SYSTEMS Pt. 3
Poles and zeros, Bode plot

Transfer functions

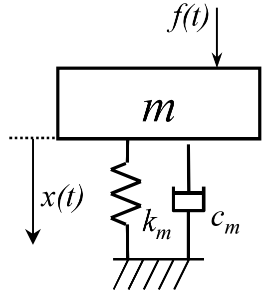
- In the last lecture we looked at how we could derive the transfer function of many common systems.

- We looked at how we could transform into the Laplace domain using differential operators;

$s = \frac{d}{dt}$ and for higher orders; $s^n = \frac{d^n}{dt^n}$

$\frac{1}{s} = \int x \, dt$ and for higher orders; $\frac{1}{s^n}$

Examples



Mass and spring with damping

$$f(t) = m\frac{d^2x}{dt^2} + c_m\frac{dx}{dt} + k_m x \quad \Rightarrow \quad F(s) = ms^2X(s) + \; c_m sX(s) + k_m X(s)$$

$$\frac{F(s)}{X(s)} = ms^2 + \; c_m s + k_m$$

Transfer function $= \dfrac{output}{input} = \dfrac{X(s)}{F(s)} = \dfrac{1}{ms^2 + c_m s + k_m}$

The Transfer function

- In the last lecture we looked at how we could derive the transfer function of many common systems.
- We can generalise this to;

$$\mathrm{TF} = \frac{a_n s^n + \ a_{n-1} s^{n-1} + \ldots\ldots + a_2 s^2 + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots . + b_2 s^2 + b_1 s + b_0}$$

- ‘m’ determines the order of the system
- ‘m’ is normally greater or equal to ‘n’ in a real system*

Poles and Zeros

$$\text{TF} = \frac{a_n s^n + \ a_{n-1} s^{n-1} + \ldots \ldots + a_2 s^2 + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots . + b_2 s^2 + b_1 s + b_0}$$

- The transfer function can be factorised

$$\text{TF} = \frac{(s - z_1)(s - z_2) \ldots \ldots (s - z_n)}{(s - p_1)(s - p_2) \ldots \ldots (s - p_m)}$$

- The roots of the numerator are called ‘zeros’
- The roots of the denominator are called ‘poles’

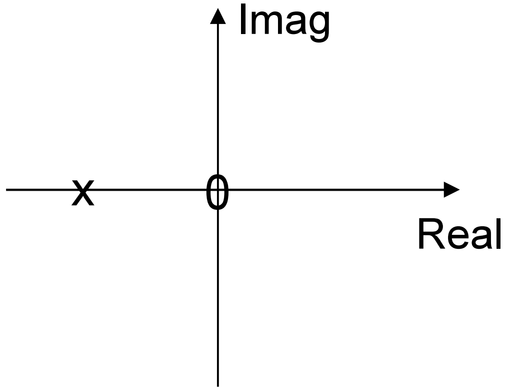
- The poles and zeros give us insight into the system behaviour.

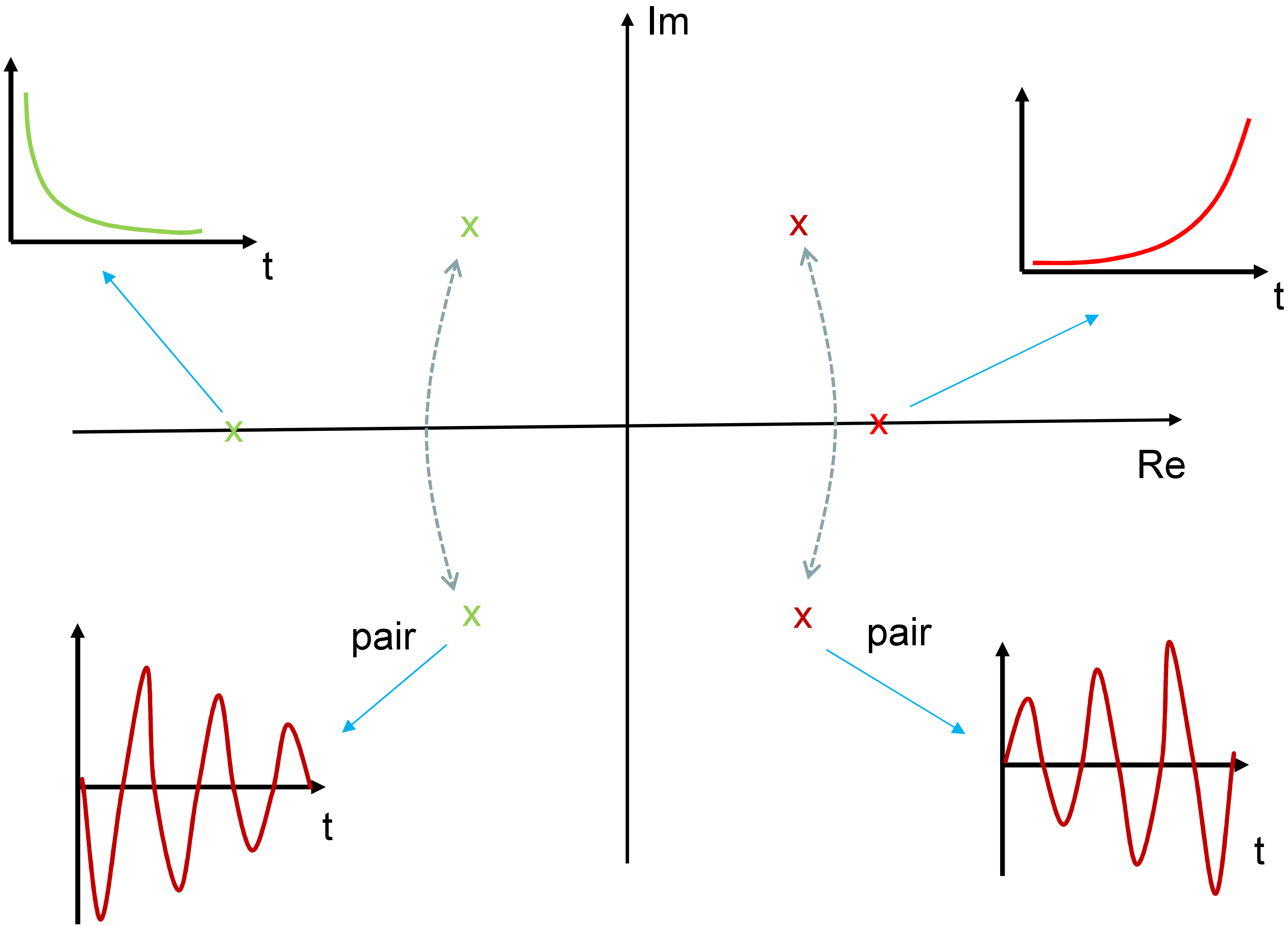
Poles and Zeros

$$\mathrm{TF} = \frac{(s - z_1)(s - z_2)\ \dots \dots\ (s - z_n)}{(s - p_1)(s - p_2)\ \dots \dots\ (s - p_m)}$$

- For systems higher than 2nd order, poles (and zeros) appear in complex conjugate pairs.
- They can be plotted on the complex plane – an important graphical tool in classical control theory.

