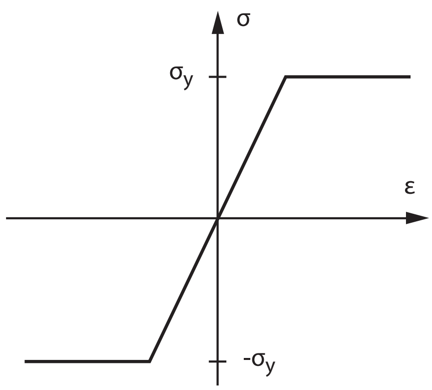
Handout 4 – Failure Criteria

This handout introduces failure criteria, to predict material failure under applied loads. In particular, we are interested in how to extrapolate the results of a uni-axial tensile test to predict the failure under a combined stress state. We shall approach the failure criteria from a macroscopic (elasticity), rather than a micro-mechanical (materials), point of view.

A common simplification is that materials have a distinct yield stress, σ_Y, after which the material fails abruptly (brittle failure) or yields plastically (ductile failure). In practise, this yield point is not sharply defined for ductile materials; nonetheless, we shall here assume an elastic/perfectly-plastic material model.

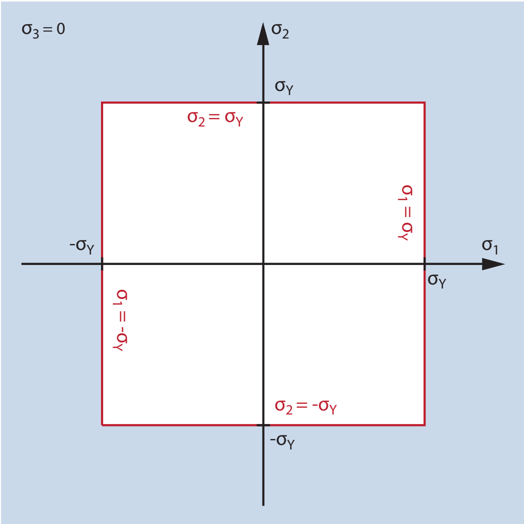


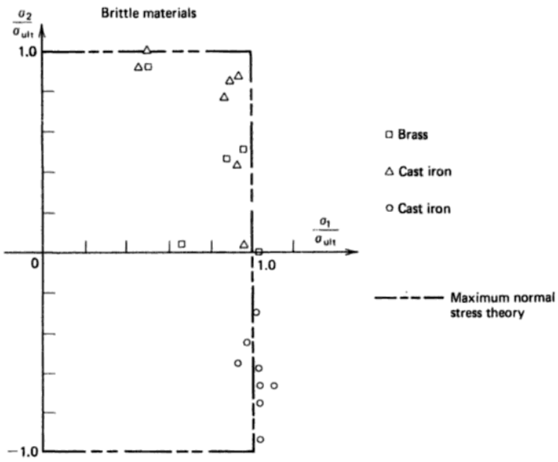
It is assumed that the material behaves linearly up to the yield point, and can therefore be described using the methods developed in this unit. To disassociate the failure criteria from a specific set of axes, these failure criteria are defined in terms of the principal stresses σ_1, σ_2, and σ_3. We shall focus on plane stress, where $\sigma_3 = 0$, but shall see that it is important to explicitly consider the out-of-plane stress in the material failure.

In the principal stress space we aim to find a **yield locus** which separates elastic from plastic deformation; any stress state within its boundaries will therefore be elastic, and failure occurs at the boundary. We are not concerned with the details of the plastic deformations after yield, and shall limit ourselves to the point of onset of failure.

4.1 Maximum Principal Stress

The simplest failure criterion, proposed by **Rankine** (1857), states that failure occurs when the **maximum principal stress** reaches a critical value. The direction of failure is then taken to be the plane of maximum principal stress.





