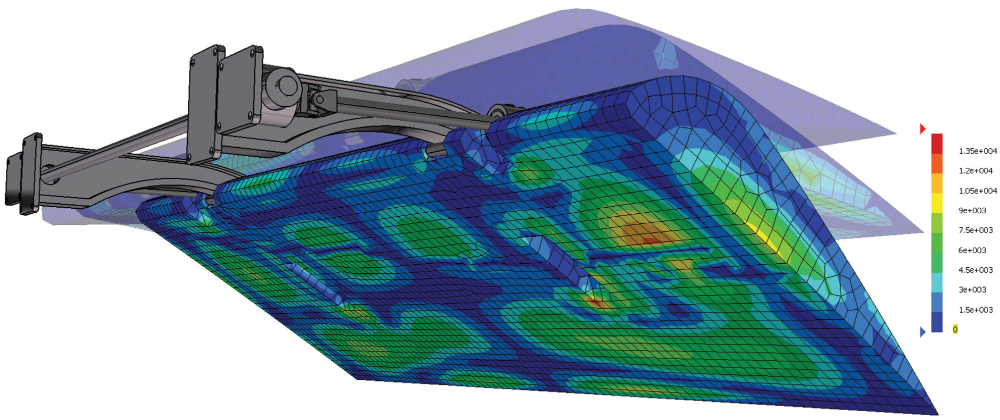
Handout 1 – 2D Stress Analysis

In this handout we shall not deal with how the stress varies within an elastic structure under the effect of external forces, *e.g.* bending of beams, stress concentrations around a hole, or aerodynamic loads on a wing flap. Instead, we shall investigate the concept and properties of stress *at a point*.



The objective is to provide a deeper understanding of stress, which in turn will enable you to interpret the results of your structural calculations. The ability to analyse combined stress states will also allow us to formulate theories of failure, which is the subject of Handout 4.

1.1 Stress Definitions

The concept of *stress* was introduced in the theory of elasticity by Cauchy (1789–1857) around 1822[1]. The following definition was later given by Saint-Venant (1797–1886):

> *"The total stress on an infinitesimal element of a plane taken within a deformed elastic body is defined as the resultant of all the actions of the molecules situated on one side of the plane upon molecules on the other side..."*

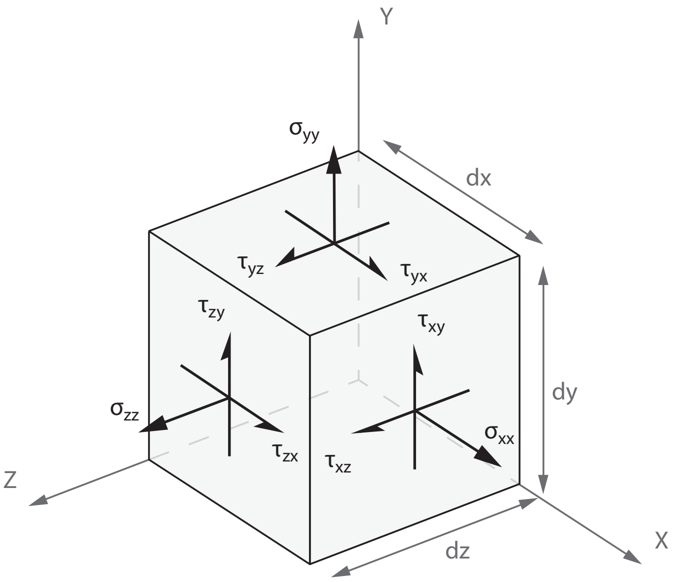
Key words are *infinitesimal element*, which enables us to define stress at a 'point' in an elastic body, and *plane*, which means that there is a orientation/direction associated with stress.

Stress is defined as force per unit area[2] with units of MPa [N/mm^2].

[1] An excellent and fascinating history of the theory of elasticity and structures up to the 1950s is found in "History of Strength of Materials" by Stephen P. Timoshenko (TA405 TIM).

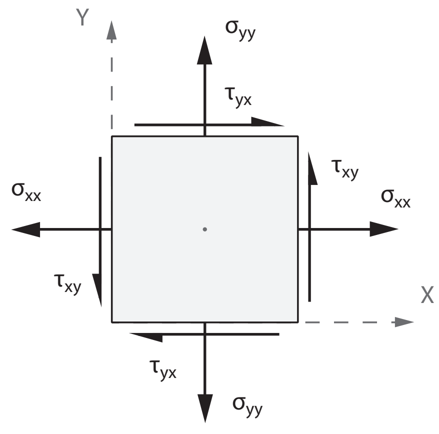
[2] One might rightfully wonder whether stress should be defined as the force per unit area of the material in its *deformed* or *underformed* configuration, since an elastic body will deform under the effect of the applied stress. For instance, in tensile tests of ductile rods the cross-sectional area will change noticeable in the 'necking' region, and the shape of the stress-strain curve will change significantly when substituting the deformed cross-sectional area. However, we shall limit ourselves to linear elasticity, where the strains are small enough for the change in area to be insignificant. Fortunately, the vast majority of engineering problems are accurately described by linear elasticity!

An *infinitesimal* element of material with dimensions $dx \times dy \times dz$ is subjected to direct stresses σ_{ii} and shear stresses τ_{ij}. Direct stresses act normal to a face, whereas shear stresses act parallel.



The first index is the face (defined by the direction of it normal vector) on which the stress acts, and the second the direction of the stress. A shear stress τ_{ij} is taken to be positive if it acts in the positive j direction on the face with positive i; similarly, on a negative face of the element, a shear stress is positive when it acts in the negative direction of an axis.

The shear stresses are not independent, by virtue of *complementary shear*. Consider an infinitesimal 2D element with dimensions $dx \times dy$ (dz is out-of-plane).



