EMAT10100 Engineering Maths I
Lecture 7: Intoduction to vectors

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+ Nikolai Bode

Scalars and Vectors

- two very different concepts in engineering:
  - SCALAR: something with magnitude only
  - VECTOR: something with magnitude and direction
- Scalar or vector?

length? force? temperature?

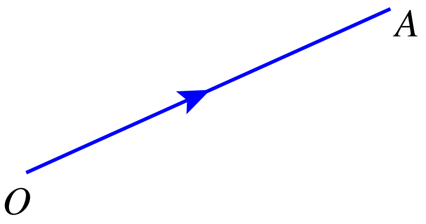
velocity? speed? voltage?

time? volume? displacement?

- Note: magnitude = size = modulus = length

Directed line segments

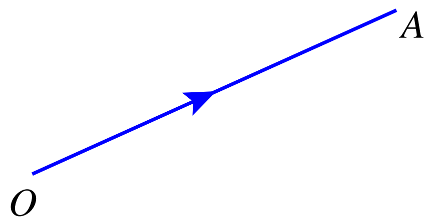
- Represent vectors as directed line segments



- magnitude is the length of line segment
- arrow is the direction from O to A (So O is as important as A.)

- Q. How do we represent the direction of a line in 3D?
- A. direction cosines (old fashined method), unit vectors (see later)

Notation



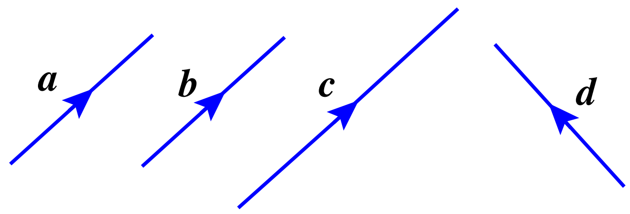
- Many alternative notations:

OA, $\overset{\mathrm{OA}}{\rightarrow}$, $\overrightarrow{\mathrm{OA}}$, $\underline{\mathrm{OA}}$, $\overline{OA}$, $\mathbf{a}$, $\underline{a}$, $\overline{a}$, $\vec{\mathrm{a}}$, $\overset{\mathrm{a}}{\rightarrow}$

- We typeset $\mathbf{a}$ and handwrite $\underline{a}$
- Magnitude of $\mathbf{a}$ written as $|\mathbf{a}|$
(pronounced 'mod $\mathbf{a}$')
  - Note $|\mathbf{a}| \geq 0$ (since lengths always ≥ 0)

Equality of vectors

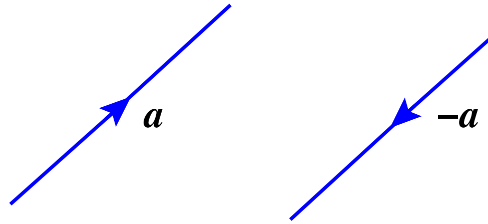
- Two vectors are equal if:
  - they have same magnitude *and* direction



- $\mathbf{a} = \mathbf{b}$, but $\mathbf{c}$ and $\mathbf{d}$ are different.
  - $\mathbf{c}$ has same direction as $\mathbf{a}$, $\mathbf{b}$ (different magnitude)
  - $\mathbf{d}$ has same magnitude as $\mathbf{a}$, $\mathbf{b}$ (different direction)

Negative vectors

- $-\mathbf{a}$ has same magnitude as $\mathbf{a}$, but has opposite direction

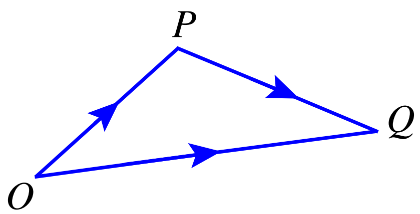


- So

$$|-\mathbf{a}| = |\mathbf{a}|,$$

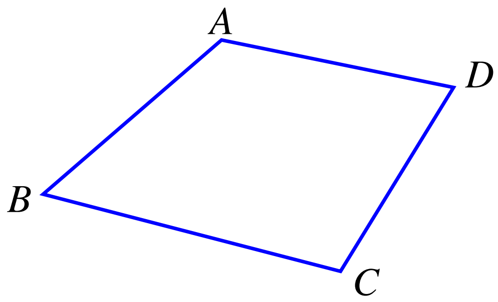
(magnitude mops up minus sign)