- Poles are denoted by 'x'
- Zeros are denoted by '0'





The denominator of the TF is the *characteristic equation,* and reveals time domain behaviour

The bode plot

- An important characteristic of LTI systems is that they only change the magnitude and phase of an input signal, not the frequency content.
- Hence if the input to a system is;

$$x(t) = \sin(\omega t)$$

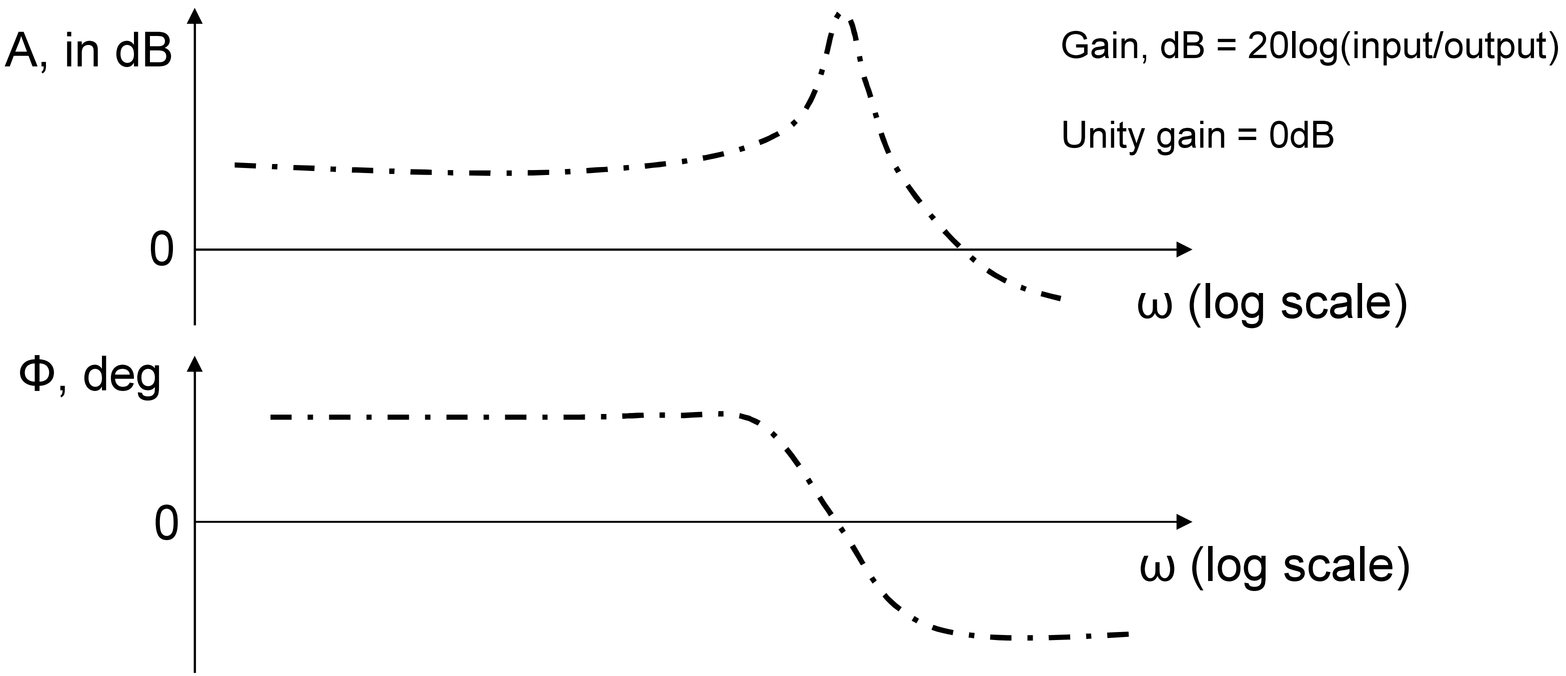
- Then the output will be of the form;

$$y(t) = A\sin(\omega t + \emptyset)$$

Gain response, A(ω). *Phase response*, ϕ(ω).

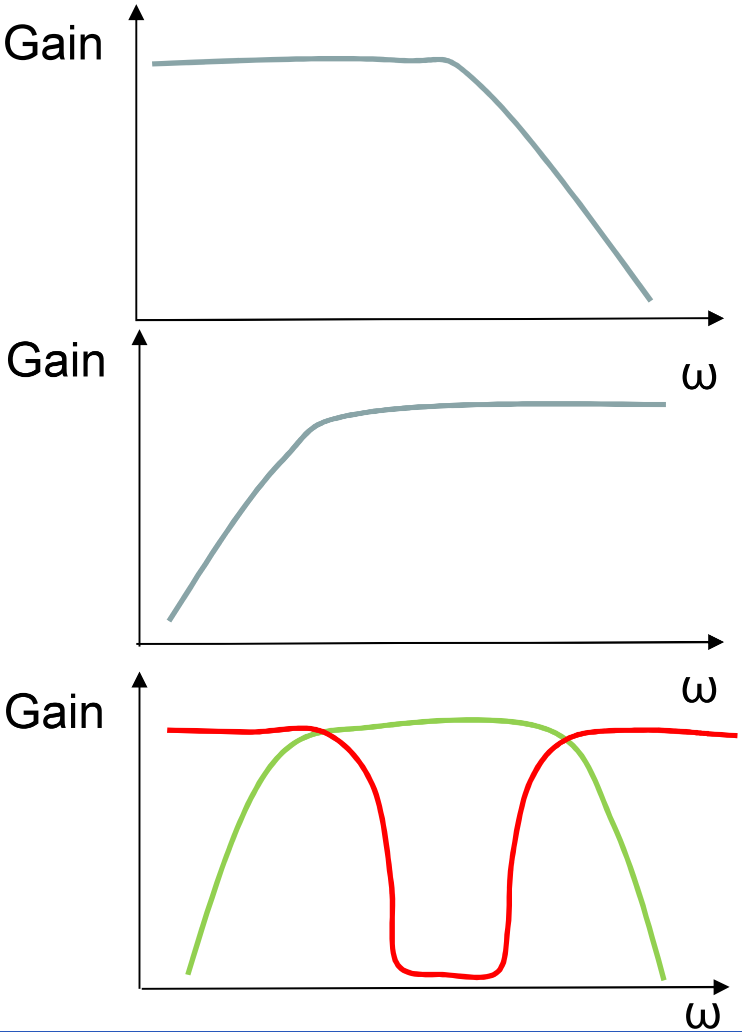
- The gain and phase response are both functions of frequency

The bode plot



- The gain and phase response of a system, plotted against frequency is called a ***bode plot***.

