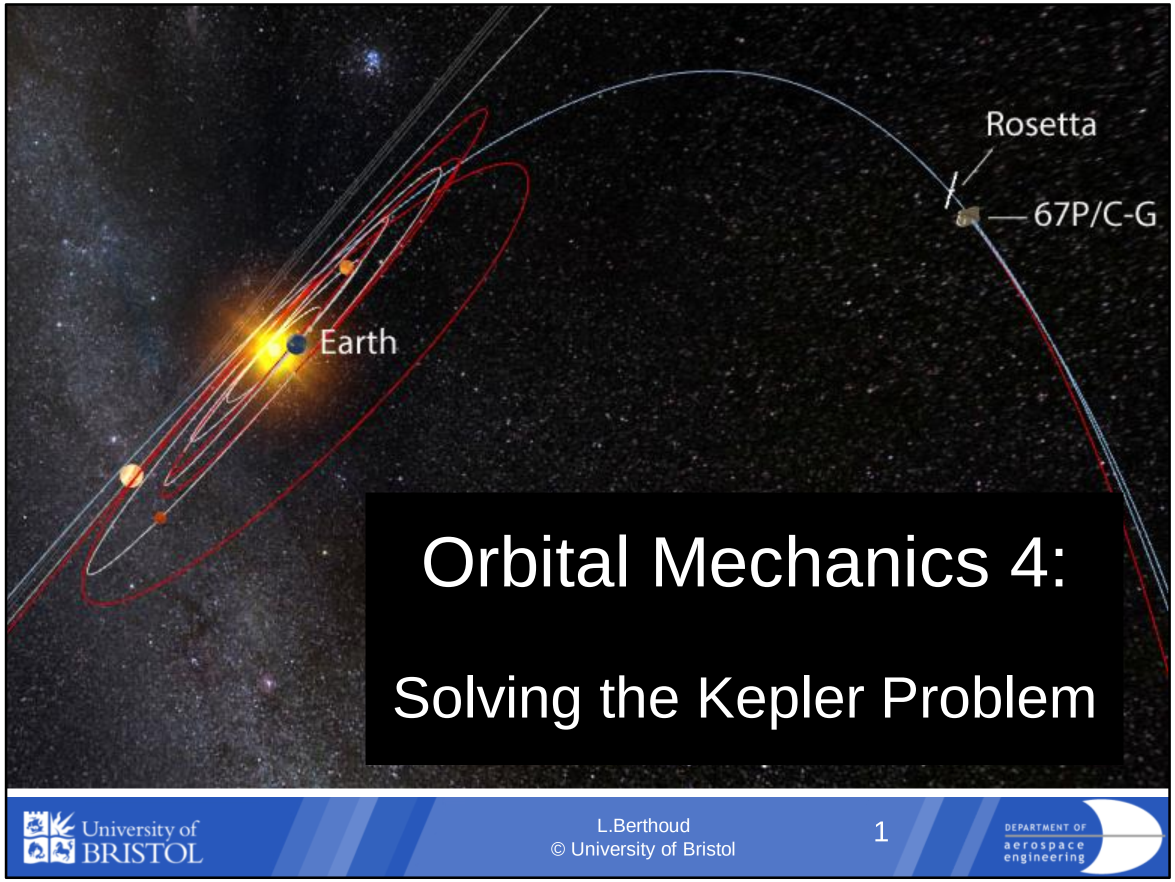


Notes about what has been said in class are provided below the slide on the following pages.

Learning Objectives

1. State why an analytical solution is useful
2. State the Kepler Problem and how to solve it
3. Use conservation of angular momentum to show that the motion is planar.
4. Find the magnitude of h in terms of r and θ
5. Derive the equation of motion of the orbiting body (in polar coordinates)
6. Resolve eq. of motion tangentially to prove K2

Revision: Examples

- We have found expressions for the acceleration of M_1 and m_2:

$$\ddot{\boldsymbol{r}}_1 = -\frac{Gm_2}{r^2}\hat{\boldsymbol{r}}$$

$$\ddot{\boldsymbol{r}}_2 = -\frac{GM_1}{r^2}\hat{\boldsymbol{r}} \qquad (3\text{-}8)$$

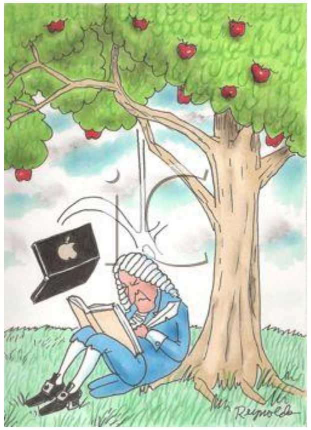
- Thus, if we know the position, $\boldsymbol{r}$, and velocity, $\dot{\boldsymbol{r}}$, of two objects at a given time, we can perform a numerical integration to determine their position after one 'time-step':

$$\dot{\boldsymbol{r}}(t+1) = \dot{\boldsymbol{r}}(t) + \ddot{\boldsymbol{r}}(\mathrm{t})\Delta t \qquad (3\text{-}11)$$

$$\boldsymbol{r}(t+1) = \dot{\boldsymbol{r}}(t)\Delta t + \frac{1}{2}\ddot{\boldsymbol{r}}(\mathrm{t})\Delta t^2$$

This is the same as v=u+at and s=ut+1/2at^2
This is numerical integration to find the position and velocity of two objects over time.