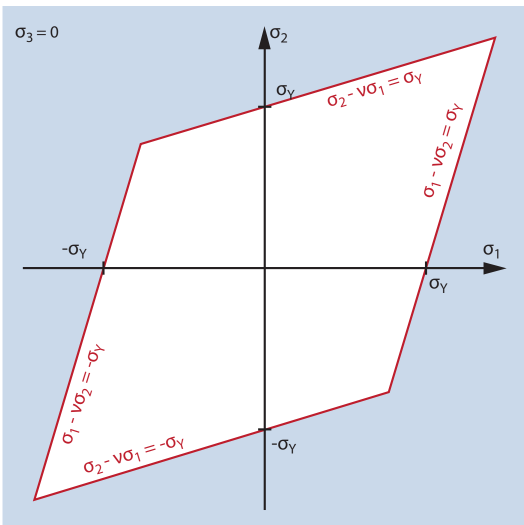
While this is not a very realistic condition for ductile failure, it is quite well suited for predicting brittle failure. This includes ceramic materials, as well as fast fracture (in tension) of metals such as cast iron. The failure in torsion of a piece of blackboard chalk also illustrates this failure mode, as the failure surface will form a 45° helix along a plane with maximum principal stress.

4.2 Maximum Principal Strain

Another theory, due to **Saint-Venant**, predicts failure at a **maximum principal strain**. Assuming that tensile and compressive strength are equal, this failure criterion can be expressed as:

$$\sigma_1 - \nu\,(\sigma_2 + \sigma_3) = \pm\sigma_Y$$
$$\sigma_2 - \nu\,(\sigma_1 + \sigma_3) = \pm\sigma_Y$$

which defines the boundaries for the yield surface, where $\sigma_3 = 0$ for plane stress. This failure criterion is mostly of historical importance, and was favoured by engineers in the 19th century.



4.3 Tresca Yield Criterion

The underlying principle of the **Tresca** yield criterion (1878) is that yielding occurs when the **maximum shear stress** reaches a critical value. This was evidenced by experiments on annealed metals that showed that spherical stress does not cause yield, and it is the *difference* in principal stresses that drives failure.

The critical shear stress can be found from a uni-axial tensile test:

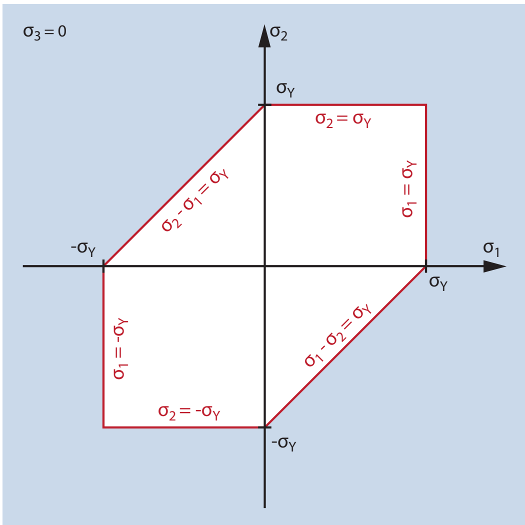
$$\tau_{\text{crit}} = \frac{|\sigma_1 - \sigma_2|}{2} = \frac{\sigma_Y}{2}$$

where $\sigma_2 = \sigma_3 = 0$, and at point of failure $\sigma_1 = \sigma_Y$.

In general, the maximum shear stress is determined by the maximum difference between the three principal stresses, and the Tresca failure criterion becomes:

$$\max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right) = \sigma_Y \tag{4.1}$$

For plane stress $(\sigma_3 = 0)$ the yield locus is drawn as:



with the following bounds:

$$\begin{aligned}
\sigma_1 \geq \sigma_2 \geq \sigma_3(=0) &\;\rightarrow\; \sigma_1 - \sigma_3 = \sigma_Y \;\therefore\; \sigma_1 = \sigma_Y \\
\sigma_2 \geq \sigma_1 \geq \sigma_3(=0) &\;\rightarrow\; \sigma_2 - \sigma_3 = \sigma_Y \;\therefore\; \sigma_2 = \sigma_Y \\
\sigma_3(=0) \geq \sigma_2 \geq \sigma_1 &\;\rightarrow\; \sigma_3 - \sigma_1 = \sigma_Y \;\therefore\; \sigma_1 = -\sigma_Y \\
\sigma_3(=0) \geq \sigma_1 \geq \sigma_2 &\;\rightarrow\; \sigma_3 - \sigma_2 = \sigma_Y \;\therefore\; \sigma_2 = -\sigma_Y
\end{aligned}$$

$$\begin{aligned}
\sigma_1 \geq \sigma_3(=0) \geq \sigma_2 &\;\rightarrow\; \sigma_1 - \sigma_2 = \sigma_Y \\
\sigma_2 \geq \sigma_3(=0) \geq \sigma_1 &\;\rightarrow\; \sigma_2 - \sigma_1 = \sigma_Y
\end{aligned}$$

