Vibrations 2, Lecture 14
Forced harmonic vibration

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Lecture 13 review

Free vibration response via modal superposition:

$$\mathbf{x} = C_1\,\mathbf{a}_1 \cos(\omega_1 t + \varphi_1) + C_2\,\mathbf{a}_2 \cos(\omega_2 t + \varphi_2) + \ldots$$

Initial conditions used to determine the unknown free response constants:

$$t = 0 : \mathbf{x}(0) = \mathbf{x}_0,\ \ \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$$

Free harmonic vibration:

$$t = 0 : \mathbf{x}(0) = \alpha\,\mathbf{a}_i,\ \ \dot{\mathbf{x}}(0) = 0$$

Lecture 14

- Harmonic excitation or forcing
- Steady-state harmonic vibration response
- Amplitude-frequency characteristics

Harmonic excitation

Consider a forced MDOF system:

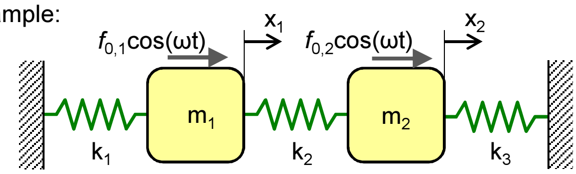
$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(\mathrm{t})$$

and assume that the harmonic load is defined as follows:

$$\mathbf{f}(\mathrm{t}) = \begin{bmatrix} f_{0,1}\cos(\omega\,\mathrm{t}) \\ \dots \\ f_{0,M}\cos(\omega\,\mathrm{t}) \end{bmatrix} = \begin{bmatrix} f_{0,1} \\ \dots \\ f_{0,M} \end{bmatrix} \cos(\omega\,\mathrm{t}) = \mathbf{f}_0\cos(\omega\,\mathrm{t})$$

where $\mathbf{f}_0$ is the vector of excitation amplitudes, ω is the excitation frequency

2DOF example:



Steady-state harmonic response

We study the steady-state response of MDOF system with harmonic excitation:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}_0 \cos(\omega t)$$

The response is assumed in the following form:

$$\mathbf{x} = \mathbf{x}_0 \cos(\omega t) \Rightarrow \ddot{\mathbf{x}} = -\omega^2\,\mathbf{x}_0 \cos(\omega t)$$

$\mathbf{x}_0$ is the vector of (*unknown*) response amplitudes, ω is the (*known*) excitation frequency. The phase angle is 0 or 90 degrees (in-phase or out-of phase). This information is be captured by the signs of the components of $\mathbf{x}_0$.

The vector of the response amplitudes is found after the following substitutions:

$$-\omega^2\mathbf{M}\;\mathbf{x}_0 \cos(\omega t) + \mathbf{K}\,\mathbf{x}_0 \cos(\omega t) = \mathbf{f}_0 \cos(\omega t)$$

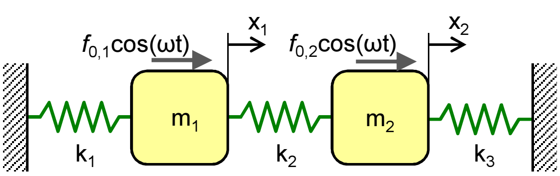
$$(\mathbf{K} - \omega^2\mathbf{M})\,\mathbf{x}_0 = \mathbf{f}_0$$

$$\mathbf{x}_0 = (\mathbf{K} - \omega^2\mathbf{M})^{-1}\,\mathbf{f}_0$$

where $(\ldots)^{-1}$ denotes inverse matrix.

2DOF example

Consider the following 2DOF system. Determine its harmonic response:



Mass matrix, stiffness matrix and excitation vector:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_{0,1} \\ f_{0,2} \end{bmatrix} \cos(\omega t)$$

2DOF example

The system of equations for this example is:

$$(\mathbf{K}-\omega^2\mathbf{M})\mathbf{x}_0=\mathbf{f}_0$$

$$\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{bmatrix}\begin{bmatrix} x_{0,1} \\ x_{0,2}\end{bmatrix}=\begin{bmatrix} f_{0,1} \\ f_{0,2}\end{bmatrix}$$

