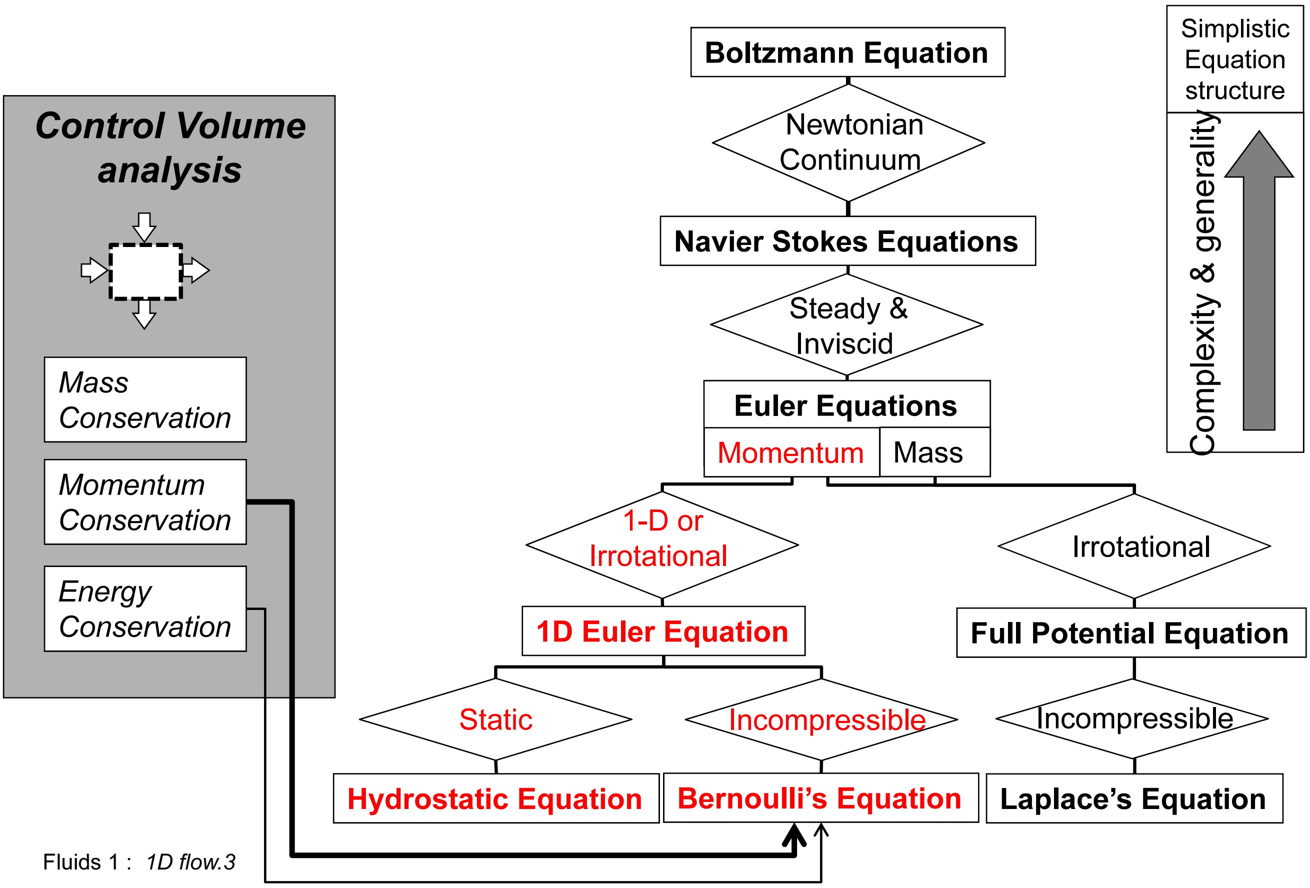
Steady Inviscid Incompressible 1-D Flow

Flow models

Real flows are usually complex and the complete flow equations are generally too complex for analytical solutions. Even the most powerful computers are incapable of computing all the scales involved.

