EMAT10100 Engineering Maths I
Lecture 18: Functions, Continuity and Derivatives

John Hogan & Alan Champneys

Mathematical Functions

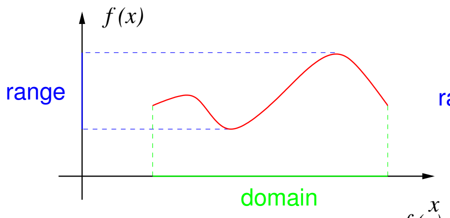
- A function f is a relationship between two sets X and Y
- For every $x \in X$, there is a unique $y \in Y$. Write as

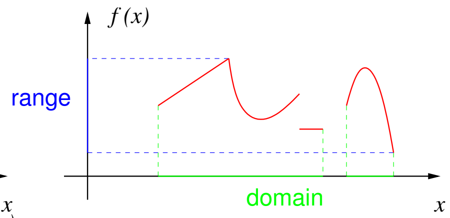
$$f : X \to Y \qquad f(x) = y$$

  - y is called the image of x under f
  - X is called the domain
  - Y is called the codomain
  - The set of all $f(x)$ is called the range
- Example: $f(x) = \sqrt{x}$
  is not a function from $\mathbb{R} \to \mathbb{R}$
  (here $\mathbb{R}$ means the set of all real numbers)
- but $f(x) = \sqrt{x}$ is a function from $[0, \infty) \to \mathbb{R}$

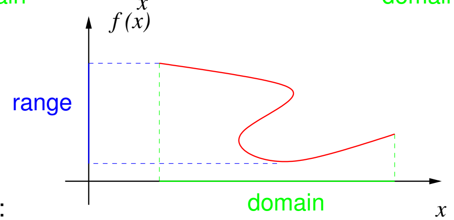
Functions as graphs

- The best way to think of functions is as a graph
- A set of points in the plane $(x, f(x))$





not a function:

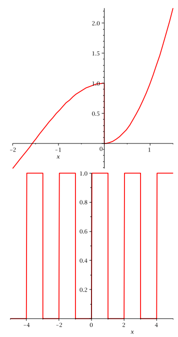


What is a continuous function?

- Continuous functions can be drawn without taking your pen off the paper (& no vertical lines)
- Functions made up from different continuous bits are sometimes called piecewise continuous, e.g.

$$f_1(x) = \begin{cases} \cos(x) & x \leq 0 \\ x^2 & x > 0 \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \sin(\pi x) \geqslant 0 \\ 0 & \sin(\pi x) < 0 \end{cases}$$



Definition of continuity

- A function f is said to be continuous at a point $x = a$ if (assuming both limits exist, of course)

$$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$$

and

$$f(a) = L$$

- A function is continuous on an interval (a, b) if it is continuous at every point of the interval
- If a function is not continuous at a point a it is called discontinuous

Exercise

Find the value of α that makes the function $f : (0, 2) \to \mathbb{R}$ continuous

$$f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 3 - \alpha x & 1 \leqslant x < 2 \end{cases}$$

All about continuous functions

- Continuous functions have very nice properties, e.g.

Intermediate value theorem: If $f : \mathbb{R} \to \mathbb{R}$ is continuous on an interval $[a, b]$ then f takes every value between $f(a)$ and $f(b)$ somewhere in $[a, b]$

- make limits easier to calculate
  - if f is continuous, and $\lim\limits_{n \to \infty} a_n = a$, then

    $$\lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right) = f(a)$$
- combinations of continuous functions are still continuous
  - if f and g are continuous, so are $f + g$, fg, $f \circ g$ and $\frac{f}{g}$ (in the last case providing $g \neq 0$)

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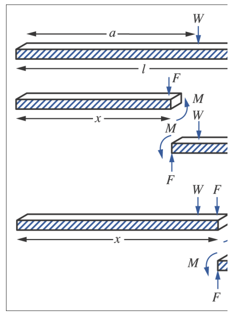
hinge-jointed beam length l with load W at $x = a$



The shear force $F(x)$ is discontinuous

$$F(x) = \begin{cases} W - W(a/l) & 0 < x < a \\ -W(a/l) & a \leq x < l \end{cases}$$

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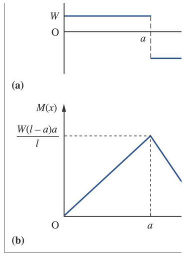


moment M is continuous (but its slope is discontinuous)

$$M(x) = \begin{cases} W(l-a)(x/l) & 0 < x < a \\ W(l-x)(a/l) & a \leqslant x < l \end{cases}$$

