# Handout 8 – Dynamics: Work & Energy

Meriam & Kraige, Dynamics: 3/6, 3/7, 6/6

Integrating the dynamic forces with respect to *displacement* leads to the concepts of **work** and **energy**. Often, dynamics problems can be solved more easily by using the results of those integrations directly, without explicitly solving for accelerations.

## 8.1 Particle Dynamics

Work and energy were previously introduced in Handout 2, and those concepts are here extended to applications in dynamics, in the form of **kinetic energy**.

### 8.1.1 Kinetic Energy

Consider a particle of mass $m$ in rectilinear motion under an applied force $F$

$$F = m\,a = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds} \qquad \rightarrow \qquad F\,ds = mv\,dv$$

This is integrated from $s_1$ to $s_2$ to give the total **work** done by the force over the displacement:

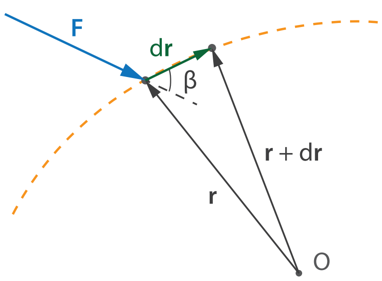
$$U_{12} = \int_{s_1}^{s_2} F ds = \int_{v_1}^{v_2} mv\,dv = \frac{1}{2}m\left(v_2^2 - v_1^2\right) = T_2 - T_1 = \Delta T \tag{8.1}$$

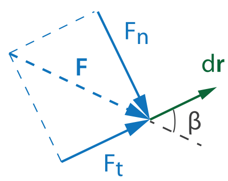
where

$$T = \frac{1}{2}mv^2 \tag{8.2}$$

is the **kinetic energy** of the particle. The work $U_{12}$ done is equal to the change in kinetic energy $\Delta T$ of the particle. Work is a *scalar* value, can be positive or negative, and has units Joule (J = Nm).

For curvilinear motion, consider the applied force $\boldsymbol{F}$ in an intrinsic coordinate system, which resolves into $F_n$ and $F_t$, normal and tangent to the curve.



The component normal to the curve (and thus displacement $d\boldsymbol{r}$) does not exert any work, and the infinitesimal amount of work done by the force $\boldsymbol{F}$ is therefore:

$$\begin{aligned} dU &= \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= |\boldsymbol{F}| \cos\beta \, |d\boldsymbol{r}| \\ &= F_t \, ds \end{aligned}$$

The total work done along the curve over a displacement from $s_1$ to $s_2$ is therefore:

$$\begin{aligned} U_{12} &= \int_{s_1}^{s_2} \boldsymbol{F} \cdot d\boldsymbol{r} = m \int_{s_1}^{s_2} \boldsymbol{a} \cdot d\boldsymbol{r} = m \int_{s_1}^{s_2} a_t ds = m \int_{v_1}^{v_2} v dv \\ &= \frac{1}{2} m \left( v_2^2 - v_1^2 \right) \end{aligned}$$

The work-energy equation relates the total work done to the change in kinetic energy:

$$U_{12} = T_2 - T_1 = \Delta T$$

Formulating the dynamics of a particle in terms of work and energy avoids the necessity to establish the accelerations, in order to calculate the velocities. Furthermore, the work-energy equation only involves those forces which do work, and thus contribute to the change in kinetic energy.

**Power** The power $P$ developed by a force $\boldsymbol{F}$ is the rate of change of work:

$$P = \frac{dU}{dt} = \frac{\boldsymbol{F} \cdot d\boldsymbol{r}}{dt} = \boldsymbol{F} \cdot \boldsymbol{v} \tag{8.3}$$

Power is a scalar quantity with units of Watt (Nm/s = J/s = W).

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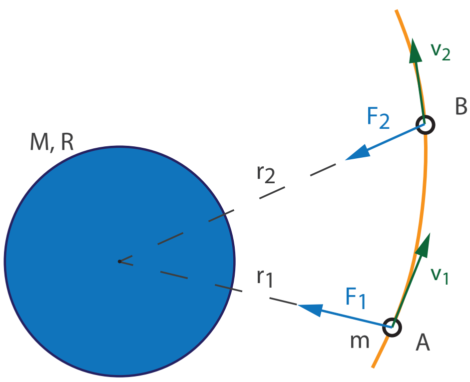
**Example 8.1 – Satellite Orbital Velocity**

A satellite of mass $m$ is placed in an elliptical orbit around the Earth. Earth has a mean radius $R = 6371$ km, and $g = 9.81$ m/s$^2$ at the surface. At point $A$ in the orbit, the satellite has an altitude $h_1 = 500$ km, and a velocity $v_1 = 30000$ km/h.




