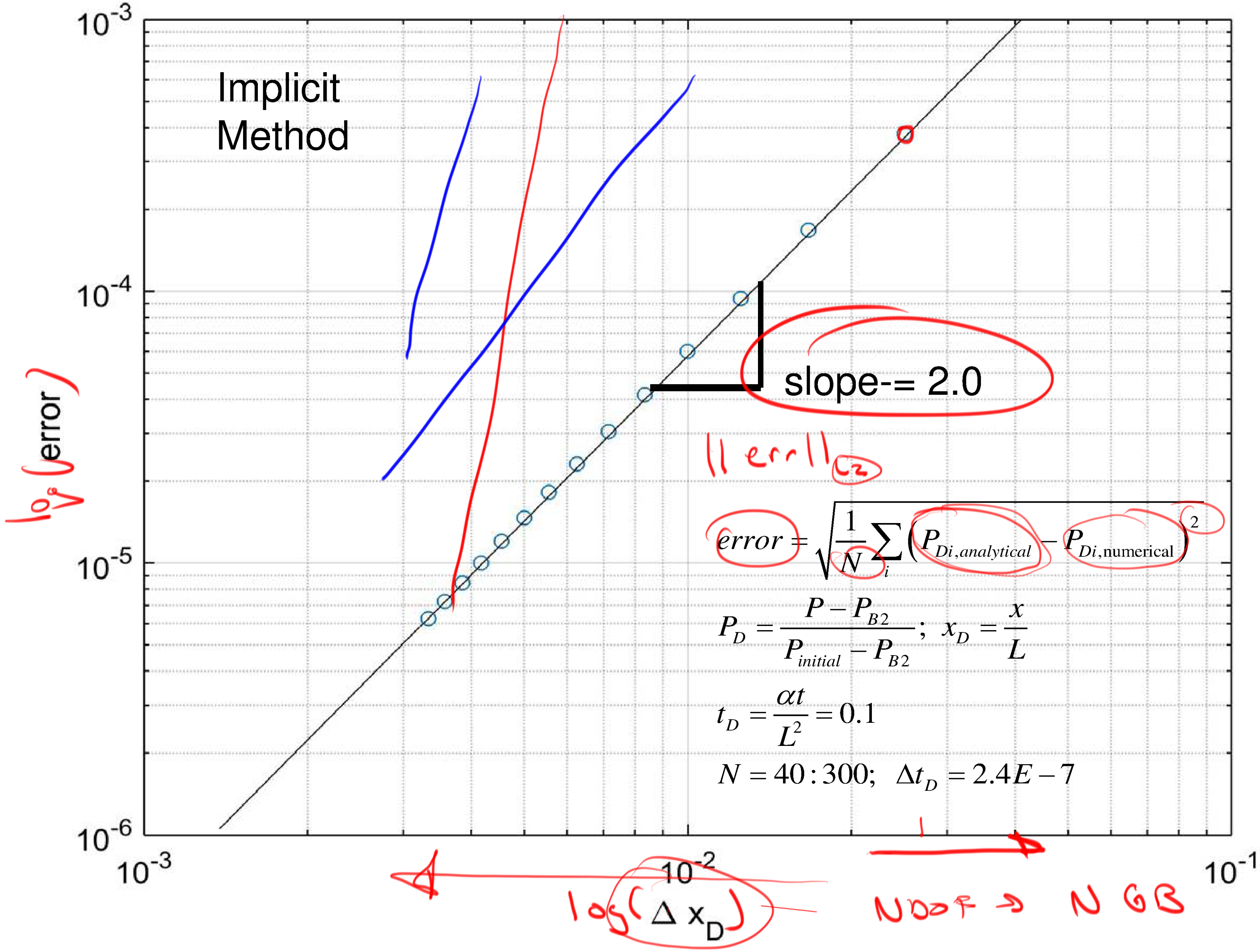
Multiply both sides of balance equation by a constant:

$$\frac{A\Delta x \phi c_t}{B_w \Delta t} = \frac{V_i \phi c_t}{B_w \Delta t}$$

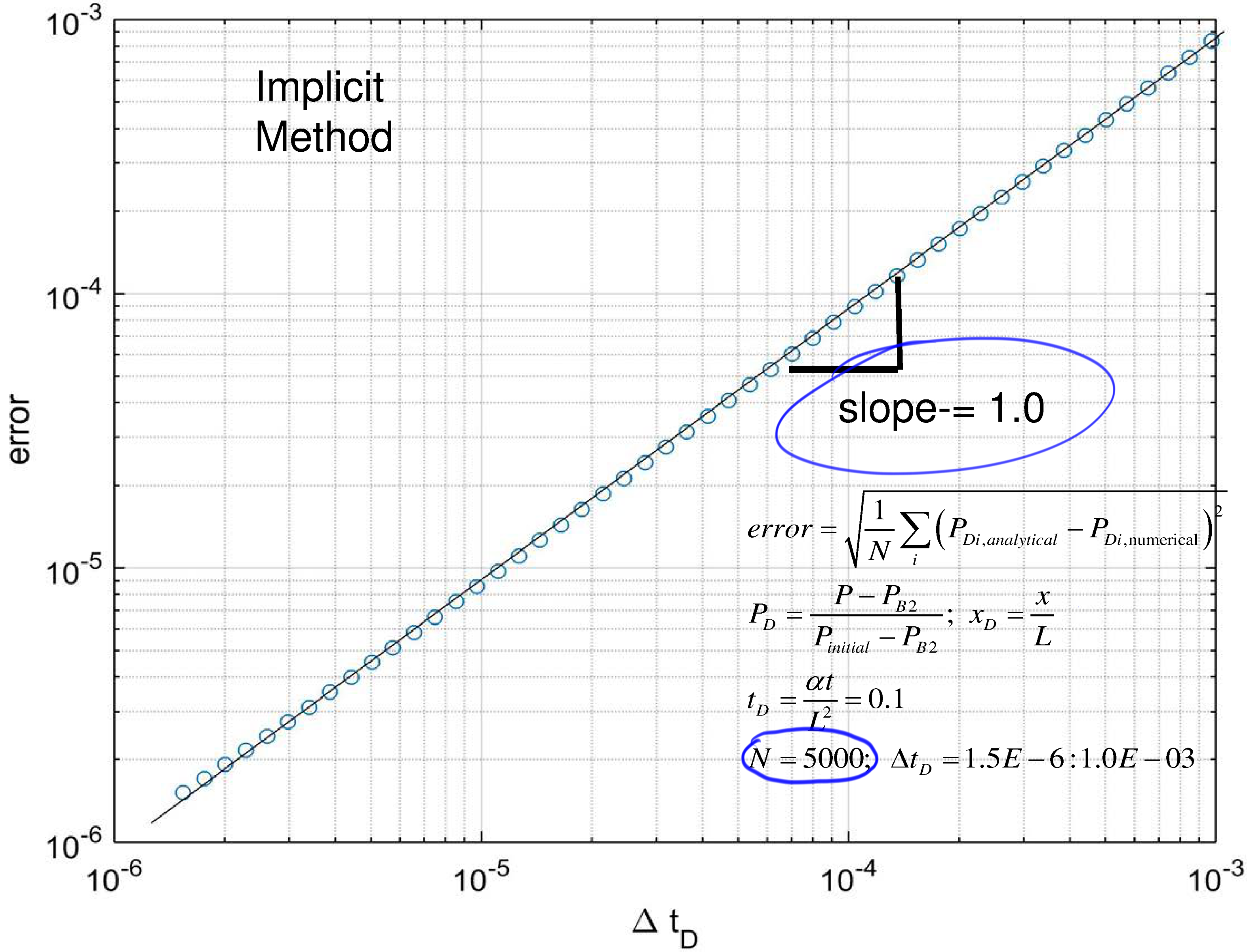
Original equation for block "i" now has units of flow rate

$$\boxed{-\frac{kA}{\mu B_w \Delta x}P_{i-1}^{n+1} + \left(\frac{V_i\phi c_t}{B_w\Delta t} + 2\frac{kA}{\mu B_w \Delta x}\right)P_i^{n+1} - \frac{kA}{\mu B_w \Delta x}P_{i+1}^{n+1} = \frac{V_i \phi c_t}{B_w \Delta t}P_i^n \qquad \left[\equiv \frac{\text{ft}^3}{\text{day}}\right]}$$