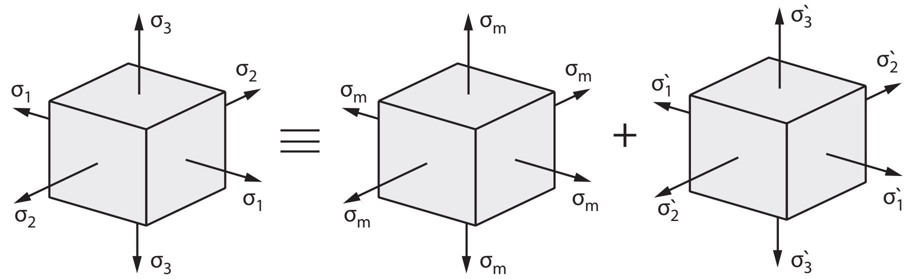
4.4 Von Mises Yield Criterion

Accurate experiments on annealed metals showed that in many cases the ratio of maximum shear stress to uni-axial yield stress was slightly greater than 0.5, and closer to 0.57. This led to the confirmation of a failure criterion based on the **maximum distortion energy**, better known as the **Von Mises** criterion (1913). It is sometimes also referred to as the *Hencky-Huber* or *Maxwell* criterion.

Consider an infinitesimal element in an elastic body subject to principal stresses, σ_1, σ_2 and σ_3. It is possible to replace this state of stress with a statically equivalent set of stresses consisting of a hydrostatic stress σ_m,

$$\sigma_m = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \tag{4.2}$$

plus deviatoric components $\sigma_i' = \sigma_i - \sigma_m$. It is these deviatoric components that induce shear, and therefore initiate yielding in ductile materials. Von Mises proposed that failure occurs when the elastic strain energy due to the deviatoric stresses reaches a critical value.



Let us consider the total energy per unit volume $\hat{U}_{\text{total}}$ of the elastic stresses:

$$\hat{U}_{\text{total}} = \sum_{i=1}^{3} \frac{1}{2}\sigma_i \varepsilon_i$$

where $\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right]$, *etc.* This gives the total elastic energy[4] as:

$$\begin{aligned}
\hat{U}_{\text{total}} &= \frac{1}{2E}\sigma_1\left[\sigma_1 - \nu\left(\sigma_2+\sigma_3\right)\right] + \frac{1}{2E}\sigma_2\left[\sigma_2 - \nu\left(\sigma_1+\sigma_3\right)\right] + \frac{1}{2E}\sigma_3\left[\sigma_3 - \nu\left(\sigma_1+\sigma_2\right)\right] \\
&= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right]
\end{aligned}$$

If we substitute σ_m (Equation 4.2) for σ_i we obtain the energy associated with the hydrostatic stress state:

$$\begin{aligned}
\hat{U}_{\text{hydrostatic}} &= \frac{3\left(1-2\nu\right)}{2E}\sigma_m^2 \\
&= \frac{1-2\nu}{6E}\left(\sigma_1+\sigma_2+\sigma_3\right)^2
\end{aligned}$$

Subtracting the hydrostatic component from the total energy, we find:

$$\begin{aligned}
\hat{U}_{\text{deviatoric}} &= \frac{1+\nu}{3E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3\right) \\
&= \frac{1}{12G}\left[\left(\sigma_1-\sigma_2\right)^2 + \left(\sigma_2-\sigma_3\right)^2 + \left(\sigma_1-\sigma_3\right)^2\right]
\end{aligned}$$

[4]The maximum total strain energy can, in fact, be considered a failure criterion in its own right, and is referred to as the *Beltrami-Haigh* failure criterion. Like the maximum principal strain criterion it has largely been superseded by other failure criteria.

