EMAT10100 Engineering Maths I
Lecture 3 of Introduction to Probability: Conditional Probability and Independence

Rosalyn Moran & Nikolai Bode

Overview

- Last time we looked at the main concepts of Probability Theory
- In particular, we learnt to compute the probability that event A *or* event B happens:
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
- But how do we calculate $P(A \text{ and } B) = P(A \cap B)$?
- By the end of this lecture we will
  - understand that sometimes knowledge that some event B has happened changes (or *conditions*) the probability that another event A will occur
  - while in other circumstances two events A and B are *independent* (the occurrence of one does not influence the probability of the other occurring)
  - learn a general formula for $P(A \cap B)$,

An Introduction to Conditional Probability (1)

See *James*: section 13.3.4.

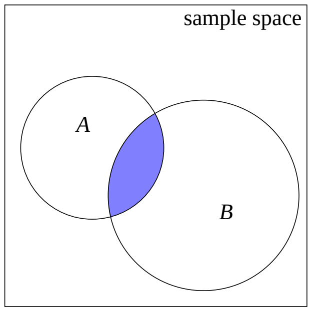
Sometimes we are interested in the probability of an event A, given that we *know* some other event B has happened. This is called *conditional probability*.

We write $P(A|B)$, pronounced “the probability of A given B”.

Examples.

- What is the probability that it will rain this PM — given that it is very cloudy?
- What is the probability that my house will be burgled today — given that I forgot to close the front door? (etc.)
- What is the probability of a black out, given that there is a fault on the main power line connecting France to the UK?

Conditional probability



If you know that B has happened, the probability that A also happens is the fraction of B that is also inside A (coloured blue).

So the *conditional probability* of A given B is
$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}, \quad = \frac{P(A \cap B)}{P(B)}.$$

Example: Conditional probability

(See Example 13.6 in James).

Suppose:

- The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$,
- the probabilty that it arrives on time is $P(A) = 0.92$,
- and the probability that it both arrives and departs on time is $P(A \cap D) = 0.78$.

Find the probability that a plane

(a) arrives on time given that it departed on time,

(b) did not depart on time given that it fails to arrive on time.

Worked solution

Solution strategy: *contingency table*

	A	not A	Total
D	0.78	0.05	0.83
not D	0.14	0.03	0.17
Total	0.92	0.08	1.00

(a) $P(A|D) = \frac{P(A\cap D)}{P(D)} = \frac{0.78}{0.83} \approx 0.940.$

(b) $P(\text{not } D|\text{not } A) = \frac{P(\text{neither } A \text{ nor } D)}{P(\text{not } A)} = \frac{0.03}{0.08} = 0.375.$

Some properties of conditional probability

(There are very natural commonsense interpretations of these Mathematical rules.)

- $P(A|A) = 1.$
- $P(B|A) \geq 0$ for all events A, B with $P(A) > 0$.
- If B and C are mutually exclusive, which means $P(B \cap C) = 0$, then $P(B \text{ or } C|A) = P(B|A) + P(C|A)$.
- Bayes' rule: $P(B|A) = P(A|B)\dfrac{P(B)}{P(A)}$.

Exercise: application of Bayes' rule

A company produces bolts using two machines, M_1 and M_2. Of the total output, machine M_1 is responsible for 25% and machine M_2 for the rest. It is known from previous experience with the machines that 5% of the output from M_1 and 4% of the output from M_2 is defective. A bolt is chosen at random from the production line and found to be defective.

- What is the probability that it came from machine M_1?

Worked Solution

- Let $D = \{$ bolt is defective $\}$; $A = \{$ bolt is from $M_1\}$; $B = \{$ bolt is from $M_2\}$;
- We know that $P(A) = 0.25$, $P(B) = 0.75$. Also $P(D|A) = 0.05$, $P(D|B) = 0.04$.
- Therefore the probability that a bolt came from M_1 (event A) given that it is defective (event D) can be computed using Bayes' rule as:

$$P(A|D) = P(D|A)\frac{P(A)}{P(D)}$$

Complete the rest as an exercise.

How to compute $P(A \cap B)$ from conditional probabilities

- Note that from the definition of conditional probability we have:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \implies P(A \cap B) = P(B)P(A|B)$$

- Or equivalently

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \implies P(A \cap B) = P(A)P(B|A)$$

- We can therefore compute $P(A \cap B)$ by using the conditional probability of an event given that the other has happened.

Independence

- Sometimes the probability of an event, say A, is not affected by the occurrence of another event, say B and vice versa. In this case we say that A and B are *independent*
- If A and B are independent then:
  - If $P(A) \neq 0$, then $P(B|A) = P(B)$.
  - If $P(B) \neq 0$, then $P(A|B) = P(A)$.
- Therefore for independent events A, B:

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B) = P(A)P(B)$$

  [this is sometimes used as a definition of independent events]
- In general, if A and B are independent, we do not gain information about the probability of one of them if we know that the other has happened.

Exercise: Independence

(Example 13.8 in James)

Items from a production line can have defects A or B. Some items have both, some just one, but most have neither. Tables (a) and (b) show two alternative sets of joint probabilities:

(a)

	B	not B	Total
A	0.02	0.08	0.10
not A	0.18	0.72	0.90
Total	0.20	0.80	1.00

(b)

	B	not B	Total
A	0.06	0.04	0.10
not A	0.14	0.76	0.90
Total	0.20	0.80	1.00

Test for independence (of defects A, B) in each case.