3. Differentiation of vector fields

Why are there two notions of differentiation of vector fields (div and curl)? What do they represent physically? Do they obey the rules of calculus I know from 1D? Which second derivatives make sense?

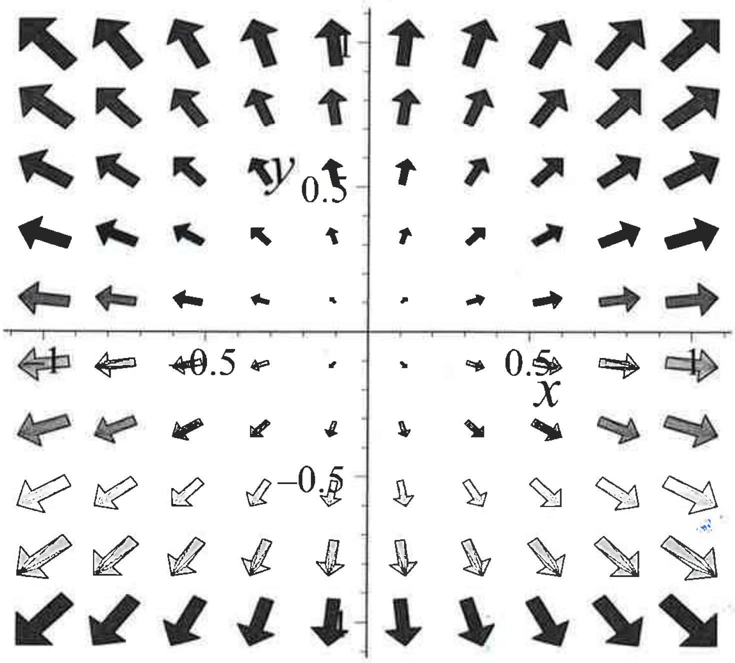
3.1 The two kinds of vector differential

$\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$

There are two key notions of differentiation of a vector field $\boldsymbol{v}(x, y, z)$. They are the so-called **divergence** 'div $\boldsymbol{v}$' and **curl** 'curl $\boldsymbol{v}$'.

Consider vector fields in 2D (think of the wind map on the weather forecast).

Loosely speaking: Divergence represents the extent to which 'nearby particles being carried by the vector field are being pulled apart'.



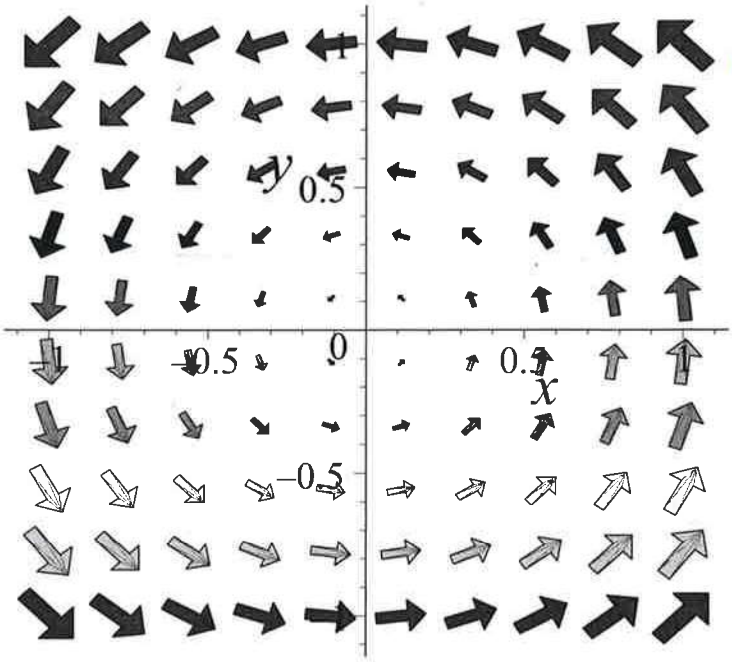
$\underline{v}(x,y) = \underline{r}$
$= (x,y)$

can show

$\text{div}\,\underline{v} = 2$

$\text{curl}\,\underline{v} = \underline{0}$

Curl represents the amount to which 'nearby particles being carried by the vector field are being rotated around each other'



$\underline{v}(x,y)$
$= r\hat{\underline{\theta}}$
$= (-y, x)$
can show
$\mathrm{div}\,\underline{v} = 0$
$\mathrm{curl}\,\underline{v} = 2\underline{k}$

We will return to precise definitions of 'pulled apart' and 'rotated' in Chapter 6, after we have dealt with integration.

Q: Why are there two ways of differentiating vector fields?

A: Recall differential operator 'del' $\underline{\nabla} = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right)$

Scalar fields $\varphi(x,y,z)$ only one way of combining "$\underline{\nabla}$" & "φ": $\mathrm{grad}\,\varphi = \underline{\nabla}\varphi = (\partial/\partial x, \partial/\partial y, \partial/\partial z)\varphi$

Vector fields $\underline{v}(x,y,z)$

(i) dot product: $\mathrm{div}\,\underline{v} \equiv \underline{\nabla}\cdot\underline{v}$ gives a scalar field 'the divergence'

(ii) cross product: $\mathrm{curl}\,\underline{v} \equiv \underline{\nabla}\times\underline{v}$ gives a vector field 'the curl'

$\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$

$\underline{v} = (v_1, v_2, v_3)$

$\underline{r} = (x, y, z)$

Calculating div $\boldsymbol{v}$ and curl $\boldsymbol{v}$.