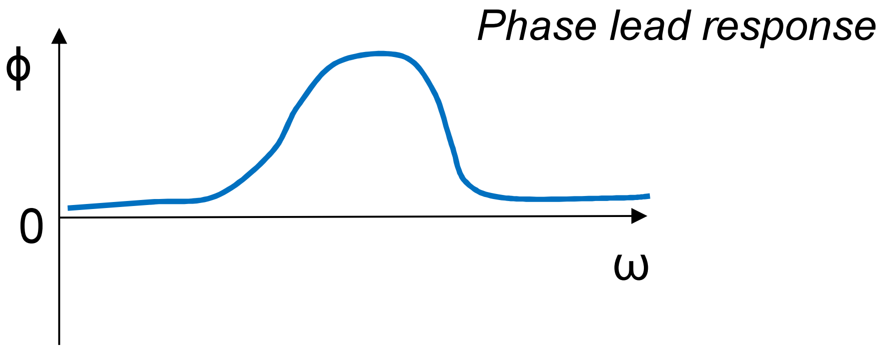
Filters - High, low and band pass

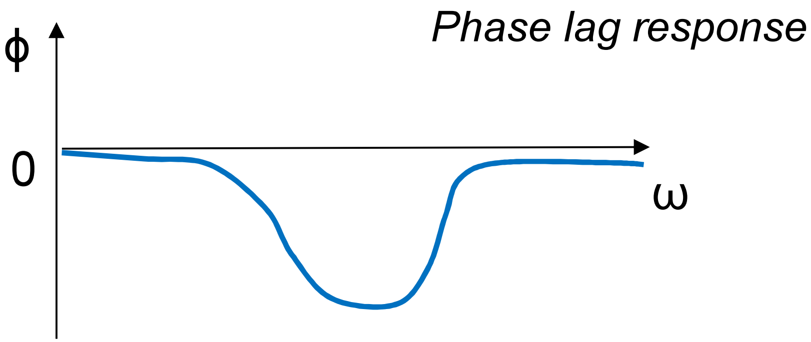


- Irrespective of the order, LTI systems are simply filters, i.e. they amplify or attenuate a signal based in frequency.

- There are several classic gain responses;
  - Low-pass
  - High-pass
  - Band-pass
  - Band-stop

Filters – Phase lead and Lag

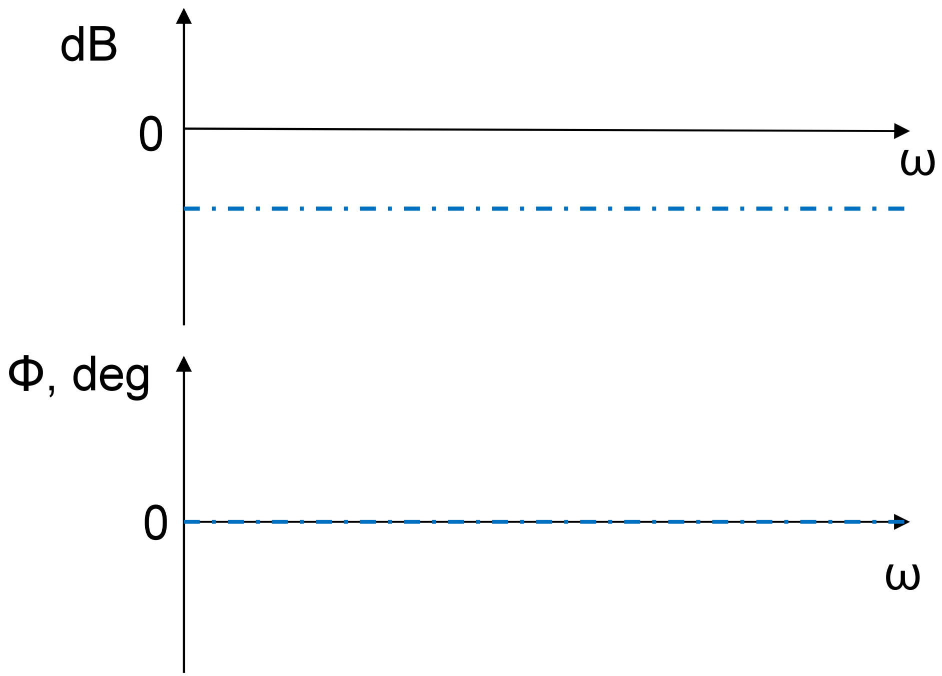




- In control the phase becomes very important and many control compensation networks are classified by their phase response….

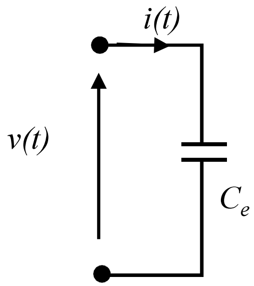
- But it is important to remember that phase and gain normally ***both*** vary with frequency (for nearly all systems)

Example systems – attenuation and gain

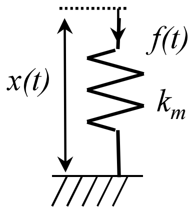


- The most basic systems have no energy storage elements, only attenuation (or amplification). There are no ‘dynamics’, no poles or zeros.
- Attenuation is produced by dashpots, resistors etc
- Amplification requires active devices e.g. electronic amplifiers, (note passive systems with resonance can amplify at a fixed frequency)

Integrators



$$\mathrm{v}(t) = \frac{1}{C_e}\int i(t)dt$$



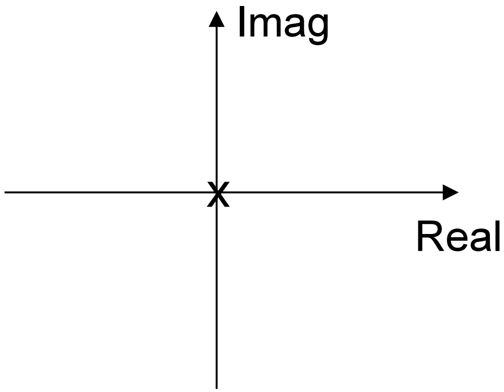
$$f(t) = k_e \int \dot{x}(t)dt$$

Capacitor integrates current

Spring integrates velocity

In the Laplace domain the general form is;

$$\frac{Y(s)}{X(s)} = \frac{const.}{s}$$



Which has a single pole at the origin

Note the input is normally 'X(s)', the output 'Y(s)' and the TF 'H(s)'

Integrators – bode plot

- The gain response is found by substituting s=jω, and evaluating the magnitude of the transfer function in dB;

$$H(s) = \frac{k}{s} \qquad\qquad H(j\omega) = \frac{k}{j\omega}$$

$$20\log\left|\frac{k}{j\omega}\right| = 20\log(k) - 20\log(j\omega)$$

$$20\log\left|\frac{k}{j\omega}\right| = \underbrace{20\log(k) - 20\log(j\omega)}_{\text{Sum is zero at } \omega=k}$$

