# Basic Compressible Flow

Aerodynamics 2
AENG21100

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*University of Bristol*

# There's just too much!

- Part of university learning is distilling and understanding a large amount of material yourself. Everyone's mind will do this differently. I can't throw information in to your brain.
- Since everyone does this differently, you have to take responsibility for your own learning.
- **You need to know what you don't know, then you need to make sure you know it. Don't have unknown unknowns.**

- I can't know what people don't know. So, you each have to tell me. Then I can help you to learn it.
- Normal English resumes

# What do I need to know?

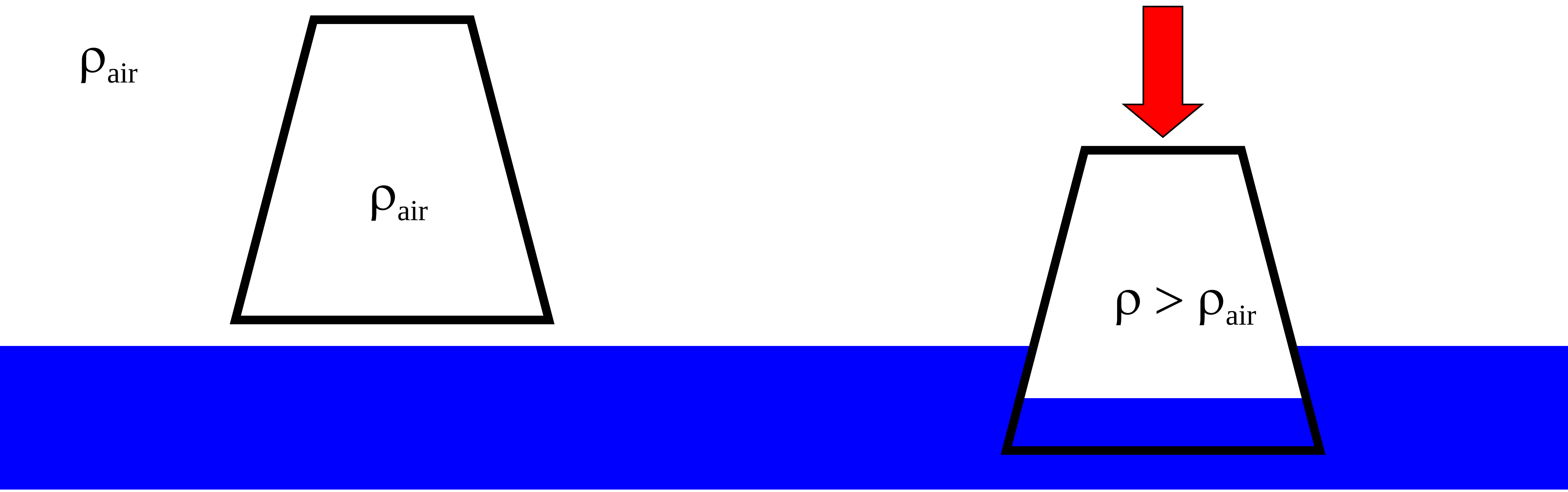
- **Well, what would you like to know?**

- There are some things you simply must know - for example, definitions of Cp, or using deflection angles to compute an expansion fan. These occur throughout the tutorial sheets you have been solving, so you will be familiar with them by the end of the course.
- In comparison, I would not expect you to derive the normal shock equations off the top of your head. However, I would expect you to understand where they come from (mass/momentum/energy conservation) and what they mean.
- Put another way, if you were given a line in their derivation, it would be reasonable to expect you to know the next line, but I would not expect you to know the entire working.

# Basic Compressible Flow

- review equations of motion of a fluid
  - conservation of energy
  - need for thermodynamics in compressible flow
- review basic thermodynamic concepts
  - energy, enthalpy and entropy …
- speed of sound
  - propagation of information
  - Mach Number
- 1D compressible flow
  - ‘compressible Bernoulli’
- isentropic duct flows with varying area
  - critical conditions

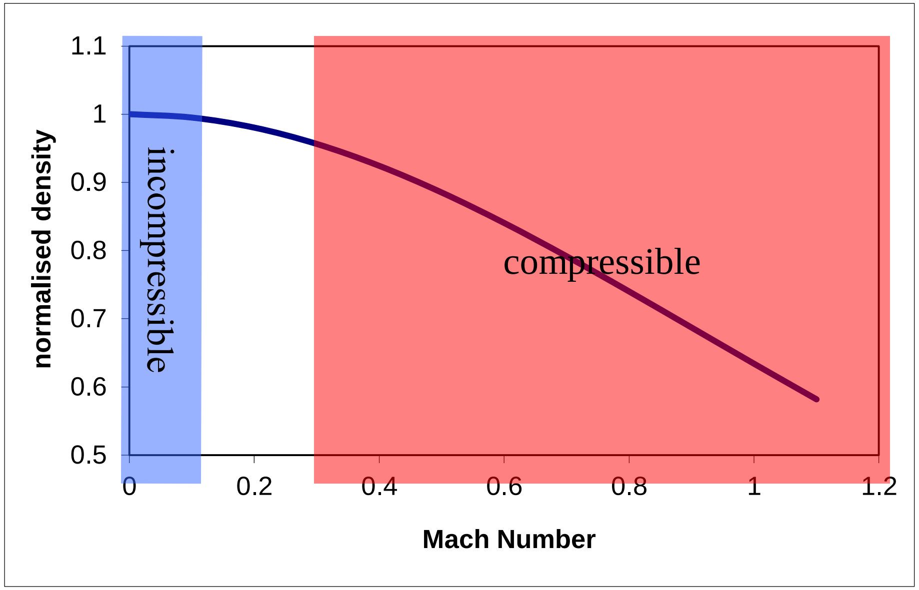
Density is a function of pressure




Density is a function of temperature

Density is a function of energy

Density is a function of flight speed



# Equations of Motion (1)

- 5 unknowns

For 1-D flow. For 3-D flows we have 7 unknowns as **V** is a vector

  - pressure $p$, density $\rho$, velocity vector $\mathbf{V}$
  - internal energy $e$, temperature $T$

- therefore 5 equations required

1. *conservation of mass* – 'continuity'

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

1 equation +

2. *conservation of linear momentum* – Newton's 2nd law

Force=rate of change of momentum

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{f} + \mathbf{F}_{\text{visc}}$$

*(VECTOR EQUATION)*

3 equations +

3. *conservation of energy + two equations of state …*

1 equation +
2 equations
=7 equations

# Equations of motion (2):
## Conservation of Energy

- 'energy cannot either be created or destroyed, merely changed in form', therefore need to balance

The idea of production balancing dissipation

- ***fluid energy***
  - internal energy
  - kinetic energy
  - potential energy

thermal

mechanical

- ***heat transfer and work done***
  - work done by body forces
  - work done by pressure forces
  - heat transfer
  - viscous dissipation

- which necessitates 2 'equations of state' (for specific internal energy $e$ and temperature $T$ ).

Typical equations are $p=\rho RT$ and $e=c_v T$.
These allow simplified equations

# Equations of motion(3):
## Thermal Energy Equation

- previously, energy equation split into *mechanical* and *thermal* energy components
- mechanical energy derived from momentum equation
  - accounts for kinetic & potential energy, body forces, work done by pressure gradient
  - can therefore be subtracted from total energy equation to give

$$\rho \frac{De}{Dt} = -p\nabla\cdot\mathbf{V} + \rho\dot{q} + \dot{Q}_{visc}$$

See Navier Stokes equations in yr1

steady Euler equations in yr1

- for adiabatic, inviscid flow this becomes $\rho \frac{De}{Dt} = -p\cancel{\nabla\cdot\mathbf{V}}$ 

$0$

continuity: conservation of mass

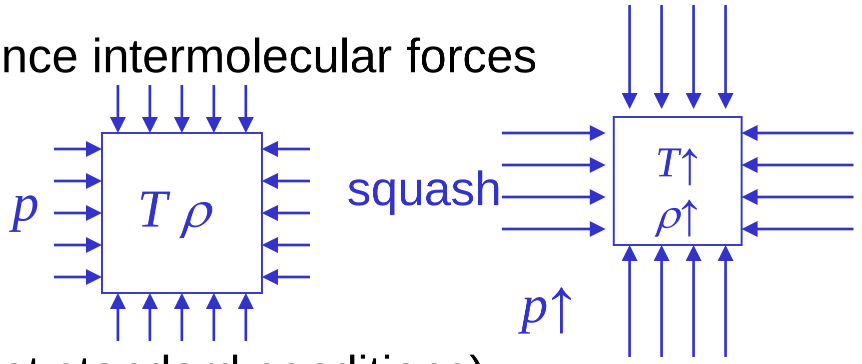
Adiabatic flow → no heat lost or gained

- for incompressible flow $\nabla\cdot\mathbf{V} = 0$ and hence internal energy $e$ is constant $\frac{De}{Dt} = 0 \rightarrow e = \text{const}$
- thermodynamics irrelevant to *incompressible* fluid flow

# Basic Thermodynamics (1)

- equation of state
  - Consider ***thermally*** perfect gas (hence intermolecular forces negligible)

$$\boxed{p = \rho R T}$$




  - $R$ = gas constant (287 *J/kgK* for air at standard conditions)

- specific internal energy $e$ *(ie energy per unit mass)*
  - sum of translational, rotational, vibrational and electronic energies
  - thermodynamic state variable = function of temperature only

$$de = c_v dT$$

therefore $e\uparrow \rightarrow T\uparrow$. note generally $de = c_v(T)dT$

  - for ***calorically*** perfect gas $c_v$ = constant, hence

$$\boxed{e = c_v T}$$

$T=0$ $e=0$ :remember $T$ in ${}^0$K

  - $c_v$ = specific heat at constant volume (717 *J/kgK* for air at standard conditions)

$c_v$ :how much energy needed to raise the temperature of 1kg by $1^0$K, with volume kept constant

