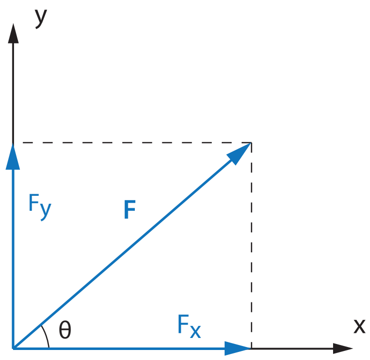
Handout 1 – Statics: Forces, Moments & Equilibrium

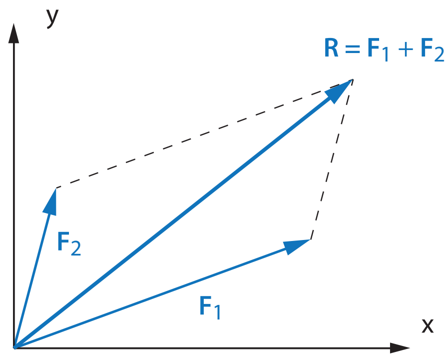
Meriam & Kraige, Statics: 2/1–2/8, 3/1–3/4

At the heart of mechanics is an understanding of forces and moments, and how they can be manipulated to help you solve problems. These are the core concepts that underpin mechanics and structures.

1.1 Forces

Forces can be classified as *contact* or *body* forces. The first are produced by direct physical contact between bodies, and the second by virtue of the position of a body in a force field (*e.g.* gravity, magnetic). A second distinction is between *distributed* or *concentrated* forces; often concentrated forces are an idealisation of a distributed force, for example contact or friction between objects.





Forces are vectors, and are defined by a *magnitude*, *direction*, and *point of application*. A force in two dimensions can be described as:

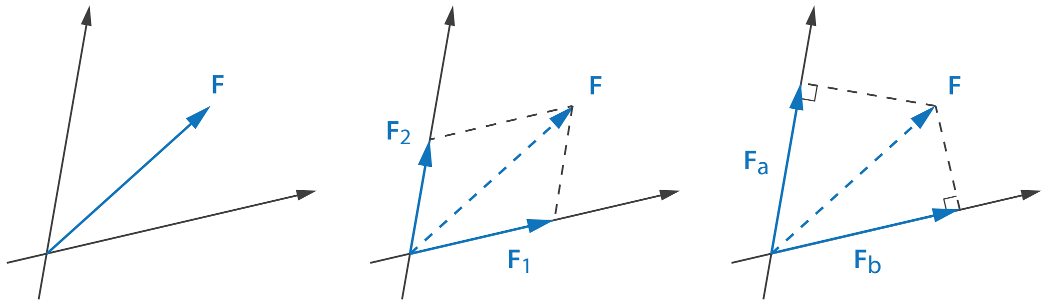
$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \tag{1.1}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in x, y directions. The magnitude of the force is:

$$F = |\boldsymbol{F}| = \sqrt{F_x^2 + F_y^2}$$

Forces are added up vectorially. In other words, the sum of the components of two forces is equal to the component of their resultant $\boldsymbol{R}$ in that direction.

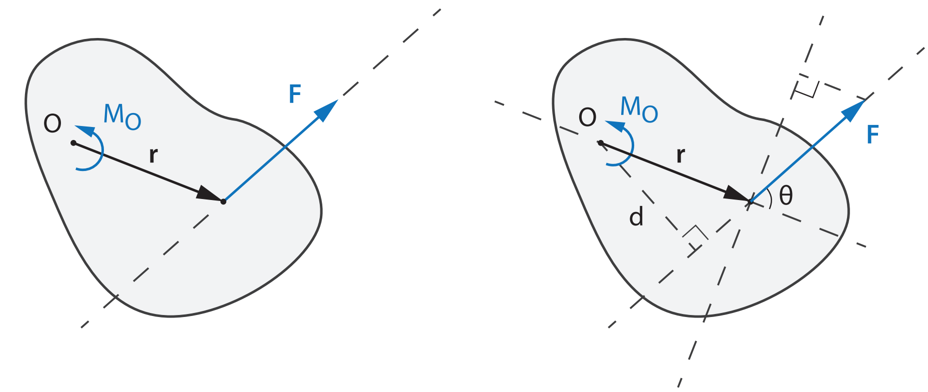
$$\begin{aligned} \boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F_2} &= (F_{1x} + F_{2x})\,\boldsymbol{i} + (F_{1y} + F_{2y})\,\boldsymbol{j} \\ &= R_x\boldsymbol{i} + R_y\boldsymbol{j} \end{aligned}$$



It is important to distinguish between *resolving* a force $\boldsymbol{R}$ into its vector components, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, along two specified axes, and their *projections* $\boldsymbol{F}_a$ and $\boldsymbol{F}_b$ along those axes. Only for an orthogonal coordinate system are the two equivalent.

