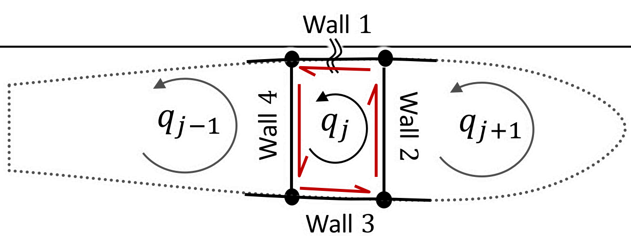
Multi-Cell Sections – Notation

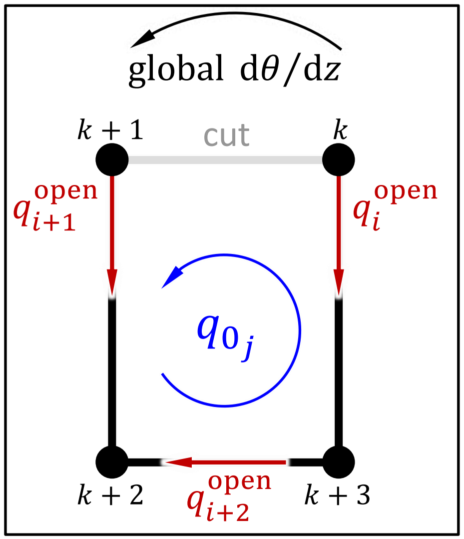
Meaning	Notation StM2	Notation Megson	Constant within cell?
Wall (*i.e.* skin segment) index	i	i	
Cell index	j	R	
Boom index	k	r	
Area of cell j	A_j	A_R	Constant
Area of boom k	A_k	B_r	Variable
Length of wall i	b_i	l_i	Variable
True closed-cell shear flow within cell j	$q_{s_j}^{\mathrm{closed}}$	q	Variable
'Open-cell' shear flow around cell j	$q_{s_j}^{\mathrm{open}}$	q_b	Variable
'Closed-cell' constant for cell j	q_{0_j}	$q_{s,0,R}$	Constant
'Equivalent' closed-cell shear flow within cell j (for torque calcs.)	q_j	q_R	Constant
True shear flow along wall i	q_i	q_i	Variable
Local moment arm for wall i (for torque calcs.)	r_i	p_O	Variable
Local shear flow (generic term)	q_s	q	Variable
Local moment arm (generic term)	r_s	p	Variable

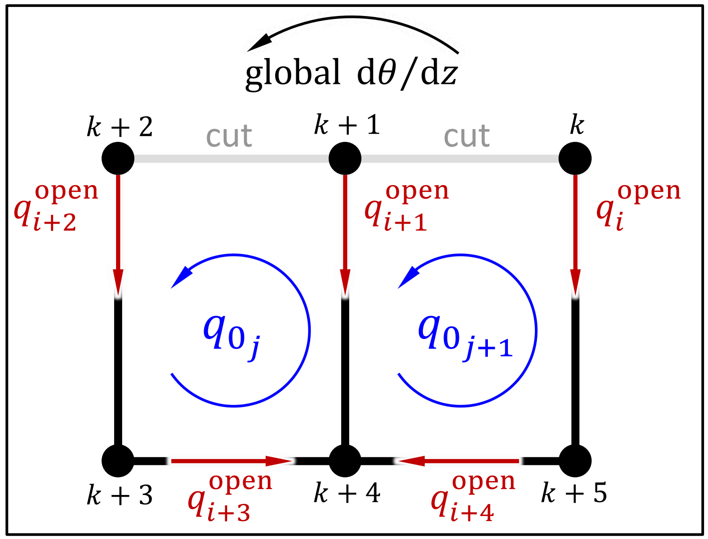


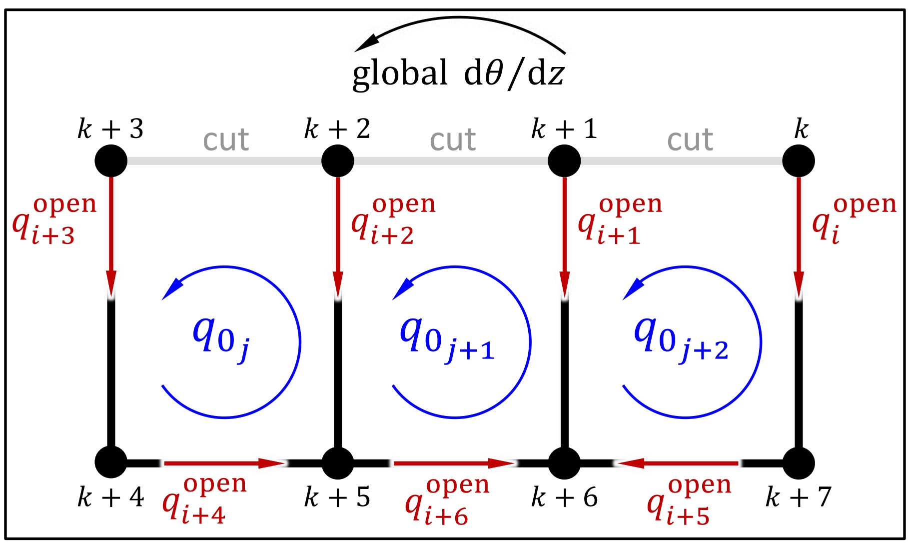
Multi-Cell Sections – Notation

- Remember: a section with n cells will have $n+1$ unknowns:

$$q_{0_1}, q_{0_2} \dots q_{0_n} \text{ and } \mathrm{d}\theta/\mathrm{d}z$$







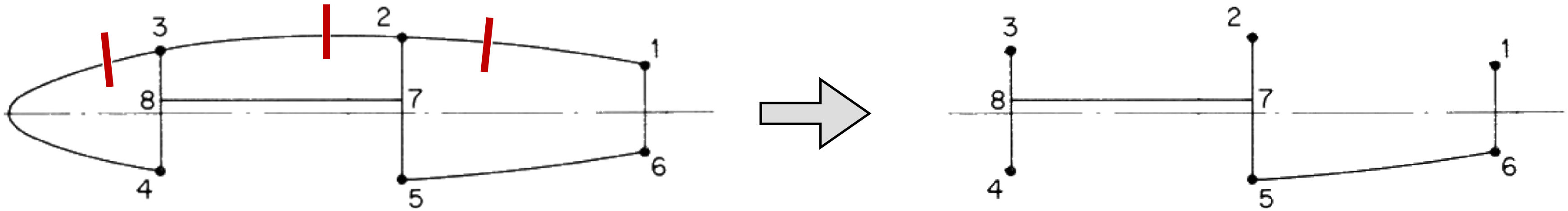
and so on...

Note:

- Numbering of booms and walls are arbitrary – shown here only to highlight the meaning of the indices (i, j, k)
- The orientations of 'open' shear flow vectors q_i^{open} are also arbitrary – to be chosen wisely (*i.e.* 'flowing from known values')
- In multi-cell sections the assumed sense of the constants q_{0_j} should be consistent – CCW sense is most commonly used

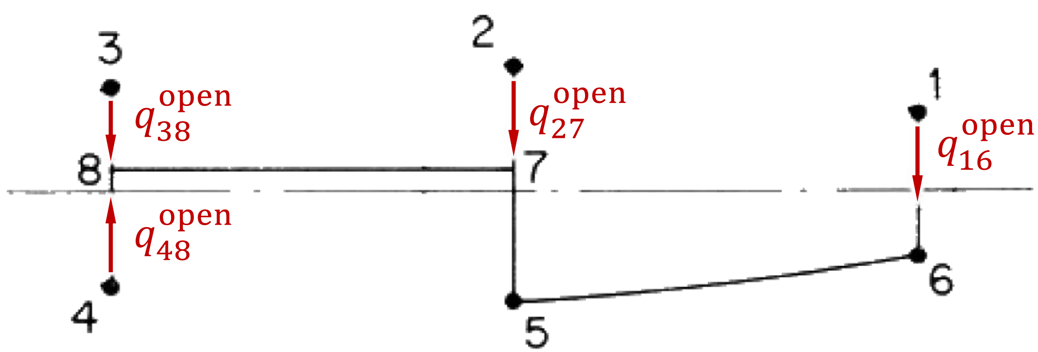
Example 3.5 – Step 1 – Cut Cells Open

- ‘Cut’ **each closed cell once** – either top or bottom skin



- Walls which are cut will have **zero open shear flow** ($q_i^{\text{open}} = 0$) and can be **removed from the section** (for now)

- Use new ‘free edges’ as starting points for the next q_i^{open}, assuming shear flow directions as if ‘flowing from’ free edges



- Note: order of indices shows the direction, *i.e.* $q_{16}^{\text{open}} = -q_{61}^{\text{open}}$

Example 3.5 – Step 2 – 'Open' Shear Flow

- Open-section shear flow for idealised sections:

$$-q_s^{\text{open}} = \left(\frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \sum_{k=1}^{n_B} x_k\, A_k + \left(\frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \sum_{k=1}^{n_B} y_k\, A_k$$

k = boom index

- Booms are symmetric through the thickness $\therefore\ I_{xy} = 0$
- No horizontal shear force is applied $\therefore\ S_x = 0$
- The equation becomes:

y-coordinate of boom

$$q_s^{\text{open}} = -\frac{S_y}{I_{xx}} \sum_{k=1}^{n_B} y_k\, A_k$$

boom area

- Remember: I_{xx} is calculated from boom properties only

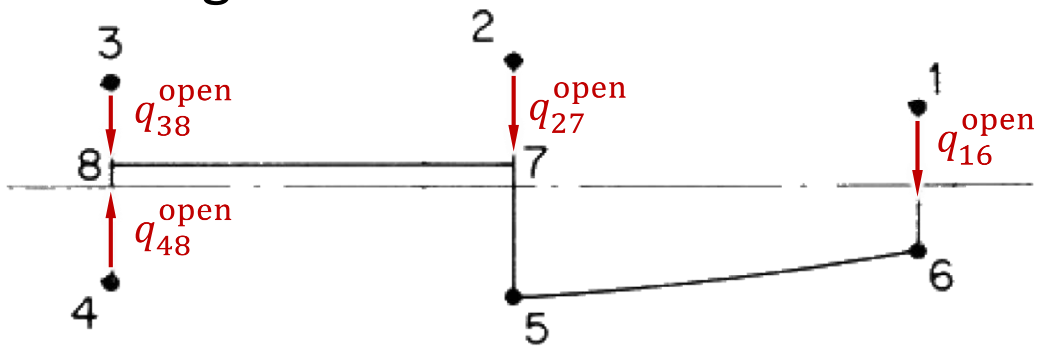
$$I_{xx} = \sum_{k=1}^{n_B} (A_k)(y_k)^2$$

$$\begin{aligned} I_{xx} &= 2 \times (2580\text{ mm}^2)(165\text{ mm})^2 \\ &+ 2 \times (3880\text{ mm}^2)(230\text{ mm})^2 \\ &+ 2 \times (3230\text{ mm}^2)(200\text{ mm})^2 \end{aligned}$$

$$I_{xx} = 809{,}385 \times 10^3\text{ mm}^4$$

Example 3.5 – Step 2 – 'Open' Shear Flow (1/3)

- Starting from free edges:



$$\frac{S_y}{I_{xx}} = \frac{86{,}600\ \mathrm{N}}{809,385 \times 10^3\ \mathrm{mm}^4} = 1.072 \times 10^{-4}\ \mathrm{N/mm}^4$$

Starting from free edge so initial value is **zero**

$$q_{16}^{open} = 0 - \frac{S_y}{I_{xx}} A_1\, y_1 = -\left(1.072 \times 10^{-4} \frac{\mathrm{N}}{\mathrm{mm}^4}\right)(2580\ \mathrm{mm}^2)(165\ \mathrm{mm}) = -45.65 \frac{\mathrm{N}}{\mathrm{mm}}$$

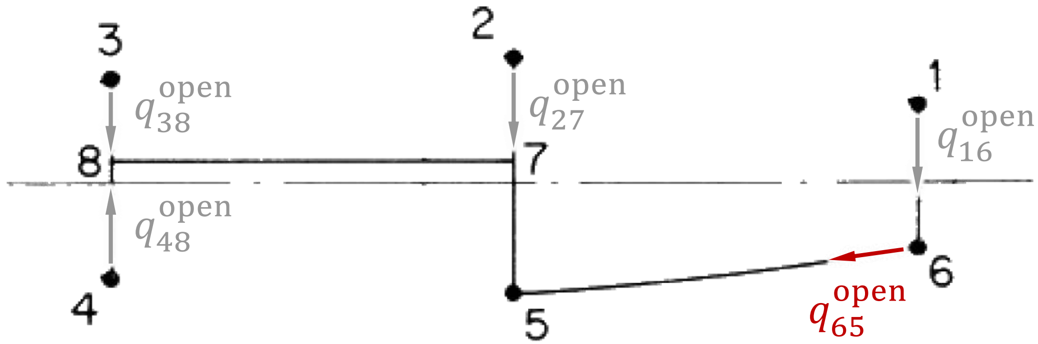
$$q_{27}^{open} = 0 - \frac{S_y}{I_{xx}} A_2\, y_2 = -\left(1.072 \times 10^{-4} \frac{\mathrm{N}}{\mathrm{mm}^4}\right)(3880\ \mathrm{mm}^2)(230\ \mathrm{mm}) = -95.70 \frac{\mathrm{N}}{\mathrm{mm}}$$

$$q_{38}^{open} = 0 - \frac{S_y}{I_{xx}} A_3\, y_3 = -\left(1.072 \times 10^{-4} \frac{\mathrm{N}}{\mathrm{mm}^4}\right)(3230\ \mathrm{mm}^2)(200\ \mathrm{mm}) = -69.28 \frac{\mathrm{N}}{\mathrm{mm}}$$

$$q_{48}^{open} = 0 - \frac{S_y}{I_{xx}} A_4\, y_4 = -\left(1.072 \times 10^{-4} \frac{\mathrm{N}}{\mathrm{mm}^4}\right)(3230\ \mathrm{mm}^2)(-200\ \mathrm{mm}) = 69.28 \frac{\mathrm{N}}{\mathrm{mm}}$$

