7. Tying it all together

In 1D we know how integration and differentiation relate: 'the fundamental theorem of calculus'. How are differential and integral vector calculus related? 1) Integration can help with the physical interpretation of div and curl. 2) The 'fundamental theorem of calculus' grown up = Stokes' theorem + Gauss' theorem.

7.1 Fundamental Theorems of vector calculus

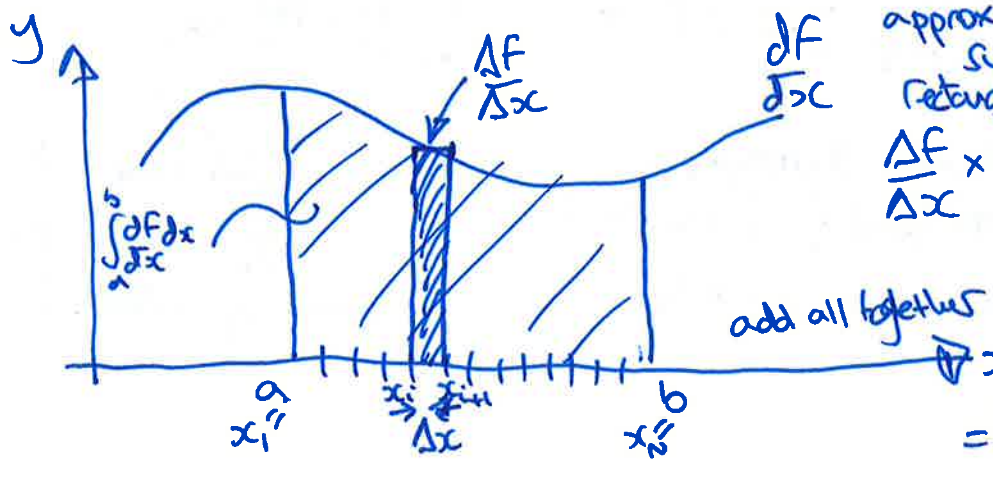
Fundamental theorem of calculus (one dimension)

Recall from single variable calculus that integration and differentiation are linked:

1D Fund. thm of calculus.

$$\boxed{\int_a^b \frac{dF}{dx}dx = F(b) - F(a).}$$

In other words, the integral of a derivative gives the original function. In this chapter, we will consider related results for vector calculus.



approx to integral sum of area of rectangles

$\frac{\Delta F}{\Delta x} \times \Delta x = \Delta F$

$= F(x_{i+1}) - F(x_i)$

$\sum_{i=1}^{N} \frac{\Delta F}{\Delta x} \Delta x$

$= \sum_{i=1}^{N} \Delta F = \sum_{i=1}^{N} F(x_{i+1}) - F(x_i)$

The fundamental theorem of calculus begs the question of whether there are related results for vector calculus. In fact there are many such results, but two are especially important. These respectively link volume integrals to surface integrals, and surface integrals to curve integrals.

cancellation of internal terms to just leave boundary terms

$\cancel{F(x_2)} - F(x_1) +$

$F(x_3) - \cancel{F(x_2)} + \dots$

$+ \dots + F(x_N) - \cancel{F(x_{N-1})}$

$= F(x_N) - F(x_1)$

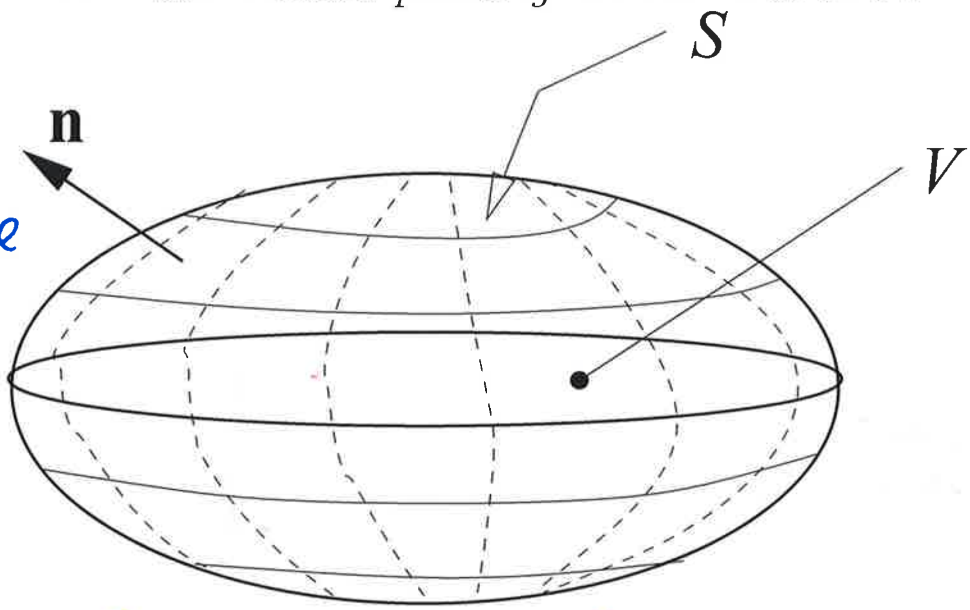
$= F(b) - F(a)$

Gauss' Divergence Theorem *If S is a* **closed** *surface bounding a volume V and $\boldsymbol{F}$ is a vector field then*

$$\boxed{\int\!\!\int\!\!\int_V \boldsymbol{\nabla}\cdot\boldsymbol{F}\, dV := \int\!\!\int\!\!\int_V \mathrm{div}\boldsymbol{F}\, dV = \int\!\!\int_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}} dA =: \int\!\!\int_S \boldsymbol{F}\cdot d\mathbf{A}.}$$

where $\hat{\boldsymbol{n}}$ is the outward *pointing unit normal to V.*

closed surface S enclosing a volume V



$$\iiint_V \underline{\nabla}\cdot\underline{F}\, dV = \iint_S \underline{F}\cdot d\underline{A}$$

total divergence of vector field $\underline{F}$ integrated over a volume V

total flux of $\underline{F}$ through the boundary surface S to V

Worked example 7.1 Verify Gauss' Divergence Theorem for

$$\int\!\!\int_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}} dA, \quad \text{where} \quad \boldsymbol{F} = 4xz\boldsymbol{i} - y^2\boldsymbol{j}. = (4xz, -y^2, 0)$$

