# APM: Structural models

IRF
1.12.2013

Note Title
29/01/2009



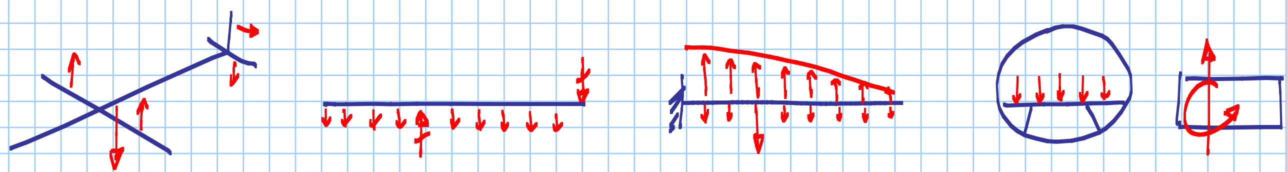
## Part Models:

To model a structure we need to idealise:
- Structural parts as representative elements
- Element sections as effective areas
- Applied loading as point or distributed loading
- Reaction supports as roller, pinned, spring or fixed, etc.

Begin with simplest models evaluated by hand calcs!

E.g. Simple line elements:

For Complete Aircraft   Fuselage   Wing   Sub Assy.



I.e. start from simplest possible representations then refine

Considering the relative conservatism of each model.





Start with simple line elements and consider elements according to the type of loading they carry, e.g.:

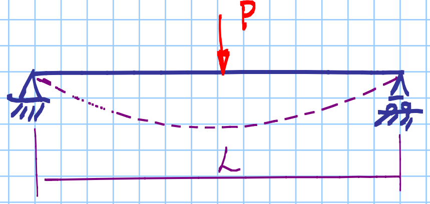
# Bars (rods, ties, struts) carrying axial loads in tension or compression

# Beams-carrying transverse loads and bending moments

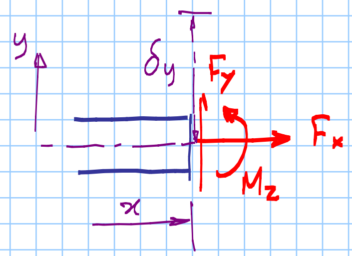
# Shafts- carrying torsion

# Elements carrying combined loadings.

E.g. Simple 2D beam element




At a given location the part model provides the deflection and the internal forces and moments "Stress resultants". E.g.:




$$\delta_y = f(k, P, L^3, E, I, x)$$

"deflection constant"

$$F_x, F_y, M_z = f(k, P, L, x)$$

"Force / Mmt constant"

See SM 1,2

Deflection and force or moment constants depend on the type of loading and supports

Lists of expressions for standard configurations can be found in numerous texts.

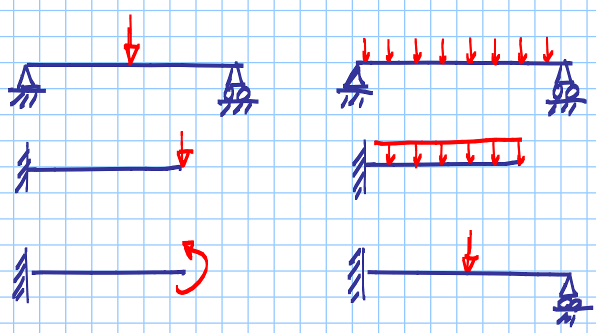




Standard beam models and formulae:

Simplify the configuration to a standard beam model to provide an initial check before analysing with more complex models.

E.g.



Look up standard beam formulae in validated sources.

Note assumptions + approximations!

Assuming the response is linear with respect to the material and structure we can use superposition of the results of different loading configurations.

# Constitutive equations



| | "Rigidity" | Constitutive Relationship | | Deformation | |
|---|---|---|---|---|---|
| Axial Stiffness: | $EA$ | $\sigma = \frac{F}{A} = \frac{Ed}{L}$ | : | $\frac{d}{L} = \varepsilon = \frac{F}{EA}$ | $\varepsilon$ = strain |
| Bending Stiffness: | $EI$ | $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$ | : | $\frac{1}{R} = \kappa = \frac{M}{EI}$ | $\kappa$ = "Curvature" |
| Torsional Stiffness: | $GJ$ | $\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$ | : | $\frac{\theta}{L} = \theta' = \frac{T}{GJ}$ | $\theta'$ = Rate of twist |

Note "stiffness" consists of material and geometric components

Modulus  Area + Length

Design not to exceed an allowable deformation, usually at limit load.

