Advanced Bending and Torsion
Unsymmetric Bending – Definitions

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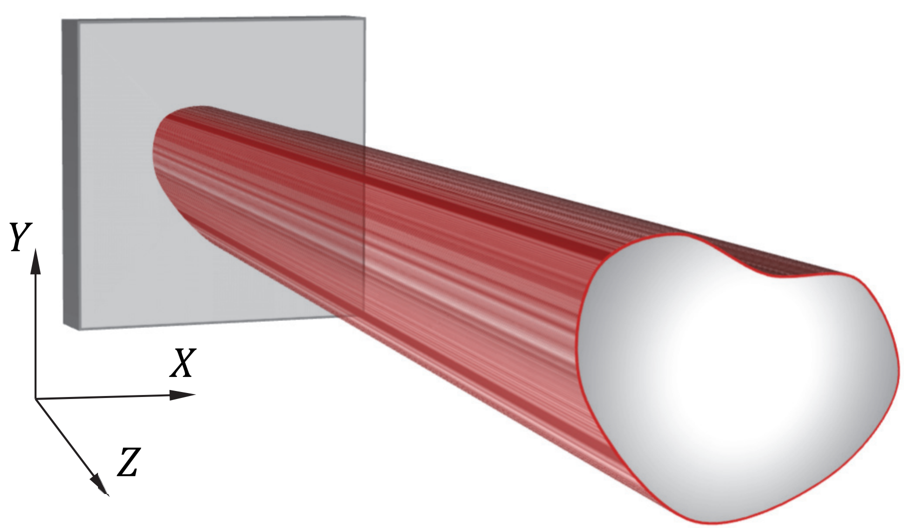
Definitions

1. Axes
    1. Global structural axes (X, Y, Z)
    2. Local section axes (x, y, z)
    3. Section principal axes $(1, 2)$
    4. Loading axes
    5. Neutral axis (NA)
2. Area properties
    1. Area, centroid, 1st moments of area
    2. 2nd moments of area (w.r.t. x, y axes)
    3. Transformed 2nd moments of area (w.r.t. x', y'axes)
    4. Principal 2nd moments of area (w.r.t. $1, 2$ axes)
    5. Product 2nd moments of area

Global (Structural) Axes (X, Y, Z)

- A global, structural coordinate system (*Global CS*) must be defined
- The structure is described in space based on the *Global CS*, which we name XYZ (uppercase roman typeface)
- The origin and orientation of the *Global CS* are somewhat arbitrary; here we will always align Z with the beam's length, so that all cross-section planes are parallel to the XY plane



Global coordinate system (*Global CS*)

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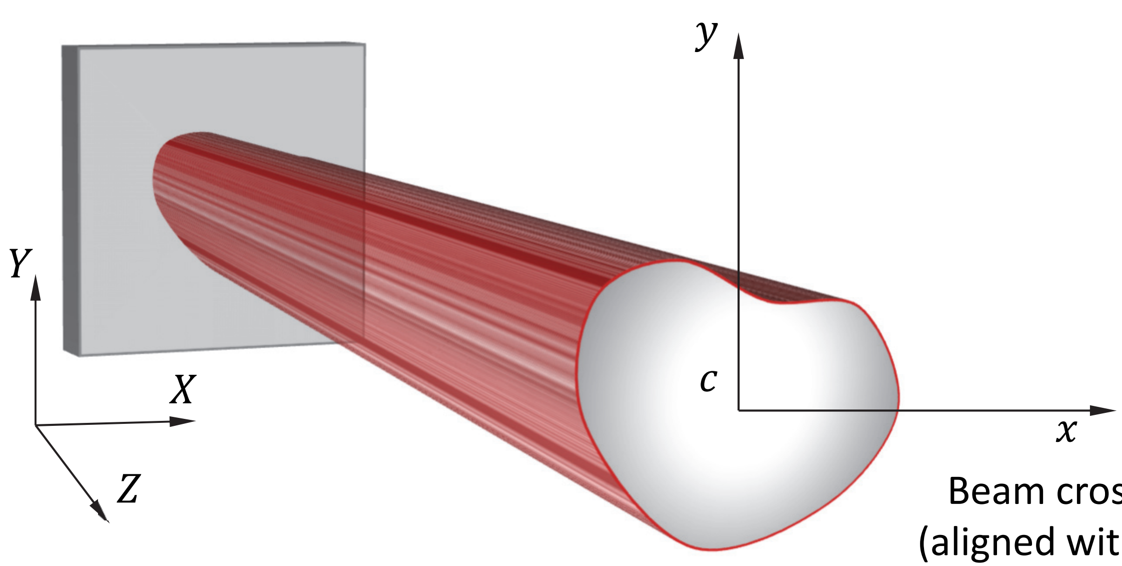
Cross-Section Axes (x, y)