**Q**: What is the velocity $v_2$ of the satellite as it reaches point $B$, at an altitude of $h_2 = 1200$ km?

**A:** the only force acting on the satellite is a gravitational force

$$F = m\frac{GM}{r^2} = m\frac{gR^2}{r^2}$$

Here, the substitution $GM = gR^2$ is made (using values for the surface of the Earth, $F = mg$).

The work done by $F$ is due to the radial component of motion, $dr$, and is negative for increasing $r$.

$$U_{12} = -\int_{r_1}^{r_2} F dr = -mgR^2 \int_{r_1}^{r_2} \frac{dr}{r^2} = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

From the work-energy equation:

$$U_{12} = \Delta T$$

$$mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

find the velocity at point B

$$v_2^2 = v_1^2 + 2gR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

From the equation, $v_2$ will be smaller than $v_1$, if $r_2 > r_1$. In other words, the further away from Earth, the slower the satellite travels. Substituting numerical values: $v_2 = 7663$ m/s (or 27590 km/h).

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### 8.1.2 Potential Energy

By introducing **potential energy** to describe the work done by gravity and elastic spring forces, the work-energy analysis is often simplified. Importantly, the potential energy of conservative forces is defined by the *end points* of the displacement, rather than the *path* taken.

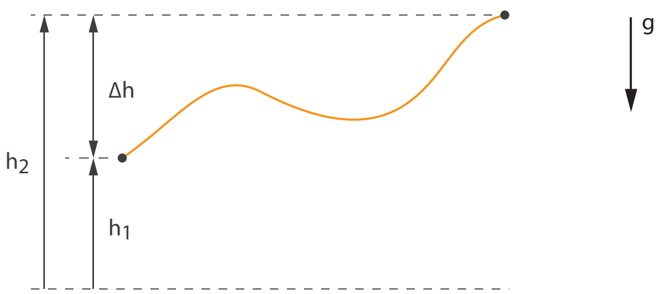
**Gravitational potential energy** is defined as the work done on a particle, against the gravitational field:

$$V_g = mgh \tag{8.4}$$

where $h$ is measured from an arbitrary datum. The change in gravitational potential energy:

$$\Delta V_g = mg\left(h_2 - h_1\right) = mg\Delta h$$

The change in potential energy is independent of the path, and only defined by $\Delta h$. Work done by gravitational force on the particle is the negative of the change in gravitational potential energy. In other words, a particle moving in a gravitational field will gain kinetic energy corresponding to the loss in potential energy.



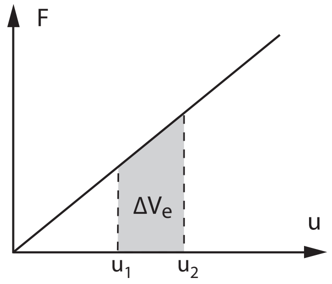
In the case of large vertical displacements, for example in orbital mechanics, the magnitude of the gravitational acceleration $g$ will vary with altitude, and integration is required; see Example 8.1.

**Elastic potential energy** is stored in a spring (or other elastic element). For a linear spring, with stiffness $k$ and extension $u$, the elastic energy is:

$$V_e = \int_0^u F\,du = \int_0^u k\,u\,du = \frac{1}{2}ku^2 \tag{8.5}$$

and the change in spring energy:

$$\Delta V_e = \frac{1}{2}k\left(u_2^2 - u_1^2\right)$$




The work exerted on a particle by the spring is the negative of the change in elastic potential energy.

