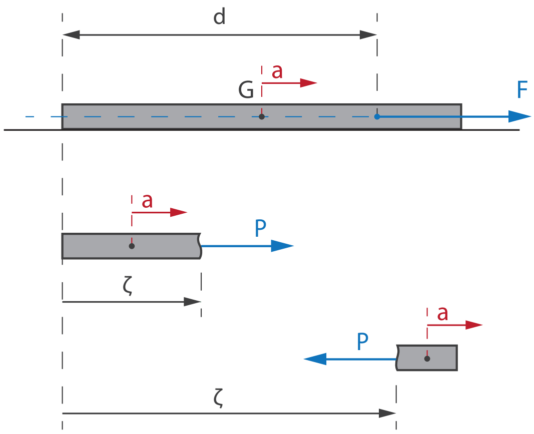
Handout 7 – Inertial Stresses

In this mechanics course we generally consider the dynamics of *rigid* bodies. Unlike in your structures unit, we therefore assume that the bodies *do not deform* under the applied loads. Nonetheless, it is important to realise that dynamic motion will result in inertial stresses inside the bodies. For example, in the Bloodhound Super Sonic Car the wheels will spin at approximately 10200 rpm, generating huge radial stresses which required specialised manufacturing techniques to prevent failure!

7.1 Sliding Rod

Consider a rod with mass m, length L and uniform cross-sectional area A on a frictionless surface. A force F is applied at a distance d from the end. The acceleration of the centre of mass G is: $a = F/m$.

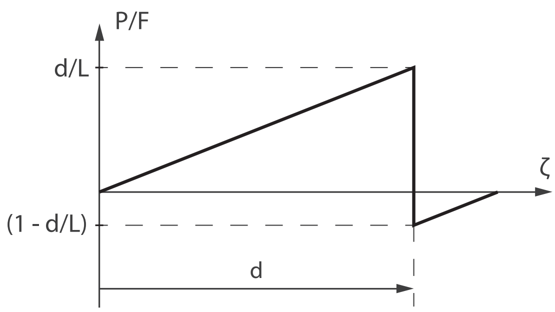


To find the axial stress inside the beam as a result of the acceleration, we make a cut at a distance ζ from the end of the bar. The acceleration a of the cut section will be the sam as that of the rigid body, allowing us to find the internal force P (and stress $\sigma = P/A$) at any point along the bar. For $0 < \zeta < d$ this gives a tensile force:

$$P = \frac{\zeta}{L} m\, a$$

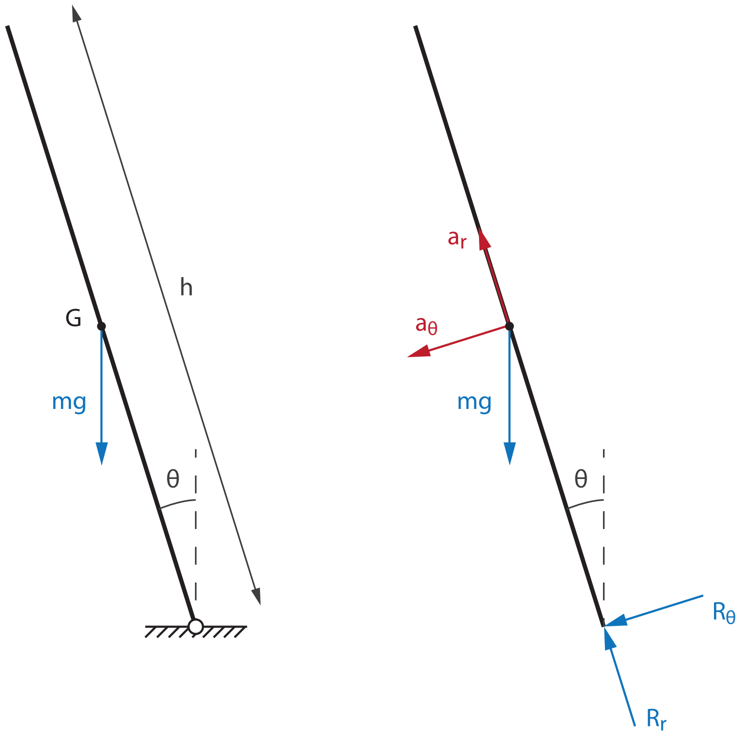
and for $d < \zeta < L$ we find a compressive force:

$$P = -\left(1 - \frac{\zeta}{L}\right) m\, a$$



7.2 Falling Chimney

Looking at recordings of tall masonry chimneys being demolished, a trend can be observed where they break at approximately one-third from the base – we can analyse this phenomenon using the knowledge you have acquired in mechanics and structures!