Error is Proportional to Δx^2 for Implicit Method



Error is Proportional to Δt for Implicit Method



Summary

- PDEs describing flow in transport in porous media can be solved numerically using finite difference approximations

- Explicit method is relatively easy to solve because new pressures are calculated "explicitly", but solution is only conditionally stable (if time steps are too large, or grids are too small it becomes unstable)

- Implicit method requires solving a system of *N* equations, but is unconditionally stable. Both explicit and implicit are second order accurate in space and first order in time

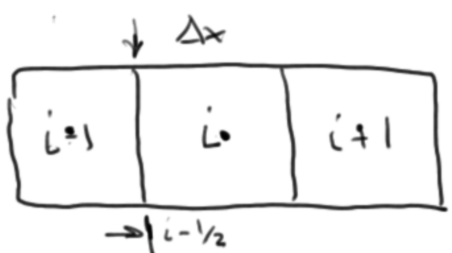
- Both the implicit and explicit equations can be written in "rate" units, introducing transmissibility (T), accumulation (B), and source (Q) terms

- Mixed methods are "hybrids" between implicit and explicit. Crank Nicholson is a special mixed method (θ=1/2) that is second order accurate in BOTH space and time

- Derived PDE's
- Discretized PDE's via FD's
- Multiplied by constant to convert to eq'ns $\qquad T = \frac{kA}{\mu B_w \Delta x}$

  → Transmiss. form

"Control Volume"



write conservation of mass directly on grid block i

in - out + injected/produced = accummulation

$$\underbrace{\rho^{RC} q_{i-1/2} \Delta t}_{\text{in}} - \underbrace{\rho^{RC} q_{i+1/2} \Delta t}_{\text{out}} + \underbrace{\rho^{RC} q_i^{RC} \Delta t}_{\text{inj/prod}} = \underbrace{m_i\Big|_{t+\Delta t} - m_i\Big|_{t}}_{\text{accum.}}$$

$$m_i = \rho^{RC} \phi V_i$$

$$q_{i-1/2} = \frac{B_w}{B_w} \frac{k_{i-1/2}}{\mu} A \frac{P_{i-1} - P_i}{\Delta X} = T_{i-1/2} B_w \left(P_{i-1} - P_i\right)$$

$$\rho^{RC} B_w T_{i-1/2} (P_i - P_{i-1}) \Delta t + \rho^{RC} B_w T_{i+1/2} (P_i - P_{i+1}) \Delta t + \rho^{RC} B_w Q_i^{sc} \Delta t = V_i \left[(\rho^{RC} \phi)^{n+1} - (\rho^{RC} \phi)^{n} \right]$$

Divide by ρ^{sc} + Δt

$$T_{i-1/2} (P_i - P_{i-1}) + T_{i+1/2} (P_i - P_{i+1}) + Q_i^{sc} = \frac{V_i}{\Delta t} \underbrace{\left[\left(\frac{\phi}{B_w}\right)^{n+1} - \left(\frac{\phi}{B_w}\right)^{n} \right]}$$

$$\left[\left(\frac{\phi}{B_w}\right)^{n+1} - \left(\frac{\phi}{B_w}\right)^{n} \right] = \left(\frac{\phi}{B_w}\right)^{n+1} - \underbrace{\left[\frac{\phi^{n+1}}{B_w^n} - \frac{\phi^{n+1}}{B_w^n} \right]}_{=0} - \left(\frac{\phi}{B_w}\right)^{n}$$

$$= \phi^{n+1} \left(\frac{1}{B_w^{n+1}} - \frac{1}{B_w^{n}} \right) + \frac{1}{B_w^n} \left(\phi^{n+1} - \phi^{n} \right)$$