### 8.1.3 Work-Energy Equation

Total mechanical energy $E$ of a mechanical system:

$$E = T + V_g + V_e$$

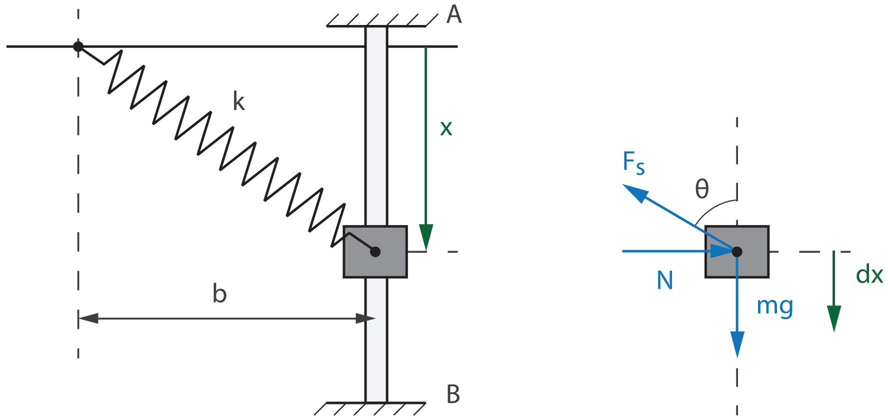
A change in total energy is due to applied external work:

$$U_{12} = \Delta\left(T + V_g + V_e\right) = \Delta E \tag{8.6}$$

For no applied external forces, the total amount of energy is conserved, and thus $\Delta E = 0$.

### Example 8.2 – Spring-Loaded Sliding Collar

A collar of mass $m$ slides over a smooth, vertical linear guide. It is connected to a spring with stiffness $k$ and rest length $L_0$.




**Q**: If the collar starts from rest at $A$, what is its velocity at position $x$?

**A:** From the FBD, integrate the net force on the collar over the displacement $dx$

$$U_{12} = \int_0^x \left(-F_s \cos\theta + mg\right) dx = \int_0^x \left(-kx + \frac{kL_0 x}{\sqrt{x^2+b^2}} + mg\right) dx$$

$$= -\frac{1}{2}kx^2 + kL_0\sqrt{x^2+b^2} - kL_0 b + mgx$$

where:

$$F_s = k\left(\sqrt{x^2+b^2} - L_0\right) \qquad \cos\theta = \frac{x}{\sqrt{x^2+b^2}}$$

Using the work-energy balance (and $T_1 = 0$):

$$U_{12} = \Delta T = \frac{1}{2}m\dot{x}^2$$

we find:

$$\dot{x} = \sqrt{-\frac{kx^2}{m} + \frac{2}{m}kL_0\sqrt{x^2+b^2} - \frac{2}{m}kL_0 b + 2gx}$$

A more efficient approach is to consider the change in potential and kinetic energy of the mechanical system:

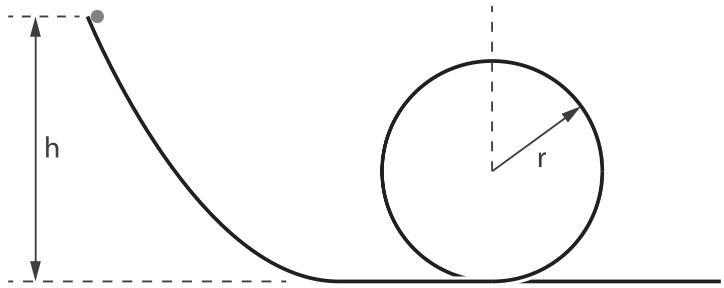
$$\Delta T + \Delta V_g + \Delta V_e = 0$$

$$\left(\frac{1}{2}m\dot{x}^2 - 0\right) + (0 - mgx) + \left(\frac{1}{2}k\left(\sqrt{x^2+b^2} - L_0\right)^2 - \frac{1}{2}k\left(b - L_0\right)^2\right) = 0$$

which only requires algebra to find the identical answer!

**Example 8.3 – Roller Coaster Loops**

A roller coaster car starts from rest at height $h$, before entering a circular loop with radius $r$.

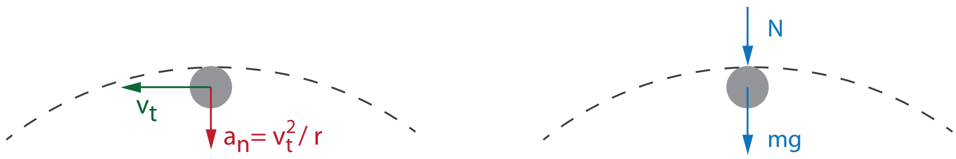



The track is assumed to be frictionless and aerodynamic drag is neglected. From the conservation of energy,

$$\Delta E = \Delta T + \Delta V_g = \left(\frac{1}{2}mv_t^2 - 0\right) + (2\,mgr - mgh) = 0$$

we find the velocity $v_t$ of the car at the top of the loop:

$$v_t = \sqrt{2g\,(h - 2r)}$$




To ensure contact between car and the roller coaster, the contact force $N$ must be greater than zero:

$$N = m\,\frac{v_t^2}{r} - mg \geq 0$$

and therefore $v_t \geq \sqrt{gr}$, and the initial height of the car must thus be: $h \geq 2.5r$