Moment equilibrium around the centre of the infinitesimal element:

$$\underbrace{\underbrace{\tau_{xy}\, dy\, dz}_{\text{force}}\; dx}_{\text{CCW couple}} = \tau_{yx}\, dx\, dz\, dy$$

$$\tau_{xy}\, \cancel{dx\, dy\, dz} = \tau_{yx}\, \cancel{dx\, dy\, dz}$$

$$\tau_{xy} = \tau_{yx}$$

Repeating for the remaining axes gives $\tau_{ij} = \tau_{ji}$, reducing the number of independent stress variables.

Thus, a 3D stress state is described by 6 variables, conveniently expressed in a *symmetric* 3×3 matrix:

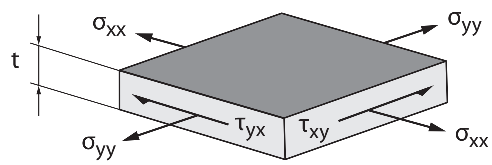
$$\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

The symmetry is a result of the complementary shear, and means that $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^T$. This formulation is known as the **Cauchy stress tensor**, and is used in more mathematical descriptions of the theory of elasticity.

NB: in many structural engineering texts $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ are simply referred to as $(\sigma_x, \sigma_y, \sigma_z)$.

1.2 Plane Stress

Many engineering applications make use of thin-walled structures, for example aircraft fuselages and wing skins, where the wall thickness is much smaller than other dimensions of the structure. This allows us to make a simplifying assumption that the stress state is uniform across the section.



Furthermore, consider a small element cut through the thickness of the surface. Since it is a free surface there can be no out-of-plane stresses, and thus:

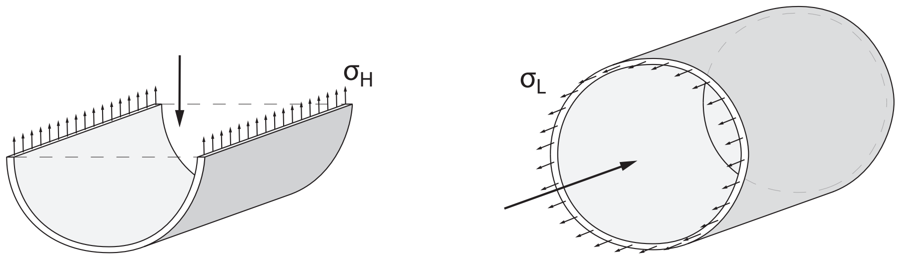
$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

The three-dimensional stress state thus reduces to a two-dimensional stress state known as **plane stress**, with three independent stress variables: σ_{xx}, σ_{yy}, and τ_{xy}. For many engineering applications plane stress is a very useful approximation.

NB: this is different from *plane strain* where out-of-plane deformations are zero $\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$. Plane strain is used to represent conditions deep within a material, *i.e.* inside thick structures, whereas plane stress is used to represent conditions at the surface of a material and in thin-walled structures.

Example 1.1 – Cylindrical Pressure Vessel

In general, determining the 2D stress distribution in a structure can be challenging. For a cylindrical pressure vessel, however, the membrane stresses can straightforwardly be derived from equilibrium considerations. The pressure vessel has radius r, wall thickness t and gauge pressure p.

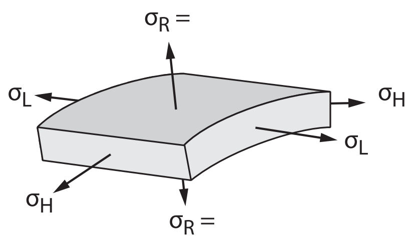


Hoop stress σ_H and longitudinal stress σ_L are given by:

$$\sigma_H =$$

$$\sigma_L =$$

The stress on the inner surface is non-zero ($\sigma_R = -p$).

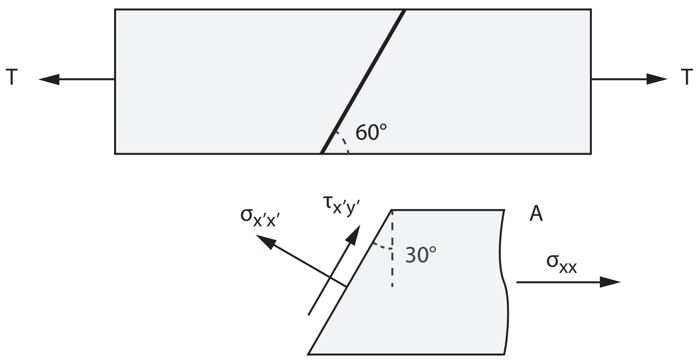


Nonetheless, we can approximate the stress state as plane stress. For thin-walled structures where $r/t \gg 1$ the membrane stresses, σ_H and σ_L are much greater than the internal pressure, and the through-thickness stress is therefore assumed to be negligible.

Example 1.2 – Stresses in a Weld

A thin plate with cross-sectional area A is loaded in tension with T. The plate consists of two welded parts, with the weld inclined at 60° to the direction of loading.

Q: What are the tensile and shear stresses in the weld?



Direct stress $\sigma_{x'x'}$ on the inclined section is at $\theta = 30°$ to σ_{xx}.