- A local section coordinate system (*Section CS*) is defined with origin at the centroid of the cross-section, but aligned with the *Global CS*

- The Section CS is defined by (x, y) (lowercase roman typeface)

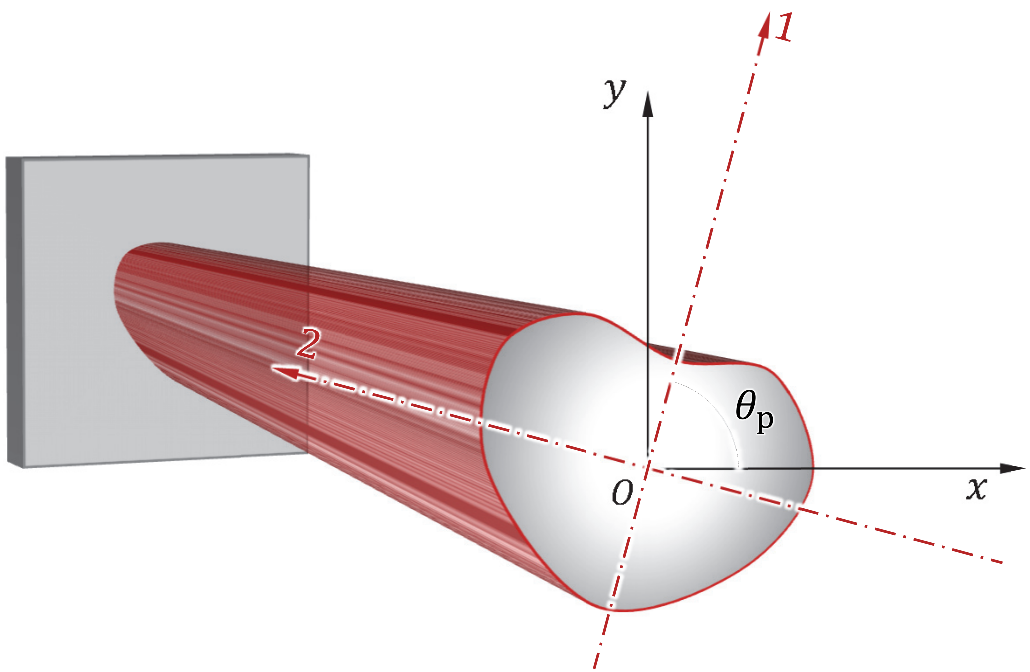


Beam cross-section axes (aligned with the *Global CS*)

Global coordinate system (*Global CS*)

