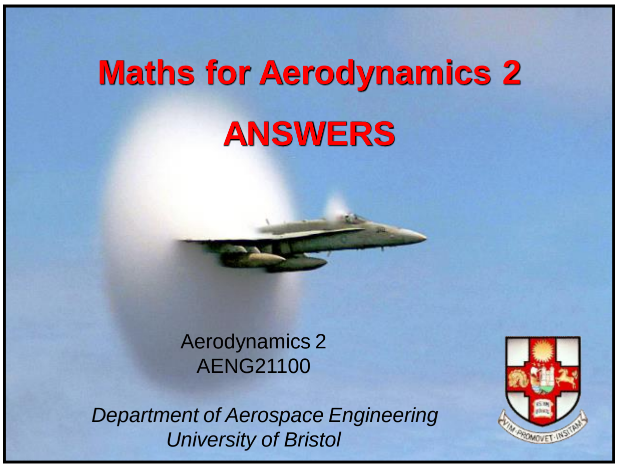


Differentiate the following functions with respect to x

x	$[1]$	$\sin x$	$[\cos x]$
x^2	$[2x]$	$\cos x$	$[-\sin x]$
x^n	$[nx^{n-1}]$	$\tan^{-1} x$	$\left[\dfrac{1}{1+x^2}\right]$
$\dfrac{1}{x}$	$\left[-\dfrac{1}{x^2}\right]$	$\dfrac{1}{x^2}$	$\left[-\dfrac{2}{x^3}\right]$
$\ln x$	$\left[\dfrac{1}{x}\right]$	$x\sin x$	$[\sin x + x\cos x]$
$x^2 \ln x$	$[2x\ln x + x]$	$\dfrac{1}{x}\ln x$	$\left[-\dfrac{1}{x^2}\ln x + \dfrac{1}{x^2} = \dfrac{1}{x^2}(1-\ln x)\right]$

Note: $\dfrac{d}{dx}\big(u(x)v(x)\big) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Aero2: Ans Handout2.*2*

Function of a function differentiation-differentiate with respect to x

$\ln(x^2)$	$\left[\dfrac{2}{x}\right]$	$\sin(x^2)$	$[2x\cos(x^2)]$
$(1+x^2)^2$	$[4x(1+x^2)]$	$\cos(1/x)$	$\left[\dfrac{1}{x^2}\sin(1/x)\right]$
$\sin(x^n)$	$[nx^{n-1}\cos(x^n)]$	$\tan^{-1}(x^3)$	$\left[\dfrac{3x^2}{1+x^6}\right]$
$\dfrac{1}{(1+\ln x)}$	$\left[-\dfrac{1}{x(1+\ln x)^2}\right]$	$\dfrac{1}{(1+\sin x)^2}$	$\left[\dfrac{-2\cos x}{(1+\sin x)^3}\right]$

Note: $\dfrac{d}{dx}\big(f(u(x))\big) = \dfrac{df}{du}\dfrac{du}{dx}$

Example: $\dfrac{d}{dx}\big(\sin(\ln x)\big) = \cos(\ln x)\left(\dfrac{1}{x}\right)$

Aero2: Ans Handout2.*3*

Partial Differentiation-Differentiate each function with respect to x and y

	x	y
$x+y$	$[1]$	$[1]$
$x^2y + xy^3$	$[2xy + y^3]$	$[x^2 + 3xy^2]$
$\sin(x+y)$	$[\cos(x+y)]$	$[\cos(x+y)]$
$\cos(xy)$	$[-y\sin(xy)]$	$[-x\sin(xy)]$
$\tan^{-1}\left(\dfrac{y}{x}\right)$	$\left[\dfrac{-y}{x^2+y^2}\right]$	$\left[\dfrac{x}{x^2+y^2}\right]$
$\dfrac{1}{xy}$	$\left[-\dfrac{1}{yx^2}\right]$	$\left[-\dfrac{1}{xy^2}\right]$
$\dfrac{1}{y}\cos(x)$	$\left[-\dfrac{1}{y}\sin x\right]$	$\left[-\dfrac{1}{y^2}\cos x\right]$
$y^2\sin(x^2)$	$[2xy^2\cos(x^2)]$	$[2y\sin(x^2)]$