Resolving forces perpendicular to the weld:

$$\sigma_{x'x'} \frac{A}{\cos\theta} =$$

$$\sigma_{x'x'} =$$

Resolving forces parallel to the weld:

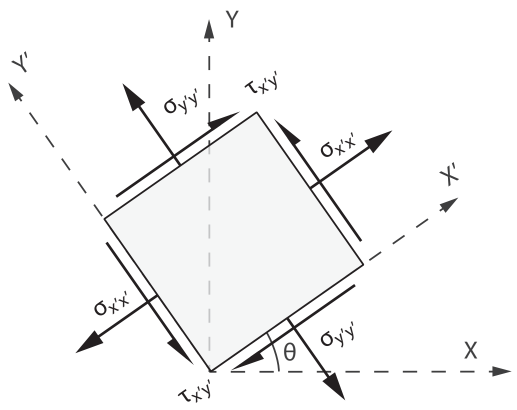
$$\tau_{x'y'} \frac{A}{\cos\theta} +$$

gives:

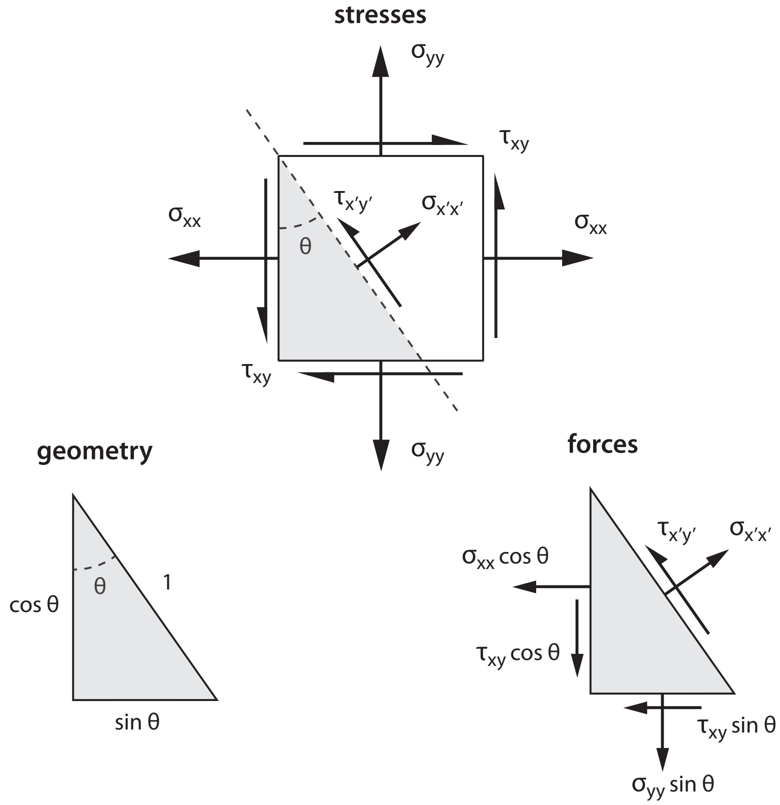
$$\tau_{x'y'} =$$

1.3 Stress Transformations

Up to this point we have defined the stresses in a convenient, but arbitrary, XY coordinate system. What are the stresses in another set of axes, $X'Y'$, at an angle θ to the original coordinate system?



To find the stresses in the new coordinate system, consider the equilibrium of an infinitesimal element (with unit depth) cut at an angle θ (measured CCW from the Y-axis) on which a normal stress $\sigma_{x'x'}$ and shear stress $\tau_{x'y'}$ act. Note that we are looking at *force* equilibrium, so the stresses must be multiplied with the area over which they act.



Resolving forces perpendicular and parallel to the cut plane:

$$\sigma_{x'x'} = \quad (\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta)\cos\theta + (\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)\sin\theta$$

$$\tau_{x'y'} = -(\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta)\sin\theta + (\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)\cos\theta$$

Reshuffle and combine into a transformation matrix $\boldsymbol{T}$:

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{1.1}$$

where $\sigma_{y'y'}$ is found using $\theta_{y'} = \theta_{x'} + \pi/2$.

These **stress transformation** equations will enable you to calculate the stresses in any given direction, and form the core of 2D stress analysis.

The stress transformation equations can be rewritten using the standard trigonometric double-angle relationships to find an alternative formulation:

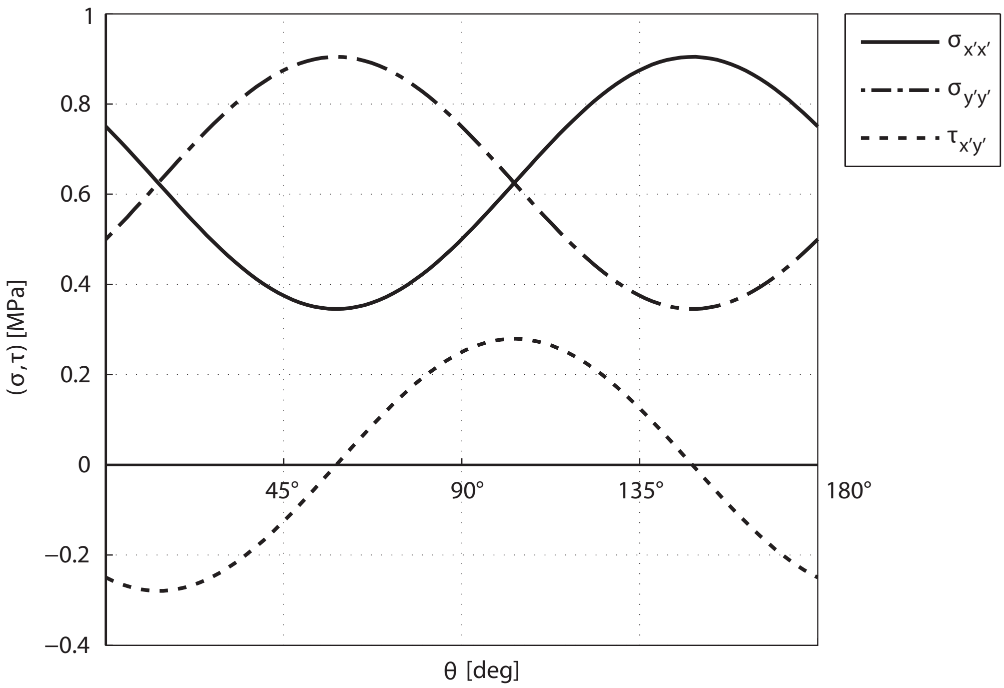
$$\sigma_{x'x'} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \tag{1.2}$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta \tag{1.3}$$

which will turn out to be particularly useful later in this handout.

