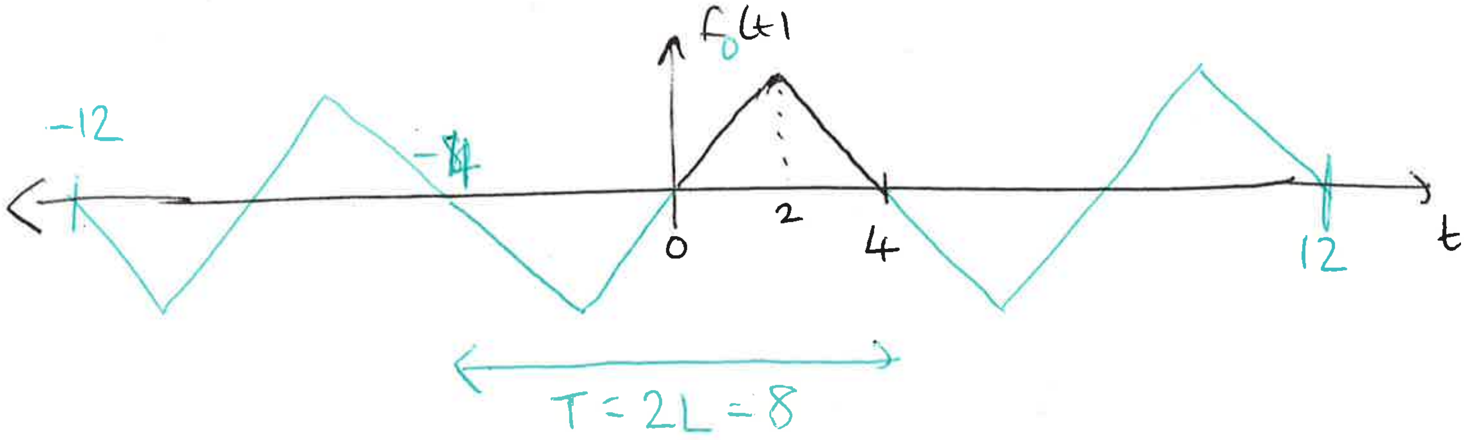
Worked example 1.5

Consider the function

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 4 - t, & 2 < t \le 4 \end{cases}$$

$L = 4$
$T = 2L = 8$

Plot its even and odd extensions. Compute its half-range cosine and sine Fourier series expansions.



Summary (1. Fourier series)

Fourier series (f periodic, period T, $f : [-T/2, T/2] \mapsto \mathbb{R}$):

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

where

$$\omega = \frac{2\pi}{T}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t) dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t) dt$$

Ex 1.5 $\frac{1}{2}$-range Fourier sin series.

$$f_o(t) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{4}\right) \qquad L=4$$

$$b_n = \frac{2}{4}\int_0^4 f(t)\sin\left(\frac{n\pi t}{4}\right)dt$$

$$= \frac{1}{2}\left\{\int_0^2 \underbrace{t}_{u}\,\underbrace{\sin\left(\frac{n\pi t}{4}\right)}_{v'}dt + \int_2^4 \underbrace{(4-t)}_{u}\,\underbrace{\sin\left(\frac{n\pi t}{4}\right)}_{v'}dt\right\}$$

$$= \frac{1}{2}\left\{\left[\underbrace{t}_{u}\,.\,\underbrace{-\frac{4}{n\pi}\cos\left(\frac{n\pi t}{4}\right)}_{v}\right]_0^2 - \int_0^2 \underbrace{1}_{u'}\,.\,\underbrace{-\frac{4}{n\pi}\cos\left(\frac{n\pi t}{4}\right)}_{v}dt\right.$$

$$\left. + \left[\underbrace{(4-t)}_{u}\,.\,\underbrace{-\frac{4}{n\pi}\cos\left(\frac{n\pi t}{4}\right)}_{v}\right]_2^4 - \int_2^4 \underbrace{-1}_{u'}\,.\,\underbrace{-\frac{4}{n\pi}\cos\left(\frac{n\pi t}{4}\right)}_{v}dt\right\}$$

$$= \frac{1}{2}\left\{\left(\cancel{-\frac{8}{n\pi}\cos\left(\frac{n\pi}{2}\right)} - 0\right) + \left(0 + \cancel{\frac{8}{n\pi}\cos\left(\frac{n\pi}{2}\right)}\right)\right.$$

terms cancel.

