Ordinary Differential Equations

Lecture 3: Classification of ODEs

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Recap

Last time ...

We introduced ordinary differential equations (ODEs) and identified some methods for their solution

- Direct integration
- Solution by Inspection (not really a method)
- Separation of Variables

Today: Classification of ODEs, a trick to help with separation of variables.

Next time: More of this plus another (pretty awesome) method for solving more complicated ODEs.

Classification of ODEs

Classification of ODEs

Lets define some useful categories that can help us to decide how to solve a given ODE.

Classification of ODEs

Order of the differential equation

We have already encountered the order of differential equations.

The order of a differential equation is the order of the highest derivative (with respect to an independent variable).

For example

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = 5xt \qquad \text{First order differential equation}$$

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} = t + 3sin(5t) \qquad \text{Third order differential equation}$$

(These two should be easy to solve)

Classification of ODEs

Some more examples

$$\boxed{\frac{\mathrm{d}\,x}{\mathrm{d}\,t}} + 3x = 0 \qquad \text{1st order}$$

$$\boxed{\frac{\mathrm{d}\,y}{\mathrm{d}\,x}} + 2y = 5x^3 \qquad \text{1st order}$$

$$\boxed{\frac{\mathrm{d}^2\,y}{\mathrm{d}\,t^2}} + 3\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - 7y = \cos(t) \qquad \text{2nd order}$$

$$\frac{\mathrm{d}\,f}{\mathrm{d}\,x} + \boxed{\frac{\mathrm{d}^3\,f}{\mathrm{d}\,x^3}} - 2\frac{\mathrm{d}^2\,f}{\mathrm{d}\,x^2} = -3 \qquad \text{3rd order}$$

The order is important because it tells us the number of constants of integration, i.e. the number of initial conditions needed to find a particular solution.

Classification of ODEs

Complexity in the dependent variable

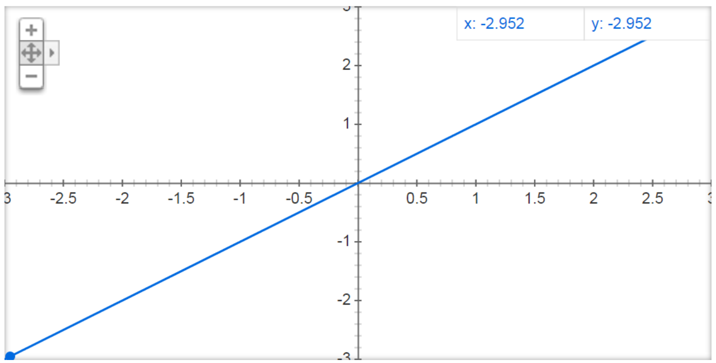
An important property of a differential equation is the complexity of the dependence on the dependent variable.

$$\underbrace{\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x^2 + xt + 5}_{\text{How complex is the dependence on x?}}$$

Lets recall what we know about the complexity of functions . . .

Classification of ODEs

Graph for x



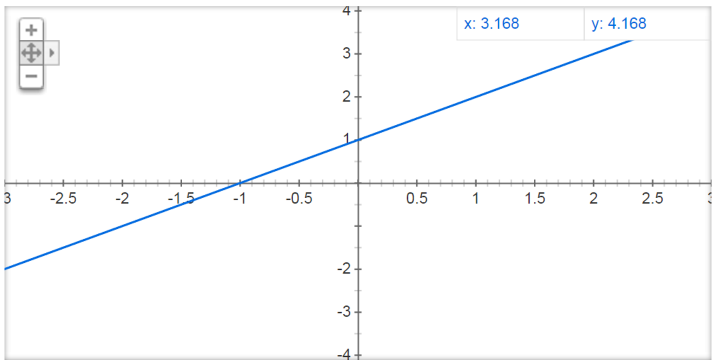
$y = x$

Linear Function

(plotted by google “y=x where x is from -3 to 3”)