1.4 Properties of Stress

The stress transformation equations enable us to calculate the stresses in any direction. As we rotate through different angles $\theta \in [0, \pi]$ the direct stresses $\sigma_{x'x'}$ and $\sigma_{y'y'}$ will vary periodically.



Can we make general observations that will apply to any stress state?

Maximum/Minimum Direct Stress

To find directions in which the direct stresses are *maximum* or *minimum*, differentiate the stress transformation equation for $\sigma_{x'x'}$ (Equation 1.2) with respect to 2θ:

$$\frac{d\sigma_{x'x'}}{d(2\theta)} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

to find

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{1.4}$$

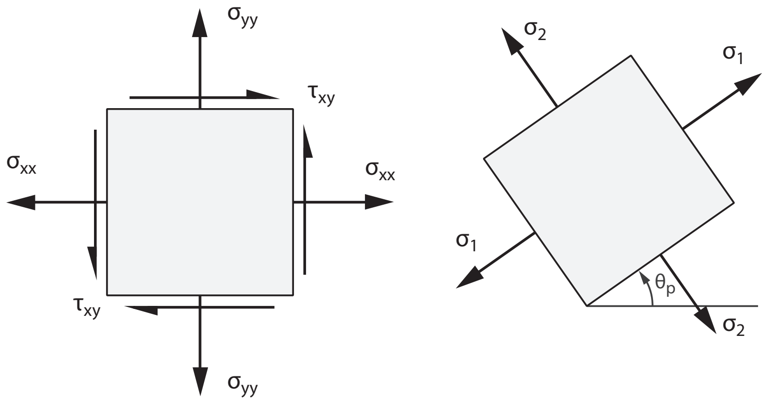
with two solutions: θ_p and $\theta_p + \pi/2$.

These angles describe two *perpendicular* planes where the direct stress is either maximum or minimum; these are known as the **principal directions**. The **principal stresses** are found by substituting θ_p into the stress transformation equation (Equation 1.1).

To find the magnitude of the shear stress in those directions, substitute $\sin 2\theta = \tan 2\theta \cos 2\theta$ into Equation 1.3, and use the result for the principal directions:

$$\begin{aligned}
\tau_{x'y'} &= -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta_p + \tau_{xy}\cos 2\theta_p \\
&= -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\cos 2\theta_p + \tau_{xy}\cos 2\theta_p \\
&= -\tau_{xy}\cos 2\theta_p + \tau_{xy}\cos 2\theta_p \\
&= 0
\end{aligned}$$

In other words, the planes where the direct stress $\sigma_{x'x'}$ is maximum or minimum carry no shear stress!



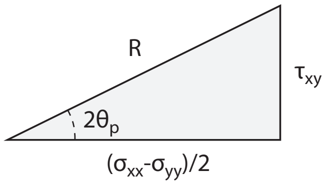
This is a very powerful result, as it holds true for any[3] state of stress!

[3] Fascinatingly, this result also extends to any three-dimensional state of stress: it is always possible to find three orthogonal directions in which the shear stresses vanish — these are the directions of principal stress. In fact, the principal stresses and the principal directions can be found as the eigenvalues and eigenvectors of the Cauchy stress tensor. Recall that a symmetric matrix will have real eigenvalues, and can be diagonalised by pre- and post-multiplying with a transformation matrix $\boldsymbol{U}$:

$$\boldsymbol{U}' \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \boldsymbol{U} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

where $\boldsymbol{U} = \begin{bmatrix} \hat{\boldsymbol{u}}_1 & \hat{\boldsymbol{u}}_2 & \hat{\boldsymbol{u}}_3 \end{bmatrix}$ with $\hat{\boldsymbol{u}}_i$ the normalized eigenvectors that define the planes of principal stress.

From Equation 1.4 we can derive further relationships.



The hypotenuse is expressed as:

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

and the other geometric relationships are:

$$\frac{\sigma_{xx} - \sigma_{yy}}{2} = R\,\cos 2\theta_p$$

$$\tau_{xy} = R\,\sin 2\theta_p$$

Substituting these into Equation 1.2 yields:

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{1.5}$$

This allows us to calculate both principal stresses directly, but does not let us know which principal stress acts on which principal plane.

Mean Direct Stress

From the stress transformation equations (Equation 1.1)

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

it is seen that the mean direct stress:

$$\frac{\sigma_{x'x'} + \sigma_{y'y'}}{2} = \frac{\sigma_{xx}\left(\cos^2\theta + \sin^2\theta\right) + \sigma_{yy}\left(\cos^2\theta + \sin^2\theta\right)}{2}$$

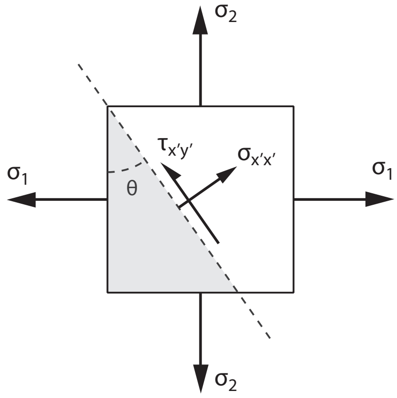
$$= \frac{\sigma_{xx} + \sigma_{yy}}{2} = C$$

is *constant* for any transformed coordinate system.

