Signals 1.3 The complex exponential

Context

- Many of the mathematical techniques and ideas we touch on in this lecture series come from a well–developed field

  - Linear Time Invariant (LTI) systems theory

- These techniques are applicable to a wide range of engineering problems and central to control.

- You will have already come across elements of this ‘toolbox’ - Fourier and Laplace transforms, and systems that conform – Mass/spring/dampers in vibrations, electrical circuits etc.

Complex exponential signals

- A useful place to start our analysis is with the complex exponential;

$$f(t) = Ce^{at}$$

Where C and A can be real or complex numbers

- It is useful because it can describe many common signals, including the basis functions used in the frequency and Laplace domains.

Complex exponential

$$f(t) = Ce^{at}$$

- 1) When C is real it is a *scaling* term;
    - If a is real and >0 then *f(t)* is a growing exponential
    - If a is real and <0 then *f(t)* is a decaying exponential
    - If a is real and =0 then *f(t)* is a constant signal

Complex exponential

$$f(t) = Ce^{at}$$

- 2) If a is complex and purely imaginary then:

$$f(t) = Ce^{j\omega t}$$

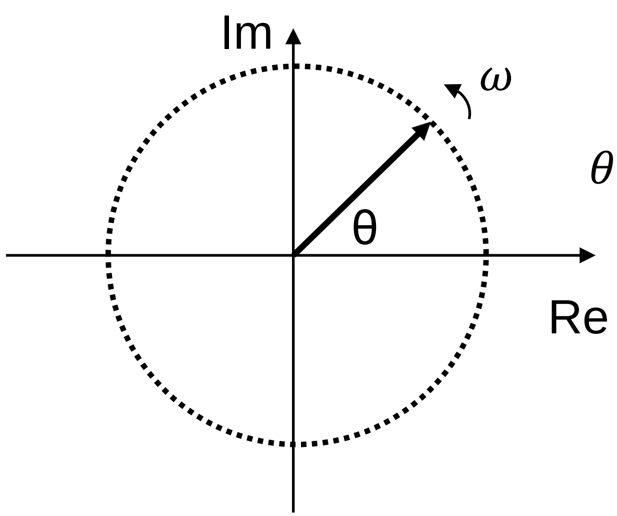
and from Euler's identity we know that ;

$$f(t) = Ccos(\omega t) + jCsin(\omega t)$$

Complex exponential continued

$$f(t) = C\cos(\omega t) + jC\sin(\omega t)$$

- *So what is this?*
  - *It is a vector on the complex plane that rotates (it is a Phasor)*



$\theta = \omega t$

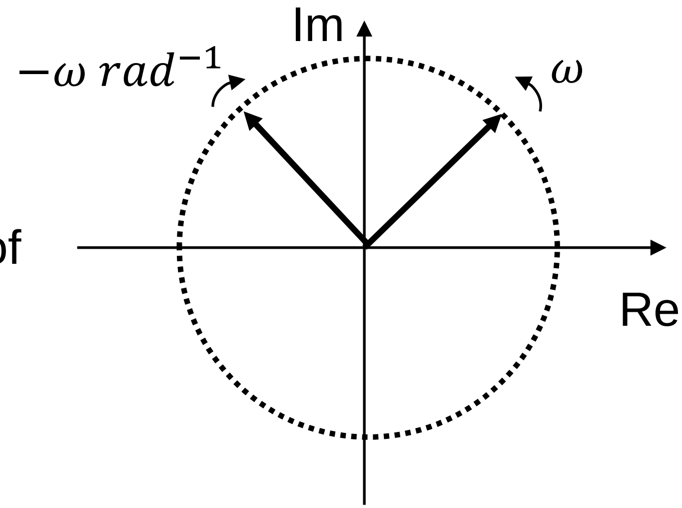
$e^{j\omega t}$ Rotation CCW

$e^{-j\omega t}$ Rotation CW

Complex exponential continued

- *So if a rotation is described by* $e^{j\omega t}$*, what is a sine/cosine describing?*
- *We know that:* $cos(\omega t) = Re(e^{j\omega t})$
- *And in general:* $Re(z) = \frac{1}{2}(z + \bar{z})$
- *So:*

$$cos(\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$



Hence a sine (or cosine) is actually the sum of two phasors of equal magnitude rotating in opposite directions!