Constant

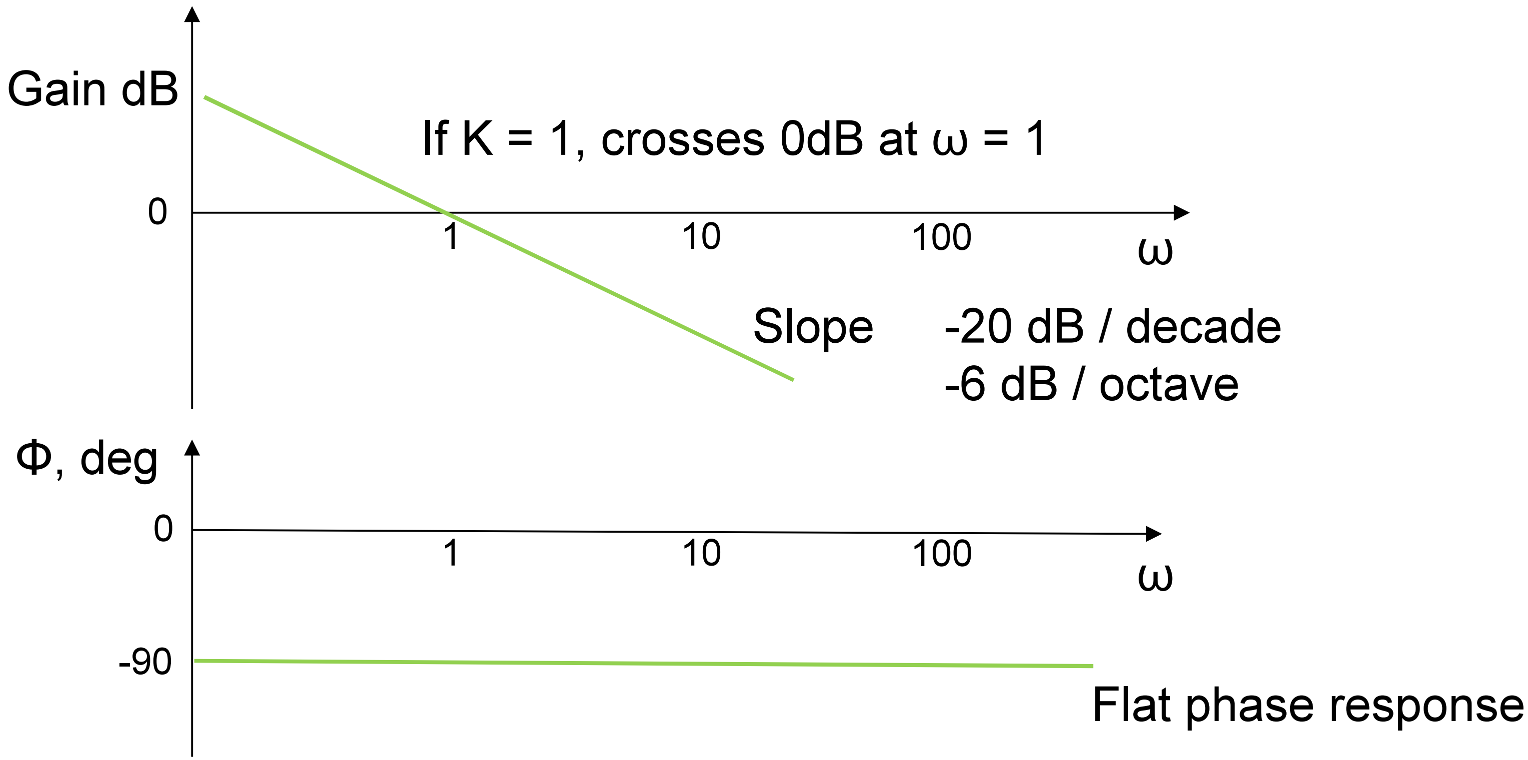
Slope of -20 db / decade

Sum is zero at
ω=k

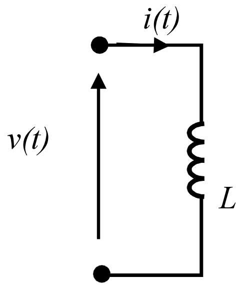
Integrators – bode plot

- The phase response is found from the argument of the transfer function;

$$\arg(H(j\omega)) = \tan^{-1}\left(\frac{imag}{real}\right) \qquad \arg\left(\frac{k}{j\omega}\right) = -90°$$

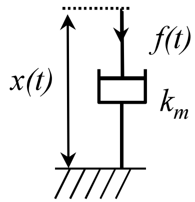


Differentiators



$$\mathrm{v}(t) = L\frac{di}{dt}$$

Inductor differentiates current



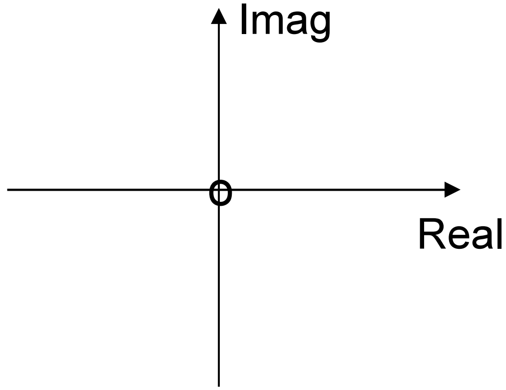
$$f(t) = C_m\frac{dx}{dt}$$

Dashpot differentiates position

In the Laplace domain the general form is;

