Lecture 15
Small Perturbation Equations

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eurowings
G-UKAG
Typical Longitudinal Responses:
Phugoid and Short Period Modes

Last Lecture - The Longitudinal Equations

- Replacing the three external forces X, Z, M in Eqn. 7 with the expanded form

$$\underbrace{\begin{bmatrix} ms - X_u & -X_w & mg - X_q s \\ -Z_u & ms - Z_w & -mU\,s - Z_q s \\ -M_u & -M_{\dot{w}} s - M_w & I_{yy} s^2 - M_q s \end{bmatrix}}_{\textit{aircraft dynamics}} \underbrace{\begin{bmatrix} u \\ w \\ \theta \end{bmatrix}} = \underbrace{\begin{bmatrix} X_\eta & X_\delta \\ Z_\eta & Z_\delta \\ M_\eta & M_\delta \end{bmatrix}}_{\textit{control surface derivatives}} \begin{bmatrix} \eta \\ \delta \end{bmatrix} - w_g \underbrace{\begin{bmatrix} X_w \\ Z_w \\ M_w \end{bmatrix}}_{\textit{gust terms}}$$

- The aerodynamic derivatives which are associated with the motion variables (u, w, q) have been taken to the LHS and now appear with negative signs, whereas those still on the RHS have retained their original signs.

Longitudinal Dynamic Stability

- We do not need to look at the time–solution following a specific control deflection but can aim to find the nature of the free response of the system.

- The solution to post-disturbance dynamics is assumed to take the form:

$$\begin{bmatrix} u(t) \\ w(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{w} \\ \bar{q} \end{bmatrix} e^{st}$$

- where $\bar{u}, \bar{w}, \bar{q}$ are the amplitudes of three simultaneous responses, each of these three potentially being a complex number, i.e. the amplitudes can also display relative phase information of one variable compared with another.

Longitudinal Dynamic Stability

- The unknown exponent factor s governs the stability characteristics and with s having the form:

$$s = \sigma \pm j\omega \qquad (1)$$

- the exponential factor e^{st} in each solution can allow for various responses: convergent or divergent depending on the sign of σ, and oscillatory or non-oscillatory depending on whether or not ω=0.

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Longitudinal Dynamic Stability

- The basic matrix equation that describes the free response of the aircraft after a disturbance will appear as follows if we insert the assumed solutions:

$$\begin{bmatrix} ms - X_u & -X_w & mg - X_q s \\ -Z_u & ms - Z_w & -mU\ s - Z_q s \\ -M_u & -M_{\dot{w}} s - M_w & I_{yy} s^2 - M_q s \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{w} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

- Then for a non-trivial solution where the responses $\bar{u}, \bar{w}, \bar{\theta}$ are not zero, we must have the determinant of the square matrix equal to zero.

Longitudinal Dynamic Stability

- An expansion of the determinant will then yield the characteristic equation; in this case it is a quartic

$$p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0 = 0 \qquad (3)$$

- where, for example,

$$p_4 = m^2 I_{yy} \qquad \text{and} \qquad p_0 = mg(Z_u M_w - M_u Z_w).$$

- The roots of this equation are the few specific values of s which can be used in the assumed solution to make it a valid solution.

Approximate Solution of the Quartic

- The exact roots of Eqn. (2) are easily obtained using a computer (e.g. Matlab) but quick approximate methods are useful for hand calculations.

- The following can be used when the correct polynomial is known, but when approximate roots are acceptable.

  [We shall shortly look at a method for finding an approximate polynomial for one of the component modes, from which an exact solution will still provide only approximate roots to the real dynamics problem.]

- We assume that the quartic has the form of Eqn. (4), namely

$$p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0 = 0 \qquad (4)$$

Approximate Solution of the Quartic

- For the two oscillatory longitudinal modes, the magnitudes of the two pairs of roots are usually so different that the larger pair (short period) dominates and it can be found approximately from only the first three terms of (4), i.e.

$$p_4 s^2 + p_3 s + p_2 = 0 \qquad (5)$$

- because any large s raised to a power will dominate terms having lesser powers of s, but for the phugoid it is not generally sufficient (in practice) to use the last three terms of (4) on the basis of the converse argument.

