Advanced Bending and Torsion
Unsymmetric Bending – Direct Stresses

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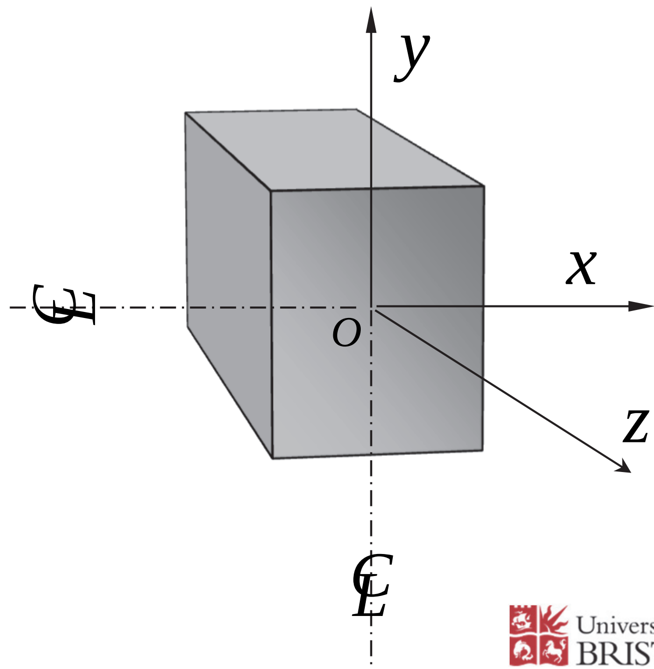
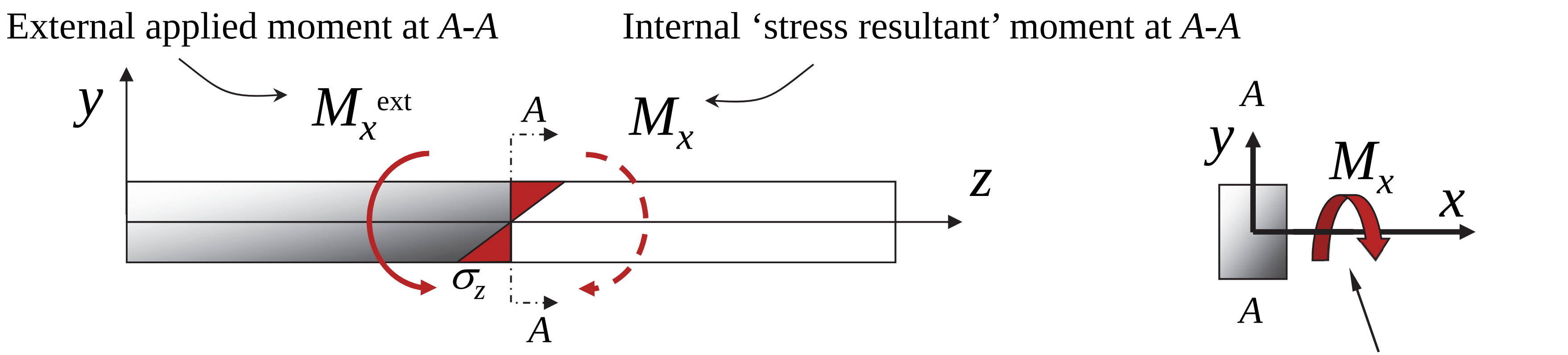
25 October 2018

Symmetric bending

- Loading is applied in a principal plane of the cross section so that the neutral axis is coincident with a principal axis and the resulting deflection is in plane of loading only
- Example: beam of rectangular cross-section loaded in a plane of symmetry
- Assumptions:
  - Loading is applied in a principal plane
  - Deformations are small
  - Plane sections remain plane
  - Shear deformation is neglected
  - Material response is linear elastic



Simple (2D) bending stresses



- We recall the simple bending relation from StM1:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

- Note that it only applies to **symmetric bending**
  - *i.e.* **bending about principal axes**
- Therefore we need to 'expand' our beam theory to account for off-axis bending