Resultant of two displacement vectors



- Sum (resultant) of $\overrightarrow{\mathrm{OP}}$ and $\overrightarrow{\mathrm{PQ}}$ is $\overrightarrow{\mathrm{OQ}}$ (obvious!)
- Needs end of previous vector to coincide with next

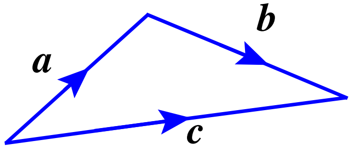
Exercises



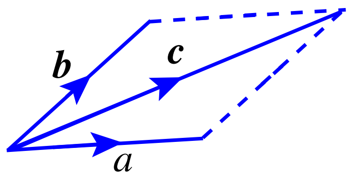
- Simplify $\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} - \overrightarrow{\mathrm{DC}}$
- Simplify $\overrightarrow{\mathrm{AC}} - \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{BD}}$

Adding vectors

- Vectors are added in the same way as 'journeys'; two ways of thinking about it: $\mathbf{c} = \mathbf{a} + \mathbf{b}$
- triangle law



- parallelogram law



Properties of addition

- Commutativity

$$\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$$

- Associativity

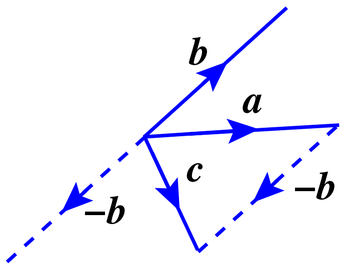
$$(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$$

Subtracting vectors

- Just add the negative!

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

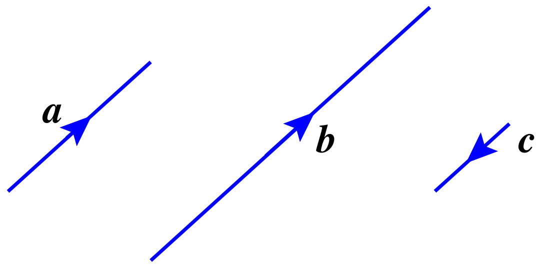
- Graphically:



Here $\mathbf{c} = \mathbf{a} - \mathbf{b}$

Scalar multiplication

- Scalar multiplication: multiply vector by scalar
  - direction stays same
  - magnitude stretched by given scalar
  - (negative scalar reverses direction)



- Parallel vectors: $\mathbf{a} = \lambda\mathbf{b}$ where λ is a scalar

Unit vectors and the zero vector

- Unit vectors
  - If $|\mathbf{a}| = 1$, $\mathbf{a}$ is a 'unit vector'
  - Given any old vector $\mathbf{b}$, a unit vector may be made from it:

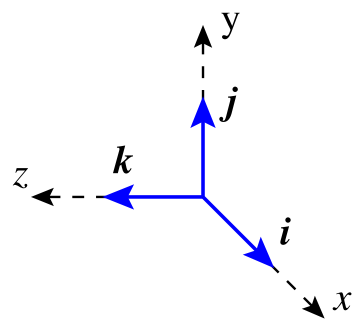
$$\hat{\mathbf{b}} = \frac{\mathbf{b}}{|\mathbf{b}|}, \qquad \text{so that } |\hat{\mathbf{b}}| = 1$$

    $\hat{\mathbf{b}}$ is pronounced 'b hat'
  - sometimes we use the symbol : $\mathbf{n}$: to mean unit vector

- Zero vector

  If $|\mathbf{b}| = 0$, then $\mathbf{b}$ is the zero vector

Coordinate vectors: $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$



- $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors parallel to x, y, z coordinate axes (not to be confused with i or $j = \sqrt{-1}$)
- Follows that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$.
- Note: hats are not used

Components of a vector

- big idea: express all 3D vectors as a combination of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- This is the position vector (or displacement) of the point (x, y, z) relative to the origin $(0, 0, 0)$
- Sometimes write

$$\mathbf{r} = (x, y, z) \qquad \text{or} \qquad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Row and column vector notation.
  For now: use is interchangeable. But not when we consider matrices

Engineering HOT SPOT

- Can vectors have more than three components?
  Why should they?
- Chemistry of swimming pools
  - reaction between chlorine and 'pollutants'
  - 6–14 key reactants, depending on level of detail
  - concentrations listed in big long vector
- What's the added value? No geometry.
- Qualitative system behaviour (equilibrium, blow-up) governed by eigenvalues of matrices involved in vector formulation.

