Lecture 7

Response to F(*t*)=F_0sin(ω*t*)

$$x = x_H + x_P$$

$$x = X\, e^{-\zeta\omega_0 t}\sin(\omega_D t + \psi) + X\sin(\omega t - \phi)$$

Frequency Response Function representation

$$|H(\omega)| = \frac{X}{F_0} = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$-\angle H(\omega) = \phi = \tan^{-1}\left(\frac{c\,\omega}{k - m\omega^2}\right)$$

Lecture 8

- FRF and its properties
- Magnification Factor (normalized FRF)
- Unbalance excitation
- Unbalance Response function

FRF derivation using complex numbers

Complex number: $C = a + ib = |C|e^{i\alpha} = |C|(\cos\alpha + i\sin\alpha)$

1 DOF damped system with harmonic force using complex notation:

$$m\ddot{x} + c\dot{x} + kx = F_0 \exp(i\omega t)$$

$$x(t) \approx x_p(t) = X \exp(i\omega t)$$

Both, F_0 and X can be complex!

Differentiate x(*t*) and substitute into the EOM:

$$(-\omega^2 m + i\omega c + k) X \exp(i\omega t) = F_0 \exp(i\omega t)$$

$$(-\omega^2 m + i\omega c + k) X = F_0$$

$$X = (-\omega^2 m + i\omega c + k)^{-1} F_0 = H(\omega) F_0$$

$$x(t) = X \exp(i\omega t) = H(\omega) F_0 \exp(i\omega t)$$

Complete FRF

H(ω) is a complex function. It has the magnitude and phase angle for each ω, H(ω)=|H(ω)|*exp(i∠H(ω)). H(ω) depends on ω and parameters m, c, k.

$$H(\omega) = \frac{1}{(k - \omega^2 m) + i(\omega c)} = A + iB$$

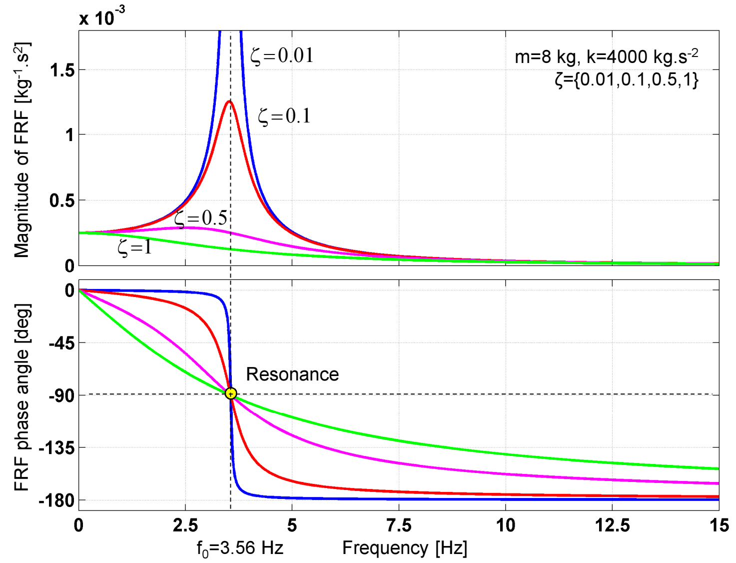
FRF magnitude and phase angle: $|H(\omega)| = \sqrt{A^2 + B^2}$ $\angle H(\omega) = B/A$

Input $F_0 \exp(i\omega t)$ → $\boldsymbol{H(\omega) = \dfrac{1}{k - \omega^2 m + i\omega c}}$ → **Output** $H(\omega) F_0 \exp(i\omega t)$