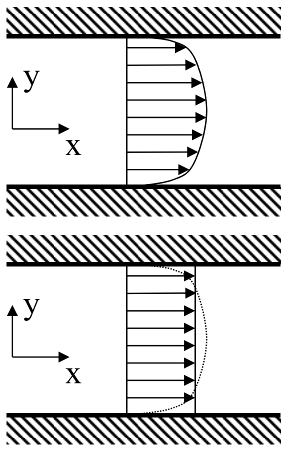
We must make reasonable assumptions to allow analytical solutions

Fluid Equations: Reduction or Construction?



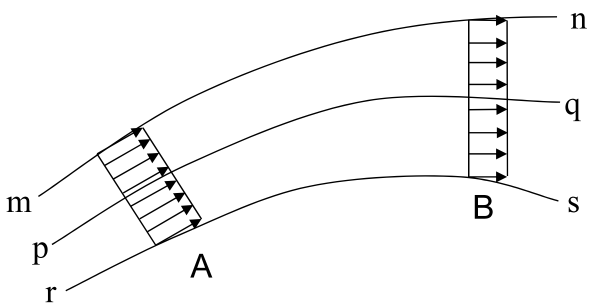
Quasi-one-dimensional Approximation

- Consider flow between two parallel flat plates
- Real flow has zero velocity at the wall
- Velocity variation in the cross-stream direction (y)



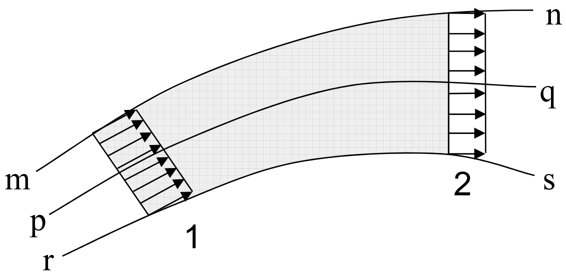
- 1D-approx takes average velocity across the profile

- General curved stream-tube
- Flow averaged, A and B not necessarily equal
- 1-D approximation good when
  - Area variation is gradual
  - Curvature is small



Conservation of Mass

- Consider 1-D approximation applied to volume defined by "1" and "2"



- Mass flow rate through section 1: $\rho_1 V_1 A_1$
- Mass flow rate through section 2: $\rho_2 V_2 A_2$
- Principle of mass conservation for quasi 1-D steady flow:

Rate of mass flow in = Rate of mass flow out

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

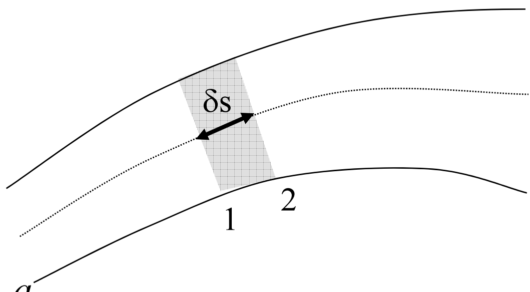
- For incompressible flow

$$A_1 V_1 = A_2 V_2$$

- Valid for viscous as well as inviscid flows

Acceleration of a Fluid Particle

- Velocity in quasi-1D flow is a function of position and time
- As fluid moves from 1 to 2 in time δt, velocity changes because
  - The velocity changes with position
  - Velocity at both 1 and 2 is unsteady



- Total change in velocity for particle

$$\delta V = \frac{\partial V}{\partial s}\delta s + \frac{\partial V}{\partial t}\delta t$$

- As δt tends to 0, stream-wise acceleration a_s

$$\underbrace{a_s = \frac{dV}{dt}}_{Lagrangian} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t} = \underbrace{\frac{\partial V}{\partial s}V + \frac{\partial V}{\partial t}}_{Eulerian}$$

- Lagrangian description follows the particle
- For Steady flow streamline=pathline and $a_s = \frac{\partial V}{\partial s}V$
- For steady flow around a streamline curve (of radius r_s) there is also an acceleration normal to flow a_n

$$a_n = \frac{V^2}{r_s}$$

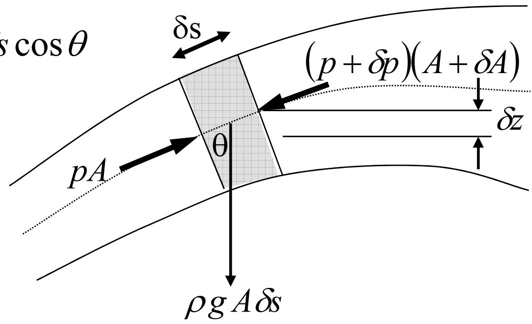
Conservation of Linear Momentum

- Consider forces in flow direction
- Forces in the flow direction due to pressure on side wall

$$(p + k\delta p)\,\delta A$$ where k is a coefficient related dA/ds

- Total force on fluid element in flow direction

$$pA - (p+\delta p)(A+\delta A) + (p+k\delta p)\,\delta A - \rho g A\,\delta s\cos\theta$$



- Neglecting products of small terms

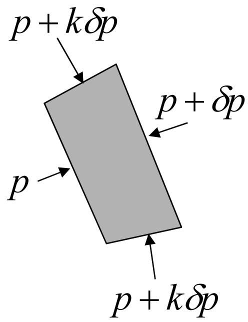
$$-A\delta p - \rho g A\,\delta s\cos\theta$$

- Force=mass x acceleration

$$-A\delta p - \rho g A\,\delta s\cos\theta = (\rho A\delta s)\left(V\frac{\partial V}{\partial s}\right)$$

- Substitute $\delta z = \delta s\cos\theta$ and dividing by $(A\delta s)$

$$\frac{dp}{ds} + \rho V\frac{dV}{ds} + \rho g\frac{dz}{ds} = 0$$



- Or in differential form (Euler's equation)

$$dp + \rho V dV + \rho g dz = 0$$

Bernoulli's equation

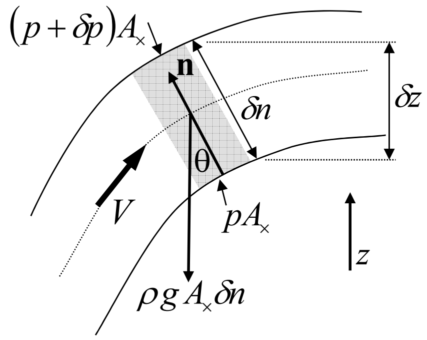
- Euler's equation is valid for steady quasi-1D inviscid flow
- Euler's equation is valid for compressible and incompressible flow
- Euler's equation can only be integrated if the variation of density with pressure is known.
- Assuming the density is constant Euler's equation can be integrated to

$$p + \tfrac{1}{2}\rho V^2 + \rho g z = \text{constant}$$

- Bernoulli's equation. Valid for steady, inviscid, constant-density flow along a streamline

Conservation of Momentum: Cross Stream

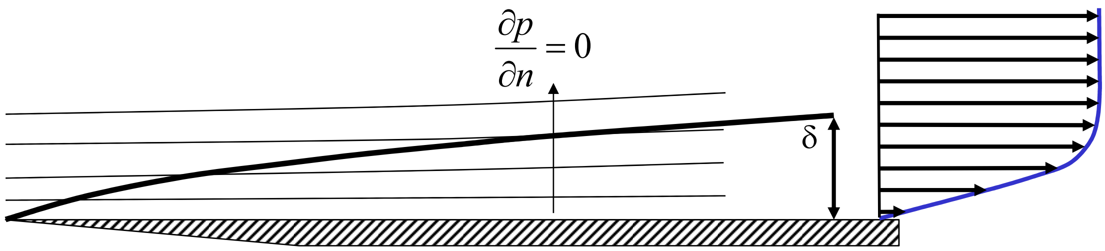
- Consider forces in cross-stream direction due to acceleration normal to streamline with radius of curvature r_s: $a_n = V^2/r_s$
- Consider shaded fluid element where r_s and therefore $A_\times$ are constant
- Net force towards centre of curvature due pressure is
  $(p+\delta p)A_\times - pA_\times = \delta p A_\times$
- Total force on fluid element towards centre of curvature
  $\delta p A_\times + \rho g A_\times \delta n \cos\theta = \delta p A_\times + \rho g A_\times \delta z$
- Force=mass x acceleration
  $$\delta p A_\times + \rho g A_\times \delta z = (\rho A_\times \delta n) a_n = (\rho A_\times \delta n)\frac{V^2}{r_s}$$
- dividing by $A_\times \delta n$ and taking limit $\delta n \to 0$
  $$\frac{\partial p}{\partial n} + \rho g \frac{\partial z}{\partial n} = \rho \frac{V^2}{r_s}$$



