FLUIDS I
Example sheet 3: 1D Fluid Flow
SOLUTIONS

- **Q1**

$$p_o = p + \frac{1}{2}\rho_o V^2$$

$$= 10^5 + \frac{1}{2} \times 1.2 \times (30)^2 \ \text{N/m}^2 = \boxed{100540 \ \text{N/m}^2}$$

$$h = \frac{p_o - p}{g\rho_{mercury}} = \frac{1/2 \times 1.2 \times (30)^2}{9.81 \times 13560} \ \text{m} = \boxed{4.06 \ \text{mm}}$$

- **Q2**

Mach number $M = \frac{V}{a} = 0.25$

Speed of sound $a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 268.65}$ m/s

$= 328.55$ m/s

$\therefore$ Speed of aircraft $V = Ma = 0.25 \times 328.55$ m/s

$= 82.14$ m/s

Density of air $\rho_o = \frac{p}{RT}$

The problem could be solved by substituting ρ_o, V and p. But we present a more elegant solution.

$$\therefore \quad \frac{p_o}{p} = 1 + \frac{\frac{1}{2}\rho_o V^2}{p} = 1 + \frac{1}{2}\frac{V^2}{RT}$$

$$= 1 + \frac{1}{2}\gamma M^2$$

$$= 1 + \frac{1}{2} \times 1.4 \times (0.25)^2 = \boxed{1.044}$$

This method of solution is not applicable if $M = 0.7$, because $p_o \neq p + \frac{1}{2}\rho_o V^2$. At such high Mach number compressibility of the gas cannot be neglected and Bernoulli's equation is not valid.

The general result for <u>subsonic</u> flow, to be presented later in the course, is

$$\frac{p_o}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}$$

• Q3 This problem can be solved by directly substituting the values in the expression for discharge derived in the lecture notes. We take here a more fundamental approach.

$\Delta p^* =$ difference in piezometric pressure across the orifice

$=$ vertical displacement of the manometric liquid $\times g \times$ density of liquid

$= (0.271 \times 0.1) \times 9.81 \times 800 \quad N/m^2$

$= 212.7 \ N/m^2$

Density of air $\rho_o = \frac{p}{RT} = \frac{0.775 \times 9.81 \times 13560}{287 \times 288.95} \ kg/m^3$

$= 1.2432 \ kg/m^3$

Apply Bernoulli's equation between the atmosphere (where velocity is zero) and the vena contracta.

The ideal velocity at the vena contracta, V

$$= \sqrt{\frac{2 \Delta p^*}{\rho_o}} = \sqrt{\frac{2 \times 212.7}{1.2432}} \ m/s = 18.498 \ m/s$$

Discharge $Q = C_d \times$ (Area of orifice) $\times$ ideal velocity V

$= 0.602 \times (\pi \times 0.025^2) \times 18.498 \ m^3/s$

$= \boxed{0.0219 \ m^3/s}$

• Q4

Continuity $\quad Q = A_1 V_1 = A_2 V_2 \quad$ ----- (1)

Bernoulli's eqn $\quad p_1^* + \frac{1}{2} \rho_o V_1^2 = p_2^* + \frac{1}{2} \rho_o V_2^2 \quad$ --- (2)

where p^* is the piezometric pressure.

From (1), $V_2 = \frac{A_1}{A_2} V_1$ Substitute this into (2), and solve for V_1^2.

$$V_1^2 = \frac{2 (p_2^* - p_1^*) / \rho_o}{1 - (A_1/A_2)^2}$$

This is the ideal velocity. The actual discharge may be calculated by substituting this velocity in equation (1)

- Q5

Continuity equation between section 1 and 2:

$$V_1 A_1 = V_2 A_2$$

$$\text{or} \quad V_1 = \frac{A_2}{A_1} V_2 = \left(\frac{75}{50}\right)^2 V_2 = 2.25\, V_2 \quad \text{------} (1)$$

Apply Bernoulli's equation between the free surface and section 1

$$\frac{p_a}{\rho_o} = \frac{p_v}{\rho_o} + \frac{V_1^2}{2} - 1.8\,g \quad \text{......} (2)$$

Apply Bernoulli's equation between the free surface and section 2

$$\frac{p_a}{\rho_o} = \frac{p_a}{\rho_o} + \frac{V_2^2}{2} - (1.8+h)\,g \quad \text{.....} (3)$$

Solving (1), (2) and (3),

$$h = \frac{(p_a - p_v)/\rho_o - \{(2.25)^2 - 1\}1.8\,g}{(2.25)^2 g}$$

$$= \frac{(10^5 - 2.39\times10^3)/1000 - \{(2.25)^2 - 1\}1.8\times 9.81}{(2.25)^2 \times 9.81}\ m$$

$$= \boxed{0.521\ m}$$

- Q6

Considering unit depth perpendicular to the plane of the drawing, the total volume flow rate at the outlet is:

$$Q_{out} = \int_0^{z_o} V\,dz = \int_0^{z_o} a z (z_o - z) = a\left[\frac{z^2 z_o}{2} - \frac{z^3}{3}\right]_0^{z_o}$$

$$= a\,\frac{z_o^3}{6}$$

By the requirement of continuity, this is equal to the volume flow rate at inlet, $Q_{in} = z_o \cdot V_o$

$$Q_{out} = Q_{in}$$

$$\text{or} \quad \boxed{a = \frac{6 V_o}{z_o^2}}$$

• Q7 The flow is quasi-steady. Therefore Bernoulli's eqn is applicable at any moment.

The ideal jet velocity is calculated by applying Bernoulli's equation between the free surface and the vena contracta

$$\frac{p_a}{\rho_o} + 0 + 0 = \frac{p_a}{\rho_o} + \frac{V_{jet}^2}{2} - gh \quad \text{---- (1)}$$

where h is the instantaneous level in the reservoir.

$$\therefore \quad V_{jet} = \sqrt{2gh}$$

Actual discharge through the orifice $Q = C_d\, a \sqrt{2gh}$.

Q changes with time, because h changes. However, if this rate of change of Q is very slow then equation (1) is almost true at every moment. (Though the magnitude of $\frac{V_{jet}^2}{2}$ is changing, it always remains close to gh if the rate of change is small.)

If the level in the reservoir is changing at a rate $\frac{dh}{dt}$ then by the requirement of continuity we have,

$$-A\frac{dh}{dt} = Q = C_d\, a \sqrt{2gh} \quad \text{.....(2)}$$

Integrating equation (2) between heights h_1 and h_2,

$$t = \int_{h_1}^{h_2} \frac{-A\,dh}{C_d\, a\sqrt{2gh}} = \frac{2A}{C_d\, a\sqrt{2g}}\left[\sqrt{h_1} - \sqrt{h_2}\right]$$

It should be noted that the velocity of the free surface, $V_s = -\frac{dh}{dt}$, is neglected while writing equation (1) but it has been fully accounted for in the continuity equation. This is a common trick applied in Fluid Mechanics and should be fully understood. (Very few texts, if any, would discuss this explicitly.)

$$V_s = -\frac{dh}{dt} = \left(\frac{a}{A}\right) C_d\, V_{jet}$$

Consequently, when the ratio $\left(\frac{a}{A}\right) \rightarrow 0$, {i.e. $A \rightarrow \infty$}

$$V_s \rightarrow 0.$$

The kinetic energy term in Bernoulli's equation $\left(\frac{V_s^2}{2}\right)$ then can be, perfectly legitimately, neglected.

Care is needed, however, while applying the continuity equation. Although V_s indeed tends to zero, A simultaneously tends to infinity, thus the product AV_s remains finite. This is the basis of writing equation (2).