You should compile and expand your own list of useful formulae, e.g.:



# "Back of the envelope" formulae

Compile your useful equations!



## AXIAL "$\frac{EA}{L}$"

$$\sigma = \frac{F}{A}$$

$$\sigma = E\varepsilon$$

$$\varepsilon = e/L$$

$$\varepsilon^T = \alpha \Delta T$$

$$F = kd$$

$$k = \lambda = \frac{EA}{L}$$

$$G = \frac{E}{2(1+\nu)}$$

## BENDING "$EI$"

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

$$\sigma = EyK$$

$$I = \frac{bd^3}{12}$$

$$I = \frac{bd^3}{3}$$

$$I = \frac{\pi R^4}{4}$$

$$I = \frac{\pi}{4}(R^4 - r^4)$$

$$I \approx \pi R^3 t$$

$$\bar{y} = \Sigma(A_i y_i)/\Sigma A_i$$

$$I = \Sigma(I_i + A_i y_i^2)$$

## SHEAR "$GA$"

$$\tau = \frac{S}{A_s}$$

$$\tau = G\gamma$$

$\gamma =$ rad

$$q = \tau t$$

$\bar{y}_1$ NA $y_1$

$$q_1 = \frac{S}{I} A_1 \bar{y}_1$$

S NA S

$$q = \frac{S}{I} \int t y \, ds$$

$$A_s \approx \tfrac{5}{6} A$$

$$A_s \approx A_w$$

$$A_s \approx \frac{A}{2} = \pi R t$$

## TORSION "$\frac{GJ}{L}$"

$$\frac{\tau}{R} = \frac{T}{J} = \frac{G\theta}{L}$$

$$J = \frac{\pi R^4}{2} = I_{xx} + I_{yy}$$

$$J = \frac{\pi}{2}(R^4 - r^4)$$

$$J = 2\pi R^3 t$$

$$q = \frac{T}{2A}$$

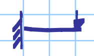
$$K_{T\,\text{OPEN}} = \Sigma \frac{bt^3}{3}$$

$$K_{T\,\text{CLOSED}} = \frac{4A^2}{\oint \frac{ds}{t}} = \frac{4A^2}{\Sigma(b/t)}$$

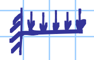


ctd ↓

$d = \frac{F}{EA/L}$ $\delta_b = k_b \frac{PL^3}{EI}$ $\delta_s = k_s \frac{PL}{GA_s}$ $\theta = \frac{TL}{GJ}$

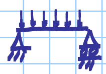


$k_b = 1/3$ , $k_s = 1$



$k_b = 1/8$ , $k_s = 1/2$

$k_b = 1/48$ , $k_s = 1/4$



$k_b = \frac{5}{384}$ , $k_s = \frac{1}{8}$

Buckling: Strut: $P_{CRIT} = \frac{\pi^2 EI}{L^2}$ Panel: $\sigma_{CRIT} = kE\left(\frac{t}{b}\right)^2$

PV: $\sigma_H = \frac{PR}{t}$ , $\sigma_L = \frac{PR}{2t}$ $\varepsilon_{x_{2D}} = \varepsilon_x - \nu\varepsilon_y = \frac{1}{E}(\sigma_x - \nu\sigma_y)$ etc.

Panel poisson constraint correction : $(1-\nu^2)$

etc. !





## Section Models:

### Section Areas

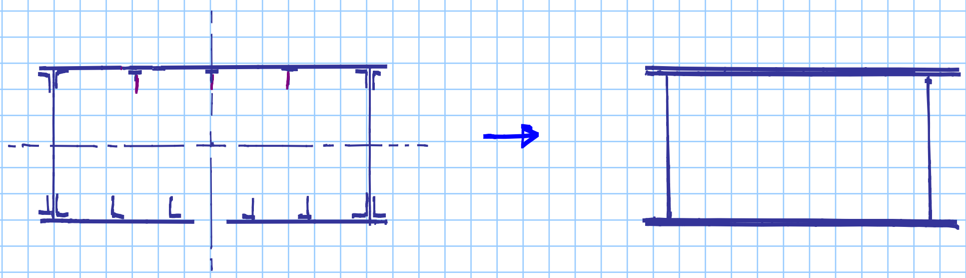
To start we need to define the effective section area, which is often less than the actual area, particularly for light thin-wall structures.