Rigid Body Dynamics Consider a chimney freely pivoted around its base (as a result of the demolition explosion) which has rotated by an angle θ. It has a height h and mass m, and has a uniform cross-section. The moment of inertia of the chimney around the pivot point is:

$$I_O = \frac{mh^2}{3}$$

Considering moments around the base:

$$\sum M : \qquad mg\sin\theta\frac{h}{2} = I_O\ddot{\theta}$$

we find the equation of motion:

$$\ddot{\theta} = \frac{3}{2}\frac{g}{h}\sin\theta$$

To find the angular velocity, we use $\dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta$ and integrate:

$$\int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \int_0^{\theta} \ddot{\theta}\,d\theta = \int_0^{\theta} \frac{3}{2}\frac{g}{h}\sin\theta\,d\theta$$

$$\frac{1}{2}\dot{\theta}^2 = \frac{3}{2}\frac{g}{h}(1-\cos\theta)$$

$$\dot{\theta} = \sqrt{3\frac{g}{h}(1-\cos\theta)}$$

From these equations we find the accelerations of the centre of mass:

$$a_r = -\dot{\theta}^2 \, \frac{h}{2}$$

$$a_\theta = \ddot{\theta} \, \frac{h}{2}$$

which are used to find the reaction forces at the base of the chimney:

$$\sum F_r : \qquad R_r - mg\cos\theta = m\,a_r$$

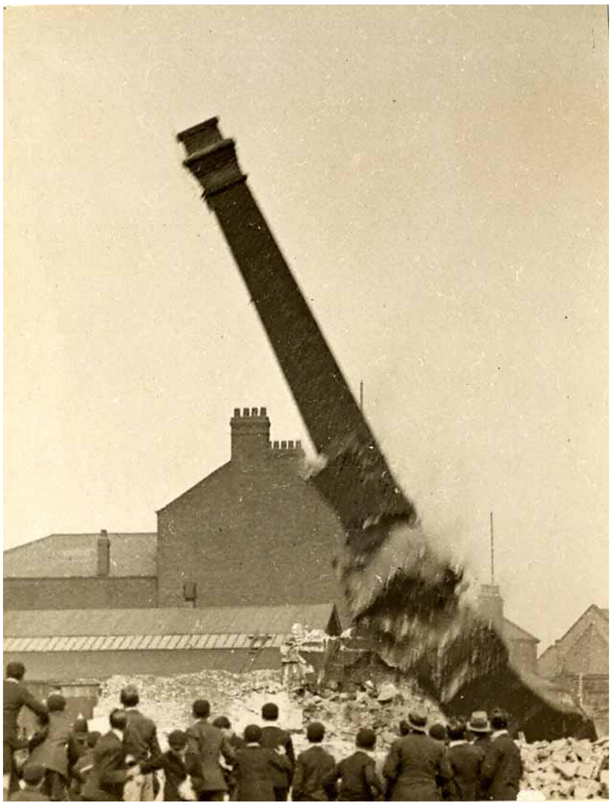
$$\sum F_\theta : \qquad R_\theta + mg\sin\theta = m\,a_\theta$$

After substituting the earlier results, we find:

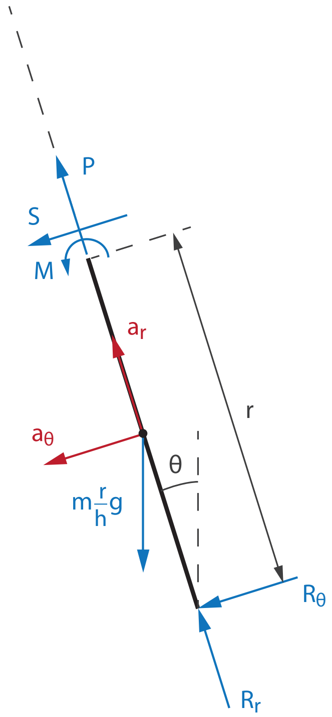
$$R_r = mg\left(\frac{5}{2}\cos\theta - \frac{3}{2}\right)$$

$$R_\theta = -\frac{1}{4}mg\sin\theta$$

Thus far the analysis has not differed from that of a compound pendulum. In the next step, however, we shall consider internal stresses in the falling chimney, which will cause it to break.



