Energy Methods for Pin-Jointed Trusses

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Energy Methods – What are they?

- **Energy methods** are alternatives to the 'conventional method' of solving forces and displacements under static equilibrium
  - Energy (positive scalar) is a function of both forces and displacements (which are vector fields)
- They enable the solution of forces and displacements in structures which cannot be analysed by **equations of static equilibrium alone** (*e.g.* statically indeterminate problems)
- Examples:
  - Castigliano's Theorems
  - Principle of Stationary Potential Energy (PSPE)
  - Principle of Virtual Work (Virtual Displacements or Virtual Forces)
  - *Etc.*

Solution Methods for Statics – Arguments

Conventional method:

- Equations of static equilibrium
- Compatibility of displacements
- Constitutive relations
  - Force-displacement $F = k\,e$ or stress-strain $\sigma = E\,\varepsilon$

Energy methods:

- Equations of static equilibrium
- Strain energy theorems

Energy Methods – Application

- Can be applied to pin-jointed trusses, beams, rigid-jointed frames

- Convenient means of analysing **statically indeterminate structures**

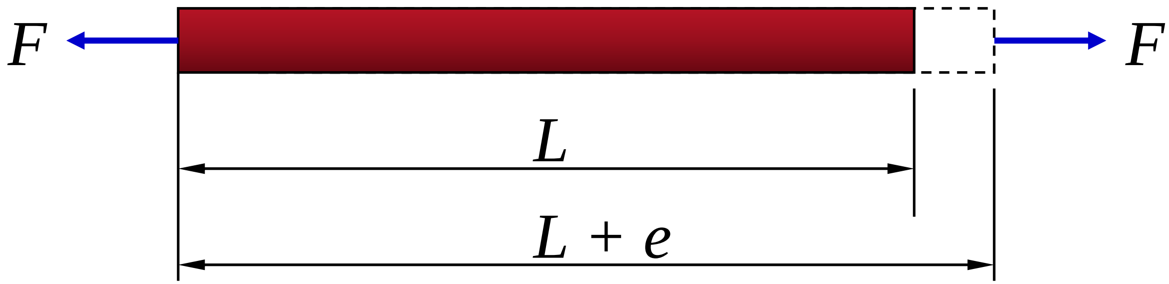
- Simple solution of deflections

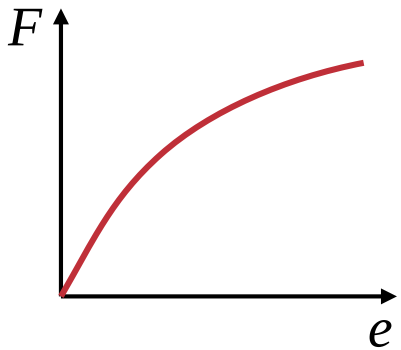
- Results in rapid approximate numerical solutions

  – For problems which do not have exact closed-form analytical solutions, *e.g.* complex geometries, nonlinearities *etc.*

Key Concepts – Strain Energy

- Consider an axially-loaded member:



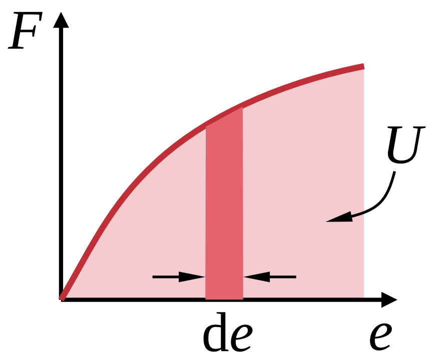


- Assuming a non-linear elastic material:

- Remember: work = force × distance

- For this 'nonlinear spring': $U = \int_0^e F \, \mathrm{d}e$

- *I.e.* strain energy = area **<u>under</u>** the F-e curve



Energy Methods
Castigliano's Theorems

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1. Introduction: Strain energy

2. Castigliano's theorems

    2.1 Key concepts

    2.2 Application: Displacement due to an external force *Q*

    2.3 Application: Displacements at any joint (virtual force method)

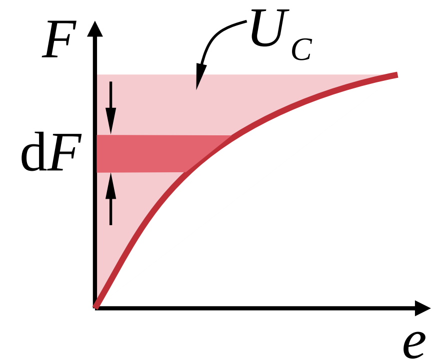
3. Principle of Stationary Potential Energy (PSPE)