Recall for small compressibility) ($\rho^n(p^o + \Delta p) \approx$ $p^n - p^o$

$$\frac{1}{B_w^n} \approx \frac{1}{B^o} \left[1 - c_f (p^n - p^o) \right] \qquad \phi^n \approx \phi^o \left[1 + c_R (p^n - p^o) \right]$$

$$\left(\frac{1}{B_w^{n+1}} - \frac{1}{B_w^{n}}\right) \approx \frac{1}{B_w^{0}}\left[\cancel{1} + c_f\left(p^{n+1} - p^{0}\right)\right] - \frac{1}{B_w^{0}}\left[\cancel{1} + c_f\left(p^{n} - p^{0}\right)\right]$$

$$\approx \frac{c_f}{B_w^{0}}\left(p^{n+1} - p^{n}\right)$$

$$\phi^{n+1} - \phi^{n} \approx \phi^{0}\left[\cancel{1} + c_R\left(p^{n+1} - p^{0}\right)\right] - \phi^{0}\left[\cancel{1} + c_R\left(p^{n} - p^{0}\right)\right]$$

$$\approx \phi^{0} c_f \left(p^{n+1} - p^{n}\right)$$

$$\left(\frac{\phi}{B_w}\right)^{n+1} - \left(\frac{\phi}{B_w}\right)^{n} = \frac{c_f}{B_w^{0}}\phi^{n+1}\left(p^{n+1} - p^{n}\right) + \frac{c_R\,\phi^{0}}{B_w}\left(p^{n+1} - p^{n}\right)$$

$$= \frac{\phi}{B_w}\underbrace{\left(c_f + c_R\right)}_{c_t}\left(p^{n+1} - p^{n}\right)$$

$$= \frac{\phi c_t}{B_w}\left(p^{n+1} - p^{n}\right)$$

$$T_{i-1/2}\left(p_{i-1} - p_i\right) + T_{i+1/2}\left(p_{i+1} - p_i\right) = \frac{V_i\,\phi c_t}{B_w \Delta t}\left(p_i^{n+1} - p_i^{n}\right) - Q_i^{sc}$$

$$\vec{p}^{\,n+1} = \left(\vec{\vec{T}} + \frac{1}{\Delta t} \vec{\vec{B}} \right)^{-1} \left(\frac{1}{\Delta t} \vec{\vec{B}}\, p^n + \vec{Q} \right) \qquad \text{Implicit}$$

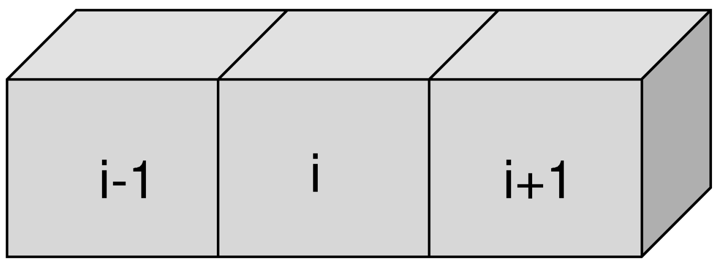
$$\vec{p}^{\,n+1} = \vec{p}^{\,n} + \Delta t\, \vec{\vec{B}}^{-1} \left(\vec{Q} - \vec{\vec{T}}\, p^n \right) \qquad \text{Explicit}$$

CHAPTER 4.
“CONTROL VOLUME” APPROACH

Control Volume Approach

Goal: Write a mass balance on each grid block “i” at reservoir conditions (RC) and attempt to recover finite difference matrix equations

mass in - mass out + injected/produced = accumulation

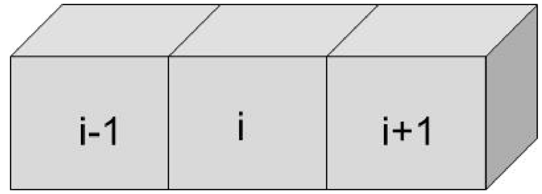


$$\underbrace{\rho^{RC} q_{i-1/2} \Delta t}_{\text{mass into CV}} - \underbrace{\rho^{RC} q_{i+1/2} \Delta t}_{\text{mass out of CV}} + \underbrace{\rho^{RC} q_i^{RC} \Delta t}_{\text{mass injected/produced}} = \underbrace{m_i^{n+1} - m_i^{n}}_{\text{accumulation n to n+1}}$$

$\rho \equiv Density \left[\frac{M}{L^3}\right]$ | $q \equiv Flow\ Rate \left[\frac{L^3}{T}\right]$ | $m \equiv Mass\ [M]$