Addition and scalar multiplication again

- Revisited in component form. If

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$

- For any scalar λ

$$\lambda\mathbf{a} = (\lambda a_1)\mathbf{i} + (\lambda a_2)\mathbf{j} + (\lambda a_3)\mathbf{k}$$

- These seem clear from geometry

Magnitude etc. in component form

- Suppose

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

in component form. Then

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \text{(3D version of Pythagoras)}$$

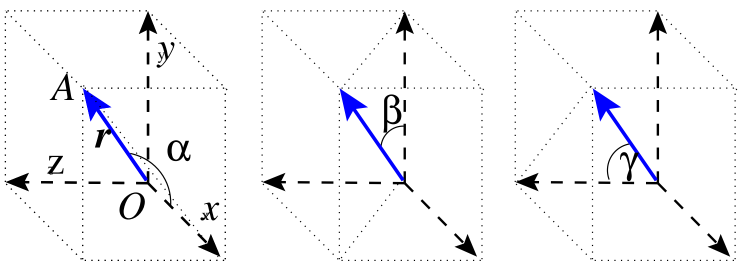
- It follows that

$$\begin{aligned} \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}, \\ &= \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}. \end{aligned}$$

- Exercise. Calculate $|\mathbf{r}|$ and $\hat{\mathbf{r}}$ when $\mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Angular interpretation of components

- To represent the direction of a vector OA



- Define the direction cosines

$$\cos\alpha = \frac{x}{r} \qquad \cos\beta = \frac{y}{r} \qquad \cos\gamma = \frac{z}{r}$$

- The vector $(\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector prove it!

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Lecture 8: Two types of vector products

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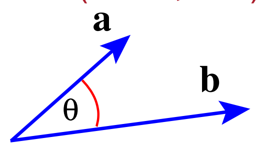
Scalar (inner, dot) product



- vector · vector = scalar
- Definition:
  $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
  (pronounced '$\mathbf{a}$ dot $\mathbf{b}$')
- Useful result (for proof, see section 4.2.7 of James):
  In components, if $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, then

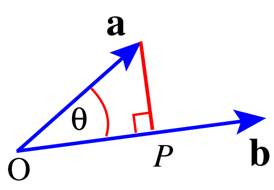
$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Gives a method for calculating angle between vectors!
- Exercise: Find the angle between $(1, 2, 2)$ and $(1, 4, 8)$

Useful properties of scalar product

- $\mathbf{a}$, $\mathbf{b}$ perpendicular same as $\mathbf{a}\cdot\mathbf{b} = 0$
  - in particular $\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$
- $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ (sometimes written a^2)
  - in particular $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$
- $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
- $(\lambda\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(\lambda\mathbf{b}), = \lambda(\mathbf{a}\cdot\mathbf{b})$
  - for any scalar λ
- $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

Projection of one vector onto another



- Projection of $\mathbf{a}$ onto $\mathbf{b}$ is defined as $|\overrightarrow{\mathrm{OP}}|$
- Also called: component of $\mathbf{a}$ in direction of $\mathbf{b}$
- How to find projection?

$$|\overrightarrow{\mathrm{OP}}| = |\mathbf{a}|\cos\theta \qquad \text{and} \qquad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

- So, Projection $|\overrightarrow{\mathrm{OP}}| = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$ (learn this)
- Exercise: Find the projection of $(3, 5, 7)$ onto $(1, 2, 2)$

Work done using scalar product

- *"Work done by a force is the product of the distance moved by the point of application of the force and the component of the force in this direction"*
- Force $\mathbf{F}$. Displacement of point of application $\mathbf{d}$
- Distance moved by point of application is $|\mathbf{d}|$
- Component of $\mathbf{F}$ in direction of $\mathbf{d}$ is