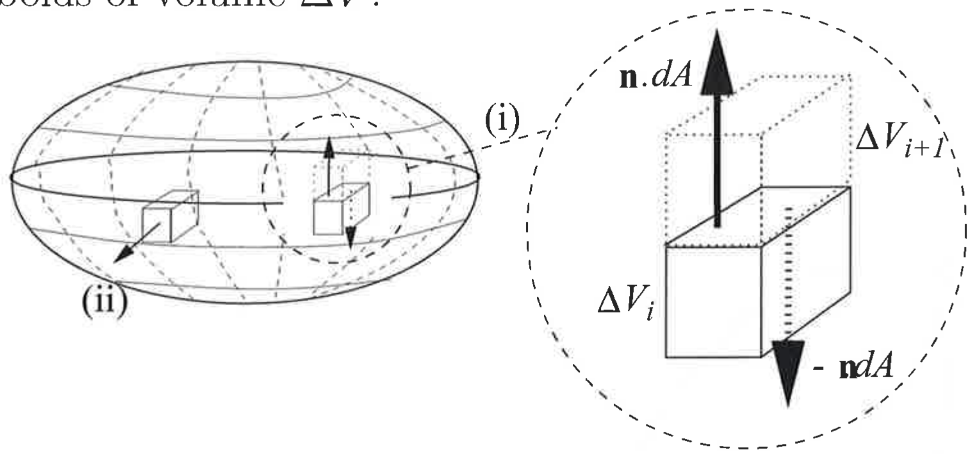
and S is the surface of the unit cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

7.2 The Divergence Theorem and the physical meaning of div $\boldsymbol{F}$

Sketch of proof of the divergence theorem.

$$\int\!\!\int\!\!\int_V \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dV = \int\!\!\int_S \boldsymbol{F}\cdot d\mathbf{A}.$$

Let $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, and let us divide the volume up into N little cuboids of volume ΔV.

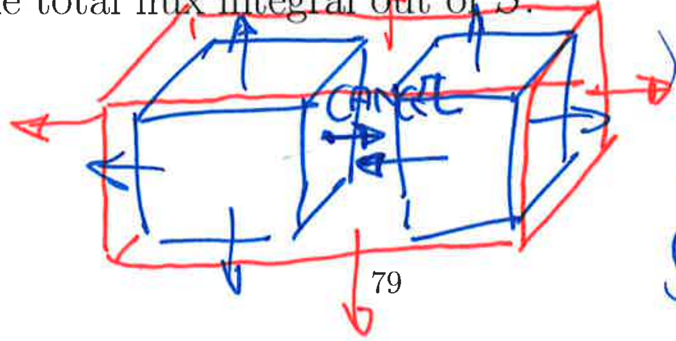


The proof follows the lines:

1. using small volumes ΔV_i with 6 surfaces ΔS oriented along the x, y and z axes, by basic algebra and the fundamental theorem of calculus you can show $\int\!\!\int\!\!\int_{\Delta V} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dV = \int\!\!\int_{\Delta S} \boldsymbol{F}\cdot d\mathbf{A}$.
2. the total volume integral sums up the volumes in each cuboid

$$\int\!\!\int\!\!\int_V \text{div}\,\boldsymbol{F}\,dV = \sum_{i=1}^{N}\int\!\!\int\!\!\int_{\Delta V_i} \text{div}\,\boldsymbol{F}\,dV$$

3. the surfaces of each ΔV_i are either: **(i)** in the interior of V or **(ii)** form part of the outside surface S. In case (i) the flux integral (out of ΔV_i) is exactly canceled by the flux integral out of adjacent faces of the neighbouring cuboids. In case (ii), all these contributions sum to the total flux integral out of S.



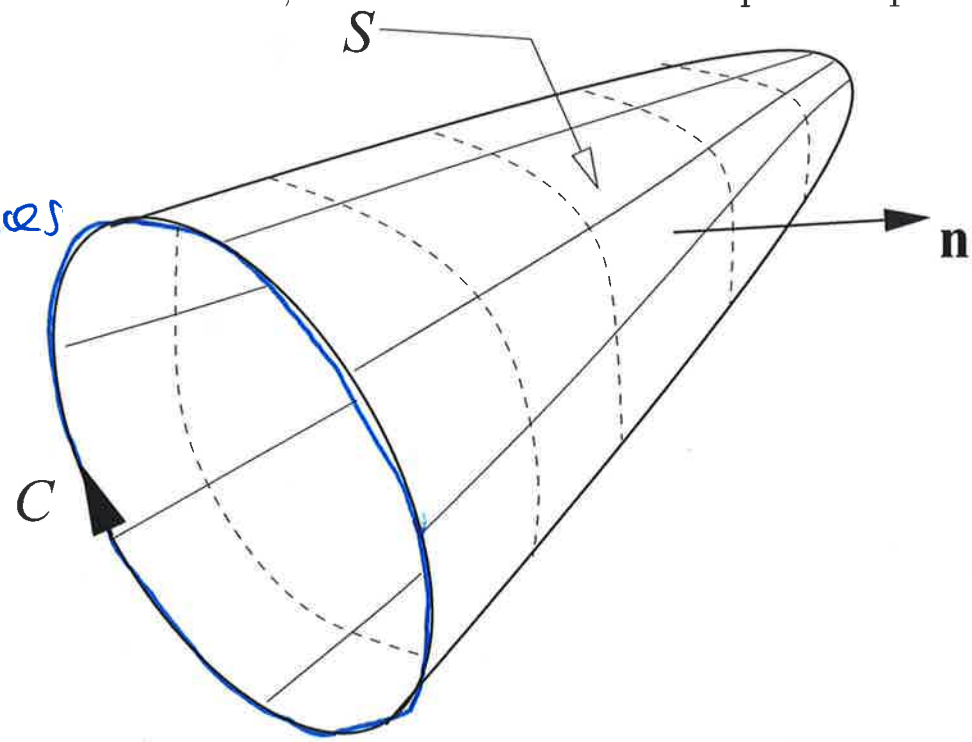
Flux Cancellation on internal surfaces to just leave flux through outer surface