Classification of ODEs

Graph for x+1



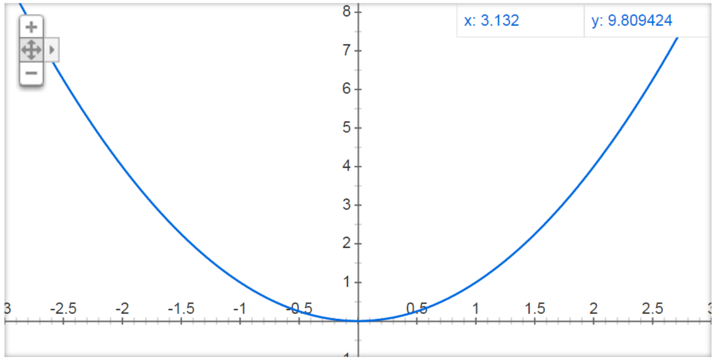
$y = x + 1$

Linear Function with Offset

(plotted by google “y=x+1 where x is from -3 to 3”)

Classification of ODEs

Graph for x^2



$$y = x^2$$

Nonlinear Function

(plotted by google “y=x^2 where x is from -3 to 3”)

Classification of ODEs

All differential operators count as linear terms with respect to the dependent variable

$$\underbrace{\frac{\mathrm{d}\,x}{\mathrm{d}\,t} \quad \frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} \quad \frac{\mathrm{d}^3\,x}{\mathrm{d}\,t^3}}_{\text{All of these are linear in } x!}$$

Linear or non-linear?

Linearity is an important property: linear equations are generally much easier to solve than non-linear ones.

An ODE is *linear* if the *dependent variable* and *its derivatives* do not appear as *products*, *raised to powers*, or as *part of nonlinear functions* (sin, cos, exponents, etc).

$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + 5y = \cos(t)$	Linear
$\frac{\mathrm{d}^2\,y}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - y = 0$	Linear
$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + 5y + \boxed{\cos(y)} = 0$	Non-linear
$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - \boxed{5y\frac{\mathrm{d}\,y}{\mathrm{d}\,t}} = 2t$	Non-linear

Classification of ODEs

A linear ODE is a linear equation connecting the dependent variable and its derivatives. Standard form of e.g. 1st, 2nd, 3rd order *linear* ODEs

$$\begin{aligned} a(t)\tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} + b(t)x &= f(t) \\ a(t)\tfrac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + b(t)\tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} + c(t)x &= f(t) \\ a(t)\tfrac{\mathrm{d}^3\,x}{\mathrm{d}\,t^3} + b(t)\tfrac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + c(t)\tfrac{\mathrm{d}\,x}{\mathrm{d}\,t} + d(t)x &= f(t) \end{aligned}$$

$a(t)$, $b(t)$ etc. are the coefficients and may in general depend on the independent variable.
Note: we can divide through by $a(t)$ so the coefficient on the highest derivative is not needed.

Classification of ODEs

We write a linear ODE in standard form with all terms involving the dependent variable on the left hand side. We can then say that the linear ODE is

homogeneous if the right hand side is zero.

non-homogeneous if the right hand side is not zero.

Some examples

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5xt \qquad \text{homogeneous}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t + tx \qquad \text{non-homogeneous}$$

(Hint: subtract terms involving x to the left-hand side.)

Classification of ODEs

More examples

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} + 5y &= \boxed{\cos(t)} && \text{linear non-homogeneous}\\
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - y &= \boxed{0} && \text{linear homogeneous}\\
\frac{\mathrm{d}y}{\mathrm{d}t} + 5y + \boxed{\cos(y)} &= 0 && \text{nonlinear}\\
\frac{\mathrm{d}y}{\mathrm{d}t} - \boxed{5y\frac{\mathrm{d}y}{\mathrm{d}t}} &= 2t && \text{nonlinear}
\end{aligned}$$

Classification of ODEs

Exampercise Classify each of the following equations

1. $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{y}{x^2-1} = 0$$
2. $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} - \sin(t) = 0$$
3. $$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + 2x = t^2$$
4. $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2x = t^2$$

ToDo: Order? Linear? Homogeneous?