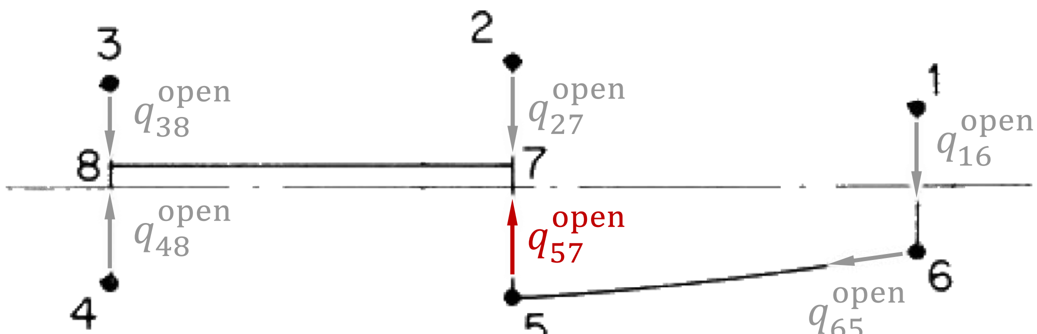
Example 3.5 – Step 2 – 'Open' Shear Flow (2/3)

- Carrying on:



Known value

$$q_{65}^{\text{open}} = q_{16}^{\text{open}} - \frac{S_y}{I_{xx}} A_6\, y_6 = -45.65\,\frac{\text{N}}{\text{mm}} - \left(\frac{S_y}{I_{xx}}\right)(2580\ \text{mm}^2)(-165\ \text{mm}) = 0$$

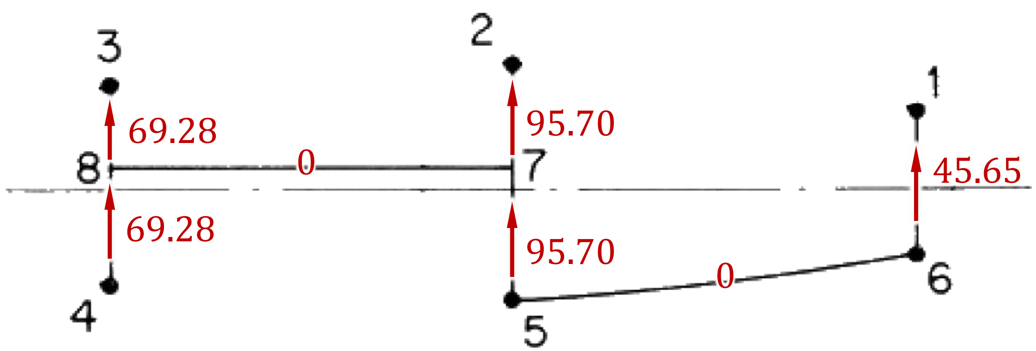


Known value

$$q_{57}^{\text{open}} = q_{65}^{\text{open}} - \frac{S_y}{I_{xx}} A_5\, y_5 = 0 - \left(\frac{S_y}{I_{xx}}\right)(3880\ \text{mm}^2)(-230\ \text{mm}) = 95.70\,\frac{\text{N}}{\text{mm}}$$

Example 3.5 – Step 2 – ‘Open’ Shear Flow (3/3)

- Now all ‘open’ shear flow values are known
- Vectors can now be drawn in their correct orientations
  - A negative value means that the assumed orientation needs to be flipped
- Therefore:



Example 3.5 – Step 3 – 'Closed' Shear Flow Equations

- n equations: twist rate of each cell

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2\,A_j\,G_{\mathrm{ref}}} \oint_j q_s^{\mathrm{closed}} \frac{\mathrm{d}s}{t_{\mathrm{eff}}} \quad \Rightarrow \quad \oint_j q_s^{\mathrm{closed}} \frac{\mathrm{d}s}{t_{\mathrm{eff}}} - \left(2\,A_j\,G_{\mathrm{ref}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}z} = 0$$

$$\sum_{i\in j}\left[q_i^{\mathrm{closed}} \frac{b_i}{(t_{\mathrm{eff}})_i}\right] - \left(2\,A_j\,G_{\mathrm{ref}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}z} = 0$$

$$\boxed{\sum_{i\in j}\left[\left(q_i^{\mathrm{open}} + q_{0_j}\right) \frac{b_i}{(t_{\mathrm{eff}})_i}\right] - \left(2\,A_j\,G_{\mathrm{ref}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}z} = 0}$$

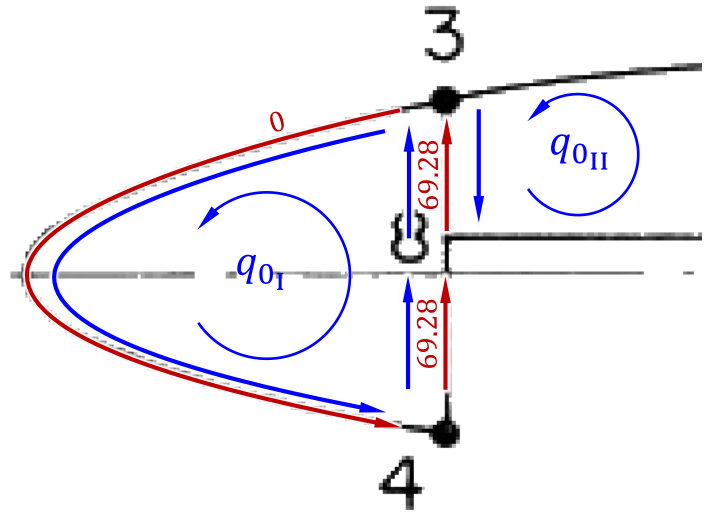
$j = 1 \ldots n$

- Plus one equation: balance of moments

$$S_y\,e_x - S_x\,e_y = \int q_s\,r_s\,\mathrm{d}s \quad \Rightarrow \quad \boxed{S_y\,e_x - S_x\,e_y = \underbrace{\sum_{\mathrm{all}\ i}\left(q_i^{\mathrm{open}}\,r_i\,b_i\right)}_{\text{'open' flow along walls}} + \underbrace{\sum_{j=1}^{n}\left(2\,A_j\,q_{0_j}\right)}_{\text{'closed' flow around cells}}}$$