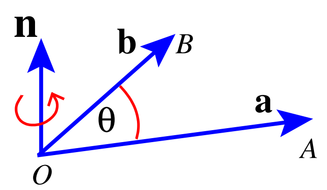
$$\frac{\mathbf{F}\cdot\mathbf{d}}{|\mathbf{d}|}$$

- Therefore, work done is

$$W = \mathbf{F}\cdot\mathbf{d}$$

  - grown up version of "work done = force $\times$ distance"

Vector (cross) product



- vector $\times$ vector = vector,
  vector $\wedge$ vector = vector
  (alternative notation)
- Definition:

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

- (pronounced '$\mathbf{a}$ cross $\mathbf{b}$')
- $|\mathbf{a}\times\mathbf{b}| = 2\times$ area of triangle OAB
- $|\hat{\mathbf{n}}| = 1$ (a unit vector)
- $\hat{\mathbf{n}}$ perpendicular to $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}}$ form right-handed set
- Erm, right-handed set?

  Also known as the right-hand rule

Component formula for cross product

Useful result (not proved here):

- If $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3)$, i.e. if

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \text{then}$$

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

- How to remember component formula: (1) try to the 'Cover-up' approach

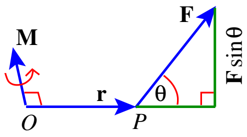
  (2), the determinant formula $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

  Don't be scared by this — crops up again later in course

Uses of cross product

- Special case: coordinate vectors
  - $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$
  - $\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$
  - sort of 'cyclic alphabetical order'
- Cross product gives a method for calculating
  - sine of angle between vectors
  - a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- Exercise: find a unit vector $\perp$ to both $(1, 2, 2)$ and $(1, 1, 1)$.

Application of vector product to moments



- Moment $\mathbf{M}$ about origin of force $\mathbf{F}$ acting at point P
- Naive (A-level Physics, Mechanics)

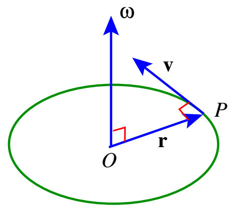
$$\text{moment} = \text{distance times normal component of force,}$$
$$= |\mathbf{r}||\mathbf{F}|\sin\theta$$

- More sophisticated (Degree level)

$$\text{moment } \mathbf{M} = \mathbf{r} \times \mathbf{F} \qquad \text{(gives same magnitude)}$$

- NB: moment is a vector
- direction of $\mathbf{M}$ gives axis that is twisted about

Engineering HOT SPOT



- Motion in a circle (about origin: i.e. simple case)
- Key characteristics:
  velocity $\mathbf{v} \perp$ displacement $\mathbf{r}$
  $\mathbf{r}$ and $\mathbf{v}$ in plane of circle
- So: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$
- $\boldsymbol{\omega} \perp$ to plane of circle
- More general motions also:
  $\boldsymbol{\omega}$ known as angular velocity
- NB: angular velocity is a vector
- Used in the definition of vorticity (important for why planes fly!)

Properties of the cross product

- LEARN THESE (all proved from component formula)
  1. $\lambda(\mathbf{a} \times \mathbf{b}) = (\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b})$
  2. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
  3. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ is equivalent to $\mathbf{a} \,||\, \mathbf{b}$
  4. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any $\mathbf{a}$
  5. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (careful!)
- Exercise: Prove 5. using component form
- Q. Is it *associative*: i.e. is $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$?
- A. No! think about it (I'll tell you why next lecture).

Homework

- Read *James* section 4.2
  (not last subsection on triple products)
- Do: (5th edition)
  - Exercises 4.2.7 Q11,13,14,26
  - Exercises 4.2.9 Q27,28,33
  - Exercises 4.2.11 Q41,48
- Do: (4th edition)
  - Exercises 4.2.6 Q1,3,4,15
  - Exercises 4.2.8 Q17,19,21
  - Exercises 4.2.10 Q31,34
- don't forget to go to a Drop-in Session if you get stuck!