# Enthalpy – its usefulness will become clear…

$$Fds = pAds = pdV = mpdv$$

Work associated with expansion

$$pv = RT$$

$$pdv + vdp = RdT$$

dp=0 – constant pressure

$$pdv = RdT$$

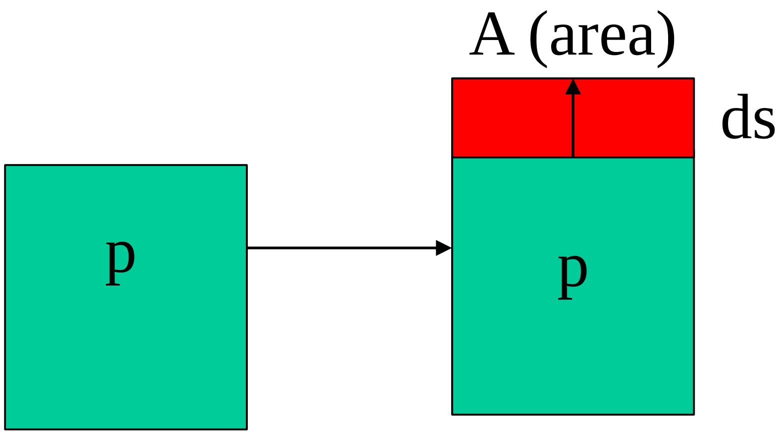
$$mdh = mde + mpdv$$

work in per unit mass (specific) quantities

$$dh = de + pdv = de + RdT = C_v dT + RdT = C_p dT$$

$$h = C_p dT$$

$$R = C_p - C_v$$



# Basic Thermodynamics (2)

- specific enthalpy $h$
  - defined as

$$\boxed{h = e + \frac{p}{\rho} = e + RT}$$

using $p = \rho RT$

  - second term can be thought of as 'pressure energy'
  - as for $e$, a thermodynamic state variable – hence **for a perfect gas**

$$\boxed{h = c_p T}$$

we could use as an alternative state equation

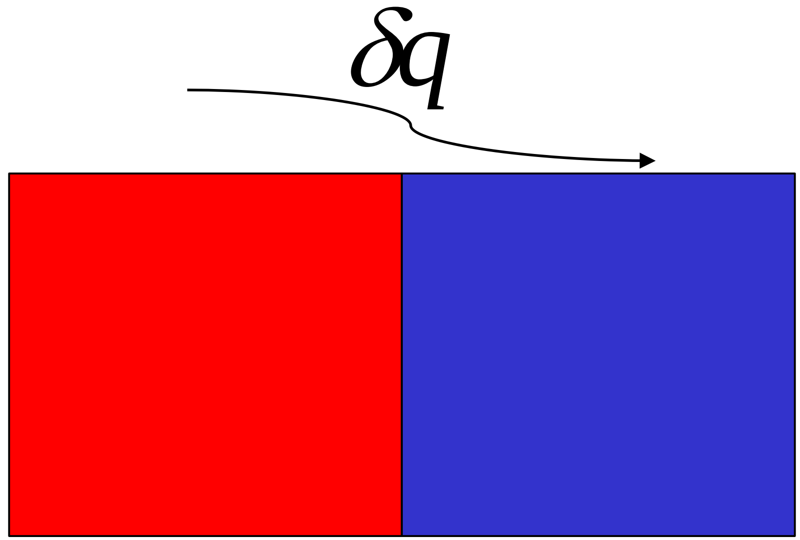
  - $c_p$ = specific heat at constant pressure (1004 *J/kgK* for air at standard conditions)

- from definition of $e$ and $h$

$$\boxed{R = c_p - c_v}$$

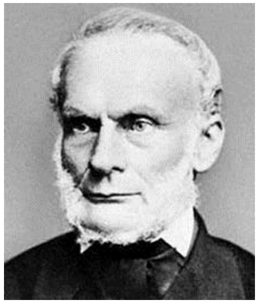
$$h = e + RT = c_v T + RT = c_p T \Rightarrow RT = c_p T - c_v T$$

# Entropy




$$\frac{-\delta q}{T_{hot}} + \frac{\delta q}{T_{cold}} \geq 0$$

(could only be = if both temperatures almost the same…)



Rudolf Clausius

# Basic Thermodynamics (3)

- ratio of specific heats $\gamma$

$$\boxed{\gamma = \frac{c_p}{c_v}}$$

useful shorthand.

may not always be the same value but we can usually consider it constant.

  - $\gamma \approx$ 1.4 for air at standard conditions (more accurately, 1.403)

- entropy $s$ represents degree of disorder   actual value of $s$ not important.
  - determines ***direction*** of thermodynamic process
  - defined as

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{irrev}$$

$2^{nd}$ law of thermodynamics.

  - $\delta q$ = amount of heat added to system at temperature T
  - $ds_{irrev}$ is entropy ***increase*** due to dissipative phenomena (viscosity, thermal conductivity and mass diffusion) occurring within the system – always positive

IREVERSIBLE PROCESS *"non-isentropic"* $ds>0.$

REVERSIBLE PROCESS *"isentropic"* $ds=0.$

this is an idealised situation

# Rearranging

1st Law

$$de = \delta q - \delta w$$

$$\delta q = T ds \quad \text{2nd Law}$$

Ideal gas law $\quad pv = RT$

Differentiate $\quad pdv + vdp = RdT$

$$pdv = RdT - vdp$$

$$Tds = de + \delta w = C_v dT + pdv$$

$$Tds = C_v dT + RdT - vdp = C_p dT - vdp$$

$$R = C_p - C_v$$

$$\frac{v}{T} = \frac{R}{p}$$

$$\boxed{ds = C_p \frac{dT}{T} - \frac{R}{p} dp}$$

# Basic Thermodynamics(4): Isentropic Processes and entropy

- change in specific entropy is for a flow from position 1 to 2.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

  - derived from integration of 1st law in terms of entropy

- for a reversible process $s_2 - s_1 = 0 \rightarrow$ ***'Isentropic'***

- with some algebra, this gives

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}} \Rightarrow p = \mathrm{const.}\rho^{\gamma} \Rightarrow \boxed{\frac{dp}{d\rho} = \frac{\gamma p}{\rho}}$$

- relates pressure, temperature & density for an isentropic process
  - representative of many practical compressible flow problems

# Basic Thermodynamics(4):
## Isentropic Processes and entropy

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  - derived from integration of 1st law in terms of entropy

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p} = 0 \quad \textit{as flow is assumed isentropic}$$

$$\frac{c_p}{R}\frac{dT}{T} = \frac{dp}{p} \rightarrow \textit{Integrate from 1 to 2} \rightarrow \ln\left(\left(\frac{T_2}{T_1}\right)^{C_p/R}\right) = \ln\left(\frac{p_2}{p_1}\right)$$

$$\left(\frac{T_2}{T_1}\right)^{C_p/R} = \left(\frac{p_2}{p_1}\right) \quad \textit{from previous} \quad \frac{c_p}{R} = \frac{c_p}{c_p - c_v} = \frac{\gamma}{\gamma - 1} \rightarrow \left(\frac{p_2}{p_1}\right) = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$\textit{using} \quad p = \rho RT \quad \left(\frac{p_2}{p_1}\right) = \left(\frac{R\rho_1 p_2}{R\rho_2 p_1}\right)^{\gamma/(\gamma-1)} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma/(\gamma-1)}\left(\frac{p_2}{p_1}\right)^{\gamma/(\gamma-1)}$$

$$\left(\frac{p_2}{p_1}\right)^{1-\left(\gamma/(\gamma-1)\right)} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma/(\gamma-1)} \qquad \left(\frac{p_2}{p_1}\right) = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

# Basic Thermodynamics(4):
## Isentropic Processes and entropy

- change in specific entropy is for a flow from position 1 to 2.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right)$$

  - derived from integration of 1st law in terms of entropy

- for a reversible process $s_2 - s_1 = 0 \rightarrow$ ***'Isentropic'***

differentiating $w.r.t\ \rho$

$$\frac{dp}{d\rho} = \text{const} \times \gamma\rho^{\gamma-1} = (\text{const} \times \rho^{\gamma})\gamma\rho^{-1}$$

- with some algebra, this gives

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}} \Rightarrow p = \text{const.}\rho^{\gamma} \Rightarrow \boxed{\frac{dp}{d\rho} = \frac{\gamma p}{\rho}}$$

- relates pressure, temperature & density for an isentropic process
  - representative of many practical compressible flow problems

# Basic Thermodynamics(5): Total Temperature (1)

- $T_0$ - also known as 'stagnation temperature' and reservoir temperature
  - similar concept to total pressure $p_0$ in potential flow
- derive from 'full' conservation of energy equation, making the assumptions
  - body forces negligible
  - adiabatic – no heat addition
  - inviscid – no ***external*** viscous losses
  - steady flow

still have internal dissipation.
No isentropic assumption.

- in terms of enthalpy, gives will take more manipulation

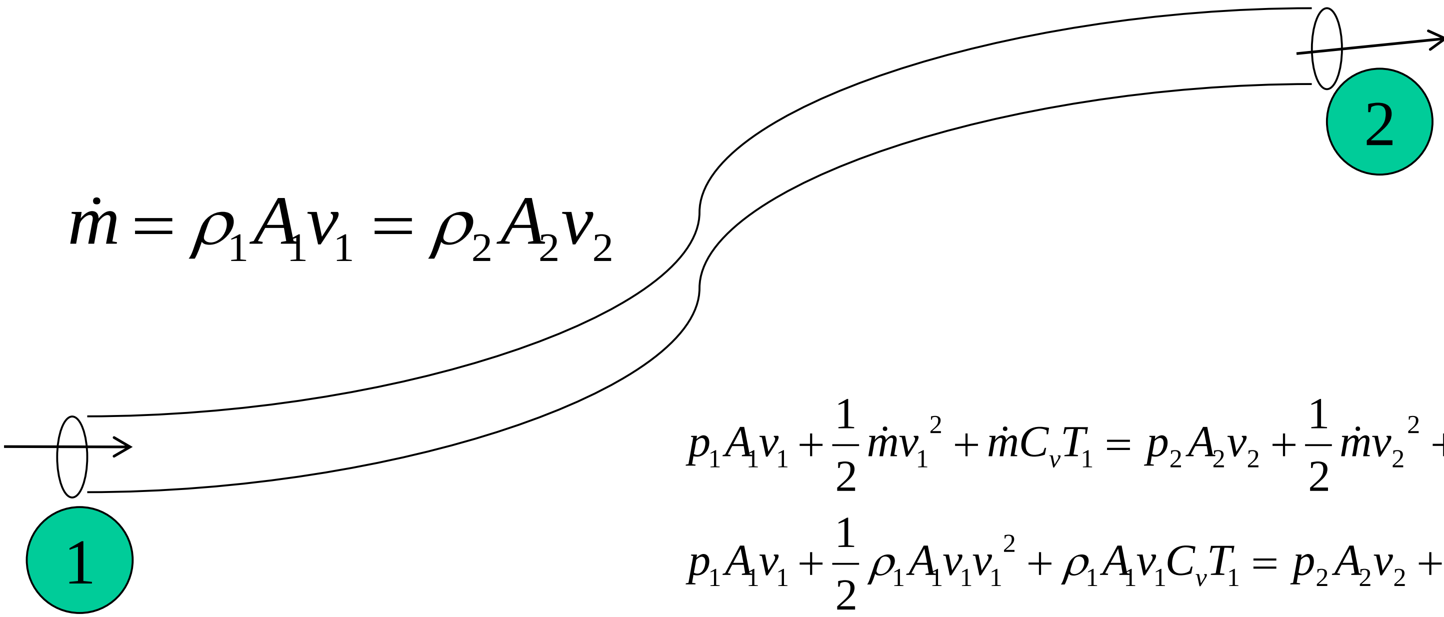
$$\frac{D\left(h+V^2/2\right)}{Dt}=0 \quad \Rightarrow \quad \boxed{h+\frac{V^2}{2} = \text{constant} = h_0}$$

enthalpy of a flow brought to rest adiabatically

  - **along a streamline** (compare with Bernoulli's Equation)

Substantial derivative is the time rate of change of a fluid element moving with the flow ie along a streamline.