Stokes theorem *Let C be a closed curve and S be an* **open** *surface with C as its boundary, then for any vector field $\boldsymbol{F}$ we have*

$$\boxed{\int\!\!\int_S \nabla\times\boldsymbol{F}\cdot d\mathbf{A} = \int\!\!\int_S (\mathrm{curl}\boldsymbol{F})\cdot d\mathbf{A} = \int\!\!\int_S (\mathrm{curl}\boldsymbol{F})\cdot\hat{\boldsymbol{n}}dA = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r}.}$$

(NOTE the question of orientation. If we traverse C in the opposite direction, we get an answer with the opposite sign. The sign convention is that C is oriented anti-clockwise when looking at the surface from above, and that the normal $\boldsymbol{n}$ points upwards.)

open surfaces have a boundary



In both Theorems, we require S to be a 'nice' surface – piecewise smooth – and $\boldsymbol{F}$ and its first partial derivatives to be continuous and bounded.

$$\iint_S \underline{\nabla}\times\underline{F}\cdot d\underline{A} = \oint_C \underline{F}\cdot d\underline{r}$$

total flux integral of $\underline{\nabla}\times\underline{F}$ through a surface — circulation (line integral) of $\underline{F}$ around a loop C

Worked example 7.2 *Verify Stokes' Theorem for*

$$\boldsymbol{F} = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k} = (y, z, x)$$

and S the paraboloid

$$z = f(x, y) = 1 - (x^2 + y^2), \quad z \geq 0.$$

Applications of the Divergence Theorem

1. **The physical meaning of div F.**

   If we take a small cuboid of volume ΔV and bounding surface S, and let $\Delta V \to 0$, then we have

   flux $$\int\int_S \boldsymbol{F} \cdot d\hat{\boldsymbol{A}} = \int\int\int_{\Delta V} \text{div } \boldsymbol{F} dV \approx \text{div } \boldsymbol{F}(\boldsymbol{r}_i)\Delta V$$

   where $\boldsymbol{r}_i$ is the centre of the cuboid. if ΔV is small enough so that $\underline{\nabla}\cdot\underline{F}$ approx. const.

   Hence

   $$\boxed{\underline{\nabla}\cdot\underline{F} = \text{div } \boldsymbol{F}(\boldsymbol{r}_i) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int\int_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dA}$$

   In other words, the divergence at a point is the amount of outward flux at that point per unit volume. i.e. 'The amount that the vector field is pulling apart small particles.'

   $$\underline{\nabla}\cdot\underline{F} \simeq \frac{1}{\Delta V}\iint_S \underline{F}\cdot d\underline{A}$$

2. **Use to evaluate 'difficult' surface integrals** hard surface integrals → easy volume integrals [or vice versa]

   (see Question 2 on Example Sheet 7).

3. **Fluid mechanics** Consider a steady fluid flow with velocity $\boldsymbol{v}(\boldsymbol{r})$. If div $\boldsymbol{v} \equiv 0$ ($\Rightarrow$ 'incompressible' Chapter 2). From the Divergence Theorem, for *any* volume V bounded by S

   incompressible $$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \qquad \Rightarrow \text{total flux} = \int\int_S \boldsymbol{v} \cdot \hat{\boldsymbol{n}}\, dA = 0$$ for any surface in fluid.

   That is, there is no average flux (= volume of fluid per unit time) into or out of V (here S can be an imaginary surface rather than a physical one). In particular, there can be no sources or sinks. This is true in general of water, and is the assumption behind hydraulics. Not necessarily true of a gas (explosions, compression, shock waves etc., can have div $\boldsymbol{v} \neq 0$).

4. **Gauss' law in electrostatics:**

"*Flux of electric displacement density* $\boldsymbol{D}$ *through a surface* S *equals the total charge enclosed inside the surface* S."

Taking the case of a point charge q situated at the origin O, this gives

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} = \frac{q\,\boldsymbol{r}}{4\pi|\boldsymbol{r}|^3} = \frac{q\hat{\underline{r}}}{4\pi|\underline{r}|^2}$$

and the results states that for any volume V surrounded by a closed surface S containing the origin

$$\int\int_S \boldsymbol{D} \cdot \hat{\boldsymbol{n}} dA = q. \tag{5.1}$$

or $\iint_S \underline{E} \cdot d\underline{A} = \varepsilon_0 q$

Worked example 7.3 *Use the Divergence Theorem to prove* (5.1)

HINT: show $\underline{\nabla} \cdot \underline{D} = 0$ everywhere $\underline{r} \neq \underline{0}$

7.3 Stokes' Theorem and the meaning of curl.

Sketch of proof of Stokes' theorem.

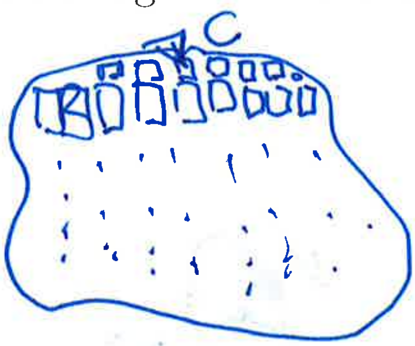
$$\int\int_S \text{curl}\, \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r},$$

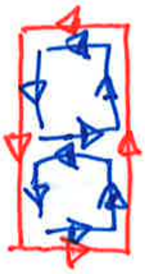
where C is a closed curve forming the boundary of an open surface S.

It is easier to prove in 2D when S is a region R in the (x, y) plane, so $(\text{curl}\, \boldsymbol{F})_3 = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$, $\quad \hat{\boldsymbol{n}} = \boldsymbol{k}$, $dA = dxdy$ and $d\boldsymbol{r} = (dx, dy, 0)$. Then Stokes' theorem becomes **Greens' Theorem**

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C F_1 dx + F_2 dy.$$

Let us split the region R into many infinitesimal rectangles ΔR_i.





cancellation of internal adjacent circulations to leave circulation around total boundary

The proof then follows the lines:

1. using small rectangular surfaces ΔR_i with edges ΔC_i oriented along x, y axes, by basic algebra and fundamental theorem of calculus you can show $\int\int_{\Delta R_i} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_{\Delta C_i} F_1 dx + F_2 dy$.
2. the total surface integral sums up over each infinitesimal rectangle
$$\int\int_R \text{curl}\, \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA = \sum_i \int\int_{\Delta R_i} \text{curl}\, \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA$$
3. the edges of each infinitesimal ΔR_i are either: **(i)** in the interior of region R or **(ii)** form part of the outside boundary C. In case (i) the work integral (around ΔC_i) is exactly cancelled by parts of the adjacent work integrals around neighbouring cells. In case (ii), all these contributions sum to the total work integral around R.

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_S \underline{\nabla} \times \underline{F} \cdot d\underline{A}$$

Applications of Stokes' Theorem

1. **Physical meaning of curl**

   Consider a flat surface S, centered on a point $\boldsymbol{r}_i$, with normal direction $\hat{\boldsymbol{n}}$, surface area ΔS and boundary curve C. Applying Stokes' Theorem we have, assuming ΔS to be small

$$\oint_{\Delta C} \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_{\Delta S} \text{curl}\, \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA \approx \Delta S\, \text{curl}\, \boldsymbol{F}(\boldsymbol{r}_i) \cdot \hat{\boldsymbol{n}}$$

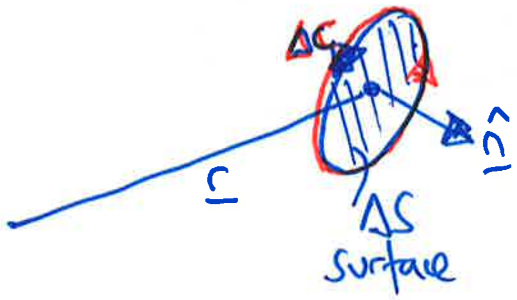
   if ΔS small enough that $\underline{\nabla} \times \underline{F}$ approx constant

   Hence, allowing $\Delta S \to 0$,

$$\boxed{\text{curl}\, \boldsymbol{F} \cdot \hat{\boldsymbol{n}} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C_n} \boldsymbol{F} \cdot d\boldsymbol{r}}$$

   This says that the component of the curl of a vector field in the direction $\hat{\boldsymbol{n}}$ is the *circulation* of the vector field per unit area about the axis $\hat{\boldsymbol{n}}$. Or the amount to which 'particles being carried by the vector field are being rotated about $\hat{\boldsymbol{n}}$'.

   curl $\underline{F}$ is $\equiv$ the circulation around an infinitesimal loop (in direction $\hat{n}$)



2. **Evaluation of 'difficult' line integrals** (see Question 2 on Example Sheet 8).

   into easy surface integrals [& vice versa]

3. **Independence of path revisited**.

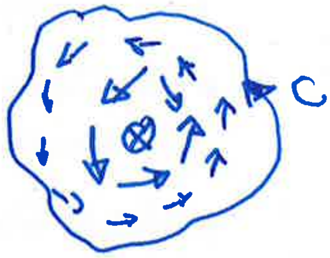
   Stokes' theorem provides a simple proof of the independence of path property for the work integral (work done) by a conservative vector (force) field. Recall the definition of a conservative field, that curl $\boldsymbol{F} \equiv 0$. Therefore, from Stokes' Theorem, for any simple curve closed curve C we have

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int\int_S (\text{curl}\, \boldsymbol{F}) \cdot \hat{\boldsymbol{n}} dA = 0$$

$\underline{\nabla} \times \underline{F} = \underline{0}$ conservative $\iff$ circulation i.e. work integral $= 0$ around a closed loop $\iff$ W.D along two paths with same ends is the same.

4. **Circulation and vortex lines.**

The **circulation** of a flow around a closed loop C is given by the line integral of the flow vector field



$$\Gamma(C) = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r}$$

Recall the flow field around a vortex line, that

$\underline{r} = (x, y)$

$\underline{\theta} = (-y, x)$

$\underline{r} \cdot \underline{\theta} = 0$

$\hat{\underline{\theta}} = \dfrac{\underline{\theta}}{|\underline{r}|}$

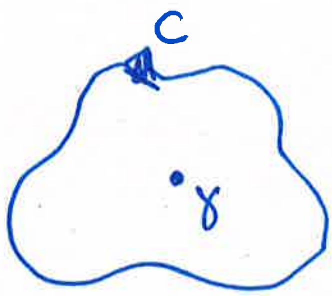
$$\boldsymbol{v} = \gamma \frac{\hat{\boldsymbol{\theta}}}{2\pi r} = \frac{\gamma}{2\pi}\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right) = \frac{\gamma}{2\pi r}(-\sin\theta, \cos\theta, 0)$$

Then the circulation around any loop C containing the vortex center is a constant

$$\Gamma(C) = \gamma \quad . \tag{5.2}$$

Worked example 7.4 *Use Stokes' Theorem to prove* (5.2)

HINT: show $\underline{\nabla} \times \underline{v} = \underline{0}$



7.4 Summary

Two theorems that link differential and integral vector calculus:

- The Gauss' Divergence Theorem links div $\boldsymbol{v}$ to the relation between volume and surface integrals.
- Stokes' Theorem links curl $\boldsymbol{v}$ to the relation between surface and line integrals.

Extra: Proof of fundamental theorem of calculus (non-examinable)

Result: From single variable calculus, integration and differentiation are linked:

$$\int_a^b \frac{dF}{dx}dx = F(b) - F(a).$$

In other words, the integral of a derivative gives the original function.

