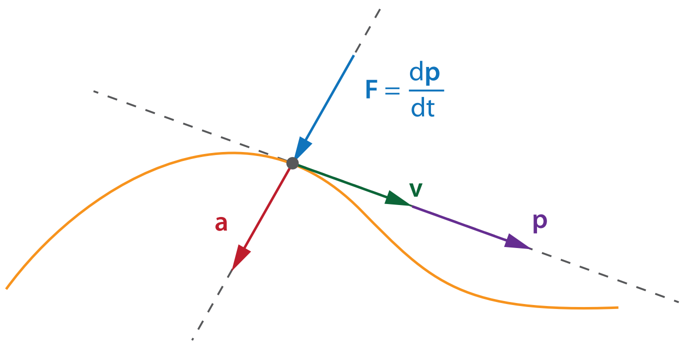
Handout 9 – Linear Momentum, Impact and Rockets

Meriam & Kraige, Dynamics: 3/9, 3/12, 4/7

In the previous handout, the concepts of work and energy were introduced to dynamics. These are integrals of the dynamic forces with respect to displacement, and applying the work-energy equations avoided having to find accelerations in order to determine velocities. Similarly, the dynamic forces can be integrated with respect to time, leading to **impulse** and **momentum** formulations. The conservation of momentum can be used to simplify the analysis of dynamics problems and offer elegant new insights.

9.1 Linear Momentum of Particles

Consider the planar motion of a particle of mass m, under applied net force $\boldsymbol{F}$.



A net force $\boldsymbol{F}$ on the particle will result in an acceleration $\boldsymbol{a}$ along the direction of the force vector.

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(m\boldsymbol{v}) = \frac{d\boldsymbol{p}}{dt} \tag{9.1}$$

where

$$\boldsymbol{p} = m\boldsymbol{v} \tag{9.2}$$

is defined as the **linear momentum** of the particle; it is a *vector quantity* with units Ns. In other words, the time derivative of the momentum vector $\boldsymbol{p}$ is equal to the force $\boldsymbol{F}$ applied to the particle. Note that the force (acceleration) and momentum (velocity) vectors will generally not be aligned!

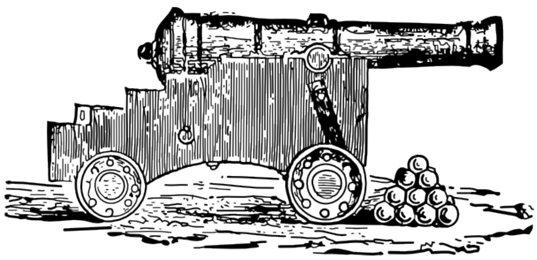
The time integral of the applied force $\boldsymbol{f}$ is known as the **linear impulse**:

$$\int_{t_1}^{t_2} \boldsymbol{F} dt = m\boldsymbol{v}_2 - m\boldsymbol{v}_1 = \boldsymbol{p}_2 - \boldsymbol{p}_1 = \Delta\boldsymbol{p} \tag{9.3}$$

and is equal to the change in linear momentum $\Delta\boldsymbol{p}$; this may involve changes in both magnitude and direction of the linear momentum. Thus, if no net external force is applied, the linear momentum must be conserved.

Example 9.1 – Cannon Recoil

Consider the recoil of a ship's cannon that fires a projectile. Let the gun be at rest, the bore aimed horizontally and the gun allowed to roll freely. As the explosive charge fires, it exerts a force on both the gun and the projectile (and thus imparts an equal and opposite impulse).

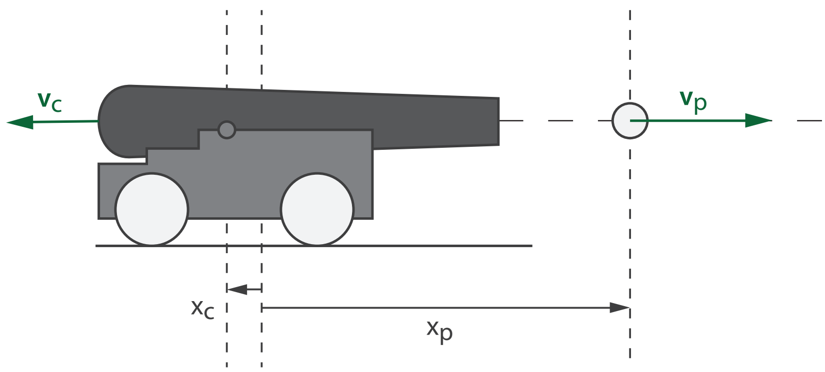


The momentum of the combined system must therefore remain constant:

$$\Delta \boldsymbol{p} = \boldsymbol{p}_p + \boldsymbol{p}_c = m_p v_p - m_c v_c = 0$$

