Advanced Bending and Torsion

Transformation of Axes – Derivations

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23 October 2018

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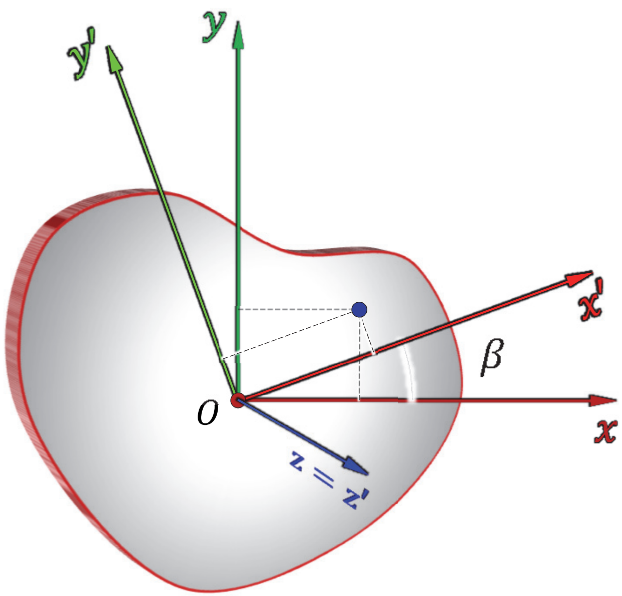
Transformation of Axes

- 2D orthogonal coordinates (x, y) can be rotated (transformed) about the out-of-plane axis (z) by any angle β by:

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$



- Where:

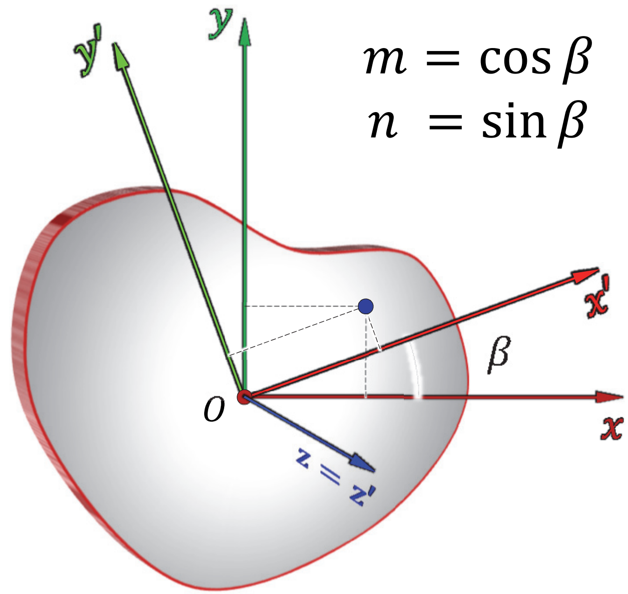
$$m = \cos\beta$$
$$n = \sin\beta$$

- Therefore:

$$x' = m\,x + n\,y$$

$$y' = -n\,x + m\,y$$

- So what happens to 2nd moments of area upon axis transformation?



$$m = \cos\beta$$
$$n = \sin\beta$$

- Transformed 2nd moments of area:

$$I_{x'x'} = \int y'^2 \, \mathrm{d}A$$

$$I_{x'x'} = \int (n\,y - m\,x)^2 \, \mathrm{d}A$$

$$I_{x'x'} = m^2 \int y^2 \, \mathrm{d}A + n^2 \int x^2 \, \mathrm{d}A - 2\,m\,n \int x\,y \, \mathrm{d}A$$

$$I_{x'x'} = m^2\,(I_{xx}) + n^2\,(I_{yy}) - 2\,m\,n\,(I_{xy})$$

- Similarly:

$$I_{y'y'} = m^2\,(I_{yy}) + n^2\,(I_{xx}) + 2\,m\,n\,(I_{xy})$$

$$I_{x'y'} = m\,n\,(I_{xx} - I_{yy}) + (m^2 - n^2)(I_{xy})$$

Transformation of Axes

- Transformed 2nd moments of area:

$$I_{x'x'} = m^2\,(I_{xx}) + n^2\,(I_{yy}) - 2\,m\,n\,(I_{xy})$$

$$I_{y'y'} = m^2\,(I_{yy}) + n^2\,(I_{xx}) + 2\,m\,n\,(I_{xy})$$

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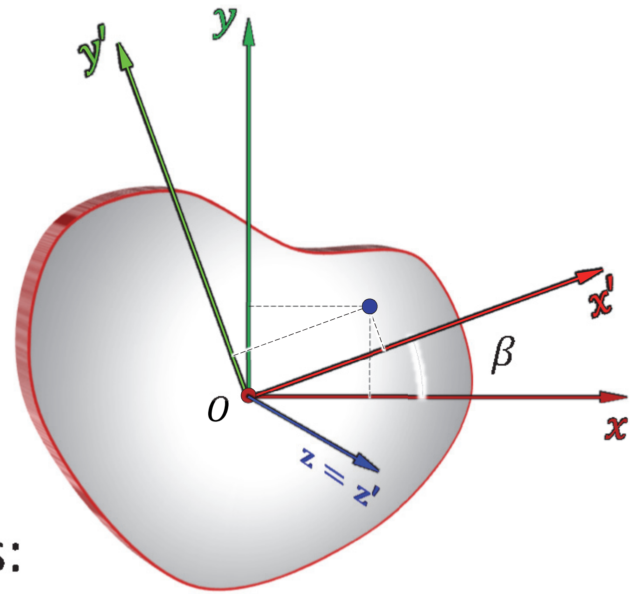


- These can be neatly written in matrix form as:

$$\begin{Bmatrix} I_{x'x'} \\ I_{y'y'} \\ I_{x'y'} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2\,m\,n \\ n^2 & m^2 & 2\,m\,n \\ m\,n & -m\,n & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} I_{xx} \\ I_{yy} \\ I_{xy} \end{Bmatrix}$$

- Where, again,

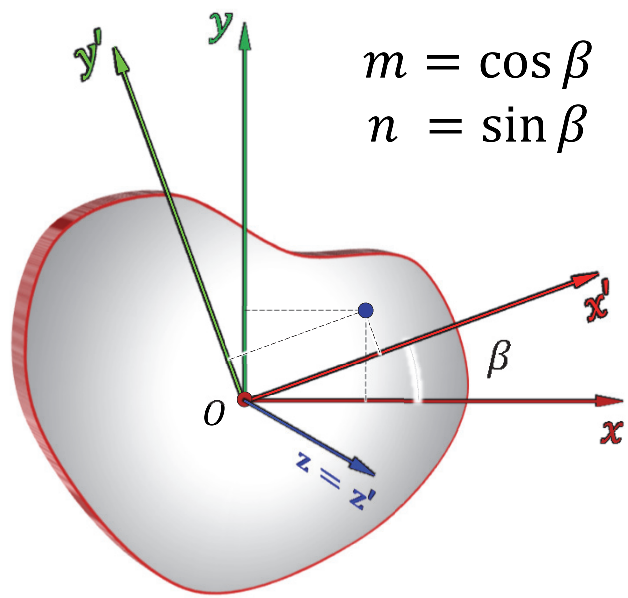
$$m = \cos\beta$$
$$n = \sin\beta$$

Transformation of Axes

$$I_{x'x'} = m^2\,(I_{xx}) + n^2\,(I_{yy}) - 2\,m\,n\,(I_{xy})$$

$$I_{y'y'} = m^2\,(I_{yy}) + n^2\,(I_{xx}) + 2\,m\,n\,(I_{xy})$$

$$I_{x'y'} = m\,n\,(I_{xx} - I_{yy}) + (m^2 - n^2)(I_{xy})$$



$$m = \cos\beta$$
$$n = \sin\beta$$

- An alternative way to express this transformation is by applying the following trigonometric identities:

$$\cos^2\beta = \frac{1 + \cos 2\beta}{2} \qquad \sin^2\beta = \frac{1 - \cos 2\beta}{2} \qquad 2\sin\beta\cos\beta = \sin 2\beta$$

- These give:

$$I_{x'x'} = \left(\frac{I_{xx} + I_{yy}}{2}\right) + \left(\frac{I_{xx} - I_{yy}}{2}\right)\cos 2\beta - (I_{xy})\sin 2\beta$$

$$I_{y'y'} = \left(\frac{I_{xx} + I_{yy}}{2}\right) - \left(\frac{I_{xx} - I_{yy}}{2}\right)\cos 2\beta + (I_{xy})\sin 2\beta$$

$$I_{x'y'} = \left(\frac{I_{xx} - I_{yy}}{2}\right)\sin 2\beta + (I_{xy})\cos 2\beta$$

