THIN AEROFOIL THEORY

AIMS

- To introduce thin aerofoil theory for uncambered and cambered aerofoils
- To introduce the lumped vortex method.

1 INTRODUCTION

A potential model of the low speed flow over a lifting aerofoil can be obtained by placing vortex sheets on the upper and lower surfaces of the aerofoil. However for an arbitrary aerofoil an analytic solution for the required vorticity distribution cannot be obtained and so a numerical panel method has to be used. Classical thin aerofoil theory takes a different approach, relying on simplifying assumptions to reduce the problem to one that may be solved analytically.

2 THIN AEROFOIL THEORY PRELIMINARIES

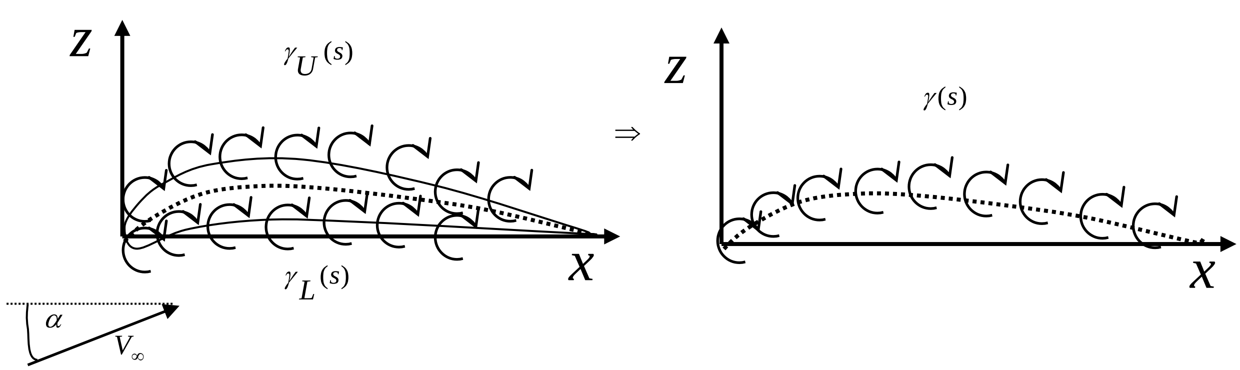
In developing thin aerofoil theory it will be assumed that the *x*-axis is the chord line of the aerofoil and that the *y*-axis passes through the leading edge.

2.1 Assumptions

(1) *For a thin aerofoil the upper and lower surfaces become indistinguishable far from the aerofoil so a single vortex sheet is used, situated on the mean camber line.*

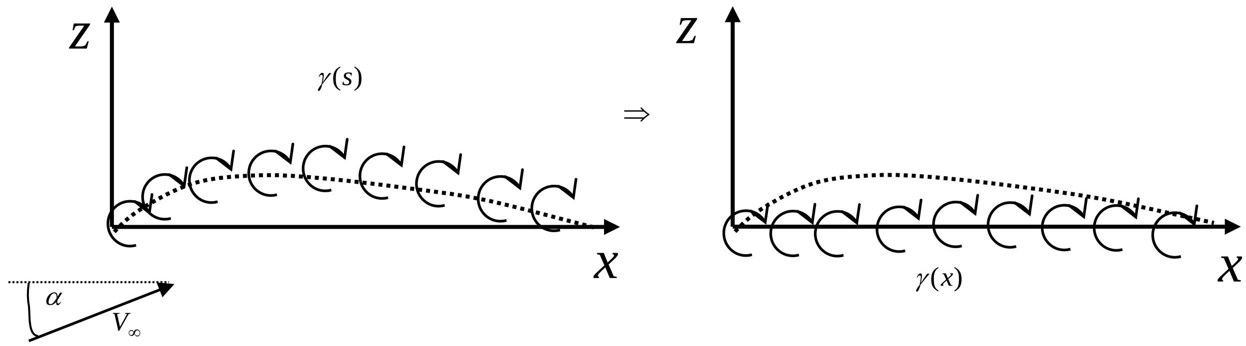
Effectively, the flow past the aerofoil is considered to be the same as the flow past an infinitely thin aerofoil, i.e. one with zero thickness.

The vorticity distribution or strength $\gamma(s)$ of the <u>single</u> vortex sheet, that makes the **camber line** a stream line of the flow must be identified (note that the camber line and the chord line are the same for a symmetric aerofoil). This is still difficult to solve analytically for a cambered aerofoil.