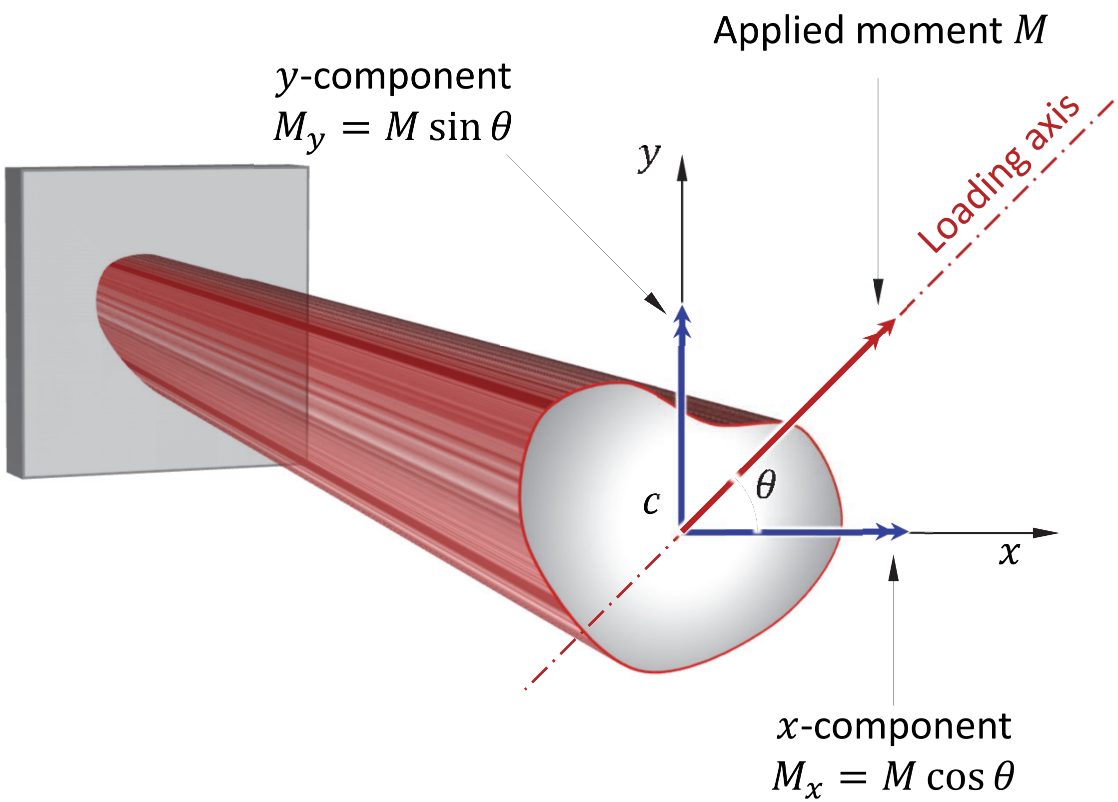
Principal Axes (1,2)

- Axes about which the **2^{nd} moment of area is maximum or minimum**
  - pass through the centroid
  - coincident with symmetry axes (if applicable)
- Principal axis of minimum I-value is the axis of least bending resistance, *i.e.* the axis about which the beam will preferentially bend