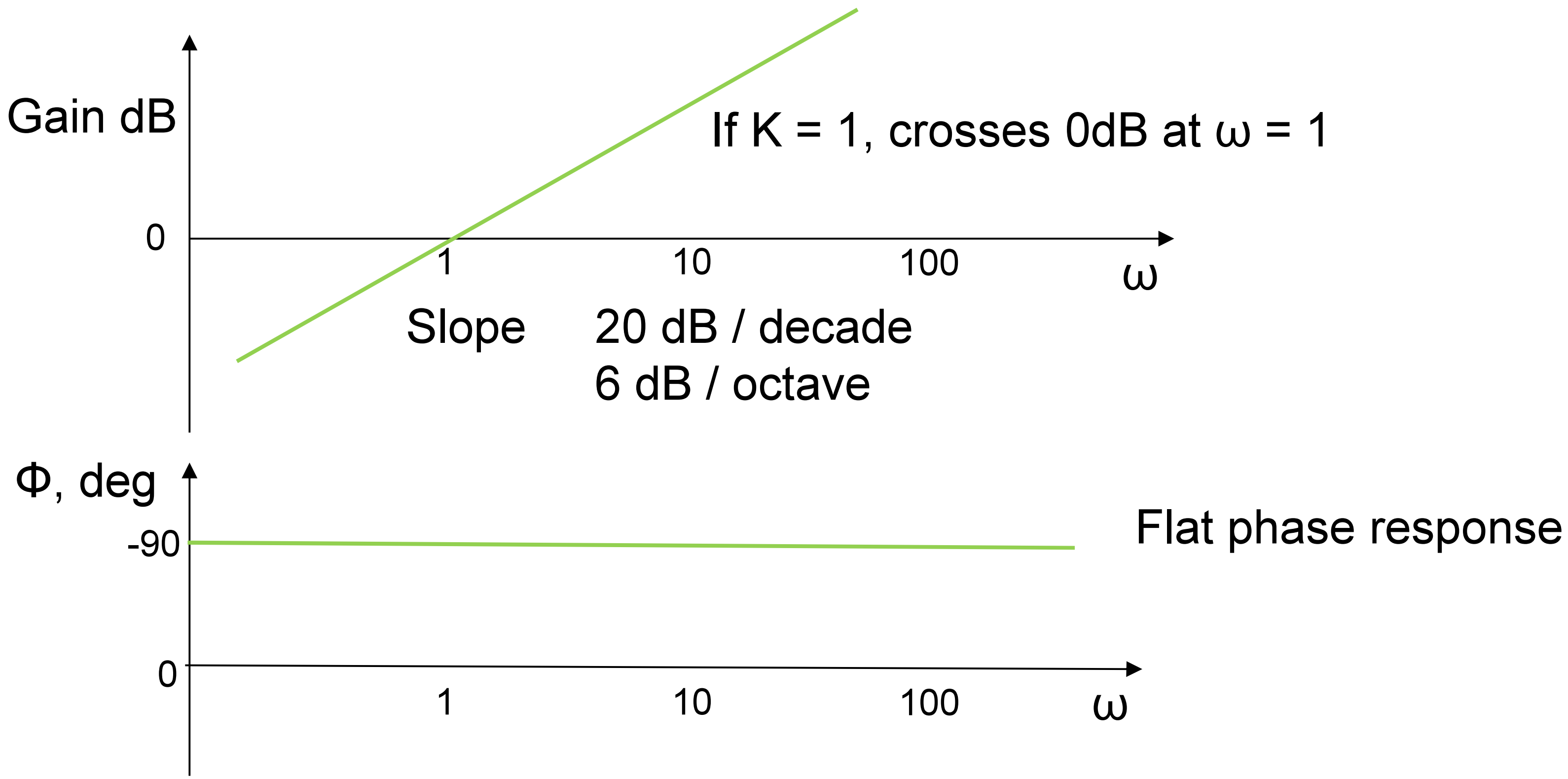
$$\frac{Y(s)}{X(s)} = const.s$$

Which has a single zero at the origin



Differentiators – bode plot

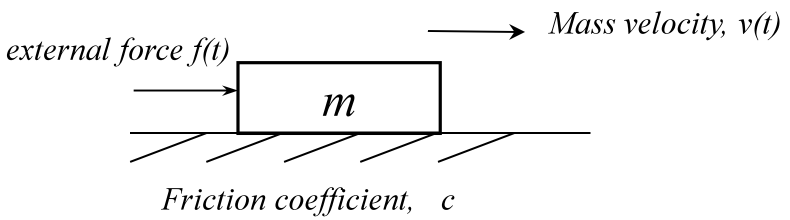
- As before we derive the bode plot by making the substitution s=jω, and evaluating the magnitude and argument of the transfer function.



Practical integration and differentiation

- Integrators behave a little like low pass filters, whilst differentiators are a little like high pass filters.
- Real implementations of these ideal systems often require a bit of care.
- Integrators can be physically realised without too many problems but the high gain at DC (low frequency) can often lead to ‘wind-up’ – saturation of the output.
- Differentiators are harder to implement because the gain become infinite as the frequency increases. This exaggerates noise in a signal. Normally compensation is applied to reduce the gain at high frequencies.

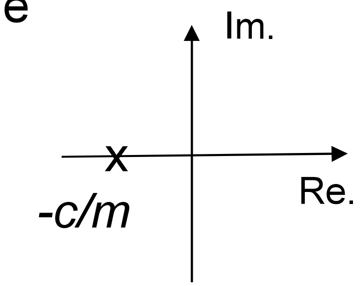
1St order systems



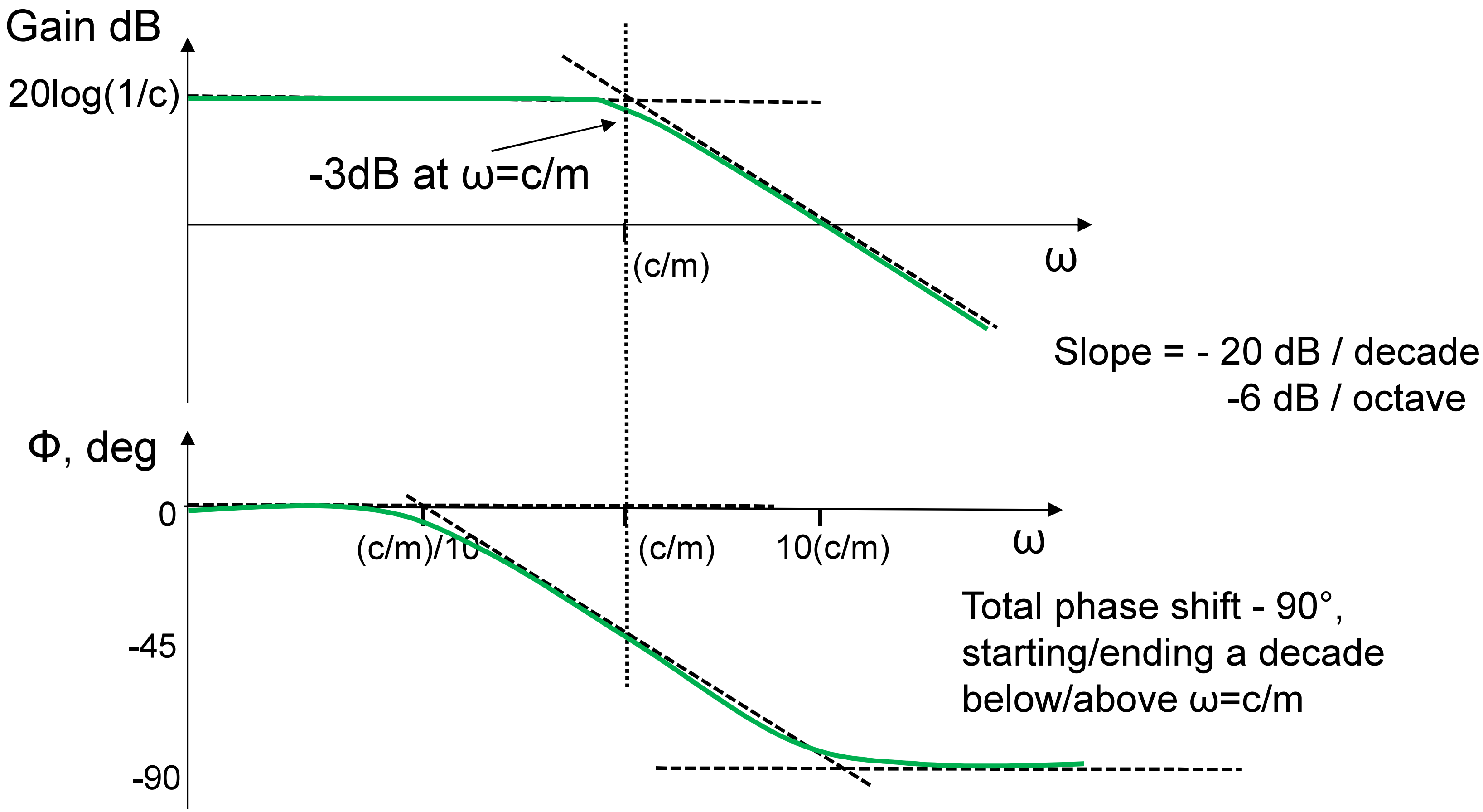
$$f(t) = m\frac{dv(t)}{dt} + cv(t) \qquad\qquad F(s) = msV(s) + cV(s)$$

$$\frac{V(s)}{F(s)} = \frac{1}{ms+c} = \frac{\frac{1}{c}}{\frac{m}{c}s+1} = \frac{\frac{1}{m}}{s+\frac{c}{m}}$$