"Thin-wall approximation" implies that the section is made up from thin elements which can reasonably be modelled as "line elements". Typically, for this to be acceptable the thickness to width ratio would be less than 1:20, although for initial design we may accept ratios up to 1:10.

Making a thin-wall assumption we can neglect the second order 2nd moment of area terms in Parallel-axis calculations for compound sections.

## Semi-monocoque smeared areas and $A_s/bt$ ratios

To assist the initial sizing of stiffened box beams (typical of aircraft wings) we can use a smeared model approach. I.e. where skins, stringers and spar caps are considered initially as a single effective thickness.

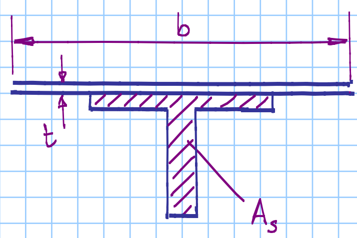


This simplified approach provides a quick estimate of the required area of material in the covers (skins + stringers + spar caps) for strength and deflection allowables assuming uniform distributions of stress.





In order to develop the smeared cover thickness into separate skin and stiffener (stringer or spar cap) items we can then refer to "$A_s/bt$" ratios. I.e.:



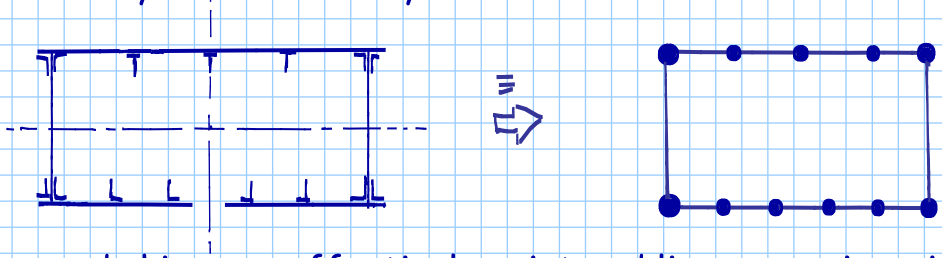

Where:
$A_s$ = stiffener area
$b$ = stiffener pitch
$t$ = skin thickness

Typically, we aim for $A_s/bt = 1.0$ to provide a reasonably efficient ratio of stiffener to skin areas, although this is just a rough guideline and the target ratio may vary. Using the chosen $A_s/bt$ ratio we can then deduce an initial sizing of skins and stiffeners.

But note, in compression this sizing will need to be reviewed for stability requirements - which must be considered at an itemised section element stage, i.e. accounting for actual skin and stiffener section geometry and column buckling of the effective skin/stiffener boom.

## Boom and Skin ideallisation

A common ideallisation for thin wall semi-monocoque (stiffened skin) structures is to represent concentrations of cross-sectional area (stiffener +skin) as booms which are assumed to carry direct end load only and skins which are assumed to carry shear load only.



The booms and skins are effectively point and line areas, i.e. with no defined shape or thickness.

The boom areas are defined in terms of a stiffener cross-sectional area plus an effective width of skin attached to the stiffener.

Effective widths will depend on whether loading is predominantly tensile or compressive. Due to stability or shear lag we tend to claim only partial effectiveness of the skins in carrying direct end load by using effective widths.