- **Divergence**: *scalar!*

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= \boldsymbol{\nabla} \cdot \boldsymbol{v} = \underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}_{\underline{\nabla}} \cdot \underbrace{(v_1, v_2, v_3)}_{\underline{v}} \\ &= \underline{\frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3} \end{aligned}$$

- **Curl**: *vector!*

$$\begin{aligned} \operatorname{curl} \boldsymbol{v} &= \boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{array}{l} \\ \leftarrow \underline{\nabla} \\ \leftarrow \underline{v} \end{array} \\ &= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) \boldsymbol{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) \boldsymbol{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \boldsymbol{k} \end{aligned}$$

[Note that the direction of curl $\boldsymbol{v}$ is the axis about which the vector field is rotating (in an anti-clockwise sense).]

Worked example 3.1 *Calculate div $\boldsymbol{v}$ and curl $\boldsymbol{v}$ for the vector fields*

1. $\boldsymbol{v} = 4xy\boldsymbol{i} + yz\boldsymbol{j} + x\boldsymbol{k}$

Fig 1 2. $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ (=*position vector* $\boldsymbol{r}$)

Fig 2 3. $\underline{v} = -y\underline{i} + x\underline{j}$ (= ~~angle~~ azimuthal / axial vector $\hat{\underline{\theta}}$)

(2) $\underline{\nabla} \cdot \underline{v} = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (x, y, z) = \partial/\partial x\, x + \partial/\partial y\, y + \partial/\partial z\, z = \underline{\underline{3}}$

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \underline{\underline{0}}$$

(3) $\underline{\nabla} \cdot \underline{v} = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (-y, x, 0) = \underline{\underline{0}}$

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 0\underline{i} - 0\underline{j} + \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y\right) \underline{k}$$
$$= 2\underline{k}$$

3.2 Applications of divergence and curl

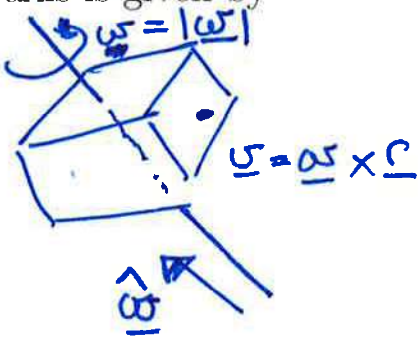
1. Rigid body mechanics

The velocity of a rigid body B rotating about a fixed axis can be described by an angular velocity vector $\boldsymbol{\omega}$, where $\omega = |\omega|$ is the angular rotation speed (rad/s) and the direction $\hat{\boldsymbol{\omega}}$ is the axis of (anti-clockwise) rotation. The velocity field at position $\boldsymbol{r}$ from the axis is given by

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{\omega} \times \boldsymbol{r}$$

$\underline{\omega} = (\omega_1, \omega_2, \omega_3)$

$\underline{r} = (x, y, z)$



Then we can show that

$$\operatorname{div} \boldsymbol{v} = 0, \qquad \operatorname{curl} \boldsymbol{v} = 2\boldsymbol{\omega} \tag{2.1}$$

that is, curl of the velocity field has the direction of the axis of rotation, and magnitude that is twice the angular speed of rotation

result holds generally
by rotating coordinate frame

$(0, 0, \omega)$

EXERCISE

Worked example 3.2 *Taking* $\boldsymbol{\omega} = \omega\boldsymbol{k}$, *show explicitly the formulae (2.1)*

$$\underline{v} = \underline{\omega} \times \underline{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (-\omega y, \omega x, 0)$$

$$\underline{\nabla} \cdot \underline{v} = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (-\omega y, \omega x, 0) = 0 + 0 + 0 = 0$$

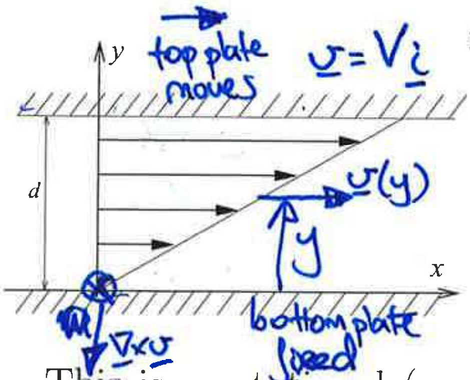
$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix} = \left(0, 0, \frac{\partial}{\partial x}(\omega x) + \frac{\partial}{\partial y}(\omega y)\right)$$

$$= (0, 0, 2\omega) = 2\underline{\omega}$$

2. Fluid mechanics

Let $\boldsymbol{v}$ be the velocity vector field of a fluid flow. Consider the following situations

Example A



Linear simple shear flow between two plates in relative motion

$$\boldsymbol{v} = \left(\frac{Vy}{d}, 0, 0\right)$$

proportional to distance from bottom plate

EXERCISE

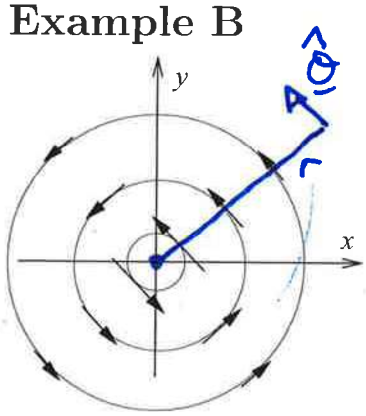
$$\Rightarrow \text{curl } \boldsymbol{v} = (0, 0, -\frac{V}{d}),$$

$$\text{div } \boldsymbol{v} = 0$$

$\underline{\nabla} \cdot \underline{v} = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)\left(\frac{Vy}{d}\ 0\ 0\right) = 0$

This is a rotational (curl $\boldsymbol{v} \neq 0$), incompressible (div $\boldsymbol{v} = 0$) flow. The rotation is perpendicular to the (x, y)-plane.