Example 3.5 – Step 3 – 'Closed' Shear Flow Equations

- Cell I:



$$\sum_{i \in j} \left[\left(q_i^{\text{open}} + q_{0_j} \right) \frac{b_i}{(t_{\text{eff}})_i} \right] - \left(2\, A_j\, G_{\text{ref}}\right) \frac{\mathrm{d}\theta}{\mathrm{d}z} = 0$$

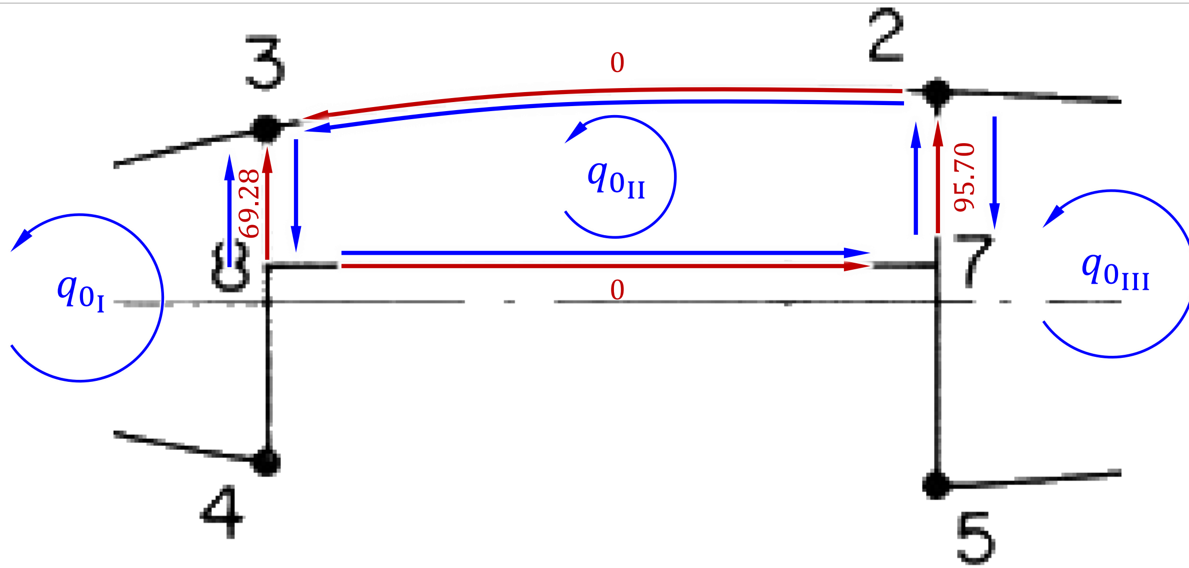
$$\left(0 + q_{0_{\mathrm{I}}}\right) \frac{b_{34}}{(t_{\text{eff}})_{34}} + \left(q_{48}^{\text{open}} + q_{0_{\mathrm{I}}}\right) \frac{b_{48}}{(t_{\text{eff}})_{48}} + \left(q_{83}^{\text{open}} + q_{0_{\mathrm{I}}} - q_{0_{\mathrm{II}}}\right) \frac{b_{83}}{(t_{\text{eff}})_{83}} - \left(2\, A_{\mathrm{I}}\, G_{\text{ref}}\right) \frac{\mathrm{d}\theta}{\mathrm{d}z} = 0$$

$$\left(q_{0_{\mathrm{I}}}\right)(1083.74) + \left(69.28 + q_{0_{\mathrm{I}}}\right)(94.70) + \left(69.28 + q_{0_{\mathrm{I}}} - q_{0_{\mathrm{II}}}\right)(56.82) - 2(2.65\ 10^5)(27.6\ 10^3) \frac{\mathrm{d}\theta}{\mathrm{d}z} = 0$$

$$\boxed{1083.74\, q_{0_{\mathrm{I}}} - 56.82\, q_{0_{\mathrm{II}}} - 1.46 \times 10^{10} \frac{\mathrm{d}\theta}{\mathrm{d}z} = -10\,488}$$

Example 3.5 – Step 3 – ‘Closed’ Shear Flow Equations

- Cell II:



$$\sum_{i\in j}\left[\left(q_i^{\mathrm{open}}+q_{0_j}\right)\frac{b_i}{(t_{\mathrm{eff}})_i}\right]-\left(2\,A_j\,G_{\mathrm{ref}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}z}=0$$