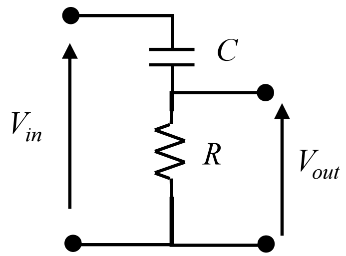
- When s=jω=0, TF reduces to *1/c* – this is the *sensitivity* of the system.
- *There is a single pole at* s=*-c/m*
- *The system 'cut-off frequency' is c/m*
- *m/c is the system time constant*



1st order low pass Bode

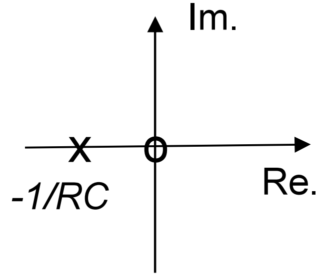


1St order systems

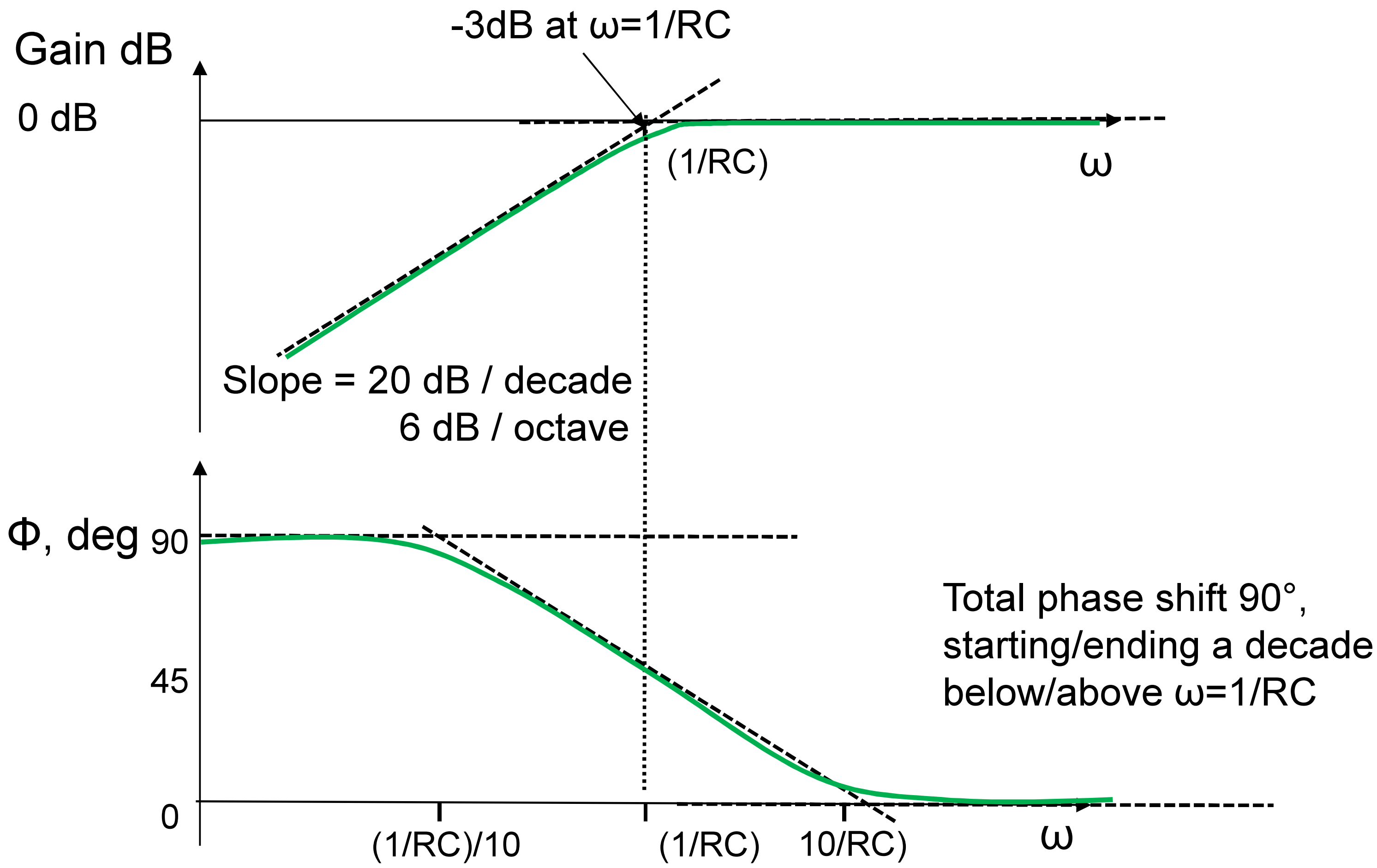


$$\frac{V_{out}}{V_{in}} = \frac{R}{R + X_c} = \frac{R}{R + \frac{1}{SC}} = \frac{s}{s + \frac{1}{RC}}$$

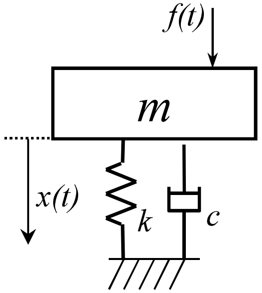
- *When s=jω=0, TF reduces to 0*
- *as s→∞, TF →1.*
- *There is a zero at s = 0, and a pole at s = -1/RC*
- *The system 'cut-off' frequency is 1/RC*
- *'RC' is the system time constant*



1st order high pass Bode



2nd order response

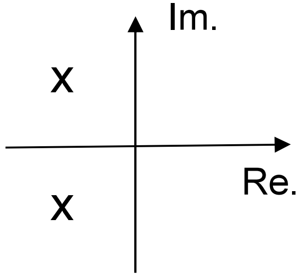


$$f(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

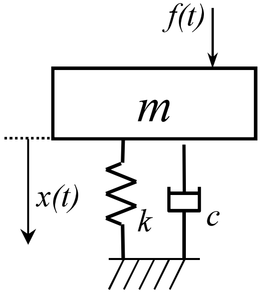
$$\frac{F(s)}{X(s)} = ms^2 + cs + k$$

$$TF = \frac{1}{ms^2+cs+k} = \frac{\frac{1}{m}}{s^2+\frac{c}{m}s+\frac{k}{m}}$$

- *When s=jω=0, TF reduces to 1/k*
- *The system has two poles, which may be complex*