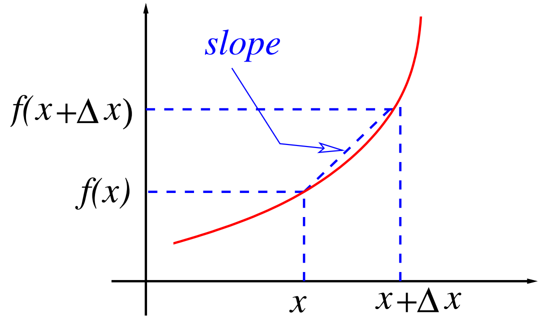
see *James* for more details

Definition of derivative

The differential or derivative of a function $f(x)$ is given by

$$\frac{df}{dx}(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Which has obvious graphical interpretation



Example from first principles

- Consider the function $f(x) = x^2$ at general point x
- form

$$\begin{aligned} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= 2x + \Delta x \end{aligned}$$

- Now take $\lim$ as $\Delta x \to 0$
- So we get $f'(x) = 2x$
- Exercise Repeat for $f(x) = x^3$

More simply

Just learn a few special functions, e.g.

function $f(x)$	derivative $f'(x)$
const.	0
x^n	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
e^{ax}	ae^{ax}
$\ln(ax)$	$1/x$
etc.	

(see formula sheet for full list)

Differentiability

- **Definition:** We say that a function $f(x)$ is differentiable at a point $x = a$ if
  - the derivative $f'(a)$ exists and is finite
  - $f'(a)$ is continuous, that is

$$\lim_{x\to a+} f'(x) = \lim_{x\to a-} f'(x) = f'(a)$$

- A function that is continuous and differentiable throughout its domain is called smooth

Exercise

Find the values of α and β that make the function $f : \mathbb{R} \to \mathbb{R}$ continuous and differentiable at $x = 1$

$$f(x) = \begin{cases} \beta x^2 & x \leq 1 \\ 3 - \alpha x & x > 1 \end{cases}$$

Second derivatives etc.

- define higher-order derivatives in an obvious way
  - 2nd derivatives

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\, x^2} = \frac{\mathrm{d}}{\mathrm{d}\, x}\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, x}\right)$$

  1st derivative = slope, 2nd derivative = curvature
  - 3rd derivatives etc.

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\, x^3} = \frac{\mathrm{d}}{\mathrm{d}\, x}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}\, x^2}\right) = \frac{\mathrm{d}}{\mathrm{d}\, x}\left[\frac{\mathrm{d}}{\mathrm{d}\, x}\left(\frac{\mathrm{d}\, f}{\mathrm{d}\, x}\right)\right] \quad \ldots$$

- notation: 2nd derivative $f''(x)$, 3rd $f'''(x)$ etc.
- special notation · for functions of time: if $x(t)$ is position
  1st derivative $\dot{x}(t)$, velocity, 2nd $\ddot{x}(t)$ acceleration

Homework

1. continuity and differentiability:
   - read *James* secs. 7.9.1-2 and sec. 8.2
   - attempt exercises 7.9.4 Qn 62, 8.2.6 Qn.1
2. 2nd derivatives, stationary points etc.
   (will also be covered in the next lecture):
   - read *James* sec. 8.5
   - attempt exercises
     - 4th edition 8.5.2 Qns. 72-75
     - 5th edition 8.5.2 Qns 72 (a),(b); 73 (a)-(c); 74-75
3. Assessed Homework (on Matrices) - see attached
   - Please completed it on the sheet (do workings elsewhere if necessary)
   - To be handed in during lecture Monday 22nd Nov (after reading week)
   - Solutions available in week 9/10.
   - Marked script will be returned to you before the end of term.
   - Mark doesn't count towards your final unit mark, but material covered does form part of the syllabus.