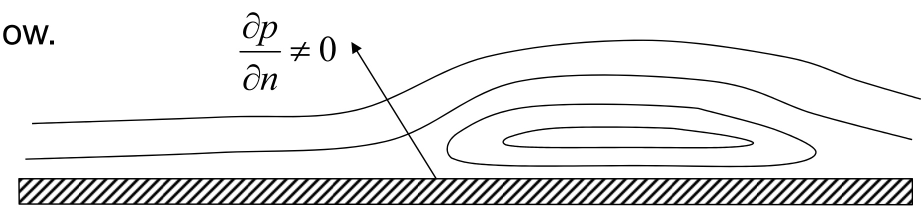
- Only assumptions made was steady flow for the equation of acceleration
  Hence Equation valid for: compressible/incompressible &
  viscous/inviscid steady flows

Cross-stream pressure variation- boundary layer

- Consider flows over approximately straight surfaces with negligible hydrostatic pressure variation
  $r_s \to \infty$ & $\rho g \frac{\partial z}{\partial n} \to 0$ hence $\frac{\partial p}{\partial n} + \underbrace{\rho g \frac{\partial z}{\partial n}}_{\to 0} = \underbrace{\rho \frac{V^2}{r_s}}_{\to 0} \quad \to \quad \frac{\partial p}{\partial n} = 0$



- Not valid for boundary layer separation, streamlines with large curvature
- Pressure gradients normal to the flow.



Cross-stream pressure variation- parallel flow

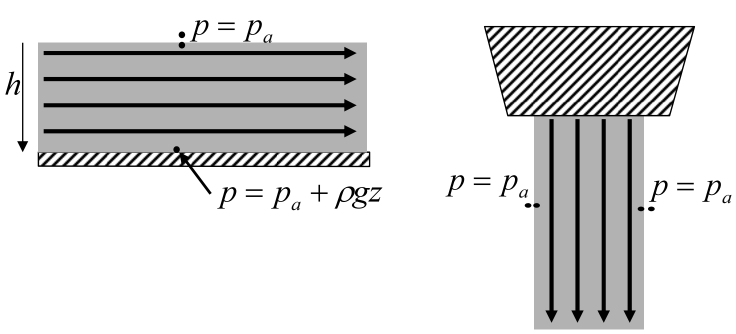
- In a lot of analysis we assume that the stream lines are parallel, hence as before

$$r_s \rightarrow \infty \quad \& \quad \rho\frac{V^2}{r_s} \rightarrow 0$$

But if we do not neglect the hydrostatic terms

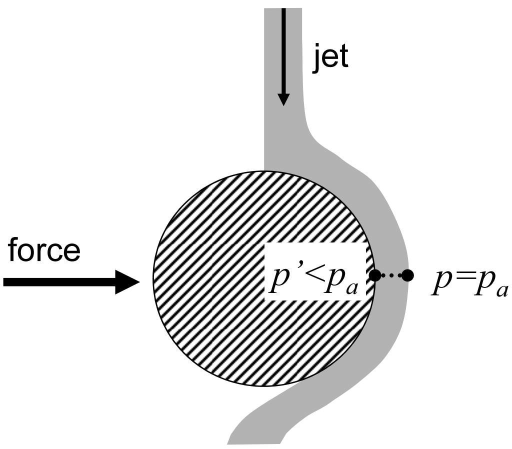
$$\frac{\partial p}{\partial n} + \rho g\frac{\partial z}{\partial n} = \underbrace{\rho\frac{V^2}{r_s}}_{\rightarrow 0} \qquad \rightarrow \qquad \frac{\partial p}{\partial z} = -\rho g$$

- So consider a number of cases shown and remember that the interface of two flows/fluids is at the same pressure



Cross-stream pressure variation- Coanda effect

- Previous theory helps to explain why jets of fluid tend to remain attached to convex bodies – the Coanda effect
- Pressure at body surface, p' , lower than atmospheric, p_a, at free surface.



- Atmospheric pressure on the unwetted surface
- Net force on the body from the imbalance in pressure