# Streamtube




$$\dot{m} = \rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

$$p_1 A_1 v_1 + \frac{1}{2}\dot{m} v_1^{\,2} + \dot{m} C_v T_1 = p_2 A_2 v_2 + \frac{1}{2}\dot{m} v_2^{\,2} + \dot{m} C_v T_2$$

$$p_1 A_1 v_1 + \frac{1}{2}\rho_1 A_1 v_1 v_1^{\,2} + \rho_1 A_1 v_1 C_v T_1 = p_2 A_2 v_2 + \frac{1}{2}\rho_2 A_2 v_2 v_2^{\,2} + \rho_2 A_2 v_2 C_v T_2$$

$$\frac{p_1}{\rho_1} + \frac{1}{2} v_1^{\,2} + C_v T_1 = \frac{p_2}{\rho_2} + \frac{1}{2} v_2^{\,2} + C_v T_2$$

Rate KE entering $\quad \frac{1}{2}\dot{m} v^2$

Rate work done by pressure forces $\quad Fv = pAv$

Rate internal energy entering $\quad \dot{m} C_v T$

Ignore internal energy and assume const density - Bernoulli

$$p + \frac{1}{2}\rho v^2 = const = p_0$$

Very rarely used in this part of the course!

$$h + \frac{1}{2} v^2 = const = h_0$$

`Compressible' Bernoulli

Used all the time!

$$\boxed{C_p T + \frac{1}{2} v^2 = const = h_0 = C_p T_0}$$

# Stagnation Temperature

$$\boxed{C_pT + \frac{1}{2}v^2 = const = h_0 = C_pT_0}$$

Velocity? Urgh! Much prefer Mach number!

$$C_pT_0 = C_pT + \frac{1}{2}M^2\gamma RT$$

Remember $v = Ma$

$a^2 = \gamma RT$

$$T_0 = T + \frac{1}{2}\frac{M^2\gamma RT}{C_p}$$

$$\frac{\gamma R}{C_p} = \frac{\gamma\left(C_p - C_v\right)}{C_p} = \gamma\left(1 - \frac{1}{\gamma}\right) = \gamma - 1$$

$$T_0 = T\left(1 + \frac{(\gamma - 1)M^2}{2}\right)$$

Surely such simple, humble equation can be of little use?

This is **NOT** heating due to friction – inviscid flow! It is analogous to heating as a result of compression in a bike pump

# Planes

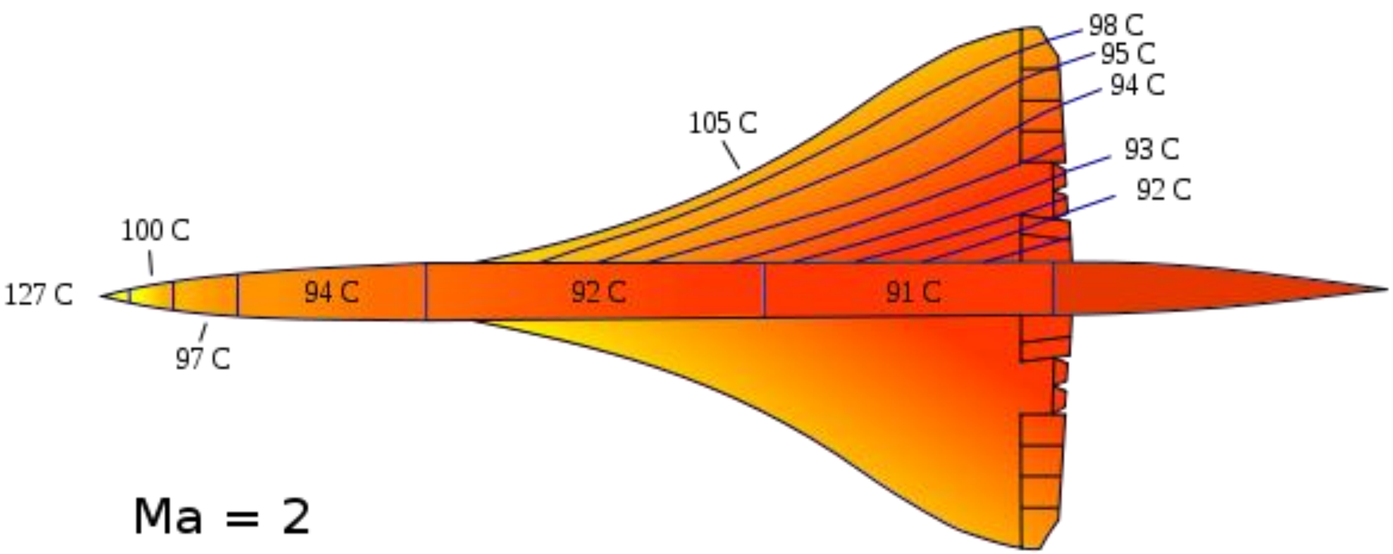
M=0.8, T=219K, T0=247K=-26C ✓

M=2.02, T=217K, T0=394K=121C ~ max allowed for aluminium 127C – so max Mach effectively determined by T0 equation ✓

M=3, T=222K, T0=622K=349C (Titanium used) ✓

M=25, T=180K?, T0=22680K=22407C? ✗

Air no longer a continuum, gamma no longer the same (or even constant), no longer adiabatic (air radiates energy) so this is **not** accurate. Actual max for shuttle (for any M) ~ 1600C


98 C
95 C
94 C
105 C
93 C
92 C
100 C
127 C
94 C
92 C
91 C
97 C
Ma = 2

# Basic Thermodynamics (6):
## Total Temperature (2)

- for a calorically perfect gas $h=c_pT$ hence *see* BC1.7

$$\boxed{c_pT+\frac{V^2}{2} = c_pT_0}$$

$$c_pT_0 = h_0$$

  - where $T_0$ is the 'total' or 'stagnation' or 'reservoir' temperature
  - an alternative form of the energy equation along a streamline
- temperature of fluid element brought to rest *adiabatically*
- no assumption about *entropy* made in derivation
- in an *isentropic* process $T_0$, $p_0$ and $\rho_0$ are constant
  - eg a sound wave
- in a ***non****isentropic* (but adiabatic) process **only** $T_0$ is constant
  - eg a shock wave

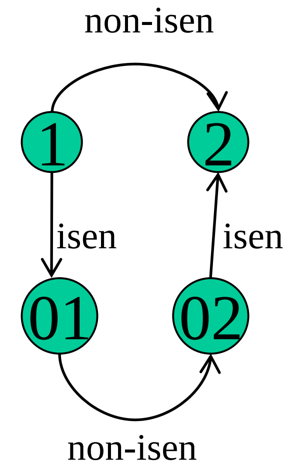
$T_o$ = *"total energy"*
$P_o$ = *"total usable energy"*

Compressible flow
$\Downarrow$
*assume no heat loss*
$\Downarrow$
Adiabatic flow
$\Downarrow$
*assume reversible*
$\Downarrow$
Isentropic flow

# Basic Thermodynamics (7):
## Stagnation pressure and entropy

- Consider an adiabatic process from condition 1 to 2
- Derive "stagnation" values corresponding to 1 and 2 using an **isentropic** deceleration.

$p_1 T_1 V_1 s_1 \longrightarrow \quad p_2 T_2 V_2 s_2 \longrightarrow$




- Entropy change between 1 and 2 equals entropy change in stagnation values.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = c_p \ln\left(\frac{T_{o_2}}{T_{o_1}}\right) - R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right)$$

*see* BC1.12

- For **adiabatic** flow $T_{01} = T_{02} \rightarrow \ln\left(\dfrac{T_{02}}{T_{01}}\right) = 0 \rightarrow -\dfrac{s_2 - s_1}{R} = \ln\left(\dfrac{p_{o_2}}{p_{o_1}}\right)$

$$\frac{p_{o_2}}{p_{o_1}} = \mathrm{e}^{-\left(\frac{s_2 - s_1}{R}\right)}$$

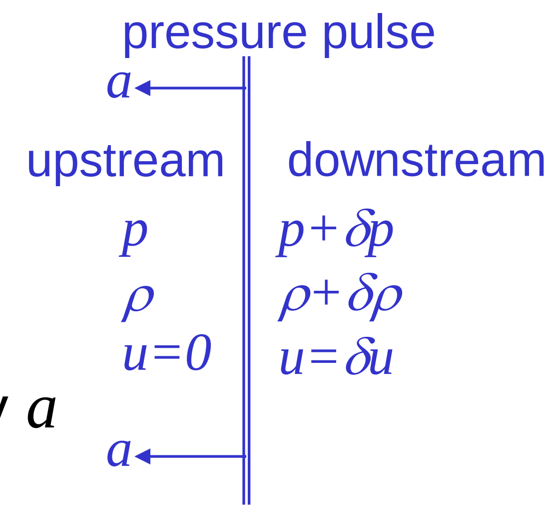
- non-isentropic – total pressure "lost"

  Isentropic – total pressure conserved

$$\frac{p_{o_2}}{p_{o_1}} < 1 \rightarrow p_{o_2} < p_{o_1}$$

$$\frac{p_{o_2}}{p_{o_1}} = 1 \rightarrow p_{o_2} = p_{o_1}$$

# Speed of Sound (1)

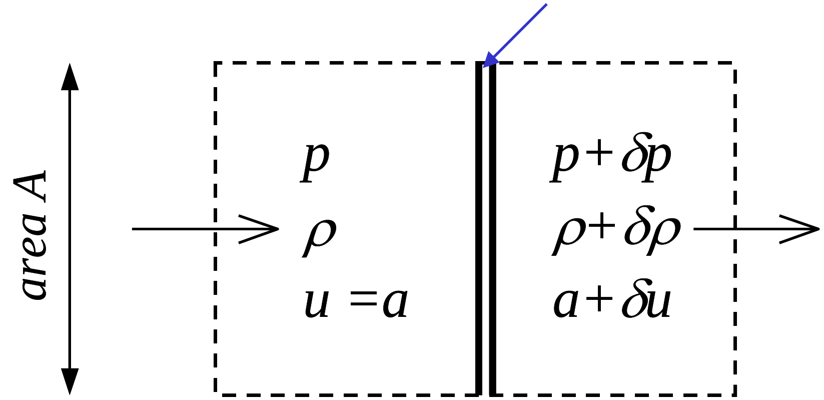



- rate of propagation of pressure 'information'
  - instantaneous in incompressible fluid
  - finite velocity $a$ in compressible flow
- start with plane pressure pulse moving at velocity $a$
- assume pulse is of infinitesimal strength, hence
  - changes in fluid properties $p$, $\rho$ and $u$ are 'small' later assume products are negligible
  - adiabatic process
  - isentropic process
- use control volume moving with the wave front Galilean Transformation
  - equivalent to superimposing freestream velocity $a$ so pulse becomes stationary
- apply momentum & continuity equations to flow through control volume next slide