Proof: Uses that a derivative can be approximated by

$$\frac{dF}{dx} = \frac{F(x_i) - F(x_i - \Delta x)}{\Delta x} = \frac{F(x_i + \Delta x) - F(x_i)}{\Delta x}$$

The Trapezium rule approximation to an integral splits the range $[a, b]$ into points a, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, ... $x_n = a + n\Delta x$, b, so

$$\int_a^b f(x)dx = \left[f(a) + f(b) + 2\sum_{i=1}^{n} f(x_i)\right]\frac{\Delta x}{2},$$

Putting it all together

$$\begin{aligned}
\int_a^b \frac{dF}{dx}dx &= \left[a + b + \sum_{i=1}^{n} \frac{F(x_i) - F(x_i - \Delta x)}{\Delta x} + \frac{F(x_i + \Delta x) - F(x_i)}{\Delta x}\right]\frac{\Delta x}{2} \\
&= \left[a + b + \sum_{i=1}^{n} \frac{F(x_i + \Delta x) - F(x_i - \Delta x)}{\Delta x}\right]\frac{\Delta x}{2} \\
&= \left[a + b + \frac{-F(a) - F(a + \Delta x) + F(b - \Delta x) + F(b)}{\Delta x}\right]\frac{\Delta x}{2} \\
&\to F(b) - F(a) \text{ as } \Delta x \to 0
\end{aligned}$$

□

Extra: Proof of divergence theorem (non-examinable)

Result: Recall that a key step in the proof of the divergence theorem was to show that on a small volume ΔV with 6 surfaces ΔS oriented along the x, y and z axes, that

$$\int\int\int_{\Delta V} \boldsymbol{\nabla} \cdot \boldsymbol{F}\, dV = \int\int_{\Delta S} \boldsymbol{F} \cdot d\mathbf{A}.$$

Proof (infinitesimal case): Consider an infinitesimal cuboid with sides of length $2\Delta x$, $2\Delta y$ and $2\Delta z$ centred on a point $\boldsymbol{r} = (x, y, z)$. The flux integral over its surface is composed of 6 bits, $\sum_{i=1}^{6} \int\int_{\Delta S_i} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA$,

Consider ΔS_1: in this case $\hat{\boldsymbol{n}} dA = \boldsymbol{i} dydz$, and

$$\int\int_{\Delta S_1} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA = \int_{z_i-\Delta z}^{z_i+\Delta z} \int_{y_i-\Delta y}^{y_i+\Delta y} F_1(x_i + \Delta x, y, z) dydz$$

Consider ΔS_2: for the opposite face, $\hat{\boldsymbol{n}} dA = -\boldsymbol{i} dydz$, and

$$\int\int_{\Delta S_2} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA = \int_{z_i-\Delta z}^{z_i+\Delta z} \int_{y_i-\Delta y}^{y_i+\Delta y} -F_1(x_i - (\Delta x), y, z) dydz$$

So, considering ΔS_1 and ΔS_2 together (ΔV volume of cuboid)

$$\begin{aligned}
\int\int_{\Delta S_1+\Delta S_2} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA &= \int\int \left[F_1(x_i + \Delta x, y, z) - F_1(x_i - \Delta x, y, z)\right] dydz \\
&= \int_{z_i-\Delta z}^{z_i+\Delta z} \int_{y_i-\Delta y}^{y_i+\Delta y} \left[\int_{x_i-\Delta x}^{x_i+\Delta x} \frac{\partial F_1}{\partial x}(x, y, z) dx\right] dydz \\
&= \int\int\int_{\Delta V} \frac{\partial F_1}{\partial x} dV
\end{aligned}$$

from the Fundamental Theorem of Calculus.

From the other two pairs of faces we get similar expressions with $x \to y$ and $x \to z$

$$\int\int_{\Delta S_3+\Delta S_4} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA = \int\int\int_{\Delta V} \frac{\partial F_2}{\partial y}(x, y, z)$$

$$\int\int_{\Delta S_5+\Delta S_6} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} dA = \int\int\int_{\Delta V} \frac{\partial F_3}{\partial z}(x, y, z)$$

Hence, since ΔS is composed of all 6 faces we have

$$\begin{aligned}\int\int_{\Delta S}\boldsymbol{F}\cdot\hat{\boldsymbol{n}}dA &= \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dV \\ &= \int\int\int_{\Delta V}\text{div}\, F\, dV.\end{aligned}$$

□

For an alternative proof of the Divergence Theorem, see Kreyszig p. 507.

Extra: Proof of Stokes' (Green's) theorem (non-examinable)

Result: Recall that a key step in the proof of Stokes' (Green's) theorem was to show that using small rectangular surfaces ΔR with edges ΔC oriented along x, y axes that Green's theorem holds

$$\int\int_{\Delta R}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy = \oint_{\Delta C}F_1dx + F_2dy.$$

Proof (infinitesimal case): Consider a small rectangle $\Delta C = \Delta C_1 + \Delta C_2 + \Delta C_3 + \Delta C_4$ centered on (x, y) with sides of length $2\Delta x$ and $2\Delta y$ aligned along the x and y axes respectively.

Taking these in pairs we find

$$\begin{aligned}\int_{\Delta C_1+\Delta C_3}\boldsymbol{F}\cdot d\boldsymbol{r} &= (F_2(x+\Delta x, y) - F_2(x-\Delta x, y))\,2\Delta y \\ &= \frac{F_2(x+\Delta x, y) - F_2(x-\Delta x, y)}{2\Delta x}2\Delta y 2\Delta x \\ &= \int_{y-\Delta y}^{y+\Delta y}\int_{x-\Delta x}^{x+\Delta x}\frac{\partial F_2}{\partial x}(x,y)dxdy\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\Delta C_2+\Delta C_4}\boldsymbol{F}\cdot d\boldsymbol{r} &= (F_1(x, y+\Delta y) - F_1(x, y-\Delta y))\,(-2\Delta x) \\ &= \int_{y-\Delta y}^{y+\Delta y}\int_{x-\Delta x}^{x+\Delta x}-\frac{\partial F_1}{\partial y}(x,y)dydx\end{aligned}$$

Hence, combining, we have

$$\int_{\Delta C}\boldsymbol{F}\cdot d\boldsymbol{r} = \int\int_{\Delta R}\left(\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\right)dydx\,,$$

which is Green's Theorem for this infinitesimal rectangle. □

Example 21

$$\iint_S \underline{F} \cdot d\underline{A} = \iiint_V \underline{\nabla} \cdot \underline{F} \, dV$$

$$\underline{\nabla} \cdot \underline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(4xz, -y^2, 0 \right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + 0$$

$$\therefore \underline{\nabla} \cdot \underline{F} = 4z - 2y$$

$$\iiint_V \underline{\nabla} \cdot \underline{F} \, dV = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (4z - 2y) \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \left[2z^2 - 2yz \right]_0^1 dy \, dx$$