$$\left. + \int_0^2 \frac{4^2}{n\pi}\cos\left(\frac{n\pi t}{4}\right)dt - \int_2^4 \frac{4^2}{n\pi}\cos\left(\frac{n\pi t}{4}\right)dt\right\}$$

$$= \left[\frac{8}{(n\pi)^2}\sin\left(\frac{n\pi t}{4}\right)\right]_0^2 - \left[\frac{8}{(n\pi)^2}\sin\left(\frac{n\pi t}{4}\right)\right]_2^4$$

$$= \left(\frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) - 0\right) - \left(0 - \frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$\boxed{b_n = \frac{16}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)}$$

$$\Rightarrow \boxed{f_o(t) \sim \sum_{n=1}^{\infty} \frac{16}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi t}{4}\right)}$$

$\frac{1}{2}$-range Fourier sin series.

Summary (2. Half-range series)

Half-range cosine series ($f : [0, L] \mapsto \mathbb{R}$, even periodic extension):

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

$$\text{where} \qquad a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

Half-range sine series ($f : [0, L] \mapsto \mathbb{R}$, odd periodic extension):

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$\text{where} \qquad b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

2. Fourier Transforms

↳ frequency analysis of non periodic functions?

How do we decompose a signal into its frequency components?

- Basic properties
- The time domain and the frequency domain
- Frequency transfer functions and frequency response functions
- What is the "fast Fourier transform (fft)?

[James Advanced MEM (4th Edn) Ch. 8]

When the signal $f(t)$ is a periodic function with period T (and frequency $\omega = 2\pi/T$), then the Fourier series: translates the signal in the *time domain* $f(t)$ into its *harmonic components*. This gives a discrete *spectrum*. The Fourier coefficients represent "how much" of the frequency $n\omega$ is "in" the signal.

↳ graph of $\frac{1}{2}\sqrt{a_n^2+b_n^2}$ vs frequency.

energy in frequency $n\omega$

What if $f(t)$ is not periodic?

If the signal is not periodic, can it still be made up of periodic functions? Does it still have a frequency spectrum? Yes!