Complex exponential continued

- *3) How about when ‘a’ has real and imaginary parts?*

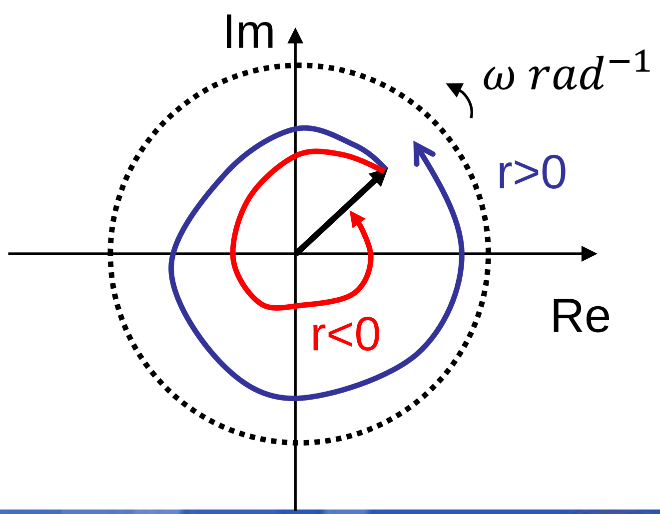
$$f(t) = Ce^{(r+j\omega)t}$$

- *rewrite as:* $f(t) = C(e^{rt}e^{j\omega t})$

Phasor with angular velocity ω

Exponential increase or decrease

Depending on the sign of ‘r’ the signal is either a growing sinusoid or a shrinking sinusoid – they look like spirals on the complex plane.



Complex exponential continued

4) How about when 'C' and 'a' have real and imaginary parts?

- *write complex 'C' in polar form* $C = |C|e^{j\theta}$
- *Then* $f(t) = |C|e^{j\theta}e^{(r+j\omega)t}$
- *rearrange* $f(t) = |C|e^{rt}e^{j(\omega t+\theta)}$

The most general form of the complex exponential introduces a phase offset, θ.

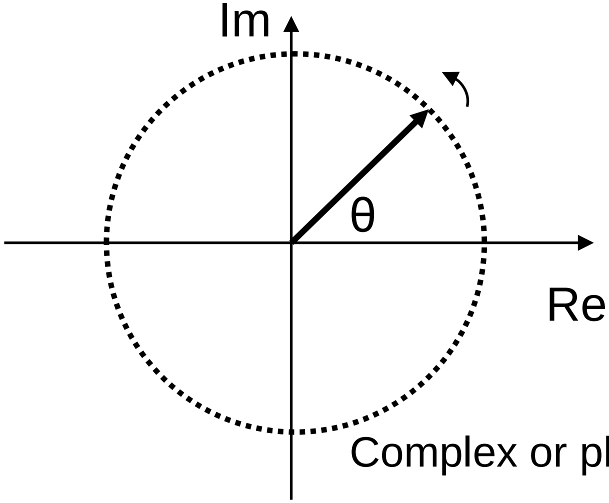
Complex exponential continued

5) We can multiply ω by an integer k:

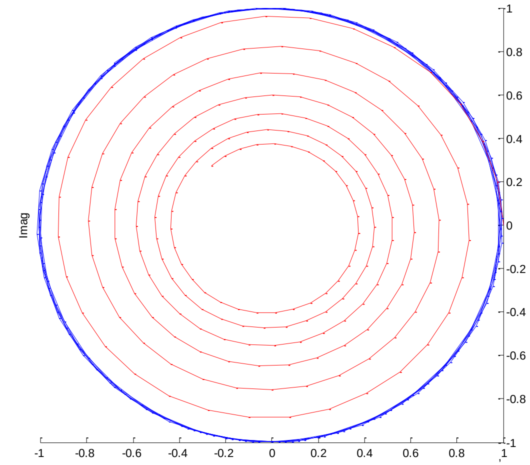
$$f(t) = e^{jk\omega t}$$ Where k = 1,2,3,4…etc..