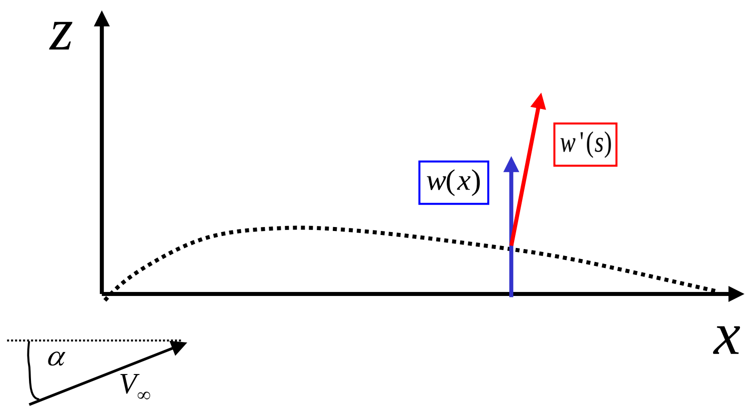
(2) *The camber of the aerofoil is small so that the vortex sheet can be moved onto the chord line.*

Again the solution process is to identify the vorticity distribution $\gamma(s)$ that makes the **camber line** a stream line (note: although the vortex sheet is moved to the chord line, it is still the camber line and NOT the chord line which is made a stream line of the flow). This assumption greatly simplifies the solution of the problem.



(3) *The camber of the aerofoil is small so that the vortex-induced component of velocity normal to the camber line can be approximated by the vertical velocity induced on the chord line with the same x coordinate*

$$w'(s) \approx w(x)$$



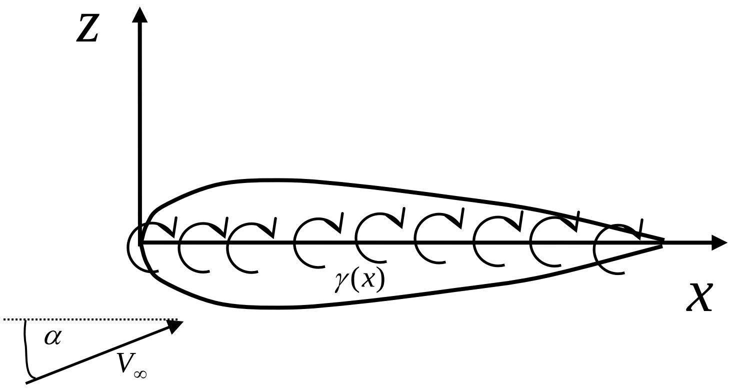
2.2 Kutta Condition

The *Kutta condition* implies that upper and lower surface velocities are equal at the trailing-edge. Since the strength of the vortex sheet is equal to the jump in velocity across the vortex sheet this means that

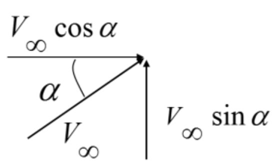
$$\gamma(TE) = 0$$

3 UNCAMBERED OR SYMMETRIC AEROFOILS
3.1 Derivation of Solution

In this section we consider the derivation of the solution for a symmetric or uncambered aerofoil. In this case the camber line is the same straight line as the chord line. This means that the assumptions (2) and (3) are unnecessary. In this case the flow past the thin aerofoil is approximated as being the same as the flow past a flat plate aerofoil. Making the camber line a stream line is the same as making the flow *parallel* to the surface of this flat plate aerofoil.



For the camber line to be a stream line of the potential flow induced by the combination of a free stream and the vortex sheet, the velocity perpendicular to the line, i.e. vertically must be zero. The vertical component due to the free stream is given by $V_\infty \sin\alpha$



and the vertical component due to the vortex sheet is $w(x)$ so

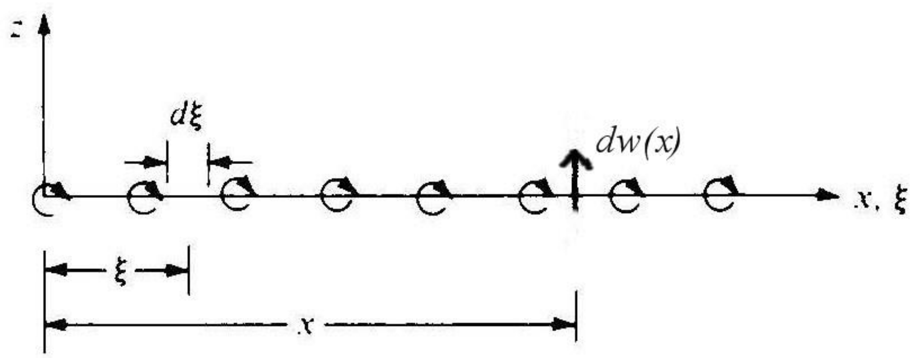
$$V_\infty \sin\alpha + w(x) = 0$$

Now for small angles of incidence α, this becomes

$$V_\infty \alpha + w(x) = 0 \qquad (1)$$

for $0 \le x \le c$.