1.2 Moments

A force tends to move a body in its direction of application, but may also cause a rotation. This is due to the **moment** of the force around a rotation axis. Consider a force $\boldsymbol{F}$ applied at a point with vector $\boldsymbol{r}$ from a point O.



The moment $\boldsymbol{M}_O$ about point O is given as:

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F} \tag{1.2}$$

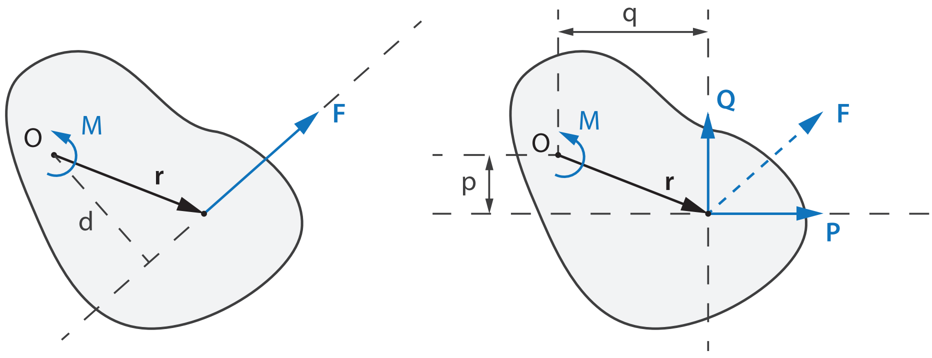
with a magnitude

$$M_O = Fd = |\boldsymbol{r}|\,|\boldsymbol{F}| \sin\theta \tag{1.3}$$

where d is the perpendicular (*i.e.* shortest) distance between point O and the line of action of force F.

Note that the *moment is a vector quantity* with magnitude and direction (using right-hand-rule). In 2D problems the moment vector is normal to the plane, and a CCW rotation is taken as positive.

Varignon's Theorem According to Varignon's Theorem, the moment of a force about a point is equal to the sum of the moments of the components of the force about that point.



The moment $\boldsymbol{M}_0$ is given as

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F}$$

where

$$\boldsymbol{F} = \boldsymbol{P} + \boldsymbol{Q}$$

Therefore

$$\begin{aligned}\boldsymbol{M}_O &= \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times (\boldsymbol{P} + \boldsymbol{Q}) \\ &= \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{r} \times \boldsymbol{Q}\end{aligned}$$

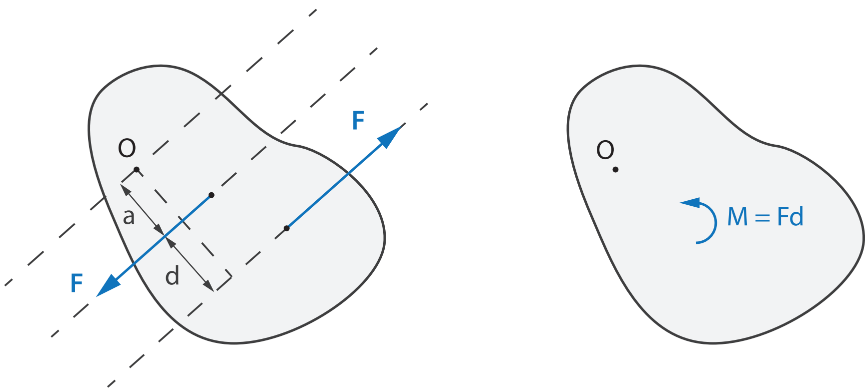
and the magnitude of the moment is found as

$$M_O = F\,d = P\,p + Q\,q$$

Using Varignon's Theorem will often greatly speed up your analysis, by resolving the force into its components in a convenient coordinate system, before calculating the moments.

1.3 Couples

Consider the special case of two equal, opposite and non-collinear forces, which forms a **couple**.