- *This results in a set of harmonics - signals with frequencies an integer multiple of each other.*

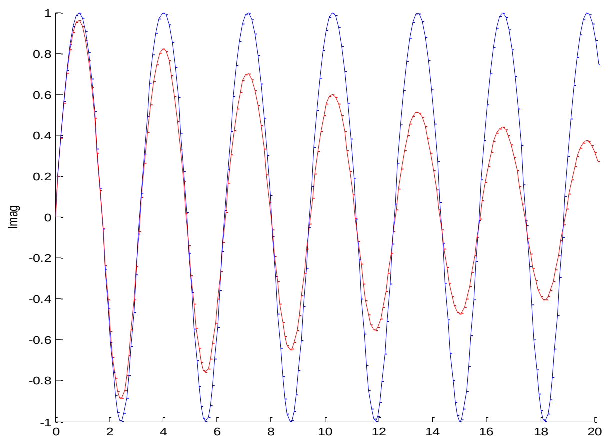
Visualising signals



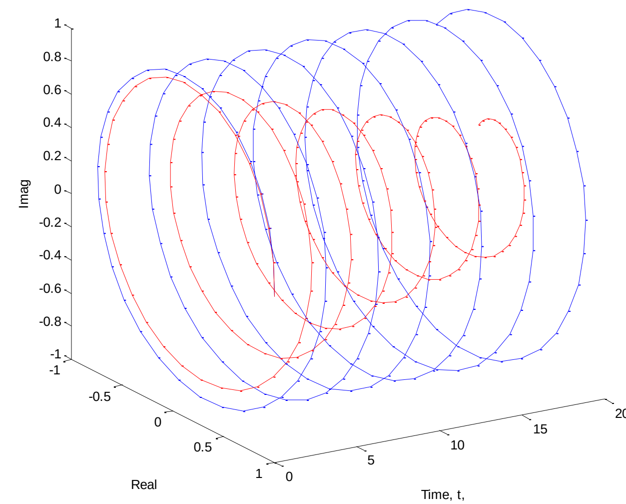
Complex or phase plane



Sinusoid (blue) and decaying sinusoid (red) on phase plane



Sinusoid (blue) and decaying sinusoid (red) in time



Analytic signals

- A signal that has both real and imaginary components defined is known as an ‘analytic signal’.

- But why do we make life more complex? *(pun intended)*
  - *Sinusoids are quite difficult to define on their own, but when thought of as the projection of a phasor along one axis they are simple – that might indicate something more fundamental about a phasor (i.e. rotation) than a sinusoid?*

- In practice, consider the energy in a mass/spring oscillator – the energy in the mass follows the real part; the energy in the spring follows the imaginary part and the energy in the system follows the analytical signal.

Analytic signals

- So analytic signals can be thought of as representing ‘real’ phenomena, but also mathematically they are arguably more elegant….

- That all said, we don’t use the analytic representation very often! But they are important consequences that need to be borne in mind when we get to transforms.

- The real and imaginary parts of an analytic signal are linked by the ‘Hilbert transform’, and this transform can be used to reconstruct an analytic signal from a real valued signal. The Hilbert transform of a sine is a cosine.

Fourier

- Fourier is credited with the idea that time domain waveforms can be represented by summations of sinusoids (his basis functions).
  - *But when we look at the maths later you will see that the basis function are analytic signals.*

- A signal made up of summed sinusoids is called a Fourier series.

- Since sinusoids have a unique frequency component hence the concept reveals the mapping between time and frequency domains.

- This works well for periodic signals that continue indefinitely or signals which we can define as constant over some time.