To obtain an expression for *w(x)* at a point with chordwise location *x*, *co*nsider the velocity induced by a small element of the sheet, located at a point with chordwise location ξ



Then the strength of the vortex element is

$$d\Gamma = \gamma(\xi)d\xi$$

and so this element induces a vertical velocity equal to

$$dw(x) = \text{"}dV_\theta = -\frac{d\Gamma}{2\pi r}\text{"}$$
$$= -\frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

Then the total vertical velocity induced by the whole vortex sheet yields

$$w(x) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)}$$

Substituting into equation (1) gives

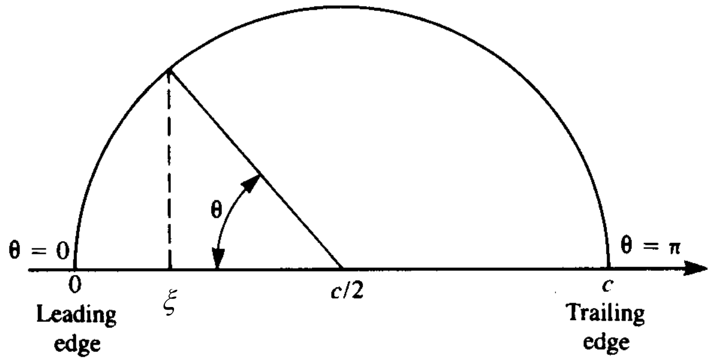
$$V_\infty \alpha = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{(x-\xi)} \qquad (2)$$

This is the *special case* of the FUNDAMENTAL THIN AEROFOIL EQUATION for a *symmetric* or *uncambered* aerofoil and expresses mathematically that "the camber line is a stream line".

This equation has been applied at a given evaluated at *x*, and if the incidence α is also given then the only unknown is γ. The central problem of thin aerofoil theory is to solve equation (2) for the vorticity distribution γ subject to the Kutta condition.

For the symmetric aerofoil the first step to a solution is to use the following transformation

$$\xi = \frac{c}{2}(1-\cos\theta) \tag{3a}$$



Then since x is a fixed on the chord line it corresponds to a particular value of θ, say θ_0.

$$x = \frac{c}{2}(1-\cos\theta_0) \tag{3b}$$

Then differentiating equation (3) gives

$$d\xi = \frac{c}{2}\sin\theta\, d\theta \tag{3c}$$

Substituting into equation (2) gives

$$V_\infty \alpha = \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{(\cos\theta - \cos\theta_0)} \tag{4}$$

This can then be solved subject to the Kutta condition via the mathematical theory of integral equations. You ARE NOT expected to be able to solve this equation. Instead given the solution which is

$$\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}$$

you ARE expected to be able to substitute it into right hand side of equation (4) and check that it equals the left hand side of the equation. This together with showing that it satisfies the Kutta condition will show that the given function is the required solution.

$$RHS_{(4)} = \frac{\alpha V_\infty}{\pi}\int_0^\pi \frac{(1+\cos\theta)\, d\theta}{(\cos\theta-\cos\theta_0)} = \frac{\alpha V_\infty}{\pi}\left(\int_0^\pi \frac{d\theta}{(\cos\theta-\cos\theta_0)} + \int_0^\pi \frac{\cos\theta\, d\theta}{(\cos\theta-\cos\theta_0)}\right)$$

These are standard integrals (which would be given on useful equation page in the exam)

$$\int_0^{\pi} \frac{\cos n\theta \, d\theta}{(\cos\theta - \cos\theta_0)} = \frac{\pi \sin n\theta_0}{\sin\theta_0}$$

Hence

$$RHS_{(4)} = \frac{\alpha V_\infty}{\pi}(\pi) = \alpha V_\infty = LHS_{(4)}$$

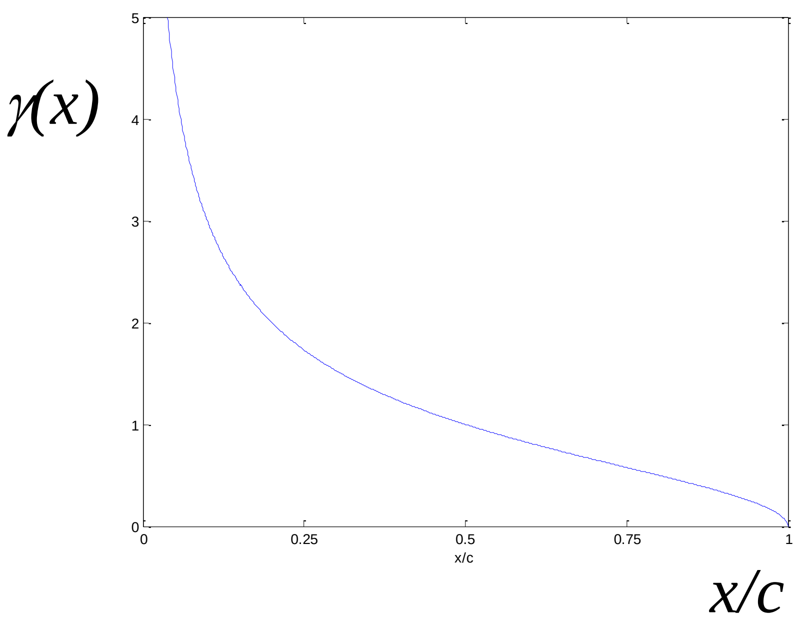
thus the given function $\gamma(\theta)$ is a solution of equation (4). Then check the Kutta condition

$$\gamma(TE) = \gamma(\pi) = 2\alpha V_\infty \frac{0}{0}$$

which is indeterminant. Need to use L'Hospital's rule, (see handout 1, Maths)

$$\gamma(TE) = 2\alpha V_\infty \frac{(-\sin\pi)}{\cos\pi} = 0$$

i.e. the given function satisfies the Kutta condition. So the given function is the required solution of the FUNDAMENTAL THIN AEROFOIL EQUATION and is shown in the figure below.



3.2 Lift Distribution

The lift predicted using thin aerofoil theory can be found using the Kutta-Joukowski theorem

$$l = \rho_\infty V_\infty \Gamma$$

where the total circulation Γ is given by

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi = \frac{c}{2}\int_0^\pi \gamma(\theta)\sin\theta\, d\theta = \pi\alpha\, cV_\infty$$

The lift is equal to

$$l = \pi\alpha\, c\rho_\infty V_\infty^2$$

The lift coefficient equals

$$c_l = a_0\alpha$$

where the 2D lift curve slope $a_0 = dc_l / d\alpha$ is given by

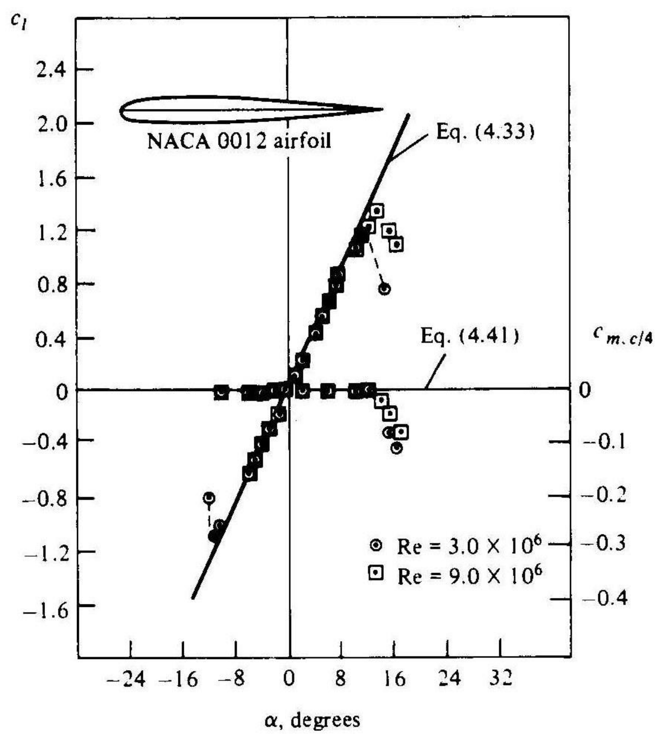
$$a_0 = 2\pi$$

Thus there is a direct linear relationship between the lift and the incidence.

An alternative way to derive this result is to consider the lift contribution of an infinitesimal section of the vortex sheet

$$dl = \rho_\infty V_\infty d\Gamma = \rho_\infty V_\infty \gamma(\xi)\, d\xi$$

This shows the physical significance of the vortex sheet strength in that it is directly related to the chordwise lift distribution on the aerofoil.



In the figure above the predicted lift is compared with experimental data for a NACA0012 aerofoil. Good agreement is found up to quite large angles of attack when the real aerofoil stalls.

3.3 Pitching Moment

The moment about the leading edge predicted by the thin aerofoil solution can be calculated as follows. Consider an element of the vortex sheet of strength $d\Gamma = \gamma(\xi)d\xi$ located a distance ξ from the leading edge. The normal force due to this element is equal to dN_{2D} and for small angles of attack $dN_{2D} \approx dl$ which is given using Kutta-Joukowski by $dl = \rho_\infty U_\infty d\Gamma$. The element creates a moment about the leading edge of size

$$dm_{LE} = -\xi dN_{2D} \approx -\xi dl$$

The moment about the leading edge is then given by

$$m_{LE} = -\rho_\infty U_\infty \int_0^c \xi\, \gamma(\xi)\, d\xi$$

Integrating and putting into non-dimensional form gives

$$c_{m_{LE}} = -\frac{\pi}{2}\alpha = -\frac{c_l}{4}$$

The two moment reference centres can be identified

$$\bar{x}_{cp} = -\frac{c_{m_{LE}}}{c_l} = 0.25$$

$$\bar{x}_{ac} = -\frac{dc_{m_{LE}}}{dc_l} = 0.25$$

i.e. both the centre of pressure and the aerodynamic centre are located at the quarter chord point.

4 CAMBERED AEROFOILS

For the exam you do not need to know details of the derivation for a cambered aerofoil, only for a symmetric aerofoil as described above, so only an overview is given below. More details of the derivation are available in RefI4. You **do need** to be able to quote equation (9), apply the theorems and formulae that are derived in section 4.2 and 4.3 and explain in words the effect of camber on lift, pitching moment and the centre of pressure.

4.1 Fundamental lifting line equation for a cambered aerofoil

Making assumptions of small angle of incidence α and small camber, the FUNDAMENTAL THIN AEROFOIL EQUATION for a *cambered* aerofoil is

$$V_\infty\left(\alpha-\frac{dz}{dx}\right)=\frac{1}{2\pi}\int_0^c\frac{\gamma(\xi)d\xi}{(x-\xi)}$$

(6)

where dz/dx defines the camber line. This equation expresses mathematically the statement that "the camber line is a stream line". This equation has been applied at a given *x*, and if the incidence α and camber *dz/dx* are also given then the only unknown is γ. The central problem of thin aerofoil theory is to solve equation (2) for the vorticity distribution γ subject to the Kutta condition. The solution is

$$\gamma(\theta)=2V_\infty\left(A_0\frac{(1+\cos\theta)}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin n\theta\right)$$

where the transformation (3a),(3b),(3c) relates ξ and θ. The Fourier coefficients are given by

$$A_0=\alpha-\frac{1}{\pi}\int_0^\pi\frac{dz}{dx}d\theta_0,\quad A_n=\frac{2}{\pi}\int_0^\pi\frac{dz}{dx}\cos n\theta_0\,d\theta_0$$

4.2 Lift Distribution

The lift predicted using thin aerofoil theory can be found using the Kutta-Joukowski theorem

$$l=\rho_\infty V_\infty\Gamma$$

where the total circulation Γ is given by

$$\Gamma=\int_0^c\gamma(\xi)\,d\xi=\frac{c}{2}\int_0^\pi\gamma(\theta)\sin\theta\,d\theta=cV_\infty\left(\pi A_0+\frac{\pi}{2}A_1\right)$$

i.e. only the first two terms of the Fourier series make any contribution.

The lift is equal to

$$l=\rho_\infty V_\infty^2c\left(\pi A_0+\frac{\pi}{2}A_1\right)$$

The lift coefficient equals

$$c_l=\pi\left(2A_0+A_1\right) \tag{8}$$

which on substituting becomes

$$c_l = 2\pi\left\{\alpha + \frac{1}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)(\cos\theta_0 - 1)d\theta_0\right\}$$

In other words

$$c_l = a_0\left(\alpha - \alpha_{l=0}\right) \tag{9}$$

Comparing this to the uncambered aerofoil results, the 2D lift curve slope a_0 is the same i.e.

$$a_0 = 2\pi$$

however there is now an offset. The zero-lift incidence $\alpha_{l=0}$ is given by

$$\alpha_{l=0} = -\frac{1}{\pi}\int_0^{\pi}\left(\frac{dz}{dx}\right)(\cos\theta_0 - 1)d\theta_0 \tag{10}$$

Notes

(1) Positive camber shifts the lift curve shifted upwards

(2) $\alpha_{l=0}$ depends on *($\cos\theta$-1)* so it is most sensitive to *dz/dx* at TE i.e. changes in camber have most effect when applied towards TE. This is the reason that TE flaps are able to increase the lift at a given geometric incidence as they effect the camber at the TE and so change the zero lift incidence.

4.3 Pitching Moment

A similar analysis to that for uncambered or symmetric aerofoils gives

$$c_{m_{LE}} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)$$

or eliminating A_0

$$c_{m_{LE}} = -\frac{c_l}{4} - \frac{\pi}{4}\left(A_1 - A_2\right) \tag{11}$$

i.e. the uncambered result plus a constant offset. The offset is the ‘zero-lift pitching moment’ c_{m_0}**.** The two moment reference centres can be identified

$$\begin{aligned} \bar{x}_{cp} &= -\frac{c_{m_{LE}}}{c_l} = 0.25\left(1 + \frac{\pi}{c_l}(A_1 - A_2)\right) \Rightarrow \textit{shifts with } \alpha \\ \bar{x}_{ac} &= -\frac{dc_{m_{LE}}}{dc_l} = 0.25 \end{aligned} \tag{12}$$

i.e. both the aerodynamic centre is located at the quarter chord point but the centre of pressure moves.

5 SIMPLIFICATION OF THIN AEROFOIL THEORY

5.1 Basic Lumped Vortex Method

The thin aerofoil theory problem has a continuous $\gamma(s)$ distribution that requires an analytic solution. For preliminary calculations when precise details of the local flow are not required rather than using full thin aerofoil theory, it is possible to use a "lumped vortex" approximation. This method will in fact be seen to be like the generalised 2D point singularity methods described in the previous handout, but with the location of singularities and control points linked to a panel representation of the geometry.

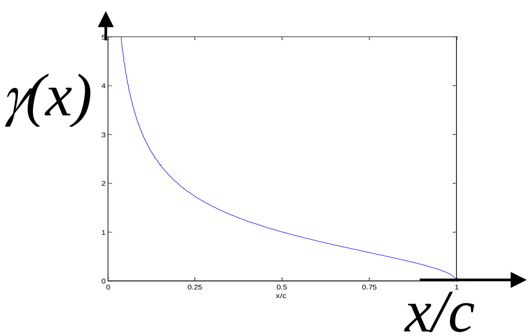
The lumped vortex method has two stages

- The camber line of the thin aerofoil is split into a series of N flat panels each with a point vortex on it.
- A "no normal flow" condition is then applied on each panel and the set of equations solved for the vortex strengths in a similar way to previous panel method solutions.

Now to get the correct lift a Kutta condition would need to be imposed, this can be done in a number of ways but only the method described in Katz and Plotkin is implemented here. In their approach the Kutta condition is not explicitly applied. It has already applied at the trailing edge in thin aerofoil theory and the lift predicted there is matched by enforced choices for the location of the singularities and control points.

The appropriate locations are established by looking at the case of a thin *symmetric* aerofoil with $N=1$. In this simple case let us force the predicted lift to be the same as thin aerofoil theory predicts, since enforcement of the Kutta condition there reveals good agreement with experimental lift and moments. This means the circulation of the point vortex must equal the total circulation of the thin aerofoil vortex sheet, because lift is related to circulation via Kutta-Joukowski. Thus the desired vortex strength is known and the question is then where should the vortex and the control point have been located to get this value as the solution of the problem if it was not known in advance. These vortex and control point locations are then used for the general case of multiple panels.

On a thin *symmetric* aerofoil the continuous $\gamma(s)$ distribution of thin aerofoil theory is replaced by a single point or "lumped" vortex of strength Γ placed at the centre of pressure (x/c = 0.25).



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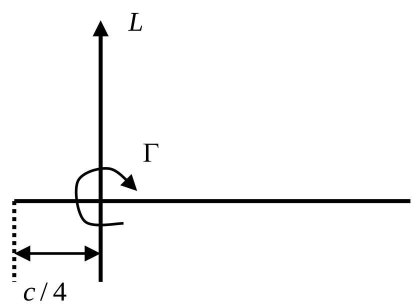


Now if we were solving the problem for the point vortex circulation there would be just one unknown, Γ, so one boundary condition of *"no normal flow"* could be imposed on the panel, i.e. would have one control point. This *"no flow normal"* would fix the circulation. But in this case Γ is known and the location is the unknown to be solved for. The choice of imposing this at the aerofoil (panel) trailing edge (i.e. where a Kutta condition is applied) which would seem to be the intuitive choice, in fact would mean the method would not predict lift to be the same as thin aerofoil theory.

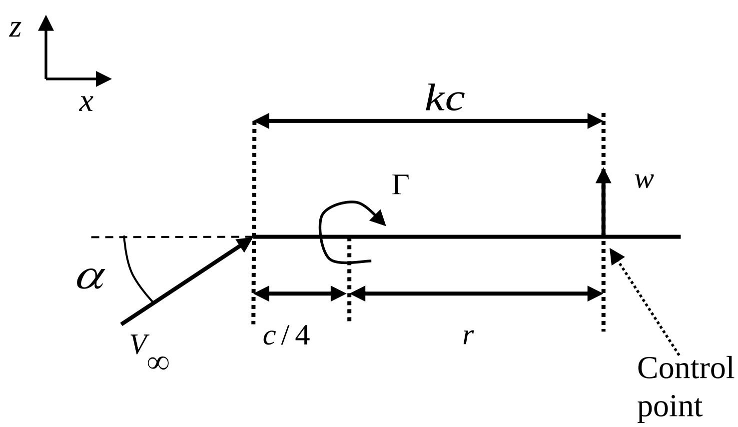
The total lift for the lumped vortex model should be the same as that which would have been predicted by thin aerofoil theory, so the total circulation must be the same i.e.

$$\Gamma = \int_0^c \gamma(x)\, dx \qquad \gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}$$

where $\gamma(x)$ is the thin aerofoil vorticity distribution for a symmetric aerofoil. So instead of imposing *"no flow normal"* at the trailing edge, the same condition is applied at a control point somewhere else on the panel so that the same lift as full thin aerofoil theory is predicted.

- (Note:details of the flow near the aerofoil won't be correct)

If the aerofoil is horizontal , then a *"no flow normal"* or *"flow tangency"* condition at the control point amounts to setting the vertical component of velocity to zero. Note that the vertical velocity w is made up of two components one due to the free stream and one due to the vortex.



Looking at the sketch of the situation, assume that the control point is a distance r from the vortex and a distance kc from the leading edge. The vertical induced velocity at the control point by the vortex only is

$$w_{vor} = v_\theta = -\frac{\Gamma}{2\pi r} = -\frac{\Gamma}{2\pi(kc - c/4)}$$

where the origin of the (r,θ) coordinate system is at the vortex. The vertical induced velocity at the control point by the free stream is

$$w_{fs} = V_\infty \sin\alpha$$

So for flow tangency

$$w = w_{vor} + w_{fs} = V_\infty \sin\alpha - \frac{\Gamma}{2\pi(kc - c/4)} = 0$$

and assuming small angles of attack

$$V_\infty \alpha - \frac{\Gamma}{2\pi(kc - c/4)} = 0$$

So far the fact that the vortex circulation must equal the same as the total circulation of thin aerofoil theory has not been enforced. This condition is necessary to ensure the lumped vortex model predicts the same lift as thin aerofoil theory.

The Kutta/Joukowski theorem applied to the lumped vortex flow is

$$l = \rho_\infty V_\infty \Gamma$$

This must be set equal to the total lift predicted by thin aerofoil theory

$$l = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2$$

So that

$$\Gamma = \pi\, \alpha\, c V_\infty$$

Then substituting into the flow tangency condition gives

$$V_\infty \alpha - \frac{\pi \alpha c V_\infty}{2\pi(kc - c/4)} = 0$$

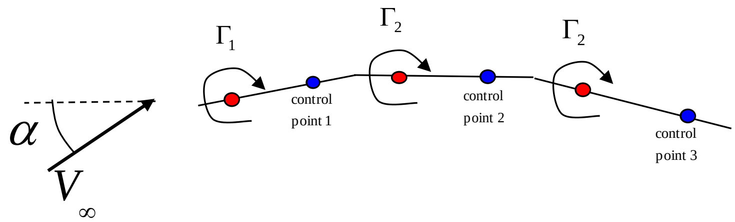
This can be solved to find that

$$k = 0.75$$

i.e. the control point is located at the *three-quarter chord* position. It is also known as the ‘collocation point’, or the ‘aft neutral point’. It effectively accounts for the Kutta condition and gives surprisingly good results. Although strictly speaking the method is *only valid in 2D,* it is widely applied ad hoc in 3D

5.2 General Lumped Vortex Method

When extending to approximate the flow past “thin aerofoils” using a set of panels; each panel has a point vortex at the panel quarter chord and a control point at the three-quarter chord.



More accurate than using a single lumped vortex, but still a cheap, simple method which can give good answers. The 3D extension of this method is called the vortex lattice method, see later notes.