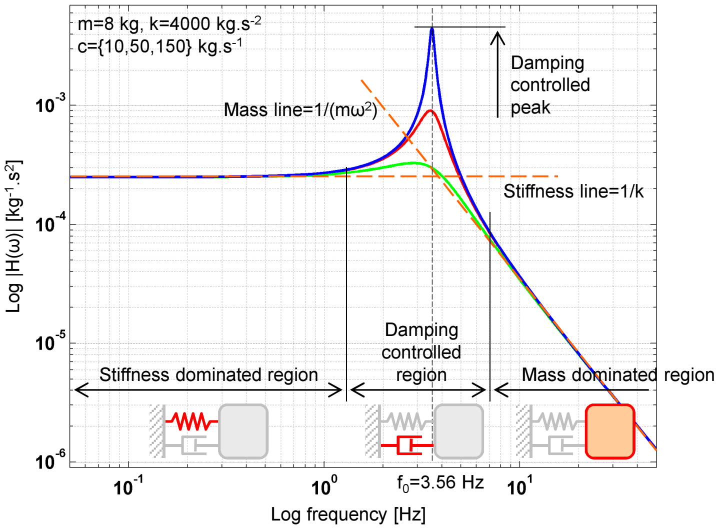
Try this in Matlab for c=0, 5, 20, 100, 380

```
» m=8; k=4000; c=10;
» ω=linspace(0,2*pi*30,1e3);
» H=1./(k-ω.^2*m+i*ω*c);
» figure, plot(ω/2/pi,abs(H)), grid
» figure, plot(ω/2/pi,angle(H)), grid
```


Example FRF



FRF properties in log-log scale



Magnification Factor

Consider the FRF magnitude where $r=\omega/\omega_0$ is the frequency ratio.

$$|H(\omega)|=\frac{|X|}{F_0}=\frac{1}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}=\frac{1}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

This form of the FRF is used to introduce the **Magnification Factor** (MF):

$$\boxed{MF=\frac{|X|}{F_0/k}=\frac{|X|}{X_{static}}=\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$$

Magnification Factor maximum:

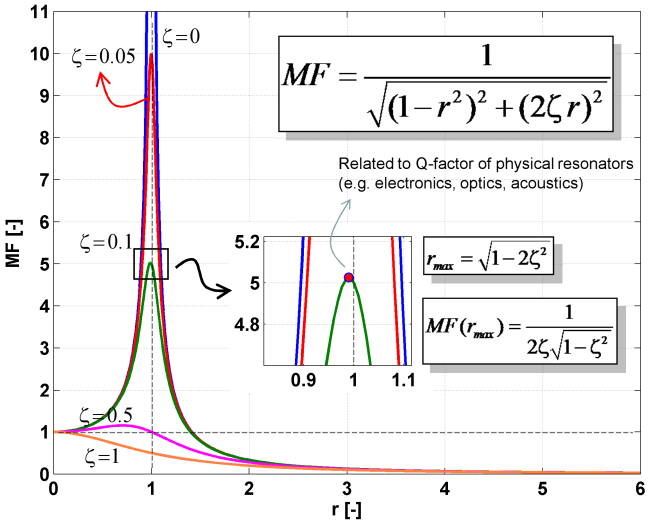
$$\frac{\partial MF}{\partial r}=\frac{\partial}{\partial r}((1-r^2)^2+(2\zeta r)^2)^{-1/2}=0$$

$$r_{max}=\sqrt{1-2\zeta^2} \qquad MF(r_{max})=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Try this in Matlab for ze=0, 0.05, 0.1, 0.5, 1

```
» m=8; k=4000; ze=0.05;
» r=linspace(0,8,1e3);
» MF=1./sqrt((1-r.^2).^2+(2*ze*r).^2);
» figure, plot(r,abs(MF)), grid
```

Magnification factor



Unbalance excitation

All systems with rotating parts can suffer from or use unbalance effects.

Useful

Vibration bio-feedback (mobiles, …)

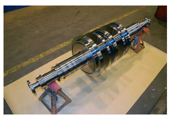
Concrete vibration compaction

Helicopter Active Vibration Control

Detrimental

Helicopter rotors

Propulsion

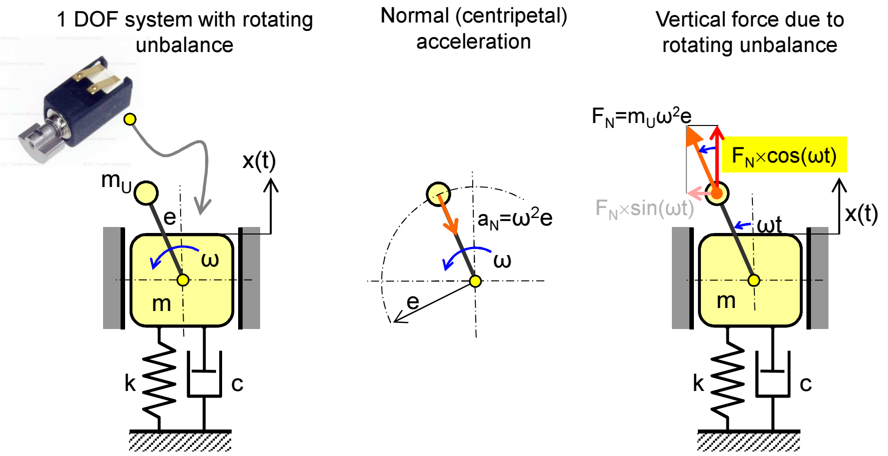


Shafts

Turbo-chargers

DAMAGE!