2nd order response



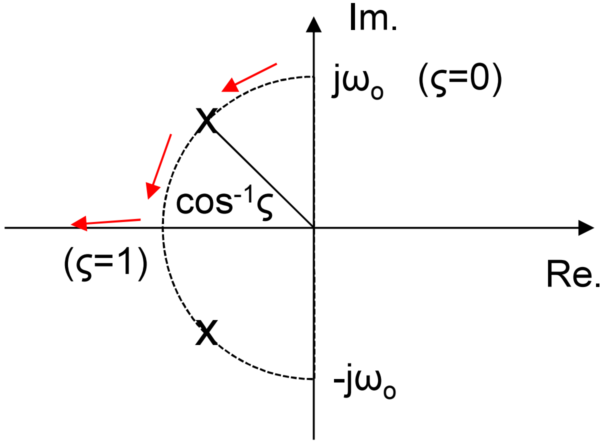
- *From vibrations 2 you should remember the CE of a second order system:*

$$\frac{d^2x}{dt^2}+2\zeta\omega_0\frac{dx}{dt}+\frac{k}{m}x$$

$$TF=\frac{1}{ms^2+cs+k}=\frac{\frac{1}{m}}{s^2+\frac{c}{m}s+\frac{k}{m}}$$

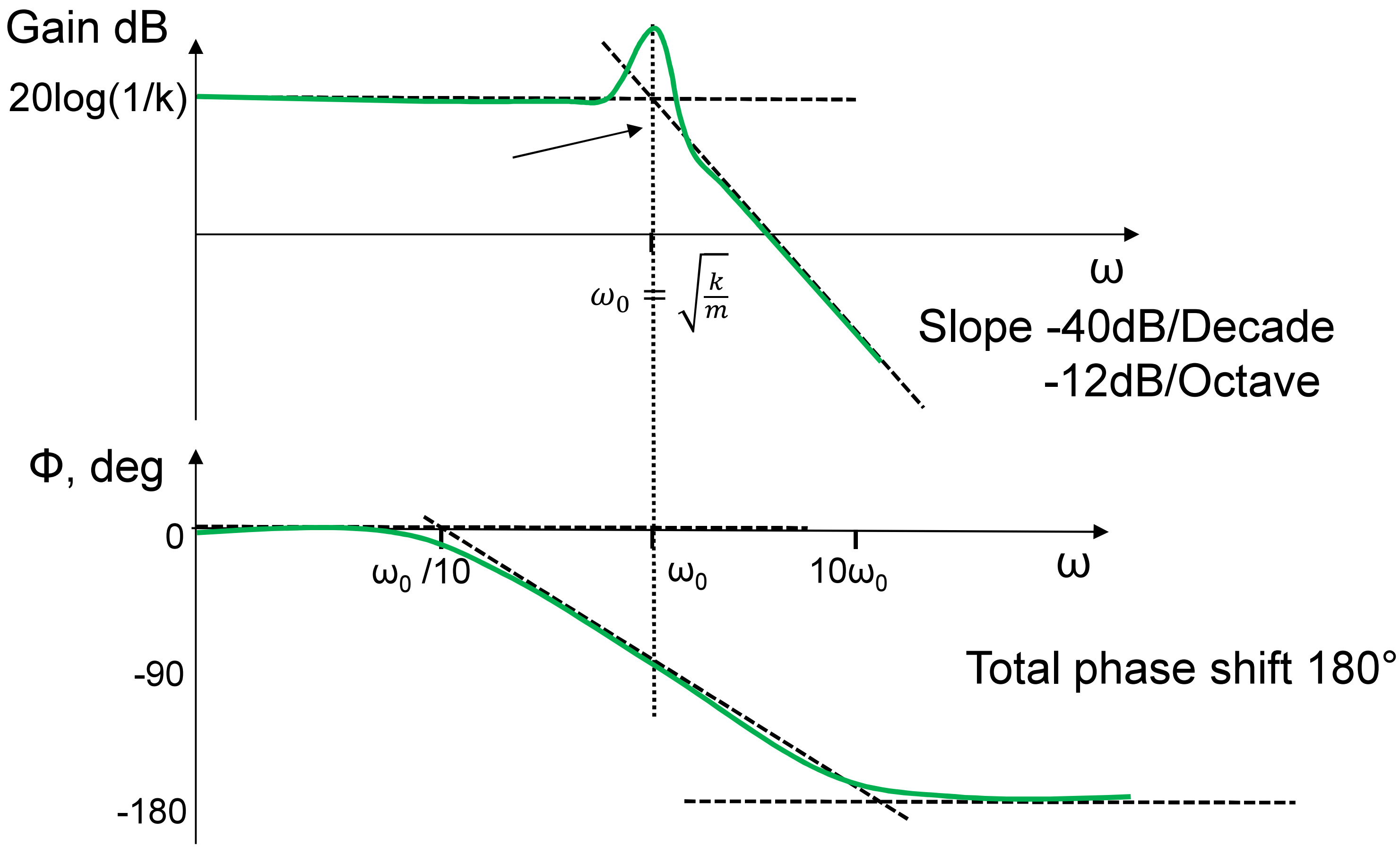
$$\omega_0=\sqrt{\frac{k}{m}}$$

Damping ratio, $\zeta=\frac{c}{2\sqrt{mk}}$



With changes in parameters, the pole pair follows a ‘root locus’. For higher levels of damping the poles become real

