EMAT10100 Engineering Maths I

Lecture 10: Introduction to Matrices

John Hogan & Alan Champneys

What is a matrix?

- It's a rectangular table of (e.g. real) numbers:

$$\mathbf{A} = \begin{pmatrix} 1.1 & 2.47 & 0 & -\pi \\ \sqrt{2} & -3.25 & 4.1 & -1.365 \\ 1.24 & 3 & -2.1 & 0 \end{pmatrix}$$

- Matrices have bold upper case names
  - or you might write $\underline{\underline{A}}$
- This is a 3×4 matrix (3 rows, 4 columns)
  - pronounced "3 by 4"
- Size of a matrix turns out to be very important

Rows and columns

- Rows of a matrix defined very simply:

$$\begin{pmatrix} 1.1 & 2.47 & 0 & -\pi \\ \color{red}{\sqrt{2}} & \color{red}{-3.25} & \color{red}{4.1} & \color{red}{-1.365} \\ 1.24 & 3 & -2.1 & 0 \end{pmatrix}$$

  - second row of the matrix is highlighted in red

- Columns also very simple:

$$\begin{pmatrix} 1.1 & 2.47 & 0 & \color{blue}{-\pi} \\ \sqrt{2} & -3.25 & 4.1 & \color{blue}{-1.365} \\ 1.24 & 3 & -2.1 & \color{blue}{0} \end{pmatrix}$$

  - fourth column is highlighted in blue

Elements and subscripts

- Elements of matrix are the individual entries:

$$\mathbf{A} = \begin{pmatrix} 1.1 & \color{red}{2.47} & 0 & -\pi \\ \sqrt{2} & -3.25 & 4.1 & \color{blue}{-1.365} \\ 1.24 & 3 & -2.1 & 0 \end{pmatrix}$$

  - Element $a_{1,2} = 2.47$ (first row, second column)
  - Element $a_{2,4} = -1.365$ (second row, fourth column)
- Subscripts are coordinates, referenced from top-left
  - row number comes first, then column number
- General notation:
  - $\mathbf{A} = \{a_{i,j}\}$ with $1 \le i \le m$, $1 \le j \le n$ (that is, talk about m rows and n columns)

Matrix transpose

- If $\mathbf{A}$ is a matrix, transpose $\mathbf{A}^{\mathrm{T}}$ obtained by swapping rows and columns
  - If $\mathbf{A}$ is $m \times n$, then $\mathbf{A}^{\mathrm{T}}$ is $n \times m$
  - If $\mathbf{B} = \mathbf{A}^{\mathrm{T}}$, $b_{i,j} = a_{j,i}$
- Exercise: Find transpose of following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 9 & 3 \\ 2 & 7 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 8 \end{pmatrix}.$$

- NB: since $\mathbf{C} = \mathbf{C}^{\mathrm{T}}$, we call $\mathbf{C}$ symmetric

Scalar multiplication of matrix

(c.f. scalar multiplication of a vector)

- To find $\lambda\mathbf{A}$, just multiply each element by scalar λ.
  - result is just the same size as $\mathbf{A}$
- Exercise: Find $\mathbf{B} = \lambda\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 0 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad \lambda = 5.$$

- Formally, we might say $b_{i,j} = \lambda a_{i,j}$

Matrix addition

(c.f. vector addition)

- To find $\mathbf{A} + \mathbf{B}$, just add elements in pairs
  - matrices $\mathbf{A}$, $\mathbf{B}$ must be same size
  - their sum is this size also
- Exercise: Find $\mathbf{C} = \mathbf{A} + \mathbf{B}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 0 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \\ 0 & 4 \end{pmatrix}.$$

- Formally we might say $c_{i,j} = a_{i,j} + b_{i,j}$

Equality of matrices and the zero matrix

- Two matrices are equal if:
  - they have the same number of rows and columns
  - all corresponding pairs of elements are equal
    i.e., $a_{i,j} = b_{i,j}$ for all i, j
- Matrix is called zero if every element is zero.
  - There are lots of zero matrices, e.g.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are different zero matrices.

Sounds a lot like vectors!