# Speed of Sound (2)

stationary pulse
flow moving at speed $a$

- apply simple continuity and momentum equations
  - neglect 2$^{nd}$ and 3$^{rd}$ order terms
  - gives Newton's result




$$\boxed{\frac{dp}{d\rho} = a^2} \sim \frac{1}{'compressibility'}$$

- now apply *isentropic* relation, followed by equation of state

$$\Rightarrow \boxed{a^2 = \frac{\gamma p}{\rho}} \Rightarrow \boxed{a = \sqrt{\gamma R T}}$$

- speed of sound is a function of static temperature $T$ only

# Speed of Sound (2)

- apply simple continuity and momentum equations
  - neglect 2nd and 3rd order terms
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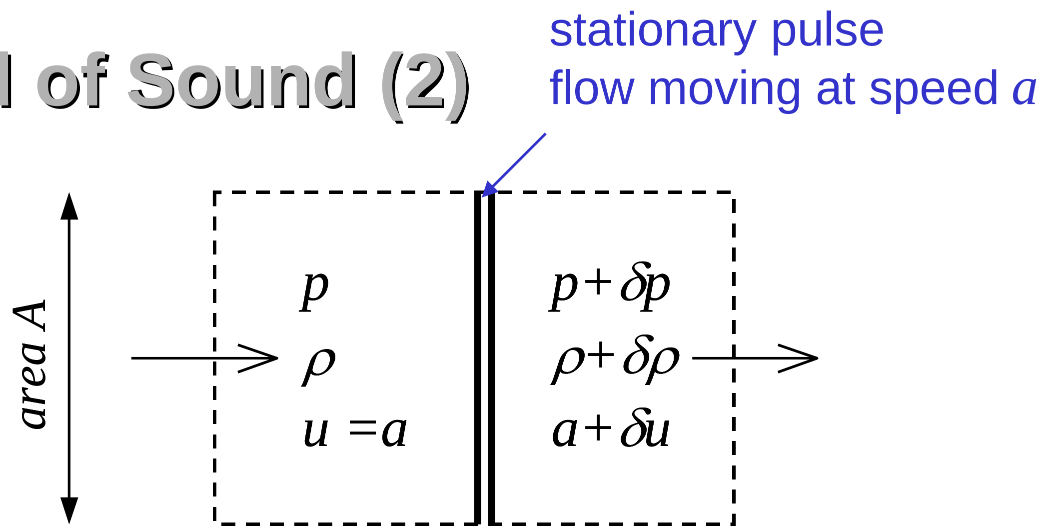



*continuity: mass flow conserved* $\dot{m} = \rho a A = (\rho + \delta\rho)(a + \delta u)A$

*neglecting products of small terms* $\rho\delta u + a\delta\rho = 0$

*momentum: force = rate of change of momentum = change in momentum rate*

$$\text{net force} = pA - (p + \delta p)A = (\rho + \delta\rho)(a + \delta u)^2 A - \rho a^2 A$$

*neglecting products of small terms* $-\delta p = 2a\rho\delta u + a^2\delta\rho$

*from continuity, sub in term for* $\rho\delta u$ *gives*

$$-\delta p = -2a^2\delta\rho + a^2\delta\rho \;\rightarrow\; \frac{\delta p}{\delta\rho} = a^2$$

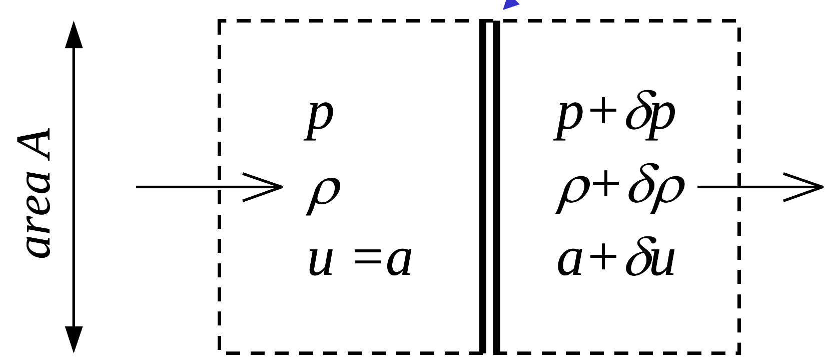
*so in the limit of infinitely small disturbances*

$$\boxed{\frac{dp}{d\rho} = a^2}$$

# Speed of Sound (2)

stationary pulse
flow moving at speed $a$

- apply simple continuity and momentum equations
  - neglect 2nd and 3rd order terms
  - gives Newton's result




$$\boxed{\frac{dp}{d\rho} = a^2} \sim \frac{1}{'compressibility'}$$

$\dfrac{d\rho}{dp}$ = compressibility

- now apply *isentropic* relation, followed by equation of state

from $BC1.9$ $\dfrac{dp}{d\rho} = \dfrac{\gamma p}{\rho} \rightarrow a^2 = \dfrac{\gamma p}{\rho}$

$$\Rightarrow \boxed{a^2 = \frac{\gamma p}{\rho}} \Rightarrow \boxed{a = \sqrt{\gamma RT}}$$

using the equation of state $p = \rho RT$

$a^2 = \gamma RT$

- speed of sound is a function of static temperature $T$ only

*For air at standard temperature and pressure* (STP:$p$=1atm $\gamma$=1.403 $R$=287.1 $T$=288°K)

$a = 340.6$

# Speed of Sound (3) :
## Mach Number

Ernst Mach

- *local* Mach Number defined as

$$\boxed{M = \frac{V}{a}} \quad \sim \frac{\text{directed KE}}{\text{random thermal energy}} \qquad M^2 = \frac{V^2}{a^2} = \frac{V^2}{\gamma RT} = \frac{V^2}{\gamma R}\frac{c_v}{e} = \left(\frac{c_v}{\gamma R}\right)\frac{V^2}{e}$$

- dynamic pressure $q$ can be given in terms of $M$

$$\boxed{q = \frac{1}{2}\rho V^2 = \frac{1}{2}p\gamma M^2} \qquad q = \frac{1}{2}\rho V^2 = \frac{1}{2}\rho M^2 a^2 = \frac{1}{2}\rho M^2 \frac{\gamma p}{\rho}$$

- from which it follows that the pressure coefficient $C_p$ is

$$\boxed{\underbrace{C_p = \frac{p - p_\infty}{q_\infty}}_{\text{definition of } c_p} = \frac{\left(p/p_\infty - 1\right)}{\frac{\gamma}{2}M_\infty^2}}$$

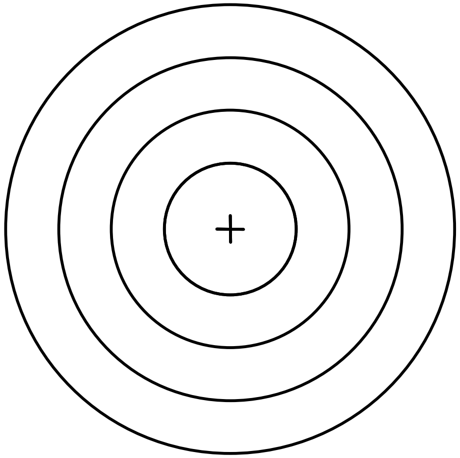
note: **use pressure ratio rather than difference**

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty v_\infty^2} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty M_\infty^2 \frac{\gamma p_\infty}{\rho_\infty}} = \frac{p - p_\infty}{\frac{1}{2}M_\infty^2 \gamma p_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

# Speed of Sound (4) : Mach Cone (1)

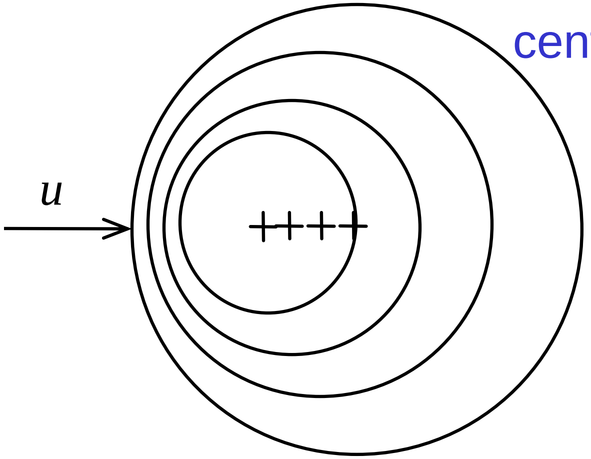
<u>note</u>: $a$=maximum velocity of pressure information for **small** disturbances. Strong pressure pulses such as shocks can move faster than sound, eg from explosions

- point sound source in 'still air'
  - pressure wave radiated in all directions
  - *spherical* wave front of radius $at$
  - all of fluid eventually disturbed



- now add freestream velocity $u < a$ (subsonic)
  - spherical wave fronts displaced downstream by distance $ut$
  - all of fluid still eventually disturbed

