ADDITIONAL BACKGROUND MATERIAL ON THE STREAM FUNCTION

AIMS FOR THIS DOCUMENT

- To give more detail on the stream function and explain its physical significance. (The stream function is introduced in lecture and handout on "Introduction to potential flow", this material is intended for self study).
- To show how the fluid velocities are related to the stream function.
- To examine the relationship between the stream function and potential and compare the limitations of each.

1 INTRODUCTION

Suppose that there is a known 2D compressible steady flow i.e. the velocity field is given by

$$u=f(x,y),\ v=g(x,y)$$

then in this case the equation for a streamline becomes

$$\frac{dy}{dx}=\frac{v}{u}=\frac{g(x,y)}{f(x,y)}$$

This is an exact (see for example Mathematics for engineers and scientists. A Jeffrey, p588-589) differential equation so can be integrated to obtain the equation of a streamline which can be written as

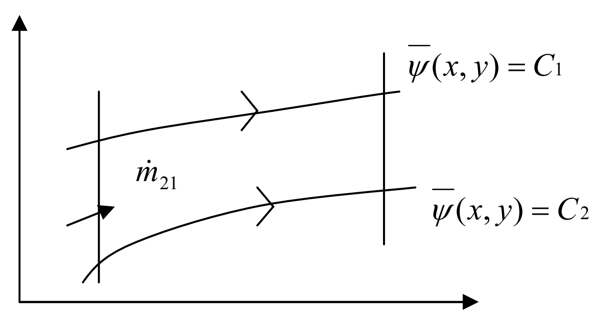
$$\overline{\psi}(x,y)=c$$

where *c* is an arbitrary constant of integration and

$$\overline{\psi}(x,y)$$

is a scalar quantity, called the *stream function*.

Now consider two streamlines defined by C1 and C2



$\dot{m}_{21}$ is the mass flow per unit depth

then since there is no flow across a streamline, the mass flow (per unit depth) between the streamlines is constant i.e.

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$$\dot{m}_{21} = \text{constant}$$

The arbitrary constant in the equation for each streamline is set so that

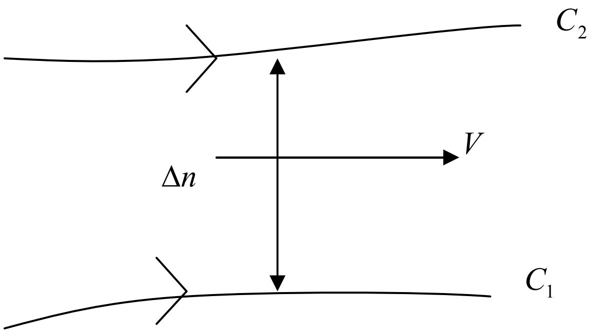
$$\dot{m}_{21} = \Delta\overline{\psi} = C_2 - C_1$$

So this means that the constant can be set arbitrarily for one streamline in the flow and the constants for the other streamlines is then fixed by the mass flow, so for example if $C_1 = 1$ then

$$C_2 = 1 + \dot{m}_{21}$$

1.1 Relationship to Velocities

Now consider the case when the two streamlines are a small distance Δn apart



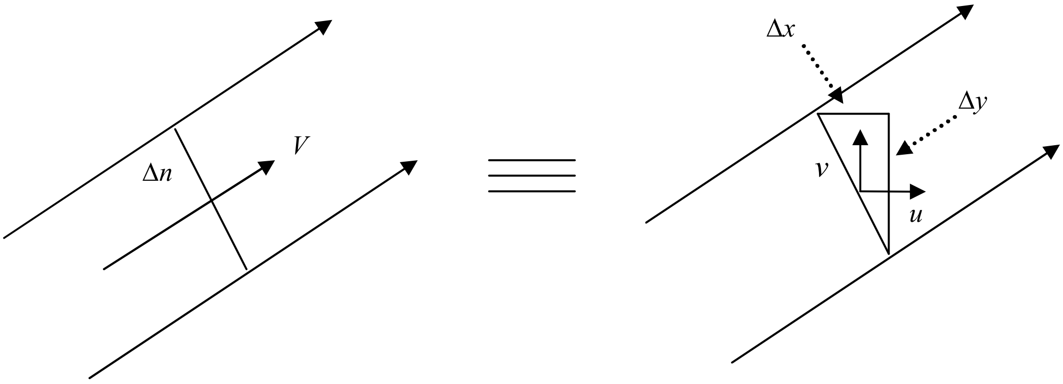
because Δn is small it can be assumed that the local velocity V is a constant so