Where $\boldsymbol{p}_p$ and $\boldsymbol{p}_c$ are the momentum of the projectile and cannon, respectively. Therefore, the velocities of the cannon and the projectile will be inversely proportional to their mass:

$$\frac{v_p}{v_c} = \frac{m_c}{m_p}$$



Now consider the kinetic energy of both the projectile and the cannon. These are scalar quantities, and the energy of both need to be added up to obtain the total energy of the system.

$$T_c = \frac{1}{2} m_c v_c^2 = \frac{1}{2} v_c \left(m_c v_c\right) = \frac{1}{2} v_c \left|\boldsymbol{p}_c\right|$$

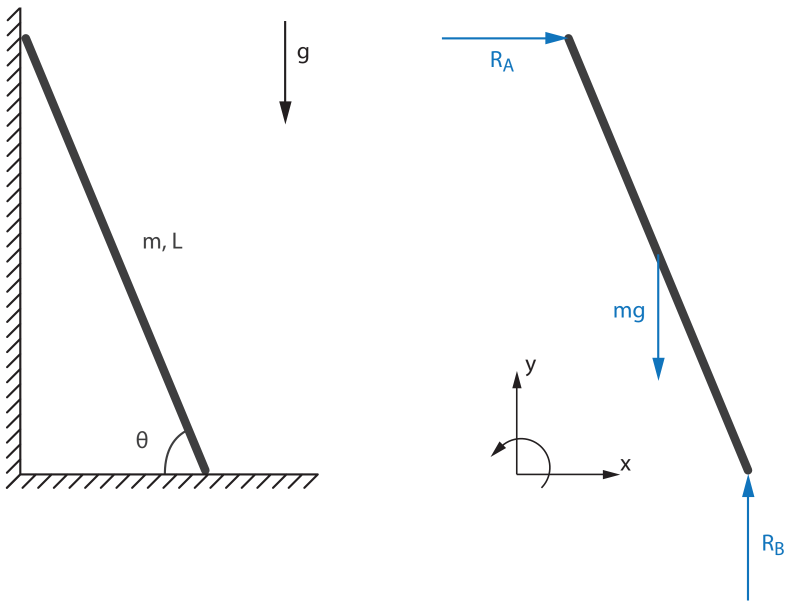
$$T_p = \frac{1}{2} m_p v_p^2 = \frac{1}{2} v_p \left(m_p v_p\right) = \frac{1}{2} v_p \left|\boldsymbol{p}_p\right|$$

The linear momentum of both is equal, and therefore the kinetic energy will be in ratio of the velocities, and therefore in inverse ratio of mass. This means that the kinetic energy of the projectile will be much greater than that of the cannon. (Which is the purpose of a cannon ball!)

Q: Why is the momentum of the cannon and projectile the same, but their kinetic energy different?

Example 9.2 – Sliding Ladder

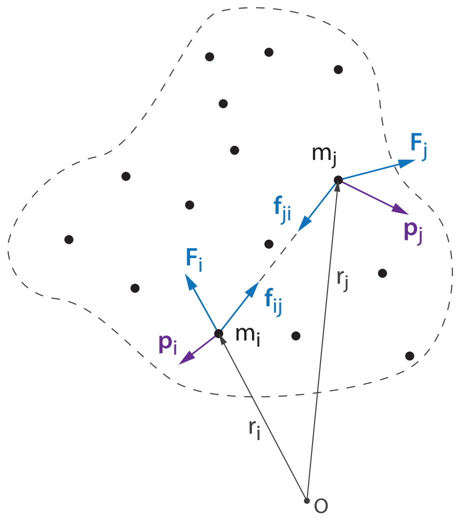
Consider a ladder standing on a smooth surface and leaning against a frictionless wall. Due to the absence of friction, the ladder will slide down and gain a horizontal, vertical and angular velocity.



The work done by the weight mg increases the kinetic energy as the ladder slides down, and although the reaction forces do no work, they impart a linear impulse: the horizontal reaction force results in a horizontal velocity of the ladder!

9.2 Momentum: System of Particles

Consider a system of n particles, each described by position vector $\boldsymbol{r}_i$, and linear momentum $\boldsymbol{p}_i$. Let $\boldsymbol{f}_{ij}$ be the internal force exerted by particle j on particle i, and $\boldsymbol{F}_i$ the external force applied to particle i.