Control Volume Approach

$$\rho^{RC} q_{i-1/2}\Delta t - \rho^{RC} q_{i+1/2}\Delta t + \rho^{RC} q_i^{RC}\Delta t = m_i^{n+1} - m_i^{n}$$



In the mass balance equation, mass of fluid in block “i”:

$$m_i = \rho^{RC}\phi V_i$$ (mass of fluid in pore volume times fluid density)

$$q_{i-1/2} = \frac{kA}{\mu\Delta x}(P_i - P_{i-1}) = T_{i-1/2}B_w(P_i - P_{i-1})$$ (Darcy’s law)

$$q_i^{RC} = B_w Q_i^{SC}$$ (formation factor converts from standard to reservoir conditions)

Substituting these variables in gives,

$$\rho^{RC} B_w T_{i-1/2}(P_i - P_{i-1})\Delta t + \rho^{RC} B_w T_{i+1/2}(P_i - P_{i+1})\Delta t + \rho^{RC} B_w Q_i^{SC}\Delta t = V_i\left[\left(\rho^{RC}\phi\right)^{n+1} - \left(\rho^{RC}\phi\right)^{n}\right]_i$$

Dividing by $\rho^{SC}\Delta t$ gives the following balance equation

$$T_{i-1/2}(P_i - P_{i-1}) + T_{i+1/2}(P_i - P_{i+1}) + Q_i^{SC} = \frac{V_i}{\Delta t}\left[\left(\frac{\phi}{B_w}\right)^{n+1} - \left(\frac{\phi}{B_w}\right)^{n}\right]_i$$

$\rho \equiv Density\left[\frac{M}{L^3}\right]$	$q \equiv Flow\ Rate\left[\frac{L^3}{T}\right]$	$m \equiv Mass\ [M]$	$T \equiv Transmissibility\left[\frac{L^4 T}{M}\right]$	$P \equiv Pressure\left[\frac{M}{LT^2}\right]$	$Q^{SC} \equiv Source\left[\frac{L^3}{T}\right]$

Transmissibility Form

$$-\underbrace{\frac{kA}{\mu B_w \Delta x}}_{T} P_{i-1}^{n+1} + \left(\underbrace{\frac{V_i \phi c_t}{B_w \Delta t}}_{B_i} + 2 \underbrace{\frac{kA}{\mu B_w \Delta x}}_{T} \right) P_i^{n+1} - \underbrace{\frac{kA}{\mu B_w \Delta x}}_{T} P_{i+1}^{n+1} = \underbrace{\frac{V_i \phi c_t}{B_w \Delta t}}_{B_i} P_i^n$$

Identify a few terms here:

$$\boxed{B_i = \frac{V_i \phi c_t}{B_w}} \longrightarrow$$

Refers to the volume **accumulation**. How much the fluid is expanded/contracted when pressure is increased/decreased

$$\boxed{T = \frac{kA}{\mu B_w \Delta x}} \longrightarrow$$

Refers to **transmissibility**. By Darcy's law, measures how much fluid flows in or out of the grid block

$$\longrightarrow \quad -TP_{i-1}^{n+1} + \left(\frac{B_i}{\Delta t} + 2T \right) P_i^{n+1} - TP_{i+1}^{n+1} = \frac{B_i}{\Delta t} P_i^n$$

*Note: Don't confuse B_w (fluid formation volume factor) with B_i (accumulation). They are different

$k \equiv Permeability\left[L^2\right]$	$\mu \equiv Viscosity\left[\frac{M}{LT}\right]$	$B_w \equiv Volumetric\ Factor\left[\frac{L^3}{L^3}\right]$	$c \equiv Compressibility\left[\frac{LT^2}{M}\right]$	$T \equiv Transmissibility\left[\frac{L^4T}{M}\right]$

Doing Some Algebra on the Right Hand Side

Expand time difference to include extra term in brackets

$$\left(\frac{\phi}{B_w}\right)^{n+1}-\left(\frac{\phi}{B_w}\right)^{n}=\left(\frac{\phi}{B_w}\right)^{n+1}-\underbrace{\left[\frac{\phi^{n+1}}{B_w^{\;n}}-\frac{\phi^{n+1}}{B_w^{\;n}}\right]}_{\text{these terms cancel}}-\left(\frac{\phi}{B_w}\right)^{n}=\phi^{n+1}\left(\frac{1}{B_w^{\;n+1}}-\frac{1}{B_w^{\;n}}\right)+\frac{1}{B_w^{\;n}}\left(\phi^{n+1}-\phi^{n}\right)$$

