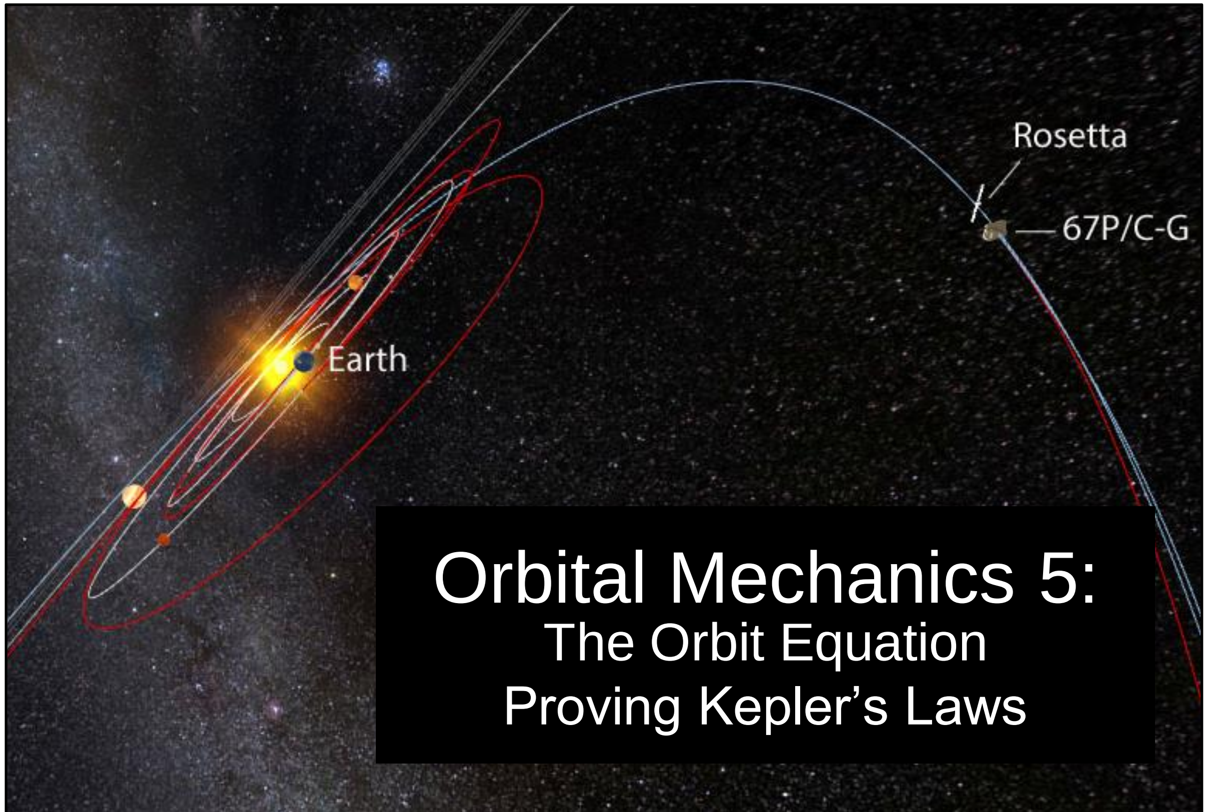


Orbital Mechanics 5:

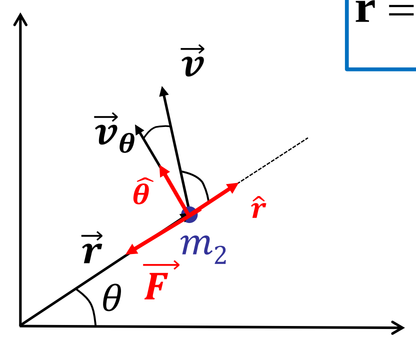
The Orbit Equation
Proving Kepler's Laws

Remember:
Steps in Solving the Kepler Problem

1. **Use conservation of angular momentum to show that the motion is planar**
2. **Find the magnitude of $\boldsymbol{h}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{\theta}$:** $h = r^2\dot{\theta}$
3. **Derive the equation of motion of the orbiting body (in polar coords):** $\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}$
4. **Resolve eq. of motion tangentially:** $\frac{dA}{dt} = \frac{h}{2} \equiv const!$
5. Resolve **radially** to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.
6. Use conservation of energy to solve for the constants of integration in 5 (annex).