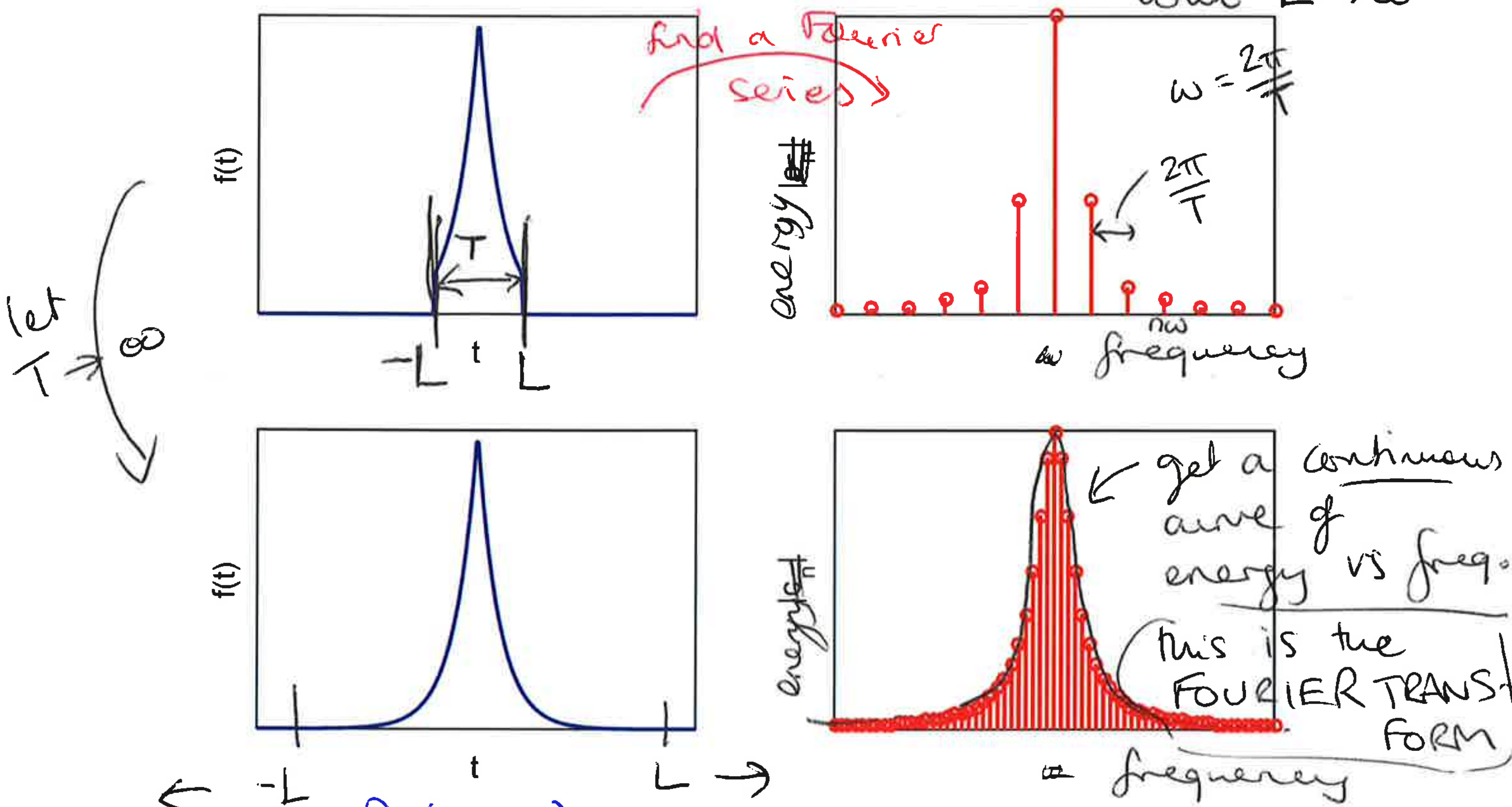
Many signals when represented in the *frequency domain* have a *continuous spectrum*. These represent signals in the time domain that do not have a single fundamental frequency. That is they are not periodic functions.

Of course, in engineering, most signals are not periodic; e.g. the accelerations in earthquake, random vibrations, chaotic outputs of simple nonlinear circuits, freak waves such as tsunamis, mosquito interaction, etc....

So, we need a technique to find the frequency content of an *arbitrary* function $f(t)$.

How? Fourier series in the limit $T \to \infty$

The idea : restrict the function to an interval $[-L, L]$ & take the limit $L \to \infty$



f (time) $\longrightarrow$ F(frequency)

What kind of functions are ok?

The only stipulation is that the signal contain a "finite amount of energy". A very conservative way of achieving this is to assume that

$$f(t) \to 0 \quad \text{as} \quad t \pm \infty.$$

e.g. $f(t) = 1$ for all t has NO Fourier transform.

In particular, we need the approach to 0 not to be "too slow":

Such functions could have a finite duration, e.g.

$$f(t) = \begin{cases} 1 & \text{for } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

or have decaying tails

$$f(t) = e^{-|t|}.$$

The Fourier Transform is the analogue of the Fourier series, but with continuous frequencies ω rather than discrete frequencies $\omega_n = 2\pi n/T$.

Definition of the Fourier transform and its inverse

Complex

output : a function of freq. input ; a function of time

Definition: The *Fourier transform* of a function $f(t)$ is defined as:

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)\underbrace{e^{-j\omega t}}_{\cos(\omega t) - j\sin\omega t} dt \qquad j = \sqrt{-1}$$

"The fourier transform of"

Combination of both a_n & b_n Fourier coeffs. in $\lim T\to\infty$

Definition: The *inverse Fourier transform* of $F(\omega)$ is defined as:

the inverse Fourier transform of

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Sometimes $f(t)$ and $F(\omega)$ are called a **Fourier transform pair** and written:

time domain $\xleftarrow{\mathcal{F}^{-1}}$ frequency domain

$$f(t) \leftrightarrow F(\omega)$$

$\mathcal{F}$ (the Fourier transform)

When does this integral even exist?