$$\dot{m}_{21} = \rho V \Delta n = \Delta\overline{\psi}$$

and therefore in the limit

$$\rho V = \frac{\partial\overline{\psi}}{\partial n} \text{ as } \Delta n \to 0$$

So ρV is given by the derivative of the stream function in the direction normal to V. Now the flow velocity V can resolved into components in the Cartesian coordinate directions



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Rewriting the expression for the mass flow rate in terms of *u* and *v* gives

$$\rho V \Delta n = \rho u \Delta y + \rho v(-\Delta x) = \Delta \overline{\psi}$$

so in the limit as $\Delta n \to 0$ this yields

$$d\overline{\psi} = \rho u dy - \rho v dx$$

Now since it is known that

$$\overline{\psi} = \overline{\psi(x, y)}$$

another expression is available for $d\overline{\psi}$

$$d\overline{\psi} = \frac{\partial \overline{\psi}}{\partial x} dx + \frac{\partial \overline{\psi}}{\partial y} dy$$

and hence the velocity components are related to the stream function via

$$u = \frac{1}{\rho}\frac{\partial \overline{\psi}}{\partial y}, \quad v = -\frac{1}{\rho}\frac{\partial \overline{\psi}}{\partial x}$$

1.2 Incompressible Flow

For incompressible flow where ρ is constant it is usual to redefine the stream function as

$$\psi = \frac{1}{\rho}\overline{\psi}$$

so the relationship of this incompressible stream function to velocities is simpler

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

The equation of a stream line is then

$$\psi = c$$

For incompressible flow the increment in stream function between stream lines $\Delta \psi$ is now in fact the volume flow rate and the mass flow rate between stream lines is

$$\rho \Delta \psi$$

Comparing the equations relating the velocities to the stream function with the equations relating them to the 2D potential function

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$$

then clearly potential and stream function are related. In fact they are ORTHOGONAL FUNCTIONS i.e. are perpendicular to each other where they cross (except at stagnation points where *u*=*v*=0)

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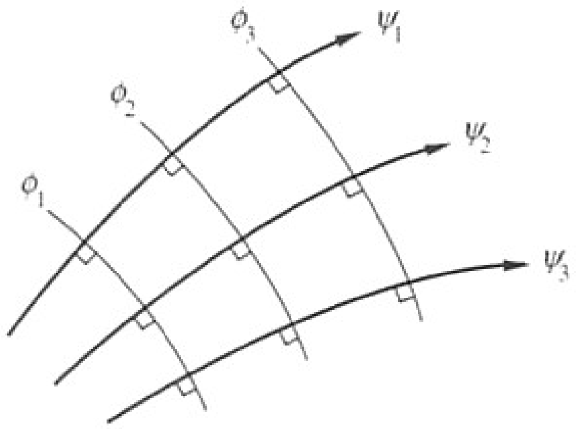

It is usually easier to visualise and draw streamlines than equi-potentials. One limitation of the stream function is that ψ is only defined in 2D, however it can be used for rotational flows. Compare this to the potential ϕ which can be used in 3D, but only exists for irrotational flows.

Both the stream function and the potential function are solutions of Laplace's equation

$$\nabla^2\psi = \nabla^2\phi = 0$$

To solve some problems it can be useful to combine the potential and stream function into a COMPLEX POTENTIAL

$$W(Z) = \phi + i\psi$$

REVISION OBJECTIVES

You should be able to:

- State that $\psi = c$ is an alternative definition of a streamline in incompressible flow.
- Explain the physical significance of the stream function
- Give expressions for u and v, the fluid velocity components in terms of ψ for an incompressible flow.
- State the limitations of stream function and the potential.

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