Example B



$|\underline{v}| = \sqrt{\frac{y^2 + x^2}{(x^2+y^2)^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

Vortex

$|\underline{v}| = \frac{1}{r}$

$$\boldsymbol{v} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = \frac{\hat{\boldsymbol{\theta}}}{r}$$

$$\Rightarrow \text{curl } \boldsymbol{v} = \boldsymbol{0},$$ confusing.

$$\text{div } \boldsymbol{v} = 0.$$ makes sense follows from being incompressible

This is an irrotational, incompressible flow.

Q: Why is $\underline{\nabla} \times \underline{v} = \underline{0}$ when looks to rotate?

A: $\underline{\nabla} \times \underline{v}$ represents a LOCAL rotation.

flow that GLOBALLY rotates

but $\underline{\nabla} \times \underline{v} = \underline{0}$ everywhere apart from one part

(origin here: $\underline{\nabla} \times \underline{v} = \infty \underline{k}$)

3. Incompressibility

Definition *A vector field is said to be* **incompressible** *if* $\boxed{\mathrm{div}\,\boldsymbol{v} = 0}$.

This has a natural interpretation in fluid mechanics, where the equation of continuity states that the fluid density $\rho(\boldsymbol{r})$ (a scalar field) and the fluid velocity $\boldsymbol{v}(\boldsymbol{r})$ are linked by the *continuity equation*

rate of change of mass = − mass outflow in some region

$$\boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{v}) = 0.}$$

CONSERVATION OF MASS

So that if the fluid has constant density (e.g. water, to good approximation, but not air) we have $\rho = \rho_0$=const. and hence

if const density everywhere

$$\rho = \rho_0 = \text{const.} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = 0$$

$$\mathrm{div}\,(\rho_0 \boldsymbol{v}) = \rho_0 \mathrm{div}\,(\boldsymbol{v}) = 0 \qquad \Rightarrow \qquad \mathrm{div}\,(\boldsymbol{v}) = 0.$$

Incompressible vector fields are also called **solenoidal**

terminology for electromagnetism

4. Irrotational flow, conservative forces

Definition *A flow whose velocity field* $\boldsymbol{v}$ *is curl free, curl*$\boldsymbol{v} = \mathbf{0}$*, is called* **irrotational**.

terminology for fluids

Definition *A force field* $\boldsymbol{F}$ *that satisfies curl* $\boldsymbol{F} = \mathbf{0}$ *is said to be* **conservative**.

the same

terminology for electromagnetics kinematics

More generally it can be shown that curl $\boldsymbol{v} = \mathbf{0}$ if and only if

$$\boldsymbol{v} = \text{grad}\,\phi, \quad \text{for some scalar field } \phi$$

$$\boxed{\underline{\nabla} \times \underline{v} = \underline{0} \iff \underline{v} = \underline{\nabla}\varphi}$$

ϕ is called the **scalar potential** of a **conservative vector field**.

Proof: The first part is easy, that: $\boldsymbol{v} = \text{grad}\,\phi \;\Rightarrow\; \text{curl}\,\boldsymbol{v} = \mathbf{0}$
Simply using direct calculation:

$$\boldsymbol{v} = \nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} \quad \Rightarrow \quad \text{curl}\,\boldsymbol{v} = \nabla \times \nabla\phi = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \begin{array}{l} \\ \leftarrow \underline{\nabla} \\ \leftarrow \underline{\nabla}\varphi \end{array}$$

$$\text{curl}\,\boldsymbol{v} = \left(\frac{\partial^2\phi}{\partial z\partial y} - \frac{\partial^2\phi}{\partial y\partial z}\right)\boldsymbol{i} + \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)\boldsymbol{j} + \left(\frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial x\partial y}\right)\boldsymbol{k} = \mathbf{0}.$$

because can reverse order of differentiation $\frac{\partial^2\varphi}{\partial z\partial y} = \frac{\partial^2\varphi}{\partial y\partial z}$

The second part of the proof is hard, that: curl$\boldsymbol{v} = \mathbf{0} \Rightarrow \boldsymbol{v} = \text{grad}\phi$.
We skip over it here, but it is given as extra (non-examinable) material at the end of this section.

Worked example 3.3 *Find scalar functions* ϕ *whose gradients* $\nabla\phi$ *are*

$$(i) \quad (2xy + z^3)\boldsymbol{i} + x^2\boldsymbol{j} + 3xz^2\boldsymbol{k}. \qquad (ii) \quad 2x\boldsymbol{i} + 4y\boldsymbol{j} + 8z\boldsymbol{k}.$$

$\underline{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

3.3 Rules of vector differentiation

We have defined three kinds of derivative involving the operator $\boldsymbol{\nabla}$.

$$\text{grad}\,\phi = \boldsymbol{\nabla}\phi, \quad \text{div}\,\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{v}, \quad \text{curl}\,\boldsymbol{v} = \boldsymbol{\nabla}\times\boldsymbol{v}$$

The good news is that you can apply all the usual formulae for differentiation with d/dx replaced by $\boldsymbol{\nabla}$ *provided you are careful.* This is because grad and curl are vectors, whereas div is a scalar. Also div and curl apply to vector fields, whereas grad applies to scalar fields.