- Fourier series are very useful for signals

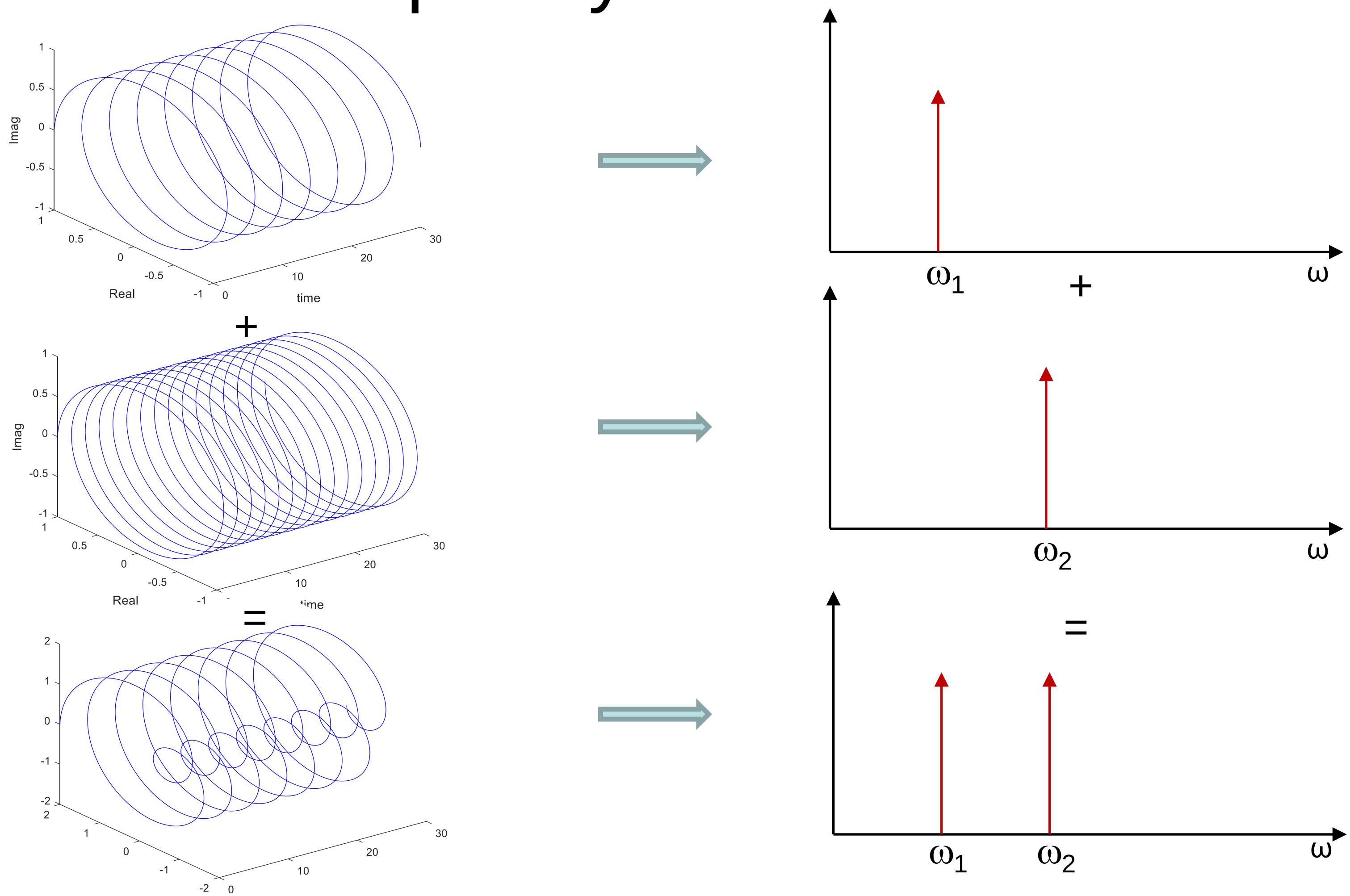
Laplace

- Laplace went one stage further than Fourier and represented his time domain functions with sinusoidal basis functions which could grow or shrink in amplitude.
  - But, as before the maths uses analytic representations!

- The physical significance of this is that the ‘S’ domain can model systems that gain or lose energy over time – e.g. the free response of a mass/spring/damper.