EMAT10100 Engineering Maths I
Special Lecture - Maple

John Hogan & Alan Champneys

Computing at University

- All degree programmes in Engineering contain some form of coding, modelling or computing
- You also get to use lots of computer packages to help with various tasks. E.g. you might learn:
  - Matlab — a general "matrix lab" for all sorts of engineering computations
  - Python — a general purpose public domain scripting language
  - C, C++, Java, . . . full industrial scale programming languages
  - Autodesk — or some other CAD/CAM Engineering Design software
  - Abacus — or some other Finite Element Analysis software for stress analysis
  - . . .
- But at bottom level all of these use the same basic ideas:
  - represent all variables as numbers
  - discretise continuous variables into discrete variables
  - do lots of lots of matrix manipulations
  - represent ("render") the output in graphics or tables of data

Maths at University

- We use a level of abstraction using algebraic symbols
- We do algebra, calculus, probability theory, geometry, trigonometry, etc. on these symbols
- We aim for general formulae that are true for all values of the variables
- Only at the end do we (sometimes) evaluate the variables and do arithmetic
- This means we can derive general engineering principles, produce design formulae, derive new computational algorithms, produce general understanding or "back of the envelope" simplifications
- This is the way of the graduate engineer. Seek generic understanding, general principles, deeper understanding . . .
- Computers (and hand calcs) are then used for specific analyses, designs etc.
- So wouldn't it be nice if there was a different form of computer package that also worked at this abstract level of algebra, calculus etc. . . .

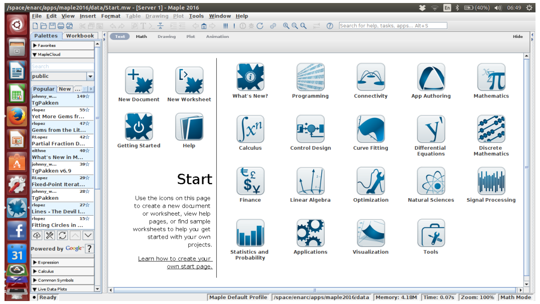
Maths on the Computer

There is!
It's called symbolic computation or Computer algebra

- There are several popular packages
  - Mathematica
  - Maple
  - many more, see wikipedia under "computer algebra"
    `en.wikipedia.org/wiki/List_of_computer_algebra_systems`
- The University has a site licence for Maple and can be downloaded for free for your *personal use* :
  `https://www.bris.ac.uk/software/software-list/maple.html`
- It is easy to use, with an intuitive GUI and help system

Getting started

On opening Maple, you should a start screen that looks something like



It is recommended to choose *"New Worksheet"* You should then get a prompt

\>

at which you can simply type.

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The rest of this lecture

- We will go through a Maple Worksheet that introduces some of the basic capabilities of Maple
- BUT You don't learn how to use a computer package through sitting through a lecture.
- This worksheet is saved as a `.mw` file on blackboard. You can play with it, use it as a reference . . .
- The final question of the homework also requires the use of Maple.
- Maple can be used as a tool to help you with all mathematical calculations throughout the rest of your degree course (and beyond!)

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Basic Commands:

The basic syntax is to define variables equal to expressions using := "is assigned to be equal to" and to end with either a semi-colon or colon. Notice the difference between

```
> f:=sin(x);
> g:=cos(x):
```

The ; puts output on the screen, the : doesn't. But both execute the command. Then you can plot things

```
> plot({f,g});
```

Note that you can also have expressions with an equals sign

```
> h:= x^2 + 5x + 6 = 0;
```

(note a slight quirk when you enter the ^key. You need to press right arrow to get back to lower level — you'll see what I mean when you try it). Then

```
> solve(h,x);
```

finds the exact solutions to this quadratic equation.

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A few Maple tips

1. Don't forget to save your work frequently!
2. Better generally to use
   `> subs(x=6,expression);`
   rather than `> x:=6;` as the latter sets $x = 6$ in all subsequent *and former* expressions
3. If all else fails type `> restart:` which unassigns all variables.
4. Another few good commands are
   `simplify( )` – use all known rules of algebra and trig to simplify
   `evalf( )` – evaluate as a decimal number
   `evalc( )` – split into real and
5. Use the help system. It is really good. Scroll to the end of each help file and see an example. A quick help on a topic (e.g. matrices) can be found using the following `> ?matrix` (which is the only command that does not need a ; or : at the end)
6. Remember Maple is CASe SEnsitiVe. The command for a matrix is `Matrix( )` not `matrix( )`
7. Some variable names are reserved for special functions or values. Usually (but not always) these start with a capital. E.g. $\sqrt{-1}$ is I. `Pi` is $3.1459\ldots$, `D` is reserved for differentiation
8. Some Maple commands are outside of the core — e.g. linear algebra is in `LinearAlgebra` differential equations are in `DETools` – you need to load them using
   ```
   > with(LinearAlgebra):
   > with(DETools):
   ```

