**Workshop Testing and Formal Methods, Week 4: Answers**

This workshop is about working with sets and relations. Sets and lists are treated in chapter 4 of "The Haskell Road". Ordered sets of pairs (HR, 4.5) are the basic ingredients of relations. Relations are treated in chapter 5 of "The Haskell Road".

If the following exercises are difficult for you, you should study the relevant material (again).

**Question 1** Consider the following relations on the natural numbers. Check their properties. The *successor* relation on $\mathbb{N}$ is the relation given by $\{(n, m) \mid n + 1 = m\}$. The divisor relation on $\mathbb{N}$ is $\{(n, m) \mid n \text{ divides } m\}$. The *coprime* relation $C$ on $\mathbb{N}$ is given by $nCm :\equiv \text{GCD}(n, m) = 1$, i.e., the only factor of $n$ that divides $m$ is 1, and vice versa.

| | $<$ | $\leq$ | successor | divisor | coprime |
|---|---|---|---|---|---|
| irreflexive | | | | | |
| reflexive | | | | | |
| asymmetric | | | | | |
| antisymmetric | | | | | |
| symmetric | | | | | |
| transitive | | | | | |
| linear | | | | | |

**Answer:**

| | $<$ | $\leq$ | successor | divisor | coprime |
|---|---|---|---|---|---|
| irreflexive | $\surd$ | | $\surd$ | | |
| reflexive | | $\surd$ | | $\surd$ | |
| asymmetric | $\surd$ | | $\surd$ | | |
| antisymmetric | $\surd$ | $\surd$ | | $\surd$ | |
| symmetric | | | | | $\surd$ |
| transitive | $\surd$ | $\surd$ | | $\surd$ | |
| linear | $\surd$ | $\surd$ | | | |

Note that the *coprime* relation is not irreflexive, for 1 and 1 are coprime.

**Question 2** Consider the relation

$$R = \{(0, 2), (0, 3), (1, 0), (1, 3), (2, 0), (2, 3)\}$$

on the set $A = \{0, 1, 2, 3, 4\}$.

1. Determine $R^2$, $R^3$ and $R^4$.
2. Give a relation $S$ on $A$ such that $R \cup (S \circ R) = S$.

**Answer:** $R^2 = \{(0,0),(0,3),(1,2),(1,3),(2,2),(2,3)\}$,
$R^3 = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$
and $R^4 = \{(0,0),(0,3),(1,2),(1,3),(2,2),(2,3)\}$.
From these results we see that $R \cup R^2$ is a good candidate for $S$. And indeed, if we put

$$S = \{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,2),(2,3)\},$$

we get that $R \cup (S \circ R) = S$.

**Question 3** The transitive closure of a relation $R$ is by definition the smallest transitive relation $S$ such that $R \subseteq S$. Notation: $R^+$

Consider again the relation

$$R = \{(0,2),(0,3),(1,0),(1,3),(2,0),(2,3)\}$$

on the set $A = \{0,1,2,3,4\}$. What is the transitive closure of $R$?

**Answer:**

$$\{(0,0),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,2),(2,3)\}.$$

Same as the $S$ we found in the previous exercise.

**Question 4** A binary relation $R$ is transitive iff $R \circ R \subseteq R$. You should check this!

Give an example of a transitive relation $R$ for which $R \circ R = R$ is false.

**Answer:** What the requirement $R \circ R \subseteq R$ expresses is that if you can get from $x$ to $y$ in two $R$-steps, then also in one. This is precisely what the transitivity requirement says.

An example of a transitive relation $R$ for which $R \circ R \neq R$ is $<$ on $\mathbb{N}$. We have that $0 < 1$, but $\neg 0 (< \circ <) 1$.

**Question 5** The reflexive transitive closure of a relation $R$ is by definition the smallest transitive and reflexive relation $S$ such that $R \subseteq S$. Notation: $R^*$.

Give the reflexive transitive closure of the following relation:

$$R = \{(n, n+1) \mid n \in \mathbb{N}\}.$$

**Answer:**

$$\{(n, n+1) \mid n \in \mathbb{N}\}^* = \leq .$$

**Question 6** The inverse of a relation $R$ is the relation $\{(y,x) \mid (x,y) \in R\}$. Notation $R^{-1}$ or $R^{\smallsmile}$.

1. if $S$ is the successor relation on the natural numbers, what is $S\breve{}$?
2. if $S$ is the successor relation on the natural numbers, what is $S \cup S\breve{}$?
3. if $S$ is the successor relation on the natural numbers, what is $(S \cup S\breve{})^*$?

**Answer:**

1. If $S$ is the successor relation on $\mathbb{N}$, then $S\breve{}$ is the predecessor relation on $\mathbb{N}$.
2. If $S$ is the successor relation on $\mathbb{N}$, then $S \cup S\breve{}$? is the relation that links 0 to 1, and every number $n + 1$ to both $n$ and $n + 2$. In a picture:

$$0 - 1 - 2 - 3 - 4 - 5 \cdots$$

3. If $S$ is the successor relation on $\mathbb{N}$, $(S \cup S\breve{})^*$ is the total relation on $\mathbb{N}$.

**Question 7** Suppose a relation $R$ satisfies $R\breve{} \subseteq R$.

1. Does it follow from this that $R$ is reflexive?
2. Does it follow from this that $R$ is symmetric?
3. Does it follow from this that $R$ is transitive?

**Answer:** If $R$ satisfies $R\breve{} \subseteq R$, then

1. it does not follow that $R$ is reflexive (counterexample: $R = \{(0, 1), (1, 0)\}$);
2. it does follow that $R$ is symmetric;
3. it does not follow that $R$ is transitive (counterexample: $\{R = \{(0, 1), (1, 0)\}\}$).

**Question 8**

1. Is $R \cup R\breve{}$ symmetric for all relations $R$? Give a counterexample if your answer is negative.
2. Is $R^* \cup R\breve{}^*$ symmetric for all relations $R$? Give a counterexample if your answer is negative.
3. Is $R^* \cup R\breve{}^*$ transitive for all relations $R$? Give a counterexample if your answer is negative.
4. Is $(R \cup R\breve{})^*$ an equivalence relation (reflexive, transitive and symmetric) for all relations $R$?

**Answer:**

1. Is $R \cup R^{\smile}$ symmetric for all relations $R$? Yes.
2. Is $R^* \cup R^{\smile *}$ symmetric for all relations $R$? Yes.
3. Is $R^* \cup R^{\smile *}$ transitive for all relations $R$? No. Counterexample: $R = \{(0,1),(0,2)\}$. This gives

$$R^* = \{(0,1),(0,2),(0,0),(1,1),(2,2)\},$$
$$R^{\smile} = \{(1,0),(2,0)\},$$
$$R^{\smile *} = \{(1,0),(2,0),(0,0),(1,1),(2,2)\},$$

and finally,

$$R^* \cup R^{\smile *} = \{(0,1),(0,2),(1,0),(1,2),(0,0),(1,1),(2,2)\},$$

which is not a transitive relation, because $(1,2)$ and $(2,1)$ are lacking.

4. Is $(R \cup R^{\smile})^*$ an equivalence relation (reflexive, transitive and symmetric) for all relations $R$? Yes.