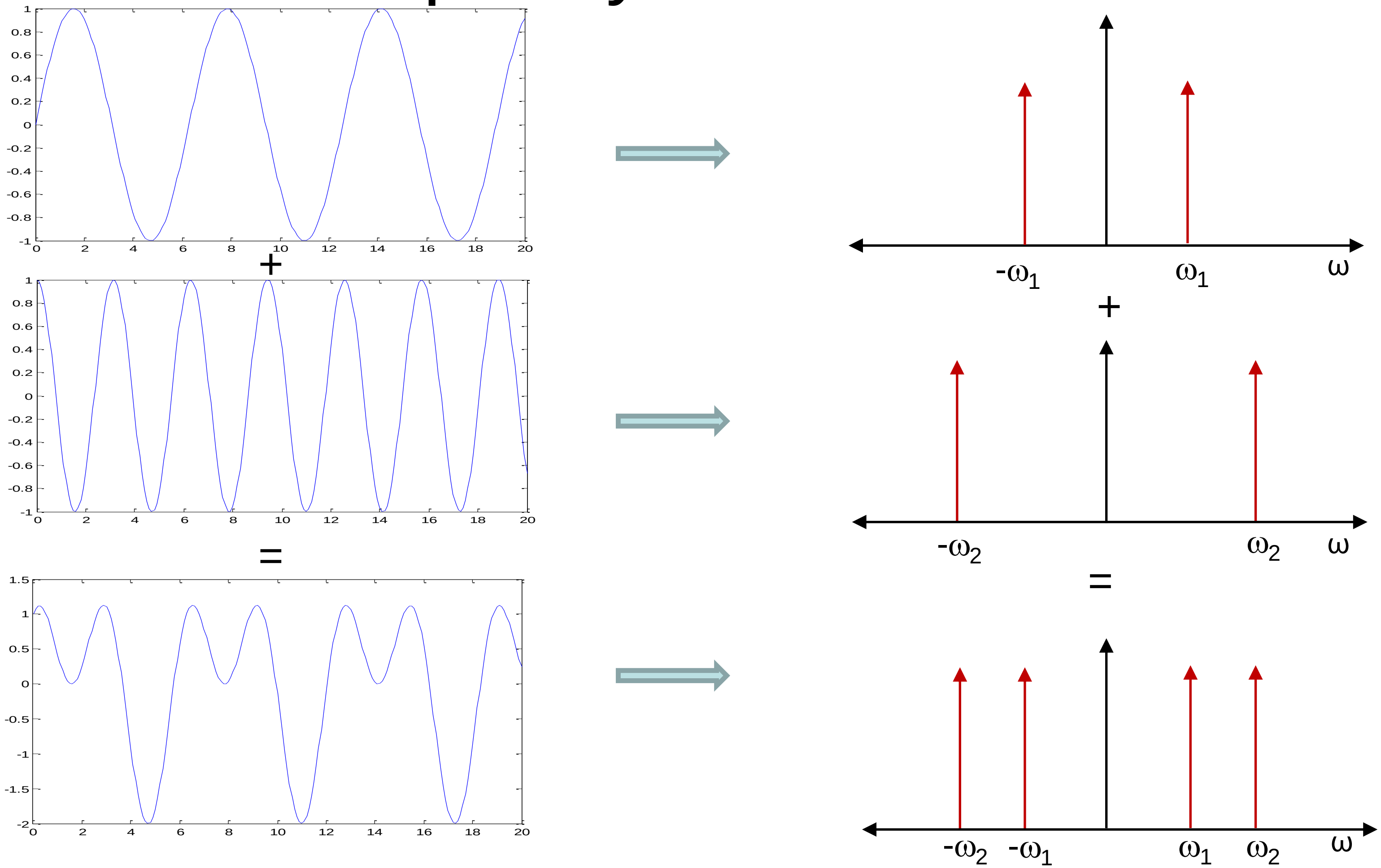
- This idea is central to the control theory you will study next term – stable systems conserve or lose ‘energy’ over time, unstable systems gain energy.

Time and frequency domains



Although both time and frequency domains are continuous, a phasor maps onto a particular frequency

Time and frequency domains



When we consider a real valued signal we don’t know if it is a phasor going CW or CCW, producing + & - ω

Questions?

In case of questions on the theory part of this unit, please do not hesitate to ask them.

If questions popped up when revising, you can contact me at this email address:

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or drop in my office, in **room 0.56** in Queen's Building.