Let $\boldsymbol{u}(\boldsymbol{r})$ and $\boldsymbol{v}(\boldsymbol{r})$ be vector fields, $f(\boldsymbol{r})$ and $g(\underline{r})$ be scalar fields and α and β be constants:

1. differentiation is linear linearity.

$$\begin{aligned}
\underline{\nabla}(\alpha f+\beta g) = \text{grad}\,(\alpha f+\beta g) &= \alpha\,\boldsymbol{\nabla}f+\beta\,\boldsymbol{\nabla}g = \alpha\,\text{grad}\,f+\beta\,\text{grad}\,g\\
\underline{\nabla}\cdot(\alpha\underline{u}+\beta\underline{v}) = \text{div}\,(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v}) &= \alpha\,\boldsymbol{\nabla}\cdot\boldsymbol{u}+\beta\,\boldsymbol{\nabla}\cdot\boldsymbol{v} = \alpha\,\text{div}\,\boldsymbol{u}+\beta\,\text{div}\,\boldsymbol{v}\\
\underline{\nabla}\times(\alpha\underline{u}+\beta\underline{v}) = \text{curl}\,(\alpha\,\boldsymbol{u}+\beta\,\boldsymbol{v}) &= \alpha\,\boldsymbol{\nabla}\times\boldsymbol{u}+\beta\,\boldsymbol{\nabla}\times\boldsymbol{v} = \alpha\,\text{curl}\,\boldsymbol{u}+\beta\,\text{curl}\,\boldsymbol{v}
\end{aligned}$$

2. product rules: multiplication by scalars $[(h(x)k(x))' = h'k + hk']$

f scalar field

$$\begin{aligned}
\text{grad}\,(fg) &= \boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g+g\boldsymbol{\nabla}f = f\text{grad}\,g+g\text{grad}\,f\\
\text{div}\,(f\boldsymbol{v}) &= \boldsymbol{\nabla}\cdot(f\boldsymbol{v}) = f\boldsymbol{\nabla}\cdot\boldsymbol{v}+\boldsymbol{\nabla}f\cdot\boldsymbol{v}+ = f\text{div}\,\boldsymbol{v}+\text{grad}\,f\cdot\boldsymbol{v}\\
\text{curl}\,(f\boldsymbol{v}) &= \boldsymbol{\nabla}\times(f\boldsymbol{v}) = f\boldsymbol{\nabla}\times\boldsymbol{v}+\boldsymbol{\nabla}f\times\boldsymbol{v} = f\text{curl}\,\boldsymbol{v}+\text{grad}\,f\times\boldsymbol{v}
\end{aligned}$$

3. a vector product rule

$$\text{div}\,(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\cdot\text{curl}\,\boldsymbol{u}-\text{curl}\,\boldsymbol{v}\cdot\boldsymbol{u}$$

$$\underline{\nabla}\cdot(\underline{u}\times\underline{v}) = \underline{v}\cdot\underline{\nabla}\times\underline{u} - (\underline{\nabla}\times\underline{v})\cdot\underline{u}$$

NOT A PROOF

Sketch proof of 3: From triple scalar product

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v}$$

and

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -(\nabla \times \boldsymbol{v}) \cdot \boldsymbol{u}.$$

But in order for the differential operator ∇ to see both $\boldsymbol{u}$ and $\boldsymbol{v}$, we must add these two forms, i.e.

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u}) \cdot \boldsymbol{v} - (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{u}$$

which gives the required result. $\square$

Alternatively can prove each of the above using co-ordinates (not the ∇ operator) - see example sheet.

EXERCISE

Worked example 3.4 *Let*

$$\boldsymbol{v} = (3xyz^2, 2xy^3, -x^2yz), \qquad \phi = 3x^2 - yz$$

Find (i) $\operatorname{div} \boldsymbol{v}$*, (ii)* $\boldsymbol{v} \cdot \operatorname{grad} \phi$*, and hence (iii)* $\operatorname{div}(\phi \boldsymbol{v})$ *at the point* $(1, -1, 1)$.

$\underline{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

3.4 Second derivatives

Having defined grad , div and curl in terms of applying ∇ to a vector or scalar field. What do we get if we apply ∇ twice?

applying $\underline{\nabla}$ ~~the~~ once there are 3 ways $\begin{pmatrix} \text{grad} \\ \text{div} \\ \text{curl} \end{pmatrix} \Rightarrow$ apply twice 9 ways

The following make no sense:

$$\begin{aligned} \times \quad \text{grad grad}\ \phi &= \nabla\nabla\phi & &\text{grad vector?} \\ \times \quad \text{div div}\ \boldsymbol{v} &= \nabla\cdot(\nabla\cdot\boldsymbol{v}) & &\text{div scalar?} \\ \times \quad \text{grad curl}\ \boldsymbol{v} &= \nabla(\nabla\times\boldsymbol{v}) & &\text{grad vector?} \\ \times \quad \text{curl div}\ \boldsymbol{v} &= \nabla\times(\nabla\cdot\boldsymbol{v}) & &\text{curl scalar?} \end{aligned}$$

The next two are identically zero

saw last lecture

$$\text{curl grad}\ \phi = \nabla\times(\nabla\phi) \equiv \mathbf{0}, \tag{2.2}$$

$$\text{div curl}\ \boldsymbol{v} = \nabla\cdot(\nabla\times\boldsymbol{v}) \equiv 0. \tag{2.3}$$

Proof. (2.2) was proved before when defining conservative vector fields (curl $\boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v} = \nabla\phi$). So consider (2.3).