Internal Stresses Finding the internal stresses in the chimney is done no different from any analysis in structural mechanics: we make a cut through the rigid body at a distance r from the base to expose the internal forces (shear S and axial P) and moment M. The values of the internal forces are then found by applying the equations of motion to the cut section of the chimney, and using the angular velocities and accelerations found from the rigid body dynamics analysis.



As the chimney is assumed to have a uniform cross-section, the mass of the cut section will be mr/h with its centre of mass at $r/2$. The moment of inertia around the pivot point is calculated accordingly. Writing the Newton equations for the acceleration of the centre of mass:

$$\sum F_r: \qquad -m\frac{r}{h}g\cos\theta + P + R_r = -m\frac{r}{h}\frac{r}{2}\dot{\theta}^2$$

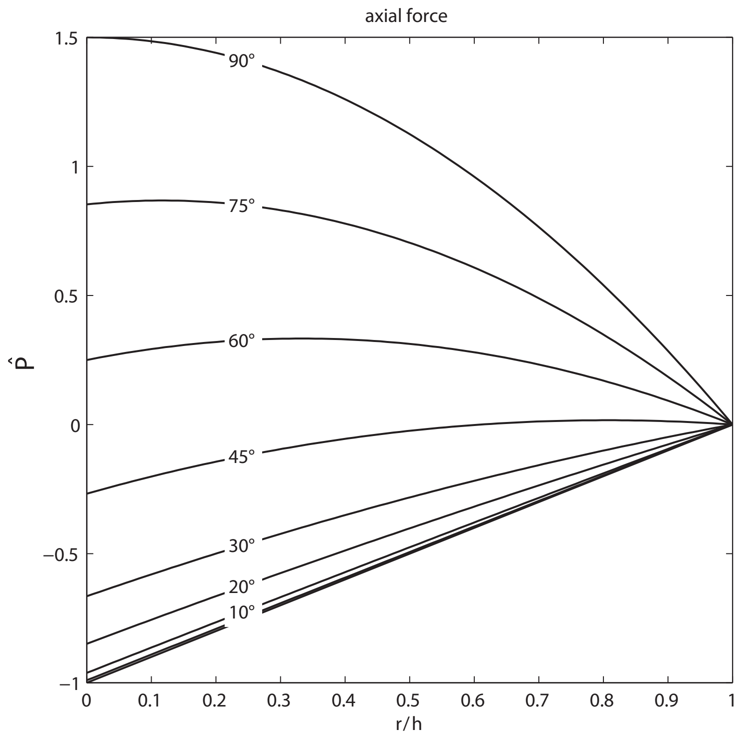
$$\sum F_\theta: \qquad m\frac{r}{h}g\sin\theta + S + R_\theta = m\frac{r}{h}\frac{r}{2}\ddot{\theta}$$

