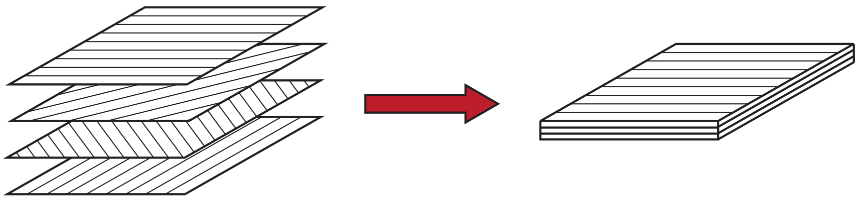
Handout 3 – Classical Laminate Theory

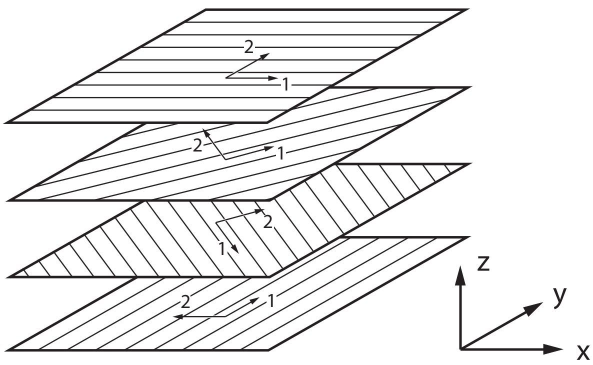
In a composite laminate, multiple composite layers are combined to form a single structural element. The laminate properties are a function of the material properties of each individual ply, their thickness, their fibre orientation with respect to the structural axes, and the ply stacking sequence. The aim is to understand the effect of the composite layup on the macroscopic mechanics of the laminate, in order to design a stacking sequence that achieves desired structural properties. Of particular interest is the ability to tailor the out-of-plane properties, *i.e.* bending and twisting, of a composite plate by selecting a suitable laminate layup.



A number of assumptions underly Classical Laminate Theory: (i) all plies are macroscopically homogeneous and linear-elastic, (b) each ply in the stack is perfectly bonded to the next plies (strain continuity), (c) the laminate is thin and wide, (d) each ply is assumed to be in plane stress (interlaminar stresses are neglected). As a result of the strain continuity, the stress distribution will be discontinuous between plies.

Furthermore, for the plate bending model we follow the Kirchoff-Love hypothesis that plane sections remain plane (equivalent to neglecting γ_{xz} and γ_{yz}), and the strain distribution in the thickness direction is linear.

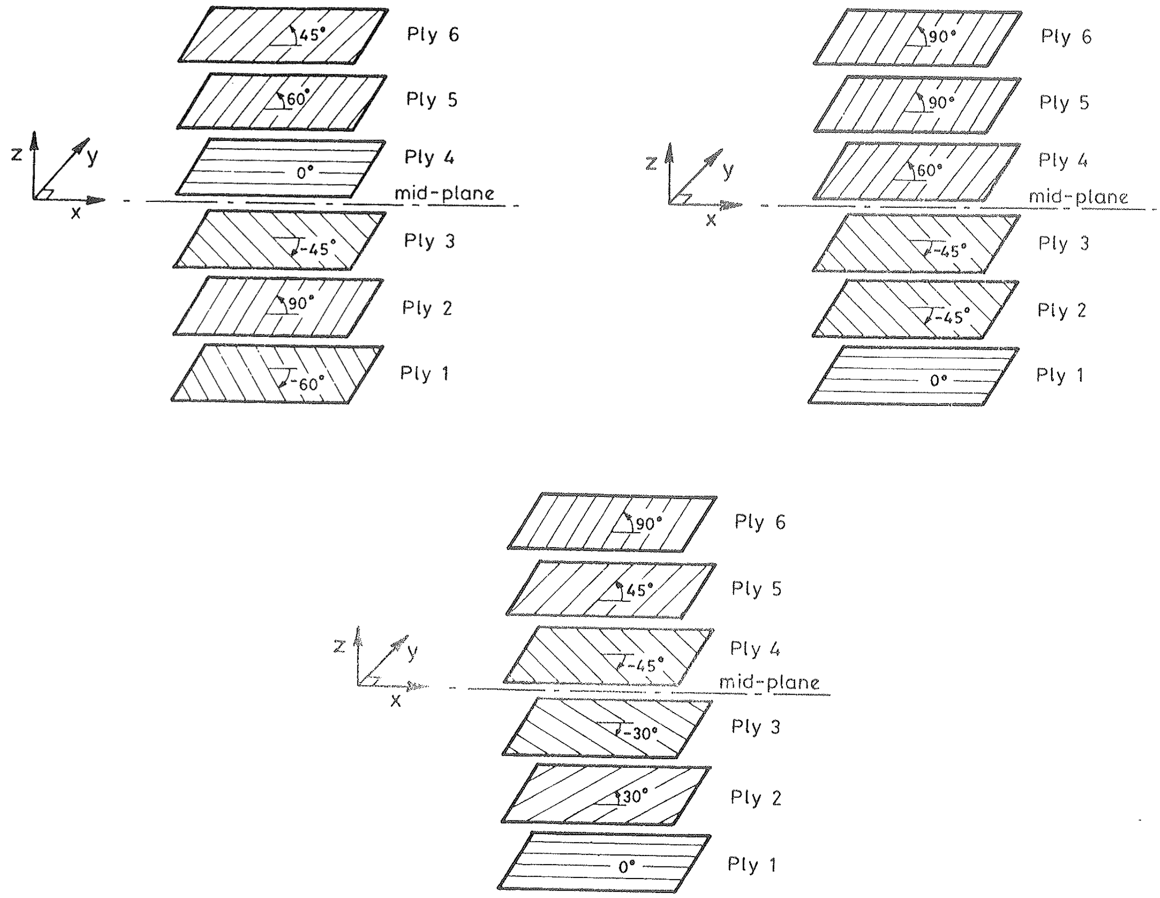
Lamination Notation Ply numbering is bottom-up in the positive z-direction. The angle θ defines the orientation of the ply material axes (fibre direction) with respect to the global x-axis, and is defined positive CCW around z.



A shorthand notation can be used:

- repeating ply angle θ in n successive layers: θ_n
- symmetry around xz-plane in successive layers: $\pm$ or $\mp$
- symmetry around the laminate mid-plane: $[0/90/90/0] = [0/90]_S$ and $[0/90/0] = [0/\bar{90}]_S$

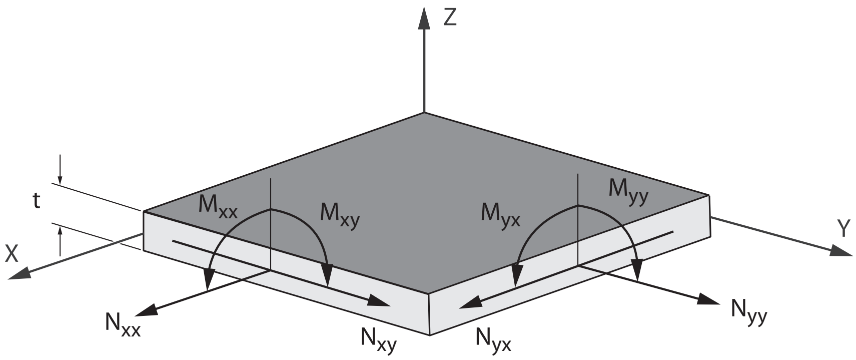
Example 3.1 – Ply Stacking Sequences



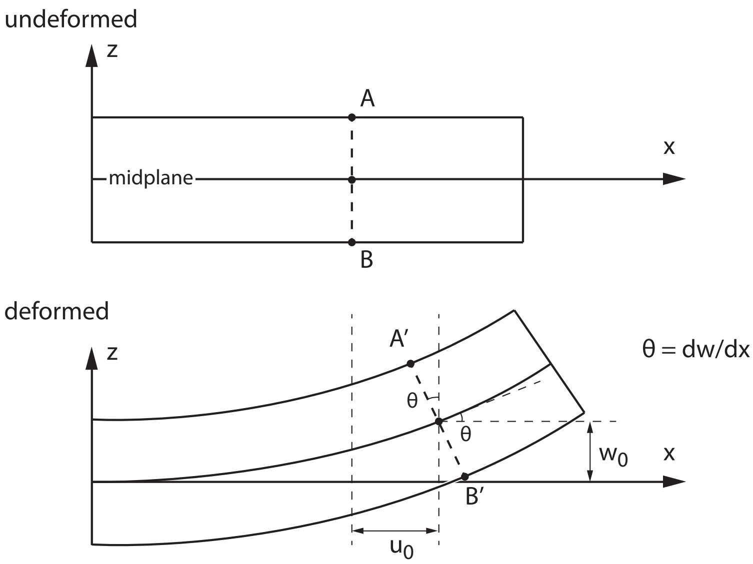
3.1 Strains and Displacements

In structural applications, the composite laminate plate is subject to in-plane loads N_{xx}, N_{yy}, N_{xy}, as well as out-of-plane bending/twisting moments M_{xx}, M_{yy}, and M_{xy}. These are defined per unit width of plate.

Note: it can be shown that $N_{xy} = N_{yx}$ and $M_{xy} = M_{yx}$.



Analogously to beam bending, the applied bending/twisting moments will result in changes in curvature of the plate: curvatures κ_{xx}, κ_{yy}, and twist κ_{xy}.



In the Kirchoff-Love plate model, it is assumed that cross-sections remain straight. For small deflections w_0, the displacement u (in x-direction) of a point on the cross-section:

$$
\begin{aligned}
u &= u_0 - z\theta \\
&= u_0 - z\frac{\partial w_0}{\partial x}
\end{aligned}
$$

where $\theta = \partial w_0/\partial x$ and z is the distance from the midplane. Similarly, for displacement v in y-direction:

$$
v = v_0 - z\frac{\partial w_0}{\partial y}
$$

The strains are then found as follows (see StM2):

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \\
&= \varepsilon^0_{xx} + z\kappa_{xx} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \\
&= \varepsilon^0_{yy} + z\kappa_{yy} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y} \\
&= \gamma^0_{xy} + z\kappa_{xy}
\end{aligned}
$$

where ε^0_{xx}, ε^0_{yy}, γ^0_{xy} are midplane strains, and κ_{xx}, κ_{yy}, κ_{xy} are the curvatures:

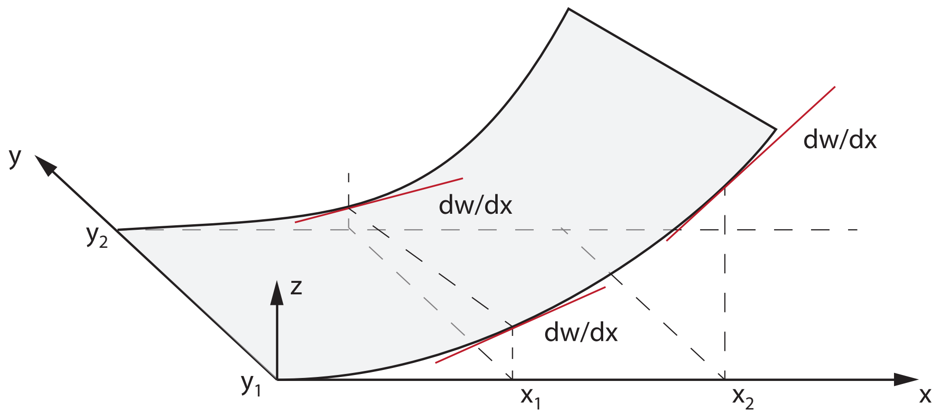
$$
\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = -\begin{bmatrix} \partial^2 w_0/\partial x^2 \\ \partial^2 w_0/\partial y^2 \\ 2\partial^2 w_0/\partial x \partial y \end{bmatrix}
$$

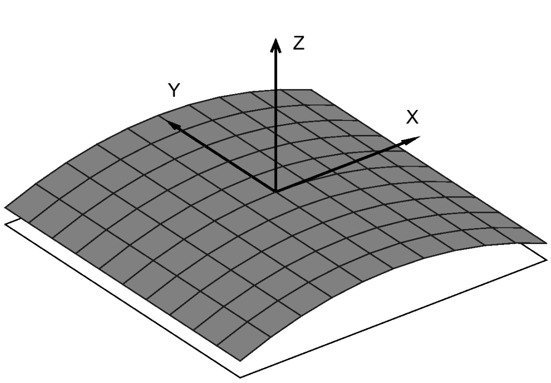
A positive curvature will result in a tensile (positive) strain in the positive z direction of the plate. The strains along the cross-section are a function of the midplane strains and curvatures, and distance from the mid-plane.

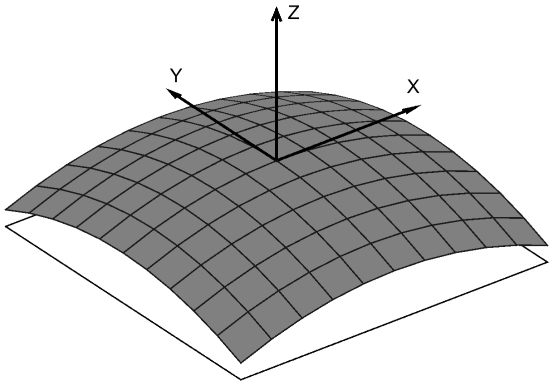
$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} \tag{3.1}
$$

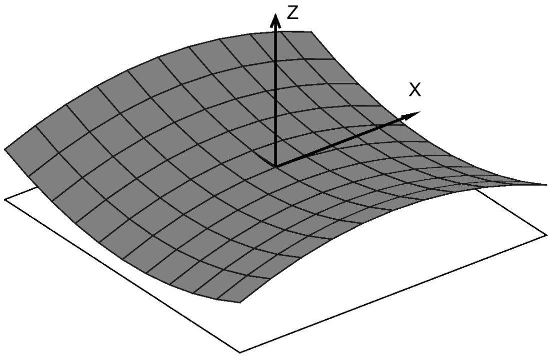
What do the curvature and twist represent geometrically?

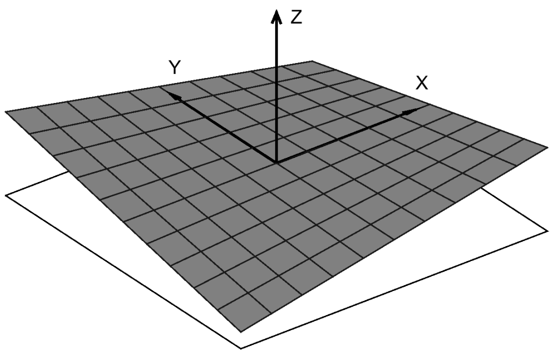
The curvature κ_{xx} is the rate of change in slope $\partial w/\partial x$ with respect to x. The twist κ_{xy} is the rate of change of slope $\partial w/\partial x$ in the y direction (and vice versa).











Fascinatingly, curvature forms a second rank tensor (like stress and strain):

$$\bar{\boldsymbol{k}} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{xy} & \kappa_{yy} \end{bmatrix}$$

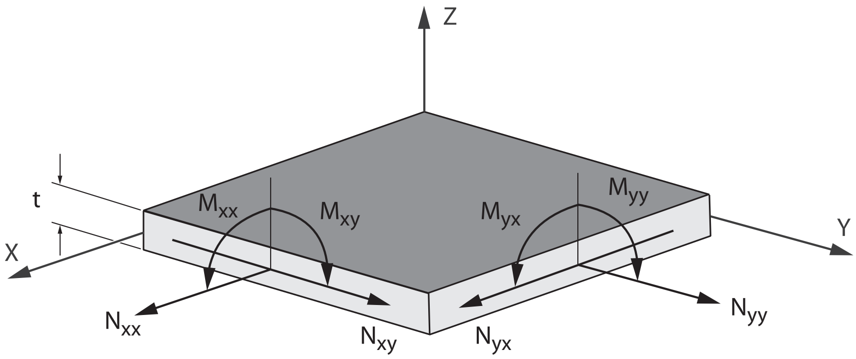
with identical coordinate transformation equations as strain:

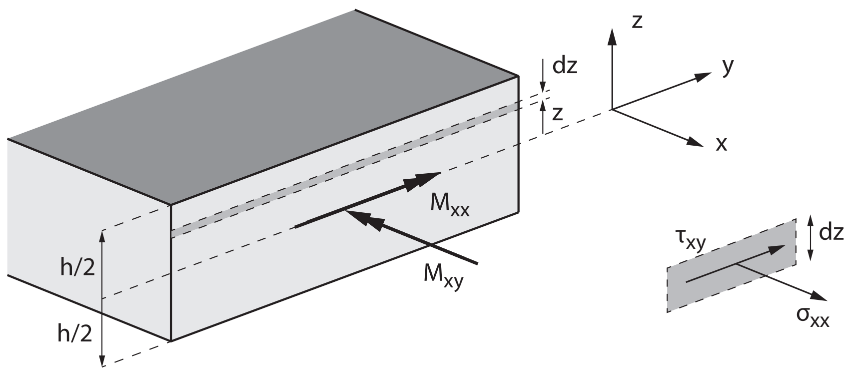
$$\begin{bmatrix} \kappa_{x'x'} \\ \kappa_{y'y'} \\ \kappa_{x'y'} \end{bmatrix} = \boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^{-1} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

For instance, a saddle shape ($\kappa_{xx} = -\kappa_{yy} \neq 0$ and $\kappa_{xy} = 0$) is the same as pure twist at 45°.

3.2 Laminate Stiffness Equations

The strains across the cross section are a function of mid-plane strains and curvatures, and distance z from the midplane. Combining the strains with the generally orthotropic material model for the composite laminae enables us to calculate lamina stresses and the plate stress resultants.





The **stress resultants** per unit width of plate are:

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} dz \qquad M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z\, dz$$

$$N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} dz \qquad M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z\, dz$$

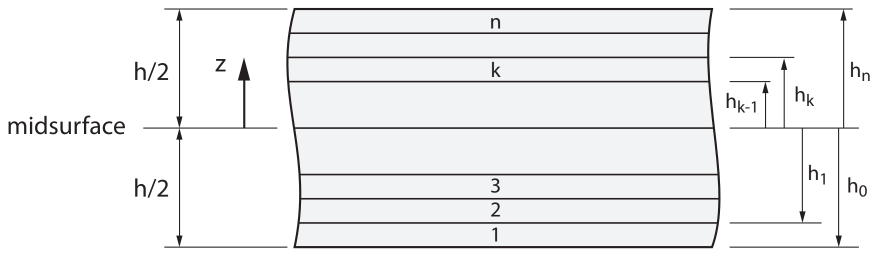
$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz \qquad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz$$

The stress resultant integrations are performed over individual plies, as there may be stress discontinuities at ply interfaces, and then summed over the n laminate plies.

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} dz = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k dz \right)$$

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} z\, dz = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k z\, dz \right)$$

A laminate with n plies has a total thickness h. The ply numbering $k = 1 \dots n$ is in positive z-direction. The position h_k of the top of each ply is measured from the geometric mid-plane of the laminate as shown. The thickness of each individual ply is therefore $h_k - h_{k-1}$.



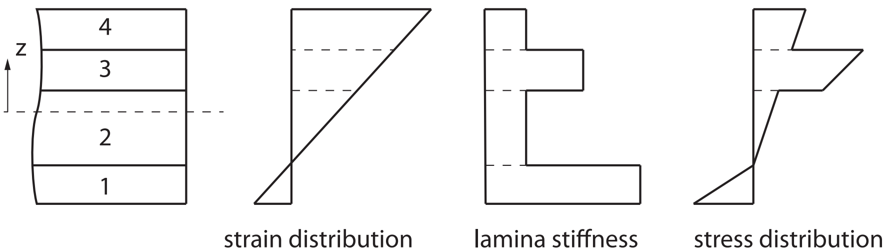
Each composite ply k is assumed to be loaded in plane stress, and is characterised by its reduced material stiffness matrix $\bar{\boldsymbol{Q}}_k$:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

where:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

Note: the stress distribution will not be continuous across the cross-section, as the in-plane stiffness changes from ply to ply, but will be piece-wise linear.



Furthermore, it is evident that the laminate midplane is not necessarily the laminate neutral axis, *i.e.* where the in-plane strain is zero.

Substitute the strains to find the in-plane stress resultants:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} dz \right) \\ \ldots + \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} z\, dz \right)$$

where the midplane strains and curvatures can be taken out of the laminate summation (constant across all plies), and $\bar{\boldsymbol{Q}}$ is taken outside the lamina integration (constant across ply k):

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} dz \right) \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} \\ \ldots + \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z\, dz \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

$$= \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k (h_k - h_{k-1}) \right) \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} \\ \ldots + \frac{1}{2} \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left(h_k^2 - h_{k-1}^2\right) \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

The in-plane stress resultants can be expressed as:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

with the extensional stiffness matrix $\boldsymbol{A}$ and coupling stiffness matrix $\boldsymbol{B}$.

Similarly for the bending and twisting moments:

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} z\, dz = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k z\, dz \right)$$

Substituting the material model for each ply:

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} zdz \right) \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} \\ \ldots + \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z^2\, dz \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

we find:

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

with the coupling stiffness matrix $\boldsymbol{B}$ and bending stiffness matrix $\boldsymbol{D}$.

ABD-matrix

The laminate stress/strain relationships are combined into a single matrix:

$$\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]$$

which is referred to as the ABD-matrix:

$$\left[\begin{array}{c} \boldsymbol{N} \\ \boldsymbol{M} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{array}\right] \tag{3.2}$$

$\boldsymbol{A}$ is the **extensional stiffness matrix**

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \tag{3.3}$$

$\boldsymbol{B}$ is the **coupling stiffness matrix**

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \tag{3.4}$$

$\boldsymbol{D}$ is the **bending stiffness matrix**

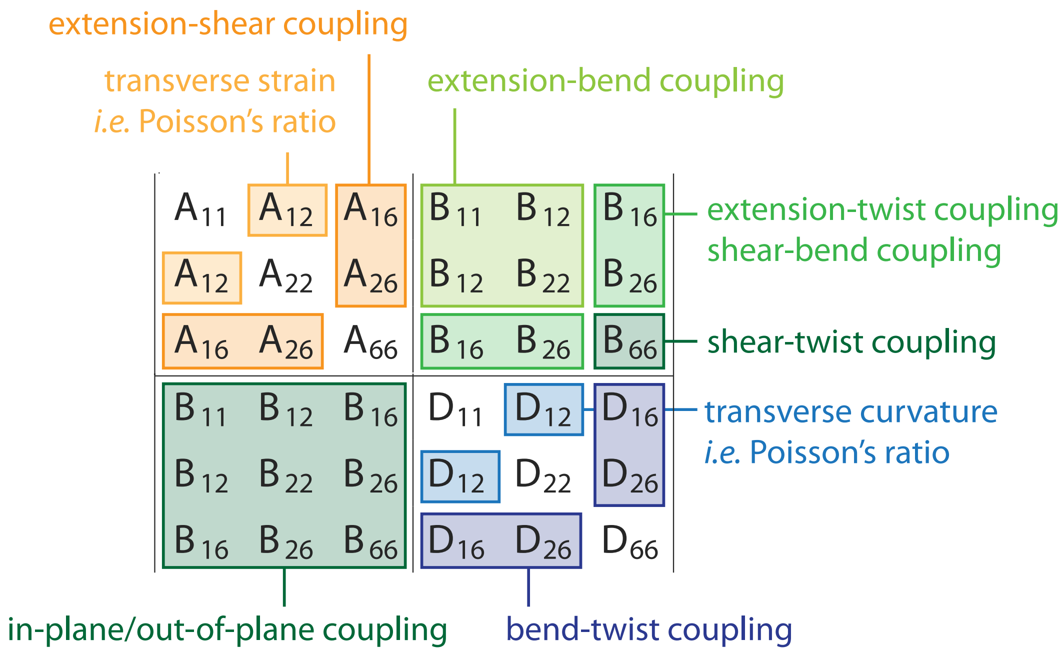
$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \tag{3.5}$$

The ABD-matrix describes the mechanical properties of a composite laminate plate. It is important to observe that a laminate is not a material, but is a *structural element* with essential features of material properties and geometric properties.

ABD-matrix coupling terms

The ABD-matrix captures the structural features of the laminate; of particular interest are the various coupling terms, that provide functionality beyond that of isotropic materials.

	ε^0_{xx}	ε^0_{yy}	γ^0_{xy}	κ_{xx}	κ_{yy}	κ_{xy}
N_{xx}	A_{11}	A_{12}	A_{16}	B_{11}	B_{12}	B_{16}
N_{yy}	A_{12}	A_{22}	A_{26}	B_{12}	B_{22}	B_{26}
N_{xy}	A_{16}	A_{26}	A_{66}	B_{16}	B_{26}	B_{66}
M_{xx}	B_{11}	B_{12}	B_{16}	D_{11}	D_{12}	D_{16}
M_{yy}	B_{12}	B_{22}	B_{26}	D_{12}	D_{22}	D_{26}
M_{xy}	B_{16}	B_{26}	B_{66}	D_{16}	D_{26}	D_{66}



ABD-matrix reformulated

The ABD-matrix equations can be reorganised to highlight the effect of the ply stiffness, ply thickness and ply location in the stacking sequence (Nettles, 1994).

extensional stiffness matrix $\boldsymbol{A}$

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \\ &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k t_k \end{aligned}$$

where t_k is thickness of k-th ply. Thus, the ply stacking order has no effect on the in-plane laminate stiffness.

coupling stiffness matrix $\boldsymbol{B}$

$$\begin{aligned} B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \\ &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \frac{(h_k + h_{k-1})}{2} \\ &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k t_k \bar{z}_k \end{aligned}$$

where $\bar{z}_k$ is distance to *middle* of k-th ply. Thus, the in-plane/out-of-plane coupling depends directly on the location of the ply with respect to the laminate midplane.

bending stiffness matrix $\boldsymbol{D}$

$$\begin{aligned} D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \\ &= \dots \\ &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(\frac{t_k^3}{12} + t_k \bar{z}_k^2\right) \end{aligned}$$

where $t_k^3/12$ is the second moment area of an individual ply, and $t_k \bar{z}_k^2$ accounts for the parallel axis contribution around the laminate midplane. Thus, the out-of-plane bending stiffness not only depends directly on the ply bending stiffness, but also on its location with respect to the laminate midplane.

Example 3.2 – Single Layer Isotropic

Consider a single sheet of isotropic material (E, ν) of thickness t. The reduced stiffness matrix is:

$$\bar{Q}_{ij} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

and the ply positions are $h_1 = t/2$ and $h_0 = -t/2$.

The following ABD matrices are found:

$$\begin{aligned} \boldsymbol{A} &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) = \bar{Q}_{ij}(h_1 - h_0) = \bar{Q}_{ij} t \\ &= \frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \end{aligned}$$

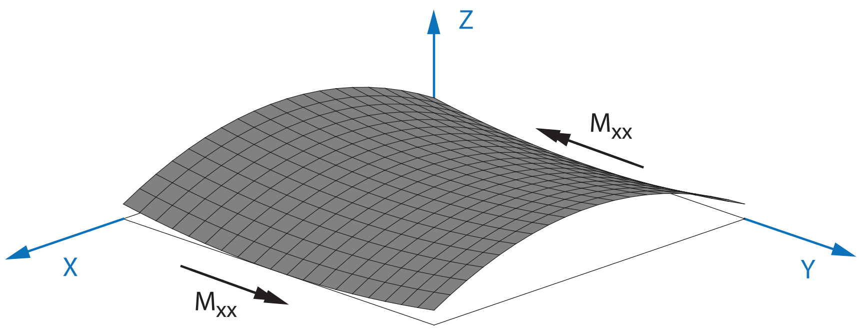
$$\begin{aligned} \boldsymbol{B} &= \frac{1}{2}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) = \frac{1}{2}\bar{Q}_{ij}\left(h_1^2 - h_0^2\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{D} &= \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) = \frac{1}{3}\bar{Q}_{ij}\left(h_1^3 - h_0^3\right) = \frac{1}{3}\bar{Q}_{ij}\frac{t^3}{4} \\ &= \frac{Et^3}{12\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \end{aligned}$$

where D is the isotropic plate flexural stiffness:

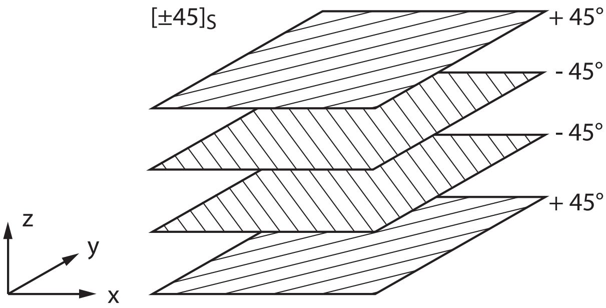
$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

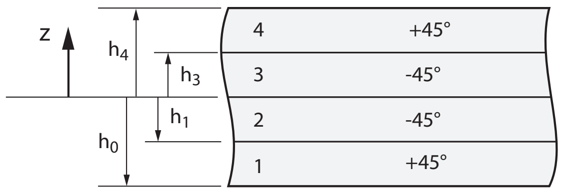
Thus, there is no coupling between bending and extension $(B_{ij} = 0)$, and no bend-twist coupling $(D_{16} = D_{26} = 0)$ for isotropic plates. There is coupling between bending in two directions (D_{12}), resulting in anticlastic curvature for an applied bending moment.



Example 3.3 – Symmetric & Balanced Laminate

Consider a *balanced* and *symmetric* laminate; for example, $[\pm 45]_S$.





symmetric: the laminate is symmetric about its mid-surface, in both lay-up and material properties. Symmetric laminates have a zero $\boldsymbol{B}$-matrix, and thus exhibit no bending-extension coupling.

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) = 0$$

Consider two plies: a ply above the midplane, and its symmetry counterpart below the midplane. The plies share equal stiffness components $\bar{Q}_{ij}$, but their lay-up terms $\left(h_k^2 - h_{k-1}^2\right)$ will be equal and opposite. Consequently, the ply stiffness contributions cancel out, and $B_{ij} = 0$.

Symmetric laminates are commonly used for simplicity of analysis, but also for manufacturing reasons. Such laminates will not bend or twist from the inevitable thermal strains that occur during cooling after the curing phase, due to the zero coupling matrix B_{ij}.

balanced: for each $+\theta$ ply, the laminate has a corresponding ply with equal stiffness and thickness oriented at $-\theta$. For balanced laminates the shear-extension components of the $\pm\theta$ layers cancel out, providing an in-plane orthotropic laminate:

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right) = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$$

This is seen by observing that the $\bar{Q}_{16}$ and $\bar{Q}_{26}$ stiffness components

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$
$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta$$

are *odd* functions in θ, and thus:

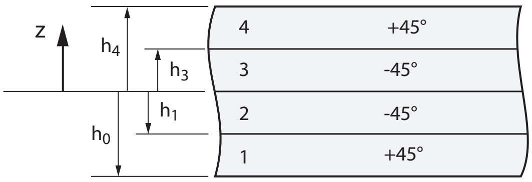
$$\bar{Q}_{16}(-\theta) = -\bar{Q}_{16}(\theta) \qquad\qquad \bar{Q}_{26}(-\theta) = -\bar{Q}_{26}(\theta)$$

The geometric term $(h_k - h_{k-1})$ represents the thickness of each ply. Therefore, in a balanced laminate the contributions of the balanced plies cancel out, and the in-plane stiffness terms A_{16} and A_{26} are zero.

In general, a balanced laminate may still exhibit bending-extension coupling ($B_{ij} \neq 0$), and bend-twist coupling (D_{16}, D_{26}) depending on the laminate lay-up.

Numerical Example: calculate the ABD-matrix for a $[\pm 45]_S$ lamiante. Each ply has thickness $t = 1$ mm, and material properties $E_{11} = 180$ GPa, $E_{22} = 10$ GPa, $G_{12} = 5$ GPa, and $\nu_{12} = 0.2$.

(i) sketch the laminate cross-section to identify the mid-plane and positions h_k of the plies:



(ii) calculate the laminate stiffness components Q_{ij} using:

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12}\nu_{21}} \qquad Q_{66} = G_{12}$$

where

$$\nu_{21} = \frac{E_{22}}{E_{11}} \nu_{12}$$

to find:

$$\boldsymbol{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \begin{bmatrix} 180.4 & 2 & 0 \\ 2 & 10.02 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ GPa}$$

(iii) calculate the generally orthotropic stiffness matrix $\bar{\boldsymbol{Q}}$ (in the structural axes) for each of the plies using:

$$\bar{Q}_{11} = Q_{11} \cos^4\theta + 2(Q_{12} + 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{22} \sin^4\theta$$

$$\bar{Q}_{22} = Q_{11} \sin^4\theta + 2(Q_{12} + 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{22} \cos^4\theta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2\theta \cos^2\theta + Q_{12} \left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2\theta \cos^2\theta + Q_{66} \left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta$$

to find:

$$\bar{\boldsymbol{Q}}_{45} = \begin{bmatrix} 53.6 & 43.6 & 42.6 \\ 43.6 & 53.6 & 42.6 \\ 42.6 & 42.6 & 46.6 \end{bmatrix} \qquad \bar{\boldsymbol{Q}}_{-45} = \begin{bmatrix} 53.6 & 43.6 & -42.6 \\ 43.6 & 53.6 & -42.6 \\ -42.6 & -42.6 & 46.6 \end{bmatrix}$$

Note that for $\pm\theta$ plies, only the sign of $\bar{Q}_{16}$ and $\bar{Q}_{26}$ changes.

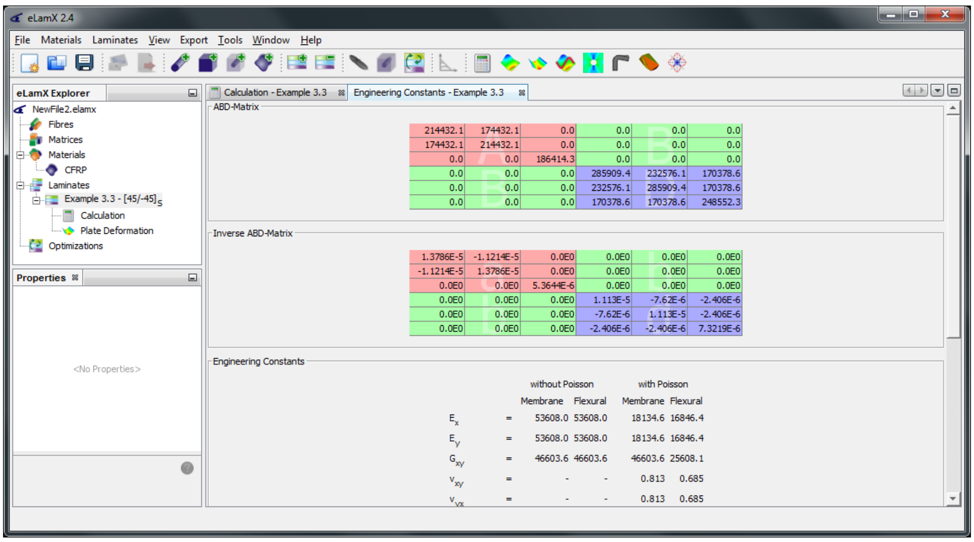
(iv) lastly, construct the ABD matrices as follows:

$$
\begin{aligned}
\boldsymbol{A} &= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right) \\
&= \bar{\boldsymbol{Q}}_{45}\left(-1-(-2)\right) + \bar{\boldsymbol{Q}}_{-45}\left(0-(-1)\right) + \bar{\boldsymbol{Q}}_{-45}\left(1-0\right) + \bar{\boldsymbol{Q}}_{45}\left(2-1\right) \\
&= \begin{bmatrix} 214 & 174 & 0 \\ 174 & 214 & 0 \\ 0 & 0 & 186 \end{bmatrix} \text{ GPa mm}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B} &= \frac{1}{2}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \\
&= \frac{1}{2}\bar{\boldsymbol{Q}}_{45}\left((-1)^2-(-2)^2\right) + \frac{1}{2}\bar{\boldsymbol{Q}}_{-45}\left(0-(-1)^2\right) + \frac{1}{2}\bar{\boldsymbol{Q}}_{-45}\left(1^2-0\right) + \frac{1}{2}\bar{\boldsymbol{Q}}_{45^\circ}\left(2^2-1^2\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{D} &= \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \\
&= \frac{1}{3}\bar{\boldsymbol{Q}}_{45}\left((-1)^3-(-2)^3\right) + \frac{1}{3}\bar{\boldsymbol{Q}}_{-45}\left(0-(-1)^3\right) + \frac{1}{3}\bar{\boldsymbol{Q}}_{-45}\left(1^3-0\right) + \frac{1}{3}\bar{\boldsymbol{Q}}_{45^\circ}\left(2^3-1^3\right) \\
&= \begin{bmatrix} 286 & 233 & 170 \\ 233 & 286 & 170 \\ 170 & 170 & 249 \end{bmatrix} \text{ GPa mm}^3
\end{aligned}
$$

As expected, the laminate has $A_{16} = A_{26} = 0$ as the lay-up is balanced, has $\boldsymbol{B} = 0$ as it is symmetric, and has a fully populated $\boldsymbol{D}$ matrix.



To check your calculations, you can use Composite Laminate Analysis software such as eLamX[2], available for free at: https://tu-dresden.de/ing/maschinenwesen/ilr/lft/elamx2/elamx

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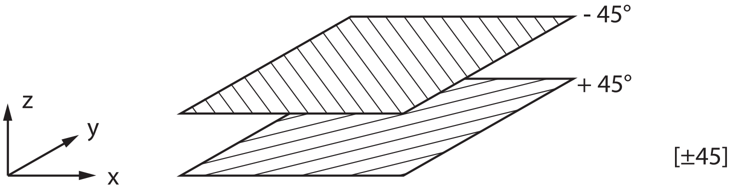
Example 3.4 – Anti-Symmetric Laminate

In an **anti-symmetric laminate** each ply above the mid-plane has a counterpart below the midplane, at equal distance, with equal material properties and thickness, but in opposite orientation ($\pm\theta$).



An anti-symmetric laminate is automatically balanced, and thus $A_{16} = A_{26} = 0$. However, the $\boldsymbol{B}$-matrix will be non-zero due to the laminate asymmetry. Next, consider the D_{ij} stiffness terms:

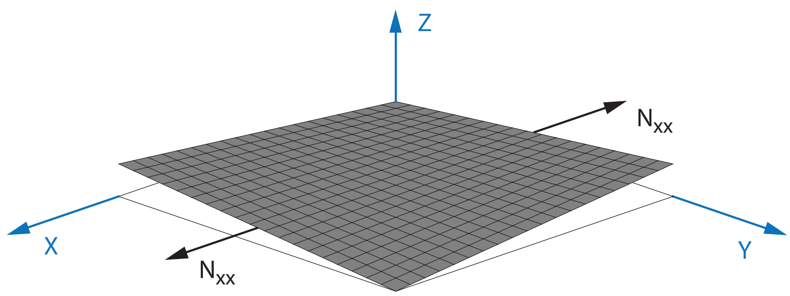
$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

The geometric component multiplying $\bar{Q}_{ij}$ is the same for two layers of equal thickness at the same distance from the midplane (above/below). As $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are odd in θ, anti-symmetric laminates will have $D_{16} = D_{26} = 0$ and therefore have no bend-twist coupling.

Anti-Symmetric Angle Ply A regular anti-symmetric angle-ply, with all plies of equal thickness and stiffness, and single ply angle $\pm\theta$ has the following ABD-matrix:

$$\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ \hline 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]$$

There is a coupling between in-plane strain and out-of-plane twist. Consider a $[\pm 45]$ laminate under N_{xx}:



Note, for laminates with $\pm\theta$ plies, it is worth remembering that:

$$\bar{Q}_{11}(-\theta) = \bar{Q}_{11}(+\theta) \qquad \bar{Q}_{22}(-\theta) = \bar{Q}_{22}(+\theta)$$
$$\bar{Q}_{12}(-\theta) = \bar{Q}_{12}(+\theta) \qquad \bar{Q}_{66}(-\theta) = \bar{Q}_{66}(+\theta)$$
$$\bar{Q}_{16}(-\theta) = -\bar{Q}_{16}(+\theta) \qquad \bar{Q}_{26}(-\theta) = -\bar{Q}_{26}(+\theta)$$

Example 3.5 – Bend-Twist Coupling

In certain applications it is desirable to minimise bend-twist coupling (D_{16}, D_{26}) of the laminate.

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

Considering the lamina stiffness terms $\bar{Q}_{16}$ and $\bar{Q}_{26}$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$
$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta$$

there are two options to achieve $D_{16} = D_{26} = 0$:

- all layers oriented at $\theta = 0°$ and $\theta = 90°$ (*i.e.* a cross-ply laminate);
- for every layer θ above the midplane, there is a layer with equal stiffness oriented at $-\theta$ below the midplane (*i.e.* an anti-symmetric laminate).

Thus, for any symmetric laminate (that is not a cross-ply) the terms D_{16} and D_{26} cannot be zero. However, by repeatedly stacking layers at $+\theta$ and $-\theta$ the terms can be minimised.

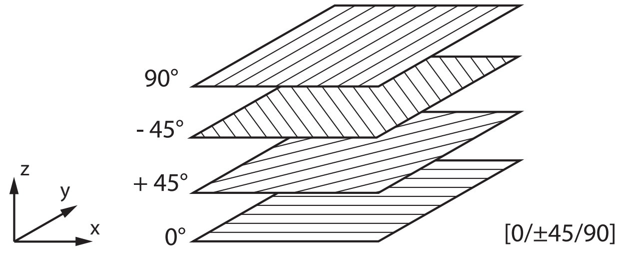
Example 3.6 – Quasi-Isotropic Laminate

By combining multiple plies, it is possible to obtain a laminate with isotropic in-plane properties:

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & (A_{11} - A_{12})/2 \end{bmatrix}$$

Such a laminate requires at least three-fold rotational symmetry (and thus at least three plies), where plies with equal Q_{ij} and t are oriented at angles π/n between layers. Note that the resulting isotropy is in-plane only, and coupling matrix B_{ij} will not necessary be zero, and there may also be non-zero bend-twist coupling (D_{16}, D_{26}). Hence the laminate is referred to as *quasi-isotropic*.

Example quasi-isotropic laminates are $[0/\pm 60]$ and $[0/\pm 45/90]$

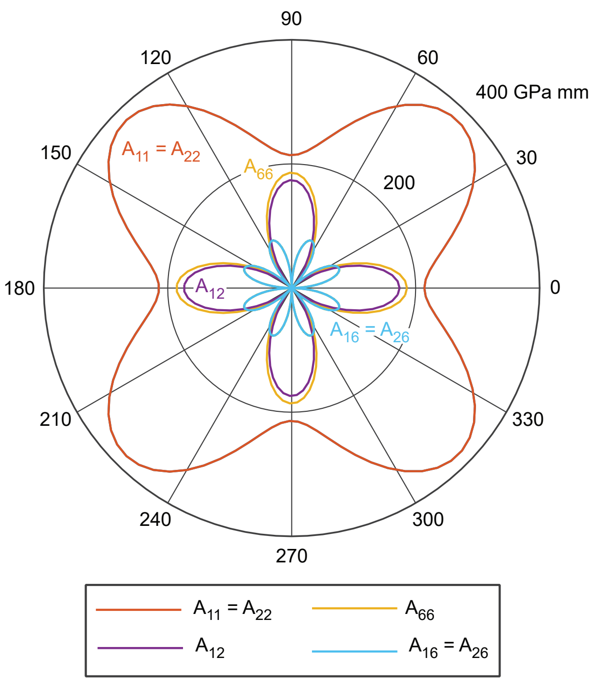


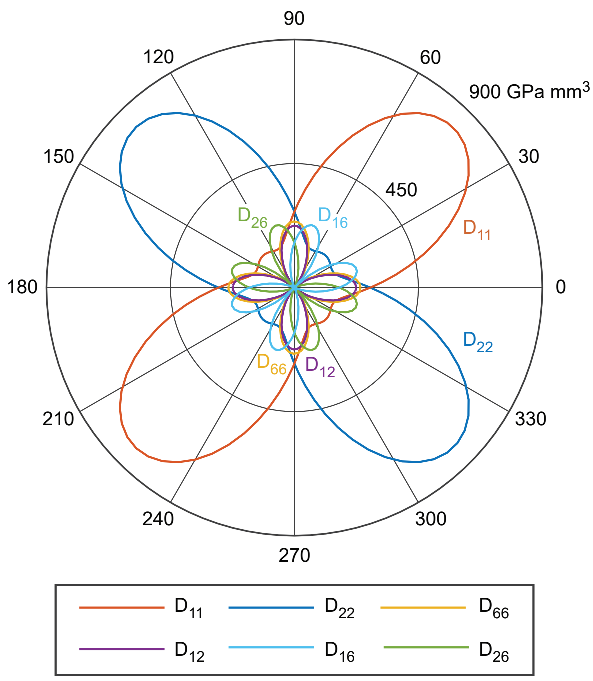
3.3 ABD Matrix Coordinate Transformation

The ABD matrix describes the laminate properties in a specific coordinate system, and the stiffness will depend on the direction of loading. The coordinate transformation of the laminate stresses and strains is done using the transformation matrix $\boldsymbol{T}$ and Reuter's matrix $\boldsymbol{R}$. This will be the same for every ply and can therefore be taken outside the summation over the plies. Thus, the ABD matrices in an $X'Y'$-coordinate system at a CCW angle θ to the original XY-coordinate system are given as:

$$\boldsymbol{A}' = \boldsymbol{TART}^{-1}\boldsymbol{R}^{-1}$$
$$\boldsymbol{B}' = \boldsymbol{TBRT}^{-1}\boldsymbol{R}^{-1}$$
$$\boldsymbol{D}' = \boldsymbol{TDRT}^{-1}\boldsymbol{R}^{-1}$$

The following polar plots show the transformation of the stiffness parameters A_{ij} and D_{ij} for the balanced, symmetric $[\pm 45]_S$ laminate in Example 3.3. The laminate is strongly anisotropic, both in-plane and out-of-plane. Note that the $\boldsymbol{B}$-matrix is zero in all directions, as the laminate is symmetric, but it will only be balanced in certain directions (where $A_{16} = A_{26} = 0$).





3.4 Laminate Compliance Equations

In general, the applied loads (membrane forces and bending moments) on the laminate are known, and the mid-surface strains and curvatures are calculated by inverting the ABD-matrix to find the **compliance matrix**.

The full compliance matrix can be found by inversion of the sub-matrices. From the in-plane stress resultants:

$$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon}^0 + \boldsymbol{B}\boldsymbol{\kappa}$$

the midplane strains can be written as:

$$\boldsymbol{\varepsilon}^0 = \boldsymbol{A}^{-1}\boldsymbol{N} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{\kappa}$$

which are substituted into the out-of-plane stress resultants:

$$\boldsymbol{M} = \boldsymbol{B}\boldsymbol{\varepsilon}^0 + \boldsymbol{D}\boldsymbol{\kappa}$$

to give

$$\boldsymbol{M} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{N} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{\kappa} + \boldsymbol{D}\boldsymbol{\kappa}$$

These are combined into the commonly used **partially inverted** form:

$$\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^* & \boldsymbol{B}^* \\ \boldsymbol{C}^* & \boldsymbol{D}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{bmatrix}$$

$$\boldsymbol{A}^* = \boldsymbol{A}^{-1} \qquad \boldsymbol{B}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}$$

$$\boldsymbol{C}^* = \boldsymbol{B}\boldsymbol{A}^{-1} \quad \boldsymbol{D}^* = \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}$$

The term $\boldsymbol{D}^*$ is known as the *reduced bending stiffness*, which accounts for the effect of a non-zero $\boldsymbol{B}$ matrix on the laminate bending stiffness. For example, in the analysis of the bistable boom in Example 3.1, the in-plane loads are assumed to be zero ($\boldsymbol{N} = 0$), and $\boldsymbol{D}^*$ may be used for the strain energy calculations.

The process is continued by solving for the curvatures:

$$\boldsymbol{\kappa} = \boldsymbol{D}^{*^{-1}}\boldsymbol{M} - \boldsymbol{D}^{*^{-1}}\boldsymbol{C}^*\boldsymbol{N}$$

which are substituted into the expression for mid-plane strain:

$$\boldsymbol{\varepsilon}^0 = \left(\boldsymbol{A}^* - \boldsymbol{B}^*\boldsymbol{D}^{*^{-1}}\boldsymbol{C}^*\right)\boldsymbol{N} + \boldsymbol{B}^*\boldsymbol{D}^{*^{-1}}\boldsymbol{M}$$

and combined to give the **compliance matrix**:

$$\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}$$

$$\boldsymbol{a} = \boldsymbol{A}^* - \boldsymbol{B}^*\boldsymbol{D}^{*^{-1}}\boldsymbol{C}^* \quad \boldsymbol{b} = \boldsymbol{B}^*\boldsymbol{D}^{*^{-1}}$$

$$\boldsymbol{c} = -\boldsymbol{D}^{*^{-1}}\boldsymbol{C}^* = \boldsymbol{b}^T \qquad \boldsymbol{d} = \boldsymbol{D}^{*^{-1}}$$

NB: these equations do not need to be memorised, and will be provided if necessary.

3.5 Laminate Engineering Constants

The laminate stiffness properties are more easily interpreted by calculating the equivalent engineering constants, which can be derived from the laminate compliances matrix:

$$\begin{aligned}
\varepsilon_{xx} &= a_{11}N_{xx} + a_{12}N_{yy} + a_{16}N_{xy} = \left(a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\tau_{xy}\right)h \\
&= \frac{\sigma_{xx}}{E_{xx}} - \nu_{yx}\frac{\sigma_{yy}}{E_{yy}} + \frac{m_{x,xy}}{G_{xy}}\tau_{xy} \\
\varepsilon_{yy} &= a_{12}N_{xx} + a_{22}N_{yy} + a_{26}N_{xy} = \left(a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\tau_{xy}\right)h \\
&= -\nu_{xy}\frac{\sigma_{xx}}{E_{xx}} + \frac{\sigma_{yy}}{E_{yy}} + \frac{m_{y,xy}}{G_{xy}}\tau_{xy} \\
\gamma_{xy} &= a_{16}N_{xx} + a_{26}N_{yy} + a_{66}N_{xy} = \left(a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\tau_{xy}\right)h \\
&= \frac{m_{xy,x}}{E_{xx}}\sigma_{xx} + \frac{m_{xy,y}}{E_{yy}}\sigma_{yy} + \frac{\tau_{xy}}{G_{xy}}
\end{aligned}$$

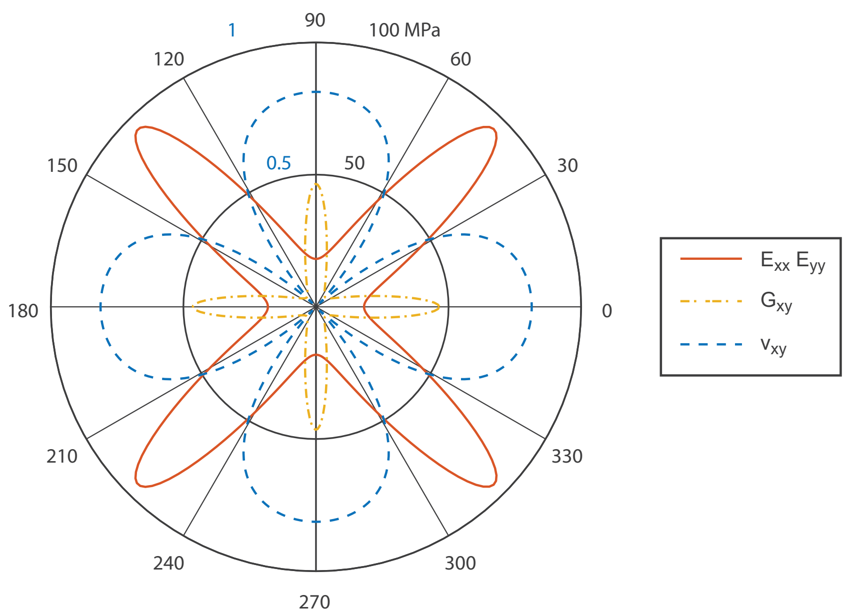
The in-plane stress resultants (N_{xx}, N_{yy}, N_{xy}) are defined per unit width of plate, and therefore the averaged laminate stresses (σ_{xx}, σ_{yy}, τ_{xy}) are found by dividing through laminate thickness h. Note that the stresses will in fact vary from ply to ply.

Thus, the equivalent laminate in-plane engineering constants are:

$$\begin{aligned}
E_{xx} &= \frac{1}{a_{11}h} & E_{yy} &= \frac{1}{a_{22}h} \\
\nu_{xy} &= -\frac{a_{12}}{a_{11}} & G_{xy} &= \frac{1}{a_{66}h} \\
m_{x,xy} &= \frac{a_{16}}{a_{66}} & m_{y,xy} &= \frac{a_{26}}{a_{66}}
\end{aligned}$$

The *coefficients of mutual influence* $m_{x,xy}$ and $m_{xy,x}$ characterise the extension-shear coupling of the laminate. Similar equivalent engineering constants could be derived for out-of-plane deformations (from $\boldsymbol{D}$ matrix).

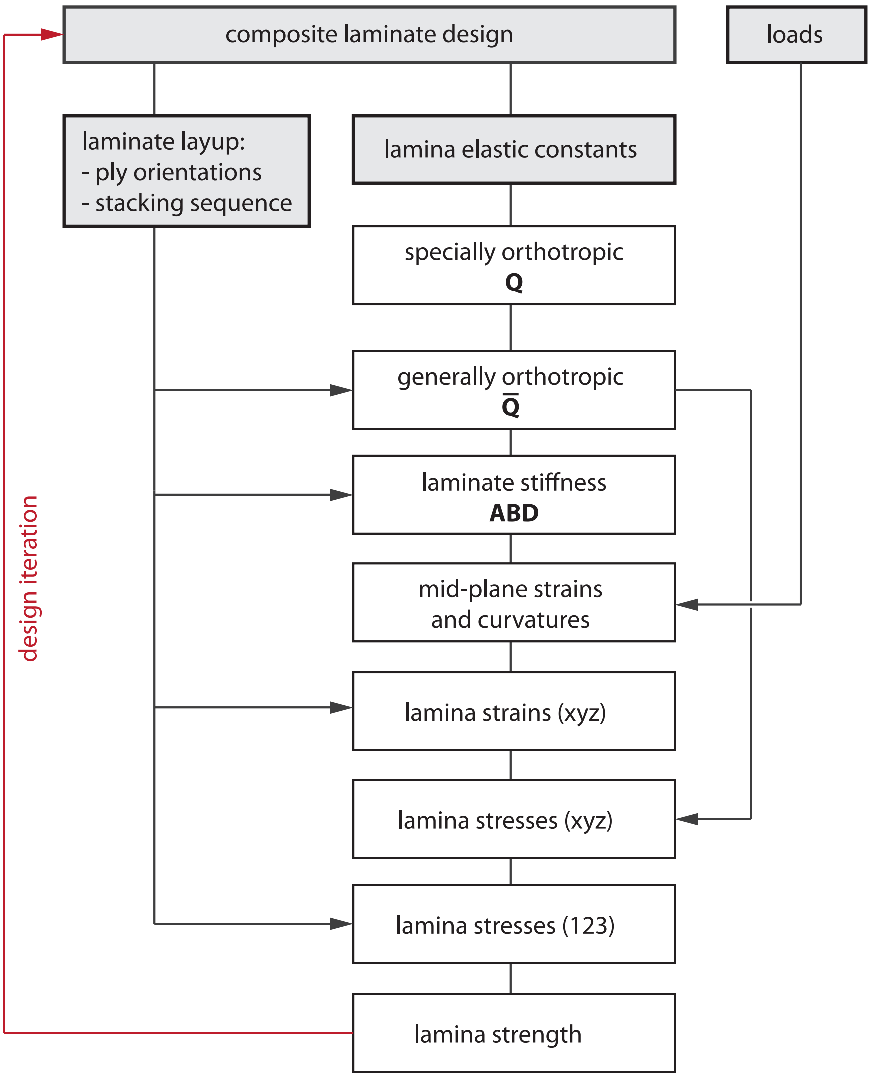
The polar plot show the variation of the engineering constants for the $[\pm 45]_S$ laminate in Example 3.3.



3.6 Laminate Strength

In Handout 1 the strength of an individual lamina was discussed, and different failure criteria were introduced: the maximum stress, maximum strain and Tsai-Hill criterion. Failure of a composite laminate is complex, and combines ply failures, ply delaminations, crack propagation, *etc.* The analysis also requires knowledge of inter-laminar stresses, which have been ignored thus far in classical laminate theory. The 4th year optional unit Advanced Composite Analysis (ACA) will go in more detail.

A common, simplified, laminate failure criterion is **first ply failure**: when the first lamina fails, the total laminate is deemed to have failed. This failure criterion is applied by calculating the stresses and strains in each individual ply (converting to the local material axes) and comparing against individual lamina failure criteria.



Designing a laminate becomes a complex iterative procedure. The freedom to choose different material properties, ply orientations, and ply stacking sequence enables tailoring of the structural properties, but also comes at the expense of design complexity! The design of composite structures is covered in the 4th year optional unit Composite Design and Manufacture (CDM).

3.7 Thermal Strains

During manufacture, composites are cured at high temperatures (*e.g.* 130°C). As the fibres and matrix have different coeffients of thermal expansion (CTEs), during cooling the plies will strain by different amounts along and transversely to the fibres. In a laminate this effect is exacerbated by the different ply orientations, and the thermal strains may cause out-of-plane deformations due to the bending-extension coupling (B_{ij}).

Consider a uniform change in temperature ΔT of an unconstrained ply; the induced thermal strains are:

$$\varepsilon_{11}^T = \alpha_{11}\Delta T$$
$$\varepsilon_{22}^T = \alpha_{22}\Delta T$$

where α_{11} and α_{22} are the coefficient of thermal expansion (CTE) in the material axes; there is no thermal shear strain in the material coordinate system. The CTEs in the structural coordinate system are found through a tensor transformation, with transformation matrix $\boldsymbol{T}$:

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{pmatrix} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1} \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{pmatrix}$$

The total ply strains are the sum of mechanical strains (superscript M), due to applied loads, and thermal strains (superscript T),

$$\begin{aligned} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} &= \begin{pmatrix} \varepsilon_{xx}^M \\ \varepsilon_{yy}^M \\ \gamma_{xy}^M \end{pmatrix} + \begin{pmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ \gamma_{xy}^T \end{pmatrix} \\ &= \begin{pmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} + \begin{pmatrix} \alpha_{xx}\Delta T \\ \alpha_{yy}\Delta T \\ \alpha_{xy}\Delta T \end{pmatrix} \end{aligned}$$

These expressions are inverted to find the ply stiffness equations:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \gamma_{xy} - \alpha_{xy}\Delta T \end{pmatrix}$$

for combined thermal and mechanical effects; hygroscopic (*i.e.* moisture) effects are similarly incorporated.

The ply stiffness equations are combined into the ABD matrix:

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} - \begin{bmatrix} \boldsymbol{N}^T \\ \boldsymbol{M}^T \end{bmatrix}$$

where

$$\left[\boldsymbol{N}^T\right] = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \boldsymbol{\alpha}_k \left(h_k - h_{k-1}\right)\Delta T$$
$$\left[\boldsymbol{M}^T\right] = \frac{1}{2}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \boldsymbol{\alpha}_k \left(h_k^2 - h_{k-1}^2\right)\Delta T$$

are the resulting thermal loads and moments. The resulting residual thermal stresses are self-equilibrating, but will result in deformations. To calculate the warping of a flat plate after thermal cure, the applied mechanical loads are set to zero; the deformations are then a result of the thermal loads $\boldsymbol{N}^T$ and $\boldsymbol{M}^T$.

Revision Objectives Handout 3:

- describe assumptions underpinning Classical Laminate Theory;
- use the lamination notation for composite laminates (including shorthand);
- interpret the geometry of bending/twisting curvatures;
- recall the sign convention for applied loads/moments;
- explain the derivation of the ABD-matrix;
- recall the equations for the ABD-matrix

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

- explain and interpret the ABD coupling terms;
- recognise standard laminate types (balanced, symmetric, anti-symmetric, quasi-isotropic) and explain the link between the laminate lay-up and their corresponding ABD-matrices;
- calculate ABD-matrices from ply properties (E_{11}, E_{22}, G_{12}, ν_{12}) and laminate lay-up;
- calculate midplane strains and curvatures, and resulting lamina stresses and strains for applied loads (given the compliance matrix inversion);
- derive expressions for ABD-matrices in offset loading directions;
- calculate equivalent laminate engineering constants;
- check for laminate failure using first-ply failure criterion;
- calculate the thermal warping of a flat laminate plate;

References:

- RM Jones (1998), "*Mechanics of Composite Materials*", 2nd edition.
- AT Nettles (1994), "*Basic Mechanics of Laminated Composite Plates*", NASA Reference Publication 1351 (NASA-RP-1351)
- D Hull and TW Clyne (2019), "*An Introduction to Composite Materials*" 3rd edition, Cambridge University Press

Appendix A: Advanced Composite Laminate Application

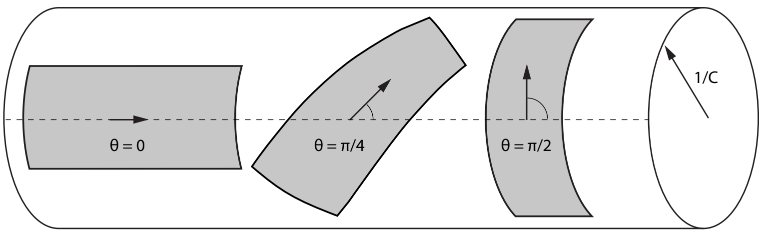
Example A.1 – Bistable Composite Deployable Boom

Composites can offer structural behaviour, which are not possible with conventional, isotropic materials. An example is a lightweight, bistable deployable CFRP boom designed for deploying solar sails. This open section, cylindrical boom can be coiled up and stowed compactly. Its unique feature is its stability in both its deployed and coiled configuration, by virtue of the laminate layup used.