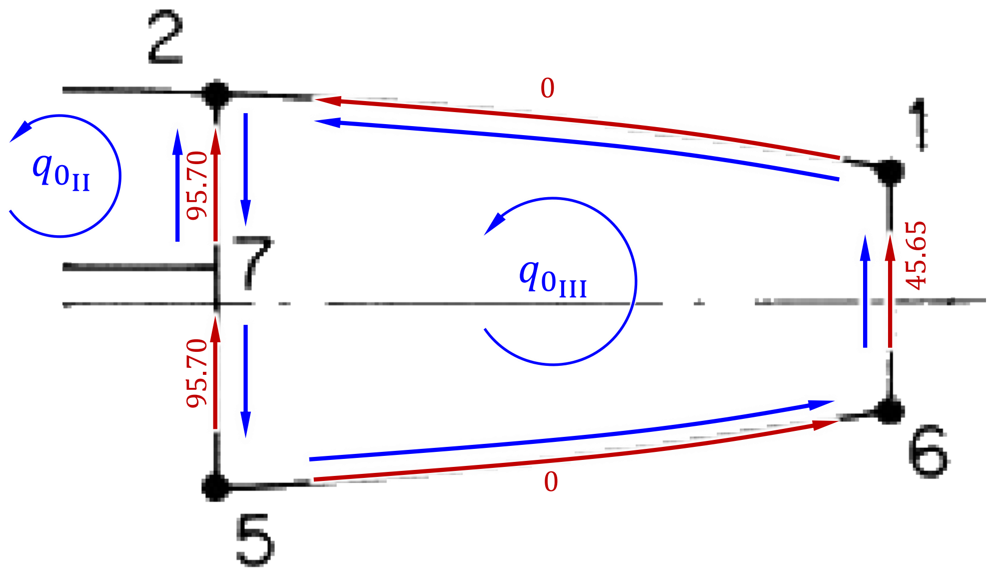
$$\left(q_{0_{\mathrm{II}}}\right)\frac{b_{23}}{(t_{\mathrm{eff}})_{23}}+\left(q_{38}^{\mathrm{open}}-q_{0_{\mathrm{I}}}+q_{0_{\mathrm{II}}}\right)\frac{b_{38}}{(t_{\mathrm{eff}})_{38}}+\left(q_{0_{\mathrm{II}}}\right)\frac{b_{87}}{(t_{\mathrm{eff}})_{87}}+\left(q_{72}^{\mathrm{open}}+q_{0_{\mathrm{II}}}-q_{0_{\mathrm{III}}}\right)\frac{b_{72}}{(t_{\mathrm{eff}})_{72}}-\left(2\,A_{\mathrm{II}}\,G_{\mathrm{ref}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}z}=0$$

$$\left(q_{0_{\mathrm{II}}}\right)(781.60)+\left(-69.28-q_{0_{\mathrm{I}}}+q_{0_{\mathrm{II}}}\right)(56.82)+\left(q_{0_{\mathrm{II}}}\right)(347.00)+\left(95.70+q_{0_{\mathrm{II}}}-q_{0_{\mathrm{III}}}\right)(68.18)-2(2.13\;10^5)(27.6\;10^3)\frac{\mathrm{d}\theta}{\mathrm{d}z}=0$$

$$\boxed{-56.82\,q_{0_{\mathrm{I}}}+1253.59\,q_{0_{\mathrm{II}}}-68.18\,q_{0_{\mathrm{III}}}-1.18\times10^{10}\frac{\mathrm{d}\theta}{\mathrm{d}z}=-2\,589}$$

Example 3.5 – Step 3 – 'Closed' Shear Flow Equations

- Cell III:



$$\sum_{i \in j}\left[\left(q_i^{\text{open}} + q_{0_j}\right)\frac{b_i}{(t_{\text{eff}})_i}\right] - \left(2\,A_j\,G_{\text{ref}}\right)\frac{d\theta}{dz} = 0$$