Approximate Solution of the Quartic

- We develop an alternative phugoid quadratic such that the product of the two approximate quadratics is a new quartic that is equivalent to Eqn. (4), i.e.

$$(p_4 s^2 + p_3 s + p_2)(s^2 + ms + n) = 0 \tag{6}$$

- If this product is expanded and compared with Eqn. (4), the constant term and the term in s appear as

$$p_0 = n p_2 \quad \text{and} \quad p_1 = m p_2 + n p_3$$

- and hence the unknown coefficients become:

$$n = \frac{p_0}{p_2} \qquad m = \frac{p_1}{p_2} - \frac{p_0 p_3}{p_2{}^2} \tag{7}$$

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- Now the new quadratic (part of Eqn. (6)) can be used for approximate phugoid roots.

Example Calculation

- The stability implied by the following quartic is to be investigated.

$$s^4 + 6.38s^3 + 17.67s^2 + 2.71s + 5 = 0 \qquad (8)$$

- The exact roots are:

$$s_{1,2} = -3.16 \pm j2.65 \quad \text{short period}$$

$$s_{3,4} = -0.025 \pm j0.541 \quad \text{phugoid}$$

Undamped natural frequency? Damping ratio?

- Using the first three terms, the approx. short-period quadratic is:

$$s^2 + 6.38s + 17.67 = 0 \qquad (9)$$

- and this has roots: $s_{1,2} = -3.19 \pm j2.74$

The Approximate Phugoid Quadratic:

- Using Eqn. (6) and (7) we have,

$$\begin{aligned} n &= \frac{p_0}{p_2} = \frac{5}{17.67} = 0.283 \\ m &= \frac{p_1}{p_2} - \frac{p_0 p_3}{{p_2}^2} = \frac{2.71}{17.67} - \frac{5 \times 6.38}{17.67^2} = 0.0511 \end{aligned} \tag{10}$$

- and these lead to the approximate quadratic:

$$s^2 + 0.0511s + 0.283 = 0 \tag{11}$$

- which has the roots: $s_{3,4} = -0.0255 \pm j0.531$

- These can also be compared with their exact counterparts on the previous slide.

- Each pair of approximate roots has a magnitude that is just over 2% different from the magnitude of the exact pair.

Approximate Short Period Equations

- A disturbance in pitch that excites the short-period motion causes the characteristic well-damped oscillation and this decays in a matter of seconds.
- The forward speed is almost unaffected and therefore $u \approx 0$ throughout the motion. Similarly, forces in the fore/aft direction (like u) are effectively constant so we can eliminate the X-row and the u-column from the 3x3 matrix to leave:

$$\begin{bmatrix} ms - Z_w & -mU - Z_q \\ -M_w - M_{\dot{w}}s & I_{yy}s - M_q \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix} = \eta \begin{bmatrix} Z_\eta \\ M_\eta \end{bmatrix} - w_g \begin{bmatrix} Z_w \\ M_w \end{bmatrix} \tag{12}$$

Approximate Short Period Equations

- For stability, we examine the determinant of the square matrix which now leads to a quadratic equation in s. It is often the case that the two derivatives $M_{\dot{w}}$ and Z_q are so small they can be assumed to be zero, under which conditions the determinant expands to:

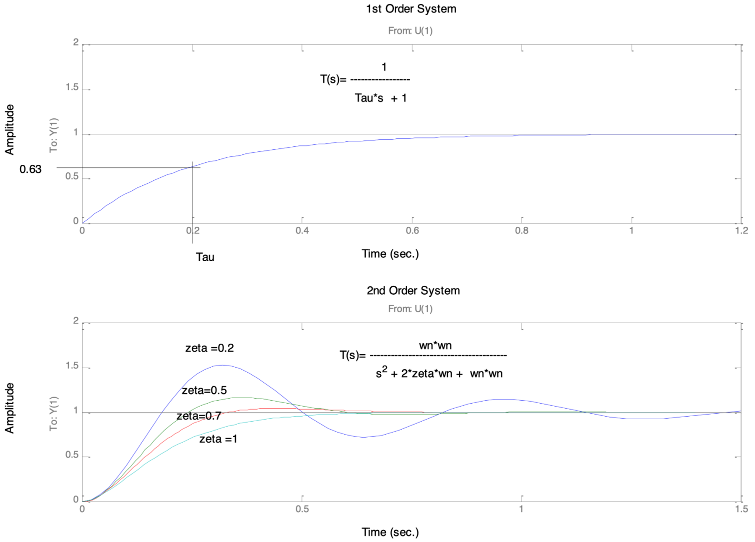
$$\left(ms - Z_w\right)\left(I_{yy}s - M_q\right) - mUM_w = 0$$

