Vibrations 2, Lecture 13
Response due to initial conditions

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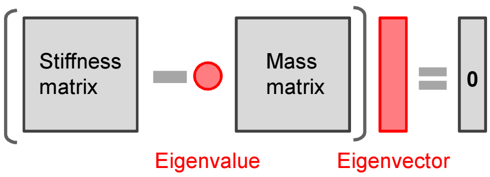
Lecture 12 review

Eigenvalue problem:

$$(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a}=\mathbf{0}$$



Characteristic equation:

$$\det(\mathbf{K}-\omega^2\mathbf{M})=0 \Rightarrow \omega_i$$

Mode shape calculation:

$$(\mathbf{K}-\omega_i^2\mathbf{M})\mathbf{a}_i=\mathbf{0} \Rightarrow \mathbf{a}_i$$

Lecture 13

- Free response via mode superposition
- Initial conditions
- 2 DOF example

Free vibrations

Consider a MDOF system without applied load (i.e. free vibration) and with non-zero initial conditions:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{0}$$

Free response of this system is expressed as a *superposition* of all modes of the system:

$$\mathbf{x} = C_1\,\mathbf{a}_1\cos(\omega_1 t + \varphi_1) + C_2\,\mathbf{a}_2\cos(\omega_2 t + \varphi_2) + \ldots$$

where C_1, C_2,…; φ_1, φ_2, … are 2M unknown constants.

Unknown constants C_i and φ_i depend on the *initial conditions* (ICs) :

- initial displacements
- initial velocities

$$t = 0:\ \mathbf{x}(0) = \mathbf{x}_0,\ \ \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$$

Free vibrations

Free response and initial conditions:

$$\mathbf{x} = C_1\,\mathbf{a}_1 \cos(\omega_1 t + \varphi_1) + C_2\,\mathbf{a}_2 \cos(\omega_2 t + \varphi_2) + \ldots$$

$$t = 0 : \mathbf{x}(0) = \mathbf{x}_0,\;\; \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$$

are used to set up 2M equations to solve for 2M unknown constants:

$$\mathbf{x}_0 = C_1\,\mathbf{a}_1 \cos(\varphi_1) + C_2\,\mathbf{a}_2 \cos(\varphi_2) + \ldots$$

$$\dot{\mathbf{x}}_0 = -\omega_1 C_1\,\mathbf{a}_1 \sin(\varphi_1) - \omega_2 C_2\,\mathbf{a}_2 \sin(\varphi_2) - \ldots$$

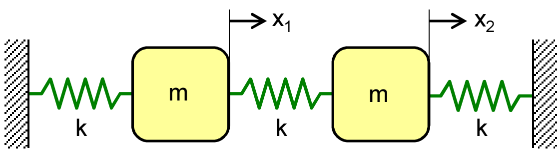
Note: the component form of these equations is:

$$\text{M equations}\begin{cases} x_1(0) = C_1\,a_{1,1}\cos(\varphi_1) + C_2\,a_{1,2}\cos(\varphi_2) + \ldots \\ \ldots \\ x_M(0) = C_1\,a_{M,1}\cos(\varphi_1) + C_2\,a_{M,2}\cos(\varphi_2) + \ldots \end{cases}$$

$$\text{M equations}\begin{cases} \dot{x}_1(0) = -\omega_1 C_1\,a_{1,1}\sin(\varphi_1) - \omega_2 C_2\,a_{1,2}\sin(\varphi_2) - \ldots \\ \ldots \\ \dot{x}_M(0) = -\omega_1 C_1\,a_{M,1}\sin(\varphi_1) - \omega_2 C_2\,a_{M,2}\sin(\varphi_2) - \ldots \end{cases}$$

2DOF example

Consider the following 2DOF system. Find its free response for the three selected initial conditions (ICs):



IC1:

$$\mathbf{x}(0) = \begin{bmatrix} a \\ a \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

IC2:

$$\mathbf{x}(0) = \begin{bmatrix} -a \\ a \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

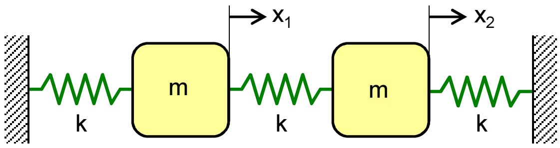
IC3:

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ a \end{bmatrix}, \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where *a* is a small number (small deflection)

2DOF example

Consider the following 2DOF system. Find its free response for the three selected initial conditions (ICs):



Equation of motion:
$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

Mass and stiffness matrices:
$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

Natural frequencies and mode shapes:
$$\omega_1 = \sqrt{\frac{k}{m}},\ \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \omega_2 = \sqrt{\frac{3k}{m}},\ \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2DOF example

Initial conditions of 2DOF system (four equations and four unknowns):
$$\mathbf{x}_0 = C_1\,\mathbf{a}_1 \cos(\varphi_1) + C_2\,\mathbf{a}_2 \cos(\varphi_2)$$
$$\dot{\mathbf{x}}_0 = -\omega_1 C_1\,\mathbf{a}_1 \sin(\varphi_1) - \omega_2 C_2\,\mathbf{a}_2 \sin(\varphi_2)$$

Substituting IC1 and the mode shapes gives the following *four* equations:
$$\begin{bmatrix} a \\ a \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\varphi_1) + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(\varphi_2)$$
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = -\omega_1 C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\varphi_1) - \omega_2 C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(\varphi_2)$$

Four equations and four unknowns:
$$a = C_1 \cos\varphi_1 - C_2 \cos\varphi_2 \qquad (1)$$
$$a = C_1 \cos\varphi_1 + C_2 \cos\varphi_2 \qquad (2)$$
$$0 = -\omega_1 C_1 \sin\varphi_1 + \omega_2 C_2 \sin\varphi_2 \qquad (3)$$
$$0 = -\omega_1 C_1 \sin\varphi_1 - \omega_2 C_2 \sin\varphi_2 \qquad (4)$$

2DOF example

Adding the two pairs of equations:

(1)+(2) $a = C_1 \cos\varphi_1 \Rightarrow C_1 = a/\cos\varphi_1$

(3)+(4) $0 = \omega_1 C_1 \sin\varphi_1 \Rightarrow 0 = \omega_1 (a/\cos\varphi_1)\sin\varphi_1$

$0 = \omega_1 a \tan\varphi_1 \Rightarrow \varphi_1 = 0, \pi, \ldots \quad \Rightarrow C_1 = a$

Subtracting the two pairs of equations:

(2)-(1) $0 = C_2 \cos\varphi_2$

(3)-(4) $0 = \omega_2 C_2 \sin\varphi_2$

$\sin\varphi_2 \;\text{ AND }\; \cos\varphi_2 \neq 0 \Rightarrow C_2 = 0$

The free response induced by the IC1 is:

$$\mathbf{x} = C_1 \mathbf{a}_1 \cos(\omega_1 t + \varphi_1) + C_2 \mathbf{a}_2 \cos(\omega_2 t + \varphi_2) = a\,\mathbf{a}_1 \cos(\omega_1 t)$$

In summary, the system vibrates at ω_1 in the shape $\mathbf{a}_1$ with zero phase angle φ_1, i.e. this is a special case of free vibration at a *single* natural frequency!

2DOF example

IC2:

$$\left.\begin{aligned} \begin{bmatrix} -a \\ a \end{bmatrix} &= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\varphi_1) + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(\varphi_2) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= -\omega_1 C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\varphi_1) - \omega_2 C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(\varphi_2) \end{aligned}\right\} \mathbf{x} = a\,\mathbf{a}_2 \cos(\omega_2 t)$$

IC3:

$$\left.\begin{aligned} \begin{bmatrix} 0 \\ a \end{bmatrix} &= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\varphi_1) + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(\varphi_2) \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= -\omega_1 C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(\varphi_1) - \omega_2 C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(\varphi_2) \end{aligned}\right\} \mathbf{x} = (a/2)\mathbf{a}_1 \cos(\omega_1 t) + (a/2)\mathbf{a}_2 \cos(\omega_2 t)$$

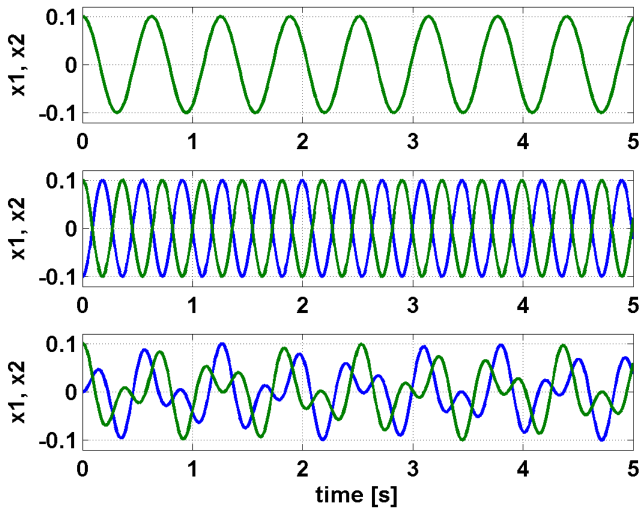
Notes: Based on the above results, it seems that we can excite *pure harmonic vibrations* in MDOF systems simply by providing suitable initial conditions. We observed that the initial static deflection $\mathbf{x}_0 = \alpha\mathbf{a}_i$ (where α is a real number) applied at t=0 produces harmonic vibrations at ω_i. This approach is sometimes used to excite only the *fundamental modes* (the lowest natural frequencies) of vibration.

2DOF example

IC1: In-phase free vibration at ω_1

IC2: Out-of-phase free vibration at ω_2

IC3: Free vibration at ω_1 and ω_2



Summary

- Free vibration response is a superposition of all modal contributions
- Unknown free response parameters are found with the help of initial conditions (initial deflections and velocities of all masses)
- Individual modes (pure harmonic vibrations) can be excited by means of special initial conditions