Next we will resolve radially…

Remember: Tangential Component ($\hat{\theta}$)



$$\boxed{\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}} \quad \text{(4-20)}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad \text{(4-21)}$$

- Recall: $h = r^2\dot{\theta}$ (4-10)
- Thus: $\dot{h} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$ (4-22)
- CoAM $\frac{\dot{h}}{r} = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$ (4-23)

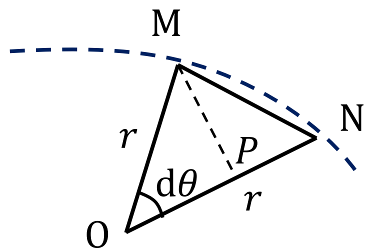
$$h = r^2\dot{\theta} \equiv const \quad \text{(4-24)}$$

This agrees with our earlier result in (4-10) and (4-6)

If we compare LHS and RHS of 4-20 we can see that there is no theta component so the tangential component is =0.
If our object is increasing its speed or slowing down, there is a non-zero tangential acceleration in direction of motion. From CoAM=Conservation of Angular Momentum, we know that hdot=0.

We can use 4-21 to prove Kepler 2nd law

Remember: Tangential Component



- We have shown that:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \qquad \text{(4-27)}$$

- Substitute using $h = r^2 \dot{\theta}$: (4-10)

$$\frac{dA}{dt} = \frac{h}{2} \equiv const! \qquad \text{(4-28)}$$

Thus we have proved Kepler's second law:
"the line joining a planet to the Sun sweeps out equal areas in equal intervals of time".

Learning Outcomes

1. Resolve radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.
2. [Use conservation of energy to solve for the constants of integration in 5 (annex)]. *Not examinable*
3. Use the orbit equation to prove K1
4. Use the orbit equation to find the true anomaly
5. Be able to prove K3
6. Be able to express K1,2 and 3 in numerical form

Radial Component ($\hat{\mathbf{r}}$)

- Our equation of motion (4-20) was:

$$\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}$$

- Resolving radially:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad \text{(5-1)}$$

- This is nonlinear and cannot be solved directly (remember we would like an expression for r in terms of θ, i.e. $r(\theta)$).
- Substitution of the variable r by $1/u$ allows us to find an analytical closed-form solution.

This is a 2nd order non linear differential equation, so we need to do something like substitution. We can see that we are going to need dθ/dt and dr/dt

Radial Component 2

- If $r = 1/u$, then:

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt} \qquad \text{(5-2)}$$

$$\boxed{\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}} \qquad \text{(5-3)}$$

- Also, rearranging $h = r^2\dot{\theta} = r^2\frac{d\theta}{dt}$ and $r = 1/u$: (4-10)

$$\boxed{\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2} \qquad \text{(5-4)}$$

- So substituting in (5-4) into (5-3)…

We are going to need to differentiate so let's rearrange. Then we differentiate dr/du.
Then we need to put du/dt in terms of dθ
Then if we rearrange (4-10) we get a useful expression for dθ/dt in terms of h and u.

Radial Component 3

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\boxed{\frac{d\theta}{dt}} \qquad \boxed{\frac{d\theta}{dt} = hu^2} \qquad \text{(5-5)}$$

- Thus:

$$\dot{r} = \frac{dr}{dt} = -h\frac{du}{d\theta} \qquad \text{(substitution) (5-6)}$$

$$\ddot{r} = \frac{d^2r}{dt^2} = -h\boxed{\frac{d}{dt}}\frac{du}{d\theta} \qquad \text{(differentiate)}$$

$$\ddot{r} = \frac{d^2r}{dt^2} = -h\boxed{\frac{d}{d\theta}}\frac{du}{d\theta}\boxed{\frac{d\theta}{dt}} = -h\frac{d^2u}{d\theta^2}\boxed{\frac{d\theta}{dt}} \qquad \text{(chain rule)}$$

$$\ddot{r} = \frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{(substitution) (5-7)}$$

We can use this in our expression for $\ddot{r}$...

To get the last line (5-7) we substitute (5-5) for dθ/dt in the chain rule line.

Radial Component 4

- Let's try and put everything in terms of u and h:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \qquad \color{red}{(5\text{-}1)} \qquad \text{-recap of (5-1)}$$

$$\ddot{r} - r\frac{h^2}{r^4} = -\frac{\mu}{r^2} \qquad \text{-substitute (4-10)} \quad h = r^2\dot{\theta}$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - r\frac{h^2}{r^4} = -\frac{\mu}{r^2} \qquad \text{-substitute (4-34)}$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\mu u^2 \qquad \text{-substitute in u=1/r}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \qquad \color{red}{(5\text{-}8)} \qquad \text{-cancel terms}$$

Do you recognise this form?

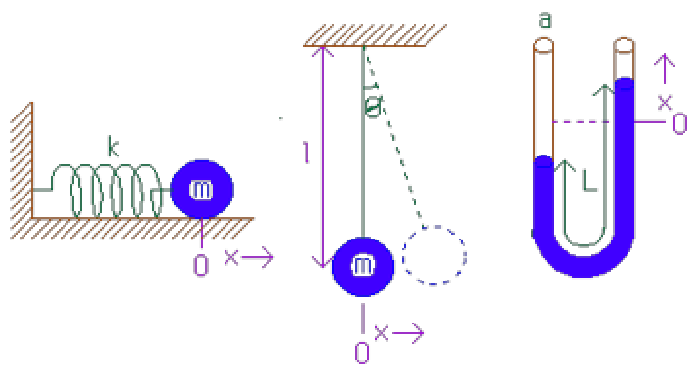
We are looking for a familiar form of equation for the acceleration. If we can find a familiar form then we may be able to work out what the integral solution (ie: position r) is.

Radial Component 5

- Compare (5-8) with a simple harmonic oscillator:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \quad \text{(5-8)}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \text{(S.H.O.)}$$



Mass on Spring   Pendulum   Fluid in U Tube

These are 3 different examples of simple harmonic oscillators. You will do more about this in vibrations.

Radial Component 6

- Compare (5-8) with a simple harmonic oscillator:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \qquad \text{(5-8)} \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{(S.H.O.)}$$

- We know the solution to the S.H.O.:

$$x(t) = A\cos(\omega t - \phi)$$

- Similarly the solution to (5-8) is of the form:

$$u(\theta) = \frac{\mu}{h^2} + A\cos(\theta - \theta_0)$$

Radial Component 7

- Compare (5-8) with a simple harmonic oscillator:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \qquad \text{(5-8)} \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad \text{(S.H.O.)}$$

- We know the solution to the S.H.O.:

$$x(t) = A\cos(\omega t - \phi)$$

- Similarly the solution to (5-8) is of the form:

$$u(\theta) = \frac{\mu}{h^2} + A\cos(\theta - \theta_0)$$

constant

Where will the orbiting body pass closest to the central body, i.e. where will u be maximal and r be minimal?
We need to define r_p where $\theta = 0$

Radial Component 8

$$u(\theta) = \frac{\mu}{h^2} + A\cos(\theta - \theta_0) \qquad (5\text{-}9)$$

- You can use conservation of energy to find the constant of integration (A) and derive the "orbit equation" which gives an expression for r in terms of θ, i.e. $r(\theta)$.
- This is some straightforward maths which you do not need to know for the exam and is done in the Annex.

Annex

Go and read the annex!

Results from annex

Position vector ‘r’:

$$r = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)}$$

For an ellipse:

$$r = \frac{a(1 - e^2)}{1 + e\ \cos(\theta - \theta_0)}$$

Specific energy ‘ε’:

$$\varepsilon = (e^2 - 1)\frac{\mu^2}{2h^2}$$

Conservation of Energy and the Orbit Equation 1

- We can use the conservation of energy to find the constant of integration, A, in (5-9):

$$A = e\frac{\mu}{h^2}$$

$$u = \frac{\mu}{h^2} + A\cos(\theta - \theta_0) \qquad \text{(5-9)}$$

- To give us:

$$u = \frac{\mu}{h^2}\left[1 + e\ \cos(\theta - \theta_0)\right] \qquad \text{(5A-16)}$$

- **ORBIT EQUATION**:

$$\boxed{r = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)}} \qquad \text{(5A-17)}$$

- For an ellipse:

$$r = \frac{a(1 - e^2)}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-20)}$$

5A means the equation comes from the Annex to lecture 5.
5A-17 is the general Orbit equation we have been looking for. For an ellipse (but not for hyperbola or parabola) we have 5A-20.

Conservation of Energy and the Orbit Equation 2

We now have an expression for r as a function of θ!

$$r = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-17)}$$

- Also, remember the expression for a conic section:

$$r = \frac{p}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-18)}$$

i.e. the motion of r is a conic section with “parameter ‘p’”

$$p = h^2/\mu \qquad \text{(5A-19)}$$

If $e < 1$, then the trajectory of r will be an ellipse. Thus we have proved **Kepler’s first law**, that the motion of the planets are ellipses with the sun at one focus.

5A-17 is sometimes known as the ‘orbit equation’. It shows that the orbit is a conic as this is an conic form. For the elliptical form ie: $p=a(1-e^2)$ there is a nice proof that this is equivalent to $x^2/a^2+y^2/b^2=1$ which you can have a go at yourselves!

Note that…

Orbit equation:

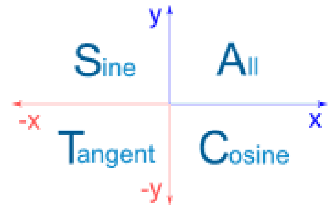
$$r = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)}$$

True anomaly

Usually = 0 but can be set to Argument of periapsis 'ω'

Looking for θ? 2 solutions!

Solution 1: θ
Solution 2: 2π-θ



1. Theta is the true anomaly which is measured from periapsis. If we set theta0 to be argument of periapsis then we can find ascending/descending nodes.
2. If we are looking for θ, there will be 2 solutions (your calculator only gives you one) and we need to use the quadrant rule to find the other one.

Steps in Solving the Kepler Problem

1. **Use conservation of angular momentum to show that the motion is planar**
2. **Find the magnitude of h in terms of r and θ**
3. **Derive the equation of motion of the orbiting body (in polar coordinates)**
4. **Resolve eq. of motion tangentially to prove K2**
5. **Resolve radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.**
6. **Use conservation of energy to solve for the constants of integration in 5 [Annex]**

Hurray!

Numerical Example

Q: Find the true anomaly of a spacecraft as it enters the Van Allen belt around the Earth at an altitude of 1500km, if its radius of perigee r_p=6500km and radius of apogee r_a=60000km.

A: $r = 6378.10^3 + 1500.10^3 = 7878.10^3 m$

$$\mathrm{e} = \frac{r_a - r_p}{r_a + r_p} = 0.8045$$

$$\mathrm{a} = \frac{r_a + r_p}{2} = 33.25\mathrm{x}10^6 m$$

$$r = \frac{a(1-e^2)}{1 + e\ \cos(\theta - \theta_0)}$$

Substituting in for r, a and e and gives…

$\theta - \theta_0 = 0.918\mathrm{rad}$ =52.6° or 5.366rad =307.4°   Check it!

Note that for positive cos (remember Quadrant rule), we could also have an answer the other side of the horizontal axis ie: true anomaly = 2π-0.0502rad.

Types of Orbits

- In the Annex we developed an expression for the SPECIFIC energy ‘ε’ of an orbit:

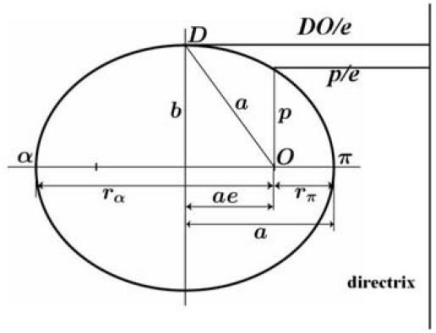
$$\varepsilon = (e^2 - 1)\frac{\mu^2}{2h^2} \qquad (5A\text{-}15)$$

- Let’s consider the following cases:

e	Conic Section	Energy ε
=0	Circle	<0
>0, <1	Ellipse	<0
=1	Parabola	=0
>1	Hyperbola	>0

If we substitute in the values for the eccentricity, we can use the equation to find the specific energy.