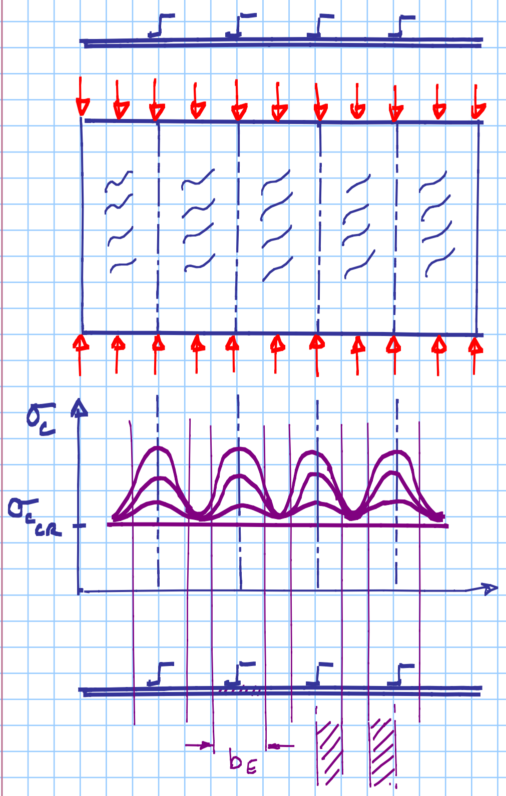
## Effective widths

Stiffeners or beams which are attached to plates along their length can be considered in combination with an effective width of the plate carrying end loading.

In tension we can assume that the majority/all the skin is effective in carrying end load.

In compression, beyond the onset of plate buckling we can only assume that only a part of the skin width next to the attached stiffener or beam is effective.

E.g. consider the behaviour of a stiffened panel carrying a compressive end load.




Up to the critical buckling strength of the panel, $\sigma_{c_{cr}}$, the skin contributes fully in carrying the compressive end load and the skin stress, $\sigma_c$, increases with the load.

However, once the skin buckles its contribution away from the stiffeners remains approximately constant and only the skin near to the stiffeners continues to carry further compressive end load as the load increases.

To account for this we use the idea of an "effective" panel width, $b_E$.





For a panel consisting of effective widths between stiffeners

using $\sigma_{c_{cr}} = K_c E \left(\frac{t}{b}\right)^2$ we can derive an expression for the effective width

as: $b_E = \sqrt{\frac{K_c E}{\sigma_c}}\, t$ where $b_E$ = effective width

and $\sigma_c$ = stiffener allowable stress (usually the stiffener crippling stress)

(Assuming stiffeners are bounded on either side by other stiffeners)

For $\frac{b}{t} \leq 40$ $K_c = 3{\cdot}62$ - i.e. stiffener providing hinged support for thick narrow panel

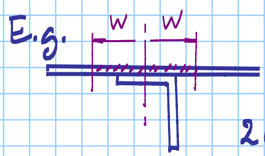
For $\frac{b}{t} \geq 110$ $K_c = 6{\cdot}32$ - i.e. stiffener providing clamped support for thin wide panel

For $40 < \frac{b}{t} < 110$ $K_c$ can be obtained by interpolation, (e.g. Niu Fig. 14.2.3)

Values obtained are often reduced according to test results, typically by a 10%. reduction or nominally rounded down.

Considering $b_e = 2w$, where $w$ = width of panel either side of a fastener joining the plate and stiffener then for a panel with b\t < 40 where the stiffener is bounded on each side by other stiffeners we get:

E.g.



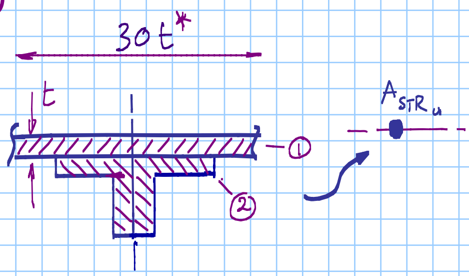

typically reduced to

$$2w = b_e = t\sqrt{\frac{3.62E}{\sigma_c}} = 1.90t\sqrt{\frac{E}{\sigma_c}} = 1.70t\sqrt{\frac{E}{\sigma_c}} \quad \text{ie } w = 0.85t\sqrt{\frac{E}{\sigma_c}}$$

Typically, for light aircraft structure an effective skin width of 30*t* is claimed with a stringer as a boom area, where *t* is the skin thickness. (Half this value, ~ 15t, is claimed for the DBT wing since only hollow pop rivets are used and the skin thicknesses are very thin)

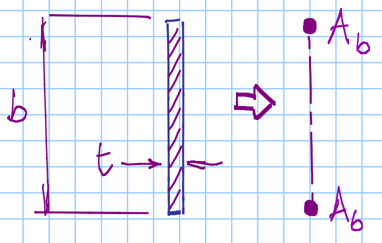
(* 15t for DBT)

E.g. Upper Stringer






To estimate the contribution of a web to end-load carrying boom area typically we use 1/6th of the web cross-sectional area which provides the equivalent 2nd moment of area as point areas separated by the web depth.