Excitation due to unbalance



Based on the analysis shown here we see that the **amplitude** of the harmonic excitation due to rotating unbalance is $m_U\omega^2 e$!

Steady state vibrations due to unbalance

From previous lecture:

Input $F_0 \exp(i\omega t)$ → $$\boxed{H(\omega) = \frac{1}{k - \omega^2 m + i\omega c}}$$ → Response $x(t) = H(\omega) F_0 \exp(i\omega t)$

$$|X| = |H(\omega)| F_0$$

We have the frequency-dependent force amplitude $F_0 = m_U\omega^2 e$. The steady-state response is:

$$|X| = |H(\omega)|(m_U\omega^2 e) = \frac{m_U\omega^2 e}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

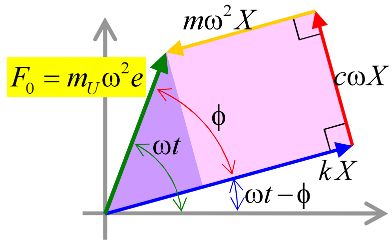
When using the frequency ratio $r = \omega/\omega_0$, the response is:

$$|X| = \frac{m_U\omega^2 e(m/k)(k/m)}{k\sqrt{(1 - m\omega^2/k)^2 + (c\omega/k)^2}} = \frac{m_U e}{m}\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Steady-state unbalance response

As before, take the ratio of the (displacement) output to the (unbalance) input.
Unbalance Response:

$$\boxed{UR = \frac{|X|m}{em_U} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}$$



Unbalance Response maximum:

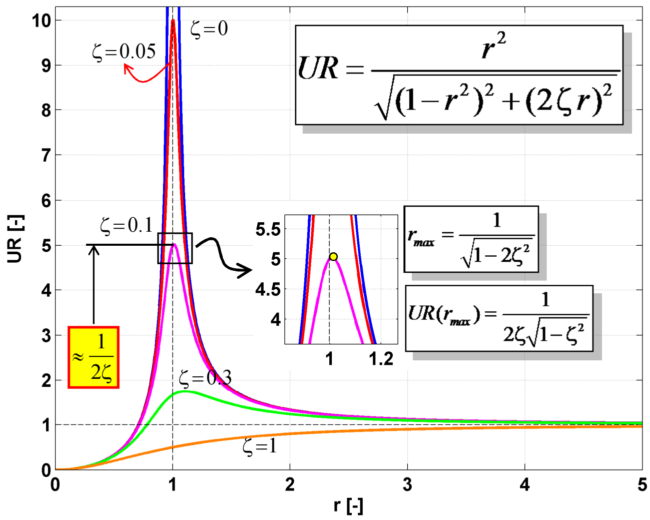
$$UR(0) = 0,\ UR(r \to \infty) = 1 \quad \frac{\partial UR}{\partial r} = 0$$

$$r_{max} = \frac{1}{\sqrt{1-2\zeta^2}} \quad UR(r_{max}) = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Try this in Matlab for ze = 0, 0.05, 0.1, 0.3, 1

```
» m=8; k=4000; ze=0.05;
» r=linspace(0,8,1e3);
» UR=(r.^2)./sqrt((1-r.^2).^2+(2*ze*r).^2);
» figure, plot(r,UR), grid
```


Steady-state unbalance response



Summary

- FRF is a complex function which contains information about the amplitude gain and the phase delays during harmonic excitation
- FRF and MF are used for dynamic analysis of steady-state harmonic vibrations
- Resonant frequency $\approx$ damped natural frequency (i.e. not =)
- Unbalance introduces harmonic excitation with frequency-dependent amplitude
- Unbalance response can be found directly using the UR function
- FRF, MF and UR functions have characteristic values for r = 0,1,Inf (see their respective graphs)