The total linear momentum of the system is the vector sum of the momentum of the individual particles:

$$\boldsymbol{p} = \sum^{n} \boldsymbol{p}_i = \sum^{n} m_i \boldsymbol{v}_i$$

The time derivative of the linear momentum $\boldsymbol{p}_i$ equals the net force on the particle (Newton's 2$^{\text{nd}}$ Law):

$$\frac{d\boldsymbol{p}_i}{dt} = \boldsymbol{F}_i + \sum^{n-1} \boldsymbol{f}_{ij}$$

For the total system of particles all *internal* forces will cancel out (Newton's 3$^{\text{rd}}$ Law):

$$\frac{d\boldsymbol{p}}{dt} = \sum^{n} \left(\boldsymbol{F}_i + \sum^{n-1} \boldsymbol{f}_{ij} \right) = \sum^{n} \boldsymbol{F}_i$$

Thus, if all *external* forces are zero, the total momentum of the system of particles will be conserved, regardless of the *internal* interactions between the particles.

The total linear momentum of the system of particles can be rewritten as

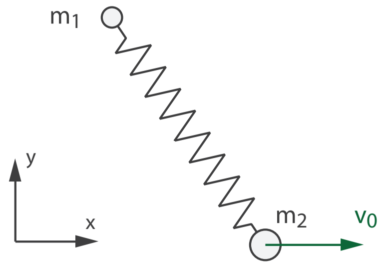
$$\boldsymbol{p} = \sum \boldsymbol{p}_i = \sum m_i \boldsymbol{v}_i = \frac{d}{dt} \left(\sum m_i \boldsymbol{r}_i \right) = \frac{d}{dt} (m \boldsymbol{r}_G) = m \boldsymbol{v}_G$$

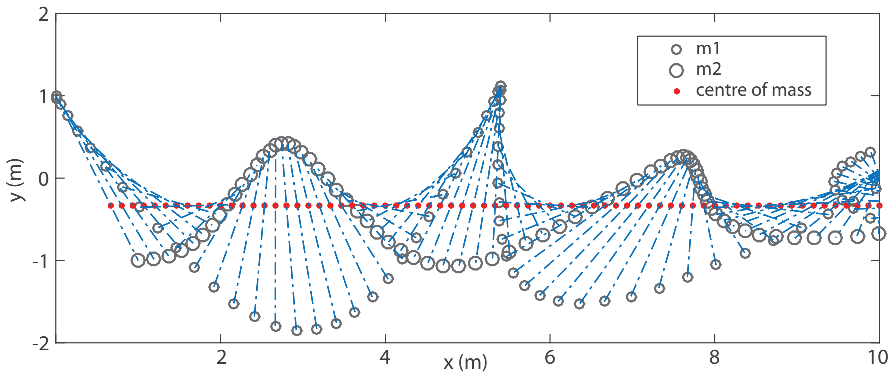
where m is the total mass, and $\boldsymbol{v}_G$ is the velocity of the centre of mass of the system. In general, in absence of *external* forces the velocity of the centre of mass of an enclosed system of particles will be constant, regardless of the interactions between the particles.

This remarkable result has elegant consequences. For the example of the cannon and the projectile, the centre of mass will remain stationary as the combined system started from rest. In a fireworks display, after the rocket explodes into its colourful pattern, the centre of mass will continue to describe a parabola, as if it were a single particle. At a larger scale, the combined centre of mass of the solar system will move irrespective of the internal gravitational forces between the sun, planets and other objects.

Example 9.3 – Two Masses and a Spring

Two masses, $m_1 = 1$ kg and $m_2 = 2$ kg, are connected by a spring with stiffness of $k = 5$ N/m with a rest length of $L_0 = 1$ m, and m_2 is given an initial horizontal velocity $v_0 = 2$ m/s. As illustrated by the simulation results, the centre of mass travels uniformly while the two masses move seemingly erratically as a result of the varying spring force.





9.2.1 Linear Momentum of Rigid Body

Clearly, the previous discussion applies to a rigid body, which can be regarded as a system of particles where the internal stresses maintain the geometry. Therefore, the linear momentum of a rigid body is:

$$\boldsymbol{p} = m\boldsymbol{v}_G \tag{9.4}$$

and

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} = m\boldsymbol{a}_G \tag{9.5}$$

where the net force $\boldsymbol{F}$ on the body determines the acceleration of its centre of mass.