$$\text{ie: } 2A_b \times \left(\frac{b}{2}\right)^2 \equiv \frac{tb^3}{12} \rightarrow A_b = \frac{1}{6}tb = \frac{1}{6}A_w$$

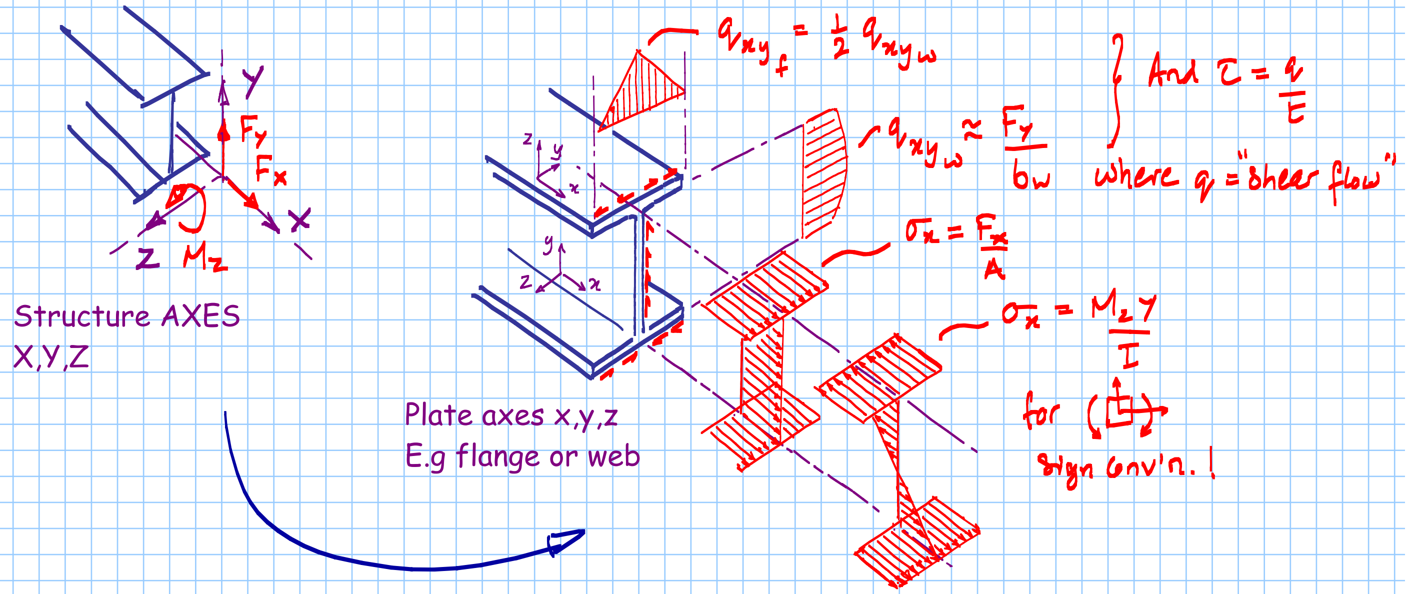
## Section Stressing



17.10.2010

The stress distribution in each element of a structural section can be obtained from basic elasticity analysis (See StM1/2 notes).

E.g. Consider an I-beam under a 2D loading system: $F_x, F_y, M_z$

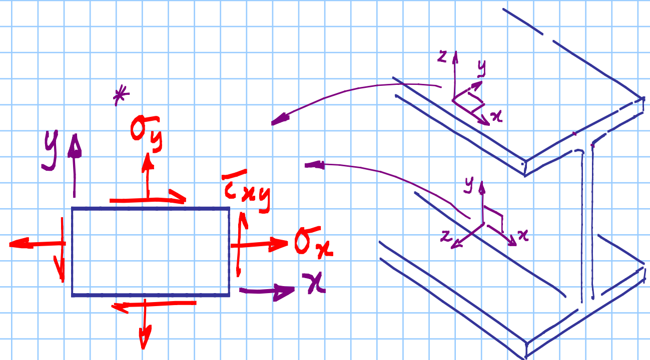



Callutate stresses from internal forces and moments "stress resultants" according to the internal deformation sign convention.





Consider the resulting 2D stress system in a flange or web element:




Note, the choice of axes labels is arbitrary, e.g. X = beam axis here and Y,Z the cross-sectional axes but this is not universal and may also not necessarily be coincident with the plate axes, e.g. see flange plate axes in this example where the y,z plate axes are transposed from the structural Y,Z axes

Note, for a simple beam subjected to transverse loading there is no transverse direct stress in the flange or web elements.