A brief introduction to some maple commands for Eng Maths I

Calculus − basic notation for differentiation, definite and indefinite integrals. Note that for indefinite integrals the answer is not written with + c

```
> diff(x^2, x);int(x^2, x);int(x^2, x = 0 .. 1):
> diff(arctan(x), x):
```

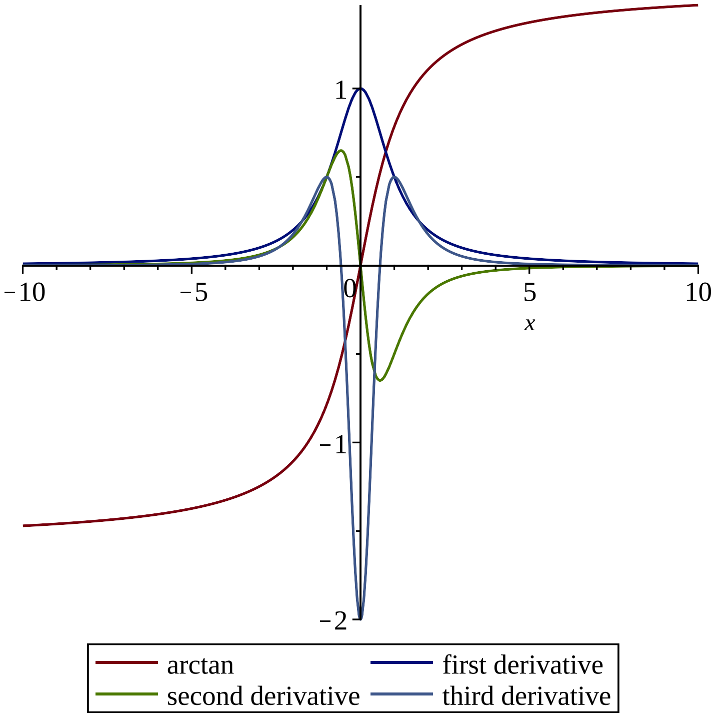
First, second, third etc derivatives (note use of := for variable asignment whereas = is just an equation, the strange $n notation means do this operation n times.

```
> f1 := arctan(x); f_x := diff(f1, x): f_xx := diff(f1, x$2): f_xxx
  := diff(f1,x$3): f_xxxx := diff(f1, x$4):
```

$$f1 := \arctan(x)$$

curve plotting

```
> plot([f1, f_x, f_xx, f_xxx], x, legend = ["arctan", "first
  derivative", "second derivative", "third derivative"]);
```



finding inflection points and points where higher derivatives are equal to zero

```
> solve(f_xx = 0, x); solve(f_xxx = 0, x);
```

Taylor series − one variable

```
> taylor(arctan(x), x = 0);
> taylor(arctan(x), x = 0, 20);
```

can do Taylor series about x other than 1. Note the use of subs to substitute y=x−1

```
> tl := taylor(ln(x), x = 1, 20);tl;
> subs(x = 1+y, tl);
```

Note taylor contains the O( ) symbol for the next term omitted. To plot the output we have to convert to a polynomial.

```
> t0 := taylor(arctan(x), x = 0, 20);
> t20 := convert(t0, polynom);
```

Note commands in Maple can stack

```
> t10 := convert(taylor(arctan(x), x = 0, 10), polynom);
```

comparing these 9th−order and 19th−order expansions in a plot, note how they become inaccurate beyond x=−1 and x=1. Note the use of x=−1.2..1.2 to set the range on the x−axis in the plot