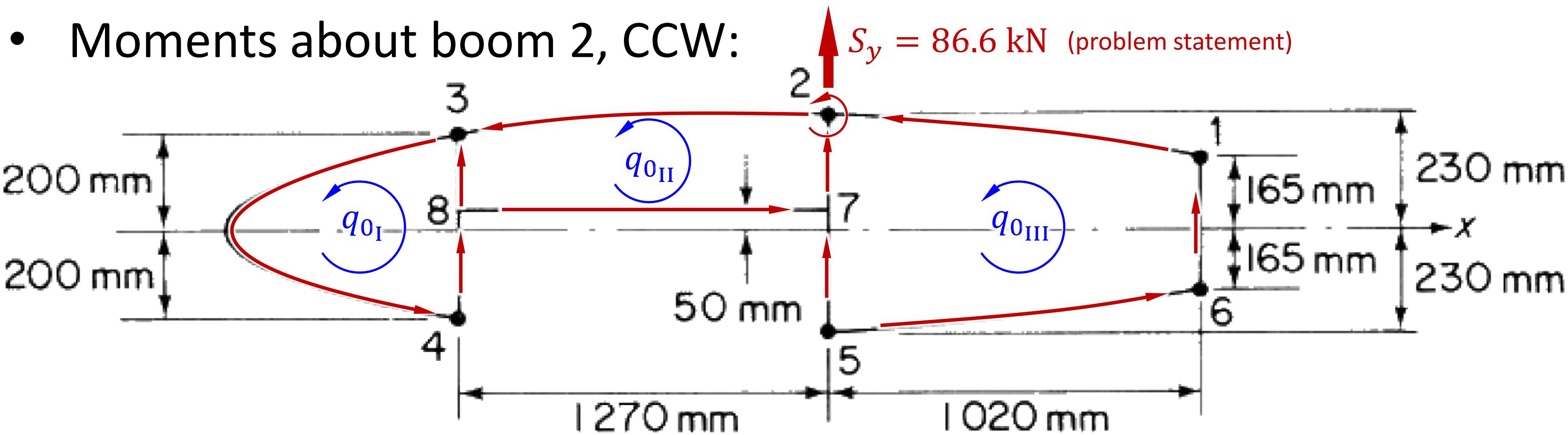
$$\left(q_{0_{III}}\right)\frac{b_{12}}{(t_{\text{eff}})_{12}} + \left(q_{27}^{\text{open}} - q_{0_{II}} + q_{0_{III}}\right)\frac{b_{27}}{(t_{\text{eff}})_{27}} + \left(q_{75}^{\text{open}} + q_{0_{III}}\right)\frac{b_{75}}{(t_{\text{eff}})_{75}} + \left(q_{0_{III}}\right)\frac{b_{56}}{(t_{\text{eff}})_{56}} + \left(q_{61}^{\text{open}} + q_{0_{III}}\right)\frac{b_{61}}{(t_{\text{eff}})_{61}} - \left(2\,A_{III}\,G_{\text{ref}}\right)\frac{d\theta}{dz} = 0$$

$$\left(q_{0_{III}}\right)(838.52) + \left(-95.70 - q_{0_{II}} + q_{0_{III}}\right)(68.18) + \left(-95.70 + q_{0_{III}}\right)(106.06) + \left(q_{0_{III}}\right)(838.52) + \left(45.65 + q_{0_{III}}\right)(202.45) - 2(4.13\ 10^5)(27.6\ 10^3)\frac{d\theta}{dz} = 0$$

$$\boxed{-68.18\,q_{0_{II}} + 2053.75\,q_{0_{III}} - 2.28 \times 10^{10}\frac{d\theta}{dz} = 7426}$$

Example 3.5 – Step 3 – ‘Closed’ Shear Flow Equations

- Moments about boom 2, CCW:



$$S_y \cancelto{0}{e_x} - \cancelto{0}{S_x} \cancelto{0}{e_y} = \sum_{\text{all } i} \left(q_i^{\text{open}}\, r_i\, b_i \right) + \sum_{j=1}^{n} \left(2\, A_j\, q_{0_j} \right)$$

$$\left(q_{61}^{\text{open}}\right)(1020)(330) + \left(q_{38}^{\text{open}}\right)(1270)(150) + \left(q_{84}^{\text{open}}\right)(1270)(250)$$
$$+2(2.65 \times 10^5)\left(q_{0_{\mathrm{I}}}\right) + 2(2.13 \times 10^5)\left(q_{0_{\mathrm{II}}}\right) + 2(4.13 \times 10^5)\left(q_{0_{\mathrm{III}}}\right) = 0$$

$$(45.65)(1020)(330) + (-69.28)(1270)(150) + (-69.28)(1270)(250)$$
$$+2(2.65 \times 10^5)\left(q_{0_{\mathrm{I}}}\right) + 2(2.13 \times 10^5)\left(q_{0_{\mathrm{II}}}\right) + 2(4.13 \times 10^5)\left(q_{0_{\mathrm{III}}}\right) = 0$$

$$\boxed{(5.30 \times 10^5)\, q_{0_{\mathrm{I}}} + (4.26 \times 10^5)\, q_{0_{\mathrm{II}}} + (8.26 \times 10^5)\, q_{0_{\mathrm{III}}} = 1.97 \times 10^7}$$

Example 3.5 – Step 3 – 'Closed' Shear Flow Equations

- Summary:

$$
\begin{array}{lllllll}
1083.74\, q_{0_{\mathrm{I}}} & -56.82\, q_{0_{\mathrm{II}}} & & -1.46\times 10^{10}\,\dfrac{\mathrm{d}\theta}{\mathrm{d}z} & = & -10\,488 \\
-56.82\, q_{0_{\mathrm{I}}} & +1253.59\, q_{0_{\mathrm{II}}} & -68.18\, q_{0_{\mathrm{III}}} & -1.18\times 10^{10}\,\dfrac{\mathrm{d}\theta}{\mathrm{d}z} & = & -2\,589 \\
 & -68.18\, q_{0_{\mathrm{II}}} & +2053.75\, q_{0_{\mathrm{III}}} & -2.28\times 10^{10}\,\dfrac{\mathrm{d}\theta}{\mathrm{d}z} & = & 7\,426 \\
5.30\times 10^{5}\, q_{0_{\mathrm{I}}} & +4.26\times 10^{5}\, q_{0_{\mathrm{II}}} & +8.26\times 10^{5}\, q_{0_{\mathrm{III}}} & & = & 1.97\times 10^{7}
\end{array}
$$