We need an analytical solution!



Newton and Kepler had no computers!

They found an analytical solution to describe the relative motion of one body relative to another where M>>m.

Special!

Because it's numerical, the method we have just described requires a computer to predict the position of the two bodies.
Newton and Kepler didn't have this luxury.
Instead, they found an analytical solution to describe the relative motion of one body to another, where the mass of one body is insignificant compared with the other, i.e. a special case of the two-body problem.

The ‘Kepler’ Problem

The *Kepler problem:*

“To find the radial and angular coordinates of an object in orbit as a function of time”.

The Orbit Equation: $r(\theta)$

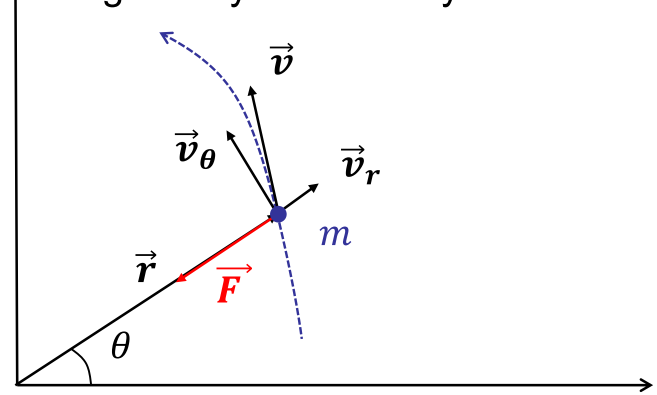
The Kepler Equation:

In classical mechanics, the Kepler problem is to find the position and speed of two bodies over time given their masses, initial positions and velocities. The solution allowed scientists to show that planetary motion could be explained by classical mechanics and Universal Law of gravitation and helped usher in the age of Enlightenment.

The "Kepler" Problem

We are going to use the conservation of angular momentum and energy to derive an expression for the motion of massless particle m.

We consider a single body acted on by a central force…



The Kepler problem consists of determining the radial and angular coordinates, of an object in a Keplerian orbit about the Sun as a function of time. $\vec{v}_r$ is the angular velocity and $\vec{v}_\theta$ is the angular velocity.

Steps in Solving the Kepler Problem

1. Use conservation of angular momentum to show that the motion is planar
2. Find the magnitude of $\boldsymbol{h}$ in terms of r and θ
3. Derive the equation of motion of the orbiting body (in polar coordinates)
4. Resolve eq. of motion tangentially to prove K2
5. Resolve radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the ‘orbit equation’.
6. Use conservation of energy to solve for the constants of integration in 5.

Law of Conservation of Momentum

"When the net external torque acting on a system about a given axis is zero, the total angular momentum of the system about that axis remains constant."

Demo: turntable and weights

$L = I\omega$

L: angular momentum
I : moment of inertia
ω : angular velocity

$p = mv$

p: linear momentum
m : mass
v : linear velocity

You may remember that linear momentum ‘p’ is proportional to mass m and linear speed v, and angular momentum L is proportional to moment of inertia I and angular velocity ω. Standing on a turntable allows us to demonstrate due to this law, if we increase the moment of inertia by spreading out our arms, the angular velocity decreases and vice versa.

Conservation of Angular Momentum 1

1. Use conservation of angular momentum to show that the motion is planar.

For large r and small m, we can write:

$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}} = \vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}}$,

or $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times m\dot{\boldsymbol{r}}$ (4-1)

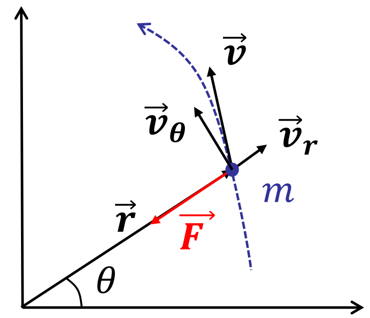
Where $\vec{\boldsymbol{L}}$ is angular momentum,

$\vec{\boldsymbol{r}}$ is position vector

$\vec{\boldsymbol{v}} = \dot{\boldsymbol{r}}$ is velocity vector

We can make an approximation for large r and small m, that L=rxmv.
Units for ang momentum are kgm2/s.
We use the RH rule to determine direction ie: L=rxmv is perpendicular to r and v.

Conservation of Angular Momentum 2



- Consider the rate of change of angular momentum:

$$\dot{\boldsymbol{L}} = \frac{d}{dt}(\boldsymbol{r} \times m\,\dot{\boldsymbol{r}}) = (\dot{\boldsymbol{r}} \times m\,\dot{\boldsymbol{r}}) + (\boldsymbol{r} \times m\,\ddot{\boldsymbol{r}}) \quad (4\text{-}2)$$

We will look at the rate of change of angular momentum, as this is also the torque and we are checking it is zero (from the law of conservation of momentum).
We use product rule to differentiate.
Mass m does not vary with time so it stays constant.