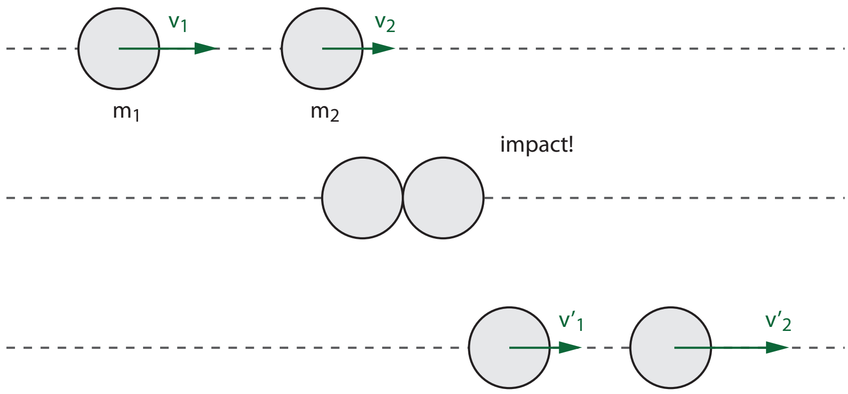
This analysis supports the assertion made in Handout 5 that the general motion of a rigid body can be described by the acceleration of its centre of mass, and the angular acceleration of the rigid body around the centre of mass. This will be elaborated on further in Handout 10, when studying angular momentum.

9.3 Impact of Particles

A collision between two particles, or *impact*, results in a relatively large contact force acting over a very short period of time. Such an impact involves material deformation (elastic and plastic), as well as dissipation of energy through the generation of heat and sound. The precise force-time relationship is complex, but is not required to determine the total impulse involved during the impact.

9.3.1 Direct Central Impact

Consider two particles with mass m_1 and m_2, and collinear velocities v_1 and v_2, with $v_1 > v_2$. The particles impact along their centre lines. During impact it is assumed that only the large contact forces are significant, and that there is a neglible change in positions of the centres of mass during the short duration of the impact.



As the contact forces are equal and opposite during impact, the combined linear momentum of the two colliding particles must remain unchanged. Therefore:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{9.6}$$

where v_1' and v_2' are the velocities of the particles after collision. Only a single equation describes the two velocities after impact, and a unique solution can therefore not be found.

The ratio between the velocities before and after impact follows from an energy argument, and is defined by the **coefficient of restitution**:

$$e = \frac{v_2' - v_1'}{v_1 - v_2} \tag{9.7}$$

Using these two equations, combining momentum and energy conditions, the velocities of the particles after impact can be found. We first consider two special cases, that of purely elastic and purely inelastic impact, before describing general impact.

purely elastic impact In a purely elastic collision, $e = 1$, all energy is conserved. This results in the following velocities after impact:

$$v_1' = \frac{(m_1 - m_2)\,v_1 + 2m_2 v_2}{m_1 + m_2}$$
$$v_2' = \frac{(m_2 - m_1)\,v_2 + 2m_1 v_1}{m_1 + m_2}$$

For the special case of equal masses, $m_1 = m_2$, this becomes:

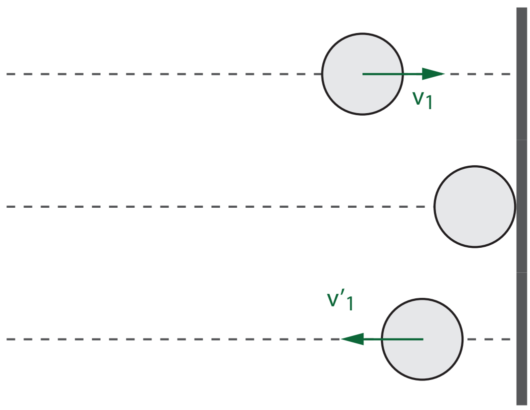
$$v_1' = v_2 \qquad\qquad v_2' = v_1$$

and therefore the velocities of the particles are exchanged.

This is elegantly demonstrated with the Newton's Cradle toy where a stationary ball is struck by a moving ball of equal mass, which then assumes the same velocity. That then passes on its linear momentum to the next, and so on and so forth.

Example 9.4 – Impact with Flat Wall

Consider the limit case of an elastic collision between a particle and a large flat body.



In that case, $v_2 = 0$, while m_2 can be considered infinite. This gives:

$$v_1' = \lim_{m_2\to\infty} \frac{(m_1 - m_2)\,v_1}{m_1 + m_2} = \lim_{m_2\to\infty} \frac{(m_1/m_2 - 1)\,v_1}{(m_1/m_2 + 1)} = -v_1$$
$$v_2' = \lim_{m_2\to\infty} \frac{2m_1 v_1}{m_1 + m_2} = 0$$

The result offers no surprises, with the ball bouncing back at the same velocity as it struck the wall.

