



## Writing and Proving Theorems:

Once one or more specifications have been established there are two things that should be done:

1. Develop properties the specifications should have
2. Prove equivalence between specifications and implementations

In future lessons, when the Software Analysis Workbench is discussed, the implementations will be mainly in the C language. Here, we develop and use Cryptol implementations to get the idea.

Consider a specification for testing whether two sequences are permutations of each other. This was developed in Lesson 2.4 and is the following:

```
remove lst n = drop {2} [w.0 | w <- res]
  where
    res = [(-1,False)]# [ if (q.1 == False /\ ~(za == n))
                          then (za, q.1)
                          else (zb, True)
                  | za <- [-1]#lst | zb <- lst#[-1] | q <- res ]

member lst n = (res ! 0)
  where
    res = [False]# [ ss \/ (p == n) | p <- lst | ss <- res ]

perm x y = if (length x) == 0 /\ (length y) == 0 then True else
            if (length x) == (length y) then (res ! 0).1 else False
  where
    res = [ ((remove q.0 p), (member q.0 p) /\ q.1)
          | q <- [(y,True)]#res | p <- x ]
```

Observe that `perm x y` is a relation that should be an equivalence relation. An equivalence relation has properties *reflexive*, *symmetric*, and *transitive*. For reflexive, write the property like this:

```
permReflexive : ([10][32] -> [10][32] -> Bit) -> Bit
property permReflexive x y = if x == y then perm x y else True
```

which says that `x` equals `y` for sequences of 32 bit numbers of length 10 implies `perm x y` is True. Small sequence lengths are used here so proofs are relatively fast. Prove this property like this in Cryptol (let the above be in file `perm.cry`):

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> :s prover=cvc4
Main> :prove permReflexive
Q.E.D.
(Total Elapsed Time: 0.121s, using "CVC4")
```

Next set up property `permSymmetric`. Here there is a problem because the algorithm used to determine that `x` is a permutation of `y` used `-1` to fill in places as equality between elements of `x` and `y` is determined. This means the following for `permSymmetric` is not going to work:

```
permSymmetric : [10][32] -> [10][32] -> Bit
property permSymmetric x y = if perm x y then perm y x else True
```

To see this, comment out property `permReflexive` and add the following type signature to `perm` (`[4][8]` is used here to get a result that is easy to see):

```
perm : [4][8] -> [4][8] -> Bit
```

Now do this in Cryptol (observe 255 is  $2^{8-1}$ ):

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> perm [0,0,0,255] [0,0,0,0]
True
```

which should not be the case but happens because 255 is considered to be `-1` for width 8. What this says is that, for the particular specification chosen for `perm`,  $2^X-1$ , where `X` is the width of the numbers in the input sequences, a test should have been added to make sure such numbers are not in the input lists. In other words, the algorithm for `perm` is wrong if all numbers up to 255 are to be allowed in input lists. Observe that without the above type signature for `perm`:

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> perm [0,0,0,255] [0,0,0,0]
False
```

which illustrates an important property about Cryptol: the requirement of monomorphic type signatures for theorems finds problems that can otherwise be easily overlooked. Specification `perm` could be changed to forbid numbers  $2^X-1$  but then `perm` would need a type signature to make such tests. It is left as an exercise to produce a specification for `perm` that allows all numbers in input sequences such that properties can be proved. In the meantime, the following works for sequences of 32 bit numbers of length 10:

```
permSymmetric : [10][32] -> [10][32] -> Bit
property permSymmetric x y =
  if ~(member x (2^32-1)) /\ ~(member y (2^32-1)) /\ perm x y
  then perm y x else True
```

Here it is tested:

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> :s prover=cvc4
Main> :prove permSymmetric
Q.E.D.
(Total Elapsed Time: 29.092s, using "CVC4")
```

For permTransitive the property is:

```
permTransitive : [6][32] -> [6][32] -> [6][32] -> Bit
property permTransitive x y z =
  if ~(member x (2^^32-1)) /\ ~(member y (2^^32-1)) /\ ~(member z (2^^32-1)) /\
  perm x y /\ perm y z
  then perm x z else True
```

where [6][32] was chosen to make the proof complete in reasonable time.

