Ordinary Differential Equations

Lecture 9: Linear systems of ODEs

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Linear systems of ODEs

Linear systems of ODEs

Last time we have seen that a system of ODEs can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\vec{x} = \mathbf{A}\vec{x}$$

where $\vec{x}$ is an N-dimensional vector and $\mathbf{A}$ is an $N \times N$ constant matrix.
We saw that this system of ODEs has solutions of the form

$$\vec{x}_i = \vec{v}_i \mathrm{e}^{\lambda_i t}$$

where λ_i and $\vec{v}_i$ are the eigenvalues and eigenvectors of $\mathbf{A}$.
So how do we make the general solution? Observe that the system of ODEs is both *linear* and *homogeneous*...

Linear systems of ODEs

Linear systems of ODEs

In any linear, homogeneous system any *linear combination* of solutions is also a solution. We can make the general solution if we have N linearly independent solutions. The well-behaved matrix $\mathbf{A}$ has N eigenvectors so...

$$\vec{x} = \sum_{i=1}^{N} \vec{v}_i \mathrm{e}^{\lambda_i t}$$

gives the general solution of

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\vec{x} = \mathbf{A}\vec{x}$$

All we need is to find the eigenvectors and eigenvalues of $\mathbf{A}$.

Linear systems of ODEs

Simple example

Consider the second-order ODE

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} - 8x = 0$$

we can rewrite it as a system of two first-order ODEs

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = y \qquad (1)$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} = -2y + 8x \qquad (2)$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\vec{x} = \mathbf{A}\vec{x}$$

Linear systems of ODEs

For the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}$$

To compute the eigenvalues we solve the characteristic polynomial $|\mathbf{A} - \lambda\mathbf{I}|$,

$$\begin{aligned} -\lambda(-2-\lambda) - 8 &= 0 && (3) \\ \lambda^2 + 2\lambda - 8 &= 0 && (4) \\ (\lambda+1)^2 &= 9 && (5) \\ \lambda &= -1 \pm 3 && (6) \end{aligned}$$

Linear systems of ODEs

We could go on to compute the eigenvectors, but we can already see that the solution will have the form

$$\vec{x}(t) = A\vec{v}_1 \mathrm{e}^{2t} + B\vec{v}_2 \mathrm{e}^{-4t}$$

This is still a vector equation, but the first component is our original x, and it will be of the form

$$x(t) = A' \mathrm{e}^{2t} + B' \mathrm{e}^{-4t}$$

where we have absorbed the (constant) entries of the eigenvectors into the constants A' and B'.

Linear systems of ODEs

So the general solution to the equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} - 8x = 0$$

is of the form

$$x(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-4t}$$

Since this is a linear homogeneous equation we could have found this more directly by finding the roots of the characteristic function

$$m^2 + 2m - 8 = 0$$

In fact this equation is identical to the characteristic polynomial from Eq. 6

$$\lambda^2 + 2\lambda - 8 = 0$$

So what's the point?

Linear systems of ODEs

The Point

For managable higher-order ODEs there is usually no point in turning them into systems as the alternative solution is quicker.

For not so managable higher-order ODEs turning it into a system of first order equations enables us to solve it numerically (next topic in this unit).

Often our modelling approach directly results in systems of ODEs. These systems can typically not be condensed to form a single higher-order ODE (however in the cases where this is possible it often leads to elegant solutions)

Linear systems of ODEs

Origin of Systems

Systems of ODEs often arise when the real system naturally breaks into units that are each described by a separate ODE

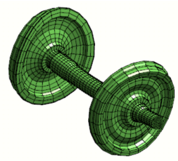
- Individual cars on a highway
- Different metabolites in a cell
- Computers in a network
- Different population in an ecosystem

Engineering Hotspot

In engineering systems of ODEs frequently arise as a result of finite element approximations.

For instance we could model a guitar string as a chain of short stiff elements connected by springs. Then every tiny bit of string would have its own set of differential equations coupled that are coupled to its neighbours.

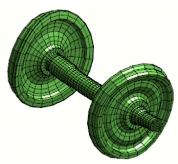
Likewise we can also put other objects together out of finite elements coupled by springs



Engineering Hotspot

Given such a system of ODEs one can solve it numerically by numerical integration (simulation, next topic).

But for linear systems one can also already compute the resonant frequencies and “eigenmodes” by numerically computing the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$, which is more efficient than numerical simulation



Summary

Summary

So, what have we learned?

Summary

1st order ODEs

Things to try (in this order)

- **Solve by inspection**, if you can ...
- ODEs that don't explicitly depend on the dependent variable (and do not contain derivatives of different orders)
  Solve by direct integration
- Linear homogeneous first-order ODEs
  Solve by separation of variables
- Linear non-homogeneous first-order ODEs
  Solve by integrating factor
- Nonlinear first-order ODEs
  Check for conservation law
  Try a clever substitution

Moments of Feedback ...

Here are some mixed exercises

Solve

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = \sin(6t) + 5$$
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 2x = \mathrm{e}^{-4t}$$
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{1}{t^2 x}$$
$$xt\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = 2(x^2 + t^2)$$
$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x = t$$
$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + x = 0$$

Summary

Higher order ODEs

- Linear, homogeneous, constant-coefficient ODEs
  Solve by Ansatz e^{mt}
- Linear, non-homogeneous, constant-coefficient ODEs
  Try to find particular integral, general solution together with homogenous part
- Nonlinear / non-constant ODEs
  Try to write as a system of first order ODEs and solve numerically (next section . . .)

Systems of ODEs

- Linear, homogeneous, constant-coefficient systems **Solution from eigenvalues and eigenvectors of the coefficient matrix.**
- Basically everything else
  Solve numerically