- Require

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

(for example, $f(t) = 1/t$ doesn't satisfy this, even though it approaches 0 as t approaches ∞).

- For technical reasons we also require that f has at most a finite number of maxima and minima and a finite number of discontinuities in any finite interval. For example $f(t) = \sin(\frac{1}{t})$ doesn't work.

[reference: James Advanced MEM (4th Edn) p. 641]

Notation, notation, notation

Warning: Some text books and webpages define the Fourier transform with an extra factor of $\frac{1}{\sqrt{2\pi}}$

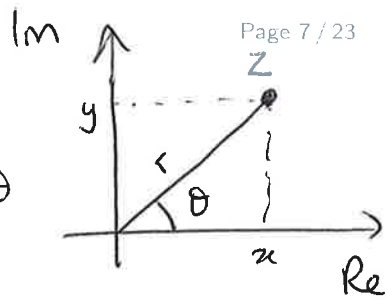
different authors choose different constants :-(

$$\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t) e^{-j\omega t} dt$$

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{j\omega t} d\omega$$

We'll stick to the definition on slide 5 ([James Advanced MEM (4th Edn) Ch. 8] uses the same definition).

Visualising the Fourier transform



$$z = \underbrace{x + jy}_{\text{Cartesian}} = re^{j\theta}$$

- The Fourier Transform $\mathcal{F}[f(t)] = F(\omega)$ is in general complex

$$F(\omega) = X(\omega) + jY(\omega) = |F(\omega)|e^{j\phi(\omega)}$$

- Visualise through the two functions $|F(\omega)|$ and $\phi(\omega)$
- $|F(\omega)|$ is the **magnitude spectrum** (or **amplitude spectrum**)
- $\phi(\omega)$ is the **phase spectrum**

|| exactly the same as the Fourier series line spectrum.

$$|F(\omega)| = \sqrt{Re(F(\omega))^2 + Im(F(\omega))^2}$$

$$\phi(\omega) = Arg(F(\omega)) = \tan^{-1}\left(\frac{Im(F(\omega))}{Re(F(\omega))}\right)$$

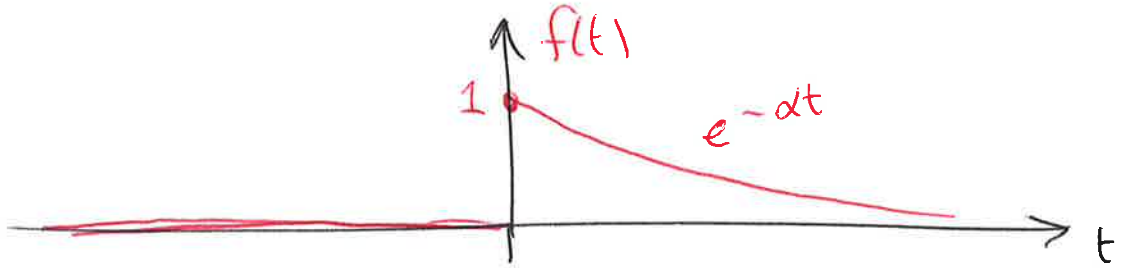
Worked example 2.1

A unit impulse is applied to an electronic circuit and is found to give an impulse response corresponding to the following function:

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-\alpha t} & \text{for } t > 0 \end{cases} \qquad (\alpha > 0).$$

Find the Fourier transform of $f(t)$ and plot the magnitude and phase spectra.

↳ means frequency analyse $f(t)$

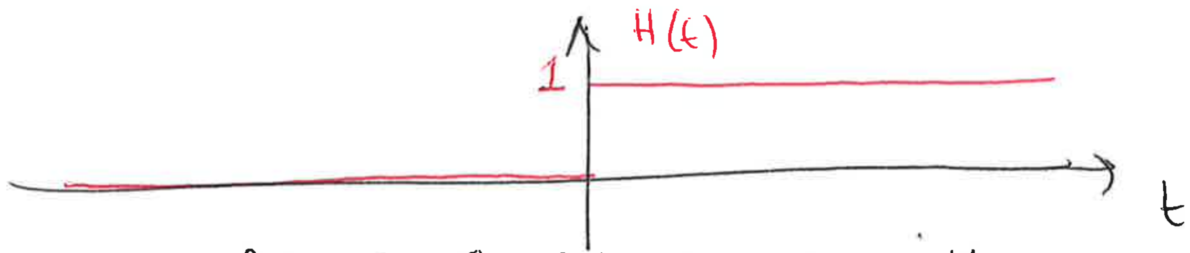


Heaviside step function

a mathematical switch.