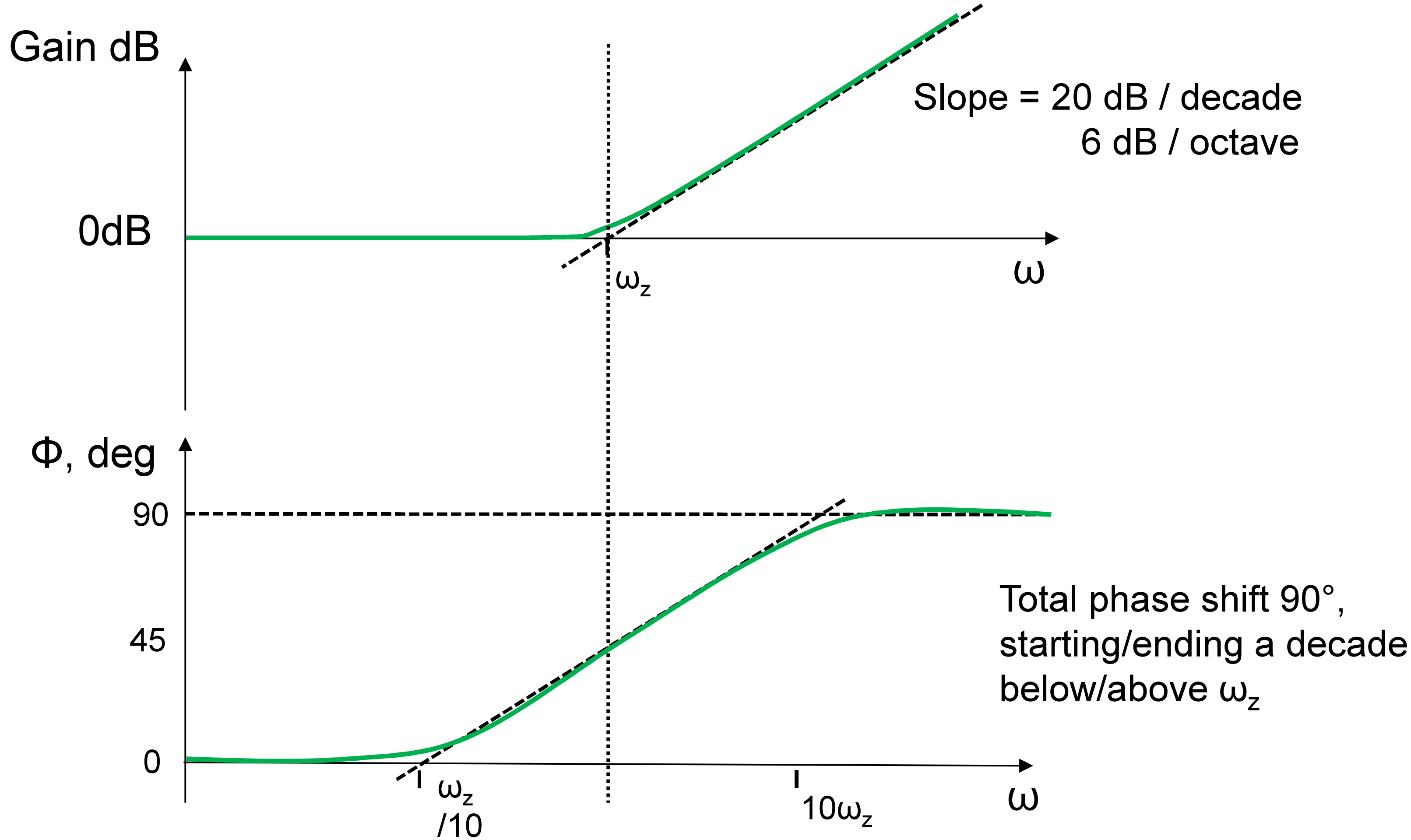
2nd order Bode



Sums of individual poles and zeros

- The systems we have seen so far were:
  - 1st order low pass - 1 pole
  - 1st order high pass – 1 pole & 1 zero
  - 2nd order low pass – pair of poles.
- Why no real system with a single zero?
- What about differentiators?

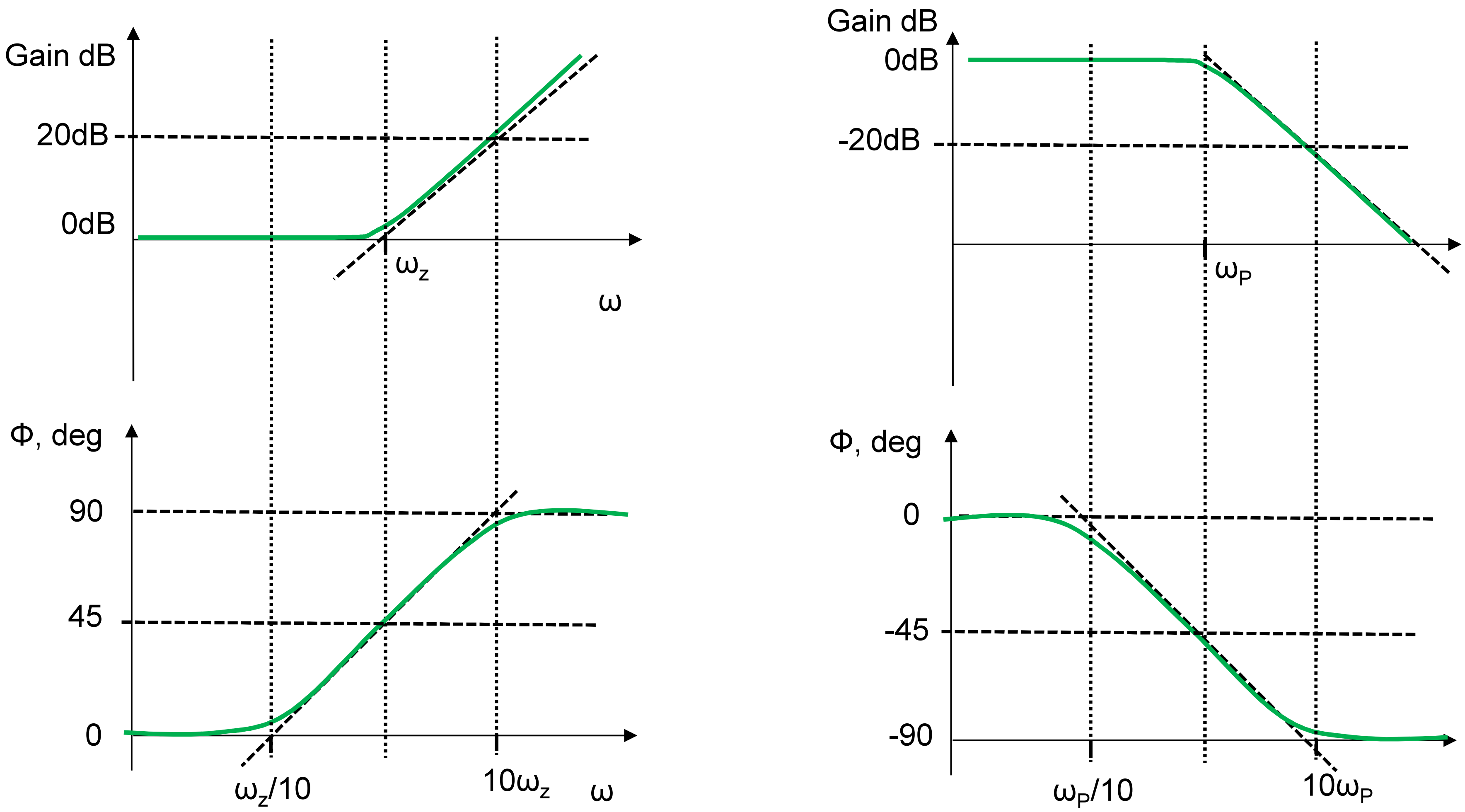
Single Zero Bode



Combining poles and zeros

- We can now describe the Bode plot of any system by adding the effects of the various individual poles and zeros
- Phase is additive
- Gain is multiplicative – but expressing it in dB means we add the values
- Real systems will always have ‘m’ $\geq$ ‘n’

Single Zero and Pole Bode plots



Combining poles and zeros

- We can use this in two ways:

- Having identified the Poles/Zeros in a system TF, we can plot the total response by summing individual contributions

- If we have several cascaded systems we can add the effects of the individual contributions of each system

- But – this only applied if the systems do not affect one another i.e. the output of the first system is unaffected by the input of the second