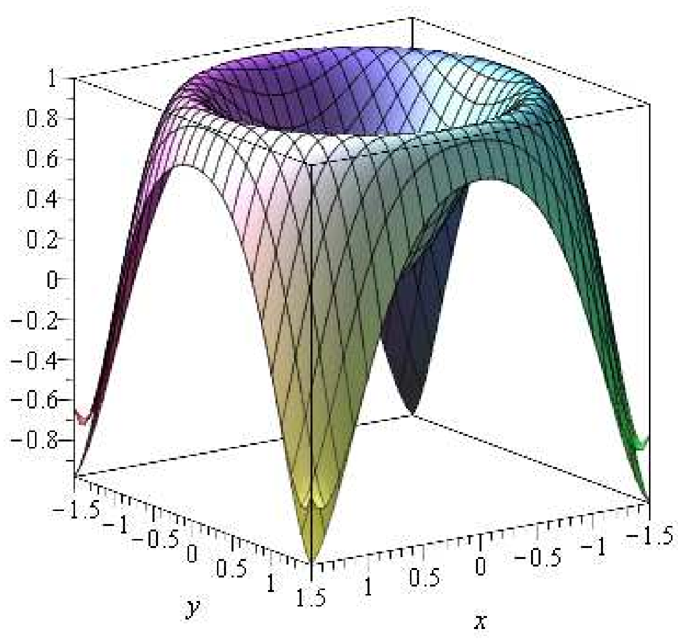
```
> plot([arctan(x), t10, t20], x = -1.2 .. 1.2, caption = "comparing
  arctan(x) and its 9th and 19th-order Taylor expansions", legend =
  ["arctan(x)", "9th-order", "19th-order"]);
```

Taylor series, two variables

```
> s20 := mtaylor(sin(x^2+y^2), [x, y], 20):
```

Note for the mtaylor command, it is assumed we are expanding about [x,y]=[0,0]. If we use plot3d we can plot the function of two variables and its Taylor series approximation. Note for x and y up to about +/− pi/2 the approximation is very good, but gets a whole lot worse after that.

```
> plot3d([sin(x^2+y^2), s20], x = -1.5 .. 1.5, y = -1.5 .. 1.5);
```




```
> plot3d([sin(x^2+y^2), s20], x = -1.5 .. 1.5, y = -1.9 .. 1.9);
```

note that you can use the mouse to rotate 3D images and by right clicking you get a lot of plot options including the ability to output the graph to standard figure formats

Some horible integrals− note the use of Int to write the integral sign and int to evaluate it

```
> Int(x^2*sin(x), x) = int(x^2*sin(x), x);
> Int(x^210*sin(x), x) = int(x^210*sin(x), x):
> Int(1/sqrt(x^2+2*x+7), x) = int(1/sqrt(x^2+2*x+7), x);
> Int(1/(sin(x)+3*cos(x)), x) = int(1/(sin(x)+3*cos(x)), x);
> Int(sin(x)^28, x) = int(sin(x)^28, x);
```

Note that this is an exact expression (with the full fractions) and if we evaluate at Pi/2 we get

```
> A := int(sin(x)^28, x): B:=subs(x=Pi,A); simplify(B);
```

Note that Maple uses upper case variables for constants so Pi =3.14159..... but pi is just a regular variable

Maple knows that sin(Pi) =0 whereas sin(3.14159...) is only approximately zero

Note that the use of colon ":" if we don't want to see the answer rather than semi−colon ";" at the end of each maple command if we do want to see the answer

Note also the very useful Maple command (simplify). Of course all of the above could have been put in one command

```
> simplify(subs(x=Pi,int(sin(x)^28, x)));
```

Note that all these answers are exact. If we want numerical answers in Maple we can always use evalf (=evaulate as a floating point decimal)

```
> CC := evalf(int(sin(x)^28, x)); simplify(subs(x = Pi, CC));
> evalf(subs(x = Pi, int(sin(x)^28, x)));
```

Note that Maple uses arbitrary precision arithmatic, 10 decimal places is just the default. look at the following and notice the difference between sin(Pi) and sin(pi) where pi is just the 10 decimal place approximation to Pi

```
> pi := evalf(Pi); 'sin(Pi)' = sin(Pi); 'sin(Pi)' = sin(pi);
```

but if we want more digits....

```
> evalf(Pi, 20);
> evalf(Pi, 200);
```

Partial fractions and general algebraic manipulation

```
> p1:=(3 *x^4+ 2*x^3 − 5 *x^2 + 6*x − 7)/(x^3−2*x+3); p2:=5*x/(
  (x^2+x+1)*(x−2)); p3:=(2*x+1)/(x^2−4*x+3);
> convert(p1, parfrac); convert(p2, parfrac); convert(p3, parfrac);
```

note that in the third example Maple knows how to factorise x^2−4x+3. In fact Maple can factorise more complex expressions. It can also expand brackets

```
> A := expand((x−1)^2*(x+3)*(x+7)*(x−4)*(x+Pi)); B := factor(A);
> convert(1/B, parfrac);
```

Maple can also solve the polynomial expression to find its roots.

```
> solve(B = 0, x);
```

It knows trig identities too, for example:

```
> A := (cos(x)^2+sin(x)^2)^8;
> B := expand(A);
> simplify(B);
```

Matrices, vectors etc. linear algebra etc.