Worked example 3.5 *Show that div curl* $\boldsymbol{v} \equiv 0$

$$\underline{\nabla}\cdot\underline{\nabla}\times\underline{v} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)\cdot\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

ASIDE: In fact, it can be shown that any incompressible (solenoidal) vector field $\boldsymbol{u}$ (i.e. one for which div $\boldsymbol{u} = 0$) can be written in the form $\boldsymbol{u} =$ curl $\boldsymbol{v}$. Such a $\boldsymbol{v}$ is called a *vector potential.*

$$\underline{\nabla}\times\underline{v} = \underline{0} \Longleftrightarrow \underline{v} = \underline{\nabla}\varphi \quad (\underline{\nabla}\times\underline{\nabla}\varphi = \underline{0})$$

$$\underline{\nabla}\cdot\underline{v} = 0 \Longleftrightarrow \underline{v} = \underline{\nabla}\times\underline{u} \quad (\underline{\nabla}\cdot\underline{\nabla}\times\underline{v} = \underline{0})$$

So we are left with only three 2nd derivatives that make sense

- div grad $\phi := \nabla^2\phi$ (=Laplacian 'del-squared')

important operator
$\nabla^2 \equiv$ Laplacian

$$\begin{aligned} \text{div grad } \phi &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \\ \nabla^2 = \underline{\nabla}\cdot\underline{\nabla} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) &= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \qquad \textit{scalar!} \end{aligned}$$

- grad div $\boldsymbol{v} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})$ *vector!*

$$= \left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_2}{\partial x\partial y}+\frac{\partial^2 v_3}{\partial x\partial z}\right)\boldsymbol{i}+\ldots$$

- curl curl $\boldsymbol{v} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{v})$ *vector!*

$$\boxed{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{v}}$$

$$= \left(\frac{\partial^2 v_2}{\partial x\partial y}-\frac{\partial^2 v_1}{\partial y^2}-\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 v_3}{\partial x\partial z}\right)\boldsymbol{i}+\ldots$$

$\underline{a}\times(\underline{b}\times\underline{c})$
$=\underline{b}(\underline{a}\cdot\underline{c})$
$-\underline{c}(\underline{a}\cdot\underline{b})$

NOTE $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{v} = \nabla^2\boldsymbol{v} := (\nabla^2\boldsymbol{v}_1, \nabla^2\boldsymbol{v}_2, \nabla^2\boldsymbol{v}_3)$ in Cartesian co-ordinates.

EXERCISE

Worked example 3.6 *Verify that*

$$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - (\nabla \cdot \nabla)\boldsymbol{v}$$

where $\boldsymbol{v} = 2xz^2\boldsymbol{i} - yz\boldsymbol{j} + 3xz^3\boldsymbol{k}$.

The Laplacian ∇^2 is an important differential operator in its own right. It can be shown that any vector field that is both irrotational and incompressible satisfies $\boldsymbol{v} = \nabla\phi$ where

$$\left.\begin{array}{l}\underline{\nabla}\cdot\underline{v}=0\\ \underline{v}=\underline{\nabla}\varphi\end{array}\right\} \Rightarrow \qquad \nabla^2\phi = 0. \tag{2.4}$$

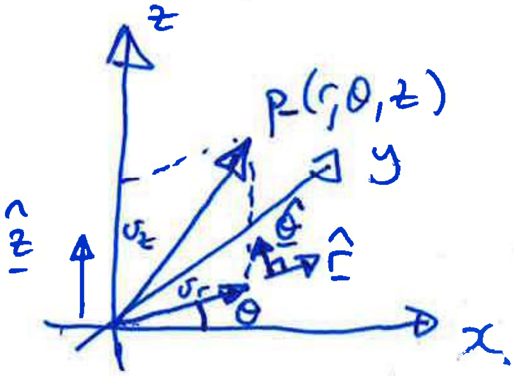
(See example sheet). The equation (2.4) is called Laplace's equation and is one of the most fundamental Partial Differential Equations in engineering science (see later in EMa II). It occurs in electrostatics, incompressible fluid dynamics (water, not air), and in stress analysis.

3.5 Grad, div and curl in polar coordinates

So far, grad , div and curl have been calculated in Cartesian coordinate (x, y, z) with a vector having components $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$. But what about other coordinate systems?

Let us consider **cylindrical polar coordinates** first. Remember that the components of a vector in cylindrical polars are $\boldsymbol{v} = v_r\hat{\boldsymbol{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\boldsymbol{k}$ where

$$\hat{\boldsymbol{r}} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}.$$



Then we have the following formulae:

$\varphi = \varphi(r, \theta, z)$

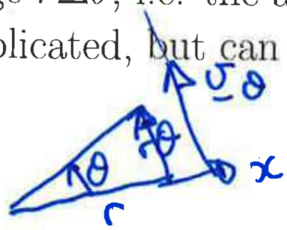
$$\text{grad}\,\phi = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

$$\text{div}\,\boldsymbol{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

$$\text{curl}\,\boldsymbol{v} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & \boldsymbol{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix}$$

(Important note: These formulae are given on sheets in the exam.)

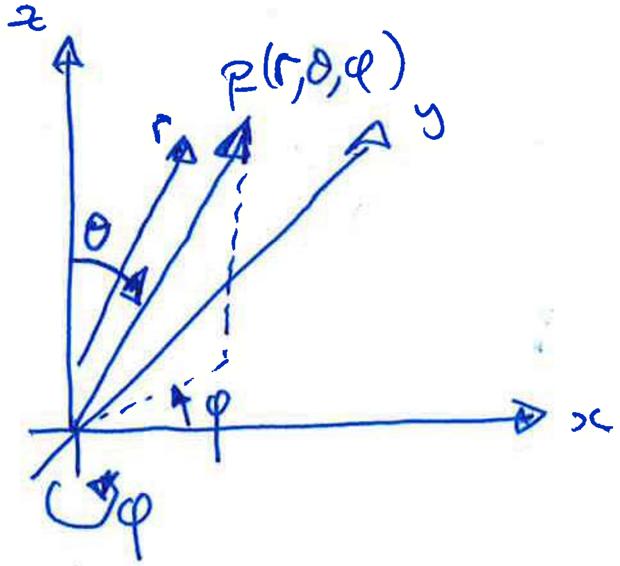
Why these formulae? Obviously, cylindrical polars will be more complicated than Cartesian coordinates. Roughly speaking, the scaling $r\partial\theta$ in grad ϕ and div $\boldsymbol{v}$ is because a small rotation $\Delta\theta$ produces a distance change $r\Delta\theta$, i.e. the arc length subtended. The other factors are more complicated, but can be derived from small differentials.