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> :s prover=cvc4
Main> :prove permTransitive
Q.E.D.
(Total Elapsed Time: 1m:12.672s, using "CVC4")
```

Conclude that, at least for sequences of 32 bit numbers of length 6, where input sequences do not contain  $2^{32}-1$ , perm defines an equivalence class. Specification perm will be used to help show that a given implementation of a proposed sorting algorithm is correct.

One such sorting algorithm is this:

```
bssort : {n, a} (Cmp a, Integral a, Literal 1 a, fin n) => [n]a -> [n]a
bssort lst = take `n` (bb (lst#[-1,-1...]))
  where
    bb seq = if (seq@0) == -1 then seq else (bbl (head seq) (bb (tail seq)))
    bbl n seq =
      if ((seq@0) == -1) then [n]#[-1,-1...] // sequence -1 ... is the padding
      else if (n < (seq@0)) then [n]#seq
      else [(seq@0)]#(bbl n (tail seq))
```

Function bbl takes a literal n and an ordered sequence seq and places n in seq so that seq remains ordered. If seq is empty [n] (plus padding) is returned. If n is less than the first element of seq then n is added to the beginning of seq. Otherwise, bbl is called recursively and returns an ordered sequence whereupon the topmost stacked literal from seq is prepended to the result and returned. Function bb takes a padded sequence seq as input and recursively calls bbl where n is the first literal of seq. The recursion sets up a stack of literals to add to the output list and begins to establish an output list when seq is only the padding. Thus, when the stacked literals are ready to be added to the output list, what they are added to is ordered and will remain ordered as stacked literals are added to the output list. Function bssort just calls bb on the infinitely long padded input sequence. The output of bb is trimmed by take. As before, since -1 has a special role, it cannot be considered an element of the input lst. The following function will ensure this is the case so proofs will complete as expected:

```
valid lst = z ! 0
  where
    z = [True]#[(x == -1) /\ y | x <- lst | y <- z ]
```

The following proves that bssort lst is a permutation of lst, one of two properties that a sorting algorithm must have, the second being that bssort lst must be non-decreasing.

```
isAPerm : [6][32] -> Bit
property isAPerm lst = if valid lst then perm lst (bssort lst) else True
```

Here is the proof for 32 bit non-negative numbers in a sequence of 6:

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> :s prover=any
Main> :prove isAPerm
Q.E.D.
(Total Elapsed Time: 1.647s, using "CVC4")
```

Since perm defines an equivalence class, every element in `lst` is also in `bsort lst` as many times as that element is in `lst`, and any element not in `lst` is not in `bsort lst`. The following is True if and only if for any two valid consecutive numbers in `lst`, the earlier one is no greater than the later one:

```
nondecreasing : [6][32] -> Bit
nondecreasing lst = (res ! 0)
  where
    res = [True]#[ if (j1 <= j2) /\ k then True else False
                  | k <- res | j1 <- lst | j2 <- drop `1} lst]
```

The second half of the proof that `bsort` sorts correctly:

```
bsortIsOrdered : [6][32] -> Bit
property bsortIsOrdered lst =
  if valid lst then nondecreasing (bsort lst) else True
```

Proving:

```
Main> :l perm.cry
Loading module Cryptol
Loading module Main
Main> :s prover=any
Main> :prove bsortIsOrdered
Q.E.D.
(Total Elapsed Time: 0.476s, using "CVC4")
```

Thus, at least for length 6 lists of 32 bit numbers, `bsort` correctly sorts its input as long as `-1` is not an element of the input list.

The “golden” specifications of `perm`, `non-decreasing`, and, if needed, `valid` can be used for other implementations of sorting algorithms including those where infinite padding with `-1` is not used. The next example uses a process similar to that of `bsort` but without the need for infinite lists. The process makes use of a function `round` that moves greater numbers at most one position to the right and lesser numbers at most one position to the left. Running `round` as many times as there are numbers in the input sequence sorts it. Here is an outline of the operation of `round`. Let `lst` be the sequence to sort and let `acc` initially be a sequence of `0s` of the same length as `lst`. In a round, elements of `lst` are compared with elements of `acc`, starting from the right (greatest elements are on or gravitate toward the right). If the `lst` element is greater than the existing `acc` element, the `acc` element moves down (to the left) by 1 and makes way for the `lst` element to be placed where the `acc` element had been. If the `lst` element is not greater then it is placed before the `acc` element it is compared with (to the left of the `acc` element where it replaces a 0) . This process has the effect that greater

elements move one position to the right and the lesser elements one position to the left on a round. Thus, running a number of rounds equal to the length of `lst` sorts `lst`.

```
round lst acc = (func lst ((length lst)-1) acc)
  where
    func lt i ac =
      if (i == -1)
      then ac
      else if ((last ac) == 0)
      then func lt (i-1) (insert ac (lt@i) i)
      else if (lt@i) > (ac@(i+1))
      then func lt (i-1) (insert (insert ac (ac@(i+1)) i) (lt@i) (i+1))
      else func lt (i-1) (insert ac (lt@i) i)
```

Function `round` makes use of function `insert` which takes a sequence `lst`, a number `m`, and an index `i` and replaces the  $i^{\text{th}}$  element in `lst` with `m`.

```
insert lst m i = [ if k==i then m else x | x <- lst | k <- [0...]]
```

The following function initializes `acc` to a sequence of 0s of length `lst`:

```
makeZeros lst = [ 0 | i <- lst ]
```

Here is the sorting wrapper:

```
sort lst = z ! 0
  where
    z = [ round x (makeZeros lst) | x <- [lst]#z | i <- lst ]
```

So `x` contains the result of the previous round and `i` controls the comprehension to complete in length of `lst` iterations.

Permutations:

```
Loading module Cryptol
Cryptol> :l a.cry
Loading module Cryptol
Loading module Main
Main> :s prover=cvc4
Main> :prove isAPerm
Q.E.D.
(Total Elapsed Time: 3.560s, using "CVC4")
```

Non-decreasing:

```
Main> :prove sortIsOrdered
Q.E.D.
(Total Elapsed Time: 1.365s, using "CVC4")
```

The valid property for the above is changed to include disallowing 0 from input sequences.

The function `round` as written recursively above may be written as a comprehension like this:

```
round : {n, b} (Literal 0 b, Cmp b, fin n) => [1 + n]b -> [1 + n]b
round lst = z ! 0
  where
    z = [q]#[ if (lst@i) > (acc@(i+1))
              then (insert (insert acc (acc@(i+1)) i) (lst@i) (i+1))
              else (insert acc (lst@i) i)
            | acc <- z | i <- (reverse (take `n` [0...])) ];
    q = insert (makeZeros lst) (last lst) ((length lst)-1)
```

There are a few interesting things about this. First, this follows immediately from the recursive round: no changes were made in the then and else snippets, and `i` corresponds completely to the `i` of the recursive version. Second, note the comprehension easily accommodates the initial state of `acc` so `acc` is not needed as an input and function `sort` looks like this for the comprehension solution:

```
sort lst = z ! 0
  where
    z = [ round x | x <- [lst]#z | i <- lst ]
```

Third, note that Cryptol does not accept statements like `i <- [0..(length n)]` and is especially fretful about statements like `i <- [n,n-1..0]`. However, the type signature of `round` is user crafted, is added as `round` is defined, and contains `n` which relates to the range of values that `i` needs to take. The above uses `(take {n} [0...])` to get `[0..n]`, which is needed, but the `i` needs to be taken in reverse order so `i <- (reverse (take {n} [0...]))` is used – pretty tricky. Finally, proofs appear to run faster with comprehensions than with recursive implementations even though they are doing the same thing from an abstract perspective. Consider these times:

comprehension solution	recursive solution
<pre>// isAPerm : [6][32] -&gt; Bit Main&gt; :l sort1.cry Loading module Cryptol Loading module Main Main&gt; :prove isAPerm Q.E.D. (Total Elapsed Time: 0.235s, using "CVC4")  // sortIsOrdered : [6][32] -&gt; Bit Main&gt; :l sort1.cry Loading module Cryptol Loading module Main Main&gt; :prove sortIsOrdered Q.E.D. (Total Elapsed Time: 0.063s, using "CVC4")</pre>	<pre>// isAPerm : [6][32] -&gt; Bit Main&gt; :l sort.cry Loading module Cryptol Loading module Main Main&gt; :prove isAPerm Q.E.D. (Total Elapsed Time: 3.187s, using "CVC4")  // sortIsOrdered : [6][32] -&gt; Bit Main&gt; :l sort.cry Loading module Cryptol Loading module Main Main&gt; :prove sortIsOrdered Q.E.D. (Total Elapsed Time: 1.395s, using "CVC4")</pre>

As a final example consider mergesort. Here is an implementation:

```
splito : {n,a} [n]a -> [inf]a
splito lst = [ lst@i | i <- [1,3...] ]
```

The input to `splito` is a sequence of numbers (as used below the sequence contains numbers to be sorted followed by infinitely many `-1`s). The output is the sequence consisting of the numbers of `lst` that are to be sorted and are located at odd indices, followed by infinitely many `-1`s.

```
splite : {n,a} [n]a -> [inf]a
splite lst = [ lst@i | i <- [0,2...] ]
```

Input to `splite` is like `splito` except the output is the sequence consisting of the numbers of `lst` that are to be sorted and are located at even indices, followed by infinitely many `-1`s.

```
merge : {a} (Literal 1 a, Ring a, Cmp a) => [inf]a -> [inf]a -> [inf]a
merge px py =
  if px@0 == -1 then py
  else if py@0 == -1 then px
  else if px@0 < py@0 then [px@0]#(merge (tail px) py)
  else [py@0]#(merge px (tail py))
```

Input to merge is two infinite sequences  $px$  and  $py$  of numbers, each in non-decreasing order, then padded with infinitely many  $-1$ s. The output is an infinite, non-decreasing sequence containing all of  $px$  and  $py$ , then padded with infinitely many  $-1$ s. The padded infinite sequences  $px$  and  $py$  are merged, up to the padding. If  $px@0$  is  $-1$ , meaning that  $px$  is only padding, then  $py$  is returned, if  $py@0$  is  $-1$ , meaning  $py$  is only padding, then  $px$  is returned, otherwise if  $px@0$  is less than  $py@0$  then  $px@0$  is concatenated with  $(\text{merge } (\text{tail } px) py)$  and returned, otherwise  $py@0$  is concatenated with  $(\text{merge } px (\text{tail } py))$  and returned.

```
mergesort : {cnt, a} (fin cnt, fin a, a >= 1, cnt >= 0) => [cnt][a] -> [cnt][a]
mergesort lst = take `cnt` (mergesrt ax)
  where
    mergesrt p =
      if (p@0 == -1) \ / (p@1 == -1)
      then p
      else merge (mergesrt (splite p)) (mergesrt (splito p))
    ax = lst#[-1,-1 ...]
```

Input to mergesort is a finite sequence  $lst$  of numbers (the empty sequence is allowed due to  $cnt \geq 0$  in the type signature). Output of mergesort is a finite sequence of numbers that is a permutation of  $lst$  and in non-decreasing order. Sequence  $ax$  is  $lst$  padded with  $-1$ s and becomes input  $p$  to function mergesrt which sorts  $p$  up to the padding, recursively merging the mergesrt of the even elements of  $p$  with the mergesrt of the odd elements of  $p$ . Here are some results:

```
mergesort [] is []
mergesort [16] is [16]
mergesort [2,1,2,3] is [1,2,2,3]
mergesort [8,2,1,9,5,4,6,7] is [1,2,3,4,5,6,7,8,9]
```

Function mergesort can be deemed correct for sequences of six 32 bit integers if the following two properties hold:

```
isAPerm : [6][32] -> Bit
property isAPerm lst = if valid lst then perm lst (mergesort lst) else True
```

and

```
mergesortIsOrdered : [6][32] -> Bit
property mergesortIsOrdered lst =
  if valid lst then nondecreasing (mergesort lst) else True
```

Here are the proofs:

```
mrgsrt> :l mrgsrt.cry
Loading module Cryptol
Loading module mrgsrt
mrgsrt> :s prover=cvc4
mrgsrt> :prove isAPerm
Q.E.D.
(Total Elapsed Time: 0.573s, using "CVC4")
mrgsrt> :prove mergesortIsOrdered
Q.E.D.
(Total Elapsed Time: 0.175s, using "CVC4")
```

## Observations:

- Proofs of a collection of carefully chosen properties for specifications can be used to prove correctness of implementations. For example, in this case, `perm` was shown to be an equivalence relation which is necessary for correctness of sorting algorithms.
- A collection of carefully chosen specifications can be used as a “gold standard” for proving properties of implementations. For example, in this case, several sorting implementations were shown correct from the same set of specifications. This is important because considerable time and effort can then be used to establish specifications and prove specification properties and those resources can be amortized over numerous implementations.
- It appears that specifications written as comprehensions will prove faster than specifications written recursively. One such example was presented but this observation applied to others that were not presented here.
- The monomorphic requirement of Cryptol for proving theorems can sometimes uncover subtle errors that would otherwise be missed. For example, in this case the specification of `perm` was wrong, as was discovered through the Cryptol typing constraints, but was compensated for by defining a function to validate inputs to `perm`.
- A type variable in a type signature may be used to return a portion of an infinite, padded sequence. For example, in this case `mergesort`, which is applied to finite sequences, was written using infinite padded sequences operated on by another procedure `mergesrt`. The signature for `mergesort` was user crafted to be:

```
mergesort : {cnt, a} (fin cnt, fin a, a >= 1, cnt >= 0) => [cnt][a] -> [cnt][a]
```

which allowed `mergesort` to be written like this:

```
mergesort lst = take `{cnt} (mergesrt ax)
```

which means, although `ax` is the infinite `lst#[-1, -1...]`, only the first `cnt` elements of the infinite output of `(mergesrt ax)` are presented as output by `mergesort` and are the elements of `lst` sorted.

- Similarly, suppose `cnt` needs to be passed to a function. It can be done like this:

```
func a b `(cnt)
```

That is, `cnt` in parens instead of curly braces.