Matrix inverse for matrices of size 2×2 is:

$$\mathbf{A}^{-1}=\begin{bmatrix} A & B \\ C & D\end{bmatrix}^{-1}=\frac{1}{\det(\mathbf{A})}\begin{bmatrix} D & -B \\ -C & A\end{bmatrix}=\frac{1}{AD-CB}\begin{bmatrix} D & -B \\ -C & A\end{bmatrix}$$

The steady-state amplitudes of response are:

$$\begin{bmatrix} x_{0,1} \\ x_{0,2}\end{bmatrix}=\frac{1}{\det(\mathbf{K}-\omega^2\mathbf{M})}\begin{bmatrix} k_2+k_3-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix}\begin{bmatrix} f_{0,1} \\ f_{0,2}\end{bmatrix}$$

$$\det(\mathbf{K}-\omega^2\mathbf{M})=(k_1+k_2-\omega^2 m_1)(k_2+k_3-\omega^2 m_2)-k_2^2$$

2DOF example

From the previous derivations, we can show that $x_{0,1}$ and $x_{0,2}$ are functions of ω

$$x_{0,1}(\omega)=\frac{(k_2+k_3-\omega^2 m_2)f_{0,1}+k_2 f_{0,2}}{\det(\mathbf{K}-\omega^2\mathbf{M})}$$

$$x_{0,2}(\omega)=\frac{k_2 f_{0,1}+(k_1+k_2-\omega^2 m_1)f_{0,2}}{\det(\mathbf{K}-\omega^2\mathbf{M})}$$

In other words, the amplitudes of steady-state vibrations depend on the excitation frequency. These functions have singular values at the points where the excitation frequencies ω are equal to the natural frequencies ω_i. These points are defined as follows (see Eigenvalue Problem section):

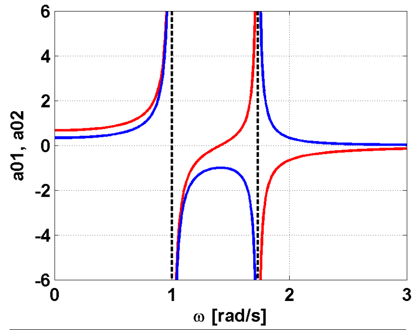
$$\det(\mathbf{K}-\omega_i^2\mathbf{M})=0$$

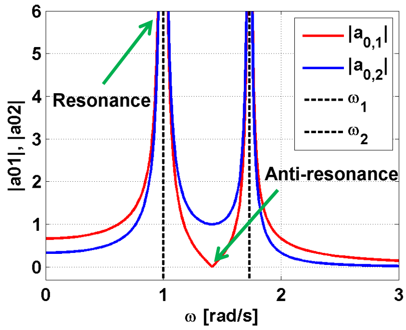
Based on this, the amplitudes of vibration reach infinite values if harmonically excited at one of the natural frequencies, i.e. *resonance/resonant excitation*:

$$x_{0,1}(\omega_i)\to\pm\infty,\ x_{0,2}(\omega_i)\to\pm\infty$$

2DOF example

The graphs of these functions are shown in real, absolute and logarithmic scales for the following system parameters: k_1=k_2=k_3=1 N/m, m_1=m_2=1 kg, f_1=1N, f_2=0N



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» m=1;k=1;f1=1;f2=0;
» M=diag([m,m]);K=[2*k,-k;-k,2*k];
» w=linspace(0,3,5e2);
» X1=zeros(1,5e2);X2=X1;
» x01=@(w)(((k+k-w^2*m)*f1+k*f2)/det(K-w^2*M))
» x02=@(w)((k*f1+(k+k-w^2*m)*f2)/det(K-w^2*M))
» for ii=1:5e2,X1(ii)=x01(w(ii));X2(ii)=x02(w(ii));end
» plot(w,abs(X1),'r',w,abs(X2),'b')
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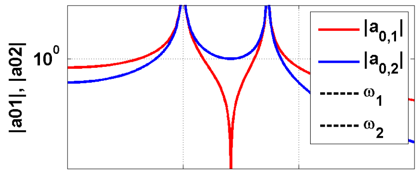

Summary

- In linear systems: harmonic excitation causes steady-state harmonic vibration at the same frequency
- Amplitude-frequency response functions have important properties, 2DOF example:
  - 2DOF – 2 resonant peaks
  - 2DOF – 2 natural frequencies
  - 2DOF – 2 mode shapes
  - 2DOF – 1 anti-resonance