Worked example 3.7 *Find the gradient $\boldsymbol{v} = grad\phi$ of a scalar field $\phi(r, \theta, z) = 1/r + z^2$ in cylindrical polar coordinates. Now find div $\boldsymbol{v}$ and curl $\boldsymbol{v}$.*

Next we consider **spherical polar coordinates**. It is the same story, but now the components of a vector are $\boldsymbol{v} = v_r\hat{\boldsymbol{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}}$ where

$$\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\boldsymbol{i} + \sin\theta\sin\varphi\boldsymbol{j} + \cos\theta\boldsymbol{k},$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\boldsymbol{i} + \cos\theta\sin\varphi\boldsymbol{j} - \sin\theta\boldsymbol{k},$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\boldsymbol{i} + \cos\varphi\boldsymbol{j}.$$



Then the formulae are:

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

$$\text{div}\,\boldsymbol{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

$$\text{curl}\,\boldsymbol{v} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ v_r & rv_\theta & r\sin\theta v_\varphi \end{vmatrix}$$

(Note: Horrible notation! Here φ is an angle and ϕ is a scalar field.)

Finally, we mentioned before that Laplace's equation $\nabla^2\phi = 0$ is a very important differential equation for engineering science (e.g. in electrostatics and fluid dynamics).

Remember in Cartesian coordinates

$$\nabla^2\phi(x,y,z) = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Now often it is useful to use Laplace's equation in cylindrical or spherical polar coordinates. For example, one might consider the electrostatics of a charged straight wire or the fluid flow around a straight vortex, which would be best represented in cylindrical polars. Equally well, one might consider the electrostatics of spherical or point-like bodies or model explosions radially outwards from a single source.

For such problems, the expressions for grad ϕ and div $\boldsymbol{v}$ in polar coordinates can be used to show

cylindrical polar coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

spherical polar coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

EXCLUDE

Worked example 3.8 *Prove these results using $\nabla^2\phi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi$ and the formulae for grad ϕ and div $\boldsymbol{v}$ in polar coordinates.*

3.6 Summary (Chapters 2 and 3)

$$\text{grad}\,\phi = \boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

(directional derivative $D_a f = \hat{\boldsymbol{a}} \cdot \boldsymbol{\nabla}\phi$)

$$\text{div}\,\boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial \boldsymbol{v}_1}{\partial x} + \frac{\partial \boldsymbol{v}_2}{\partial y} + \frac{\partial \boldsymbol{v}_3}{\partial z}$$

$$\text{curl}\,\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{vmatrix}$$

$\boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{0} \Rightarrow$ irrotational (=conservative)

$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \Rightarrow$ incompressible (=solenoidal)

Only some 2nd derivatives make sense, the most important of which for Engineering is $\nabla^2\phi$, the Laplacian.

The above expressions are in Cartesian coordinates. There are versions of grad, div and curl in cylindrical and spherical polar coordinates.

Extra: ~~Reversing order of integration in area integrals~~

Second, we show that for some scalar field $\phi(\boldsymbol{r})$

$$\text{curl}\,\boldsymbol{v} = \underline{0} \Rightarrow \boldsymbol{v} = \boldsymbol{\nabla}\phi$$

Start with curl $\boldsymbol{v} = \boldsymbol{0} \Rightarrow$

$$\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = 0 \tag{2.5}$$

$$\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = 0 \tag{2.6}$$

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \tag{2.7}$$

Now, taking $\frac{\partial}{\partial x}$ of (2.5) and $\frac{\partial}{\partial y}$ of (2.6) we get two expressions for $\frac{\partial^2 v_3}{\partial x \partial y}$. Setting these equal gives

$$\frac{\partial^2 v_2}{\partial z \partial x} = \frac{\partial^2 v_1}{\partial z \partial y} := \frac{\partial^3 \phi}{\partial x \partial y \partial z} \tag{2.8}$$

for some scalar function $\phi(x, y, z)$. This can only be true if

$$v_1 = \frac{\partial \phi}{\partial x} + b_1' y + c_1' z + d_1' \quad \text{and} \quad v_2 = \frac{\partial \phi}{\partial y} + b_2' x + c_2' z + d_2'.$$

For some functions $b_1'(x)$, $c_1'(x)$, $d_1'(x)$ and $b_2'(y)$, $c_2'(y)$, $d_2'(y)$ (where $'$ = 'differentiate'). But, by redefining ϕ to be

$$\phi \mapsto \phi - b_1 y - c_1 z - d_1 x - b_2 xy - c_2 yz - d_2 y,$$

we can without loss of generality choose $b_{1,2} = c_{1,2} = d_{1,2} = 0$.

Similarly, from (2.6) and (2.7) we find

$$\frac{\partial^2 v_3}{\partial x \partial y} = \frac{\partial^2 v_2}{\partial x \partial z} = \frac{\partial^3 \phi}{\partial x \partial y \partial z}$$

so that, without loss of generality

$$v_3 = \frac{\partial \phi}{\partial z}, \quad v_2 = \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \boldsymbol{v} = (v_1, v_2, v_3) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \boldsymbol{\nabla}\phi \qquad \square$$

Another Proof

Prove : $\underline{\nabla} \times \underline{v} = \underline{0} \Rightarrow \underline{v} = \underline{\nabla}\varphi$

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} \partial v_3/\partial y - \partial v_2/\partial z \\ -\partial v_3/\partial x + \partial v_1/\partial z \\ \partial v_2/\partial x - \partial v_1/\partial y \end{pmatrix} = \underline{0}$$

$$\Rightarrow \quad \partial v_2/\partial z = \partial v_3/\partial y \quad -①$$
$$\partial v_1/\partial z = \partial v_3/\partial x \quad -②$$
$$\partial v_2/\partial x = \partial v_1/\partial y \quad -③$$

now define $\varphi = \int^z dz\, v_3$ so $v_3 = \partial\varphi/\partial z$

① $\to$ $\partial v_2/\partial z = \partial^2\varphi/\partial y\partial z = \partial/\partial z\, \partial\varphi/\partial y \quad \therefore v_2 = \partial\varphi/\partial y + c_2(x,y)$

② $\to$ $\partial v_1/\partial z = \partial^2\varphi/\partial x\partial z = \partial/\partial z\, \partial\varphi/\partial x \quad \therefore v_1 = \partial\varphi/\partial x + c_1(x,y)$

③ $\to$ $\partial v_2/\partial x = \partial v_1/\partial y \Rightarrow \cancel{\partial^2\varphi/\partial x\partial y} + \partial c_2/\partial x = \cancel{\partial^2\varphi/\partial x\partial y} + \partial c_1/\partial y$

alltogether $\underline{v} = \underline{\nabla}\varphi + \begin{pmatrix} c_1(x,y) \\ c_2(x,y) \\ 0 \end{pmatrix}$; $\partial c_2/\partial x = \partial c_1/\partial y$

now define $\tilde{\varphi} = \varphi + \int^x dx\, c_1(x,y)$

$$\text{so } \underline{\nabla}\tilde{\varphi} = \underline{\nabla}\varphi + \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \int^x dx\, c_1(x,y)$$

$$= \underline{\nabla}\varphi + \begin{pmatrix} c_1(x,y) \\ \int^x dx\, \partial c_1/\partial y \\ 0 \end{pmatrix}$$

$$= \underline{\nabla}\varphi + \begin{pmatrix} c_1(x,y) \\ \int^x dx\, \partial c_2/\partial x \\ 0 \end{pmatrix} = \underline{\nabla}\varphi + \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$$

Hence $\underline{v} = \underline{\nabla}\tilde{\varphi}$

QED

WORKED EXAMPLE 3-1

(i) $\underline{v} = (4xy, yz, x)$

$$\underline{\nabla}\cdot\underline{v} = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right)\cdot(4xy, yz, x)$$

$$= \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(yz) + \cancelto{0}{\frac{\partial}{\partial z}(x)}$$

$$= 4y + z + 0$$

$$= \underline{\underline{4y + z}}$$

$$\frac{\partial x}{\partial x} = 1$$

$$\frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} = 0$$

$$\underline{\nabla}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy & yz & x \end{vmatrix}$$

$$= \underline{i}\left(\cancel{\frac{\partial}{\partial y}x} - \frac{\partial}{\partial z}(yz)\right) - \underline{j}\left(\frac{\partial}{\partial x}x - \cancel{\frac{\partial}{\partial z}(4xy)}\right)$$

$$+ \underline{k}\left(\cancel{\frac{\partial}{\partial x}(yz)} - \frac{\partial}{\partial y}(4xy)\right)$$

$$= -y\,\underline{i} - \underline{j} - \underline{k}\times 4x$$

$$= \underline{\underline{(-y, -1, -4x)}}$$

2. Example A

$$\underline{v} = \frac{Vy}{d}\underline{i} \Rightarrow \underline{\nabla}\cdot\underline{v} = \frac{\partial}{\partial x}\left(\frac{Vy}{d}\right) = 0$$

$$\underline{\nabla}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Vy/d & 0 & 0 \end{vmatrix} = -\frac{V}{d}\underline{k}$$

Example B

$$\underline{v} = \frac{-y\,\underline{i}}{x^2+y^2} + \frac{x\,\underline{j}}{x^2+y^2} \Rightarrow \underline{\nabla}\cdot\underline{v} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{+2yx}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

$$\underline{\nabla}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/x^2+y^2 & x/x^2+y^2 & 0 \end{vmatrix}$$

$$= \underline{k}\left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right)$$

$$= \underline{k}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right)$$

$$\therefore \nabla\times\underline{v} = \underline{k}\left(\frac{2}{x^2+y^2} - \frac{2\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}\right) = 0$$

Example 3.3(i) $\underline{v} = (2xy+z^3, x^2, 3xz^2)$

find φ s.t. $\underline{v} = \underline{\nabla}\varphi = (\partial\varphi/\partial x, \partial\varphi/\partial y, \partial\varphi/\partial z)$

first component: $\dfrac{\partial\varphi}{\partial x} = 2xy+z^3 \underset{\int dx}{\Rightarrow} \varphi = x^2y+z^3x+f(y,z)$

$\therefore \dfrac{\partial f}{\partial x} = 0$

second component: $\dfrac{\partial\varphi}{\partial y} = x^2 = \dfrac{\partial}{\partial y}(x^2y+z^3x+f(y,z))$

$= x^2 + \partial f/\partial y$

ie $x^2 = x^2 + \dfrac{\partial f}{\partial y} \Rightarrow \dfrac{\partial f}{\partial y} = 0 \underset{\int dy}{\Rightarrow} f(y,z) = g(z)$

$\therefore \varphi = x^2y+z^3x+g(z)$

third component: $\dfrac{\partial\varphi}{\partial z} = 3xz^2 = \dfrac{\partial}{\partial z}(x^2y+z^3x+g(z))$

$= 3z^2x + \partial g/\partial z$

$\therefore 3xz^2 = 3xz^2 + \partial g/\partial z \Rightarrow \partial g/\partial z = 0$

$\underset{\int dz}{\Rightarrow} g(z) = c$, const.