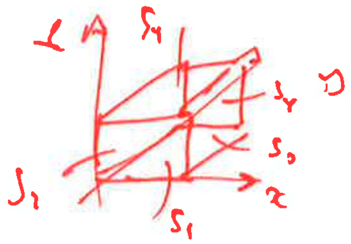
$$= \int_{x=0}^{1} \int_{y=0}^{1} (2 - 2y) \, dy \, dx$$

$$= \left[x \right]_{x=0}^{1} \left[2y - y^2 \right]_{y=0}^{1}$$

$$= 1$$

Exemplo 7.1

$$\underline{F} = (4xz, -y^2, 0)$$



$$\iint \underline{F} \cdot d\underline{A} =$$

S_1: $0 \le x \le 1$ $\quad 0 \le y \le 1$ $\quad z = 0$ $\qquad \hat{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\underline{F} = (0, -y^2, 0) \Rightarrow \underline{F} \cdot \hat{n} = 0$

S_2: $0 \le x \le 1$ $\quad 0 \le y \le 1$ $\quad z = 1$ $\qquad \hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{F} = (4x, -y^2, 0) \Rightarrow \underline{F} \cdot \hat{n} = 0$

S_3: $0 \le x \le 1$ $\quad 0 \le z \le 1$ $\quad y = 0$ $\qquad \hat{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$\underline{F} = (4xz, 0, 0) \Rightarrow \underline{F} \cdot \hat{n} = 0$

S_4: $0 \le x \le 1$ $\quad 0 \le z \le 1$ $\quad y = 1$ $\qquad \hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\underline{F} = (4xz, -1, 0) \Rightarrow \underline{F} \cdot \hat{n} = -1$

S_5: $0 \le y \le 1$ $\quad 0 \le z \le 1$ $\quad x = 0$ $\qquad \hat{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{F} = (0, -y^2, 0) \Rightarrow \underline{F} \cdot \hat{n} = 0$

S_6: $0 \le y \le 1$ $\quad 0 \le z \le 1$ $\quad x = 1$ $\qquad \hat{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

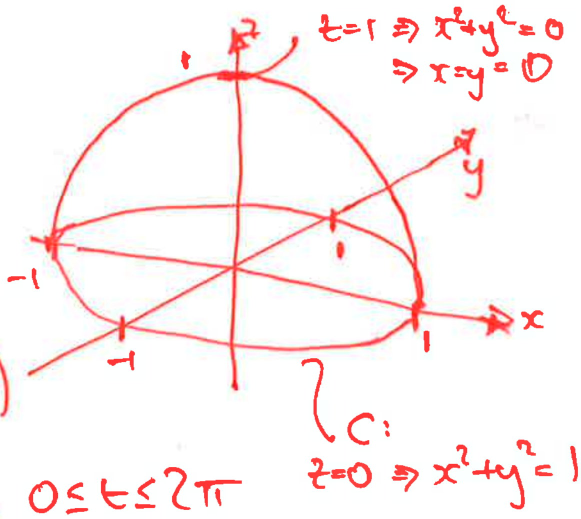
$\underline{F} = (4z, -y^2, 0) \Rightarrow \underline{F} \cdot \hat{n} = 4z$

$$\iint_S \underline{F} \cdot d\underline{A} = \int_{z=0}^{1} \int_{x=0}^{1} -1 \, dx\,dz + \int_{z=0}^{1} \int_{y=0}^{1} 4z \, dy\,dz$$

$$= -1 + 2$$

$$= 1$$

Example 7.2 $\quad \underline{F} = (y, z, x)$



verify $\displaystyle\int_C \underline{F}\cdot d\underline{r} = \iint_S \underline{\nabla}\times\underline{F}\cdot d\underline{A}$

C will be the circle $x^2+y^2=1 \ (z=0)$

ie $\underline{r}_C(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$

S is the paraboloid $z = f(x,y) = 1-(x^2+y^2)$, $z \ge 0$.

cylindrical polars : $\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{aligned} \Rightarrow z = 1-\rho^2$

ie $\underline{r}_S(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 1-\rho^2)$

with $\begin{aligned} 0 \le \theta \le 2\pi \\ 0 \le \rho \le 1 \end{aligned}$

PART I : $\displaystyle\int_C \underline{F}\cdot d\underline{r} = \int_{t=0}^{2\pi} \underline{F}\left(x_C(t), y_C(t), z_C(t)\right)\cdot \frac{d\underline{r}_C}{dt}\,dt$

$\displaystyle = \int_{t=0}^{2\pi} \underbrace{\left(\sin t, 0, \cos t\right)}_{(y,z,x)}\cdot \underbrace{(-\sin t, \cos t, 0)}_{\frac{d}{dt}(\cos t, \sin t, 0)}\,dt$

$\displaystyle = \int_{t=0}^{2\pi} -\sin^2 t\,dt$

$\displaystyle = \int_{t=0}^{2\pi} -\frac{1-\cos 2t}{2}\,dt$

$= -\pi$

PART II : $\iint_S \underline{\nabla} \times \underline{F} \cdot d\underline{A}$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= (-1, -1, -1)$$

$\underline{F} = (y, z, x)$

$\underline{r}_S = (\rho\cos\theta, \rho\sin\theta, 1-\rho^2)$

$$d\underline{A} = \frac{\partial \underline{r}_S}{\partial \rho} \times \frac{\partial \underline{r}_S}{\partial \theta} \, d\rho \, d\theta$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & -2\rho \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} d\rho \, d\theta$$

$$= \left(2\rho^2\cos\theta, +2\rho^2\sin\theta, \rho \right) d\rho \, d\theta$$

$$\therefore \iint_S \underline{\nabla} \times \underline{F} \cdot d\underline{A} = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} (-1,-1,-1) \cdot (2\rho^2\cos\theta, +2\rho^2\sin\theta, \rho) \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \left(-2\rho^2\cos\theta - 2\rho^2\sin\theta - \rho\right) d\rho \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-\frac{2\rho^3}{3}\cos\theta - \frac{2\rho^3}{3}\sin\theta - \frac{\rho^2}{2} \right]_{\rho=0}^{1} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(-\frac{2}{3}\cancelto{0}{\cos\theta} - \frac{2}{3}\cancelto{0}{\sin\theta} - \frac{1}{2} \right) d\theta$$

$$= -\pi$$

AGREES!