Suffixes are usually omitted for 2D bending:

$$M = M_x \qquad I = I_{xx} \qquad x = x_p$$

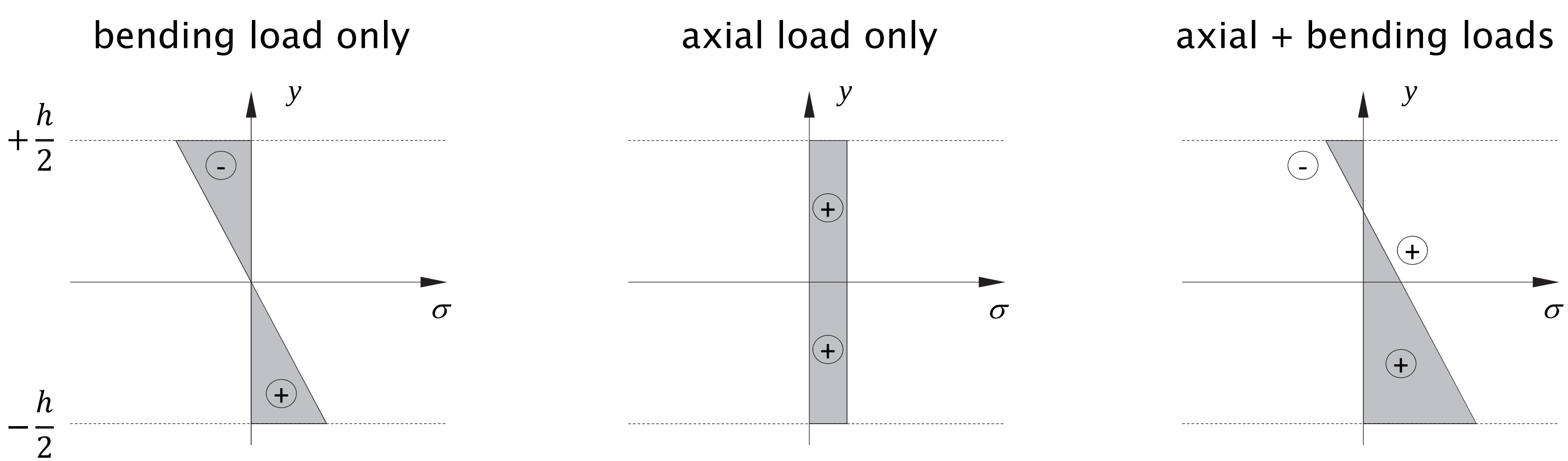
$$\sigma = \sigma_z \qquad \varepsilon = \varepsilon_z$$

StM1 Recap: Stress-Resultant Axial Forces

- For a constant width b, the resultant internal axial force F_z can be interpreted as the 'summation' of the 'areas' under the curves of stress distribution through the thickness

$$F_z = b \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z \, \mathrm{d}y$$

(Note that these areas can have different 'signs')



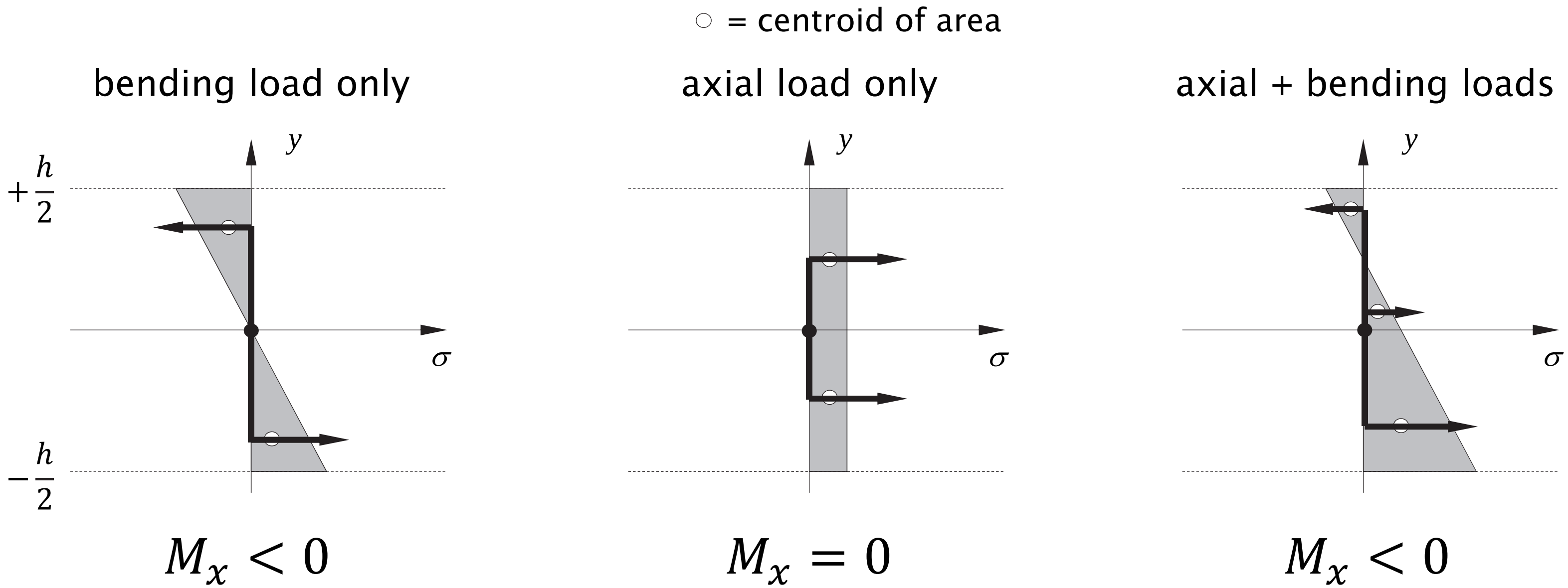
$F_z = 0$ $\quad$ $F_z > 0$ $\quad$ $F_z > 0$