+ is the apparent center of disturbance

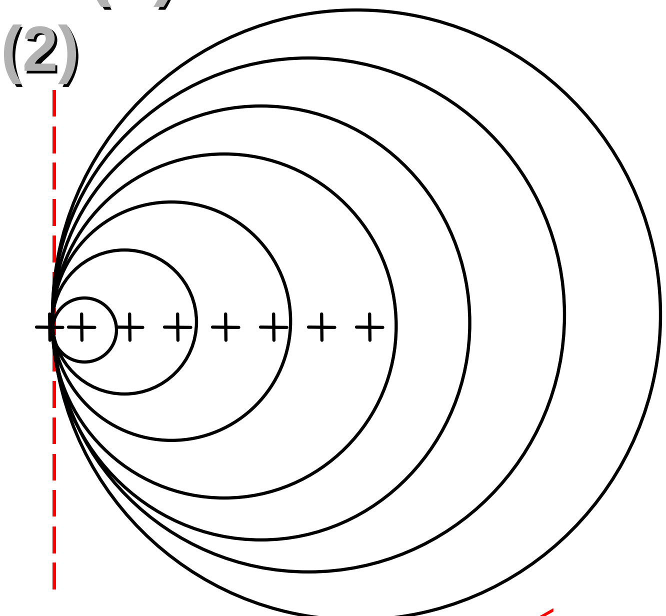



- 'ripples on a pond'
  - close analogy between shallow water waves and sound waves

different values of $\gamma$

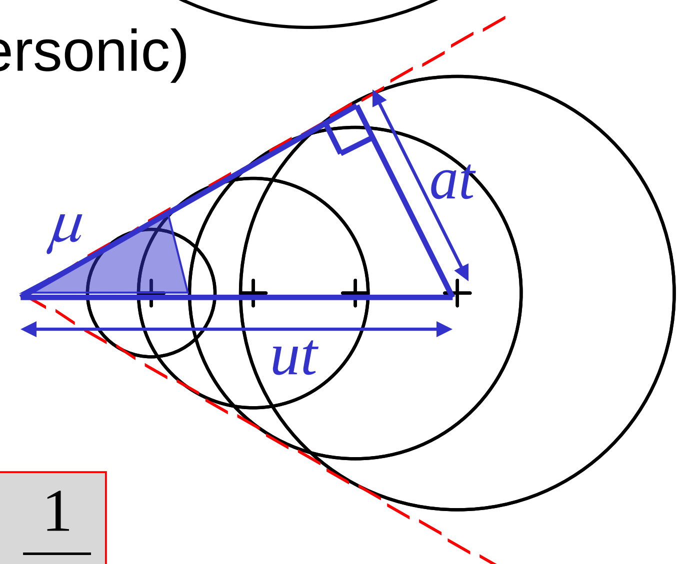
# Speed of Sound (5) :
## Mach Cone (2)

- increase freestream to $u = a$ (sonic)
  - no wavefronts propagated upstream
  - ‘zone of silence’ ahead of source
  - no ‘warning’ of downstream disturbances
  - pressure rise at front no longer infinitesimal



- finally, increase freestream to $u > a$ (supersonic)
  - spherical wavefronts swept downstream to form a *‘Mach Cone’*
  - defines lateral extent of source influence
- half-angle $\mu$ of cone = *‘Mach Angle’*
- surface of cone = *‘Mach Wave’*

$$\boxed{\sin\mu = \frac{1}{M}}$$




$$\sin\mu = \frac{at}{ut} = \frac{a}{u} = \frac{1}{M}$$

# 1D Compressible Flow (1):
## Energy Equation Revisited

- for an ***adiabatic*** process from BC1.11

$$\boxed{c_p T + \frac{V^2}{2} = c_p T_0}$$

  - where $T_0$ is the 'total' or 'stagnation' or 'reservoir' temperature

- more useful in terms of Mach Number $M$ and constants $\gamma\ R\ etc$, but not other state variables such as: $p,\ \rho\ etc.$
- so, substitute

BC2.2 $$M^2 = \frac{V^2}{\gamma RT}, \quad c_p = \frac{\gamma R}{\gamma - 1}$$ BC1.8

to obtain

$$\boxed{T_0 = T\left\{1 + \frac{\gamma - 1}{2} M^2\right\}}$$

*ADIABATIC*

fundamental energy equation in terms of $M$ & $T$

- no assumption made about entropy (yet) so equation valid through shock waves

# 1D Compressible Flow (2):
## Local Speed of Sound

- Mach Number based on *local* speed of sound $a$
- but $a$ varies with *local* temperature $T$ ..

$$a = \sqrt{\gamma RT} \quad \Rightarrow \quad a_0/a = \sqrt{T_0/T}$$

$$a_0 = \sqrt{\gamma RT_0}$$ $a_o$ = reservoir speed of sound

and hence

substitute into energy relationship
in previous slide BC2.6

$$\boxed{a_0 = a\left\{1+\frac{\gamma-1}{2}M^2\right\}^{0.5}}$$

ADIABATIC

- 'stagnation' or 'reservoir' speed $a_0$ is constant in an adiabatic flow
- ***'critical'*** value $a^*$ is speed of sound for $M = 1$
  - common reference value in duct flows
  - $a^* = 0.913a_0$ for air at STP

$$\boxed{a^* = a_0\sqrt{\frac{2}{\gamma+1}}}$$

later we will see values of $p^*, \rho^*$ *etc.*

# Map – lectures 1-3

Adiabatic – no heat lost or gained – total enthalpy constant along a streamline

Energy equation

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

Valid through shocks!

Entropy Constant

$$p = const \times \rho^{\gamma}$$

+ideal gas

$$p = \rho R T$$

Speed of sound

$$a^2 = \gamma R T$$

1st and 2nd Laws of Thermo.

Today

Isentropic flow relations

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

Not valid through shocks! Still valid in between

Cons. of mass

Mach-area Relation + Con-di nozzles + Mach-rate of area change relation

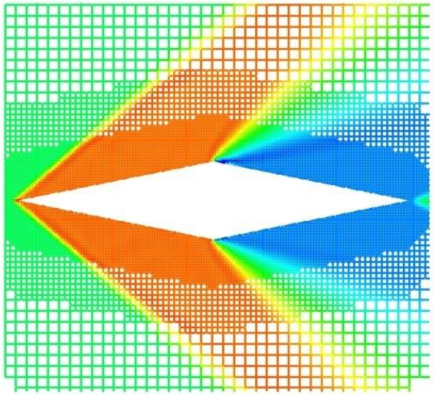
**NB we have not yet done shocks!**

# Problems we shall solve

- Working out pressure/density/temperature from Mach number. Usually done using tables.
- Working out Mach number from area ratio for nozzles, then using Mach number for pressure, density and temperature.

ONCE WE HAVE COVERED SHOCKS (next lecture)…

- Cp on supersonic wedge/bicon aerofoils – allows us to find Cl, Cd and Cm.
- Supersonic pitot probes
- Nozzles with simple shock patterns



# 1D Compressible Flow (3):
## 'Compressible Bernoulli'

- make the additional assumption of an ***isentropic*** process

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1}$$

from BC1.9

- substitute in total temperature equation to give

$$\boxed{\begin{aligned} p_0 &= p\left\{1+\frac{\gamma-1}{2}M^2\right\}^{\frac{\gamma}{\gamma-1}} \\ \rho_0 &= \rho\left\{1+\frac{\gamma-1}{2}M^2\right\}^{\frac{1}{\gamma-1}} \end{aligned}}$$

$$\left(\frac{p_0}{p}\right) = \left(\frac{T_0}{T}\right)^{\gamma/\gamma-1} = \left\{1+\frac{\gamma-1}{2}M^2\right\}^{\gamma/\gamma-1}$$

ISENTROPIC

$$\left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{p_0}{p}\right) = \left\{1+\frac{\gamma-1}{2}M^2\right\}^{\gamma/\gamma-1}$$

- compressible equivalent of Bernoulli's Equation
  - relationship between pressure, density and velocity

$p_o$ constant in an isentropic flow

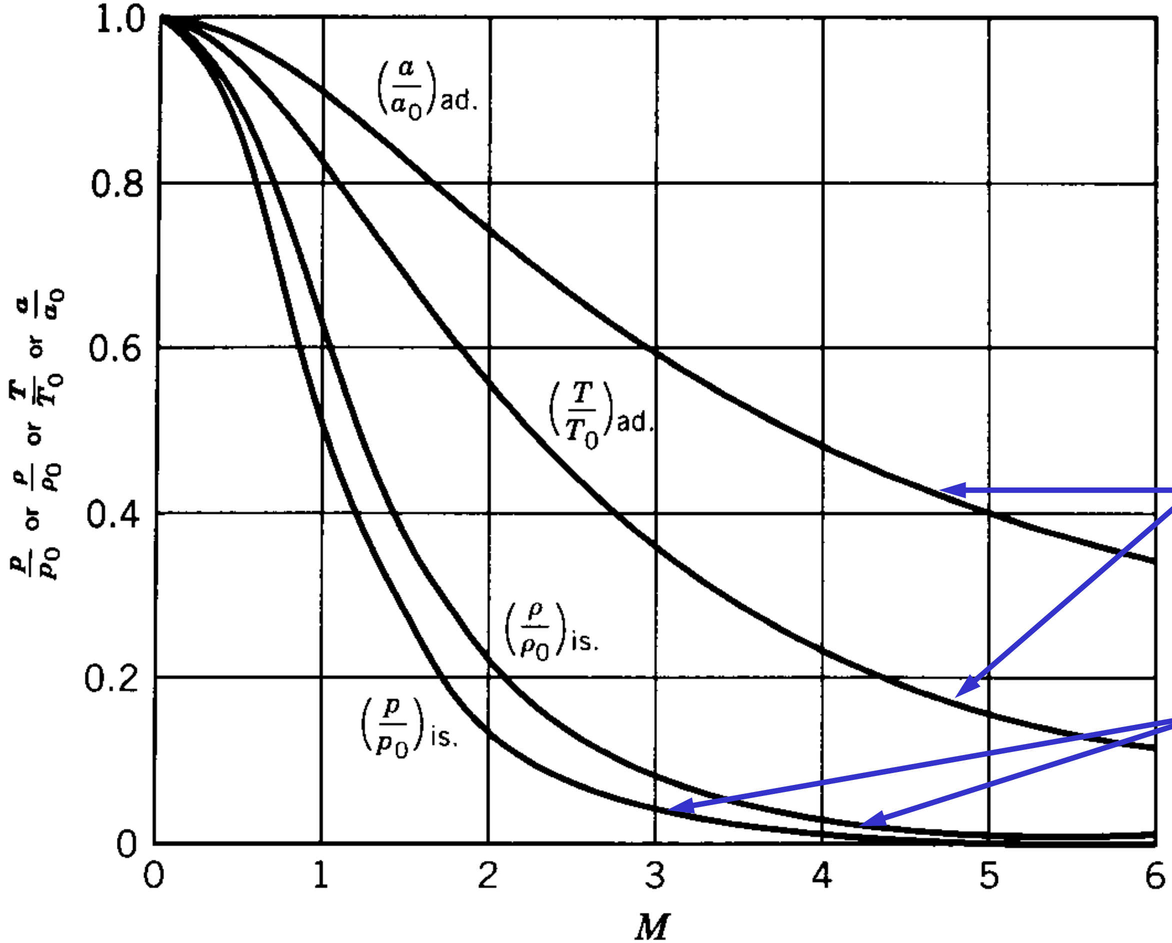
- ratios $T_0/T$, $p_0/p$ and $\rho_0/\rho$ given in compressible flow tables