From the uni-axial test with $\sigma_1 = \sigma_Y$ and $\sigma_2 = \sigma_3 = 0$ we find the critical deviatoric strain energy as:

$$\hat{U}_{\text{critical}} = \frac{1}{12G}\left[(\sigma_Y - 0)^2 + (0-0)^2 + (\sigma_Y - 0)^2\right] = \frac{\sigma_Y^2}{6G}$$

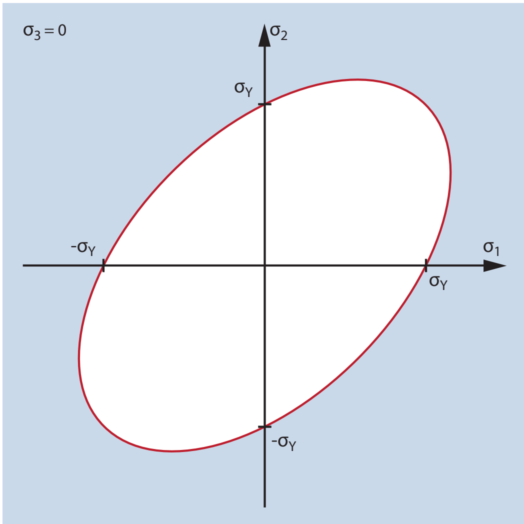
Substituting this critical value into the equation for deviatoric strain energy gives the Von Mises failure criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_Y^2 \tag{4.3}$$

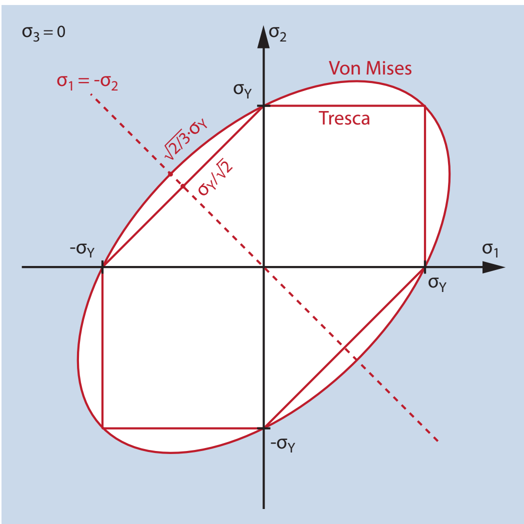
For plane stress ($\sigma_3 = 0$) this condition reduces to:

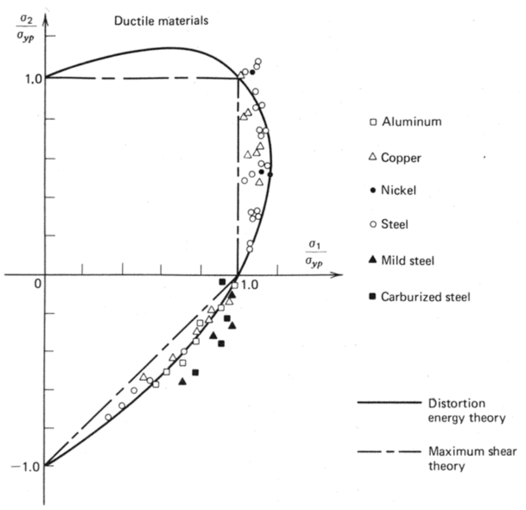
$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2 \tag{4.4}$$

which describes a yield locus in the form of an ellipse at an angle $\pi/4$ to the principal stress axes.



Comparing the Von Mises and Tresca condition, it can be seen that they agree quite closely. The largest discrepancy is for pure shear ($\sigma_1 = -\sigma_2$) where the difference is approximately 15%. The Von Mises criterion gives closer correlation with experiments, whereas the Tresca condition gives a useful conservative estimate. Often the decision which failure criterion to use will depend on which one is *easiest* to use for a calculation.





From a more theoretical perspective, it can be observed that the Von Mises criterion is related to the RMS of the principal stress differences, whereas the Tresca accounts for the maximum absolute difference.

Von Mises stress When using Finite Element software to analyse structural components, it is common practise to plot the Von Mises stress to identify the most highly stressed points. The Von Mises stress is calculated as:

$$\sigma_{\mathsf{vm}} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}} \tag{4.5}$$

and it has to remain below the material yield stress (with a suitable margin of safety) to avoid failure.