Next, consider the velocity $v_b$ of the car at the bottom of the loop. From conservation of energy:

$$\Delta E = \Delta T + \Delta V_g = \left(\frac{1}{2}mv_b^2 - 0\right) + (0 - mgh) = 0$$

we find

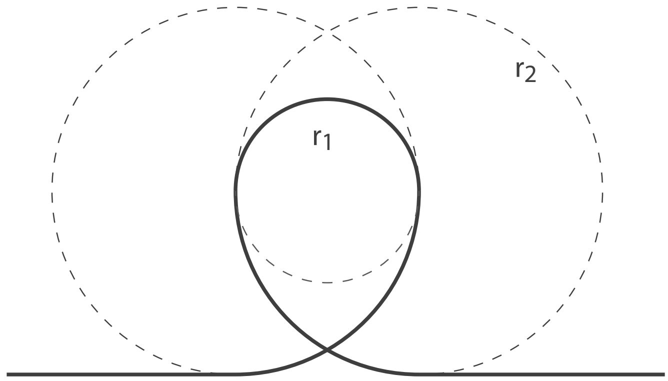
$$v_b = \sqrt{2gh}$$

Using $h = 2.5r$, the centripetal acceleration at the bottom of the loop becomes:

$$a_n = \frac{v_b^2}{r} = \frac{2gh}{r} = 5g$$

In other words, people in the car will experience $5 + 1 = 6$ g. This would be uncomfortable for fighter pilots, let alone for a family day out in Alton Towers!

The solution is to vary the curvature along the length of the track, providing a gentle curvature at the bottom of the loop and a tight curvature towards the top (thus ensuring sufficient centripetal acceleration). One solution is to construct the loop of sections of two different radii, creating a teardrop shape[1].




[1] Even more elegantly, the shape can be a 'clothoid' (or Euler spiral) where the curvature varies with the arc length. The enables the design of roller coaster loops with a constant centripetal acceleration.

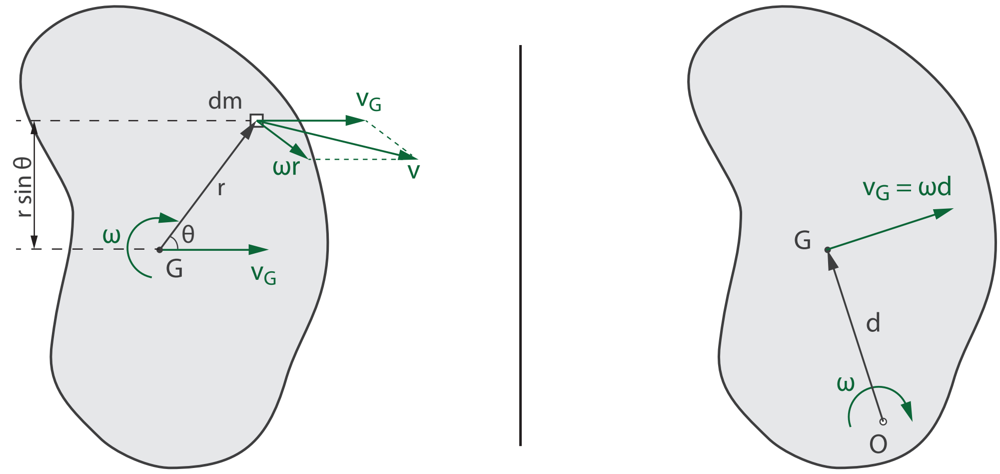
`http://gizmodo.com/why-roller-coaster-loops-are-never-circular-1549063718`

## 8.2 Rigid Body Dynamics

The kinetic energy of a rigid body is composed of translational and rotational kinetic energy:

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2 \tag{8.7}$$

where $v_G$ is the velocity of the centre of mass, $I_G$ the moment of inertia of the body, and $\omega$ its angular velocity.




Proof: consider a rigid body in plane motion, with $v_G$ and $\omega$. The kinetic energy $dT$ associated with an infinitesimal element of mass $dm$ at distance $r$ from the centre of mass is found as:

$$dT = \frac{1}{2}v^2 dm = \frac{1}{2}\left(v_G^2 + \omega^2 r^2 - 2v_G\omega r\cos\left(\frac{\pi}{2}+\theta\right)\right)dm$$

where $v$ is calculated from $v_G$ and $\omega r$ using the cosine rule. Integrate over body for total kinetic energy $T$:

$$\begin{aligned} T &= \int \frac{1}{2}v^2 dm = \int \frac{1}{2}v_G^2 dm + \int \frac{1}{2}\omega^2 r^2 dm + \cancel{v_G\omega\int r\sin\theta dm} \\ &= \frac{1}{2}mv_G^2 + \frac{1}{2}\omega^2 I_G \end{aligned}$$

where the last term is zero, by virtue of the origin being at the centre of mass.

**Fixed Axis of Rotation** For rotation around fixed point $O$ at a distance $d$ from the centre of mass:

$$\begin{aligned} T &= \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2 = \frac{1}{2}m\left(\omega d\right)^2 + \frac{1}{2}I_G\omega^2 \\ &= \frac{1}{2}\left(md^2 + I_G\right)\omega^2 = \frac{1}{2}I_0\omega^2 \end{aligned}$$

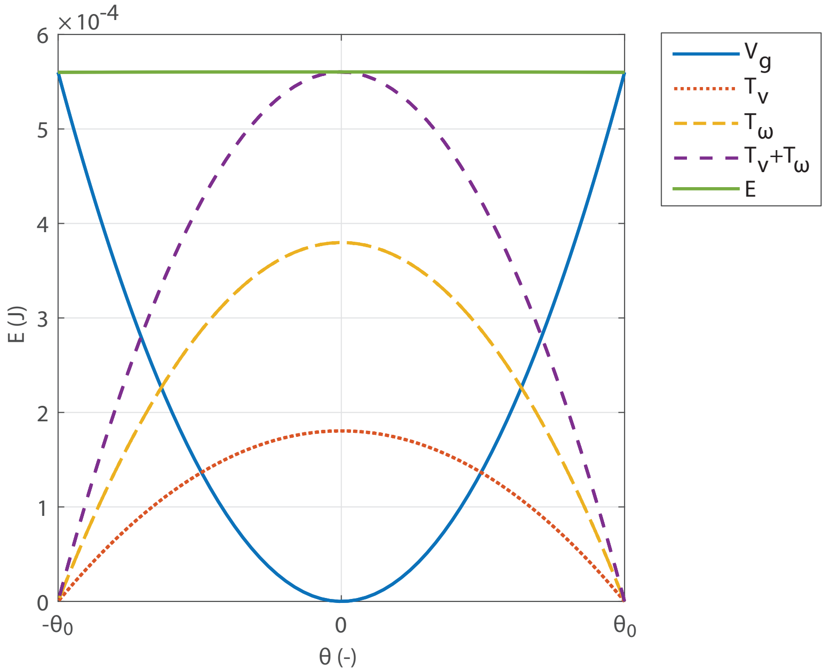
The moments around the point of rotation can be integrated with respect to angular rotation:

$$\int_{\theta_1}^{\theta_2} M_O d\theta = \int_{\theta_1}^{\theta_2} I_O\alpha d\theta = \int_{\omega_1}^{\omega_2} I_O\omega d\omega = \frac{1}{2}I_O\left(\omega_2^2 - \omega_1^2\right) \tag{8.8}$$

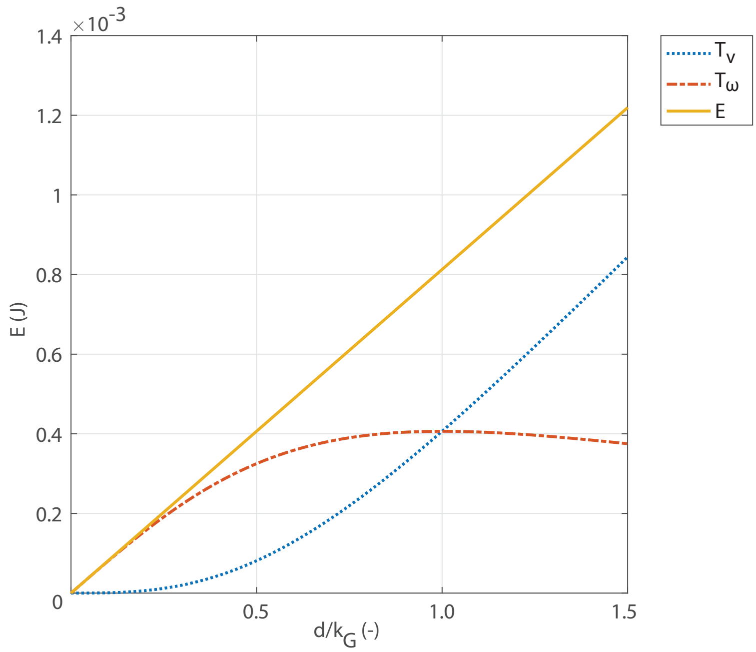
where the work done by the moment is equal to the change in kinetic energy. For example, a flywheel is designed to store and releases rotational kinetic energy, and thus spins at a high angular velocity.

### Example 8.4 – Compound Pendulum

Over the oscillation of a compound pendulum (see Handout 6), potential and kinetic energy are continuously exchanged. For the compound pendulum shown in lecture ($L = 0.5$ m, $w = 0.05$ m, $m = 0.150$ kg) with initial angle $\theta_0 = 5°$ and pivot distance $d = 0.10$ m, the energy components are shown. The kinetic energy associated with translational ($T_v = \frac{1}{2}mv_G^2$) and rotational ($T_\omega = \frac{1}{2}I_G\omega^2$) motion are plotted separately; the total sum of energy $E = V_g + T_v + T_\omega$ is constant.

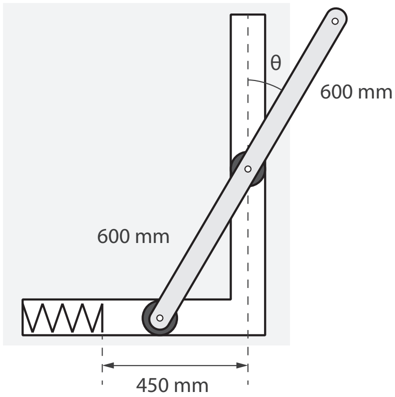



The pendulum distance $d \in [0, L/2]$ affects the proportion of translational versus rotational kinetic energy; shown is the kinetic energy components at $\theta = 0$ when potential energy is zero.



**Example 8.5 – Sliding and Falling Rod**

A uniform rod of mass $m = 20$ kg and length $L = 1200$ mm is released from near-vertical position. The centre of the bar is confined to move vertically, and one end point moves in a smooth horizontal guide. A spring with stiffness $k = 5$ kN/m is compressed as the bar falls, and slows its descent.




**Q:** What is the angular velocity of the bar at $\theta = 30°$?

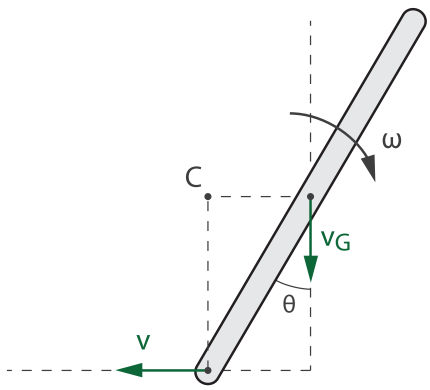
**A:** At $\theta = 30°$ the spring is not yet engaged, and the loss of potential energy is converted to kinetic energy.

$$\Delta T + \Delta V_g = 0$$

$$\left(\frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2 - 0\right) + mg\frac{L}{2}(\cos\theta - 1) = 0$$

The kinematic relationship between $v_G$ and $\omega$ is found using the point of instantaneous zero velocity:

$$v_G = \sin\theta\frac{L}{2}\omega$$




Rewriting the work-energy equation gives:

$$\omega^2 = \frac{4mgL(1 - \cos\theta)}{m\sin^2\theta L^2 + 4I_G}$$

Substituting the numerical values: $\omega = 2.74$ rad/s.

**Revision Objectives Handout 8:**

Work & Energy

- calculate kinetic energy of particles ($T = \frac{1}{2}mv^2$) and rigid bodies ($T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$)
- calculate gravitational ($V_g = mgh$) and elastic ($V_e = \frac{1}{2}ku^2$) potential energy
- apply the work-energy balance ($U_{12} = \Delta E = \Delta T + \Delta V_g + \Delta V_e$)
- recognize cases where energy is conserved ($\Delta E = 0$)
- apply the work energy balance to solving problems in dynamics of particles and rigid bodies