Lecture 8
The Inertial Terms

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February 26, 2019


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The Inertial Terms

The Inertial Terms

$$A = \sum m(y^2 + z^2) \rightarrow I_{xx} \rightarrow I_x$$

$$B = \sum m(x^2 + z^2) \rightarrow I_{yy} \rightarrow I_y$$

$$C = \sum m(x^2 + y^2) \rightarrow I_{zz}$$

$$D = \sum myz \rightarrow I_{yz} or I_{zy}$$

$$E = \sum mxz \rightarrow I_{xz} or I_{zx}$$

$$F = \sum mxy \rightarrow I_{xy} or I_{yx}$$

Note: here x, y, z are positions of masses m relative to the coordinate origin.

Introduction

- Any study of the dynamics of an aircraft must allow for inertial factors and these are not always easily understood: You need to familiar with:

  - *moments of inertia*,

  - *products of inertia*,

  - *dynamic coupling of motions and of equations*

- i.e. need to know what to expect for the inertia terms in the equations of motion when they represent a typical aircraft configuration.


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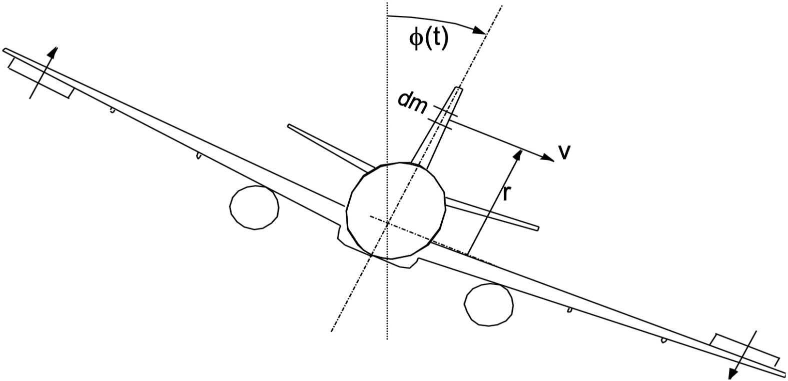
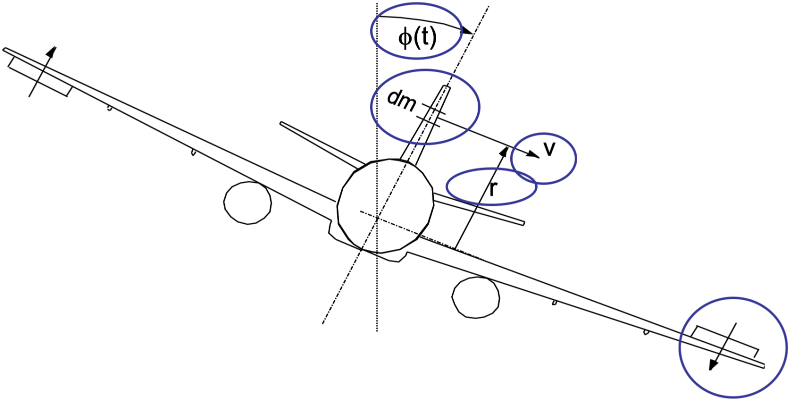
Rolling Example

Rolling Example



Rolling Example



Rolling Example

- Consider that a rolling moment has been applied via a deflection ξ [ksi] of the ailerons and is equal to $L(\xi)$. The differential element of momentum for a mass in the fin will be given by:

$$\text{d(momentum)} = dm \times \text{velocity} \qquad (1)$$

- but the local velocity (about the roll axis) is simply $v = r\dot{\phi}$ so the differential linear momentum is:

$$\text{d(momentum)} = r\dot{\phi}dm \qquad (2)$$

Rolling Example

- the differential *moment* of momentum (or angular momentum) about the roll axis is simply the product r x d(momentum) or:

$$\text{d(angular momentum)} = r^2 \dot{\phi} dm \qquad (3)$$

- Allowing for the rolling moment $L(\xi)$ being applied to the whole of the aircraft mass we can show:

$$L(\xi) = \frac{d}{dt}\left(\int_{vehicle} r^2 \dot{\phi} \, dm \right) = \ddot{\phi} \int_{veh.} r^2 \, dm \qquad (4)$$

Rolling Example

- the differential *moment* of momentum (or angular momentum) about the roll axis is simply the product r x d(momentum) or:

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Rolling Example

- *Note: The previous equation assumes that neither r (for any element dm), nor dm (for any part of the vehicle) changes with time*

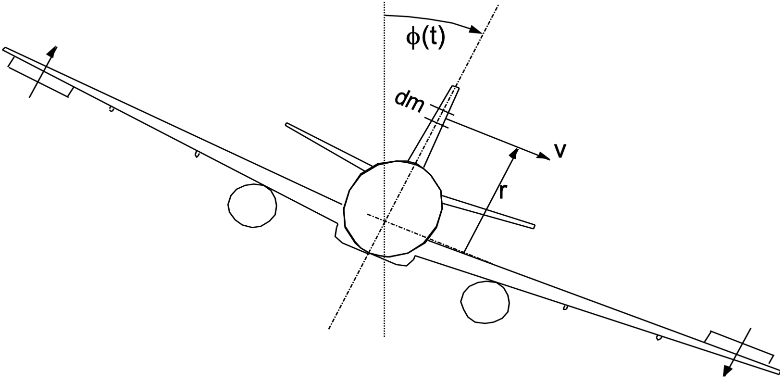
- Recognising that $\ddot{\phi}$ is an angular acceleration, it will become clear that

$$\int_{vehicle} r^2 \, dm$$

- is the inertial factor that we need for the "*angular or rotational inertia I*" quoted for Eqn.2. Clearly, it has the dimensions of

(mass) x (length)^2

Rolling Example

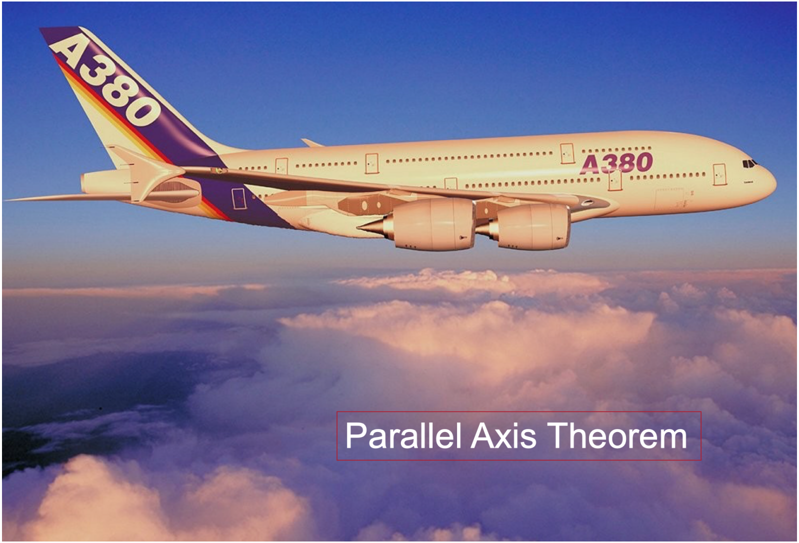


Rolling Example

- This is the general rule for determining the mass moment of inertia (i.e. the angular inertia) of mass rotating about a known axis, a value r being needed for every identifiable mass. e.g.

$$I_{roll} = I_{XX} = \sum_{vehicle} m_i r_i^2 \qquad (5)$$

- Obviously, the same set of masses with two different sets of r_i would be used for calculating *I* about the yaw (zz) or pitch axes (yy).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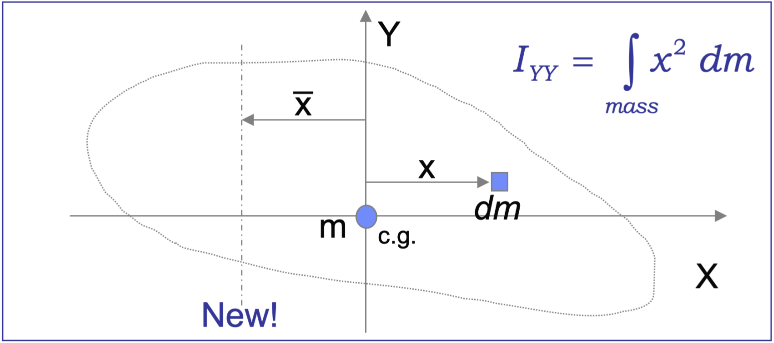
Parallel Axis Theorem