Note that the linear momentum of the particle changed as a result of reversing its velocity

$$\Delta \boldsymbol{p}_1 = -2m_1 v_1$$

and the required impulse was provided by the stationary wall. The precise contact force and contact time of the collision become irrelevant to the calculations, by considering the total impulse.

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purely inelastic impact In a purely inelastic collision, $e = 0$, the particles merge and thus have the same velocity v' after impact.

$$v' = v_1' = v_2' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

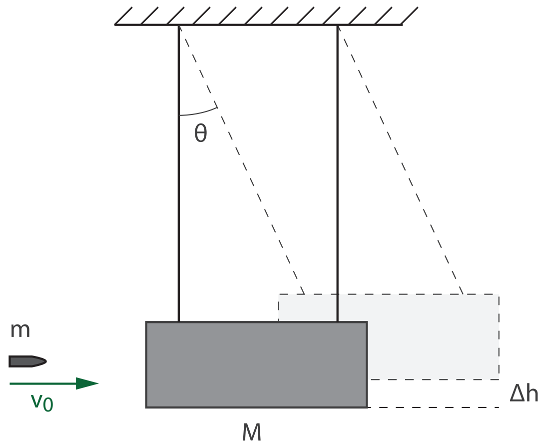
Although momentum is conserved, the kinetic energy after impact is reduced:

$$\begin{aligned}\Delta T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2} (m_1 + m_2) \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right)^2 \\ &= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} (v_1 - v_2)^2\end{aligned}$$

In fact, all kinetic energy can disappear during a collision! (When is this the case?)

Example 9.5 – Ballistic Pendulum

A ballistic pendulum is an effective method to measure the velocity of a projectile, without the need for high-speed cameras. It consists of a bifilar pendulum of length L that supports a sand box of mass M. The projectile is fired into the sand box, and its velocity can be determined from the maximum deflection angle θ.



From conservation of linear momentum:

$$m v_0 = (M + m) v_1$$

the velocity v_1 of the sand box and projectile directly after impact is found.

Determining the maximum deflection is best done through conservation of energy, as that expresses a relationship between velocity and displacement. At maximum θ the velocity, and therefore kinetic energy, will be zero. This gives the work-energy balance:

$$\Delta T + \Delta V = 0$$

$$0 - \frac{1}{2} (M + m) v_1^2 + (M + m) g \Delta h = 0$$

where

$$\Delta h = L (1 - \cos \theta)$$

and therefore:

$$\theta = \arccos \left(1 - \left(\frac{m}{M + m} \right)^2 \frac{v_0^2}{2Lg} \right)$$

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general impact In general, impact between particles will be partially elastic, as defined by the coefficient of restitution e, where $0 < e < 1$. From the conservation of momentum and the coefficient of restitution, the velocities after impact are determined as:

$$v_1' = \frac{m_1 v_1 + m_2 v_2 - e\, m_2\,(v_1 - v_2)}{m_1 + m_2}$$
$$v_2' = \frac{m_1 v_1 + m_2 v_2 - e\, m_1\,(v_2 - v_1)}{m_1 + m_2}$$

The loss of kinetic energy over the impact can be calculated, and (after some algebra) reduces to:

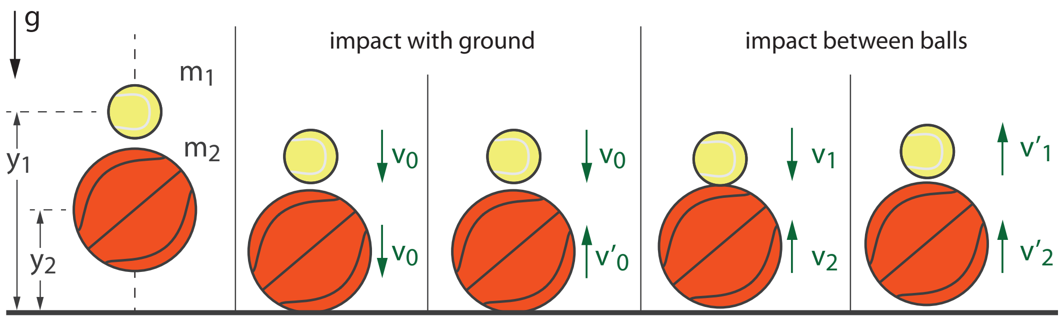
$$\Delta T = \frac{1 - e^2}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2$$

Example 9.6 – Two-Ball Bounce

In the classic two-ball bounce experiment a tennis ball is placed on top of a basket ball, and both are dropped simultaneously.