- or

$$s^2 - \frac{I_{yy}Z_w + mM_q}{mI_{yy}}s + \frac{Z_wM_q - mUM_w}{mI_{yy}} = 0. \qquad (13)$$

- Thinking exercise! – under what conditions is the short period stable? Which terms affect the damping?

System Stability



System Stability

$$T(s) = \frac{wn*wn}{s^2 + 2*zeta*wn + wn*wn}$$

$$s^2 - \frac{I_{yy}Z_w + mM_q}{mI_{yy}}s + \frac{Z_wM_q - mUM_w}{mI_{yy}} = 0.$$

- Thinking exercise! – under what conditions is the short period stable? Which terms affect the damping?

Lateral Small-Disturbance Equations

Lateral Small-Disturbance Equations

Mode	Displ.	Vel.
lat. translation (sideslip)	y	v
roll	ϕ	p
yaw	ψ	r

Translation

Rotation

Lateral Small-Disturbance Equations

- Sometimes another angle is used to represent the purely translational motion by declaring a

sideslip angle $\beta = \dfrac{v}{U}$ (1)

- Note: often use α, β, V_T

Lateral Small-Disturbance Equations

- In keeping with the convenience of having the time-variation of all disturbance variables return to zero after the disturbance, it is better to deal with a set of velocities instead of displacements.
- The choice of β, p, r includes an apparent angular displacement whereas the set v, p, r employs three obvious velocities.
- The sideslip angle β is widely used but remember that it is a velocity ratio.

The Lateral Rigid-Body Equations

We assume:

- that steady level flight exists prior to a disturbance (as with longitudinal equations)
- that longitudinal motion is not disturbed by an initiation of lateral motion
  - this leads to $\theta = 0$, $w = 0$, $q = 0$,
  - i.e. decoupled longitudinal & lateral eqns.
- that for small disturbances any product of the disturbances can be ignored, e.g. pq or qr.

The Lateral Rigid-Body Equations

The equations extracted from the 6 DoF set are:

$$
\begin{array}{rl}
\text{side-force} & \left. Y + mg\cos\theta\sin\phi = m\left(\dot{v} + ru - pw\right) \right\} \text{Translation} \\
\begin{array}{r} \text{rolling mom.} \\ \text{yawing mom.} \end{array} & \left. \begin{array}{l} L = I_x\dot{p} - I_{zx}\left(\dot{r} + pq\right) - \left(I_y - I_z\right)qr \\ N = I_{\substack{x\\z}}\dot{r} - I_{zx}\left(\dot{p} - qr\right) - \left(I_x - I_z\right)pq \end{array} \right\} \text{Rotation}
\end{array}
$$

(2)

The Lateral Rigid-Body Equations

The equations extracted from the 6 DoF set are:

$$
\begin{aligned}
&\text{side-force} && Y + mg\cos\theta\sin\phi = m\left(\dot{v} + ru - \cancelto{0}{pw}\right) && \Big\}\ \text{Translation}\\
&\text{rolling mom.} && L = I_x\dot{p} - I_{zx}\left(\dot{r} + \cancelto{0}{pq}\right) - \left(I_y - I_z\right)\cancelto{0}{qr} && \multirow{2}{*}{$\Bigg\}\ \text{Rotation}$}\\
&\text{yawing mom.} && N = I_{\substack{x\\z}}\dot{r} - I_{zx}\left(\dot{p} - \cancelto{0}{qr}\right) - \left(I_x - I_z\right)\cancelto{0}{pq} &&
\end{aligned}
$$

(2)

The Lateral Rigid-Body Equations

- One of the centripetal contributions to lateral acceleration can be retained if we recognise that the normal forward velocity U along with yaw rate r will give a valid product.

- We will also use the simplification $\sin\phi \approx \phi$.

