Integrating factors

Lecture 6: Integrating factors

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Recap

Last time ...

Things to try (in this order)

- **Solve by inspection**, if you can ...
- ODEs that don't explicitly depend on the dependent variable (and do not contain derivatives of different orders)
  Solve by direct integration
- Linear homogeneous first-order ODEs
  Solve by separation of variables
- Linear homogeneous ODEs with constant coefficients
  Use exponential ansatz

Today: How to solve non-homogeneous first-order ODEs

Solving ODEs: Integrating Factors

Introductory example

Sometimes ODEs become easier if we multiply them by a factor.

Consider for instance

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{x}{t} = t^2$$

This is a linear non-homogeneous first-order ODE .
Nice things happen when we multiply by t

$$\underbrace{t\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x}_{\frac{\mathrm{d}}{\mathrm{d}\,t}(tx)} = t^3$$

Solving ODEs: Integrating Factors

We can thus write this ODE as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(tx) = t^3$$

and solve it by direct integration

$$tx = \frac{1}{4}t^4 + C$$

or explicitly

$$x(t) = \frac{1}{4}t^3 + \frac{C}{t}$$

Solving ODEs: Integrating Factors

Exact Equations

So why were we able to solve this equation?

The key step was realising that the equation could be rewritten such that all terms that contained the dependent variable or its derivatives were absorbed into one big derivative

$$t\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\,t}(tx)$$

Equations for which this is possible are called **exact equations**.
This is basically a total derivative in reverse.

Solving ODEs: Integrating Factors

Integrating Factors

Sometimes we encounter exact equations straight away, but more often we have to multiply the equation by a factor (e.g. t) to bring it into exact form.

Factors that make an equation exact are called **integrating factors**.

Solving ODEs: Integrating Factors

Finding Integrating Factors

So how did we know that t was an integrating factor in the example?

The general rule is that for the general non-homogeneous first-order ODE

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + xp(t) = r(t)$$

the integrating factor is

$$I(t) = \mathrm{e}^{\int p(t)\mathrm{d}t}$$

Lets try this in the example ...

Solving ODEs: Integrating Factors

Back to the example

The example system was

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{x}{t} = t^2$$

which we can write in the form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + xp(t) = r(t)$$

with $p(t) = 1/t$ and $r(t) = t^2$.

So now we know $p(t)$.

Solving ODEs: Integrating Factors

Starting from

$$p(t) = \frac{1}{t}$$

our rule tells us that

$$I = \mathrm{e}^{\int \frac{1}{t}\mathrm{d}t} = \mathrm{e}^{\log t + C} = At$$

So the rule tells us that expressions of the form At are integrating factors (so we might as well chose $A = 1$, which yields the integrating factor t)

Solving ODEs: Integrating Factors

Exampercise

Find the general solution to

$$t^2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + tx = t + 1 \tag{1}$$

ToDo:

- Bring into the form $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + p(t)x = r(t)$.
- Identify $p(t)$.
- Compute $\int p(t)\mathrm{d}t$.
- Compute $I = \mathrm{e}^{\int p(t)\mathrm{d}t}$.
- Multiply Eq. (1) by I.
- Apply total derivative backwards to absorb terms.
- Solve by direct integration.

Solving ODEs: Integrating Factors

Solving ODEs: Integrating Factors

Why it works

We know the rule for finding integrating factors, but why does it actually work?

Lets start again with the general linear non-homogeneous first-order ODE.

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + xp(t) = r(t)$$

when we multiply a factor $I(t)$ this becomes

$$I(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + I(t)xp(t) = I(t)r(t) \tag{2}$$

At this point we don't care much about the right-hand-side. However, we want the left-hand-side to be an exact equation, i.e. the total derivative of a function, say $h(x(t), t)$.

Solving ODEs: Integrating Factors

Lets recall what happens if we differentiate $h(x(x),t)$.

$$\frac{\partial h}{\partial x}\frac{\mathrm{d}\,x}{\mathrm{d}\,t}+\frac{\partial h}{\partial t}$$

Now compare with the left-hand-side of Eq. (2)

$$I(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t}+I(t)xp(t)$$

We need

$$\frac{\partial h}{\partial x} = I(t) \tag{3}$$

$$\frac{\partial h}{\partial t} = I(t)xp(t) \tag{4}$$

Solving ODEs: Integrating Factors

From the blue condition

$$\frac{\partial h}{\partial x}=I(t)$$

We find a possible

$$h(x,t)=I(t)x$$

where we have ignored the constant of integration (all we need is a particular solution).

We can now substitute $h(x,t)$ into the green condition

$$\frac{\partial h}{\partial t}=I(t)xp(t)$$

to obtain

$$\frac{\partial I}{\partial t}x=I(t)xp(t)$$

but this is a differential equation for I!

Solving ODEs: Integrating Factors

Dividing by x we can write

$$\frac{\mathrm{d}\,I}{\mathrm{d}\,t}=Ip(t)$$

a linear homogeneous first order differential equation, which we solve by separation of variables

$$\int\frac{1}{I}\mathrm{d}I = \int p(t)\mathrm{d}t \tag{5}$$

$$\log(I) = \int p(t)\mathrm{d}t \tag{6}$$

$$I(t) = \mathrm{e}^{\int p(t)\mathrm{d}t} \tag{7}$$

which is the rule we used earlier.

Solving ODEs: Integrating Factors

Exampercise

Classify and solve

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}+xy=x$$

ToDo:

- Classify.
- Compute the integrating factor.
- Use integrating factor to form an exact equation.
- Solve.

Interlude

The story so far ...

Things to try (in this order)

- **Solve by inspection**, if you can ...
- ODEs that don't explicitly depend on the dependent variable (and do not contain derivatives of different orders)
  Solve by direct integration
- Linear homogeneous first-order ODEs
  Solve by separation of variables
- Linear non-homogeneous first-order ODEs
  Solve by integrating factor

What about nonlinear first-order ODEs ?
There is no universal approach but maybe a substitution...

Homework

James 5th edition
Read section 10.5.9
solve exercises from 10.5.11

James 4th edition
Read section 10.5.9
solve exercises from 10.5.11