Aero2: Ans Handout2.*4*

Integrate the following functions with respect to x

x	$\left[\frac{x^2}{2}\right]$	$\sin x$	$[-\cos x]$
x^2	$\left[\frac{x^3}{3}\right]$	$\cos x$	$[\sin x]$
x^n	$\left[\frac{x^{n+1}}{n+1}\right]$	c (constant)	$[cx]$
$\frac{1}{x}$	$[\ln x]$	$\frac{1}{x^2}$	$\left[-\frac{1}{x}\right]$
$\frac{1}{1+x^2}$	$[\tan^{-1} x]$		

Example of solving for a function given its partial derivatives

$$\phi_x = x^2 y + y\cos x \qquad \phi_y = \frac{1}{3}x^3 + \sin x + y$$

Integrate ϕ_x with respect to $x \Rightarrow \phi = \frac{1}{3}x^3 y + y\sin x + f(y)$

Differentiate ϕ with respect to $y \Rightarrow \phi_y = \frac{1}{3}x^3 + \sin x + f'(y)$

Comparing to ϕ_y $\; f'(y) = y \Rightarrow f(y) = \frac{1}{2}y^2$

So $\phi = \frac{1}{3}x^3 y + y\sin x + \frac{1}{2}y^2$

Solve for ϕ given its partial derivatives

1) $\phi_x = xy + \sin y\cos x \qquad \phi_y = \frac{1}{2}x^2 + \cos y\sin x + \frac{1}{2}y^2$

$$\phi = \frac{1}{2}x^2 y + \sin y\sin x + f(y) \;\Rightarrow\; \phi_y = \frac{1}{2}x^2 + \cos y\sin x + f'(y)$$

$$\Rightarrow f'(y) = \frac{y^2}{2} \;\Rightarrow f(y) = \frac{y^3}{6} + c \;\Rightarrow \phi = \frac{1}{2}x^2 y + \sin y\sin x + \frac{y^3}{6} + c$$

c is a constant

2) $\phi_x = \frac{\ln y}{x} + 1 \qquad \phi_y = \frac{\ln x}{y}$

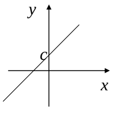
$$\phi = \ln x\ln y + x + f(y) \;\Rightarrow\; \phi_y = \frac{\ln x}{y} + f'(y)$$

$$\Rightarrow f'(y) = 0 \quad \Rightarrow f(y) = c \quad \Rightarrow \phi = \ln x\ln y + x + c$$

Sketch the following curves

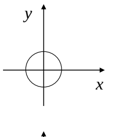
$y = mx + c \quad$ (c, m *constants)*

- Straight line
- Gradient m
- y–axis intercept c

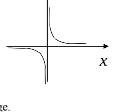


$x^2 + y^2 = c \quad$ (c, m *constant)*

- Circle
- Centre (0,0)
- Radius $\sqrt{c}$



$y = \frac{1}{x}$



Note: you should recognise these curves, if you don't check the notes on the next page.

More on curve sketching

When sketching a curve in the $x\,y$ plane which is not immediately recognisable you should:

(1) Find out what happens as $y \to 0$ & $x \to 0$ for example are there any places where the curve crosses the axes

(2) Consider the behaviour as $x \to \pm\infty$

(3) Look for any points where y tends to infinity and then consider the behaviour if this point is approached from either side

(4) Sometimes you might what to find max/mins via differentiation

(5) Choose what to plot on the axes to simplify the sketch and to make evaluation of sample points easier.

(6) Evaluate a few sample points if necessary.