Q: How high will the balls bounce after impact with the ground?

It is assumed that the mass of the basket ball is much greater than that of the tennis ball ($m_2 >> m_1$), and a small (assumed) separation between the balls ensures that the impact can be modelled as two independent collisions: first, the basket ball hits the ground, bounces back, and then strikes the tennis ball. The coefficient of restitution between the balls is e_{12} and between the basket ball and the ground e_2.



From potential energy, we find velocity v_0 of both balls at time of impact with the ground:

$$v_0 = -\sqrt{2g\,(y_2 - r_2)}$$

After impact the upward velocity of the basket ball becomes:

$$v_0' = -e_2 v_0$$

Next, the two balls travelling at v_1 and v_2 collide, giving the following velocities after impact:

$$v_1' = \frac{\mu v_1 + v_2 - e_{12}\,(v_1 - v_2)}{\mu + 1}$$

$$v_2' = \frac{\mu v_1 + v_2 - e_{12}\mu\,(v_2 - v_1)}{\mu + 1}$$

where $\mu = m_1/m_2$. For the case of zero initial separation, substitute $v_2 \approx v_0' = -e_2 v_0$ and $v_1 \approx v_0$ to find:

$$v_1' = \frac{\mu - e_2 - e_{12}\,(1 + e_2)}{\mu + 1} v_0$$

$$v_2' = \frac{\mu - e_2 + \mu e_{12}\,(1 + e_2)}{\mu + 1} v_0$$

Taking the limit of the mass ratio $\mu = m_1/m_2 \approx 0$

$$v_1' \approx -\left[e_2 + (1 + e_2)\,e_{12}\right] v_0$$

$$v_2' \approx -e_2 v_0$$

and assuming purely elastic collisions, $e_2 = e_{12} = 1$, this reduces to

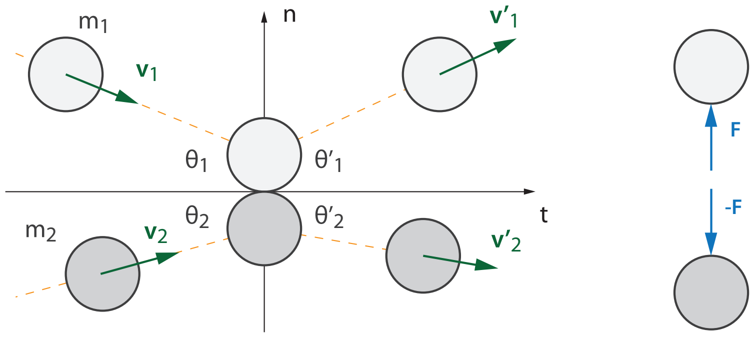
$$v_1' \approx -3v_0$$

$$v_2' \approx -v_0$$

which means that the resulting velocity of the tennis ball can be up to 3 times its initial velocity. This means its kinetic energy will be 9 times greater, and thus launch 9 times higher than it started!

9.3.2 Oblique Central Impact

The methods described for direct central impact can be extended to a case where the intial and final velocities are not collinear. Consider two colliding particles of mass m_1 and m_2 and initial velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.



The directions of the velocity vectors are measured in an nt coordinate system, which is defined by axes normal and tangent to the contact point. Note that the equal and opposite contact forces during impact only act in the n direction. Four equations are needed to solve for the velocity vectors after impact:

- conservation of momentum in n-direction:

$$m_1 v_{1n} + m_2 v_{2n} = m_1 v'_{1n} + m_2 v'_{2n}$$

- conservation of momentum in t-direction for both particles, as no impulse is applied:

$$m_1 v_{1t} = m_1 v'_{1t}$$
$$m_2 v_{2t} = m_2 v'_{2t}$$

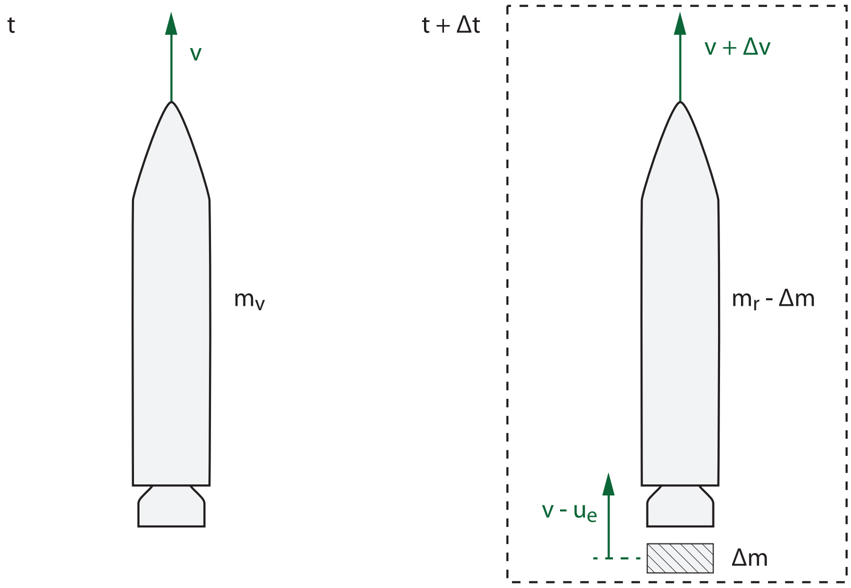
- coefficient of restitution in n-direction:

$$e = \frac{v'_{2n} - v'_{1n}}{v_{1n} - v_{2n}}$$

these equations are solved simultaneously to find the components of the velocity vectors $\boldsymbol{v}'_1$ and $\boldsymbol{v}'_2$.

9.4 Variable Mass System: The Rocket Equation

In a rocket, the fuel will account for a large fraction of its launch mass and therefore its change in mass must be taken into account when deriving its velocity. Consider a rocket at time t with mass m_r and velocity v. It burns fuel and propels exhaust gases at a relative exhaust velocity u_e and at a mass rate of $\mu = dm/dt$.



At time is $t + \Delta t$ the rocket will have expelled mass Δm and attained a velocity $v + \Delta v$. The *relative* exhaust velocity is u_e, and therefore the absolute velocity of the expelled mass is $v - u_e$. The linear momentum of the total system (here denoted by a control volume) must be conserved:

$$\begin{aligned}\Delta \boldsymbol{p} &= \boldsymbol{p}_2 - \boldsymbol{p}_1 \\ &= (m_r - \Delta m)(v + \Delta v) + (v - u_e)\Delta m - m_r v \\ &= m_r \Delta v - u_e \Delta m = 0\end{aligned}$$

where the higher-order term $\Delta m \Delta v$ was neglected. Taking the time derivative of the linear momentum:

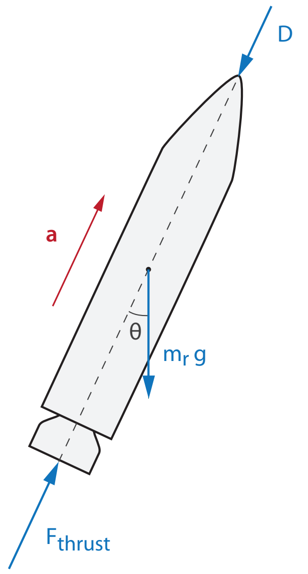
$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{p}}{\Delta t} &= m_r \frac{dv}{dt} - u_e \frac{dm}{dt} = 0 \\ &= m_r a - u_e \mu = 0\end{aligned}$$

and leads to the concept of the thrust that accelerates the rocket:

$$F_{\text{thrust}} = m_r a = u_e \mu$$

Note that the rate of change in mass of the rocket is the negative of mass flow rate, and therefore:

$$F_{\text{thrust}} = u_e \mu = u_e \frac{dm}{dt} = -u_e \frac{dm_r}{dt}$$



For a rocket launched from Earth, consider the forces acting on the rocket:

$$\sum F_t : \qquad F_{\text{thrust}} - D - m_r g \cos\theta = m_r a$$

where F_{thrust} is the rocket engine thrust, and D the drag force.

In the ideal rocket equation, neglect D, set $\theta = 0$, and assume a constant exhaust velocity u_e. This gives:

$$m_r a = u_e \mu - m_r g$$

$$m_r \frac{dv}{dt} = -u_e \frac{dm_r}{dt} - m_r g$$

$$dv = -u_e \frac{dm_r}{m_r} - g dt$$

which is integrated (taking $v = 0$ at $t = 0$) to find

$$v = -u_e \ln\left(\frac{m_r}{m_0}\right) - gt$$

where the m_0 is the inital mass, or wet mass, of the rocket. The maximum velocity that can be attained for such a single stage rocket is therefore determined by ratio of dry mass over wet mass (m_r/m_o), but also by the duration of the launch and thus the impulse due to gravity load.