Maximum/Minimum Shear Stress

Taking the derivative of the transformation equation for $\tau_{x'y'}$ (Equation 1.3) with respect to 2θ, it is found that the directions of maximum and minimum shear stress are also perpendicular.

Next, consider an element under principal stresses σ_1 and σ_2 (recall, $\tau_{12} = 0$).



Equation 1.3 gives the shear stress at an angle θ to the principal axes:

$$\tau_{x'y'} = -\frac{1}{2}\left(\sigma_1 - \sigma_2\right)\sin 2\theta$$

which has a maximum/minimum

$$\tau_{\text{max,min}} = \pm\frac{1}{2}\left(\sigma_1 - \sigma_2\right) \tag{1.6}$$

at $\theta = \pi/4$. The planes of maximum/minimum shear stress are at 45° to the planes of principal stress.

$$\theta_s = \theta_p \pm \frac{\pi}{4}$$

Using Equation 1.5 and Equation 1.6 we can write:

$$\tau_{\text{max,min}} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{1.7}$$

This completes the full toolset to calculate the principal stresses, maximum shear stress, and principal directions for any given state of stress.

Example 1.3 – Stress Calculations

A finite element calculation of a section of an aircraft fuselage has given $\sigma_{xx} = -75$ MPa, $\sigma_{yy} = 210$ MPa and $\tau_{xy} = -200$ MPa. Determine (a) the principal stresses, (b) the maximum shear stresses and associated normal stresses. Sketch a correctly oriented infinitesimal element for both load cases.

(a) The principal stresses are given by

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

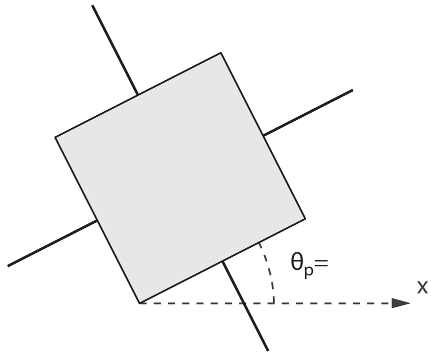
as $\sigma_1 = 313$ MPa and $\sigma_2 = -178$MPa.

The principal directions are found using:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

which gives $\theta_p = 27.3°$ and $\theta_p = 27.3° + 90° = 117.3°$

In order to find which principal stress corresponds to which principal direction, substitute θ_p into Equation 1.1.

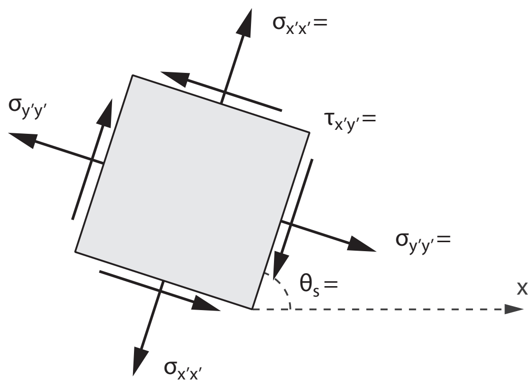


(b) The maximum shear stress is given by:

$$\tau_{\text{max,min}} = \pm \frac{\sigma_1 - \sigma_2}{2} =$$

These planes will be at 45° to the directions of principal stress, so $\theta_s = \theta_p + 45°$ and $\theta_s = \theta_p - 45°$.

To find which direction corresponds to maximum/minimum shear stress, and to find the associated direct stresses, substitute θ_s into Equation 1.1.



The direct stresses on planes with maximum/minimum shear stress are equal to $(\sigma_{xx} + \sigma_{yy})/2$.

Example 1.4 – Pure Shear

Consider a plate subjected to pure shear, where $\sigma_{xx} = \sigma_{yy} = 0$. Using Equation 1.4,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

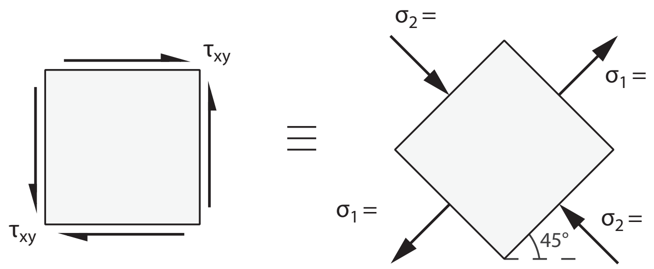
we find the principal directions as $45°$ and $135°$. Substituting into Equation 1.2,

$$\sigma_{x'x'} = \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) + \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

we find the principal stresses as:

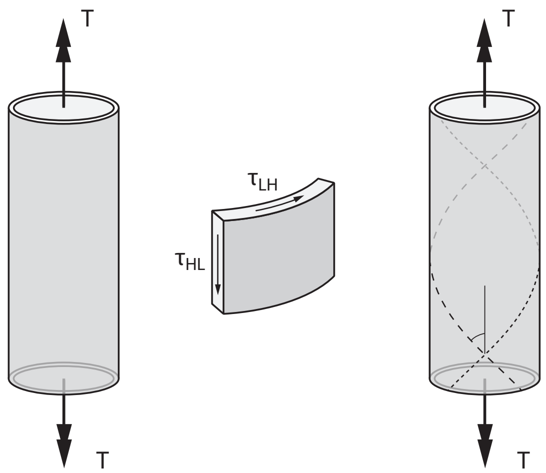
$\sigma_1 =$ at $45°$

$\sigma_2 =$ at $135°$



Thus a case of pure shear gives rise to direct tensile and compressive stresses of equal magnitude to that of the applied shear at $\pm 45°$ to the direction of shear.