StM1 Recap: Stress-Resultant Bending Moment

- The internal bending moment is given by the integral:

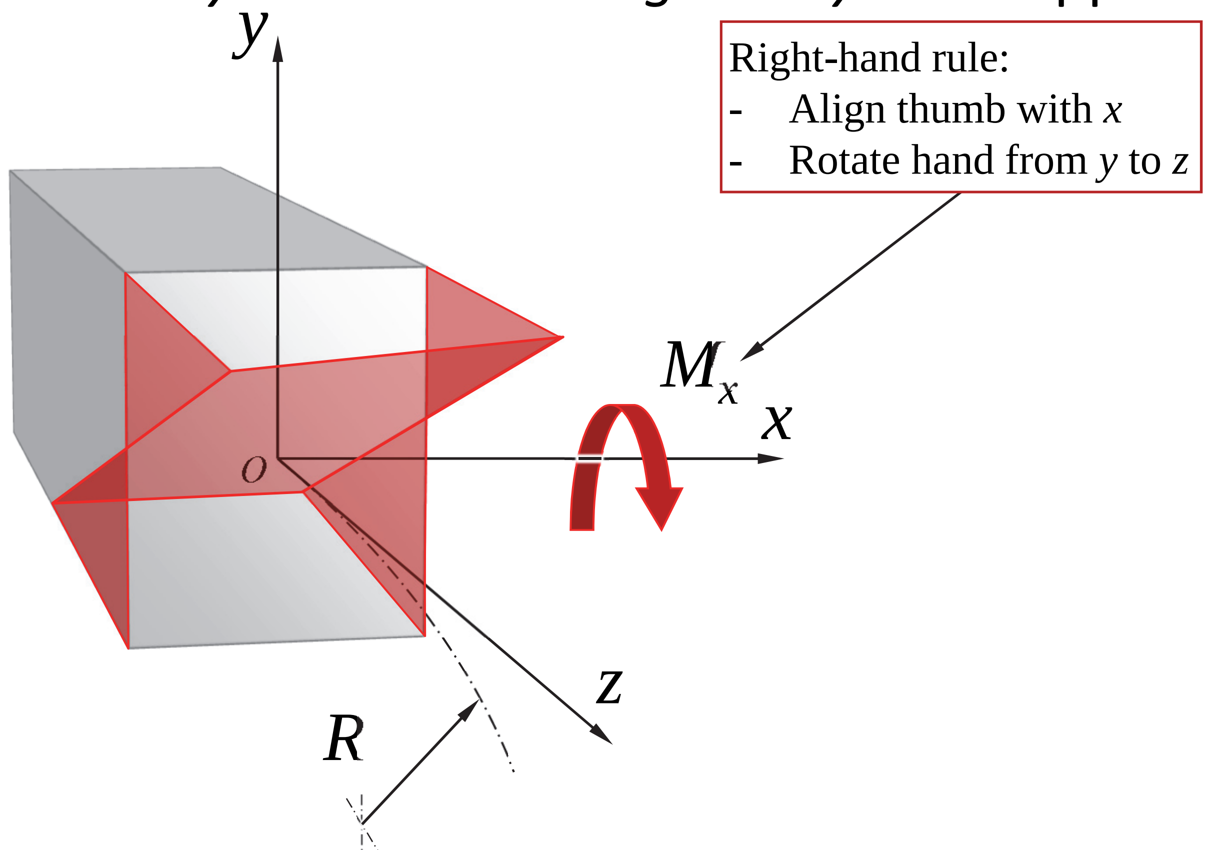
$$M_x = b \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z \, y \, \mathrm{d}y$$

- This can be interpreted as the '**moment of area**' of the stress distribution:



3D Bending Stresses

- Loading off the principal axis can be resolved into components along the principal axes to allow *symmetric bending theory* to be applied:



- Internal sign convention: tension = positive, compression = negative
- External sign convention: right-hand rule about all each axis

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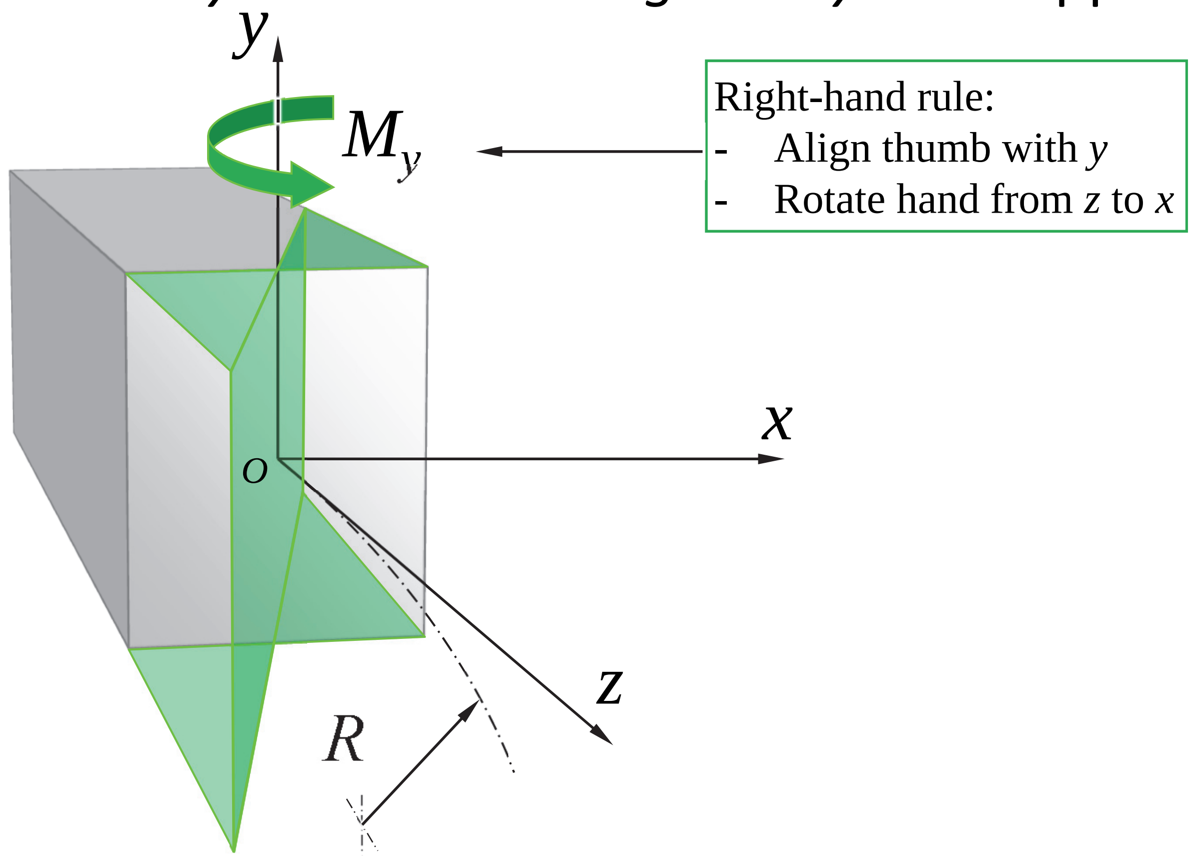

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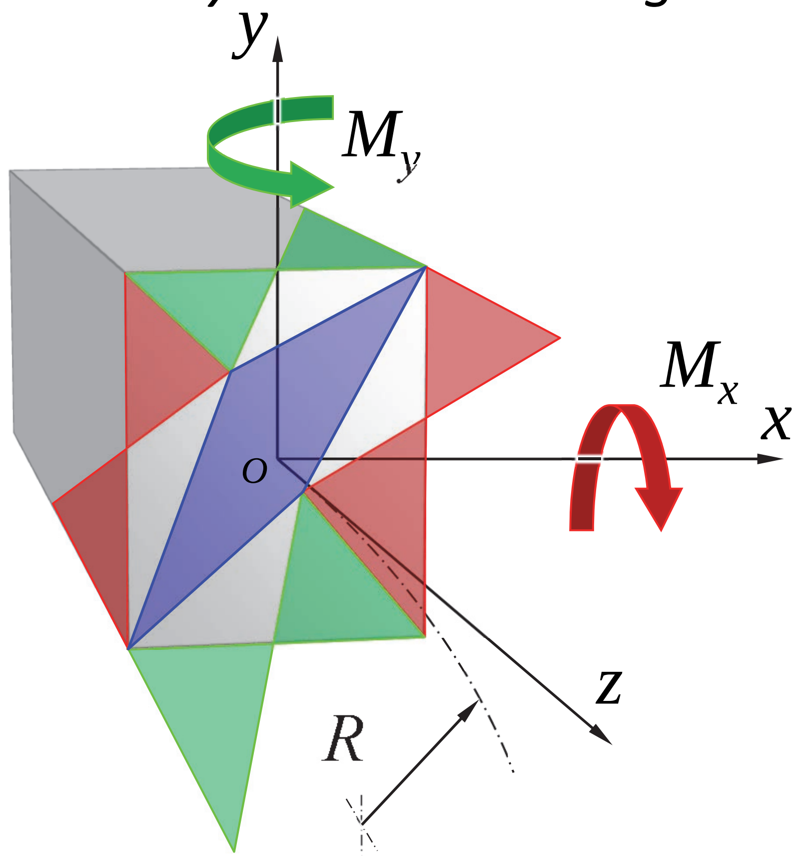

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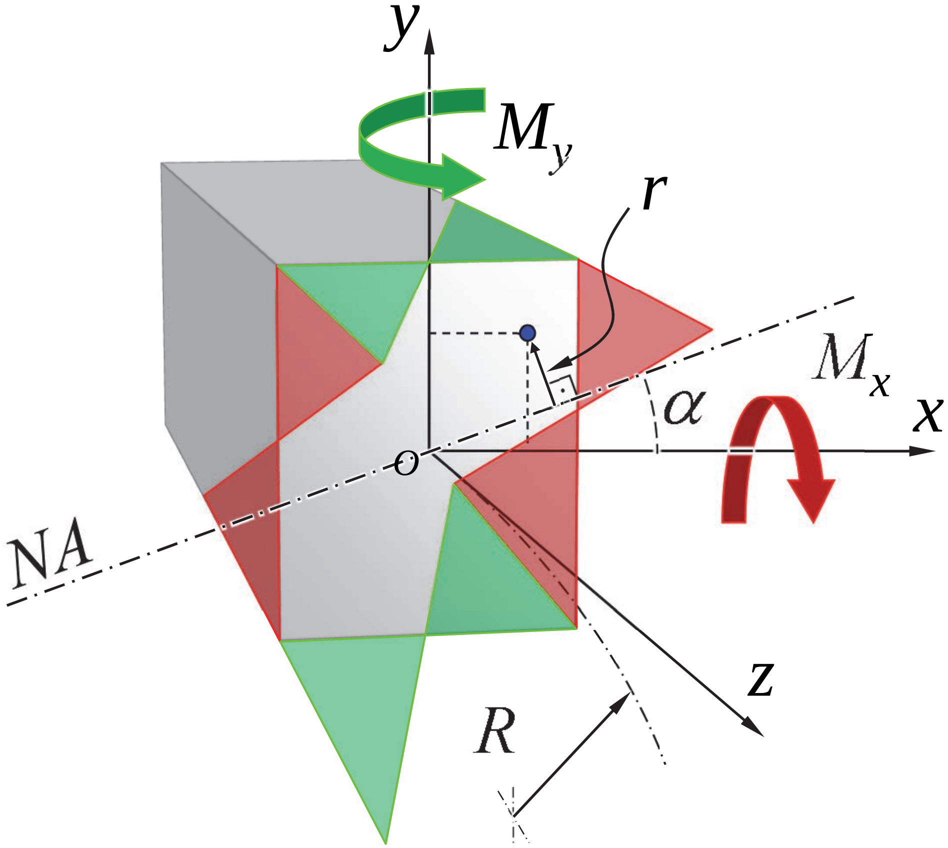

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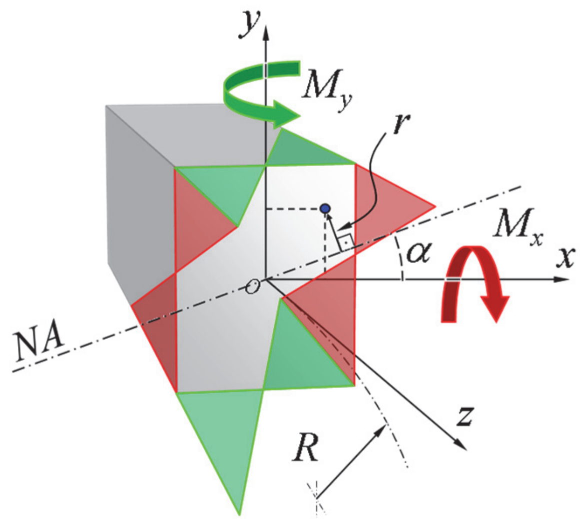
3D Bending Stresses

- Plane sections remain plane: $\varepsilon_z = \dfrac{1}{R} r$
- Linear elasticity: $\sigma_z = E\,\varepsilon_z = \dfrac{E}{R} r$
- NA rotation: $r = -x\sin\alpha + y\cos\alpha$
- Re-writing: $\sigma_z = \dfrac{E}{R}(-x\sin\alpha + y\cos\alpha)$



- Now replacing in the 'stress resultant' moment equations:

$$M_x = \int y\,\sigma_z\,\mathrm{d}A \qquad M_y = \int x\,\sigma_z\,\mathrm{d}A$$

$$M_x = \frac{E}{R}\int y\,(-x\sin\alpha + y\cos\alpha)\,\mathrm{d}A \qquad M_y = -\frac{E}{R}\int x\,(-x\sin\alpha + y\cos\alpha)\,\mathrm{d}A$$

$$M_x = -\frac{E}{R}\sin\alpha\int x\,y\,\mathrm{d}A + \frac{E}{R}\cos\alpha\int y^2\,\mathrm{d}A \qquad M_y = \frac{E}{R}\sin\alpha\int x^2\,\mathrm{d}A - \frac{E}{R}\cos\alpha\int x\,y\,\mathrm{d}A$$

$$M_x = \frac{E}{R}\cos\alpha\;I_{xx} - \frac{E}{R}\sin\alpha\;I_{xy} \qquad M_y = \frac{E}{R}\sin\alpha\;I_{yy} - \frac{E}{R}\cos\alpha\;I_{xy}$$

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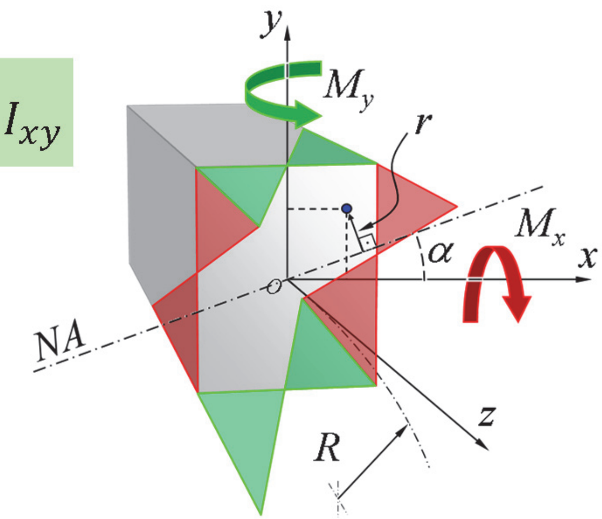


- Re-writing in terms of $\frac{E}{R}\cos\alpha$ and $\frac{E}{R}\sin\alpha$:

$$\frac{E}{R}\sin\alpha = \frac{-M_y\, I_{xx} - M_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}$$

$$\frac{E}{R}\cos\alpha = \frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}$$

- Substituting in $\sigma_z = \frac{E}{R}(-x\sin\alpha + y\cos\alpha)$ gives:

$$\sigma_z = \frac{M_y\, I_{xx} + M_x\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\, x + \frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\, y$$

3D Bending Stresses – Composite Beams

- For composite beams where I_{xx}, I_{yy} and I_{xy} are obtained using the 'scaled area' method, the assumption that 'plane sections remain plane' (*i.e.* 'iso-strain' assumption in composites terminology) gives:

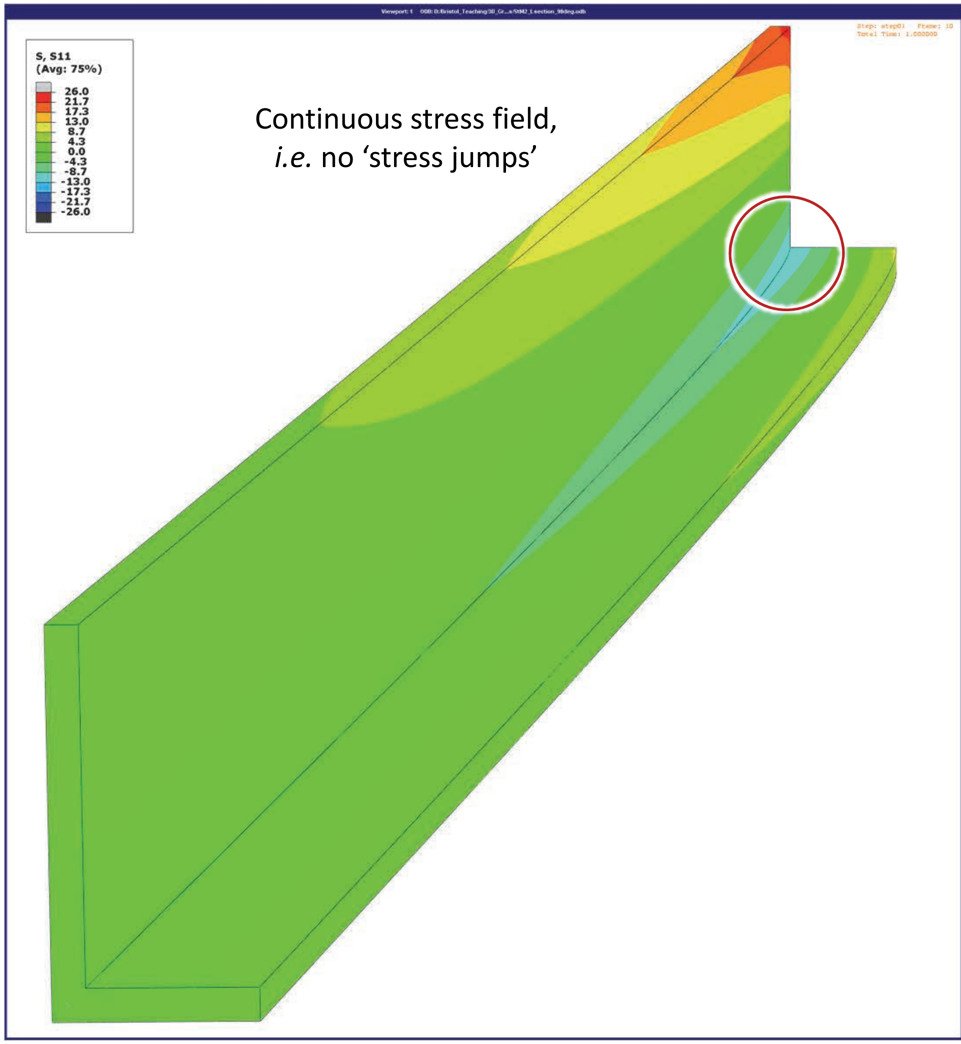
$$\sigma_z = \left(\frac{M_y\, I_{xx} + M_x\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}} x + \frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2} y \right) n_i$$

- So the stress field is now discontinuous, *i.e.* there are 'stress jumps' between domains i with different Young's moduli