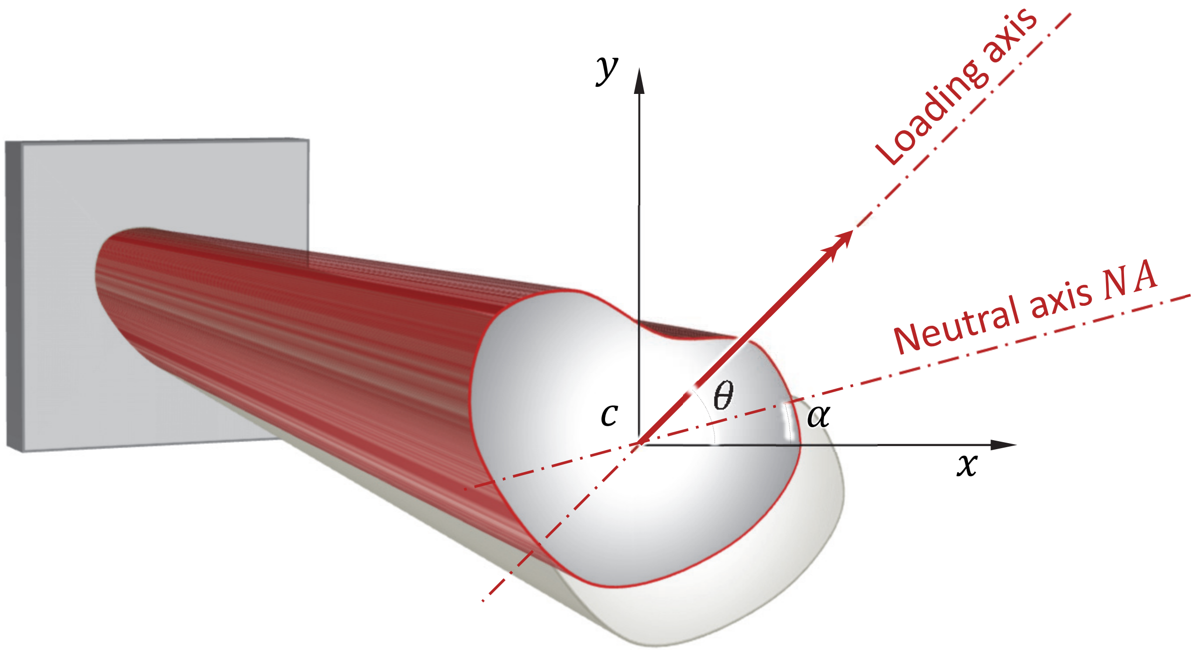
Loading Axes

- Axes of resultant applied bending moment
  - *i.e.* axis about which the beam is being forced to bend
- Moments are vectorial quantities and can be resolved onto chosen reference axis:



Neutral axis

- Axis about which the beam **actually** bends once loaded
- Passes through the centroid
- Direct stress is zero on the neutral axis and maximum at furthest distance from it, varying linearly, as before
- Bending deflection is perpendicular to the neutral axis



Area, centroid, 1st moment of area

- Area:

$$A = \int \mathrm{d}A \text{ or } \sum A_i$$

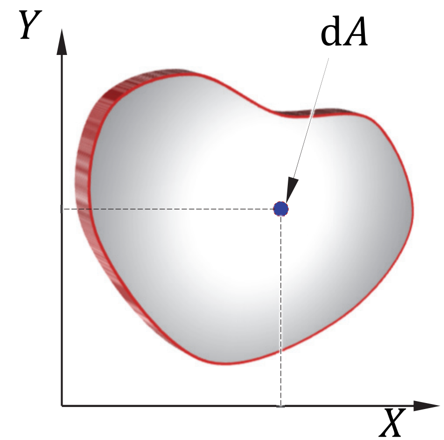
- 1st moment:

$$Q_{XX} = \int Y \, \mathrm{d}A$$

$$(= \sum \bar{Y}_i \, A_i)$$

$$Q_{YY} = \int X \, \mathrm{d}A$$

$$(= \sum \bar{X}_i \, A_i)$$



- Centroid:

$$\bar{X} = \frac{Q_{YY}}{A} \text{ or } \bar{X} = \frac{\sum \bar{X}_i A_i}{\sum A_i}$$

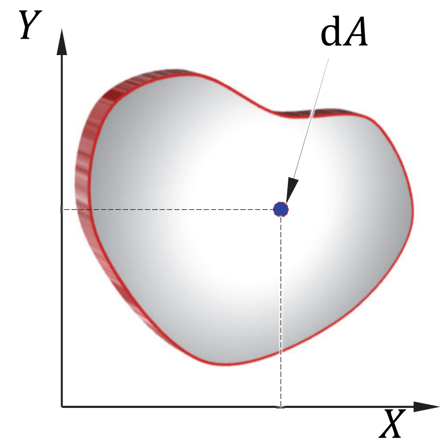
$$\bar{Y} = \frac{Q_{XX}}{A} \text{ or } \bar{Y} = \frac{\sum \bar{Y}_i A_i}{\sum A_i}$$