# Tables

- A wide variety of tables will be used in the course. The isentropic flow relations are conveniently tabulated as a function of Mach number. Be prepared to interpolate linearly in Mach number.
- Other tables describe shocks and expansions – these will be covered in the next few lectures

| $M$ | $p_o/p$ | $\rho_o/\rho$ | $T_o/T$ | $A/A^*$ |
|---|---|---|---|---|
| 0.50 | 1.187 | 1.130 | 1.050 | 1.340 |
| 0.52 | 1.203 | 1.141 | 1.054 | 1.303 |
| 0.54 | 1.220 | 1.152 | 1.059 | 1.270 |
| 0.56 | 1.238 | 1.164 | 1.063 | 1.240 |
| 0.58 | 1.257 | 1.177 | 1.068 | 1.213 |

# 1D Compressible Flow (4):
## Compressible Flow Relations




Flow parameters versus Mach number of adiabatic (ad.) and isentropic (is.) flows.

tabulated data:
must interpolate between data points

valid through strong pressure waves such as shocks
*ie* non-isentropic flows

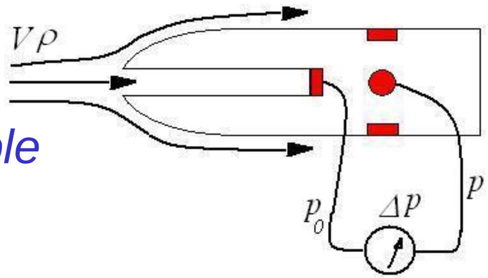
only valid for isentropic flow

*(NB ratios inverted to keep magnitude < 1)*

# 1D Compressible Flow (5): Pitot-Static Probe in Compressible Flow

*example of compressible Bernoulli*

- incompressible pitot-static equation




assume flow decelerated isentropically so $p_{pitot}=p_o$

Bernoulli $$V_{measured} = \sqrt{\frac{2(p_0 - p)}{\rho}}$$

*for compressible*
*$\rho$ variable*

- expanding the pressure ratio $p_0/p$ equation (for air) from BC2.8

$$\frac{p_0}{p} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1} = 1+\frac{\gamma M^2}{2}\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\ldots\right]$$

use binomial theorem *with* $\gamma = 1.4$

$$(1+x)^n = \left(1+nx+\frac{n(n-1)}{2}x^2+\cdots\right)$$

and hence using $q = \gamma M^2 p/2$ from BC2.3

$$p_0 - p = \frac{1}{2}\rho V^2\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\ldots\right]$$

$$p_0 = p + \frac{\gamma M^2 p}{2}\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\ldots\right]$$

- term in [ ] is basic compressibility correction

$$p_0 - p = \frac{\rho V_{measured}^2}{2}$$

$$\boxed{V = V_{measured}\Big/\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\ldots\right]^{0.5}}$$

$$V_{measured} = \sqrt{2\frac{p_0-p}{\rho}} = \sqrt{V^2\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\cdots\right]}$$

# 1D Compressible Flow (6): Airspeed Corrections

*for info-not in exam*

- ASIR *'Airspeed Indicator Reading'*
- IAS *'Indicated Airspeed'* $V_i$
- CAS *'Calibrated Airspeed'* $V_c$
- EAS *'Equivalent Airspeed'* $V_e$
- TAS *'True Airspeed'* $V$

basic calibration
(compressible flow at SL)

+ Instrument Error Correction

+ Pressure Error Correction
(position error in static reading)

+ Compressibility Correction
(altitude effect on M)

+ Density Correction
(altitude effect on density)

# Towards 1D Euler equation

- For the steady Euler equations the momentum equations can be written out in full as (cons. of mass has been substituted in)

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y}$$

$$\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}$$

- These three momentum equations can, in certain circumstances, be reduced to one equation which links pressure to velocity and density.

# Irrotational Flow

Assume irrotational flow i.e. vorticity (or elemental angular or rotational velocity) is zero.

$$\nabla \times \mathbf{V} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\boxed{\begin{aligned} \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} \\ \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} \\ \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} \end{aligned}} \quad \begin{aligned} &\times dx \\ &+ \\ &\times dy \\ &+ \\ &\times dz \end{aligned}$$

Substitute for highlighted terms

$$u = u(x, y, z)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

definition of derivative

$$udu = \frac{1}{2}d\left(u^2\right)$$

$$\frac{1}{2}d\left(u^2 + v^2 + w^2\right) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

# Alternative approach – if you're happy with 1D

$$\frac{\partial(\rho u^2 + p)}{\partial x} = 0$$ Momentum eq'n in 1D

$$\rho u \frac{\partial u}{\partial x} + u \frac{\partial(\rho u)}{\partial x} + \frac{\partial p}{\partial x} = 0$$ Expanding

$$\frac{\partial(\rho u)}{\partial x} = 0$$ Mass conservation in 1D

$$\rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0$$ Substitute mass conservation into momentum eq'n

If flow 1D, then we only have x direction, so partial notation can be dropped.

# Most compact approach

$$\rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p$$

Vector calculus identity

$$\boxed{\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) = \mathbf{V} \cdot \nabla \mathbf{V} + \mathbf{V} \times \nabla \times \mathbf{V}}$$

Option (a)

Zero if irrotational

Taking the dot product gives

$$\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) \cdot \mathbf{ds} = (\mathbf{V} \cdot \nabla \mathbf{V}) \cdot \mathbf{ds} + \underline{(\mathbf{V} \times \nabla \times \mathbf{V}) \cdot \mathbf{ds}}$$

Option (b)

Zero – bracket term is perpendicular to streamline by definition

We can now take

$$(\rho \mathbf{V} \cdot \nabla \mathbf{V}) \cdot \mathbf{ds} = -\nabla p \cdot \mathbf{ds}$$

to arrive at

$$\frac{\rho}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) \cdot \mathbf{ds} = -\nabla p \cdot \mathbf{ds}$$

Valid everywhere if flow irrotational, or only along a streamline otherwise

- Since $V^2 = \left(u^2 + v^2 + w^2\right)$ this is equivalent to

$$\boxed{dp = -\rho V\, dV}$$

*1D Euler equation*

This relationship still applies to

**compressible &**

**Incompressible flows**

As an aside, we can integrate this to get Bernoulli's equation if we assume the density is constant! Or in the compressible case we can substitute with $p = k\rho^{\gamma}$

$$\frac{1}{\rho} = \left(\frac{p}{k}\right)^{\frac{-1}{\gamma}} \longrightarrow \int \left(\frac{p}{k}\right)^{\frac{-1}{\gamma}} dp = \int -V dV$$

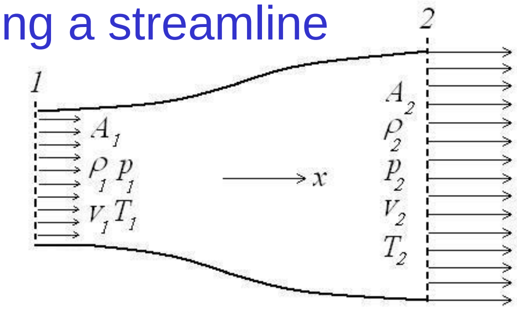
Integrate + set constant of integration so that p=p0 at stagnation

$$\left(\frac{1}{k}\right)^{\frac{-1}{\gamma}} p^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{\gamma-1} = -\frac{V^2}{2} + \left(\frac{1}{k}\right)^{\frac{-1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{\gamma-1} \longrightarrow \left(1 + \frac{M^2(\gamma-1)}{2}\right)^{\frac{\gamma}{\gamma-1}} = \frac{p_0}{p}$$

i.e. the isentropic flow relations again! Perhaps this explains the origins of `compressible Bernoulli'

# Back to work and …1D Isentropic Duct Flow

and flow along a streamline




- flow in a duct with *slowly* changing area $A$
  - angle between streamlines is 'small'
  - impact of boundary layer small (for turbulent flow)
- flow is therefore effectively one-dimensional "convergent divergent nozzles"
- assume inviscid, steady flow with negligible height variation
- momentum equation is the **1D Euler Equation**

$$dp = a^2 d\rho$$

$$dp \;=\; -\rho V\,dV \quad \Rightarrow \quad \boxed{\frac{dp}{\rho} + VdV \;=\; 0}$$

$$\frac{a^2 d\rho}{\rho} + VdV = 0$$

  - from previous work from handout 5.12

$$\frac{dV}{V} + \frac{dA}{A} - \frac{V^2}{a^2}\frac{dV}{V} = 0$$

- continuity equation given by derivative of mass flow

$$\dot{m} \;=\; \rho AV = \text{const} \quad \Rightarrow \quad \boxed{\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} \;=\; 0}$$

$$d\dot{m} = \rho AdV + \rho VdA + AVd\rho = d(\text{const}) = 0$$ then divide by $\rho AV$

# Area Velocity Variation (Adiabatic Flow)

- using Newton's result for $a$ then gives

from BC2.2 $$\frac{dp}{d\rho} = a^2$$ sub $dp = a^2 d\rho$ into 1D Euler $$\boxed{\frac{dA}{A} = \left(M^2 - 1\right)\frac{dV}{V}}$$ ADIABATIC

  – a direct relation between area variation and velocity variation

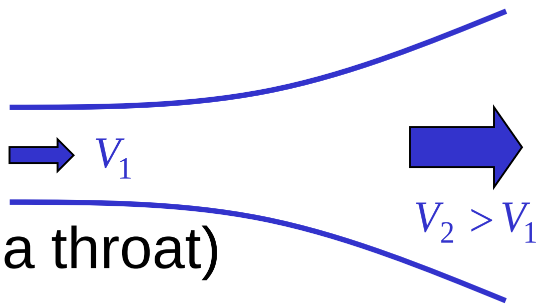
- now introduce the element of length along the duct $dx$

$$\boxed{\left(M^2 - 1\right)\frac{1}{V}\frac{dV}{dx} = \frac{1}{A}\frac{dA}{dx}}$$

or flow streamline

- consider an *accelerating* flow $dV/dx > 0$

  – $M < 1 \rightarrow dA/dx < 0 \rightarrow$ *converging* duct
  – $M > 1 \rightarrow dA/dx > 0 \rightarrow$ *diverging* duct (!)
  – $M = 1 \rightarrow dA/dx = 0 \rightarrow$ *minimum* area (ie a throat)



# Area Ratio (Isentropic Flow)

- sonic ($M$ = 1) duct flow can **only** occur at a throat – ie where the area $A_t$ is a local minimum
- since $M$ =1 here, this area is also the 'critical' area $A*$
- $A*$ is often used as a reference area in compressible flow
- use isentropic equations to relate local area $A$ andMach Number $M$ to $A*$

mass conservation $\rho AV = \rho^* A^* V^*$

$V = Ma \quad V^* = a^* \quad \Rightarrow \quad \rho AMa = \rho^* A^* a^*$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{a}\frac{1}{M}$$ then using