    3.1 Key concepts

    3.2 Application: Internal forces in statically indeterminate trusses

Key Concepts – Complementary Strain Energy

- Consider now the area **above** the F-e curve:

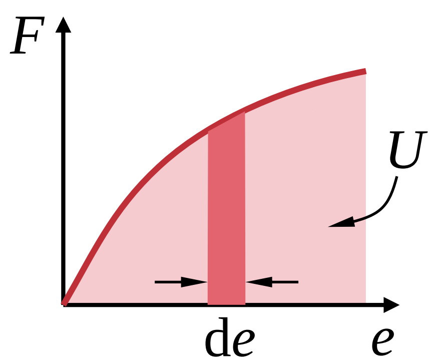


- This is the **complementary strain energy**:

$$U_C = \int_0^F e \, \mathrm{d}F$$

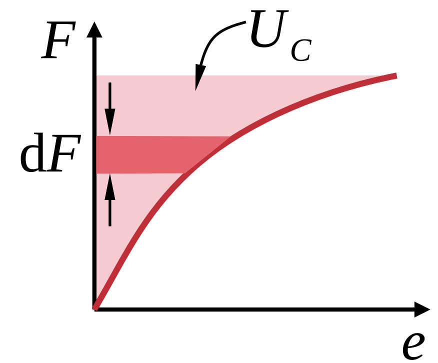
- Note that for a non-linear elastic material $U_C \neq U$

Strain Energy



$$U = \int_0^e F \, \mathrm{d}e$$

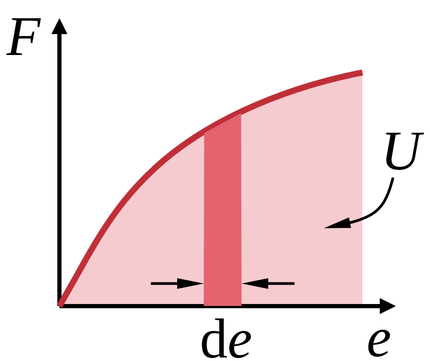
Complementary Strain Energy



$$U_C = \int_0^F e \, \mathrm{d}F$$

- Differentiating strain energy and complementary strain energy:

Strain Energy

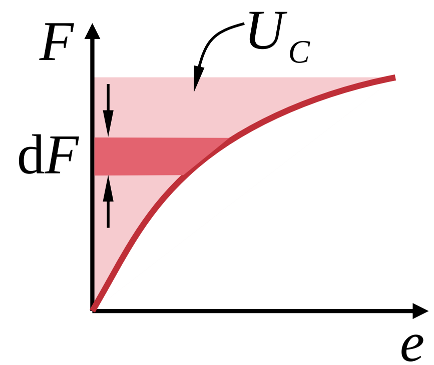


$$U = \int_0^e F \, \mathrm{d}e \longrightarrow \boxed{\frac{\partial U}{\partial e} = F}$$

The derivative of strain energy w.r.t. deflection gives the force causing this deflection

→ **Castigliano's 1st Theorem**

Complementary Strain Energy



$$U_C = \int_0^F e \, \mathrm{d}F \longrightarrow \boxed{\frac{\partial U_C}{\partial F} = e}$$

The derivative of complementary strain energy w.r.t. force gives the deflection at that particular point

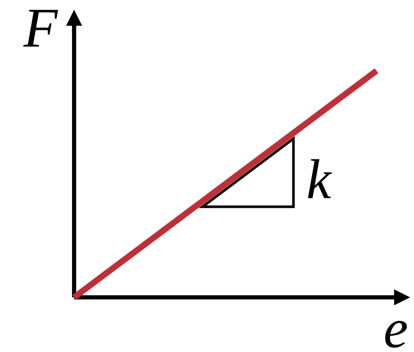
→ **Castigliano's 2nd Theorem**

- These theorems are after Carlo Alberto Castigliano (1847–1884, Milan) who proposed these in 1873 (at the age of 25)