A situation where we might encounter pure shear, is a thin-walled cylindrical shaft in torsion. In that case the principal stress directions can be interpreted as forming a helix at 45° to the longitudinal axis.



1.5 Mohr's Circle

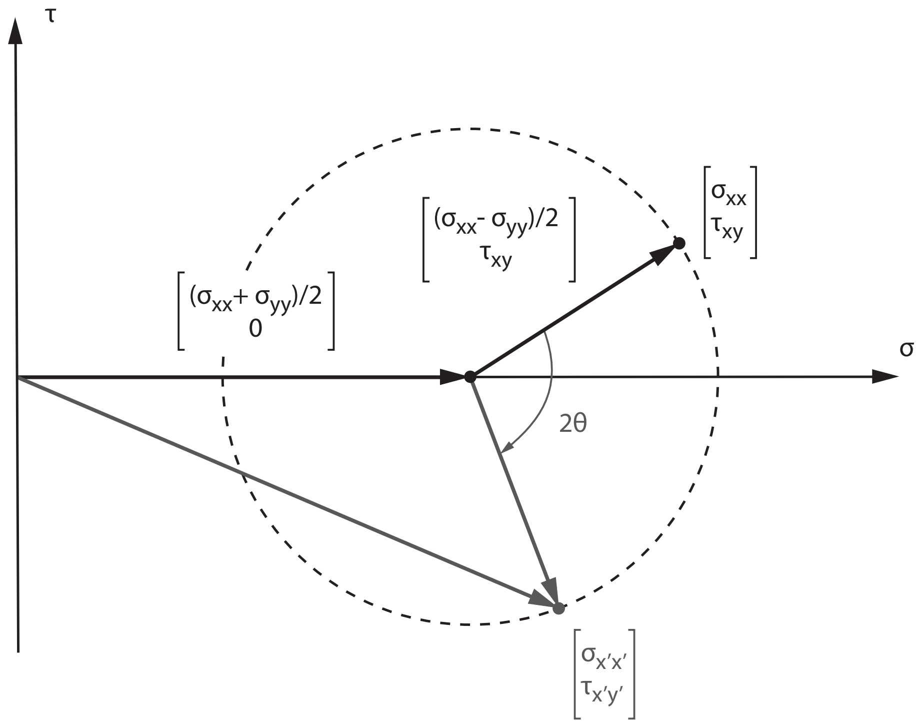
Mohr's circle is a very useful device for visualising stress, and performing quick calculations. The method was introduced by civil engineer Otto Mohr (1835–1918) in 1882 as a graphical representation of the state of stress occurring in a material.

Rewriting transformation Equations 1.2 and 1.3 into a matrix formulation:

$$\begin{bmatrix} \sigma_{x'x'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} (\sigma_{xx} + \sigma_{yy})/2 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} (\sigma_{xx} - \sigma_{yy})/2 \\ \tau_{xy} \end{bmatrix}$$

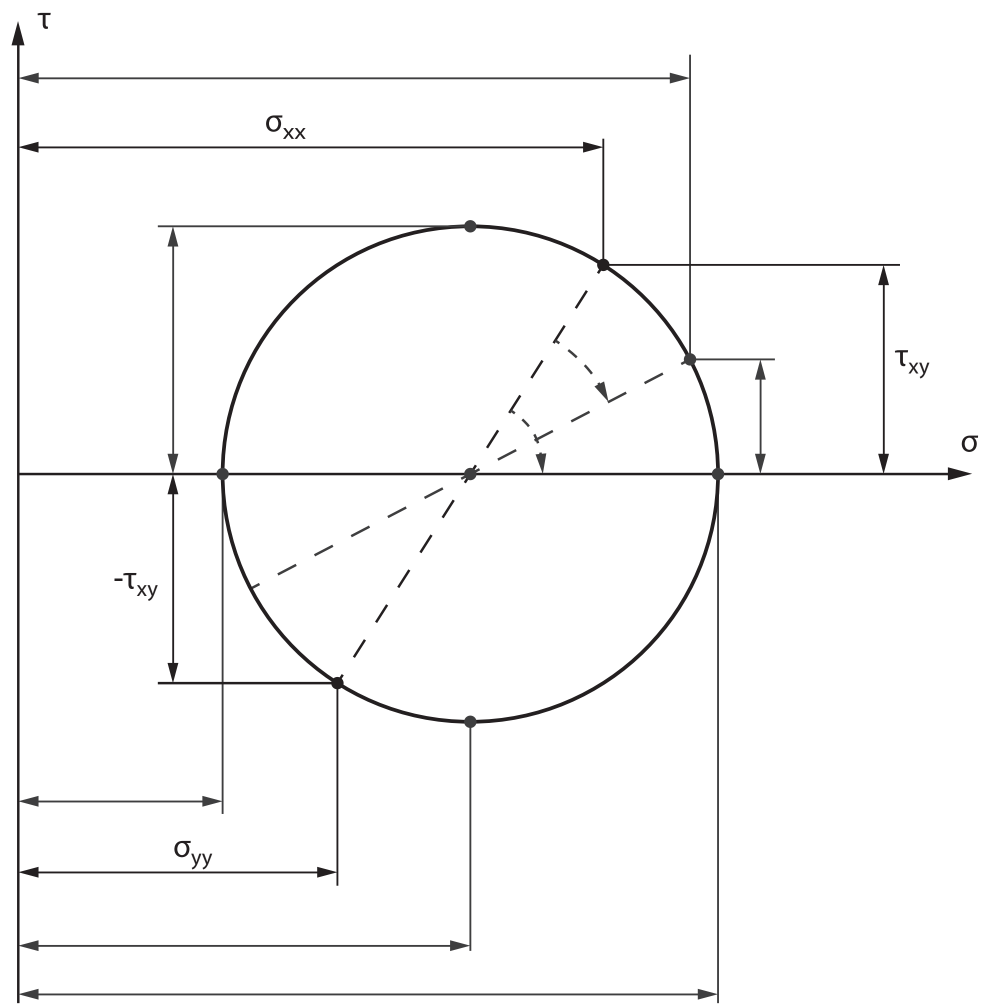
Here you will recognise a rotation matrix for a clockwise rotation by an angle 2θ.