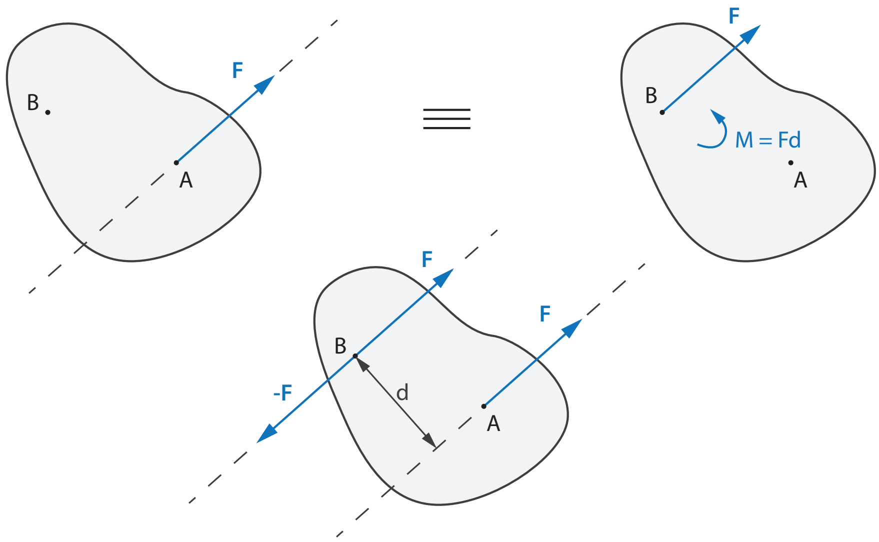
The moment around point O is given as

$$M = F\,(a + d) - F a = F d$$

This means that a couple has the same value for *any* moment centre, and is therefore a *free* vector. This means that it will have the same effect on the body, regardless of point of application. A couple is unchanged as long as its magnitude and direction remain constant, but the ratio of F/d can vary.

1.4 Force-Couple System

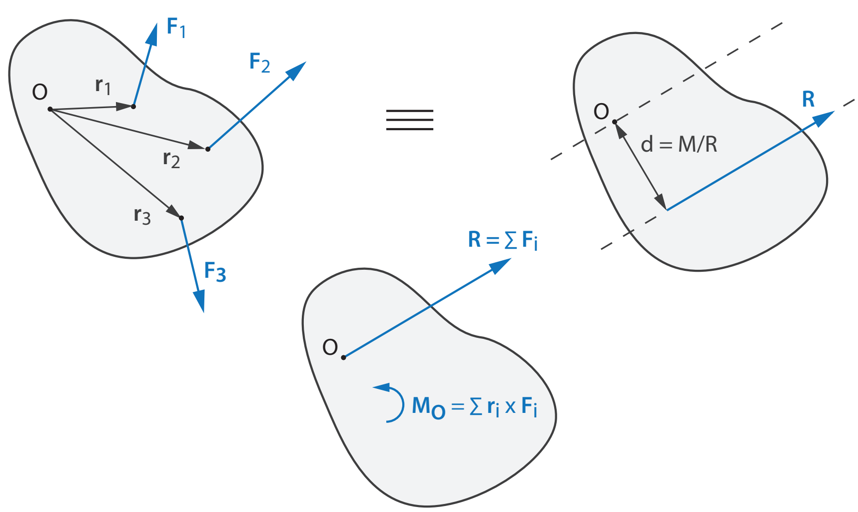
The effect of a force on a body will be a combination of translation (parallel to the direction of force) and a rotation about a fixed axis which does not intersect the line of the force. A force can be replaced by a combination of an equal parallel force, and a couple to compensate for the change in the moment of the force.



This will become critical in dynamics, where a body will rotate about is centre of mass, regardless where the force is applied. The force is therefore moved to the centre of mass, and the corresponding couple is used to calculate the rotation.

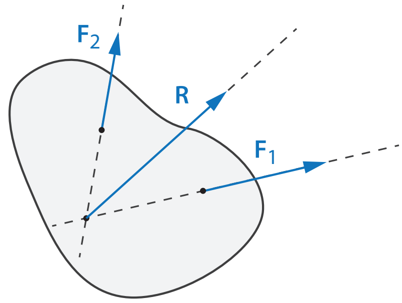
1.5 Resultant

The resultant is the simplest force combination that replaces the original set of forces acting on the rigid body.



First, choose a convenient reference point O, and move the resultant of the forces $\boldsymbol{R} = \sum \boldsymbol{F}$ to that point and add up all the moments around that point $M_O = \sum \boldsymbol{r}_i \times \boldsymbol{F}_i$. This gives a single force-couple system. Next find the line of action of the resultant which produces the same moment about point O; this will be parallel to the resultant force $\boldsymbol{R}$ and at a distance $d = M_O/R$.