Parallel Axis Theorem



- The theorem says essentially that for a new axis, parallel with that through the c.g., the new value of I can be found to be

$$I_{new} = I_{CG} + m\bar{x}^2$$

Parallel Axis Theorem

- Need to be careful applying this, easy as it may be to employ, because it must not be interpreted as

$$\cancel{I_{new} = I_{current} + m\bar{x}^2} \quad !$$

where $\bar{x}$ is the distance through which the axis is to be moved. *The true separation between a new axis and the original c.g. axis must always be found.*

- Note: the implication that whatever the orientation of the axis, e.g. pitch, roll, yaw, the minimum value of I for rotation about that axis will exist when the axis goes through the c.g.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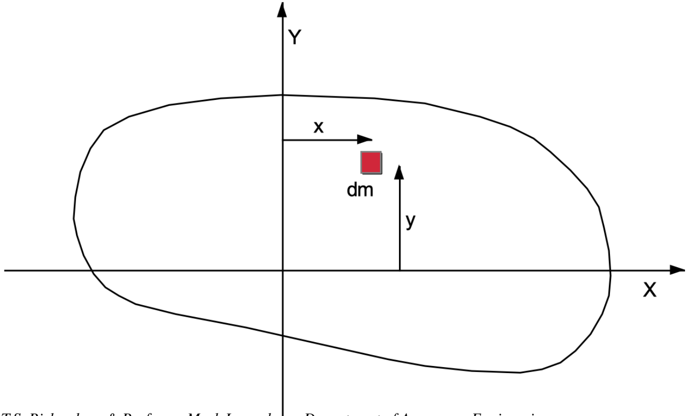
Product of Inertia

Product of Inertia

- The concept of a cross-inertia can be more difficult to understand than the moment of Inertia **I**.
- Start from known concepts.
- First and second moments.



First Moment

- The First Moment, for the body in the previous figure, is given by

$$\int_{body} y\,dm \quad or \quad \int_{body} x\,dm$$

and is associated with the task of finding a centre-of-mass or c.g.

- With regard to the problem of finding a centre-of-mass, when the axis passes through the c.g,

$$\int_{body} x\,dm = 0 \quad or \quad \sum_i m_i x_i = 0$$

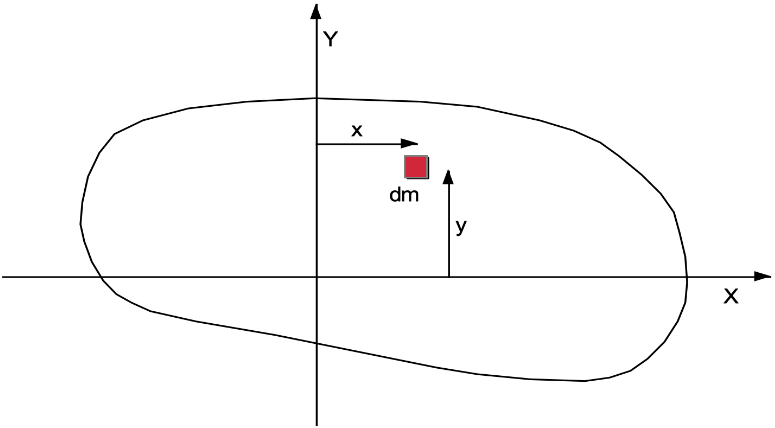
- Clearly, if both the X and Y axes give a zero moment of mass, the chosen origin would be at the c.g.

Second Moment

- in the previous section we used the definition

$$\int r^2 \, dm$$

A definition for the Product of Inertia



A definition for the Product of Inertia

- The classical definition quoted for this inertial factor is usually a form of

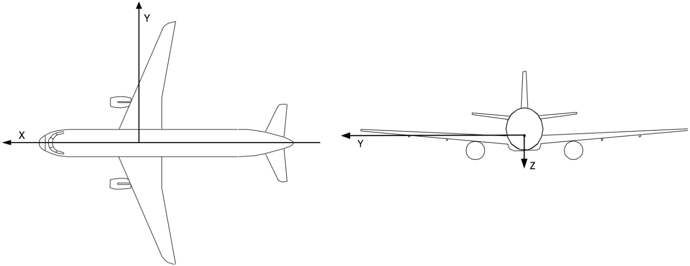
$$I_{XY} = \int_{body} xy \, dm$$

- This does not reveal the physical significance of I_{XY} and that is the challenge that remains!
- Under what conditions is this cross-inertia zero?