Rewriting in terms of mass ratio:

$$\frac{m_0}{m_r} = e^{(v+gt)/u_e}$$

shows that the fuel fraction increases exponentially with desired velocity (and thus orbit altitude). In order to minimize the fuel mass required to achieve the desired velocity v, the ratio of $(v_e + gt)/u_e$ must be minimised. The time is given by the (constant) fuel burn rate as:

$$t = \frac{m_0 - m_r}{\mu}$$

In effect, the most fuel efficient way to launch a rocket is to release as much fuel (μ) with as high a velocity (u_e) as possible. Having multi-stage rockets will also aid significantly, and will be discussed further in your second year Space Systems module.

9.4.1 General Variable Mass System

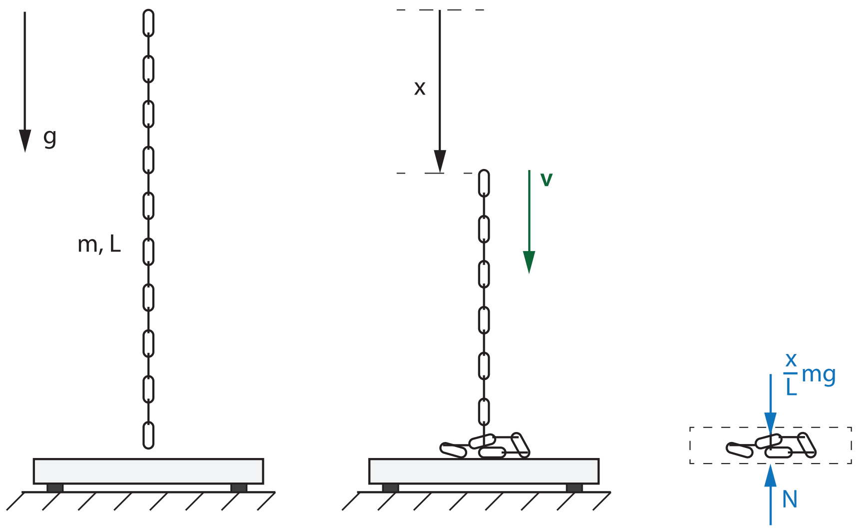
The rocket equation is an example of a general variable mass system, which could gain or lose mass. The net force on a variable mass system is derived as:

$$\sum \boldsymbol{F} = m\dot{v} + \dot{m}u$$

where $\dot{v}$ is acceleration of the system, $\dot{m}$ is the rate of change in mass, and $u = v - v_0$ the relative velocity of the expelled or accumulated mass.

Example 9.7 – Chain Falling on Scale

A chain of length L and mass m is released above a scale.



Q: What is the force measured by the scale as the chain falls?

A: The velocity v of the falling chain as it hits the scale is found from conservation of energy:

$$v = \sqrt{2gx}$$

where x is the distance the chain has fallen.

Consider the section of the chain on the scale (with length x) as a variable mass system, which has zero acceleration, but increases in mass ($\dot{m} = \rho v$, with $\rho = m/L$) with relative velocity v.

$$\sum \boldsymbol{F} = m\dot{v} + \dot{m}u$$

$$N - \frac{x}{L}mg = 0 + \sqrt{2gx}\frac{m}{L}\sqrt{2gx} \qquad \rightarrow \qquad N = 3\frac{x}{L}mg$$

In other words, the scale reads 3 times the weight of the section of chain currently lying on the scale!

Revision Objectives Handout 9:

Linear Impulse and Momentum of Particles

- calculate linear momentum of a particle ($\boldsymbol{p} = m\boldsymbol{v}$)
- use equations of linear momentum in solving problems in dynamics of particles
  - change in linear momentum equals the net force on the particle ($\boldsymbol{F} = d\boldsymbol{p}/dt$)
  - a linear impulse results in a change in linear momentum ($\int \boldsymbol{F} dt = \Delta \boldsymbol{p}$)
- appreciate the implications of conservation of linear momentum on systems of particles
- perform calculations on collinear and oblique impact of particles
  - combine conservation of momentum and coefficient of restitution
  - recall coefficient of restitution: $e = (v_2' - v_1') \,/\, (v_1 - v_2)$
  - differentiate between elastic ($e = 1$), inelastic ($e = 0$) and general ($0 < e < 1$) collisions
  - calculate the change in kinetic energy before and after collision

Note: the rocket equation and variable mass systems are **not** examinable.