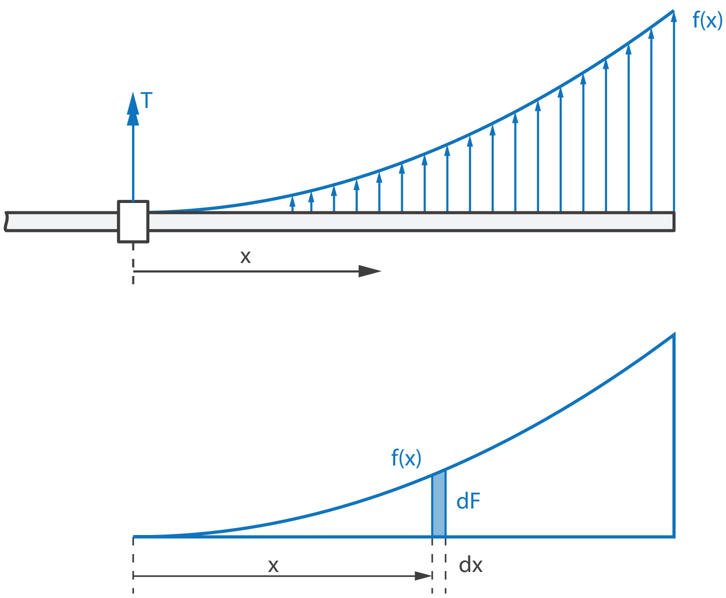
Alternatively, the resultant of two forces can be found, by placing it at the point of intersection of the two lines of action (about which they produce zero moment).



Note that for rigid bodies the points of application of forces can be changed, as long as it represents an equivalent resultant force and moment. However, the *internal* stress distribution, and resulting deformations, will be affected!

1.6 Distributed Loads

In many engineering applications one finds examples of a distributed load, where $f(x)$. For instance, consider a helicopter rotor with a constant cross-section. The aerodynamic lift force can be assumed proportional to the square of the relative air speed, $L \propto v^2$. The relative air speed along the rotor is proportional to the distance x from the hub and the angular velocity ω of the rotor: $v = \omega x$. This means that the lift force $f(x)$ will vary quadratically along the length of the rotor. (My colleagues in aerodynamics and rotorcraft dynamics will shudder at this simplification...)

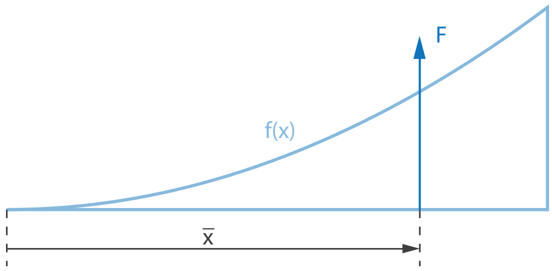


The total resultant force F for a distributed load $f(x)$ is found through integration:

$$F = \int f(x)\,dx \tag{1.4}$$

and the resulting moment M around $x = 0$ is

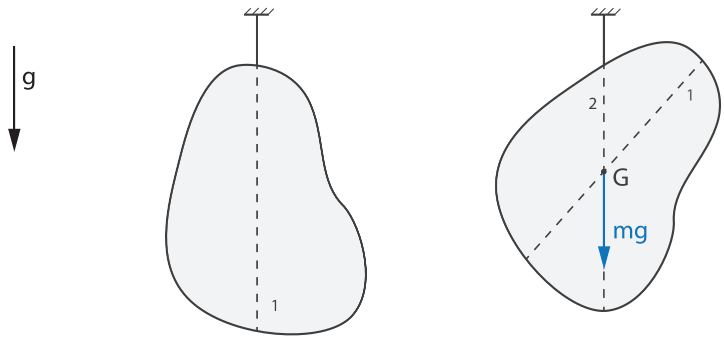
$$M = \int x\,f(x)\,dx = F\,\bar{x} \tag{1.5}$$



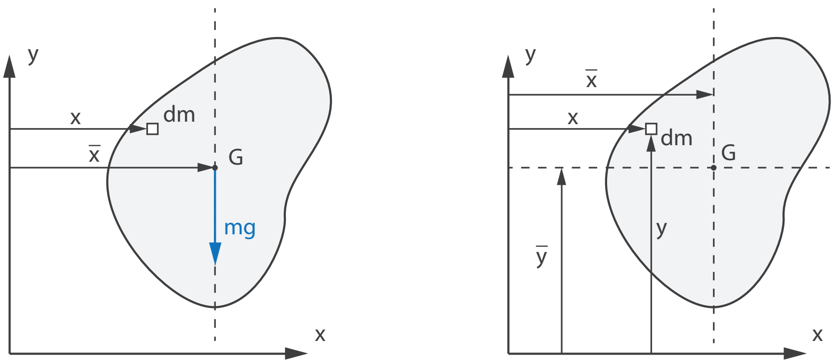
This allows us to replace the distributed load by a single resultant force F at a distance $\bar{x} = M/F$. Although the two load cases are statically equivalent, the internal stress distribution in the rotor will differ significantly!

1.7 Centre of Mass

The most common example of a distributed load is weight due to gravity. The **centre of mass** is the average position of the mass of a rigid body, and is the point where the resultant gravity force acts. Experimentally, the centre of mass of a planar object can be found by suspending the object from two points: the centre of mass will lie along the vertical from the suspension point. Choosing two (arbitrary) suspension points will thus determine the centre of mass, G.



Mathematically, the centre of mass (more precisely here, centre of gravity) is found by determining the line of action of the resultant gravity force, by looking at the distributed load dm.



The centre of mass is therefore found as:

$$\bar{x} = \frac{\int x\, dm}{m} \qquad \bar{y} = \frac{\int y\, dm}{m} \tag{1.6}$$

Note that for a coordinate system located at the centre of mass, these integrals must by definition be zero.

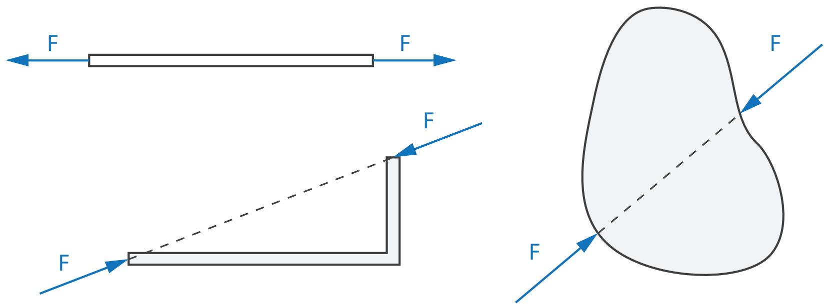
1.8 Static Equilibrium

The discussion of forces and moments has laid the ground work for the core concept of statics: **static equilibrium**. In statics a body or series of bodies is assumed to be in equilibrium, when the net sum of forces and moments is zero.

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M = 0$$

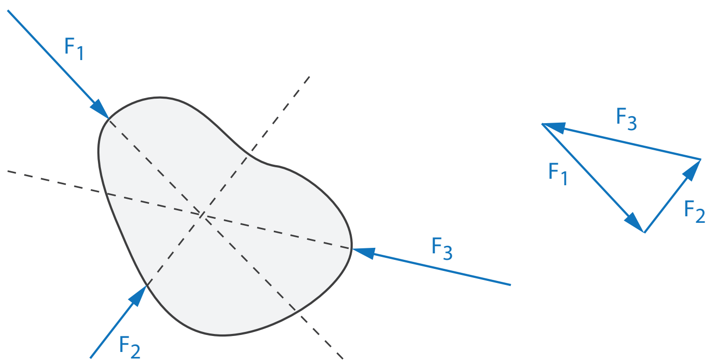
The three equilibrium equations (for planar problems) also imply that there may be no more than three unknowns; a problem where there are more unknowns is called *statically indeterminate* and cannot be solved using statics alone.

Two and Three Force Members Often members will only have two or three external forces acting on it. In these cases, some useful observations can be made. First consider a two force member, where it can be seen that the two forces must be equal, opposite and collinear to satisfy equilibrium.



A common example of such members is in trusses, where each member only carries pure tensile or compressive loads; such truss structures will be studied in greater detail in your structures unit.

In the case of three-force members, it can be seen that the lines of action of all three forces must pass through a single point, in order to satisfy moment equilibrium.



In addition, the force equilibrium must still be satisfied. This can be visualised using a polygon of forces. For static equilibrium, the applied forces must form a closed polygon: all force components in x and y direction add up to zero.

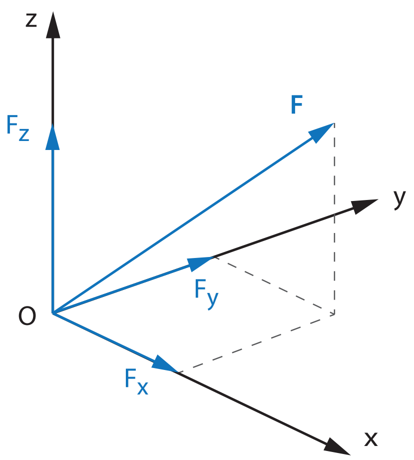
1.9 Forces, Moments and Equilibrium - 3D

The extension of **forces** to three dimensions is straightforward:

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \tag{1.7}$$

where magnitude of the force is

$$F = |\boldsymbol{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}$$



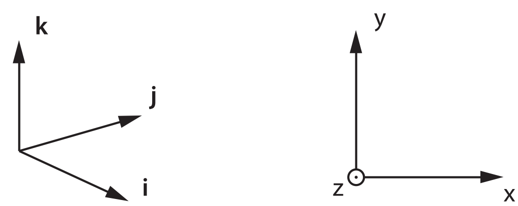
For **moments**, the formulation is more involved, but follows the same format as the 2D case, with:

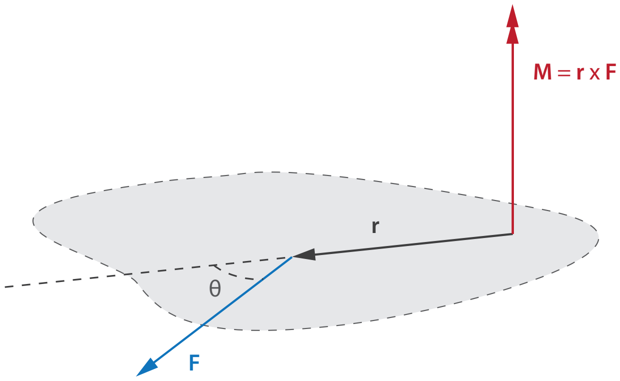
$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{r} \times \boldsymbol{F} \\ &= (r_x\boldsymbol{i} + r_y\boldsymbol{j} + r_z\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}) \\ &= r_xF_y\boldsymbol{k} - r_xF_z\boldsymbol{j} - r_yF_x\boldsymbol{k} + r_yF_z\boldsymbol{i} + r_zF_x\boldsymbol{j} - r_zF_y\boldsymbol{i} \\ &= (r_yF_z - r_zF_y)\,\boldsymbol{i} + (r_zF_x - r_xF_z)\,\boldsymbol{j} + (r_xF_y - r_yF_x)\,\boldsymbol{k} \end{aligned} \tag{1.8}$$

where use is made of:

$$\begin{array}{lll} \boldsymbol{i} \times \boldsymbol{i} = 0 & \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} & \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j} \\ \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k} & \boldsymbol{j} \times \boldsymbol{j} = 0 & \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \\ \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j} & \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i} & \boldsymbol{k} \times \boldsymbol{k} = 0 \end{array}$$

using the right-hand rule.





Note that the resulting moment axis is perpendicular to $\boldsymbol{r}$ and $\boldsymbol{F}$. Often one would like to find the moment of a force about a specific axis. If the axis λ is defined by a unit vector $\boldsymbol{n}$, the moment around this axis is found as:

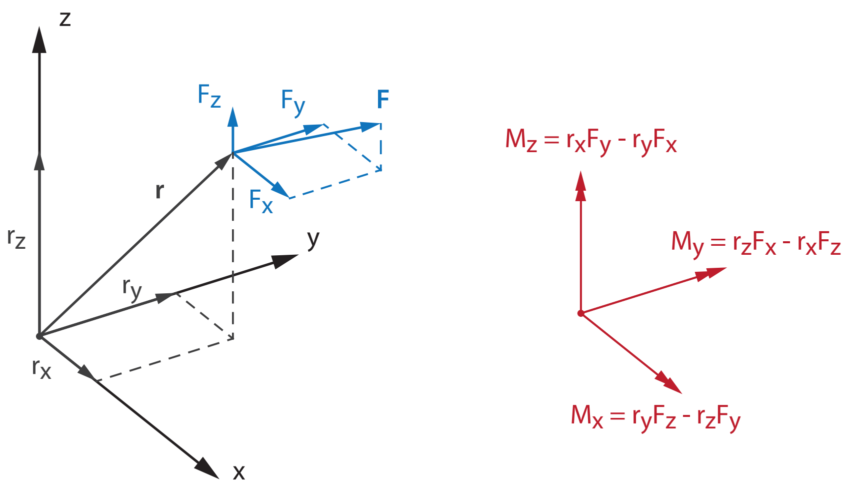
$$\boldsymbol{M}_\lambda = (\boldsymbol{r} \times \boldsymbol{F} \cdot \boldsymbol{n})\,\boldsymbol{n}$$

where the moment is projected onto the unit vector $\boldsymbol{n}$ using the dot product.

The expression for

$$\boldsymbol{M} = \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \begin{pmatrix} r_yF_z - r_zF_y \\ r_zF_x - r_xF_z \\ r_xF_y - r_yF_x \end{pmatrix} \tag{1.9}$$

can also be derived using Varignon's theorem.



For **static equilibrium** in 3D, the following six equations must be satisfied:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

1.10 Free Body Diagram

Perhaps the single most important concept of this course is the **Free Body Diagram** (FBD), and it forms the crucial first step of solving any statics and dynamics problem. To apply the equilibrium conditions (and in dynamics, equations of motion) the system must be isolated, and all forces acting on it must be represented.