$\therefore \varphi(x,y,z) = x^2y+z^3x+c$

EXAMPLES 3.3(ii) $\underline{v} = (2x, 4y, 8z)$

find φ s.t. $\underline{v} = \underline{\nabla}\varphi$

$\underline{v} = \underline{\nabla}\varphi = (\partial\varphi/\partial x, \partial\varphi/\partial y, \partial\varphi/\partial z)$

$\frac{\partial\varphi}{\partial x} = 2x \underset{\int dx}{\Longrightarrow} \varphi(x) = x^2 + f(y,z)$

$(\because \partial f/\partial x = 0)$

$\frac{\partial\varphi}{\partial y} = 4y = \frac{\partial}{\partial y}(\cancel{x^2} + f(y,z)) = \partial f/\partial y$

ie $\partial f/\partial y = 4y \underset{\int dy}{\Longrightarrow} f(y,z) = 2y^2 + g(z)$

$\partial\varphi/\partial z = 8z = \partial/\partial z(\cancel{x^2 +} \cancel{2y^2} + g(z)) = \partial g/\partial z$

ie $\partial g/\partial z = 8z \underset{\int dz}{\Longrightarrow} g(z) = 4z^2 + \text{const}$

$\therefore \varphi(x,y,z) = x^2 + 2y^2 + 4z^2 + \text{const}$

check $\underline{\nabla} \times \underline{v} = 0 \iff \underline{v} = \underline{\nabla}\varphi$

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & 4y & 8z \end{vmatrix} = \underline{0}$$

Example 3.4

$$\underline{v} = (3xyz^2, 2xy^3, -x^2yz) \qquad \varphi = 3x^2 - yz$$

at $(x,y,z) = (1,-1,1)$: $\underline{v} = (-3,-2,1)$

$\varphi = 4$

(i) $$\underline{\nabla} \cdot \underline{v} = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right) \cdot \left(3xyz^2, 2xy^3, -x^2yz\right)$$

$$= 3yz^2 + 6xy^2 - x^2y$$

$$= -3 + 6 + 1 = 4$$

(ii) $$\underline{v} \cdot \underline{\nabla}\varphi = \underline{v} \cdot \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right) 3x^2 - yz$$

$$= \underline{v} \cdot (6x, -z, -y)$$

$$= (-3,-2,1) \cdot (6,-1,1)$$

$$= -18 + 2 + 1 = -15$$

(iii) $$\underline{\nabla} \cdot (\varphi \underline{v}) = \varphi \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla}\varphi$$

$$= 4 \times 4 - 15$$

$$= 1$$

Example 3.6

$$\underline{v} = (2xz^2, -yz, 3xz^3)$$

1) find $\underline{\nabla} \times (\underline{\nabla} \times \underline{v})$:

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= (y, 4xz - 3z^3, 0)$$

$$\therefore \underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 4xz-3z^3 & 0 \end{vmatrix}$$

$$= (-4x + 9z^2, 0, 4z - 1)$$

2) find $\underline{\nabla}(\underline{\nabla} \cdot \underline{v})$ and $(\underline{\nabla} \cdot \underline{\nabla})\underline{v}$:

$$\underline{\nabla} \cdot \underline{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (2xz^2, -yz, 3xz^3)$$

$$= 2z^2 - z + 9xz^2$$

$$\therefore \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(2z^2 - z + 9xz^2)$$

$$= (9z^2, 0, 4z - 1 + 18xz)$$

$$(\underline{\nabla} \cdot \underline{\nabla})\underline{v} = \nabla^2 \underline{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(2xz^2, -yz, 3xz^3)$$

$$= (4x, 0, 18xz)$$

3) Can see directly that $\underline{\nabla} \times (\nabla \times \underline{v}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{v}) - \nabla^2 \underline{v}$

$$(-4x + 9z^2, 0, 4z - 1) = (9z^2, 0, 4z - 1 + 18xz) - (4x, 0, 18xz)$$

EXAMPLE 3.7 $\quad \varphi = 1/r + z^2 \quad$ (cyl. polars)

$$\underline{v} = \underline{\nabla}\varphi = \frac{\partial\varphi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\underline{\theta}} + \frac{\partial\varphi}{\partial z}\underline{k}$$

$$\therefore \underline{v} = \frac{-1}{r^2}\hat{\underline{r}} + 2\underline{k}z$$

$$\underline{\nabla}\cdot\underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{-1}{r}\right) + \frac{\partial}{\partial z}(2z)$$

$$\therefore \underline{\nabla}\cdot\underline{v} = \frac{1}{r^3} + 2$$

$$\underline{\nabla}\times\underline{v} = \frac{1}{r}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & \underline{k} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ v_r & rv_\theta & v_z \end{vmatrix}$$

$$= \frac{1}{r}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & \underline{k} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ -1/r^2 & 0 & 2z \end{vmatrix}$$

$$\therefore \underline{\nabla}\times\underline{v} = 0$$

ie irrotational / conservative

Examples 3.8

cylindrical polars : $\underline{v} = \underline{\nabla}\varphi = \frac{\partial \varphi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\underline{\theta}} + \frac{\partial \varphi}{\partial z}\underline{k}$

$$\underline{\nabla}\cdot\underline{\nabla}\varphi = \underline{\nabla}\cdot\underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$\therefore \nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) + \frac{\partial^2 \varphi}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

spherical polars : $\underline{v} = \underline{\nabla}\varphi = \frac{\partial \varphi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\underline{\theta}} + \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}\hat{\underline{\phi}}$

$$\underline{\nabla}\cdot\underline{\nabla}\varphi = \underline{\nabla}\cdot\underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\therefore \nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\sin\theta}{r}\frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \varphi}{\partial \phi^2}$$