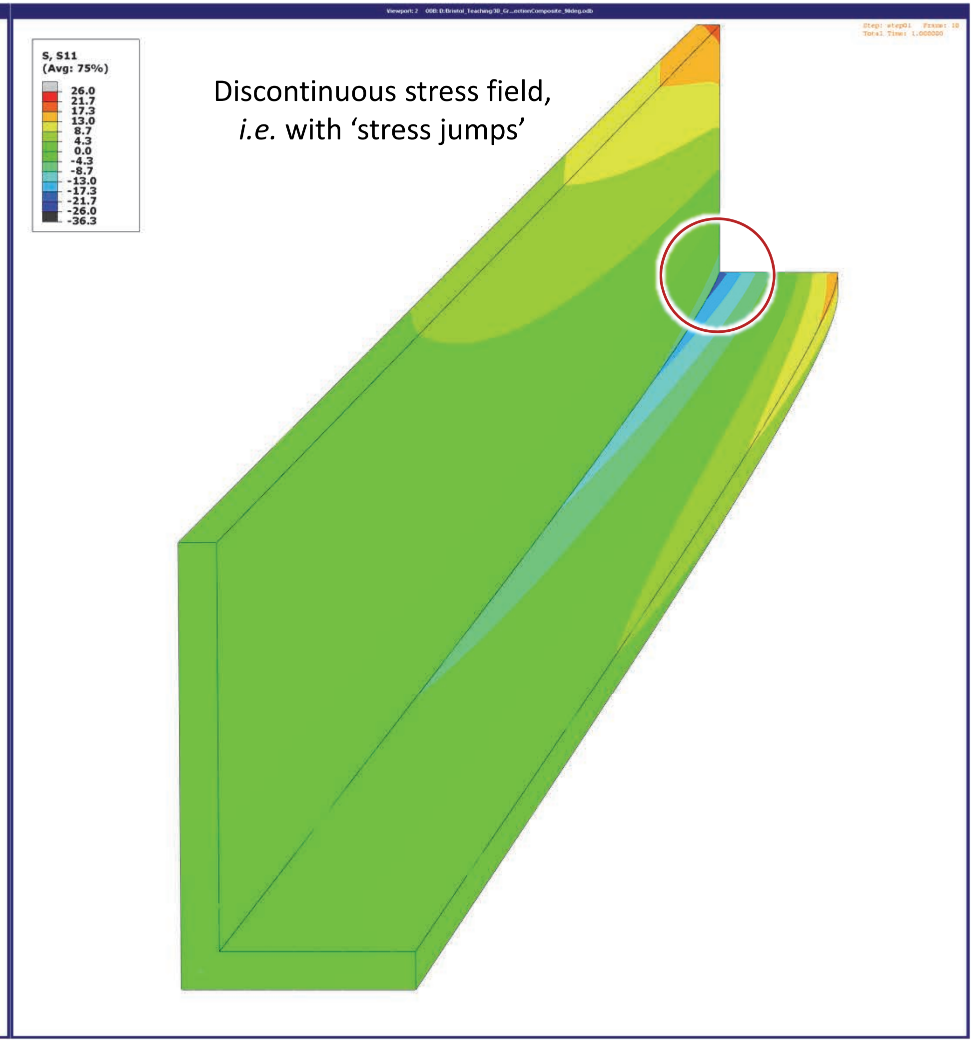
- Domains with higher modulus will have higher stresses and therefore a greater contribution towards the load carrying capacity of the structure

3D Bending Stresses – Composite Beams

Aluminium Alloy

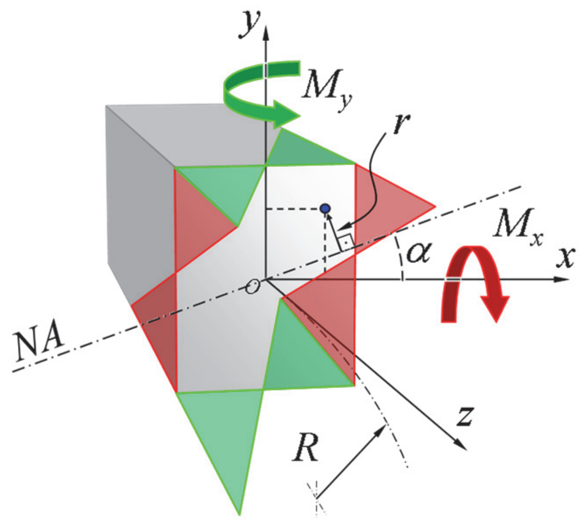


Aluminium Alloy + Steel



3D Bending Stresses – Neutral Axis

$$\boxed{\sigma_z = \frac{M_y\, I_{xx} + M_x\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\, x + \frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\, y}$$



- Bending stresses are **zero** along the neutral axis, therefore:

$$\frac{M_y\, I_{xx} + M_x\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\, x_{NA} + \frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\, y_{NA} = 0$$

$$\frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\, y_{NA} = -\left(\frac{M_y\, I_{xx} + M_x\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x_{NA}$$

$$\frac{M_x\, I_{yy} + M_y\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\, y_{NA} = -\left(\frac{-M_y\, I_{xx} - M_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) x_{NA}$$

$$y_{NA} = -\left(\frac{-M_y\, I_{xx} - M_x\, I_{xy}}{M_x\, I_{yy} + M_y\, I_{xy}}\right) x_{NA}$$

$$\tan\alpha = \frac{M_y\, I_{xx} + M_x\, I_{xy}}{M_x\, I_{yy} + M_y\, I_{xy}}$$

This also applies to composite beams since I_{xx}, I_{yy} and I_{xy} are 'scaled' properties

$$\alpha = \arctan\left(\frac{M_y\, I_{xx} + M_x\, I_{xy}}{M_x\, I_{yy} + M_y\, I_{xy}}\right)$$