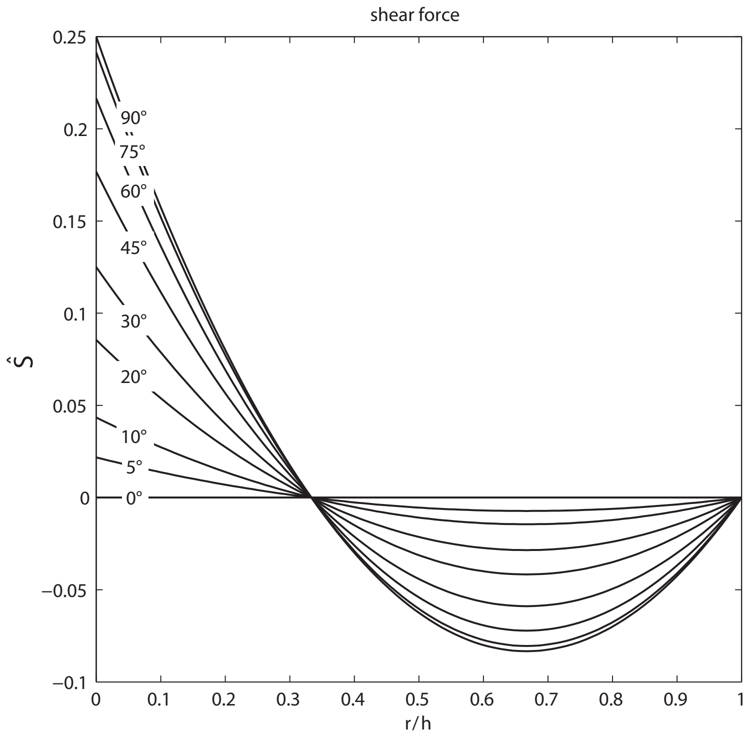
Substituting for $\ddot{\theta}$ and $\dot{\theta}$ and rearranging, we find:

$$\hat{P} = \frac{P}{mg} = \cos\theta\left(\frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{r}{h} - \frac{5}{2}\right) - \frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{3}{2}$$

$$\hat{S} = \frac{S}{mg} = \sin\theta\left(\frac{3}{4}\left(\frac{r}{h}\right)^2 - \frac{r}{h} + \frac{1}{4}\right)$$

where $\hat{P}$ and $\hat{S}$ are dimensionless, and are expressed in terms of a dimensionless parameter r/h. This makes the results applicable to any chimney!


axial force
90°
75°
60°
45°
30°
20°
10°
P̂
r/h


shear force
90°
75°
60°
45°
30°
20°
10°
5°
0°
Ŝ
r/h

Next we find consider the angular acceleration around the base of the chimney:

$$\sum M: \qquad M + mg\frac{r}{h}\sin\theta\frac{r}{2} + Sr = \frac{1}{3}m\frac{r}{h}r^2\ddot{\theta}$$

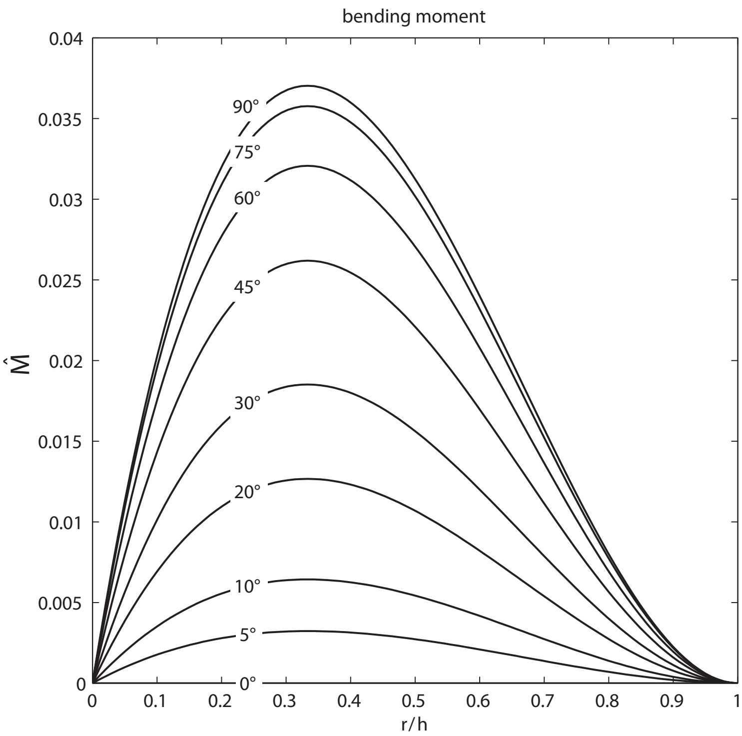
which after substituting S and $\ddot{\theta}$ results in:

$$\hat{M} = \frac{M}{mgh} = -\frac{1}{4}\sin\theta\left(1-\frac{r}{h}\right)^2\frac{r}{h} \tag{7.1}$$

To find the position r/h at which the bending moment is maximum, take the derivative:

$$\begin{aligned}\frac{d\hat{M}}{d(r/h)} &= \left(3\left(\frac{r}{h}\right)^2 - 4\frac{r}{h} + 1\right)\frac{\sin\theta}{4} = 0\\ &= \left(\frac{r}{h}-\frac{1}{3}\right)\left(\frac{r}{h}-1\right)\frac{\sin\theta}{12}\end{aligned}$$

and therefore $r/h = 1/3$ or $r/h = 1$. In other words, the maximum bending moment will always be 1/3 from the base of the chimney; for *any* chimney with a uniform cross section!