Note, such a function $f(t)$ in the above example is often written $H(t)e^{-\alpha t}$, where H is the so-called **Heaviside step function**

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$



e.g. the $f(t)$ in Ex 2.1 can be written as

$$f(t) = H(t)\,e^{-\alpha t}$$

Ex 2.1//

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\underbrace{(\alpha + j\omega)}_{\text{constants w.r.t. the integral}} t}\, dt$$

$$= \left[-\frac{1}{\alpha + j\omega} e^{-(\alpha + j\omega)t} \right]_0^{\infty}$$

$$= 0 + \frac{1}{\alpha + j\omega}$$

$$\boxed{F(\omega) = \frac{1}{\alpha + j\omega}}$$

The ~~of~~ Fourier transform of $f(t)$

To interpret: find & plot the spectra.

<u>mag</u>

$$|F(\omega)| = \left| \frac{1}{\alpha + j\omega} \right|$$

<u>hard way</u>:

$$\frac{1}{\alpha + j\omega} = \frac{1}{\alpha + j\omega} \cdot \frac{\alpha - j\omega}{\alpha - j\omega} \ldots$$

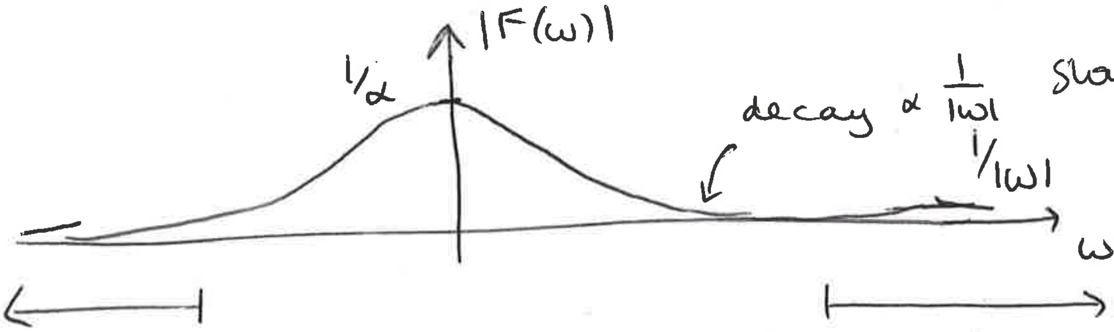
$$= \frac{1}{|\alpha + j\omega|}$$

<u>easy way</u>:

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$= \frac{1}{\sqrt{\alpha^2 + \omega^2}}$$



slow convergence.

high freq. (relatively) important.

phase

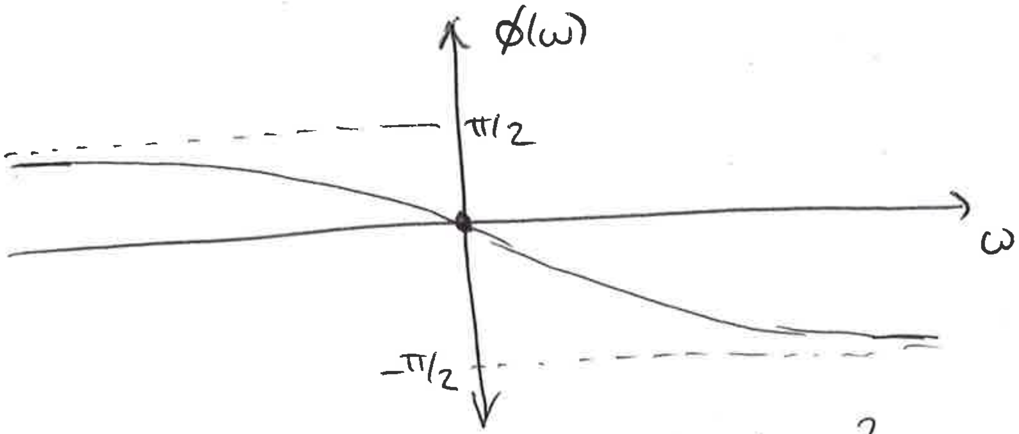
$$\phi(\omega) = \mathrm{Arg}(F(\omega)) = \mathrm{Arg}\left(\frac{1}{\alpha + j\omega}\right)$$

$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$

$\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

$$= -\mathrm{Arg}(\alpha + j\omega)$$

$$= -\tan^{-1}\left(\frac{\omega}{\alpha}\right)$$



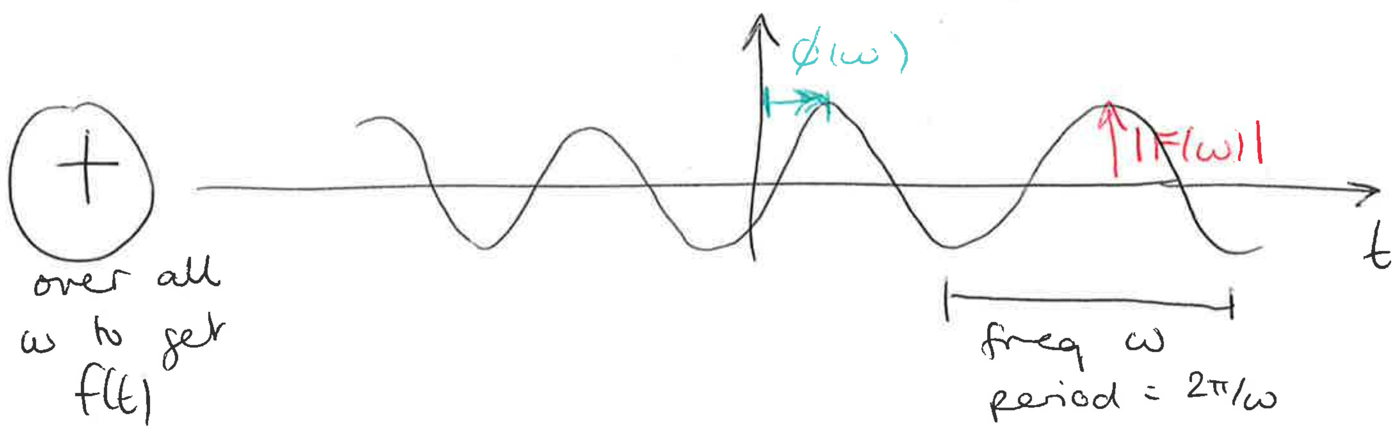
How to interpret the phase spectrum?

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|\, e^{j\phi(\omega)}\, e^{j\omega t}\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|F(\omega)|\, e^{j(\omega t + \phi(\omega))}}_{\text{oscillation frequency } \omega}\, d\omega$$

Sum over ω of oscillation frequency ω



Properties of the Fourier transform: linearity

The Fourier transform is **linear**: because integration is linear.

$$\mathcal{F}[ax(t)+by(t)] = \int_{-\infty}^{\infty} [ax(t)+by(t)]e^{-j\omega t}dt$$

$$= a\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt + b\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt$$

$$= a\,\mathcal{F}[x(t)] + b\,\mathcal{F}[y(t)]$$

the F.T. of a sum is the sum of the F.T.s.

If $x(t) \longleftrightarrow X(\omega)$ **and** $y(t) \longleftrightarrow Y(\omega)$

then $ax(t)+by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(\omega)+bY(\omega)$

Properties of the Fourier transform: symmetry

The Fourier transform has some **symmetry**:

By definition,

$$x(t) = \overbrace{\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega}_{\text{looks almost exactly like the F.T. formula}}}^{\mathcal{F}^{-1}[X(\omega)]}$$

Replacing t by $-t$:-
& multiply by 2π

$$2\pi x(-t) = \underbrace{\int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega}_{\text{exactly the F.T. formula.}}$$

Interchanging t and ω :-

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = \mathcal{F}[X(t)]$$

$$\boxed{\textbf{If } x(t) \longleftrightarrow X(\omega) \textbf{ then } X(t) \longleftrightarrow 2\pi x(-\omega)}$$