In the elegant analysis by Guest and Pellegrino (2006), it is assumed that the shell deforms inextensionally (*i.e.* no midplane strains). The change in curvature $\Delta\boldsymbol{\kappa}$ is from the initial unstressed configuration:

$$\Delta \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = \frac{C}{2} \begin{bmatrix} 1 - \cos 2\theta \\ \cos 2\theta + 1 \\ 2 \sin 2\theta \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{R} \\ 0 \end{bmatrix}$$

where $1/R$ is the original transverse curvature of the boom, C is the imposed curvature at an angle θ to the original axes.



For the inextensional deformation model, the non-dimensionalised elastic strain energy is given as:

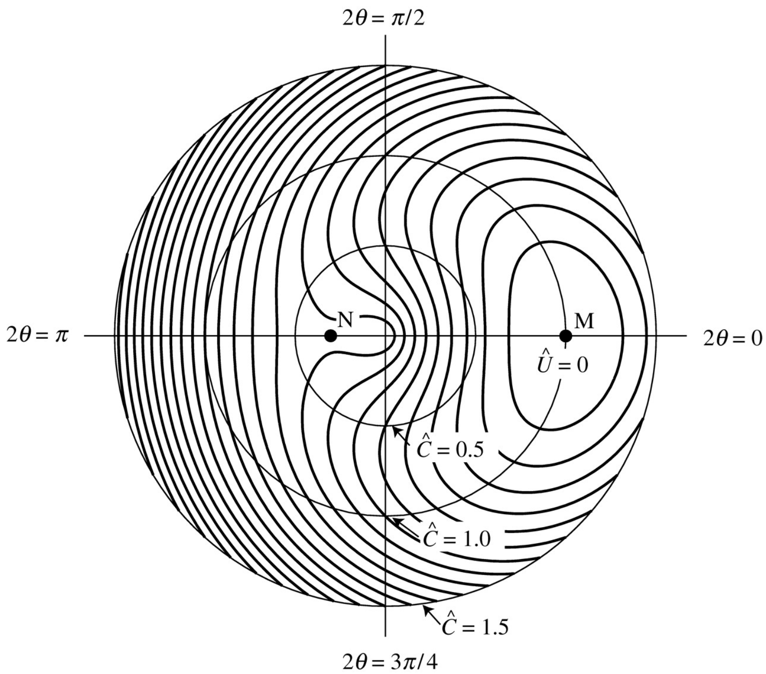
$$\hat{U} = \hat{\boldsymbol{\kappa}}^T \hat{\boldsymbol{D}} \hat{\boldsymbol{\kappa}}$$

where

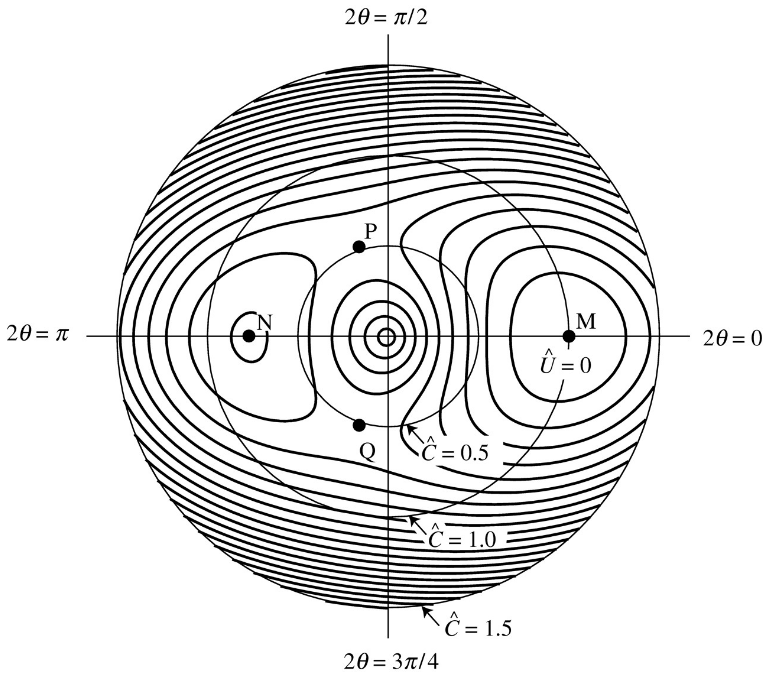
$$\hat{U} = \frac{UR^2}{D_{11}}, \qquad \hat{\boldsymbol{D}} = \frac{\boldsymbol{D}}{D_{11}}, \qquad \hat{\boldsymbol{\kappa}} = R\kappa, \qquad \hat{C} = CR$$

By plotting the energy contours in a polar plot (radial: bending radius, theta: twist angle), it can be seen where the boom is in equilibrium $\partial U/\partial\theta = \partial U/\partial C = 0$. By inspection (or calculation) it is determined if this is a local maximum (unstable) or minimum (stable).

An isotropic material has one stable (M) and unstable (N) equilibrium position.

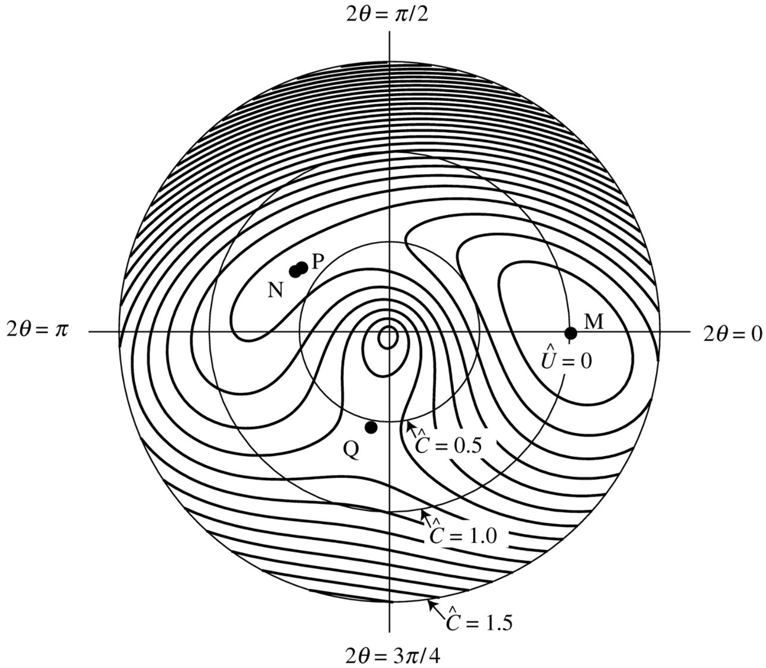


Antisymmetric 45° layup ($[\pm 45/0/\pm 45]$), where $B_{16} \neq 0$ and $B_{26} \neq 0$, and where $D_{16} = D_{26} = 0$.



This configuration is bistable, and has a second stable equilibrium point N. There are also two unstable equilibrium points P and Q.

A symmetric 45° layup ($[\pm 45/0/\mp 45]$), where $\boldsymbol{B} = 0$, but D_{16}, and D_{26} are nonzero.



This configuration is also bistable, but the symmetry is broken due to the bend-twist coupling. It is also very close an unstable equilibrium point P, which makes the structure unstable.

For this application, it is important to minimise the bend/twist coupling (D_{16}, D_{26}) as otherwise the booms would coil helically. For layups with no bend/twist coupling, the structure will always have a second equilibrium position, which was shown to be stable if:

$$4\hat{D}_{66} + 2\hat{D}_{12} - 2\frac{\hat{D}_{22}}{\hat{D}_{12}} > 0$$

For the bistable deployable booms $D_{16} = D_{26} = 0$ was achieved by using a woven fabric, where a simplification can be made that the $\pm\theta$ of the braid lie at the same distance from the midplane. This eliminates any bend-twist coupling, but also ensures $\boldsymbol{B} = 0$ by virtue of the lay-up symmetry.

NB: this example is intended to show an interesting application of composite structures; the details of the analysis are not examinable.

References:

- S. D. Guest and S. Pellegrino (2006), "*Analytical models for bistable cylindrical shells*", Proceedings of the Royal Society A, Volume 462, Issue 2067, doi:10.1098/rspa.2005.1598
- J. M. Fernandez, A. Viquerat, V. Lappas, A. J. Daton-Lovett (2014), "*Bistable Over the Whole Length (BOWL) CFRP Booms for Solar Sails*", in Advances in Solar Sailing, pp. 609-628, doi:10.1007/978-3-642-34907-2_38