5.3 Applications of the Lumped Vortex Method

The lumped vortex model is particularly useful for multi-element configurations for example

- tandem wings
- ground effect
- wind tunnel interference

When using the method for such cases there are multiple vortices because either there are several aerofoils or wings being represented or there are image vortices. The flow velocities induced by other vortices must be taken into account

- at control points – governs strength of vortices
- at vortex locations – governs lift generated by vortices

REVISION OBJECTIVES

You should be able to:

- Describe the simplifying assumptions made in "thin aerofoil" theory for both uncambered or cambered aerofoils
- Derive the fundamental equation for a symmetric aerofoil.
- Determine the lift and pitching moment distributions for a symmetric aerofoil from the vortex sheet strength distribution
- Explain the effect of camber on lift, pitching moment and the centre of pressure
- Quote and apply equation (9).
- Given the formula for $\alpha_{l=0}$, find the lift coefficient for a given cambered aerofoil at a specified incidence
- Demonstrate why the control point is placed at ¾ chord point for a lumped vortex model.
- Apply the lumped vortex model to ground effect and wing/flap calculations, see examples sheet.
- Use thin aerofoil theory to describe the effect of a flap on the lift curve slope.

EXTRA EAMPLE QUESTIONS

1 Cambered Aerofoil

Q: The circulation of the vortex sheet used to model a thin cambered aerofoil is

$$\Gamma = cV_\infty\left(\pi A_0 + \frac{\pi}{2}A_1\right)$$

where A_0 and A_1 are Fourier coefficients given by

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}d\theta_0\ , \quad A_1 = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos\theta_0 d\theta_0$$

Find an expression for the lift in terms of the Fourier coefficients and hence evaluate the lift for an aerofoil whose camber line is given by

$$\frac{z}{c} = 0.001 - 0.004\left(\frac{x}{c} - 0.5\right)^2$$

A: The lift is related to the circulation via Kutta-Joukowsky theorem

$$l = \rho_\infty V_\infty \Gamma$$

So substituting the given expression for the circulation of the vortex sheet gives

$$l = \rho_\infty c V_\infty^2\left(\pi A_0 + \frac{\pi}{2}A_1\right)$$

For the specific aerofoil given

$$\frac{z}{c} = 0.001 - 0.004\left(\frac{x}{c} - 0.5\right)^2$$

And so

$$\frac{dz}{dx} = -2\times 0.004\left(\frac{x}{c} - 0.5\right)$$

The expressions for the Fourier coefficients are in terms of θ_0, hence need to rewrite this gradient in terms of θ_0.

Then $x = \frac{c}{2}(1-\cos\theta_0)$ so $\frac{x}{c} - 0.5 = -\frac{1}{2}\cos\theta_0$, which gives

$$\frac{dz}{dx} = 0.004\cos\theta_0$$

Then using given formulae

$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi}\frac{dz}{dx}d\theta_0$$

$$= \alpha - \frac{0.004}{\pi}\int_0^{\pi}\cos\theta_0 d\theta_0$$

$$= \alpha - \frac{0.004}{\pi}\left[\sin\theta_0\right]_0^{\pi}$$

$$= \alpha$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}\cos\theta_0 d\theta_0$$

$$= \frac{2\times 0.004}{\pi}\int_0^{\pi}\cos^2\theta_0 d\theta_0$$

$$= \frac{2\times 0.004}{\pi}\int_0^{\pi}\left(\frac{1+\cos 2\theta_0}{2}\right)d\theta_0$$

$$= \frac{0.004}{\pi}\left[\theta_0 + \frac{\sin 2\theta_0}{2}\right]_0^{\pi}$$

$$= 0.004$$

So finally on substituting into lift formula

$$l = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 c\pi(\alpha + 0.002)$$

2 Cambered Aerofoil

Q: Consider the NACA23012 aerofoil. If $\alpha_{l=0} = -1.09^o$ calculate (a) the lift coefficient when $\alpha = 4^o$ (b) the moment coefficient about the leading edge when $\alpha = 4^o$. You may use without proof

$$A_1 = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}\cos\theta\, d\theta = 0.0954 \quad \text{and} \quad A_2 = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}\cos 2\theta\, d\theta = 0.0792$$

Note additional useful expressions would be given in an exam in the list of useful equations or on standard data sheets

A: (a) The formula for the lift coefficient is

$$c_l = 2\pi(\alpha - \alpha_{l=0})$$

Then since $\alpha_{l=0} = -1.09^o$ then as $\alpha = 4^o$

$$c_l = 2\pi(4 - (-1.09))\frac{\pi}{180} = 0.5582$$

(b) The moment coefficient about the leading edge is given by

$$c_{m_{LE}} = -\frac{c_l}{4} - \frac{\pi}{4}\left(A_1 - A_2\right)$$

And substituting the given values for the Fourier coefficients

$$c_{m_{LE}} = -0.13955 - \frac{\pi}{4} 0.0162 = -0.1523$$