$\mathcal{F}[x(t)] = X(\omega)$ then $\mathcal{F}[X(t)] = 2\pi x(-\omega)$

e.g. we know that

$$\mathcal{F}\left[\underbrace{H(t)e^{-\alpha t}}_{x(t)}\right] = \underbrace{\frac{1}{\alpha + j\omega}}_{X(\omega)}$$

~~CORRECTED~~

SYMMETRY

Symmetry result =>

$$\mathcal{F}\left[\frac{1}{\alpha + jt}\right] = 2\pi .\ H(-\omega)\, e^{-\alpha . -\omega}$$

$$= 2\pi\, H(-\omega)\, e^{\alpha\omega}$$

A Fourier Transform for free!
(that would have been hard to find using the defn)

Properties of the Fourier transform: time delay

What about **time-delay**?

time delay t_0 $\qquad t'+t_0$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{t'})e^{-j\omega t}dt$$

the effect of a delay in time domain ≡ multiplication by an exponential in the freq. domain → a phase shift

Put $t' = t - t_0$ so $dt' = dt$ and

$$\mathcal{F}\left[x(t')\right] = \int_{-\infty}^{\infty} x(t')e^{-j\omega(t'+t_0)}dt'$$

constant w.r.t. the integral

$$= e^{-j\omega t_0}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt'}_{\text{F.T. defn.}} = e^{-j\omega t_0}X(\omega)$$

If $x(t) \longleftrightarrow X(\omega)$ **then** $x(t-t_0) \longleftrightarrow X(\omega)e^{-j\omega t_0}$

$\mathcal{F}[x(t)] = X(\omega) \qquad \mathcal{F}[x(t-t_0)] = X(\omega)e^{-j\omega t_0}$

Convolution

The convolution of two functions $x(t)$ and $y(t)$ is defined by the function:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(v)y(t-v)dv$$

The Convolution Theorem

$$\mathcal{F}[x(t) * y(t)] = X(\omega)Y(\omega)$$

the convolution of.

The Fourier transform of a "special product" the convolution (in the time domain) ≡ the product of the F.T.s (in the freq. domain)

or

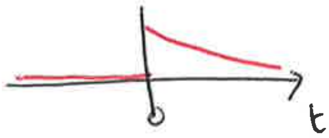
$$\mathcal{F}^{-1}[X(\omega)Y(\omega)] = x(t) * y(t)$$

most useful version for system analysis.