Insertion of the External Forces

Placing the external forces on the RHS, this results in:

$$
\begin{aligned}
m(\dot{v} + Ur) &= mg\phi + Y \\
I_x \dot{p} - I_{zx} \dot{r} &= L \\
I_z \dot{r} - I_{zx} \dot{p} &= N
\end{aligned}
\qquad (3)
$$

Terms

$$\left.\begin{array}{l} Y \\ L \\ N \end{array}\right\} \text{Aerodynamic Forces \& Moments}$$

$$\left.\begin{array}{l} \dot{v} \\ Ur \end{array}\right\} \text{Lateral Accelerations}$$

$$\left.\begin{array}{l} \dot{p} \\ \dot{r} \end{array}\right\} \text{Rotational Accelerations}$$

Insertion of the External Forces

We express the external forces Y, L, N in terms of aerodynamic derivatives as we did for the longitudinal set:

$$\begin{bmatrix} Y \\ L \\ N \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r \\ L_v & L_p & L_r \\ N_v & N_p & N_r \end{bmatrix} \begin{bmatrix} v \\ p \\ r \end{bmatrix} + \begin{bmatrix} Y_\xi & Y_\zeta \\ L_\xi & L_\zeta \\ N_\xi & N_\zeta \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \tag{4}$$

in which we also use control surface deflections ξ and ζ.

Insertion of the External Forces

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e.g

e.g

in which we also use control surface deflections ξ and ζ.

Insertion of the External Forces

Insertion of (4) into (3), along with a rearrangement to the general form seen for the longitudinal equations, gives:

$$\begin{bmatrix} ms - Y_v & -Y_p - mgs^{-1} & mU - Y_r \\ -L_v & I_x s - L_p & -I_{zx} s - L_r \\ -N_v & -I_{zx} s - N_p & I_z s - N_r \end{bmatrix} \begin{bmatrix} v \\ p \\ r \end{bmatrix} = \begin{bmatrix} Y_\xi & Y_\zeta \\ L_\xi & L_\zeta \\ N_\xi & N_\zeta \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \quad (5)$$

Note: Recognise & use – no need to derive.

Insertion of the External Forces

- This set of equations (5) is the linearised small-disturbance set of equations for lateral motions.
- We could have included, on the RHS, a column for lateral gusts, i.e. forces depending on v_g, as we did for the longitudinal set.

An Alternative Set of Variables

- Note also that when the velocity co-ordinate p (roll rate) is used, there is one term in the square matrix having s^{-1} because in (3) it appears as $mg\phi$.
- If in (5) we had used the co-ordinates v, ϕ, r instead, the whole of the central column would have required an extra factor s applied to each term, i.e.

$$\begin{bmatrix} v \\ p \\ r \end{bmatrix} \Rightarrow \begin{bmatrix} v \\ \phi \\ r \end{bmatrix}$$

$$\left[\begin{array}{c|c|c} & -Y_p s - mg & \\ \hline & I_x s^2 - L_p s & \\ \hline & -I_{zx} s^2 - N_p s & \end{array}\right]\begin{bmatrix} v \\ \phi \\ r \end{bmatrix} = \qquad (6)$$

An Alternative Set of Variables

In this form (when the full equations are shown) it is clearer that the determinant of the square matrix will produce a quartic characteristic polynomial because the principal diagonal elements then include the product $(ms)\ (I_x s^2)(I_z s)$.

$$p = \dot{\phi} = s\phi$$

$$\dot{p} = \ddot{\phi} = s^2\phi$$

Lateral Dynamic Stability

The procedure for studying stability, as opposed to solving the equations explicitly to study response (i.e. time simulation), is as used in the longitudinal case, namely to assume a form of exponential solution:

$$\begin{bmatrix} v(t) \\ p(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \bar{v} \\ \bar{p} \\ \bar{r} \end{bmatrix} e^{st}$$

Lateral Dynamic Stability

- And thereby convert the differential equations to a set of algebraic equations in s from which the characteristic equation is found as:

$$p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0 = 0$$

$$\boxed{s = \sigma + j\omega}$$

- But whereas the longitudinal characteristic roots were two pairs of complex roots, the lateral quartic normally leads to one complex pair and two real roots, these defining the three lateral modes.

