EMAT10100 Engineering Maths I
Lecture 6 of Introduction to Probability: Continuous Random Variables

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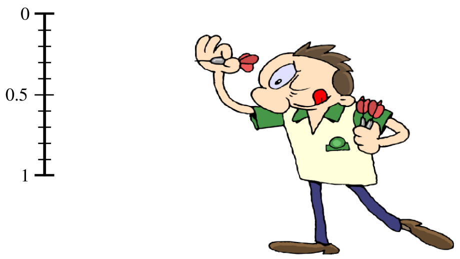
Overview

- Last time we discussed *discrete* random variables (RVs).
- Discrete RVs assume either finitely many or somehow 'well separated' values (like the integers) — Mathematicians call this *countably many*
- But some things instead need to be modelled by *continuous* random variables:
  - E.g.: the height of an individual selected at random in this lecture theatre — it could take a value anywhere on the continuous range $0\,\mathrm{m}$ to $3\,\mathrm{m}$

  But obviously, different parts of this range are more likely than others!
- Most of what we learnt for discrete RVs applies to continuous RVs, with the sum sign replaced by an integral — except the concept of a *probability function* needs more care

For illustration: uniform random variable

- Consider generating a random real number in the interval $[0, 1]$, for example $X = 0.144457237559393803985911\ldots$, where every decimal is equally likely.
- This is like throwing darts randomly at the unit interval:



- Such an RV X is called "uniform in $[0, 1]$".

Why are continuous random variables different?

- What is the probability of obtaining exactly $X = 0.144457237559393803985911\ldots$?
- It is the probability that the
  - first decimal equals 1,
  - second decimal equals 4,
  - third decimal equals 4,
  - $\ldots$
- All decimals are independent, so $P(X = 0.144457237559393803985911\ldots) = \frac{1}{10} \times \frac{1}{10} \times \ldots = 0$
- If the probability of hitting any one number is zero, how can they add up to one? (the probability that you hit *somewhere* on the interval)

Resolution: define Probabilities on intervals

The probability that X is in the interval

- $[0, 0.5]$ equals 0.5,
- $[0.7, 0.8]$ equals 0.1,
- $[x_1, x_2]$ equals $x_2 - x_1$ (provided $0 \le x_1 \le x_2 \le 1$).

In other words

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x)\,\mathrm{d}x,$$

where

$$f_X(x) = \begin{cases} 1 \text{ if } 0 < x < 1, \\ 0 \text{ otherwise.} \end{cases}$$

$f_X(x)$ is called the *probability density function* of X, or *pdf*.

It is to continuous RVs what the probability function $P(X = x)$ is to discrete RVs.

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Properties of the *Probability density function*

Generally, a continuous RV X can be characterised by a probability density function f_X so that for $x_1 \le x_2$

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x)\,\mathrm{d}x.$$

Properties of f_X:

- $f_X \ge 0$

  because probabilities are non-negative

- $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$

  because "probabilities must add up to one"
  (in technical terms, because $P(S) = 1$).

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Examples for *probability density functions*

- *uniform distribution*

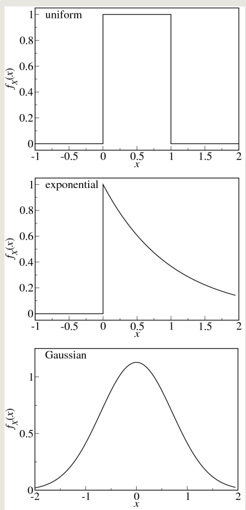
$$f_X(x) = \begin{cases} \frac{1}{u-l} \text{ if } l < x < u, \\ 0 \text{ otherwise.} \end{cases}$$

- *exponential distribution*

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) \text{ if } x \ge 0, \\ 0 \text{ otherwise.} \end{cases}$$

- *Gaussian distribution* a.k.a. *normal distribution*

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$



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Cumulative distribution functions

As we noted, a continuous RV X is defined on intervals. An important special case are intervals $-\infty < X \le x$.

$F_X(x) = P(X \le x)$ is called the *cumulative distribution function* of X (just like it is for discrete RVs).