- Row vector
$$\mathbf{a} = \begin{pmatrix} 1 & 3 & -2 & 5 \end{pmatrix}$$
is a 1×4 matrix
- Column vector
$$\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix} \quad \text{is a } 4\times1 \text{ matrix}$$
- NB: as matrices, you can't add $\mathbf{a}$ and $\mathbf{b}$
(different numbers of rows and columns)
or say that they are equal

How to multiply matrices

(the big question)

- To work out $\mathbf{AB}$:
  - multiply the rows of $\mathbf{A}$ by the columns of $\mathbf{B}$
  - NEED: num cols of $\mathbf{A}$ = num rows of $\mathbf{B}$
- Example:
$$\begin{pmatrix} 4 & 1 & 4 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 4 & 1 \\ 1 & 2 & 2 & 1 \\ 3 & 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 6 & 26 & 17 \\ 3 & 8 & 14 & 5 \end{pmatrix}$$
- (2×3)×(3×4) gives a 2×4
- inner dimensions match, outer dimensions give size

Exercises

- If possible, find $\mathbf{AB}$ and $\mathbf{BA}$ when
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}$$
- NB: Here $\mathbf{AB} \neq \mathbf{BA}$.
We say that $\mathbf{A}$ and $\mathbf{B}$ do not commute
- Find $\mathbf{Ax}$ where
$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Properties of matrix multiplication

- Associative:
$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$
- Distributive in various ways:
  - $(\lambda\mathbf{A})\mathbf{B} = \lambda(\mathbf{AB}) = \mathbf{A}(\lambda\mathbf{B})$
  - $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
  - $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- However, $\mathbf{AB}$ and $\mathbf{BA}$ can be different
- Also:
$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

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Lecture 11: Matrices as transformations

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Square matrices and transformations

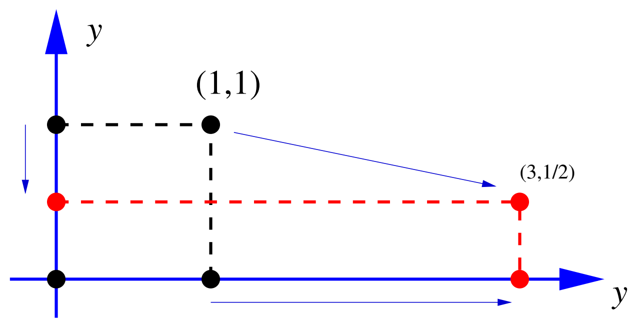
- Square matrix: number of columns = number of rows
- Let $\mathbf{A}, \mathbf{B}$ be square $n \times n$ matrices
  Let $\mathbf{x}$ be a column n-vector, i.e. an $n \times 1$ matrix
- Then
  - $\mathbf{Ax}$ is an $n \times 1$ matrix
  - $\mathbf{AB}$ is an $n \times n$ matrix
- For $n = 2, 3$: $\mathbf{A}$ 'maps' points in space
- It's linear: straight lines $\mapsto$ straight lines etc.
- What does matrix multiplication mean?

Transformation of unit square

- Let $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$.
- What does $\mathbf{A}$ do to the square with vertices $(0,0)^{\mathrm{T}}$, $\mathbf{e}_1 = (1,0)^{\mathrm{T}}$, $\mathbf{e}_2 = (0,1)^{\mathrm{T}}$, $\mathbf{e}_1 + \mathbf{e}_2 = (1,1)^{\mathrm{T}}$?
- Note $\mathbf{A}(0,0)^{\mathrm{T}} = (0,0)^{\mathrm{T}}$. Also:

$$\begin{aligned}
\mathbf{A}\mathbf{e}_1 &= \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix}, \\
\mathbf{A}\mathbf{e}_2 &= \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}, \\
\mathbf{A}(\mathbf{e}_1 + \mathbf{e}_2) &= \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,1} + a_{1,2} \\ a_{2,1} + a_{2,2} \end{pmatrix}.
\end{aligned}$$

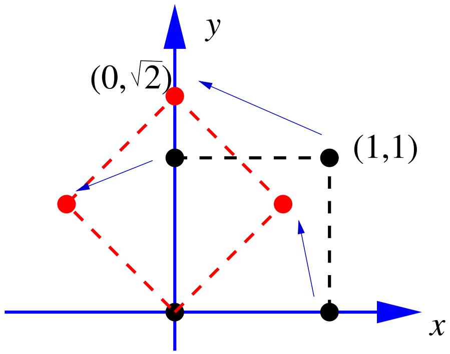
Example 1



- $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$
- x-direction gets stretched by factor of 3
- y-direction gets condensed by factor of $1/2$
- Area of square is scaled by $3/2$

Example 2

- $\mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} +1 & -1 \\ +1 & +1 \end{pmatrix}$
- Anticlockwise rotation of $\pi/4$ about origin
- Area of square unchanged
- How to do rotation by other angles?