Examples

1) $y = \dfrac{x-3}{(x-1)^2}$

$(i)\quad y=0 \Rightarrow x=3,\ \ x=0 \Rightarrow y=-3$

$(ii)\ \ x \to \pm\infty\ \ |x-3| << (x-1)^2 \quad so \quad y \to \pm 0$

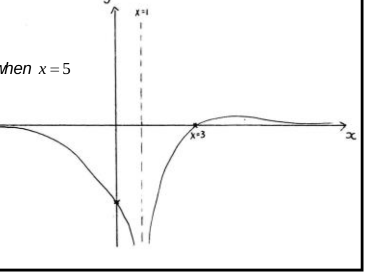
$(iii)\ y \to \infty$ *when* $x = 1.\ For\ \ x = 1 \pm \delta$

$(x-1)^2 > 0$

$(x-3) < 0$

$y < 0$

$(iv)\ \dfrac{dy}{dx} = \dfrac{5-x}{(x-1)^3} \Rightarrow \dfrac{dy}{dx} = 0$ *when* $x = 5$



Examples

2) $y = -\dfrac{1}{2}\left(1 + \dfrac{\sqrt{1+(x/2)^2}}{(x/2)}\right)$ *or* $y = -\dfrac{1}{2}\left(1 + \dfrac{\sqrt{1+(x')^2}}{(x')}\right)$ *where* $x' = x/2$

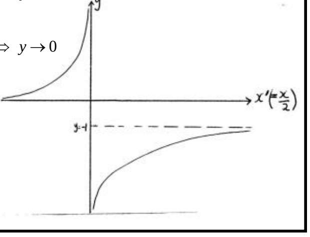
$(i)\quad x' \to 0 \Rightarrow \dfrac{\sqrt{1+(x')^2}}{(x')} \to \dfrac{1}{0} \to \infty \Rightarrow\ y \to \infty$

$x' \to +\delta \quad y < 0$ *and* $x' \to -\delta \quad y > 0$

No solution for $y = 0$

$(ii)\ \ x \to +\infty\ \ \dfrac{\sqrt{1+(x')^2}}{(x')} \to \dfrac{\sqrt{(x')^2}}{(x')} \to +1 \Rightarrow\ y \to -1$

$x \to -\infty\ \ \dfrac{\sqrt{1+(x')^2}}{(x')} \to \dfrac{\sqrt{(x')^2}}{(x')} \to -1 \Rightarrow\ y \to 0$



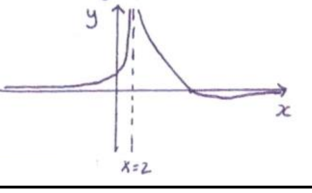
Sketch the following curve

1) $y = \dfrac{8-x}{(x-2)^2}$ $\quad (i)\ \ y=0 \Rightarrow x=8, \quad x=0 \Rightarrow y=2$

$(ii)\ x \to \pm\infty \quad |8-x| << (x-2)^2$ so $y \to \mp 0$

$(iii)\ y \to \infty$ when $x = 2$. For $x = 2 \pm \delta,\ \ (x-2)^2 > 0$ and $(8-x) > 0 \Rightarrow y > 0$

$(iv)\ \dfrac{dy}{dx} = \dfrac{x-14}{(x-2)^3} \Rightarrow \dfrac{dy}{dx} = 0$ when $x = 14$



Miscellaneous

(1) Fill in the brackets

$$\ln a + \ln b = \ln(\quad ab \quad)$$
$$\ln a - \ln b = \ln(\quad a/b \quad)$$

(2) If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$ where A and B are either both zero or both infinite then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

is called indeterminate $0/0$ or ∞/∞. However the limit can be evaluated using L'Hospital's Rule if the following limit is determinate.

.

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{df(x)/dx}{dg(x)/dx}$$

Aero2: Ans Handout2.*13*