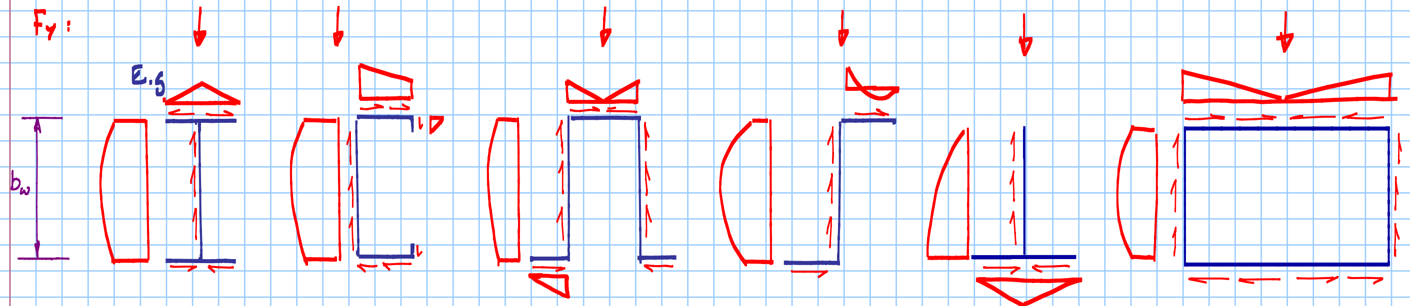
*I.e. $\sigma_y = 0$ because a simple beam reacts transverse loads by the generation of longitudinal direct stress $\sigma_x$ and in-plane shear stress $\tau_{xy}$

But this still leaves $\sigma_x$ and $\tau_{xy}$ to consider as combined stresses.

To account for these combined stresses in our failure analysis we need to use an "interactive failure criterion".

At the initial sizing stage, before proceeding to a full integration of shear flow in the beam section it is reasonable to estimate shear from the average shear flow in the webs for thin wall sections. It is therefore useful to have an idea of the expected shear flow distribution for various common sections.

E.g. For vertical shear force through the shear centre:




Start with $q_w \approx \dfrac{F_y}{n.b_w}$ then consider continuity of shear flow, decreasing to zero at a free edge. Design for max $q$ in each element.

where $n$ = No. of shear webs of height $bw$

Note, this level of shear flow estimation will be adequate throughout the DBT exercise

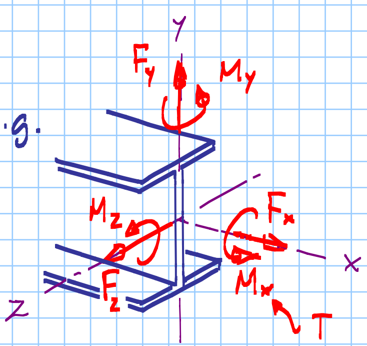
Shear flow due to torsion, e.g. $q = T/2A$ for closed sections will be constant and can be added directly and algebraically to shear due to transverse loads.



Expanding our consideration to a <u>3D</u> loading system we may need to include include Fz as orthogonal transverse shear, *My* as a bending moment about the lateral axis and *Mx* as torsion ( *T* ).

Assuming the material and structural responses are linear we can account for combined loadings by superposition. I.e. carry out a separate analysis for each loading and simply add the results by superposition. We can then add the direct stresses due to bending about different axes and axial load. We can also add the shear flows due to transverse load and torsion. But of course we can only add stresses of the same type, i.e. we can <u>not</u> add direct stress to shear stress. To account for combined direct and shear stresses we will need to consider Von Mises stress as an effective scalar quantity and combined stress failure criteria.

E.g. E.g.




$$\sigma_x = \sigma_{F_x} + \sigma_{M_z} + \sigma_{M_y}$$

$$= \frac{F_x}{A} + \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\tau = \tau_{F_y} + \tau_{F_z} + \tau_T$$

References and further background reading:

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McGraw-Hill 1989, ISBN 0-07-100375-8

Gere + Timoshenko "Mechanics of Materials" 4th Ed. PWs Publishing Co., 1997, ISBN 0-534-93429-3

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Ryder "Strength of Materials"

etc.!