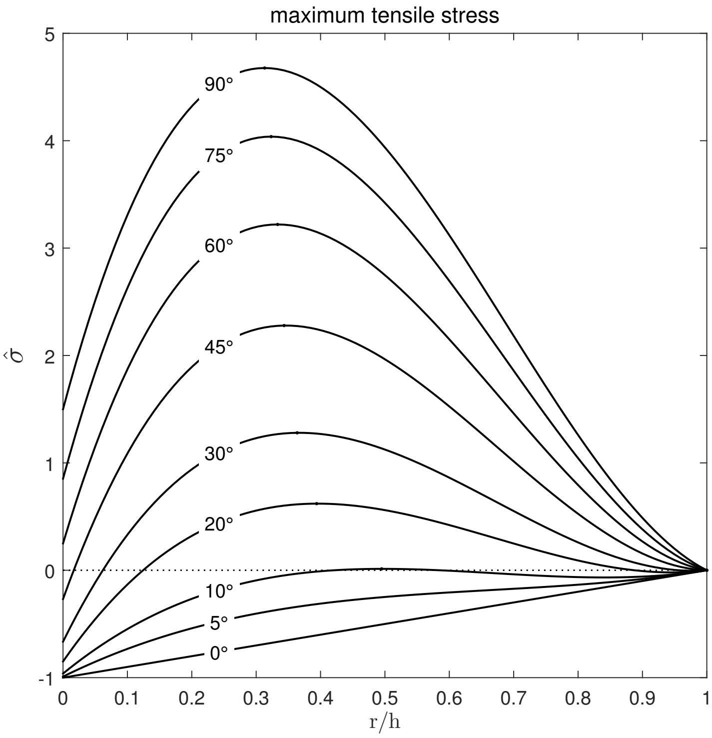


Failure : Maximum Tensile Stress Often chimneys are constructed of masonry, with a low tensile strength. A potential point of failure could be the location with maximum tensile stress. This would be a combination of axial load P and bending moment M. If we assume a solid square cross-section with dimensions a, the maximum tensile stress will be:

$$\sigma_{\text{max}} = \frac{P}{A} + \frac{Ma}{2I_{xx}}$$

where the I_{xx} is the second moment of area of the chimney cross-section, here given as: $I_{xx} = \frac{a^4}{12}$ for a solid square cross-section. Again using a dimensionless formulation, this gives:

$$\hat{\sigma} = \frac{\sigma_{\text{max}}a^2}{mg} = \cos\theta\left(\frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{r}{h} - \frac{5}{2}\right) - \frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{3}{2} + \sin\theta\frac{3}{2}\frac{r}{h}\frac{h}{a}\left(1-\frac{r}{h}\right)^2$$



The precise point at which a falling chimney breaks will depend on the strength of the material. Shear failure will occur closer to the base, whereas the location of tensile failure (combined axial and bending loading) will depend on the rotation of the chimney.

For the demonstration model used in the lecture[1], using a stack of loose blocks, the failure is expected to be in tension as the friction between blocks will provide some resistance to shear. For a slenderness of $h/a = 15$ the results are plotted; the first tensile stress is observed at $r/h \approx 0.5$ at 10° while for greater rotations, it shifts closer to the limit of $r/h = 1/3$.

Revision Objectives Handout 7:

Inertial Stresses

- appreciate how dynamic motion generates stresses inside structures
- calculate internal stresses for simple dynamic systems

Note: details of the calculations for the moments/stresses in the falling chimney are **not** examinable.

[1] The analysis presented here follows that described in: G. Varieschi and K. Kamiya (2003), "Toy models for the falling chimney", American Journal of Physics, Volume 71, Issue 10. doi:10.1119/1.1576403