2$^{\text{nd}}$ moments of area

- Again, same as before:

$$I_{XX} = \int Y^2 \, \mathrm{d}A$$

$$I_{YY} = \int X^2 \, \mathrm{d}A$$



- Now, the **product second moment of area** is:

$$I_{XY} = \int XY \, \mathrm{d}A$$

Product 2nd moments of area

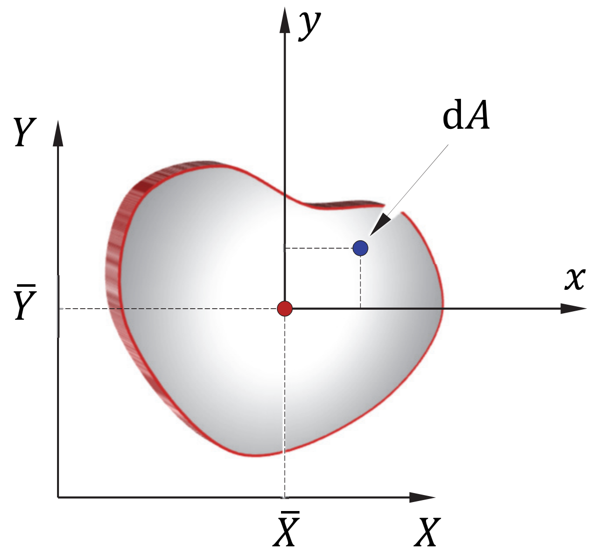
- Named I_{XY}
- Can be positive, negative or zero, depending on which 'quadrant' of the XY plane the section is in
- Zero about principal axes (areas in opposite quadrants cancel out)
- Can be calculated by parallel axis theorem

Parallel Axis Theorem: 2nd moments of area

$$I_{XX} = \int Y^2 \, \mathrm{d}A = \int (\bar{Y} + y)^2 \, \mathrm{d}A$$

$$I_{XX} = \bar{Y}^2 \int \mathrm{d}A + 2\,\bar{Y} \int y \, \mathrm{d}A + \int y^2 \, \mathrm{d}A$$

$$I_{XX} = \int y^2 \, \mathrm{d}A + A\,\bar{Y}^2$$



- Similarly:

$$I_{YY} = \int x^2 \, \mathrm{d}A + A\,\bar{X}^2$$

- Note that these are identical to the expressions seen in StM1 for compound sections:

$$I = \sum I_i + A_i\, d_i^2$$

(N.B. StM1 notation differs from StM2)

Parallel Axis Theorem: Product 2nd moments of area

- The same principles apply to I_{XY}:

$$I_{XY} = \int XY \,\mathrm{d}A = \int (\bar{X} + x)(\bar{Y} + y)\,\mathrm{d}A$$

$$I_{XY} = \bar{X}\,\bar{Y}\int \mathrm{d}A + \bar{X}\int y\,\mathrm{d}A + \bar{Y}\int x\,\mathrm{d}A + \int x\,y\,\mathrm{d}A$$

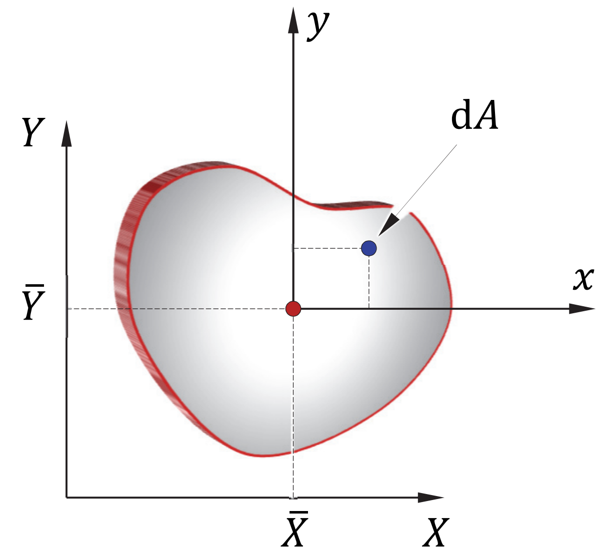
- But by definition:

$$\int x\,\mathrm{d}A = 0 \text{ and } \int y\,\mathrm{d}A = 0$$

- Therefore:

$$I_{XY} = \bar{X}\,\bar{Y}\int \mathrm{d}A + \int x\,y\,\mathrm{d}A$$

$$I_{XY} = I_{xy} + \bar{X}\,\bar{Y}\int \mathrm{d}A$$



Principal Axes

Parallel axes theorem

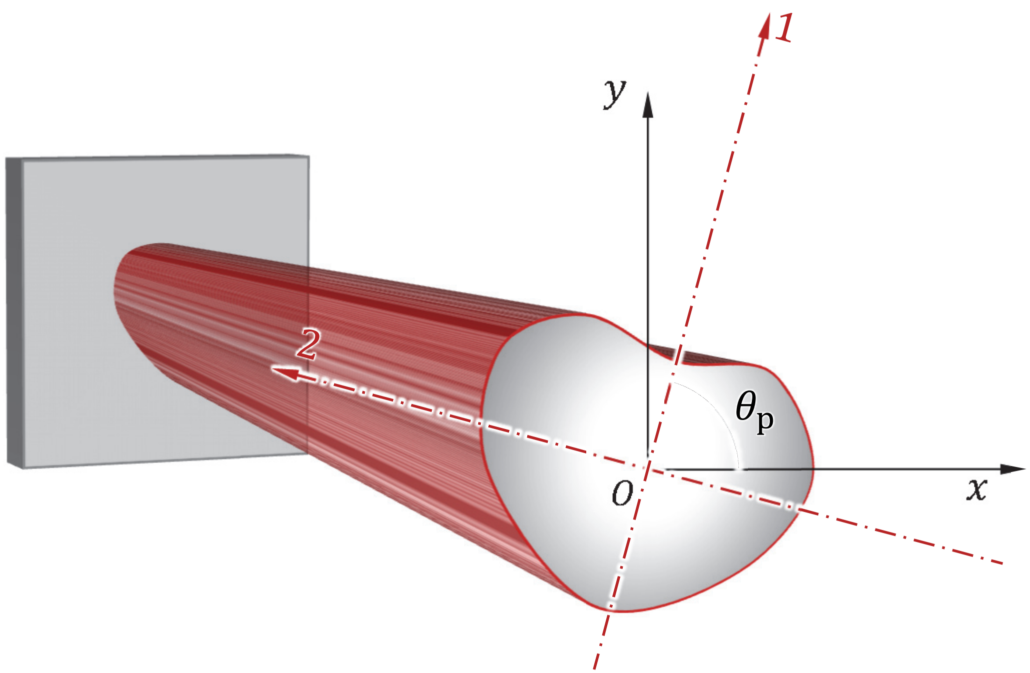
$$I_{XX} = \sum_{i=1}^{n} \left(I_{xx_i} + y_i^2 A_i \right)$$

$$I_{YY} = \sum_{i=1}^{n} \left(I_{yy_i} + x_i^2 A_i \right)$$

$$I_{XY} = \sum_{i=1}^{n} \left(I_{xy_i} + x_i \, y_i \, A_i \right)$$

Principal axes

$$\theta_p = \frac{1}{2} \tan^{-1} \left[\frac{2 I_{XY}}{I_{YY} - I_{XX}} \right]$$