Plotting the transformed stresses for $\theta \in [0, \pi]$ on a set of axes with $\sigma_{x'x'}$ on the abscissa and τ_{xy} on the ordinate, traces out Mohr's circle for stress:



NB: please note the sign convention used for Mohr's circle: a θ **CCW** rotation of the coordinate system is represented by a 2θ **CW** rotation on Mohr's circle! You may find an alternative sign conventions in certain textbooks, where the shear stress is plotted negative upwards to let the rotation of the vector to be counter-clockwise.

The key thing to remember is that Mohr's circle is nothing more than a graphical representation of the stress transformation equations, but one that can provide new insights and improve intuition of the state of stress.



The midpoint of Mohr's circle:

$$C = \frac{\sigma_{xx} + \sigma_{yy}}{2} \tag{1.8}$$

represents the mean value of the normal stresses, and is *invariant* to the choice of coordinate system.

The radius of Mohr's circle:

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{1.9}$$

is equal to the maximum/minimum shear stress:

$$\tau_{\text{max,min}} = \pm R$$

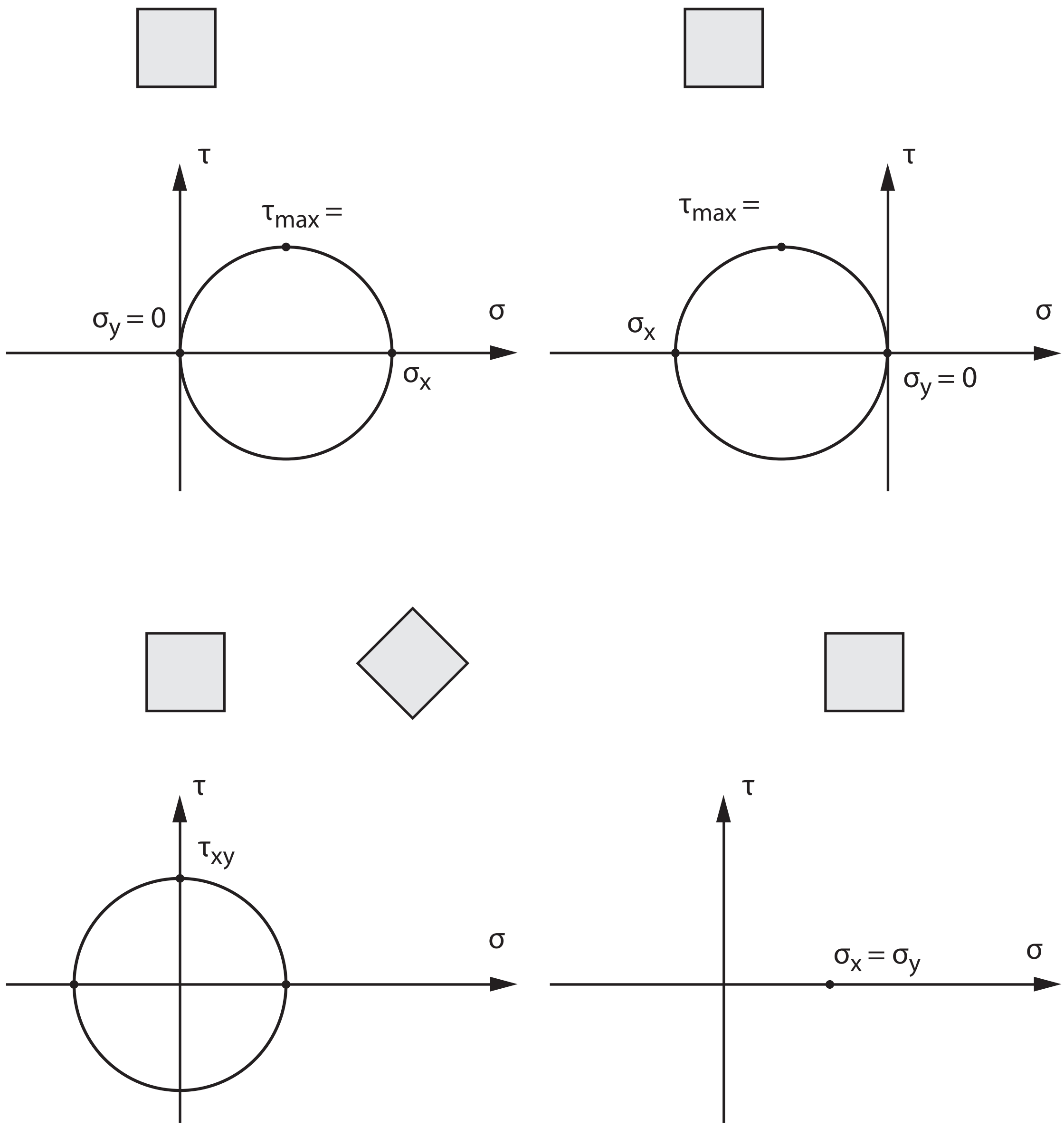
The points where Mohr's circle crosses the horizontal axis represent the principal stresses:

$$\sigma_{1,2} = C \pm R$$

The principal directions can then be found from geometry.