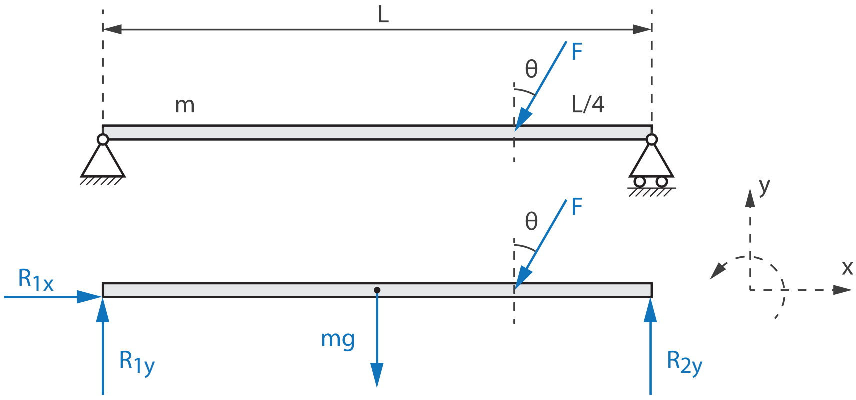
Steps for constructing an FBD:

1. decide which system to isolate;
2. isolate the chosen system by drawing a diagram which represents its complete external boundary; the boundary defines the isolation of the system from all other bodies, which are considered removed;
3. identify all forces acting on the isolated system as applied by the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system; use vector arrows, and if sense is unknown (+/-) assign arbitrarily;
4. show the choice of coordinate system on the diagram

The steps for a Free Body Diagram are deceptively simple, but often this is where the critical mistakes are made in mechanics problems!

Example 1.1 – Simply Supported Beam

Consider a beam of length L and mass m, with a point load F at an angle θ to the vertical acting at $L/4$ from the right-hand support.



What are the support reactions? We write the equilibrium equations:

$$\sum F_x: \quad R_{1x} - F\sin\theta = 0$$

$$\sum F_y: \quad R_{1y} + R_{2y} - F\cos\theta - mg = 0$$

$$\sum M: \quad LR_{2y} - \frac{L}{2}mg - \frac{3}{4}LF\cos\theta = 0$$

where moments were taken about the left-hand support. These equations can be solved by substitution:

$$R_{1x} = F\sin\theta$$

$$R_{1y} = \frac{1}{2}mg + \frac{1}{4}F\cos\theta$$

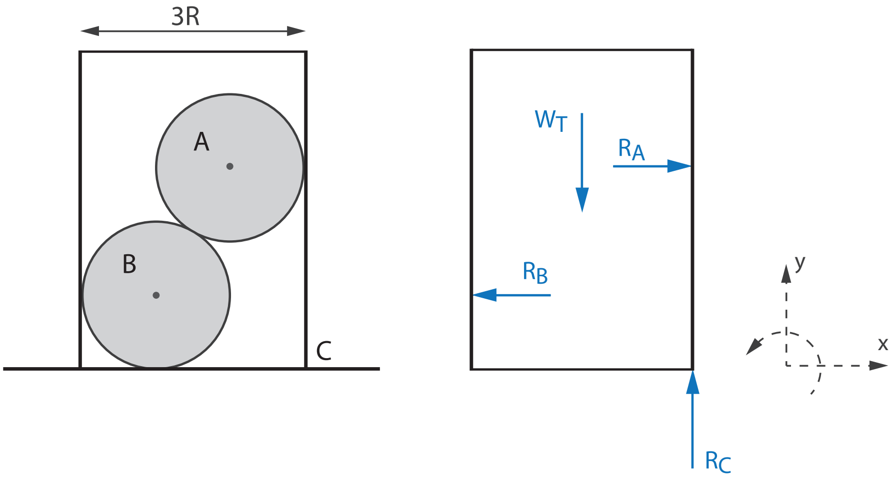
$$R_{2y} = \frac{1}{2}mg + \frac{3}{4}F\cos\theta$$