EXAMPLE 7.3

$$\underline{D} = \frac{q\,\hat{\underline{r}}}{4\pi|\underline{r}|^2} = \frac{q\,\underline{r}}{4\pi|\underline{r}|^3}$$

to show for any surface S enclosing origin $\iint_S \underline{D}\cdot d\underline{A} = q$

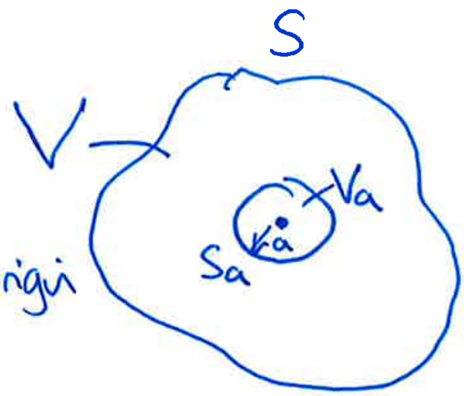
(I) first consider spherical surface S_a (radius a)

$$d\underline{A} = \hat{\underline{r}}\, a^2 \sin\theta \, d\theta \, d\varphi$$

$$\iint_{S_a} \underline{D}\cdot d\underline{A} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{q}{4\pi \cancel{a^2}} \overbrace{\hat{\underline{r}} \cdot \hat{\underline{r}}}^{1} \cancel{a^2} \sin\theta \, d\theta \, d\varphi$$

$$= \frac{q}{4\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\varphi$$

$$= q$$



(II) for any surface S enclosing origin

$$\iint_S \underline{D}\cdot d\underline{A} = \iiint_V \underline{\nabla}\cdot\underline{D}\, dV$$

$$= \iiint_{V_a} \underline{\nabla}\cdot\underline{D}\, dV + \iiint_{V-V_a} \underline{\nabla}\cdot\underline{D}\, dV$$

$$= \iint_{S_a} \underline{D}\cdot d\underline{A} + \iiint_{V-V_a} \underline{\nabla}\cdot\underline{D}\, dV$$

$$\iint_S \underline{D}\cdot d\underline{A} = q + \iiint_{V-V_a} \underline{\nabla}\cdot\underline{D}\, dV$$

(iii) show $\underline{\nabla} \cdot \underline{D} = 0$ everywhere $\underline{r} \neq \underline{0}$

$$\underline{D} = \frac{q\,\underline{r}}{4\pi|\underline{r}|^3} = \frac{q}{4\pi}\left(\frac{x}{|\underline{r}|^3}, \frac{y}{|\underline{r}|^3}, \frac{z}{|\underline{r}|^3}\right)$$

$$|\underline{r}| = (x^2+y^2+z^2)^{1/2}$$

$$\underline{\nabla} \cdot \underline{D} = \frac{q}{4\pi}\left(\frac{\partial}{\partial x}\frac{x}{|\underline{r}|^3} + \frac{\partial}{\partial y}\frac{y}{|\underline{r}|^3} + \frac{\partial}{\partial z}\frac{z}{|\underline{r}|^3}\right)$$

$$\partial/\partial x\left(x/|\underline{r}|^3\right) = \partial/\partial x\left(x/(x^2+y^2+z^2)^{3/2}\right)$$
$$= 1/(x^2+y^2+z^2)^{3/2} - 3/2 \times 2x^2/(x^2+y^2+z^2)^{5/2}$$
$$= 1/r^3 - 3x^2/r^5$$

similarly $\partial/\partial y\left(y/|\underline{r}|^3\right) = 1/r^3 - 3y^2/r^5$

$\partial/\partial z\left(z/|\underline{r}|^3\right) = 1/r^3 - 3z^2/r^5$

$$\therefore\ \underline{\nabla} \cdot \underline{D} = \frac{q}{4\pi}\left(3/r^3 - 3(x^2+y^2+z^2)/r^5\right)$$
$$= 0 \qquad (\text{for } \underline{r} \neq \underline{0})$$

Putting it all together:

$$\iint_S \underline{D} \cdot d\underline{A} = q$$ for any surface S enclosing the origin.

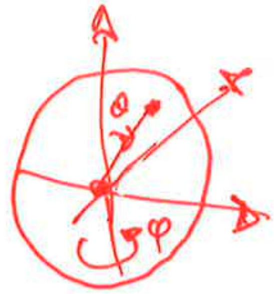
EXAMPLE 7.3 CONT

evaluating $\iint_{S_a} \underline{D} \cdot d\underline{A}$ for S_a sphere radius a

either (i) remember $d\underline{A} = \hat{\underline{r}}\, a^2 \sin\theta \, d\theta \, d\varphi$

or (ii) work it out :

parameterize
$$x = a\cos\varphi \sin\theta$$
$$y = a \sin\varphi \sin\theta$$
$$z = a\cos\theta$$



$$d\underline{A} = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \varphi} \, d\theta \, d\varphi$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a\cos\varphi\cos\theta & a\sin\varphi\cos\theta & -a\sin\theta \\ -a\sin\varphi\sin\theta & a\cos\varphi\sin\theta & 0 \end{vmatrix} d\theta \, d\varphi$$

$$= \left(a^2 \sin^2\theta \cos\varphi , \; a^2 \sin^2\theta \sin\varphi , \; a^2 \sin\theta\cos\theta \right)$$

$$= a^2 \sin\theta \, \hat{\underline{r}} \, d\theta \, d\varphi$$

rest of proof the same . . . .