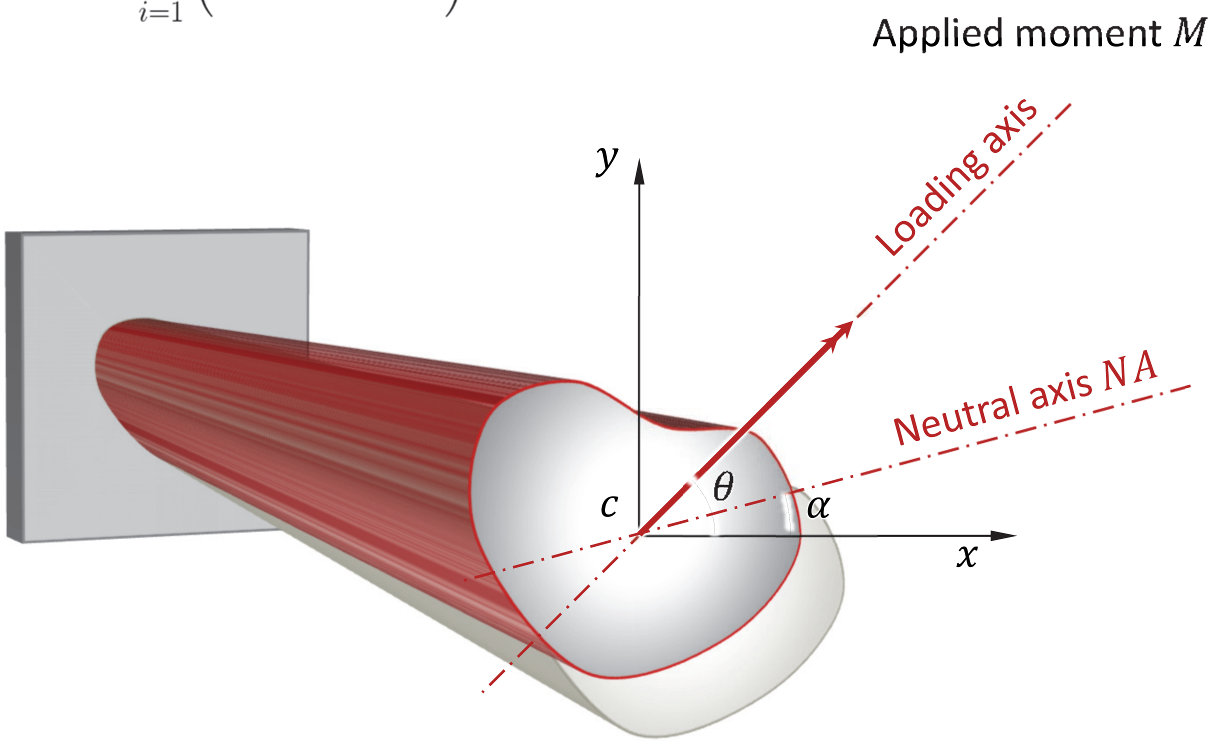


Parallel axes theorem

$$I_{XX} = \sum_{i=1}^{n} \left(I_{xx_i} + y_i^2 A_i \right)$$
$$I_{YY} = \sum_{i=1}^{n} \left(I_{yy_i} + x_i^2 A_i \right)$$
$$I_{XY} = \sum_{i=1}^{n} \left(I_{xy_i} + x_i\, y_i\, A_i \right)$$

Neutral axes

$$\tan\alpha = \frac{Y_{NA}}{X_{NA}} = \left[\frac{M_X I_{XY} - M_Y I_{XX}}{M_X I_{YY} - M_Y I_{XY}} \right]$$



Transformation (rotation) of axes

Coordinates,

$$\begin{Bmatrix} X_p \\ Y_p \end{Bmatrix} = \begin{bmatrix} \cos\theta_p & \sin\theta_p \\ -\sin\theta_p & \cos\theta_p \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

Properties,

$$\begin{Bmatrix} I_{11} \\ I_{22} \\ I_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_p & \sin^2\theta_p & -2\sin\theta_p\cos\theta_p \\ \sin^2\theta_p & \cos^2\theta_p & 2\sin\theta_p\cos\theta_p \\ \sin\theta_p\cos\theta_p & -\sin\theta_p\cos\theta_p & \cos^2\theta_p - \sin^2\theta_p \end{bmatrix} \begin{Bmatrix} I_{XX} \\ I_{YY} \\ I_{XY} \end{Bmatrix}$$

Alternatively,

$$I_{ii} = \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta,$$

$$I_{jj} = \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta + I_{xy}\sin 2\theta,$$

$$I_{ij} = \frac{1}{2}(I_{xx} - I_{yy})\sin 2\theta + I_{xy}\cos 2\theta.$$