- In matrix form:

$$
\begin{bmatrix} 1083.74 & -56.82 & 0 & -1.46\times 10^{10} \\ -56.82 & 1253.59 & -68.18 & -1.18\times 10^{10} \\ 0 & -68.18 & 2053.75 & -2.28\times 10^{10} \\ 5.30\times 10^{5} & 4.26\times 10^{5} & 8.26\times 10^{5} & 0 \end{bmatrix}
\begin{bmatrix} q_{0_{\mathrm{I}}} \\ q_{0_{\mathrm{II}}} \\ q_{0_{\mathrm{III}}} \\ \mathrm{d}\theta/\mathrm{d}z \end{bmatrix}
=
\begin{bmatrix} -10\,488 \\ -2\,589 \\ 7\,426 \\ 1.97\times 10^{7} \end{bmatrix}
$$

- And:

$$
\begin{bmatrix} q_{0_{\mathrm{I}}} \\ q_{0_{\mathrm{II}}} \\ q_{0_{\mathrm{III}}} \\ \mathrm{d}\theta/\mathrm{d}z \end{bmatrix}
=
\begin{bmatrix} 1083.74 & -56.82 & 0 & -1.46\times 10^{10} \\ -56.82 & 1253.59 & -68.18 & -1.18\times 10^{10} \\ 0 & -68.18 & 2053.75 & -2.28\times 10^{10} \\ 5.30\times 10^{5} & 4.26\times 10^{5} & 8.26\times 10^{5} & 0 \end{bmatrix}^{-1}
\begin{bmatrix} -10\,488 \\ -2\,589 \\ 7\,426 \\ 1.97\times 10^{7} \end{bmatrix}
$$

Example 3.5 – Step 4 – Invert Matrix & Solve

- Using software (*e.g.* Excel or Matlab):

$$\begin{bmatrix} 1083.74 & -56.82 & 0 & -1.46\times10^{10} \\ -56.82 & 1253.59 & -68.18 & -1.18\times10^{10} \\ 0 & -68.18 & 2053.75 & -2.28\times10^{10} \\ 5.30\times10^{5} & 4.26\times10^{5} & 8.26\times10^{5} & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5.43\times10^{-4} & -1.93\times10^{-4} & -2.48\times10^{-4} & 6.02\times10^{-7} \\ -1.93\times10^{-4} & 6.04\times10^{-4} & -1.87\times10^{-4} & 5.15\times10^{-7} \\ -2.48\times10^{-4} & -1.87\times10^{-4} & 2.56\times10^{-4} & 5.59\times10^{-7} \\ -2.18\times10^{-11} & -1.87\times10^{-11} & -2.02\times10^{-11} & 4.88\times10^{-14} \end{bmatrix}$$

- And finally:

$$\begin{bmatrix} q_{0_{\mathrm{I}}} \\ q_{0_{\mathrm{II}}} \\ q_{0_{\mathrm{III}}} \\ \mathrm{d}\theta/\mathrm{d}z \end{bmatrix} = \begin{bmatrix} 4.84\ \mathrm{N/mm} \\ 9.25\ \mathrm{N/mm} \\ 16.02\ \mathrm{N/mm} \\ 1.09\times10^{-6}\ \mathrm{rad/mm} \end{bmatrix}$$

Example 3.5 – Step 5 – Find True Shear Flow

- The 'closed' (*i.e.* 'true') shear flow vectors are then:

$$q_{12}^{\text{closed}} = q_{12}^{\text{open}} + q_{0_{\text{III}}} \qquad q_{87}^{\text{closed}} = q_{87}^{\text{open}} + q_{0_{\text{II}}} \qquad q_{83}^{\text{closed}} = q_{83}^{\text{open}} + q_{0_{\text{I}}} - q_{0_{\text{II}}}$$

$$q_{23}^{\text{closed}} = q_{23}^{\text{open}} + q_{0_{\text{II}}} \qquad q_{75}^{\text{closed}} = q_{75}^{\text{open}} + q_{0_{\text{III}}} \qquad q_{72}^{\text{closed}} = q_{72}^{\text{open}} + q_{0_{\text{II}}} - q_{0_{\text{III}}}$$

$$q_{34}^{\text{closed}} = q_{34}^{\text{open}} + q_{0_{\text{I}}} \qquad q_{56}^{\text{closed}} = q_{56}^{\text{open}} + q_{0_{\text{III}}}$$

$$q_{48}^{\text{closed}} = q_{48}^{\text{open}} + q_{0_{\text{I}}} \qquad q_{61}^{\text{closed}} = q_{61}^{\text{open}} + q_{0_{\text{III}}}$$

- Plotting these on the cross-section (in units of N/mm):