Example 4.1 – Experimental Yield Envelope

A novel Ceramic Matrix Composite material (CMC) is under development as a possible material for turbine blades in jet engines. Its yield and failure criteria need to be characterised. It is found to yield in uni-axial compression (albeit for small strains) at a stress of 1500 MPa, but is found to fail by a cleavage (brittle) mechanism in uni-axial tension at a stress of 700 MPa.

Q: Sketch the yield and cleavage surfaces in principal stress space and determine combinations of σ_1 and σ_2 where there is a transition from cleavage failure to yielding.

A: In tension a brittle failure is observed, and the Rankine criterion, with $\sigma_Y = 700$ MPa, is used.

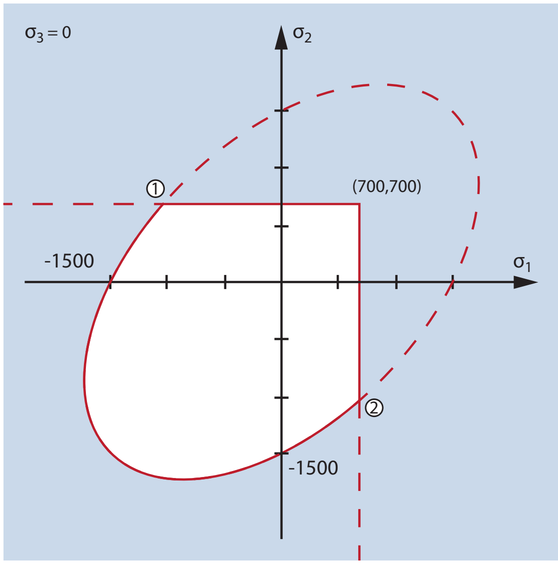
To capture the ductile compressive failure, the Von Mises criterion is assumed, with $\sigma_Y = 1500$ MPa.

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_Y^2$$

which for plane stress ($\sigma_3 = 0$) results in:

$$\sigma_Y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

The combined yield locus is therefore:



At the two points where the lines intersect, the damage mechanism changes. To find point 1, insert $\sigma_2 = 700$ in the equation for the Von Mises failure envelope and solve for σ_1 (similarly for point 2) to find (-1022,700) and (700,-1022).

Example 4.2 – Strength Calculation

Q: A finite element calculation of a thin-walled wing section under plane stress provided the following stresses: $\sigma_{xx} = 50$ MPa, $\sigma_{yy} = 100$ MPa and $\tau_{xy} = -125$ MPa. The material used is Aluminium 6061-T6, with a yield strength $\sigma_Y = 240$ MPa. Using a suitable failure criterion, verify the strength of the structure.

A: Aluminium is a ductile material, and either *Tresca* or *Von Mises* would therefore be a suitable failure criterion. First determine the principal stresses:

$$\sigma_1 = \qquad\qquad\qquad \sigma_2 =$$

Using the Von Mises failure criterion for plane stress:

$$\sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} =$$

we find that the Aluminium: fails / does not fail

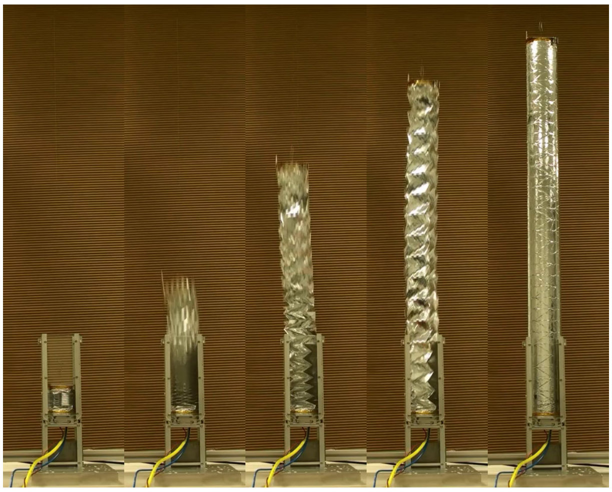
The Tresca criterion considers the maximum difference in principal stresses (in plane stress, $\sigma_3 = 0$)

$$|\sigma_1 - \quad| =$$

and predicts that the Aluminium: fails / does not fail.