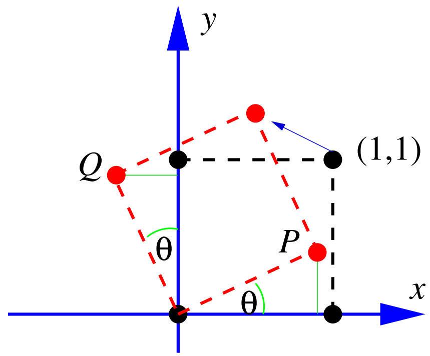
Rotation through angle θ

- Need

$$P = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \; Q = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

- Rotation through angle θ: multiply by

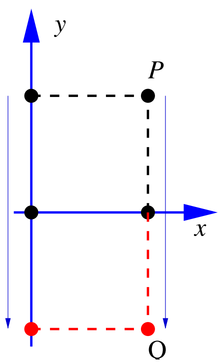
$$\mathbf{A}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$



- Exercise Calculate the matrix product $\mathbf{A}(\phi)\mathbf{A}(\theta)$. What does it correspond to geometrically?

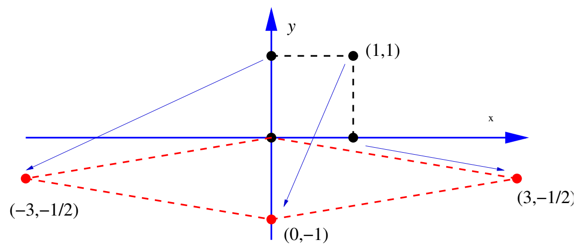
Example 3

- $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection in x-axis
- Area of square unchanged
- Is it possible to reflect about other lines?



Example 4

- $\mathbf{A} = \begin{pmatrix} +3 & -3 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$



- Mix of shear, rotation, reflection, magnification
- How to analyse, and what is new area?

Example 4 continued

- It turns out (matrix multiplication is associative)

$$\begin{pmatrix} +3 & -3 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} +1 & -1 \\ +1 & +1 \end{pmatrix}$$

- Transformation corresponds to (in order):
  1. Rotation by $\pi/4$, and stretch both axes by $\sqrt{2}$
  2. Stretch x-axis by factor of 3
     Condense y-axis by factor of $1/2$
  3. Reflect in x-axis

- Area magnification factor is $\sqrt{2} \times \sqrt{2} \times 3 \times \frac{1}{2} = 3.$

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In general?

- What if you are given a matrix and you don't know a nice decomposition in the manner of Example 4?

- Area scale factor: calculate determinant
  (next slide & next lecture)

- general structure: calculate eigenvalues and eigenvectors (lectures in two weeks time)

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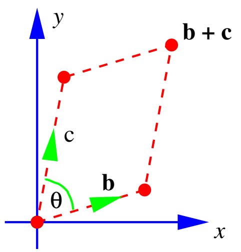
Area of the parallelogram image

- Area red parallelogram

$$\begin{aligned} &= |\mathbf{b}|\,|\mathbf{c}|\,\sin\theta \\ &= |\mathbf{b} \times \mathbf{c}| \end{aligned}$$

(give or take a sign) and

$$\mathbf{b} = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}.$$



- Here

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ 0 \end{pmatrix} \times \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \end{pmatrix}.$$

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Determinant: two by two case

- LEARN THIS: The determinant of

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

is defined by

$$\det \mathbf{A} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

- Alternative notation:

$$\det(\mathbf{A}) \equiv \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}.$$

- Size of determinant gives area scale factor
- Negative value indicates reflection

Zero determinants

- What does it mean if a matrix has zero determinant?
- Exercise Examine the transformation defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

(Observe $\det \mathbf{A} = 0$.)

- $\mathbf{A}$ destroys area: transformation cannot be undone
- We say: $\mathbf{A}$ is singular or non-invertible

Identity matrix

- Identity matrix: $n \times n$ matrix which leaves $n \times 1$ and other $n \times n$ matrices unchanged under multiplication

$$\text{E.g.} \quad \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{etc.}$$

(omit subscript when context obvious)

- Leaves areas unchanged, so $\det \mathbf{I} = 1$
- NB: can only have square identity matrices

Diagonal matrices

- Diagonal matrix: square matrix with non-zero entries on 'diagonal' only
- E.g.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3.1 & 0 & 0 \\ 0 & 1.7 & 0 \\ 0 & 0 & -1.3 \end{pmatrix}, \quad \text{and so on}$$

- Corresponds to simple scaling in each axis direction
- Determinant: simply product of diagonal entries
- $n = 3$: determinant gives volume scaling
  How to find $\det$ for more general 3×3 matrices?

Homework

- Read James 5.1–5.2
- Do the following exercises in James (4th edition)
  - Do Exercises 5.2.3 Qs 1–3
  - Do Exercises 5.2.5 Qs 11, 12, 14, 16
  - Do Exercises 5.2.7 Qs 17, 22, 32
- Do the following exercises in James (5th edition)
  - Exercises 5.2.3 Qs 1–3
  - Exercises 5.2.5 Qs 12,13, 16, 18
  - Exercises 5.2.7 Qs 19, 24, 32
- Next time, determinants of more general $n \times n$ matrices