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho} \qquad \frac{a^*}{a} = \frac{a^*}{a_0}\frac{a_0}{a}$$

$$\frac{\rho^*}{\rho} = \left(\frac{1+\frac{\gamma-1}{2}.1^2}{1+\frac{\gamma-1}{2}M^2}\right)^{\frac{-1}{\gamma-1}} \quad and \quad \frac{a^*}{a} = \left(\frac{1+\frac{\gamma-1}{2}.1^2}{1+\frac{\gamma-1}{2}M^2}\right)^{\frac{-1}{2}}$$

using isentropic density/speed of sound relationships

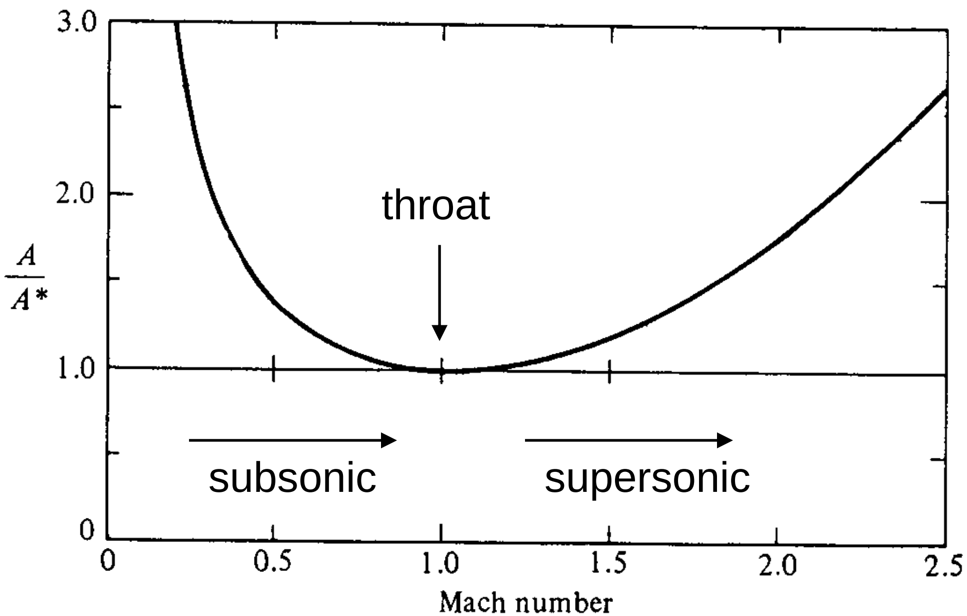
$$\boxed{\frac{A}{A^*} = \frac{1}{M}\left\{\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}}$$

2 solutions: $M>1, M<1$

- given in standard tables

# Area Ratio (Isentropic Flow)

- sonic ($M$ = 1) duct flow can **only** occur at a throat – ie where the area $A_t$ is a local minimum
- since $M$ =1 here, this area is also the 'critical' area $A^*$
- $A^*$ is often used as a reference area in compressible flow
  - use isentropic equations to relate local area $A$ and Mach Number $M$ to $A^*$
  - given in standard tables




$$\frac{A}{A^*} = \frac{1}{M}\left\{\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}$$

*ISENTROPIC*

$M=0.38$: area reduction to accelerate to sonic flow $(M=1)$ is $1/1.658$

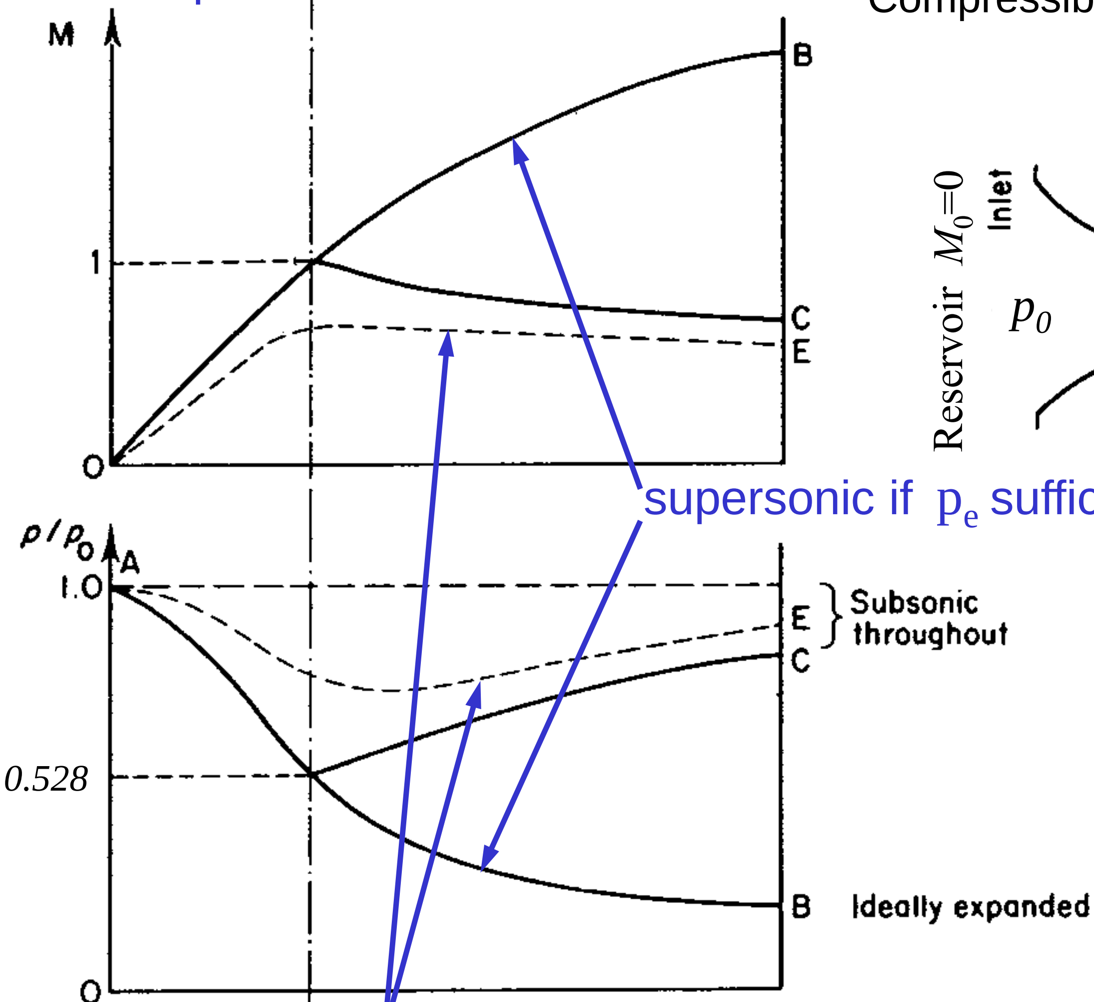
$M=1.98$: area reduction to decelerate to sonic flow $(M=1)$ is $1/1.658$

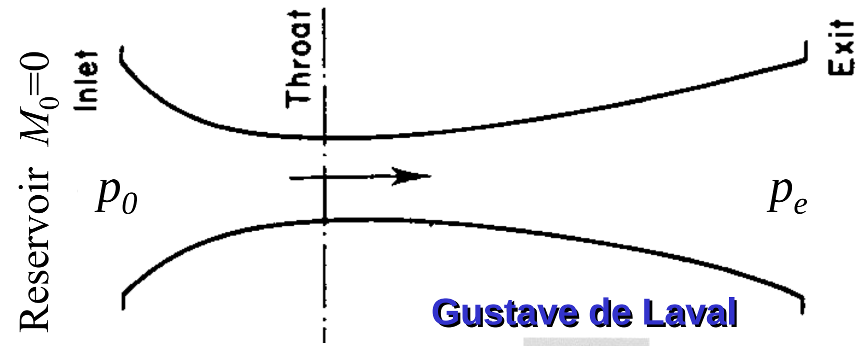
# Laval Nozzle

*19th* Century steam turbine designed using isentropic area ratio and ..

Infinitely many solutions depending on $p_e$ but only 2 isentropic solutions sonic at throat

Compressible Bernoulli $\quad p_0 = p\left\{1+\frac{\gamma-1}{2}M^2\right\}^{\frac{\gamma}{\gamma-1}}$




supersonic if $p_e$ sufficient

subsonic if $p_e$ insufficient

**Gustave de Laval**

*isentropic flow*

*isentropic flow not possible*

Shock system

in the nozzle

*isentropic flow* ***(in nozzle)***

but shock system outside the nozzle

# 'Choking'

- once M=1 reached at the throat further reductions in $p_e$ have **no** effect on subsonic flow upstream
  - no pressure 'information' can propagate past the throat
- therefore mass flow through duct also unaffected
  - sonic throat $\rightarrow$ duct is 'choked'
- mass flow can be written in non-dimensional form

$$\boxed{\frac{\dot{m}\sqrt{RT_0}}{Ap_0} = \sqrt{\gamma}\,\frac{p/p_0}{\sqrt{T/T_0}}\,M} \quad = fn(\gamma, M)\ only$$

$$m = \rho AV \qquad M = \frac{V}{a}$$

$$a^2 = \gamma RT \qquad p = \rho RT$$

and rearrange

  - with maximum value of 0.686 (for air) at $M$ = 1
- maximum (choked) mass flow is then

$$\dot{m}_{\max} = 0.686\frac{A_t p_0}{\sqrt{RT_0}} = 0.0404\frac{A_t p_0}{\sqrt{T_0}}$$

# 'Choking'

$$\dot{m} = \rho AV = \frac{p}{RT} AMa = \frac{p}{RT} AM\sqrt{\gamma RT}$$

$$\dot{m} = \sqrt{\frac{\gamma}{RT}} \frac{p}{p_0} p_0 AM = \sqrt{\frac{\gamma}{R\frac{T}{T_0}T_0}} \frac{p}{p_0} p_0 AM$$