EXAMPLES 7.4: $\underline{v} = \gamma \dfrac{\underline{\hat{\theta}}}{2\pi r} = \dfrac{\gamma}{2\pi}\left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}, 0\right)$

to show for any loop C enclosing $x=y=0$ $\quad \oint_C \underline{v}\cdot d\underline{r} = \gamma$

(I) first consider circular loop C_a (radius a)

$$\underline{r}(t) = (a\cos t,\ a\sin t,\ 0) \qquad 0 \le t \le 2\pi$$

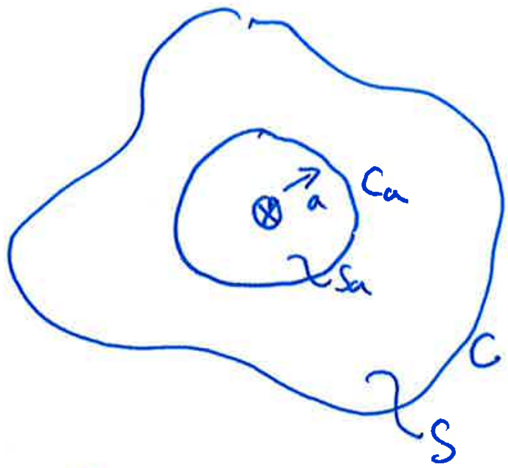
$$\underline{v}(\underline{r}(t)) = \gamma \frac{\underline{\hat{\theta}}}{2\pi a} = \frac{\gamma}{2\pi a}(-\sin t, \cos t, 0)$$

$$\oint_{C_a} \underline{v}\cdot d\underline{r} = \int_{t=0}^{2\pi} \underline{v}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\, dt$$

$$= \int_{t=0}^{2\pi} \frac{\gamma}{2\pi a}(-\sin t, \cos t, 0)\cdot(-a\sin t, a\cos t, 0)dt$$

$$= \int_{t=0}^{2\pi} \frac{\gamma}{2\pi}(\sin^2 t + \cos^2 t)$$

$$\therefore \oint_{C_a} \underline{v}\cdot d\underline{r} = \gamma$$



(II) for any loop C

$$\oint_C \underline{v}\cdot d\underline{r} = \iint_S \underline{\nabla}\times\underline{v}\cdot d\underline{A}$$

$$= \iint_{S_a} \underline{\nabla}\times\underline{v}\cdot d\underline{A} + \iint_{S-S_a} \underline{\nabla}\times\underline{v}\cdot d\underline{A}$$

$$= \oint_{C_a} \underline{v}\cdot d\underline{r} + \iint_{S-S_a} \underline{\nabla}\times\underline{v}\cdot d\underline{A}$$

$$\therefore \oint_C \underline{v}\cdot d\underline{r} = \gamma + \iint_{S-S_a} \underline{\nabla}\times\underline{v}\cdot d\underline{A}$$

(iii) show $\underline{\nabla} \times \underline{v} = \underline{0}$ everywhere $r \neq 0$ (cyl polars)

$$\underline{v} = \gamma \frac{\hat{\underline{\theta}}}{2\pi r} = \frac{\gamma}{2\pi}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

$$\underline{\nabla} \times \underline{v} = \frac{\gamma}{2\pi} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/x^2+y^2 & x/x^2+y^2 & 0 \end{vmatrix}$$

$$= \frac{\gamma}{2\pi}\left(0, 0, \partial/\partial x\left(x/x^2+y^2\right) + \partial/\partial y\left(y/x^2+y^2\right)\right)$$

$$\partial/\partial x\left(x/x^2+y^2\right) = 1/x^2+y^2 - 2x^2/(x^2+y^2)^2$$

$$\partial/\partial y\left(y/x^2+y^2\right) = 1/x^2+y^2 - 2y^2/(x^2+y^2)^2$$

$$\partial/\partial x\left(x/x^2+y^2\right) + \partial/\partial y\left(y/x^2+y^2\right) = 2/x^2+y^2 - \frac{2(x^2+y^2)}{(x^2+y^2)^2}$$

$$= 0$$

$$\therefore \underline{\nabla} \times \underline{v} = \underline{0} \qquad (r \neq 0)$$

Putting it all together

$$\oint_C \underline{v} \cdot d\underline{c} = \gamma$$

for any loop C enclosing the vortex line at $x=y=0$

Application (non-examinable)

MAXWELL'S EQUATIONS

$\underline{E}(\underline{r})$ = electric field $\qquad$ $\underline{B}(\underline{r})$ = magnetic field

$\rho(\underline{r})$ = charge density $\qquad$ $\underline{j}(\underline{r})$ = current density

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_0} \qquad \underline{\nabla}\cdot\underline{B} = 0$$

$$\underline{\nabla}\times\underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{\nabla}\times\underline{B} = \mu_0\left(\underline{j} + \varepsilon_0\frac{\partial \underline{E}}{\partial t}\right)$$

Electrostatics : $\dfrac{\partial \underline{B}}{\partial t} = \underline{0} \Rightarrow \underline{\nabla}\times\underline{E} = \underline{0}$

$\therefore\ \underline{E} = \underline{\nabla}\varphi$

ie defined from electric potential !

if in addition in a region with $\rho = 0$

$$\underline{\nabla}\cdot\underline{E} = 0 \Rightarrow \nabla^2\varphi = 0$$

Laplaces equation

Magnetic potential : $\underline{\nabla}\cdot\underline{B} = 0 \Rightarrow \underline{B} = \underline{\nabla}\times\underline{A}$

ie defined from vector potential

$\underline{B} = \underline{0}$ outside a solenoid

$\underline{0}$ field

then $\underline{\nabla}\times\underline{A} = \underline{0}$

solenoidal !

electric flux

$$\underline{\Phi}_E = \int_S \underline{E} \cdot d\underline{A} = \int_V \underline{\nabla} \cdot \underline{E} \, dV = \int_V \frac{\rho}{\varepsilon_0} \, dV$$

Gauss's law of electrostatics.

magnetic flux

$$\underline{\Phi}_B = \int_S \underline{B} \cdot d\underline{A} = \int_V \underline{\nabla} \cdot \underline{B} \, dV = 0 .$$

~~No~~ or Magnetic flux always zero through closed surface

electric circulation

$$C_E = \int_C \underline{E} \cdot \partial \underline{r} = \int_S \underline{\nabla} \times \underline{E} \cdot dA = \frac{\partial}{\partial t} \int_S \underline{B} \cdot dA$$

$$= \partial/\partial t \, \underline{\Phi}_B .$$

ie $\partial/\partial t$ magnetic flux through open surface.

magnetic circulation

$$C_B = \int_C \underline{B} \cdot \partial \underline{r} = \int_C \underline{\nabla} \times \underline{B} \cdot dA = \mu_0 \underline{\Phi}_j + \overbrace{\mu_0 \varepsilon_0}^{1/c^2} \frac{\partial}{\partial t} \underline{\Phi}_E$$

through open surface.

displacement current.