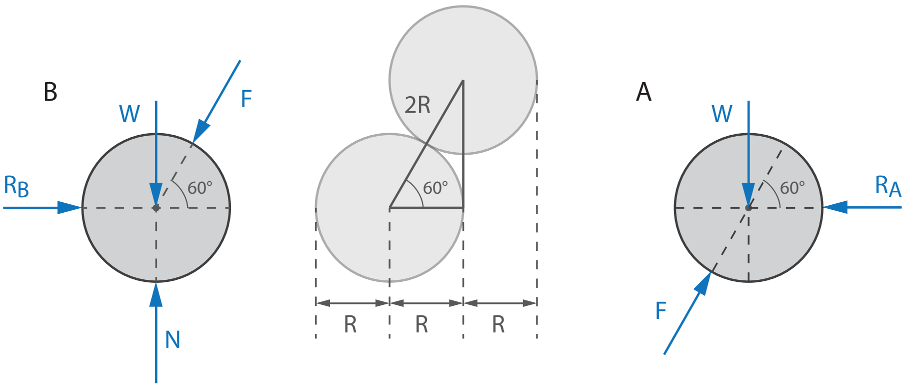
Q: What would happen if the right-hand support had not been allowed to move freely along the x-axis?

A: It would have introduced a further reaction force, R_{2x}, and raised the total number of unknowns to four. With only three equilibrium equations, the problem becomes *statically indeterminate* and cannot be solved with statics alone. It requires knowledge of the material properties of the beam, to provide the additional equation, and is therefore the topic of structures.

Example 1.2 – Spheres in Cylinder

Two smooth spheres of radius R and weight W are at rest inside a smooth vertical cylindrical tube of diameter $3R$ which has open ends. The spheres and tube are supported on a smooth horizontal surface. Find the contact force N between the bottom sphere and the table, and find the minimum weight of the tube to prevent it from toppling over; assume it is rotating around point C.





To find the contact force N, start with the FBD for sphere B. The contact force F between the two spheres must be perpendicular to the point of contact of the two cylinders, and the line of action must therefore pass through the centre of the sphere; angle follows from geometry as 60°.

Note that the moment equilibrium is automatically satisfied (Why?) and thus:

$$\sum F_x : \qquad R_B - \frac{1}{2}F = 0$$

$$\sum F_y : \qquad N - W - \frac{\sqrt{3}}{2}F = 0$$

With three unknowns in two equations this cannot be solved yet. Next, consider the equilibrium of sphere A:

$$\sum F_x : \qquad \frac{1}{2}F - R_A = 0$$

$$\sum F_y : \qquad \frac{\sqrt{3}}{2}F - W = 0$$

These equations can be solved to find:

$$R_A = R_B = \frac{W}{\sqrt{3}}$$

$$F = \frac{2W}{\sqrt{3}}$$

$$N = 2W$$

Lastly, consider the FBD of the cylinder at the point of tipping. From vertical equilibrium $R_C = W_T$, and the moment equilibrium is written as:

$$\sum M_C : \qquad W_T\,\frac{3}{2}R - \frac{W}{\sqrt{3}}\,R\sqrt{3} = 0$$

which gives the required weight of the cylinder $W_T = \frac{2}{3}W$.

That was a lot of work... Is there a faster approach to find the requested answers?

Revision Objectives Handout 1:

Forces & Moments

- combine force vectors to find a resultant force, graphically and mathematically
- resolve force vectors into two (orthogonal) components, graphically and mathematically
- express and manipulate force vectors in i, j, k notation
- calculate the moment of a force in 2D and 3D, using vector notation and Varignon's Theorem
- recognise that a moment is a vector (and thus has direction and magnitude)
- determine the resultant force-couple system of a set of forces acting on a 2D rigid body
- determine centre of mass for simple planar geometries
- calculate resultant force (magnitude and position) for a distributed load

Static Equilibrium

- isolate appropriate bodies in multi-body systems and draw a Free Body Diagram (FBD)
- apply static equilibrium equations to 2D and 3D problems to calculate reaction forces
- recognise two-force and three-force systems, and exploit this observation

Appendix – Greek characters

The Greek alphabet is commonly used for notations. Here a brief reminder of the characters, and some of their common uses in mechanics and structures.

α	alpha	$\alpha = \ddot{\theta}$ denotes angular acceleration
β	beta	
Γ, γ	gamma	γ denotes shear strain
Δ, δ	delta	
ϵ, ε	epsilon	ε denotes strain
ζ	zeta	
η	eta	
Θ, θ	theta	θ denotes angle
ι	iota	
κ	kappa	κ denotes curvature
Λ, λ	lambda	
μ	mu	μ denotes friction coefficient
ν	nu	ν denotes Poisson's ratio
Ξ, ξ	xi	
o	omicron	
Π, π	pi	
ρ	rho	ρ denotes density
Σ, σ	sigma	σ denotes stress
τ	tau	τ denotes shear stress
Υ, υ	upsilon	
Φ, ϕ, φ	phi	φ denotes angle
χ	chi	
Ψ, ψ	psi	
Ω, ω	omega	$\omega = \dot{\theta}$ denotes angular velocity