Note that these are all contained within the LinearAlgebra package which you activate by typing

```
> with(LinearAlgebra):
```

There are several ways of entering matrices

```
> A := Matrix(3, 3, [1, 2, 3, 4, 5, 6, 7, 8, 9]); B := Matrix(3,
  shape = identity); C := <<1, 2, 3>|<2, 3, 4>|<7, 7, 9>>;
```

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$B := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 2 & 7 \\ 2 & 3 & 7 \\ 3 & 4 & 9 \end{bmatrix}$$

Standard matrix operations like determinants, inverses, transposes, multiplication etc is possible.

```
> Determinant(A); Determinant(B); Determinant(C);
> MatrixInverse(C); Transpose(C);
> 'A.B' = A.B; 'C.MatrixInverse(C)' = C.MatrixInverse(C);
```

Eigenvalues and eigenvectors are also perfectly possibe.

```
> Eigenvalues(A): CharacteristicPolynomial(A, lambda); Eigenvectors
  (A);
```

$$\lambda^3 - 15\,\lambda^2 - 18\,\lambda$$

$$\begin{bmatrix} \frac{15}{2} + \frac{3}{2}\sqrt{33} \\ \frac{15}{2} - \frac{3}{2}\sqrt{33} \\ 0 \end{bmatrix}, \begin{bmatrix} \dfrac{4}{\frac{11}{2} + \frac{3}{2}\sqrt{33}} & \dfrac{4}{\frac{11}{2} - \frac{3}{2}\sqrt{33}} & 1 \\ \dfrac{1}{2}\,\dfrac{\frac{19}{2} + \frac{3}{2}\sqrt{33}}{\frac{11}{2} + \frac{3}{2}\sqrt{33}} & \dfrac{1}{2}\,\dfrac{\frac{19}{2} - \frac{3}{2}\sqrt{33}}{\frac{11}{2} - \frac{3}{2}\sqrt{33}} & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

Vectors are also easily written as column vectors, with * used for scalar multiplication and . for dot productor for matrix times vector

```
> v := Vector([1, 3, 4]); v := <1, 3, 4>; '6*v' = 6*v;
> evalm(A)*v = A.v;
```

Cross product is CrossProduct or &x

```
> w := <2, 3, 7>; CrossProduct(v, w); '&x'(v, w);
```

EMAT10100 Eng Maths 1. Homework I

Name: Degree Course:

Name of Tutor: Write answers and some working on this sheet.

Hand in noon 22nd Nov 2017 in Lecture

1 Show that the characteristic polynomial for the matrix $\mathbf{A} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{5}{2} \\ -3 & 2 & 3 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$ is $P(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$. Explicitly verify that $\lambda = 1$ is an eigenvalue (that is $P(1) = 0$). Find the other eigenvalues.

2 Find the *unit* eigenvectors corresponding to each eigenvalue in Q1.

3 Verfiy by explicit calculation that the matrix $\mathbf{A}$ in Q1 satisfies its own characteristic polynomial. That is, $P(\mathbf{A}) = \mathbf{0}$ where the polynomial $P(\mathbf{A})$ is written as an expression involving matrix multiplication. [This result is true for all square matrices and is known as the Cayley-Hamilton Theorem]

4 Multiply both sides of the matrix equation $P(\mathbf{A}) = 0$ obtained in Q3 by $\mathbf{A}^{-1}$. Hence find a simple expression for $\mathbf{A}^{-1}$ involving $\mathbf{A}$ and $\mathbf{A}^2$. Use this expression to compute the inverse of the matrix $\mathbf{A}$. Check your answer by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_3$

5 Use **Maple** to repeat Q1-4 (that is find the eigenvalues, eigenvectors and characteristic polynomial and show the matrix satisfies its own characteristic polynomial) for the 5×5 matrix $\begin{pmatrix} 34 & -68 & 16 & -10 & -62 \\ -17 & 46 & -8 & 15 & 41 \\ 40 & -120 & 44 & -24 & -112 \\ -13 & 30 & -8 & 27 & 37 \\ 39 & -98 & 24 & -29 & -91 \end{pmatrix}$ There is no need to append your Maple output. But write the eigenvalues here: