StM3 – Composite Laminate Analysis

Lecture 2 :
- macromechanics of uni-directional lamina

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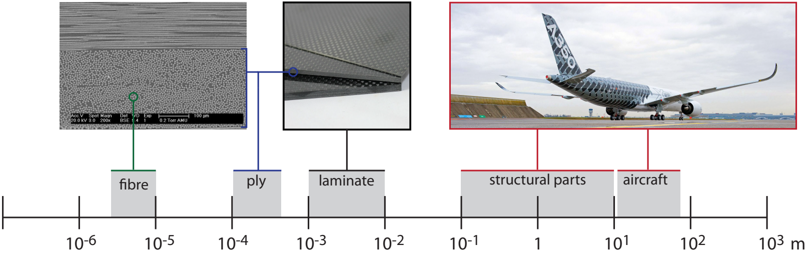
Mark Schenk – M.Schenk@bristol.ac.uk

Unit Content

Composite Laminate Analysis (CLA)

learning outcome: able to describe and analyse *structural* properties of a fibre reinforced composite laminate material

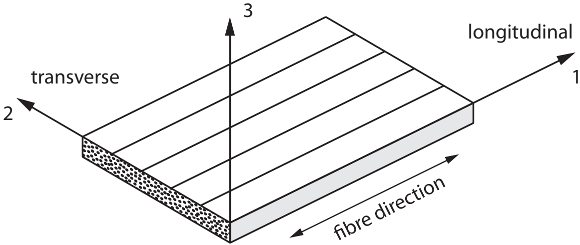


Lecture Outline

- derive material model for a specially orthotropic ply
- revise stress/strain transformation equations (StM2)
- derive material model for an angled ply

Composite Lamina: Plane Stress – I

a unidirectional composite lamina is **anisotropic**: mechanical properties differ along direction of fibres and perpendicular to fibres



formulate constitutive equations in natural/material axes (123)

Composite Lamina: Plane Stress – II

each individual lamina is assumed to be loaded under *plane stress*

constitutive equations reduce to *specially orthotropic* material:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

Q_{ij} components are known as **reduced stiffnesses**

Composite Lamina: Plane Stress – III

find $\boldsymbol{Q}$ by inverting the plane stress **compliance** matrix $\boldsymbol{S}$:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix}$$

to give:

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} \qquad Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2} \qquad Q_{66} = \frac{1}{S_{66}}$$

Composite Lamina: Plane Stress – IV

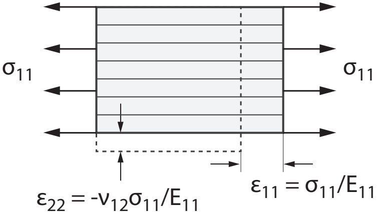
compliance matrix is constructed by considering three load cases:

1. uni-axial stress σ_{11} (with $\sigma_{22} = \tau_{12} = 0$):

$$\varepsilon_{11} = \frac{\sigma_{11}}{E_{11}}$$

$$\varepsilon_{22} = -\nu_{12}\varepsilon_{11} = -\nu_{12}\frac{\sigma_{11}}{E_{11}}$$

$$\gamma_{12} = 0$$



definition of the Poisson's ratio ν_{ij}

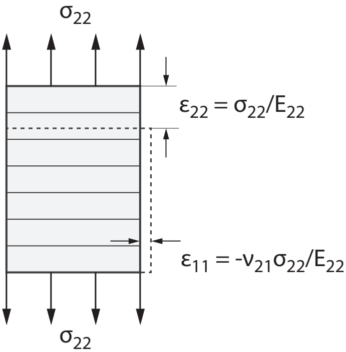
Composite Lamina: Plane Stress – V

2. uni-axial stress σ_{22} (with $\sigma_{11} = \tau_{12} = 0$):

$$\varepsilon_{11} = -\nu_{21}\varepsilon_{22} = -\nu_{21}\frac{\sigma_{22}}{E_{22}}$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E_{22}}$$

$$\gamma_{12} = 0$$



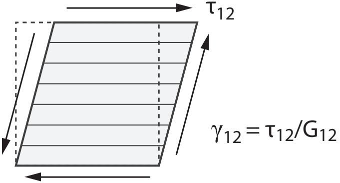
Composite Lamina: Plane Stress – VI

3. pure shear τ_{12} (with $\sigma_{11} = \sigma_{22} = 0$):

$$\varepsilon_{11} = 0$$

$$\varepsilon_{22} = 0$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}}$$



Composite Lamina: Plane Stress – VII

each load case provides column of $\boldsymbol{S}$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & 0 \\ -\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix}$$

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symmetry of compliance matrix $(S_{12} = S_{21})$ gives:

$$\frac{\nu_{21}}{E_{22}} = \frac{\nu_{12}}{E_{11}}$$

specially orthotropic material in plane stress: E_{11}, E_{22}, ν_{12}, G_{12}

Summary: Specially Orthotropic Lamina – I

specially orthotropic: coordinate system aligned with natural axes

compliance matrix $\boldsymbol{S}$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix}$$

with

$$S_{11} = \frac{1}{E_{11}} \qquad S_{22} = \frac{1}{E_{22}}$$

$$S_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \qquad S_{66} = \frac{1}{G_{12}}$$

Summary: Specially Orthotropic Lamina – II

reduced stiffness matrix $\boldsymbol{Q}$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

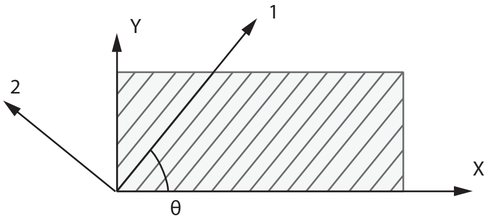
with

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21} E_{11}}{1 - \nu_{12}\nu_{21}} \qquad Q_{66} = G_{12}$$

Angled Plies

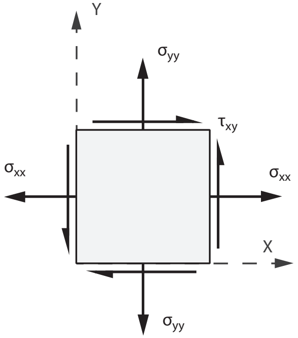
angled plies: generally, in composites the material axes (123) are not aligned with structural axes (*xyz*)

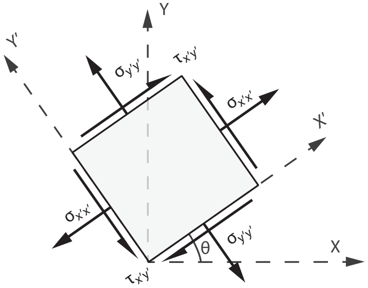


transform stress/strain between natural and structural axes

Revision: Stress/Strain Transformation – I

stresses in $x'y'$ coordinate system, at CCW angle θ to xy axes





Revision: Stress/Strain Transformation – II

calculate using transformation matrix $\boldsymbol{T}$:

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

coordinate transformation matrix for second-rank tensor

Revision: Stress/Strain Transformation – III

remarkably, in-plane strain transforms identically:

$$\begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

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convert *stress* and *strain* between **structural axes** (defined by applied loads) and **material axes** (defined by fibre orientations)

Revision: Stress/Strain Transformation – IV

NB: tensor transformation for *tensorial* strain $\varepsilon_{xy} = \gamma_{xy}/2$

introduce Reuter's matrix $\boldsymbol{R}$

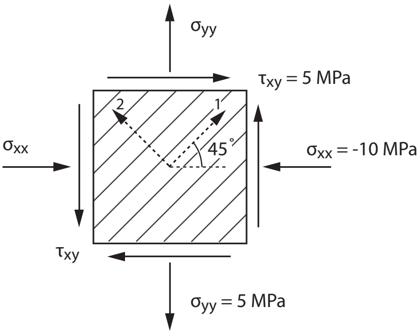
$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

engineering strain transformation becomes:

$$\begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \gamma_{x'y'} \end{bmatrix} = \boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^{-1} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Example 1.1 - Lamina Stress/Strain – I

angled composite ply with $E_{11} = 180$ GPa, $E_{22} = 10$ GPa, $\nu_{12} = 0.2$, $G_{12} = 5$ GPa, is subject to the following loads:



Q: what are the resulting strains?

Example 1.1 - Lamina Stress/Strain – II

A: first, transform applied stresses from structural into material coordinate system:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 2.5 \\ -7.5 \\ 7.5 \end{bmatrix} \text{ MPa}$$

using transformation matrix $\boldsymbol{T}$ with $\theta = 45^\circ$

$$\boldsymbol{T} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ -0.5 & 0.5 & 0 \end{bmatrix}$$

Example 1.1 - Lamina Stress/Strain – III

next, using compliance matrix $\boldsymbol{S}$ and Reuter's matrix $\boldsymbol{R}$

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1/E_{11} & -\nu_{21}/E_{22} & 0 \\ -\nu_{12}/E_{11} & 1/E_{22} & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix}$$

calculate strains in material coordinate system:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \boldsymbol{S} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 22 \\ -753 \\ 1500 \end{bmatrix} \mu\varepsilon$$

Example 1.1 - Lamina Stress/Strain – IV

convert strains back into structural reference frame:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} -1115 \\ 385 \\ 775 \end{bmatrix} \mu\varepsilon$$

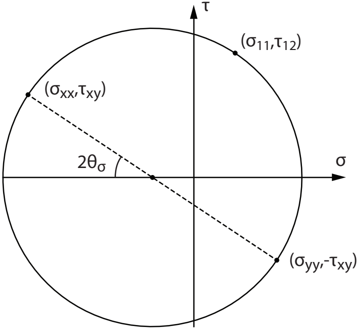
NB: calculate $\boldsymbol{T}^{-1}$ as a CW θ rotation – do not invert!

$$\boldsymbol{T} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}; \qquad \boldsymbol{T}^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix}$$

where $s = \sin\theta$ and $c = \cos\theta$

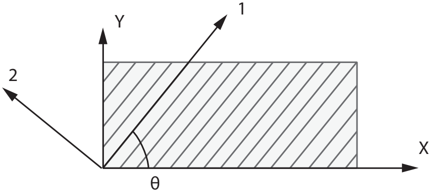
Example 1.1 - Lamina Stress/Strain – V

Mohr's circle for stress *and* strain are insightful



principal directions for stress and strain are not aligned!

Generally Orthotropic Material – I



constitutive equations in structural axes: *generally orthotropic*

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}
$$

$$
\bar{\boldsymbol{Q}} = \boldsymbol{T}^{-1} \boldsymbol{Q} \boldsymbol{R} \boldsymbol{T} \boldsymbol{R}^{-1}
$$

Generally Orthotropic Material – II

with components $\bar{Q}_{ij}$

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta$$

$$\bar{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta$$

Generally Orthotropic Material – III

ply stiffness matrix $\bar{\boldsymbol{Q}}$ is symmetric and fully populated:

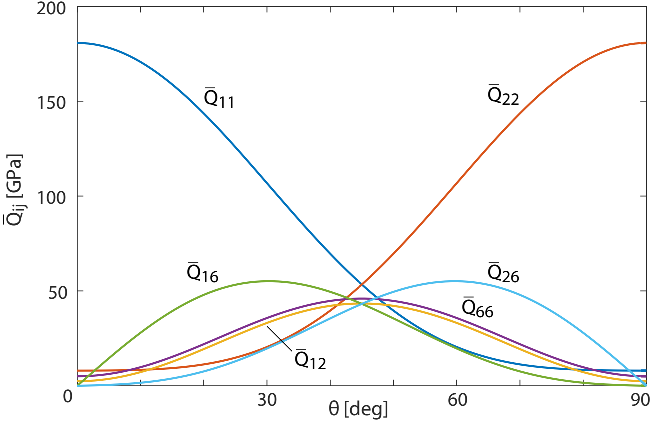
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

where $\bar{Q}_{16}$ and $\bar{Q}_{26}$ provide direct/shear coupling

NB: $\bar{\boldsymbol{Q}}$ matrix contains only four independent mechanical constants of a specially orthotropic material (E_{11}, E_{22}, G_{12} and ν_{12})

Example 1.2 - Stiffness Transformation

carbon/epoxy composite with reduced stiffness $Q_{11} = 180.7$ GPa, $Q_{12} = 2.41$ GPa, $Q_{22} = 8$ GPa and $Q_{66} = 5$ GPa



Generally Orthotropic Material – I

similar transformation for compliance matrix:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

$$\bar{\boldsymbol{S}} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}\boldsymbol{S}\boldsymbol{T}$$

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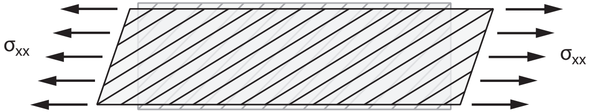
due to non-zero $\bar{S}_{12}$ and $\bar{S}_{16}$ direct stress results in shear strain:



Generally Orthotropic Material – II

with components $\bar{S}_{ij}$

$$\bar{S}_{11} = S_{11}\cos^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta + S_{22}\sin^4\theta$$

$$\bar{S}_{22} = S_{11}\sin^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta + S_{22}\cos^4\theta$$

$$\bar{S}_{12} = (S_{11} + S_{22} - S_{66})\sin^2\theta\cos^2\theta + S_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{S}_{66} = 2\,(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\sin^2\theta\cos^2\theta + S_{66}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})\sin\theta\cos^3\theta - (2S_{22} - 2S_{12} - S_{66})\cos\theta\sin^3\theta$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})\cos\theta\sin^3\theta - (2S_{22} - 2S_{12} - S_{66})\sin\theta\cos^3\theta$$