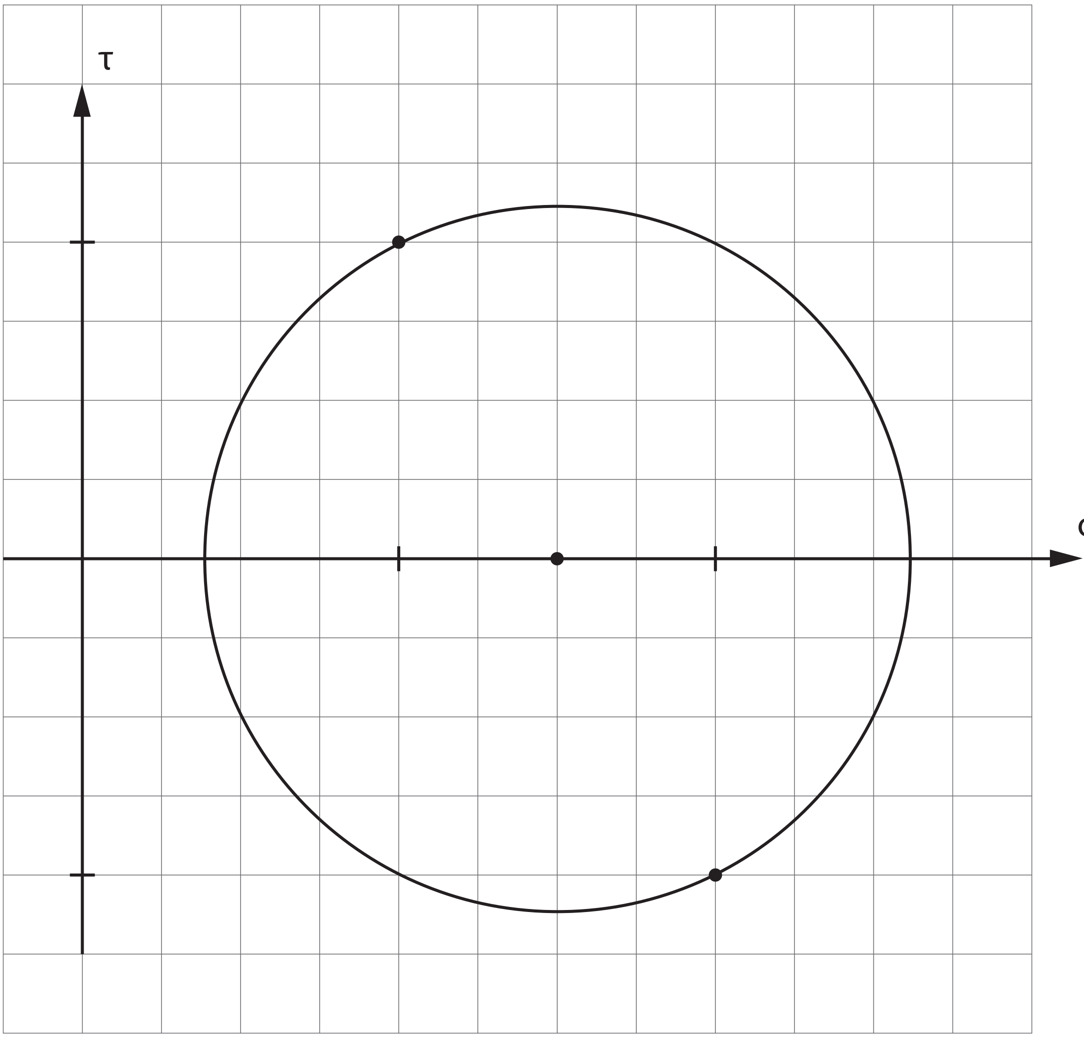
Example 1.5 – Stress States

For the Mohr's circles shown, draw the corresponding stress states.



Example 1.6 – Constructing Mohr's Circle

For a stress state $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = 200$ MPa, and $\tau_{xy} = 100$ MPa, find the principal stress state using Mohr's circle.



Recipe for constructing Mohr's circle:

1. draw a coordinate system with σ on the horizontal and τ on the vertical axis;
2. plot points (σ_{xx}, τ_{xy}) and $(\sigma_{yy}, -\tau_{xy})$;
3. draw a straight line connecting these points to find the centre of Mohr's circle where the line crosses the horizontal-axis: $C = (\sigma_{xx} + \sigma_{yy})/2$
4. draw Mohr's circle through the points, with radius $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$
5. the points where the circle crosses the horizontal axis are the principal stresses;
6. find the angle θ_p for maximum/minimum principal stress with respect to the x-axis

1.6 Summary

In this handout we have discussed the concept and definitions of stress, and focused on plane stress: σ_{xx}, σ_{yy} and τ_{xy}. This is a useful engineering approximation for the stress state in thin-walled structures, or at the surface of thick structure.

By considering the equilibrium of an infinitesimal element, we derived the stress transformation equations, which enable us to find the magnitudes of the stress in different directions. Mohr's circle is a valuable tool to visualise the stress transformation equations, and help perform calculations.

Studying the stress transformation equations revealed the concept of principal stresses: for any state of 2D stress we can find two perpendicular directions where the direct stresses are at maximum and minimum. What is more, the shear stress is zero on those planes.

Revision Objectives Handout 1:

- be familiar with stress notations, and the concept of 2D plane stress: σ_{xx}, σ_{yy}, τ_{xy}
- derive the stress transformation equations

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

- calculate stresses in a Cartesian coordinate system which has undergone a rotation from the original;
- explain the concept of principal stresses and directions;
- recall and apply the equations for principal stresses, maximum shear stress and directions;

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\text{max,min}} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

- draw a Mohr's circle for a 2D state of stress, and use it to support calculations using stress transformation quations;