Properties of F_X common to discrete and continuous RVs:

- F_X is an increasing function. If $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$.
- $\lim_{x \to +\infty} F_X(x) = 1$.
- $\lim_{x \to -\infty} F_X(x) = 0$.

Properties of F_X particular to continuous RVs:

- $F_X(x) = \int_{-\infty}^{x} f_X(\tilde{x})\,\mathrm{d}\tilde{x}$.
- $F'_X(x) = f_X(x)$.

Exampercise: exponential distribution

13.13 in James, p. 995 (5th ed., pp. 1021):

The lifetime of an electronic component is a continuous RV with probability density function

$$f_X(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{x}{2}\right) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Plot the cumulative distribution function and probability density function.

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Solution

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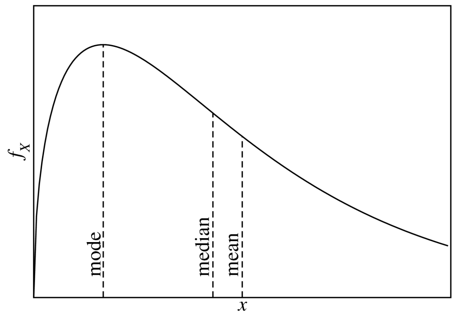
Representative Measures of a Continuous RV

- mean $\mu = E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,\mathrm{d}x$: the average of X
- median m implicitly defined by $F_X(m) = \frac{1}{2}$
  i.e., equal probability to be less or greater than m
- mode M implicitly defined by $f_X(M) \geq f_X(x)$ for all x
  i.e., value with the largest probability density



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Example: exponential distribution

13.16c in James, p. 1000 (5th ed., pp. 1026).

Find the mean, median and mode for the lifetime distribution

$$f_X(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{x}{2}\right) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

- Mean: $\mu = \int_0^{\infty} \frac{x}{2}\exp\left(-\frac{x}{2}\right)\,\mathrm{d}x \overset{\text{by parts}}{=} \int_0^{\infty} \exp\left(-\frac{x}{2}\right)\,\mathrm{d}x = 2.$
- Median: $F_X(m) = 1 - \exp\left(-\frac{x}{2}\right) = \frac{1}{2}. \Rightarrow m = 2\ln(2) \approx 1.386.$
- Mode: $f_X(x)$ is a decreasing function. $\Rightarrow$ Maximum of f_X at $M = 0$.

Variance and standard deviation of a continuous RV X

- The *variance*, usually denoted by $\mathrm{Var}(X)$ or σ^2, is defined as
$$\mathrm{Var}(X) = \sigma^2 = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$$
just like for discrete RVs.
- But now
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x.$$
- (As for discrete RVs) the square root of the variance is called the *standard deviation*,
$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$
The standard deviation is a measure for the spread of the distribution around the mean μ.

Example: exponential distribution

13.17c in James, p. 1001 (5th ed., pp. 1027).

Find the variance and standard deviation of the lifetime distribution

$$f_X(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{x}{2}\right) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note: we already know $\mu = 2$.

Example: exponential distribution

Solution:

$$\mathrm{Var}(X) = \int_0^{\infty} \frac{x^2}{2}\exp\left(-\frac{x}{2}\right)\,\mathrm{d}x - 4$$

The first integral can be calculated with two integrations by parts.

$$\begin{aligned}\int_0^{\infty} \frac{x^2}{2}\exp\left(-\frac{x}{2}\right)\,\mathrm{d}x &= 2\int_0^{\infty} x\exp\left(-\frac{x}{2}\right)\,\mathrm{d}x = \\ &= 4\int_0^{\infty} \exp\left(-\frac{x}{2}\right)\,\mathrm{d}x = 8.\end{aligned}$$

$\Rightarrow \mathrm{Var}(X) = 4$

$\Rightarrow \sigma = 2.$

Summary

- Keep reading James and practising the exercises.
- Next week (the final week of probability) will be focussed entirely on some particularly special RVs (discrete and continuous) and learning how to model with them.