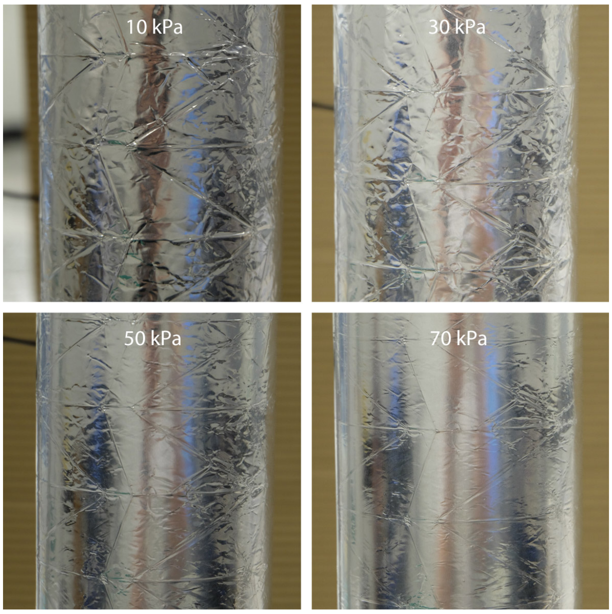
Example 4.3 – Inflatable-Rigidisable Cylinder

For a CubeSat technology demonstration mission, called InflateSail, an inflatable deployable mast was developed. In its deployed configuration the mast has a length $L \approx 1$ m, radius $r = 45$ mm and it is folded down for launch to approximately 65 mm using an origami pattern.



The cylindrical mast is made of a thin laminate, constructed of two layers of Aluminium foil sandwiching a Mylar film (14.5 μm and 16 μm respectively). The aluminium layers provide stiffness and strength, and the polymer layer offers toughness against crack propagation.

After inflation the skin of the cylinder will be wrinkled and creased as a result of the packaging process. To remove these residual creases — and thereby recover the stiffness and strength of the deployed boom — the pressure is increased until the skin material yields and smoothens out. This proces is referred to as strain-rigidisation, and its purpose is to provide stiffness to the boom, even after the inflation gas has been vented.



Q: What is the minimum required inflation pressure to achieve strain-rigidisation? Uni-axial tests suggested a yield stress of approximately 50 MPa for the laminate material.

A: From Example 1.1 recall the hoop stress σ_H, longitudinal stress σ_L and radial stress σ_R equations:

$$\sigma_H = \frac{pr}{t}$$
$$\sigma_L = \frac{pr}{2t} = \frac{1}{2}\sigma_H$$
$$\sigma_R \approx 0$$

with radius r, wall thickness t and gauge pressure p. These are also the principal stresses, as the stress state does not produce shear stresses in the chosen coordinate system.

Assuming plane stress Von Mises conditions, we can use:

$$\sigma_Y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

and substituting $\sigma_1 = \sigma_H$ and $\sigma_2 = \sigma_L = \frac{\sigma_H}{2}$, we find:

$$\sigma_Y^2 = \sigma_H^2 - \frac{1}{2}\sigma_H\sigma_H + \frac{1}{4}\sigma_H^2$$
$$= \frac{3}{4}\sigma_H^2$$

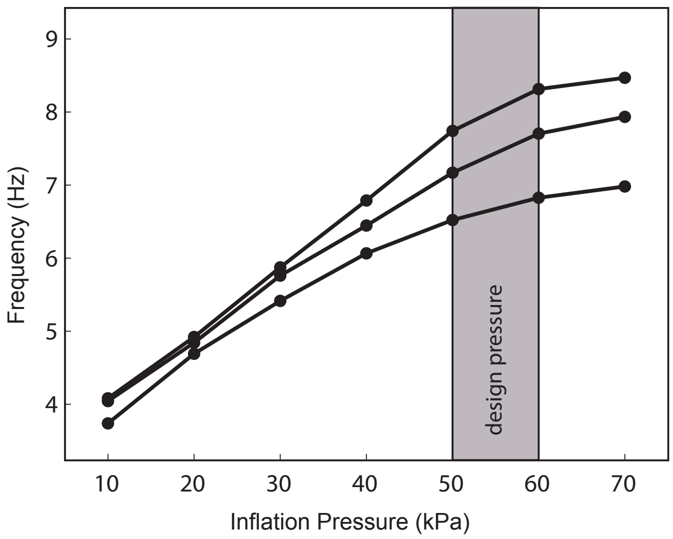
and therefore:

$$p = \sqrt{\frac{4}{3}}\frac{\sigma_Y t}{r}$$

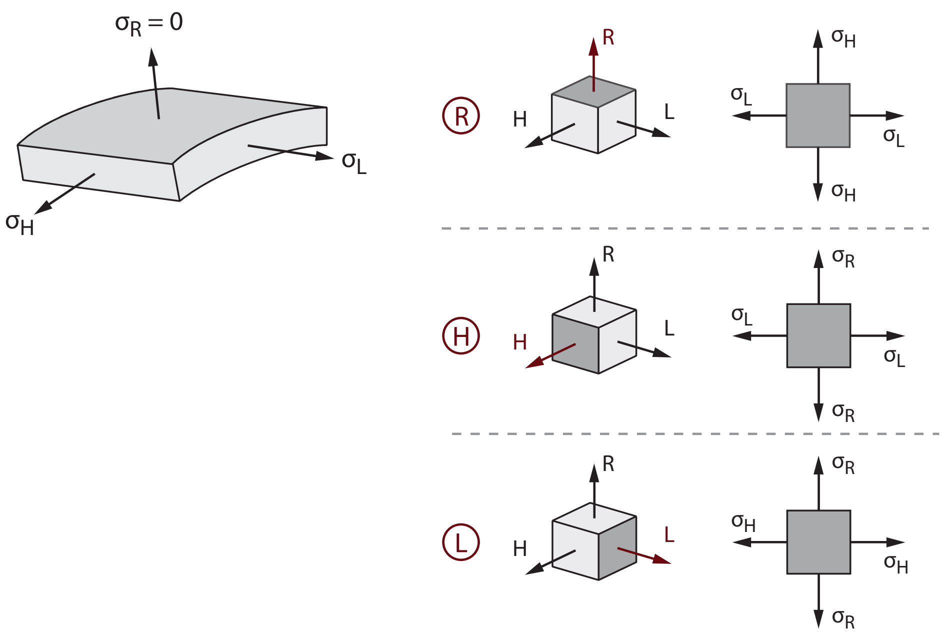
Substituting the values for our specific cylinder gives:

$$p = \sqrt{\frac{4}{3}} \cdot \frac{55 \cdot 10^6 \cdot 45 \cdot 10^{-6}}{45 \cdot 10^{-3}} = 57 \text{ kPa}$$

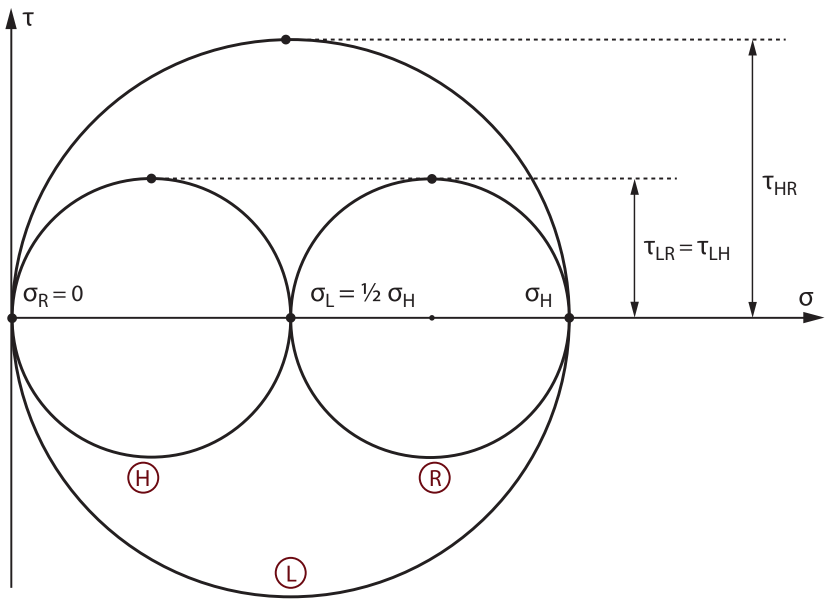
This simple analysis formed the basis of experiments done to verify the recovery of the boom stiffness, for different inflation pressures. This was done non-destructively by measuring the natural frequency of the boom, using small accelerometers attached to the boom tip. It was found that near the predicted rigidisation pressure the boom stiffness no longer increased for greater inflation pressures, indicating that the creases were effectively removed.