And recalling linear approximations for fluid and rock compression…

$$\frac{1}{B_w^{\;n}}\approx\frac{1}{B_w^{\;0}}\left[1+c_f\left(P^n-P^0\right)\right]\qquad \phi^n\approx\phi^0\left[1+c_r\left(P^n-P^0\right)\right]$$

Change in formation volume factor and porosity then simplify

$$\left(\frac{1}{B_w^{\;n+1}}-\frac{1}{B_w^{\;n}}\right)\approx\frac{1}{B_w^{\;0}}\left[\cancel{1}+c_f\left(P^{n+1}-\cancel{P^0}\right)\right]-\frac{1}{B_w^{\;0}}\left[\cancel{1}+c_f\left(P^{n}-\cancel{P^0}\right)\right]=\frac{c_f}{B_w^{\;0}}\left(P^{n+1}-P^n\right)$$

$$\left(\phi^{n+1}-\phi^n\right)\approx\phi^0\left[\cancel{1}+c_r\left(P^{n+1}-\cancel{P^0}\right)\right]-\phi^0\left[\cancel{1}+c_r\left(P^{n}-\cancel{P^0}\right)\right]=\phi^0c_r\left(P^{n+1}-P^n\right)$$

Therefore,

$$\left(\frac{\phi}{B_w}\right)^{n+1}-\left(\frac{\phi}{B_w}\right)^{n}=\frac{c_f\phi^{n+1}}{B_w^{\;0}}\left(P^{n+1}-P^n\right)+\frac{\phi^0c_r}{B_w^n}\left(P^{n+1}-P^n\right)$$

And Some More Manipulation…

$$\left(\frac{\phi}{B_w}\right)^{n+1}-\left(\frac{\phi}{B_w}\right)^{n}=\frac{c_f\phi^{n+1}}{B_w^{\ 0}}\left(P^{n+1}-P^{n}\right)+\frac{\phi^0 c_r}{B_w^n}\left(P^{n+1}-P^{n}\right)$$

If B_w^0 and ϕ^0 are evaluated at a reference pressure, P^0, near P^n then $B_w^0 = B_w$ and $\phi^0 = \phi$

$$\left(\frac{\phi}{B_w}\right)^{n+1}-\left(\frac{\phi}{B_w}\right)^{n}=\frac{\phi}{B_w}\left(c_f+c_r\right)\left(P^{n+1}-P^{n}\right)=\frac{\phi c_t}{B_w}\left(P^{n+1}-P^{n}\right)$$

Mass balance then becomes the same as was found from finite differences

n+1

$$\boxed{T_{i-1/2}(P_{i-1}-P_i)+T_{i+1/2}(P_{i+1}-P_i)=\frac{V_i\phi c_t}{B_w\Delta t}(P_i^{n+1}-P_i^n)-Q_i^{SC}}$$

$T \equiv Transmissibility\left[\frac{L^4T}{M}\right]$ | $P \equiv Pressure\left[\frac{M}{LT^2}\right]$ | $Q^{SC} \equiv Source\left[\frac{L^3}{T}\right]$

Implicit Solution for all Grid Blocks

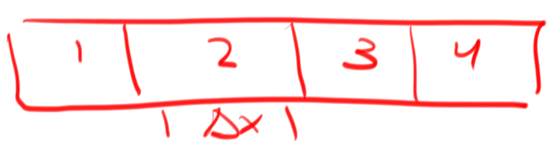
Consider a 4-block system with Dirichlet BC on left, Neumann on right

$$\text{block \#1}\quad \left(\frac{B_1}{\Delta t}+3T\right)P_1^{n+1}-TP_2^{n+1}=\frac{B_1}{\Delta t}P_1^{n}+2TP_{B1}$$

$$\text{block \#2}\quad -TP_1^{n+1}+\left(\frac{B_2}{\Delta t}+2T\right)P_2^{n+1}-TP_3^{n+1}=\frac{B_2}{\Delta t}P_2^{n}$$

$$\text{block \#3}\quad -TP_2^{n+1}+\left(\frac{B_3}{\Delta t}+2T\right)P_3^{n+1}-TP_4^{n+1}=\frac{B_3}{\Delta t}P_3^{n}$$

$$\text{block \#4}\quad -TP_3^{n+1}+\left(\frac{B_4}{\Delta t}+T\right)P_4^{n+1}=\frac{B_4}{\Delta t}P_N^{n}$$