The physical significance of a Product of Inertia

Symmetric Bodies

- The easy part of this exercise is to consider how this relates to a body which shows symmetry.



- consider a normal aircraft configuration

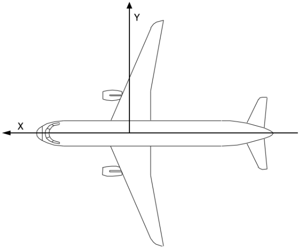
The physical significance of a Product of Inertia

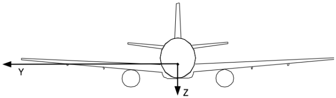
Symmetric Bodies

- If we think of the integral

$$I_{XY} = \int_{vehicle} xy \, dm$$

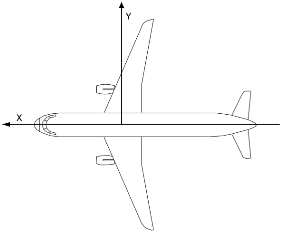
- and recognise that both x and y can be of either sign then the following must be true:





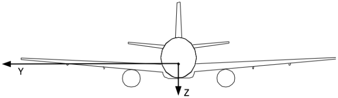
Symmetric Bodies

- - if, for any x, there are equal masses at $\pm$ y then the summation of (16) must be zero.



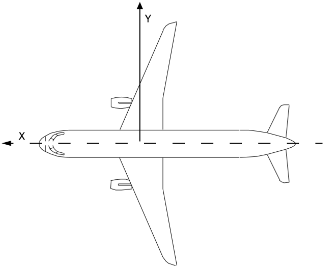
- Similarly, - if there are equal masses at $\pm$ y for any chosen z then

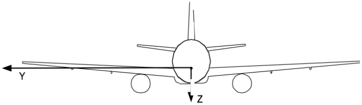
$$I_{YZ} = \int_{vehicle} yz\, dm = 0$$



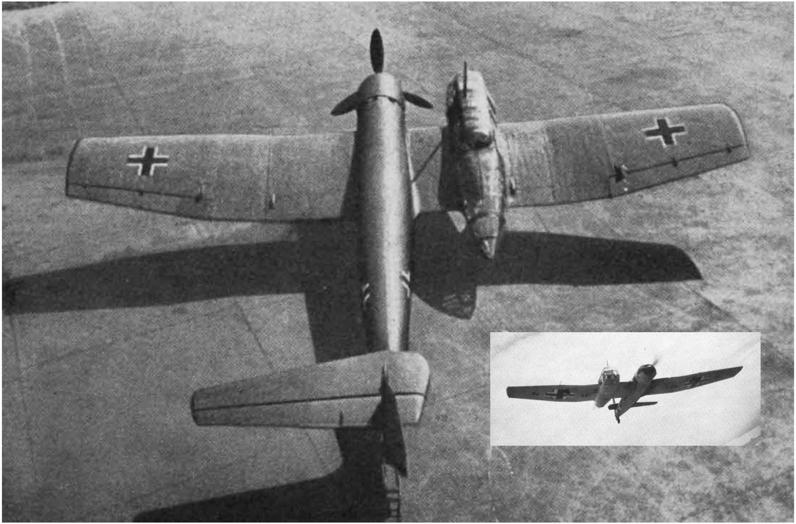
Symmetric Bodies

- Thus we have a plane of symmetry for a normal aircraft
- What is the physical significance?





Note: Bv 141 - a rare exception

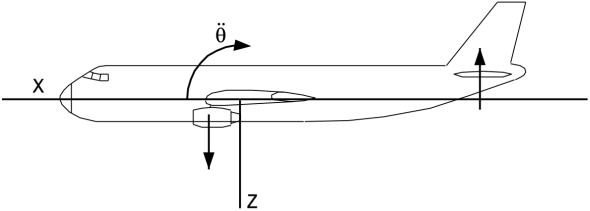


Symmetric Bodies

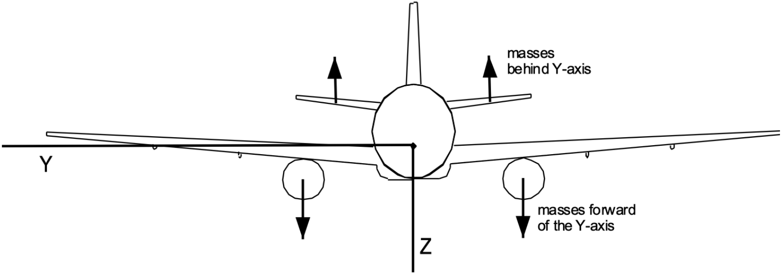
- if we were to impose a pitching acceleration $\ddot{\theta}$ about the Y-axis and then look at

$$\ddot{\theta} I_{XY} = \int_{veh.} \ddot{\theta} xy \, dm \quad or \quad \int_{veh.} y \, \ddot{\theta} x \, dm$$