TIME DELAY

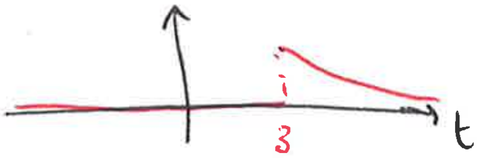
e.g. we know that

$$\mathcal{F}\left[H(t)\, e^{-\alpha t} \right] = \frac{1}{\alpha + j\omega}$$



if instead we have a time delay of $3 = t_0$

$$\mathcal{F}\left[H(t-3)\, e^{-\alpha(t-3)} \right]$$



$$= \frac{1}{\alpha + j\omega} \cdot e^{-3j\omega}$$

$$= \frac{e^{-3j\omega}}{\alpha + j\omega}$$

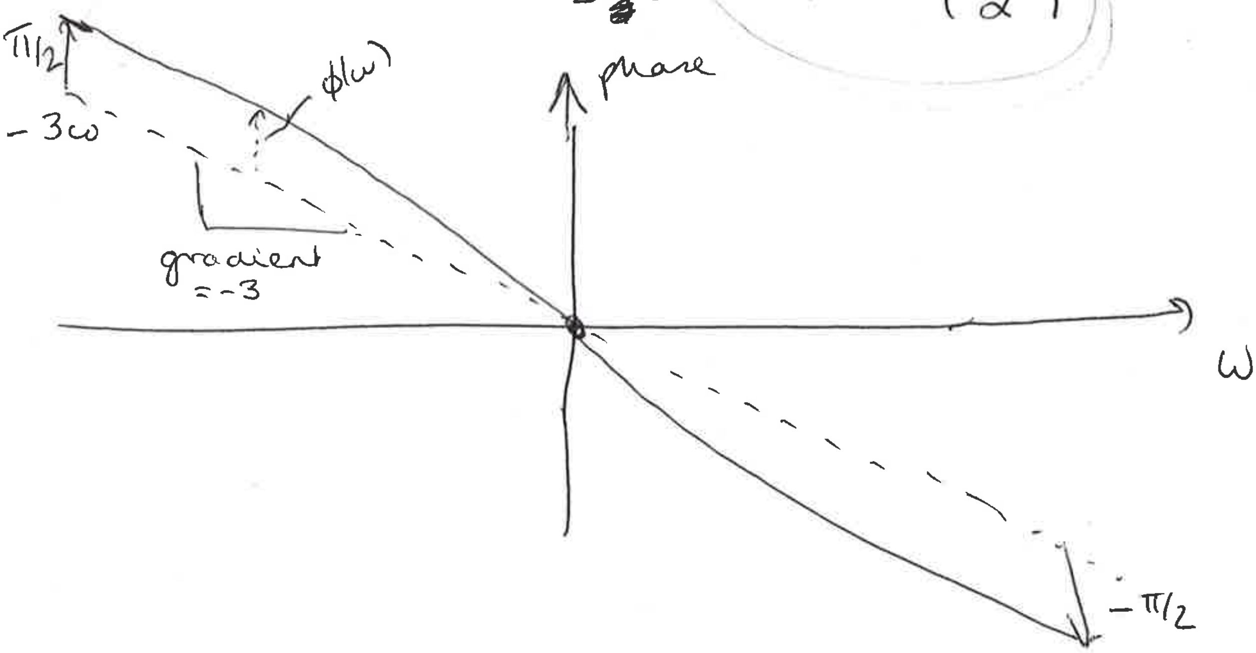
mag

$$\left| \frac{e^{-3j\omega}}{\alpha + j\omega} \right| = \frac{\overbrace{|e^{-3j\omega}|}^{=1}}{|\alpha + j\omega|} = \frac{1}{|\alpha + j\omega|} = |F(\omega)|$$

phase

$$\mathrm{Arg}\left(\frac{e^{-3j\omega}}{\alpha + j\omega} \right) = \underbrace{\mathrm{Arg}\left(e^{-3j\omega}\right)}_{-3j\omega} - \underbrace{\mathrm{Arg}\left(\alpha + j\omega\right)}_{\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{\alpha}\right)}$$

$$= -3\omega - \tan^{-1}\left(\frac{\omega}{\alpha}\right)$$



Proof of the convolution theorem (1)

$\mathcal{F}[(x*y)(t)]$

Start by taking the Fourier transform of a convolution:

$$\mathcal{F}\underbrace{\left[\int_{-\infty}^{\infty} x(u)y(t-u)du\right]}_{\mathcal{F}[(x*y)(t)]} = \int_{-\infty}^{\infty} e^{-j\omega t}\underbrace{\left(\int_{-\infty}^{\infty} x(u)y(t-u)du\right)}_{(x*y)(t)}dt$$

by definition of the Fourier transform

Swap.

$$= \int_{-\infty}^{\infty} x(u)\left[\int_{-\infty}^{\infty} e^{-j\omega t}y(t-u)dt\right] du$$

where we have now switched order of integration.

Now we use the result for delays to write the integral in brackets in terms of the Fourier transform of y:

$\mathcal{F}[y(t-u)] = Y(\omega)e^{-j\omega u}$

F.T of a delayed signal.

Proof of the convolution theorem (2)

$$\left[\int_{-\infty}^{\infty} e^{-j\omega t}y(t-u)dt\right] = e^{-j\omega u}Y(\omega)$$

so that

$$\begin{aligned}\mathcal{F}\left[\int_{-\infty}^{\infty} x(u)y(t-u)du\right] &= \int_{-\infty}^{\infty} x(u)\left[e^{-j\omega u}Y(\omega)\right] du \\ &= Y(\omega)\underbrace{\int_{-\infty}^{\infty} x(u)e^{-j\omega u}du}_{\mathcal{F}[x(t)]} \\ &= Y(\omega)X(\omega)\end{aligned}$$

constant wrt. integral.

and we can invert this as well:

$$\mathcal{F}^{-1}[X(\omega)Y(\omega)] = \int_{-\infty}^{\infty} x(u)y(t-u)du = x(t)*y(t)$$