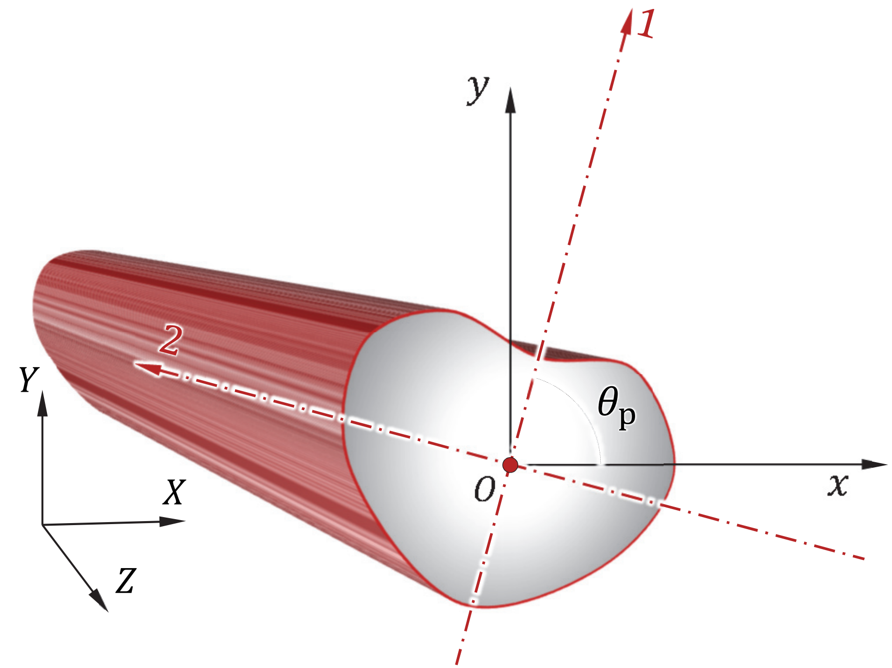
Principal Axes

By definition:

- The first principal 2nd moment of area, I_{11}, is the <u>maximum value</u> obtained when the reference frame is rotated

- Conversely, the second principal 2nd moment of area, I_{22}, is the <u>minimum value</u> that can be obtained

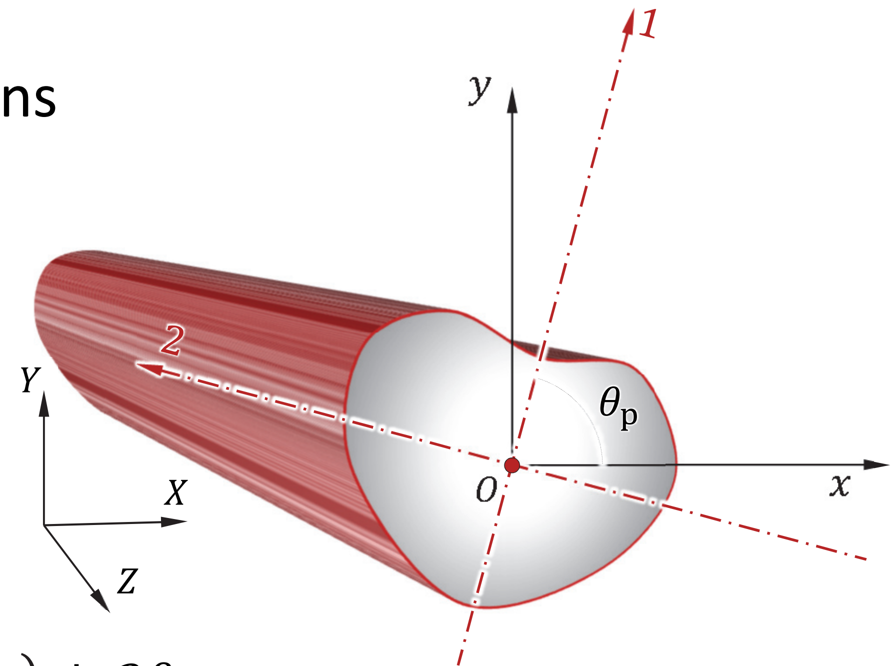
- The principal product 2nd moment of area, I_{12}, must be **<u>zero</u>**



Principal Axes

- Therefore we can find the angle θ_{p} which satisfies these three conditions by searching for a rotation angle β which gives the maximum $I_{x'x'}$ overall

- Remember: a maximum is characterised by a zero derivative:



$$I_{x'x'} = \left(\frac{I_{xx} + I_{yy}}{2}\right) + \left(\frac{I_{xx} - I_{yy}}{2}\right)\cos 2\beta - \left(I_{xy}\right)\sin 2\beta$$

$$\frac{\mathrm{d}\, I_{x'x'}}{\mathrm{d}\,\beta} = \frac{\mathrm{d}}{\mathrm{d}\,\beta}\left[\left(\frac{I_{xx} - I_{yy}}{2}\right)\cos 2\beta\right] - \frac{\mathrm{d}}{\mathrm{d}\,\beta}\left[\left(I_{xy}\right)\sin 2\beta\right] = 0$$

$$\frac{\mathrm{d}\, I_{x'x'}}{\mathrm{d}\,\beta} = -\left[\left(I_{xx} - I_{yy}\right)\sin 2\beta\right] - 2\left(I_{xy}\right)\cos 2\beta = 0$$

$$\tan 2\beta = \frac{2\, I_{xy}}{I_{xx} - I_{yy}}$$

$$\theta_{\mathrm{p}} = \frac{1}{2}\arctan\left(\frac{2\, I_{xy}}{I_{xx} - I_{yy}}\right)$$