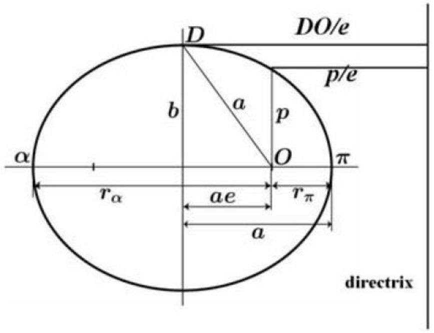
Some Properties of Elliptical Trajectories 1



- Kepler's Third Law relates the square of the period of a planet's orbit, T^2, to the cube of its semi-major axis, a^3.
- To prove this law we need to find an expression with T and a.
- We can construct a geometrical argument to prove Kepler was correct.

We will now consider the case of an ellipse in some detail before comparing some properties with the other classes of orbit.

Reminder of Elliptical orbits

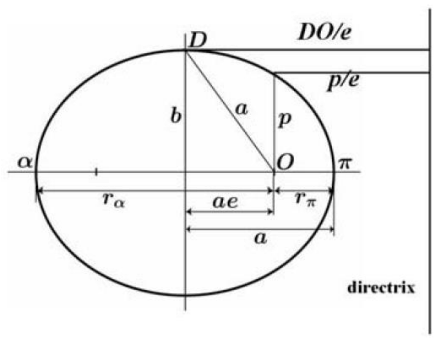


- If a is the semi-major axis:

$$2a = r_p + r_a \qquad (1\text{-}1)$$

- r_p is called periapsis.
- r_a is called apoapsis.

Some Properties of Elliptical Trajectories 2



- If a is the semi-major axis:

$$2a = r_p + r_a \qquad \text{(1-1)}$$

- r_p and r_a occur at θ=0 and π.
- Using $r = \dfrac{a(1-e^2)}{1+e\cos(\theta-\theta_0)}$

We have:

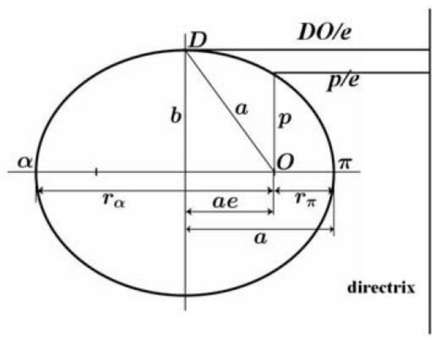
$$r_p = \frac{a(1-e^2)}{1+e} = a(1-e)$$

$$r_a = \frac{a(1-e^2)}{1-e} = a(1+e) \qquad \text{(5-10)}$$

Note: $(1-e^2) = (1+e)(1-e)$

Our orbit equation can be applied to our two useful points of the ellipse: the periapse and apoapse. We can derive equations for these radii. These can be turned around to find a and e.

Some Properties of Elliptical Trajectories 3



- The area of an ellipse is given by:

$$A = \pi ab \qquad \text{(1-5)}$$

- We have also previously shown that:

$$\frac{dA}{dt} = \frac{h}{2} \qquad \text{(4-28)}$$

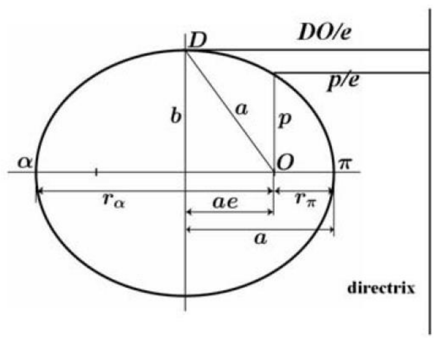
Integrating (w.r.t. time) we have:

$$A = \frac{h}{2}T \qquad \text{(5-11)}$$

Where T is the period of the orbit.

Now let's consider the period T.
We know from K2 that the line from the focus of the ellipse sweeps out equal areas in equal time, so lets start by considering the area of the ellipse.

Some Properties of Elliptical Trajectories 4



- Equating (1-5) and (5-3):

$$\pi ab = \frac{h}{2} T \qquad \text{(5-12)}$$

- Previously (in the Annex) we have shown that:

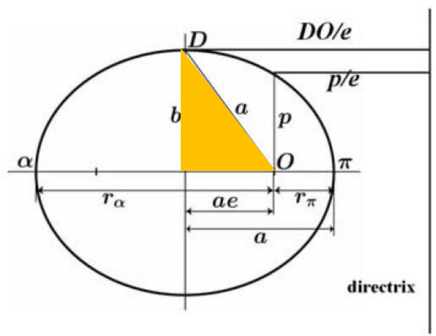
$$\frac{h^2}{\mu} = a(1 - e^2) \qquad \text{(5A-19)}$$

- Thus:

$$h^2 = \mu a(1 - e^2) \qquad \text{(5-13)}$$

We have two expressions for the area of the ellipse (1-5) and (5-3), so we can equate them. Remember we are looking for an equation relating T and a for Kepler's 3rd law.

Some Properties of Elliptical Trajectories 5



- Equating (1-5) and (5-3):

$$\pi ab = \frac{h}{2} T$$

- We have shown that:

$$h^2 = \mu a(1 - e^2) \qquad \text{(5-13)}$$

- So:

$$\pi a . a\sqrt{1 - e^2} = \frac{\sqrt{\mu a}\sqrt{1 - e^2}}{2} T$$

$$2\pi a^2 = T\sqrt{\mu a}$$

Pythagoras gives:

$$b = a\sqrt{1 - e^2}$$

Kepler's 3rd Law: $$\boxed{T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}} \qquad \text{(5-14)}$$

We can get two expressions for h2 and equate them, in order to develop an expression for T and a. Note that we need to put b (the semi-minor axis of the ellipse) in terms of a and e. The SQRT(1-e2) terms cancel and we can extract T (the period). If we do this then we can see that it proves Kepler's 3rd Law.

And now… (drum roll)

Kepler's Laws in mathematical formulation:

1st Law: $$r = \frac{a(1-e^2)}{1 + e\ \cos(\theta - \theta_0)}$$

2nd Law: $$\frac{dA}{dt} = \frac{h}{2}$$

3rd Law: $$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

Hurray!

Numerical example for K3

What is the altitude of an orbit with a period equal to the Earth's rotational period?

T=23.934hrs=23.934x3600s= 86162.4s [sidereal day],

Earth gravitational parameter μ=3.986e14 Nm^2kg^{-1}

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$$

so rearranging: $a^{3/2} = T\frac{\sqrt{\mu}}{2\pi} = 86162.4\frac{\sqrt{3.986e14}}{2\pi}$

So a=42.164 e6 m or 42,164km

Assuming a circular orbit,

Altitude=a - radius of Earth

So altitude= **35,786km**

We can use K3 to work out the altitude of an orbit with the same period as Earth's sidereal (relative to stars) period. What is this type of orbit called if it is also in the equatorial plane? The science fiction author Arthur C Clarke worked this out in 1945.

Summary

1. We have resolved the equation of motion radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.

$$r = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)}$$

2. The $\mathrm{r} = \frac{a(1-e^2)}{1+e\ \cos(\theta-\theta_0)}$ form of the orbit equation shows that orbits are elliptical and proves K1.
3. We have an expression for the specific energy:

$$\varepsilon = (e^2 - 1)\frac{\mu^2}{2h^2}$$

4. From 2 and ellipse geometry we have shown K3: $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$

References

1. Orbital Mechanics for Engineering Students, Howard D Curtis. 1st Edition, Publ Elsevier. Chapter 3.4, pp109-117
2. MIT Physics 8.01, Lecture 17
   http://web.mit.edu/8.01t/www/materials/modules/guide17.pdf

   MIT Physics 8.01, Lecture 17, Appendices

   http://web.mit.edu/8.01t/www/materials/modules/guide17Appendix.pdf

Test Yourself! (Feedback)

1. What does $\theta - \theta_0$ in the orbit equation represent?
2. Assuming argument of periapsis is 0, what is the value of true anomaly for the apogee point?
3. If the argument of perigee is 150, what value of $\theta - \theta_0$ would you use to find the ascending or descending node?
4. For a satellite with semimajor axis 7500000m, e=0.1, AOP=45 degrees, calculate the length of its position vector at the descending node.