Engineering Constants – I

compliance matrix $\bar{\boldsymbol{S}}$ for angled ply

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

discussing properties of ply in terms of $\bar{S}_{11}$, $\bar{S}_{12}$, *etc.* not intuitive

use effective engineering constants:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_{xx}} - \nu_{yx}\frac{\sigma_{yy}}{E_{yy}} + m_{x,xy}\frac{\tau_{xy}}{G_{xy}}$$

$$= \bar{S}_{11}\sigma_{xx} + \bar{S}_{12}\sigma_{yy} + \bar{S}_{16}\tau_{xy}$$

Engineering Constants – II

effective engineering constants:

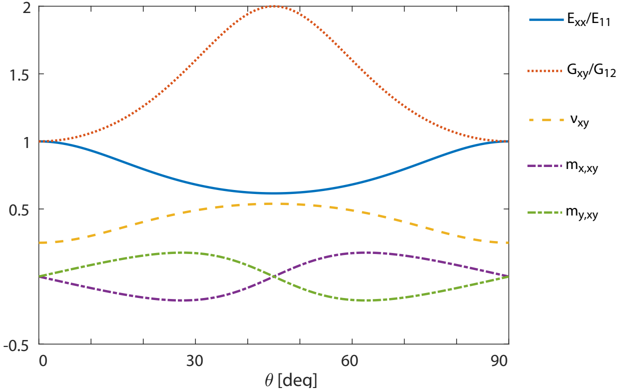
$$E_{xx} = \frac{1}{\bar{S}_{11}} \qquad E_{yy} = \frac{1}{\bar{S}_{22}}$$

$$G_{xy} = \frac{1}{\bar{S}_{66}} \qquad \nu_{xy} = -\frac{\bar{S}_{12}}{\bar{S}_{11}}$$

coefficients of mutual influence m describe direct/shear coupling

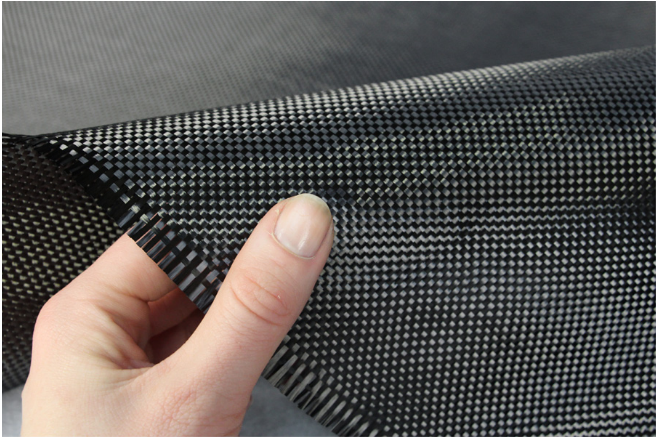
Example 1.3 – Engineering Constants Transformation – I

$E_{11} = E_{22} = 100$ GPa, and $\nu_{12} = \nu_{21} = 0.25$, and $G_{12} = 20$ GPa



Does not behave isotropically! Why?

Example 1.3 – Engineering Constants Transformation – II



Summary

- derived reduced stiffness matrix $\boldsymbol{Q}$ and compliance matrix $\boldsymbol{S}$
- revised plane stress and strain coordinate transformation
- derived generally orthotropic material model ($\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{S}}$)
- able to calculate stress/strain for an angled composite lamina

Revision Objectives

Revision Objectives:

- recognise and use composite subscript conventions $(i, j = 1 \ldots 6)$;
- derive lamina compliance matrix $\boldsymbol{S}$ in terms of E_{11}, E_{22}, G_{12}, and ν_{12};
- recall relationship between Poisson's ratio: $\nu_{12}/E_{11} = \nu_{21}/E_{22}$;
- recall reduced stiffness matrix $\boldsymbol{Q}$ in terms of 4 elastic constants;
- apply tensor transformation $\boldsymbol{T}$ for plane stress and strain;
- derive transformed reduced stiffness matrix: $\bar{\boldsymbol{Q}} = \boldsymbol{T}^{-1}\boldsymbol{QRTR}^{-1}$;
- derive transformed compliance matrix: $\bar{\boldsymbol{S}} = \boldsymbol{RT}^{-1}\boldsymbol{R}^{-1}\boldsymbol{ST}$;
- calculate stiffness/compliance in non-fibre directions using $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{S}}$;
- calculate effective engineering constants from ply stiffness matrix;