Approximate Approaches to the Lateral Roots

As with the longitudinal equations, there are two possible approaches:

- try to form a reduced set of equations that reasonably describes one mode at a time, but approximately:
  → a smaller-order characteristic polynomial should result in each case.
- try to extract approximate roots from the full characteristic polynomial.

Lateral Roots – Roll Convergence

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} v \\ p \\ r \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}\begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

If there were pure roll, not involving the other freedoms, we could retain only the central element of the square matrix and also eliminate any rudder action so as to leave only the following:

$$\left(I_x s - L_p\right) p = L_\xi \xi \qquad (8)$$

Lateral Roots – Roll Convergence

a. As soon as a sudden aileron deflection is applied, and before a roll rate can build up, the initial acceleration in roll comes from:

$$I_x \dot{p} = L_\xi \xi$$

$$\text{or} \quad \dot{p}(t=0) = \frac{L_\xi}{I_x}\xi \qquad (9)$$

Lateral Roots – Roll Convergence

b. Long after imposing a constant value for ξ, the steady roll rate that develops, i.e. when $\dot{p}(t \to \infty) = 0$ is given by:

$$\underbrace{I_x \dot{p}}_{\substack{now\\zero}} - L_p p = L_\xi \xi$$

$$\text{or} \quad p = -\frac{L_\xi}{L_p}\xi \qquad (10)$$

which we saw earlier when looking at rolling power.

Lateral Roots – Roll Convergence

c. The roll convergence root comes from (Eqn. (8)):

$$(I_x s - L_p) = 0$$

$$\text{so} \quad s_{rc} = \frac{L_p \;\leftarrow \text{neg.}}{I_x \;\leftarrow \text{pos.}} \qquad (11)$$

Lateral Roots – Roll Convergence

- Using the second approach to finding the root for this mode, i.e. to find an approximate root from the full polynomial, we can obtain a single large root from the first two terms of the quartic, i.e. from:

$$p_4 s^4 + p_3 s^3 \approx 0$$

$$\text{which leads to } s_{rc} \approx -\frac{p_3}{p_4} \qquad (12)$$

- Note: could solve the full quartic to find all the exact roots.

Lateral Roots – Spiral Root

- A simplified set of equations is not easily formed because all freedoms contribute, as mentioned above.
- However, since this root is known to be quite small in magnitude it can be found, approximately, from the last two terms of the quartic:

$$p_1 s + p_0 \approx 0$$

$$\text{or} \quad s_{spi} \approx -\frac{p_0}{p_1}. \tag{13}$$

Lateral Roots – Spiral Root

- This root can be positive or negative, and in practice this means having to look at the sign of p_0. (For a stable spiral we want s_{spi} to be negative.)
- The constant term is found to be:

$$p_0 = mg[L_v N_r - N_v L_r] \tag{14}$$

Rolling moment

Yawing moment

- and both products in [ ] are normally positive whereas their difference can be of either sign, though small.

Lateral Roots – Dutch Roll

Oddly, this lateral mode does *not* involve a lot of roll, and a decent approximation can often be found by eliminating the influence of p from the full set of equations (5) and then also eliminating the whole roll equation.

$$\left[\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \end{array}\right]\begin{bmatrix} v \\ p \\ r \end{bmatrix} = \left[\begin{array}{c|c} & \\ \hline & \\ \hline & \end{array}\right]\begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

Lateral Roots – Dutch Roll

- The consequent quadratic from the reduced set of equations will be in the form

$$s^2 + bs + c = 0 \qquad (15)$$

where $b = -\left[\frac{N_r}{I_z} + \frac{Y_v}{m}\right]$ (16)

- and, as both aerodynamic derivatives in [ ] are normally negative, there is normally positive damping in the Dutch Roll.

Lateral Roots – Summary

- The signs, at least, of the various aerodynamic derivatives can be deduced from a knowledge of the meaning of the short-hand forms. In some cases the relative magnitude of some derivatives can be deduced.
- The completed set of lateral equations is given in (5) and you must be able to place the various derivatives in their correct rows and columns, LHS or RHS, properly signed.
- Be able to describe the three characteristic motions and suggest typical positions for their roots in the complex plane.
- As with the longitudinal equations, the full characteristic equation is a quartic. Approximate methods are given for finding the four roots and you are to show that you can use these methods and that you understand the logic behind their use.