For M=1 T/T0=0.833
p/p0=0.528

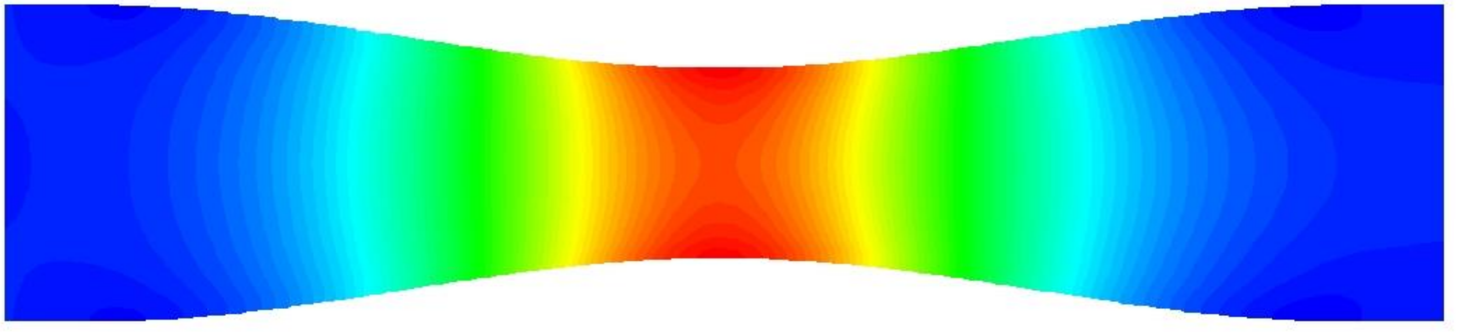
$$\frac{\dot{m}\sqrt{RT_0}}{Ap_0} = \sqrt{\frac{\gamma}{T/T_0}} \frac{p}{p_0} M = \sqrt{\frac{1.4}{0.833}} 0.528 \times 1 = 0.685$$

# Basic Compressible Flow Review

- review equations of motion of a fluid
  - conservation of energy – fundamental!
  - 1D Euler equation
  - need for thermodynamics in compressible flow – provides additional equations to close or simplify the problem
- review basic thermodynamic concepts
  - energy, enthalpy and entropy … you should know these from your 1st year
- speed of sound
  - propagation of information
- 1D compressible flow
  - ‘compressible Bernoulli’ – builds on energy equation with the isentropic assumption
- isentropic duct flows with varying area
  - critical conditions – possibility of choking

Fully subsonic
(flood plot shows Mach number)

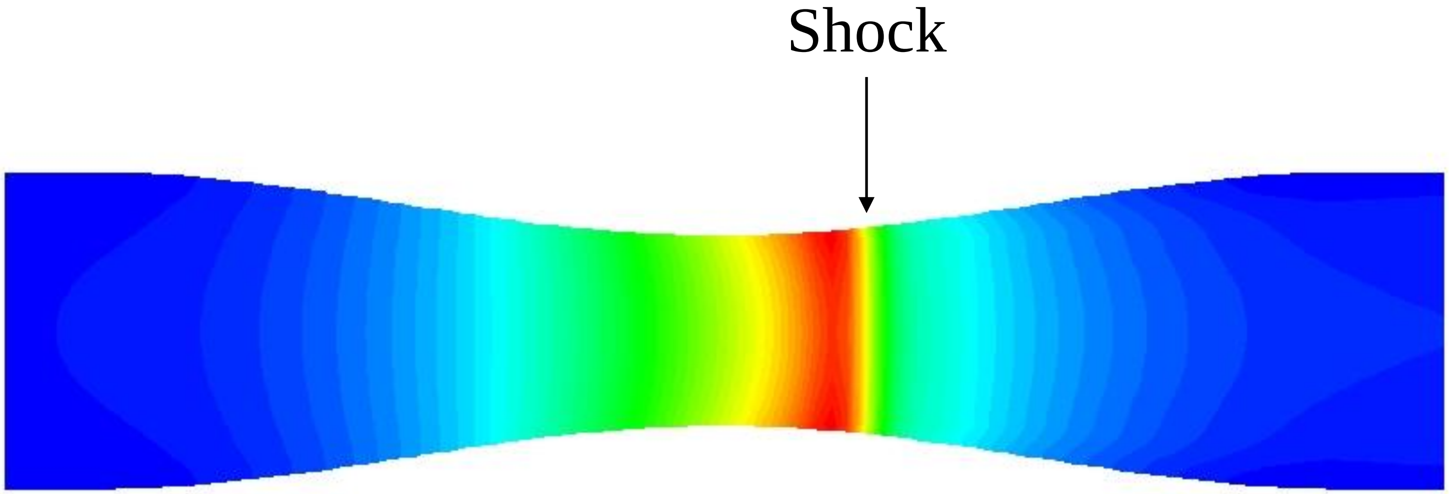
Here Athroat/Aexit=0.6
Remember Athroat is only = to A* if the nozzle is choked



Isentropic (and symmetric)

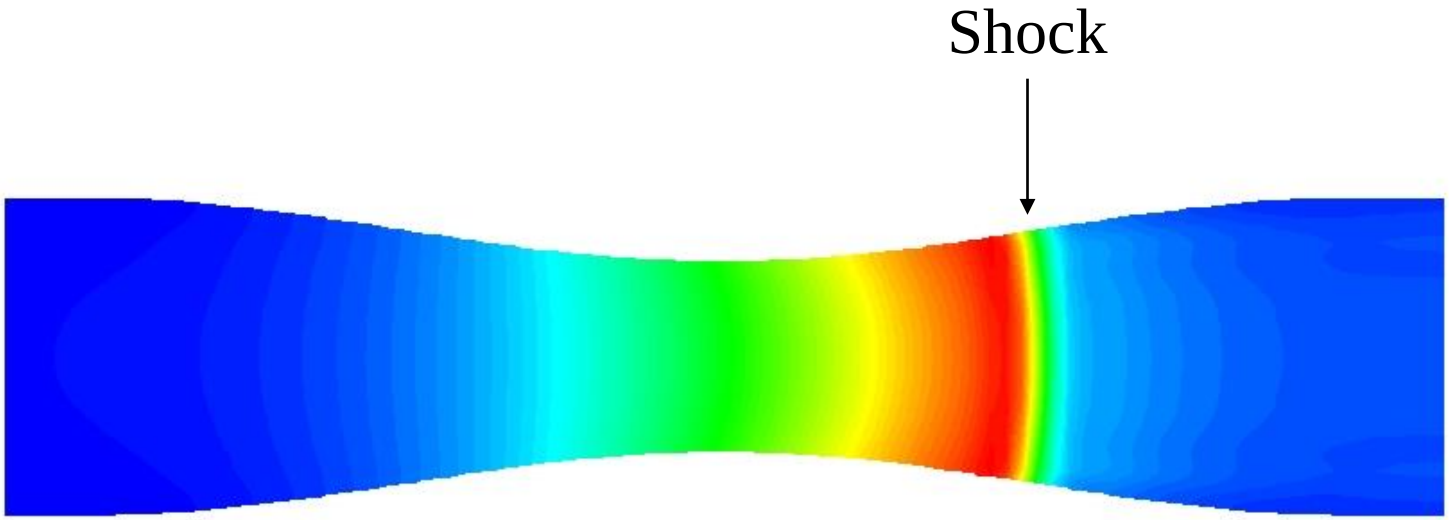
NB these are 2D CFD images, so not completely uniform across nozzle

# Reduce downstream pressure – now choked




Non-isentropic (and non-symmetric – mirroring does not produce a valid solution)

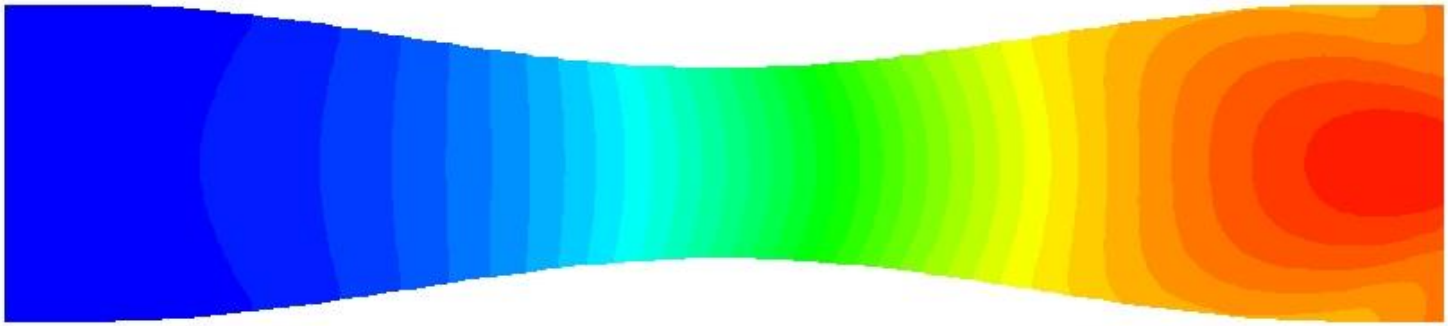
Reduce downstream pressure further – now shock moves further downstream




Non-isentropic (and non-symmetric – mirroring does not produce a valid solution)

Subsequent pressure drop downstream produces fully supersonic flow

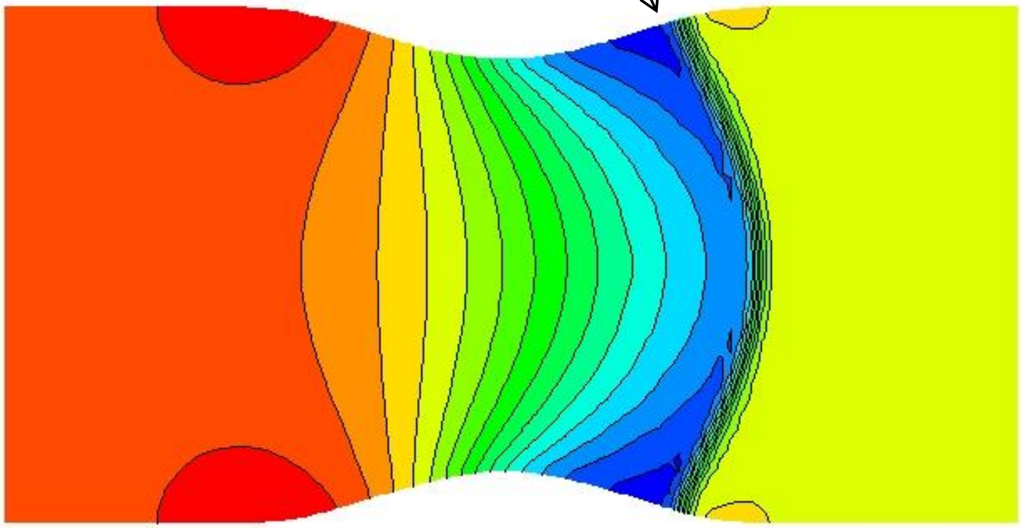
Shock system might exist at nozzle exit



Isentropic
(but may or may not remain isentropic downstream)

This case is non-symmetric, but a left-right mirror is also a valid solution, so a link between entropy and symmetry still holds

Next 3 lectures will focus on shockwaves



- You should now be able to attempt the 1st tutorial sheet!
- Tutorial session – in 2 weeks time?
- Make sure you attempt all questions before you come – this will clarify in your own mind what you know and what you need to practice.