- the product $\ddot{\theta} x$ being the local vertical acceleration, then $\ddot{\theta} x \, dm$ would be a local inertial force.

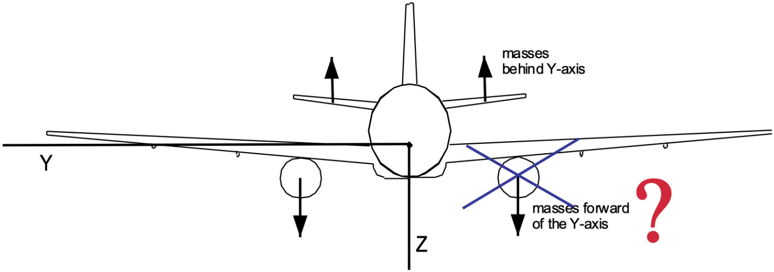


Symmetric Bodies



Reversed-effective forces are shown, a consequence of $\ddot{\theta}$

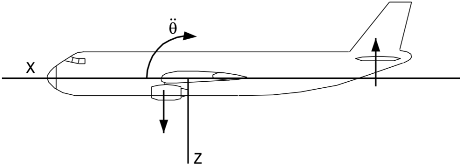
Symmetric Bodies



Symmetric Bodies

- Using a similar argument:

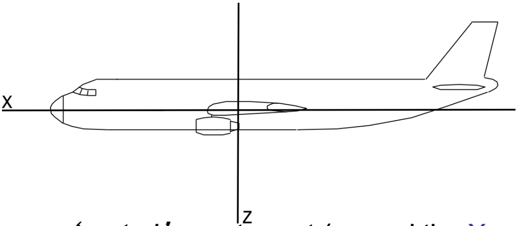
  the imposition of a pitching acceleration about the Y-axis does not induce a yawing moment about the Z-axis.



- These are special cases for a body which displays some symmetry when we consider the directions YZ and XY.

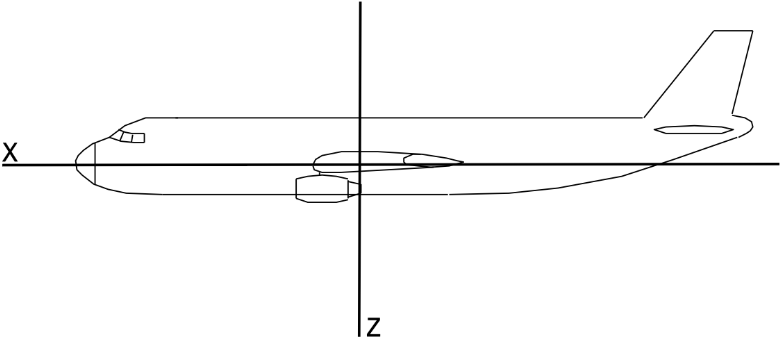
An Asymmetric Body

- An aircraft viewed from the side has several asymmetries:



- - the fin has no *'ventral'* counterpart (around the X- axis),
- - the wing and engines below X are not repeated above,
- - both of the above have no counterpart for the opposite sign of X (around the Z-axis),
- i.e. Asymmetry is present!