Boundary conditions can be used to eliminate two imaginary grid pressures and we are left N linear equations and N unknown pressures

System of Equations Written in Matrix Form:

$$\left(\frac{1}{\Delta t}\underbrace{\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix}}_{\mathbf{B}} + \underbrace{\begin{bmatrix} 3T & -T & 0 & 0 \\ -T & 2T & -T & 0 \\ 0 & -T & 2T & -T \\ 0 & 0 & -T & T \end{bmatrix}}_{\mathbf{T}} \right) \underbrace{\begin{bmatrix} P_1^{n+1} \\ P_2^{n+1} \\ P_3^{n+1} \\ P_4^{n+1} \end{bmatrix}}_{P^{n+1}} = \frac{1}{\Delta t}\underbrace{\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} P_1^{n} \\ P_2^{n} \\ P_3^{n} \\ P_4^{n} \end{bmatrix}}_{P^n} + \underbrace{\begin{bmatrix} 2TP_{in} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{Q}$$

$$\underbrace{\left(\mathbf{T} + \frac{1}{\Delta t}\mathbf{B} \right)}_{A} \vec{P}^{n+1} = \overbrace{\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n} + \vec{Q}}^{\vec{b}}$$

$A\,p^{n+1} = \vec{b} \Rightarrow p^{n+1} = A^{-1}\vec{b}$

- T (transmissibility) matrix is tri-diagonal for 1D problems
- B (accumulation) matrix is diagonal
- Q (source) vector includes boundary conditions and wells (discussed later)

$T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$ | $P \equiv Pressure \left[\frac{M}{LT^2}\right]$ | $c \equiv Compressibility \left[\frac{LT^2}{M}\right]$

All Units are Now Rates

$$\left(\mathbf{T}+\frac{1}{\Delta t}\mathbf{B}\right)\vec{P}^{n+1}=\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n}+\vec{Q}$$

$$\left(\left[\frac{L^4T}{M}\right]+\left[\frac{1}{T}\frac{L^4T^2}{M}\right]\right)\left[\frac{M}{LT^2}\right]=\left[\frac{1}{T}\frac{L^4T^2}{M}\frac{M}{LT^2}\right]+\left[\frac{L^3}{T}\right]$$

$$\left[\frac{L^3}{T}\right]=\left[\frac{L^3}{T}\right]+\left[\frac{L^3}{T}\right]\equiv\frac{\mathrm{ft}^3}{\mathrm{day}}$$

Explicit vs. Implicit in Transmissibility form

Explicit Method

$$\frac{B_i}{\Delta t} P_i^{n+1} = T P_{i-1}^{n} + \left(\frac{B_i}{\Delta t} - 2T \right) P_i^{n} + T P_{i+1}^{n}$$

$$\Downarrow$$

$$\mathbf{T}\vec{P}^{n} + \frac{1}{\Delta t}\mathbf{B}\vec{P}^{n+1} = \frac{1}{\Delta t}\mathbf{B}\vec{P}^{n} + \vec{Q}$$

$$\Downarrow$$

$$\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n+1} = \left(-\mathbf{T} + \frac{1}{\Delta t}\mathbf{B} \right)\vec{P}^{n} + \vec{Q}$$

$$\Downarrow$$

$$\boxed{\vec{P}^{n+1} = \vec{P}^{n} + \Delta t \mathbf{B}^{-1}\left(\vec{Q} - \mathbf{T}\vec{P}^{n} \right)}$$

