EMAT10100 Engineering Maths I Lecture 14: Geometry of Linear Systems

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Looking back looking forward

- Last lecture:
  - Linear systems of equations: $\mathbf{A}\mathbf{x} = \mathbf{b}$
  - Solution by Gaussian row operations and back substitution
- This lecture: why it works
  - geometric interpretation
  - what does it mean if $\mathbf{A}$ is singular?
  - can you still solve the equations?

Geometric interpretation in 3D

- Equation for a plane:

$$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{a}, \qquad \text{or} \qquad n_1x + n_2y + n_3z = c$$

- So three equations in three unknowns, e.g.

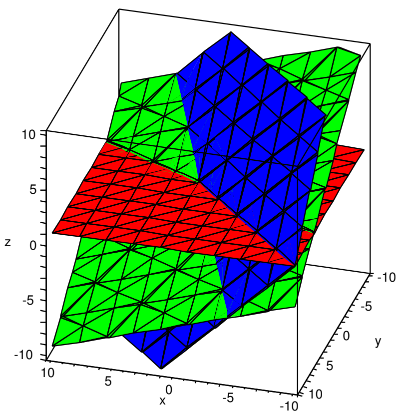
$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 2 \\ z &= 1 \end{aligned}$$

  is really the intersection between 3 planes.
- which we write in matrix form as $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\text{where} \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

- Exercise find the solution $(x, y, z)^{\mathrm{T}}$ to this example.

Geometric interpretation



- We can draw the three planes easily
- The "solution" to the system of linear equations is the UNIQUE point of intersection between the three planes.

What to do when $\det \mathbf{A} = 0$?

- If $\det \mathbf{A} \neq 0$:
  - $\mathbf{A}^{-1}$ exists
  - solution of $\mathbf{Ax} = \mathbf{b}$ is unique:
    given as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
    (even if we don't find it like that)
- What if $\det \mathbf{A} = 0$?
  (find row of zeros after doing row operations)
  - $\mathbf{A}^{-1}$ does not exist
  - solution of $\mathbf{Ax} = \mathbf{b}$ is NOT unique
  - and might not exist at all!

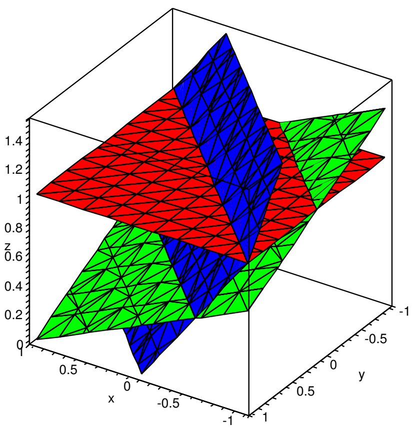
What does this mean? Case I.

- Consider a small modification to previous example

$$\mathbf{Ax} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \mathbf{b}$$

- Exercise
  - Calculate det **A**.
  - What happens when we try to solve this system using row elimination?
- We say in this case, the equations are INCONSISTENT
- . . . and they do not have a solution

Geometric interpretation



- In this case the there is no solution
- i.e. no point where all three planes intersect
- instead they intersect in pairs along three parallel lines

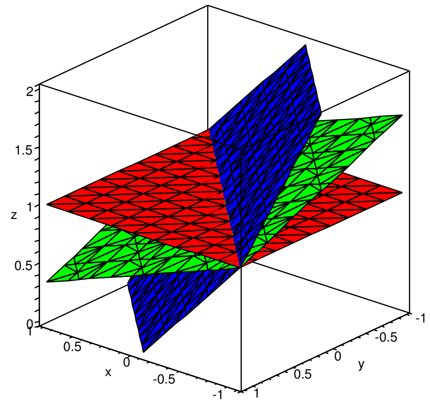
What does this mean? Case II.

- Consider another small modification to previous example

$$\mathbf{Ax} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \mathbf{b}$$

- Exercise
  - Calculate det **A**
  - What happens when we try to solve this system using row elimination?
- We say in this case, the equations are DEGENERATE or under-determined
- . . . and they have a whole family of solutions

Geometric interpretation



- In this case the planes all intersect along the same line
- so there are infinitely many solutions
- each point on the line is a possible solution to the system

What is the difference?

- How do we determine the difference between cases I and II?
- i.e. between
  - inconsistent equations; no solution
  - degenerate (under-determined) equations; infinitely many solutions
- The difference came after row elimination:
  - in case I, we got the answer $0 = 1, \Rightarrow$ INCONSISTENT
  - in case II, we got the answer $0 = 0, \Rightarrow$ DEGENERATE
- How can we formalise this?

Rank of a matrix

Formal definition for an $n \times n$ square matrix $\mathbf{A}$ (practical examples next lecture)

- We say a singular matrix does not have full rank.
- Rank is the number of independent pieces of information carried by the equations represented by rows of the matrix
- How to compute rank?
  - Do row eliminations on $n \times n$ matrix $\mathbf{A}$ to obtain upper triangular form
  - Compute the Nullity Null$(\mathbf{A})$ which is number of entirely zero rows left at end of elimination process
  - it is dimension of solution set of $\mathbf{Ax} = \mathbf{0}$
  - Rank of $\mathbf{A}$, Rank$(\mathbf{A}) := n -$ Null$(\mathbf{A})$
  - it is the dimension of the image of space under $\mathbf{A}$
- Rank can also be computed for non-square matrices
  - Key result: Rank$(\mathbf{A}) =$ Rank$(\mathbf{A}^{\mathrm{T}})$
- What?

So, I still don't get the difference!

- For an $n \times n$ system of equations $\mathbf{Ax} = \mathbf{b}$, there are three possibilities:
  1. If det **A** $\neq 0$, then A has full rank, Rank$(\mathbf{A}) = n$.
     - Hence there is a UNIQUE solution
  2. If det **A** $= 0$, Rank$(\mathbf{A}) < n$:
     - Case I, if Rank$(\mathbf{A}|\mathbf{b}) >$ Rank$(\mathbf{A})$
       then there is no solution, INCONSISTENT
     - Case II, Rank$(\mathbf{A}|\mathbf{b}) =$ Rank$(\mathbf{A})$
       then there are family of solutions, DEGENERATE
- We will return to this next time . . .
- No homework this lecture. Please catch-up on matrices so far.
- But don't forget the QMP Tests for instant feedback on progress