Key Concepts – Complementary Strain Energy

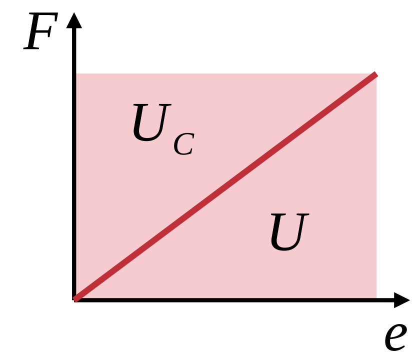
- Note that for a **linear elastic material**:

$$F = k\,e \;\therefore\; e = \frac{F}{k}$$



$$U = \int_0^e F\,\mathrm{d}e = \int_0^e k\,e\,\mathrm{d}e = \frac{k\,e^2}{2}$$

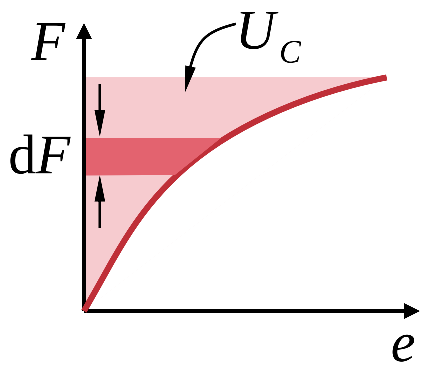
$$U_C = \int_0^F e\,\mathrm{d}F = \int_0^F \frac{F}{k}\,\mathrm{d}F = \frac{F^2}{2k} = \frac{k\,e^2}{2}$$



- Therefore **for linear elastic materials U_C=U**
- *i.e.* U_C and U become interchangeable

Castigliano's 2nd Theorem

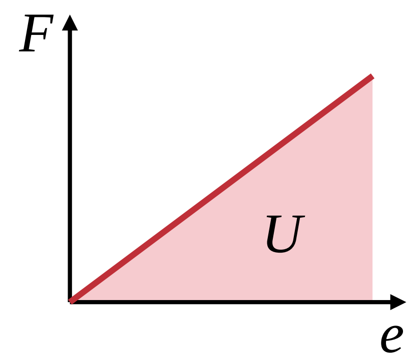
Complementary Strain Energy



$$U_C = \int_0^F e\,\mathrm{d}F \longrightarrow \boxed{\frac{\partial U_C}{\partial F} = e}$$

For a truss made of axial members:

$$\left(U_C\right)_i = \frac{1}{2}F_i\,e_i = \frac{1}{2}F_i\left(\frac{F_i}{k_i}\right) = \frac{1}{2}\frac{F_i^{\,2}}{k_i} \quad \therefore \quad U_C = \sum\left(\frac{1}{2}\frac{F_i^{\,2}}{k_i}\right)$$



$$\frac{\partial U_C}{\partial Q} = \sum\left(\frac{\partial U_C}{\partial F_i}\cdot\frac{\partial F_i}{\partial Q}\right) = e$$

Finally: $$\boxed{\frac{\partial U_C}{\partial Q} = \sum\left(\frac{F_i\,L_i}{A_i\,E_i}\cdot\frac{\partial F_i}{\partial Q}\right) = e}$$

where: $$\frac{\partial F_i}{\partial Q} = F_i^{'}$$

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Using Castigliano's 2nd Theorem

Most common problem: for a statically determinate or indeterminate truss where the internal forces F_i are **known**, find the **displacements** at the joint where the known external force Q is applied

1. Tabulate the values of F_i, L_i, A_i and E_i
2. Find the derivatives $F_i' = \frac{\partial F_i}{\partial Q}$
   – Since F_i and Q are constants, simply divide one by the other
3. Perform the summation $\sum \left(\frac{F_i L_i}{A_i E_i} \cdot \frac{\partial F_i}{\partial Q} \right) = e$ to find the displacement e
4. Note that the direction and sense of e are relative to the force vector Q !

Application of Castigliano's 2nd Theorem

KEY POINTS

General procedure: For a statically determinate or indeterminate truss where the internal forces F_i are **known**, find the **displacements at any joint**

1. Tabulate the values of F_i, L_i, A_i and E_i
2. Go back to the truss and temporarily remove the external force Q
3. Now introduce a unit force P (a.k.a. a **virtual force**) at the joint of interest and pointing in the direction of interest
4. Using the **method of joints** or **method of sections**, find the **virtual internal forces** F_i' due to the virtual force P
5. Perform the summation $\sum\left(\frac{F_i L_i}{A_i E_i}\cdot F_i'\right)=e_P$
   - The displacement e_P is the real displacement at the joint where the virtual force P was applied