Q: A next question would be how the inflatable cylinder *deforms* under the applied pressure, which is where the Tresca yield condition provides more insight. It is important to consider all three principal stresses, to identify the plane in which the maximum shear stress occurs.



Rotating the stress state around the hoop (H), longitudinal (L) and radial (R) vector will provide three Mohr's circles of stress, which can be plotted in a single diagram:

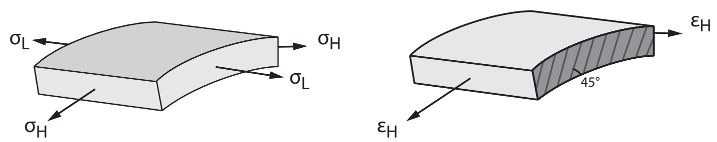


A similar diagram can be constructed for any state of (plane) stress. NB: it is very important to only rotate around vectors of principal directions, as otherwise you lose information about the shear stress on that face!

Looking at the three Mohr's circles of stress, we can see that the greatest shear stress is found in hoop/radial plane (*i.e.* rotating around longitudinal direction):

$$\tau_{\text{max}} = \frac{\sigma_H - \sigma_R}{2} = \frac{\sigma_H}{2}$$

Thus plastic deformation initiates as shear in the hoop direction.



In other words, the diameter increases through plastic deformation, but the cylinder does not permanently change length axially. This has a strong effect on the rigidisation process, as it means that the longitudinal creases will be flattened out most as a result of the strain rigidisation.

References

A Viquerat, M Schenk, B Sanders & VJ Lappas (2014), "*Inflatable Rigidisable Mast For End-Of-Life Deorbiting System*", European Conference on Spacecraft Structures, Materials and Environmental Testing (SSMET) 2014, April 1–4, Braunschweig, Germany

A Viquerat, M Schenk, VJ Lappas & B Sanders (2015), "*Functional and Qualification Testing of the InflateSail Technology Demonstrator*", 2nd AIAA Spacecraft Structures Conference, SciTech 2015, 5–9 January 2015, Kissimmee, FL, USA

C Underwood, A Viquerat, M Schenk, S Fellowes, B Taylor, C Massimiani, R Duke, B Stewart, C Bridges, D Masutti, A Denis (2017), "*Development of the InflateSail (QB50 GB06) 3U CubeSat Technology Demonstrator and First Flight Results*" 9th European CubeSat Symposium, 29 November – 1 December, Oostende, Belgium

C Underwood, A Viquerat, M Schenk, B Taylor, C Massimiani, R Dukea, B Stewart, S Fellowes C Bridges, G Aglietti, B Sanders, D Masutti, A Denis (2018), "*InflateSail De-Orbit Flight Demonstration Results and Follow-On Drag-Sail Applications*" 69th International Astronautical Congress (IAC), 1-5 October 2018, Bremen, Germany. IAC-18-A6.5.2.

4.5 Summary

A number of failure criteria have been introduced and compared. The criteria are defined in terms of principal stresses, to disassociate the results from a specific set of axes.

The *Rankine* criterion is appropriate for brittle failure, whereas *Tresca* and *Von Mises* apply to ductile materials such as metals. The Tresca criterion assumes yield to occur at the plane of maximum shear stress, while the Von Mises condition uses a maximum distortion energy. In practise both will give usable results, and the selection of the yield criterion depends on which is most convenient for the problem at hand. An example problem highlighted the importance of thinking in 3D when considering failure criteria, even if we are working with plane stress.

It should be noted that in addition to the failure criteria described in this handout, there exist various other theories, to capture the failure modes of specific materials, such as fibre-reinforced composites, concrete or soils.

Revision Objectives Handout 4:

- describe different failure criteria (Rankine, St Venant, Tresca, Von Mises), and draw their failure envelopes in principal stress space;
- select appropriate failure criteria for brittle/ductile materials;
- recall expressions for failure criteria and evaluate the failure load of a structure;