Implicit Method

$$-T P_{i-1}^{n+1} + \left(\frac{B_i}{\Delta t} + 2T \right) P_i^{n+1} - T P_{i+1}^{n+1} = \frac{B_i}{\Delta t} P_i^{n}$$

$$\Downarrow$$

$$\mathbf{T}\vec{P}^{n+1} + \frac{1}{\Delta t}\mathbf{B}\vec{P}^{n+1} = \frac{1}{\Delta t}\mathbf{B}\vec{P}^{n} + \vec{Q}$$

$$\Downarrow$$

$$\left(\mathbf{T} + \frac{1}{\Delta t}\mathbf{B} \right)\vec{P}^{n+1} = \frac{1}{\Delta t}\mathbf{B}\vec{P}^{n} + \vec{Q}$$

$$\Downarrow$$

$$\boxed{\vec{P}^{n+1} = \underbrace{\left(\mathbf{T} + \frac{1}{\Delta t}\mathbf{B} \right)}_{A}^{-1}\left(\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n} + \vec{Q} \right)}$$

Both the explicit and implicit method are $O(\Delta x^2)$ and $O(\Delta t)$ accurate. The explicit method is conditionally stable while the implicit method is unconditionally stable

$T \equiv Transmissibility \left[\frac{L^4T}{M}\right]$	$P \equiv Pressure \left[\frac{M}{LT^2}\right]$	$\underline{B} \equiv Compressibility\ Matrix \left[\frac{L^4T^2}{M}\right]$	$\vec{Q} \equiv Source\ Vector \left[\frac{L^3}{T}\right]$

Mixed Methods: Combination of Implicit and Explicit Methods

Mixed methods weight explicit method by 0<θ<1 and implicit by 1-θ

$$\theta\left[\mathbf{T}\vec{P}^{n}+\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n+1}=\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n}+\vec{Q}\right] \quad \text{Explicit Method}$$

$+$

$$(1-\theta)\left[\mathbf{T}\vec{P}^{n+1}+\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n+1}=\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n}+\vec{Q}\right] \quad \text{Implicit Method}$$

$$\theta\mathbf{T}\vec{P}^{n}+(1-\theta)\mathbf{T}\vec{P}^{n+1}+\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n+1}=\frac{1}{\Delta t}\mathbf{B}\vec{P}^{n}+\vec{Q} \quad \text{Mixed Method}$$

After some rearrangement we get,

$$\boxed{\underbrace{\left((1-\theta)\mathbf{T}+\frac{1}{\Delta t}\mathbf{B}\right)}_{A}\vec{P}^{n+1}=\left(\frac{1}{\Delta t}\mathbf{B}-\theta\mathbf{T}\right)\vec{P}^{n}+\vec{Q}}$$

* If θ=1/2, it is called "Crank Nicholson"

$T \equiv Transmissibility \left[\frac{L^4T}{M}\right]$	$P \equiv Pressure \left[\frac{M}{LT^2}\right]$	$\underline{B} \equiv Compressibility\ Matrix \left[\frac{L^4T^2}{M}\right]$	$\vec{Q} \equiv Source\ Vector \left[\frac{L^3}{T}\right]$

Crank-Nicholson is More Accurate in Time

Both explicit and implicit methods are $O(\Delta x^2)$ accurate in space because a centered difference was used but are only $O(\Delta t)$ accurate in time because forward/backward difference was used

$$\frac{\partial P}{\partial t} = \frac{P^{n+1} - P^{n}}{\Delta t} + O(\Delta t)$$

Equation is a forward difference for the explicit method (current time is "n") but a backward difference for the implicit method (current time is n+1)

Alternatively, we could take the current time to be "n+1/2" and then the approximation is a "centered" difference. The spatial derivative would be a hybrid of the "n" and "n+1" time level

$$\frac{\partial P^{n+\frac{1}{2}}}{\partial t} = \frac{P^{n+1} - P^{n}}{\Delta t} + O(\Delta t^2) \qquad \frac{\partial^2 P^{n+\frac{1}{2}}}{\partial x^2} \approx \frac{1}{2}\frac{P_{i-1}^{n} - 2P_{i}^{n} + P_{i+1}^{n}}{(\Delta x)^2} + \frac{1}{2}\frac{P_{i-1}^{n+1} - 2P_{i}^{n+1} + P_{i+1}^{n+1}}{(\Delta x)^2} + O(\Delta x^2)$$

Crank-Nicholson is second-order accurate in both time and space!!

Implicit and Analytical Solutions to 1D Diffusivity Equation