Classification of ODEs

Autonomous and separable ODEs

Two further definitions:
Autonomous and Separable

Besides the standard classification introduced so far that are two additional bit of terminology that are very helpful.

Autonomous

Autonomous ODEs

An ODE is **autonomous** if it does not explicitly depend on the independent variable (i.e. time).

If it is linear this means that it has **constant coefficients**. (What is meant by that is terms that are constant with respect to the independent variable).

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = 5x^2 \qquad \text{autonomous / constant-coefficients}$$

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \sin(t) + x \qquad \text{non-autonomous}$$

Separable ODEs

Separable ODEs

We say that an ODE is separable if it can be solved by separation of variables which means when it can be written in the form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = g(x)h(t) \tag{1}$$

with arbitrary functions $g()$ and $h()$

Examples

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = 5\sin(x)\mathrm{e}^t \qquad \text{separable}$$

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \sin(t) + x \qquad \text{not separable}$$

(Separability is harder to decide for higher order differential equations, so we use the term only for first order equations)

Linear 1st-order homogeneous ODEs

General *first-order linear* ODE

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + b(t)x = f(t)$$

General first-order linear *homogeneous* ODE

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + b(t)x = 0$$

We can rewrite this as

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -b(t)x \tag{2}$$

So, this type of equation is always separable!

Autonomous and Separable ODEs

By contrast, non-homogeneous ODEs such as

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + t \tag{3}$$

are typically not separable.

(The only exceptions are trivial examples that are really easy to solve.)

Interlude

Interlude

Classification is important because it can help us to decide on the best way to solve a given ODE.

Interlude

The story so far ...

Things to try (in this order)

- **Solve by inspection**, if you can ...
- ODEs that don't explicitly depend on the dependent variable (and do not contain derivatives of different orders)
  Solve by direct integration
- Linear homogeneous first-order ODEs
  Solve by separation of variables
- Nonlinear first-order ODEs
  Might be separable (if you're lucky) or not (usually).
- Linear non-homogeneous ODEs
  Other tricks needed.
- Higher order ODEs
  Other tricks needed.

Substitution

Substitution

Some ODEs try to hide their true identity.
By some simple transformations we can reveal their true identity and discover simple solutions.

Substitution

Note

When we classify ODEs we really want to classify them in the simplest possible form.

Trivial example:

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = 3t + tx - t - 2t \tag{4}$$

looks non-homogeneous and hence not separable, but it is identical to

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = xt \tag{5}$$

which is homogeneous, first-order and linear and hence separable.
It is not always easy to find the simplest form of an ODE, and finding such a form can be the key step for the solution. There are some standard cases, which are useful to recognize. Lets look at the first one. . .

Substitution

Substitution

ODEs of the form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = f(x/t) \tag{6}$$

are always separable if we use the substitution $y = x/t$.

It is useful to note that $yt = x$ and hence

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{\mathrm{d}\,yt}{\mathrm{d}\,t} = y + t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} \tag{7}$$

Lets use this ...

Substitution

Example

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \left(\frac{x}{t}\right)^2 + 3\frac{x}{t} + 1 \tag{8}$$

becomes

$$y + t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = y^2 + 3y + 1 \tag{9}$$

$$t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = y^2 + 2y + 1 \tag{10}$$

$$t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = (y+1)^2 \tag{11}$$

which becomes $\int \frac{1}{(y+1)^2}\mathrm{d}y = \int \frac{1}{t}\mathrm{d}t$.
Still not completely easy, but much better than what we started with.

Substitution

Exampercise

Solve

$$t^2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x^2 + xt \tag{12}$$

To do:

- Write as function of x/t.
- Use the substitution.
- Separate
- Solve the integrals
- Find the solution in terms of $x(t)$

Substitution

University of BRISTOL
EngMaths I: ODEs lectures 3: Classification of ODEs
Spring Semester 2018

Homework

James 5th edition
Read section 10.3, 10.5.5
solve exercises from 10.5.6

James 4th edition
Read section 10.3, 10.5.5
solve exercises from 10.5.6