An Asymmetric Body



$$\dot{r}\ I_{XZ} = \int_{veh.} \dot{r}\, xz\, dm \qquad or \qquad \int_{veh.} z\ \dot{r}x\, dm$$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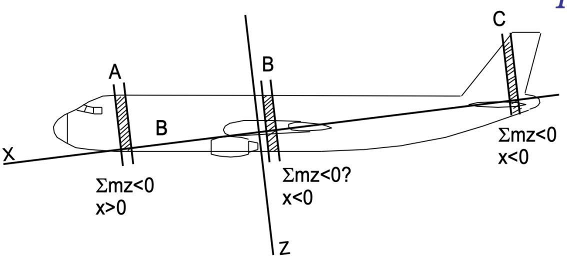
Principal Axes

An Asymmetric Body

- For convenience, we shall assume that our axis origin is at the c.g.,
- We shall concentrate on the inertial properties of only a few slices of the aircraft, each at a different value of x, while trying to establish the significance of:

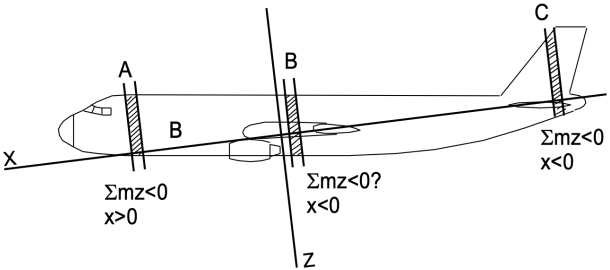
$$I_{xz} = \int_{vehicle} xz \, dm$$



An Asymmetric Body

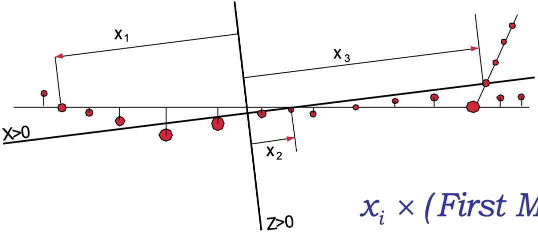
- Might use a practical version of this to evaluate I_{xz} using a list of component masses and their x,z positions:

$$I_{xz} = \sum_{vehicle} xz\,\Delta m$$



An Asymmetric Body

- We can then reduce the aircraft to a skeletal "equivalent". Locally, every First Moment is found (*some positive, some negative with respect to the chosen X-Z-axes*) and then "weighted" using:



$$x_i \times (First\ Moment)_i$$

- where the slices farthest away from the origin (largest x) have the greatest influence on the sum and on the ultimate sign of I_{XZ}

An Asymmetric Body

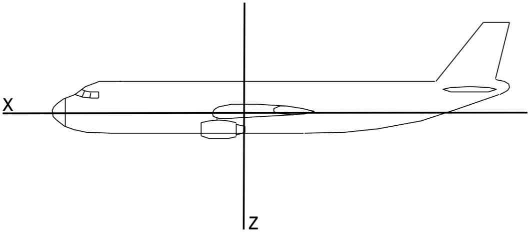
- It should be clear that some particular inclination of the X-Z axes will produce

$$I_{xz} = 0$$

- even though the likely choice of orientation for these axes for flight mechanics studies is not going to produce a zero value for this cross-inertia.
- In most cases a small inclination away from the horizontal (*for the X-axis*) will give that zero value.

Principal Axes

- all off-diagonal terms of the square matrix are coupling terms: the products of inertia.
- The previous slides showed that I_{XY} and I_{YZ} were zero, whereas a very careful choice of orientation for the X-axis could also produce $I_{XZ} = 0$
- Normally, however, the more convenient choice for the X-axis (along the fuselage) would produce $I_{XZ} \neq 0$.



Principal Axes

- When the choice of orientations for the axes produces zero values for all three products of inertia, the axes are said to be **principal axes** and clearly the square matrix below would become diagonal.

$$\begin{bmatrix} I_{XX} & -\cancel{I_{XY}} & -\cancel{I_{XZ}} \\ -\cancel{I_{YX}} & I_{YY} & -\cancel{I_{YZ}} \\ -\cancel{I_{ZX}} & -\cancel{I_{ZY}} & I_{ZZ} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

- The three rotational dynamic freedoms would then be de-coupled. *Every acceleration around one of these three 'special axes' would lead to an inertial reaction around only that one axis,* i.e. there would be direct inertial restraint against the acceleration. No other rotational response would be induced about another axis.

Conclusions

- It should be evident now! that for the standard set of rigid body aircraft equations of motion, we *should* expect to find

$$I_{XZ} \neq 0$$

- whereas symmetry will cause the other two cross-inertias to be zero. The positions of the two factors I_{XZ} *and* I_{ZX} (equal though having reversed subscripts) in the equations imply that

  a. when a rolling acceleration exists, there will be a yawing moment induced

  b. when a yawing acceleration exists, there will be a rolling moment induced

Next Lecture

Elevator Angle to Trim