Conservation of Angular Momentum 3

$$\dot{\boldsymbol{L}} = \frac{d}{dt}(\boldsymbol{r} \times m\,\dot{\boldsymbol{r}}) = \cancel{(\dot{\boldsymbol{r}} \times m\,\dot{\boldsymbol{r}})} + (\boldsymbol{r} \times m\,\ddot{\boldsymbol{r}}) \qquad \text{(4-2)}$$

- The cross-product of velocity $\dot{\boldsymbol{r}}$ and momentum m$\dot{\boldsymbol{r}}$ is zero, because these vectors are parallel, so we have:

$$\dot{\boldsymbol{L}} = \boldsymbol{r} \times m\,\ddot{\boldsymbol{r}} = \boldsymbol{r} \times \boldsymbol{F} \qquad \text{(4-3)}$$

- As $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ are also parallel (directed to the origin) we have that $\dot{\boldsymbol{L}} = 0$, i.e. the angular momentum vector, $\vec{\boldsymbol{L}}$, is constant and perpendicular to $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$.

Therefore, the motion of m is planar.

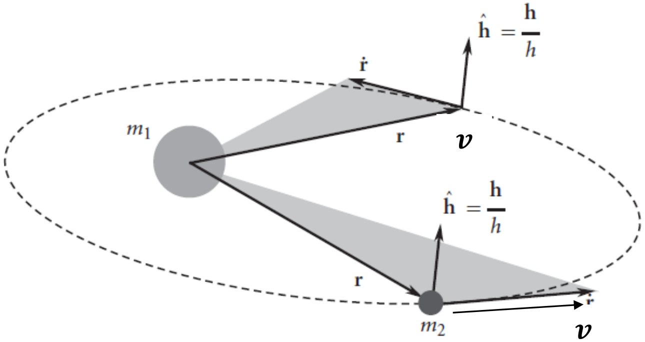
Acceleration is defined as being the change in velocity over time. Therefore, if acceleration is 0, then velocity isn't changing over time. Velocity must therefore be a constant. As the velocity is a constant, the quantities m and v are constants so L=mvrsinα=constant. So angular momentum is constant and as r and v describe motion of m, you can see that m's motion is in one plane.

Specific Angular Momentum

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \qquad \text{(4-4)}$$

$\boldsymbol{h}$ is the specific angular momentum $m^2 s^{-1}$
$\boldsymbol{r}$ is the relative orbital position vector m
$\boldsymbol{v}$ is the relative orbital velocity vector ms^{-1}

The magnitude of the vector $\boldsymbol{h}$ is denoted by h :



$$h = \|\boldsymbol{h}\|$$

If we divide L by m, we get 'specific' angular momentum. Note that some books and references use 'L' and some use 'H' for angular momentum, all use 'h' for specific angular momentum.

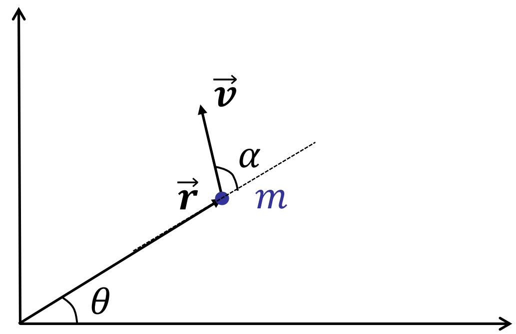
Steps in Solving the Kepler Problem

1. **Use conservation of angular momentum to show that the motion is planar**
2. Find the magnitude of h in terms of r and θ
3. Derive the equation of motion of the orbiting body (in polar coordinates)
4. Resolve eq. of motion tangentially to prove K2
5. Resolve radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.
6. Use conservation of energy to solve for the constants of integration in 5.

Magnitude of h…1

Question: What is the magnitude of $\vec{\boldsymbol{h}}$?

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$$



Answer:

Remember:

1. $a = \|\vec{\boldsymbol{a}}\|$
2. $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\alpha\, \hat{\boldsymbol{n}}$

So:

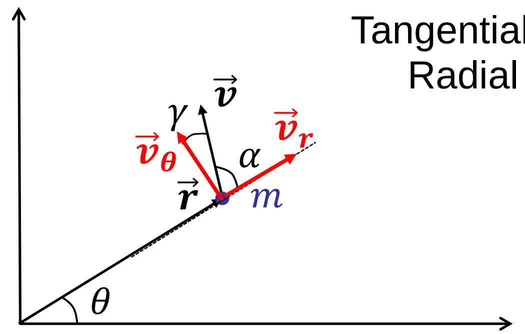
$$h = |\boldsymbol{r}||\boldsymbol{v}| \sin\alpha \qquad (4\text{-}5)$$

Alpha is the angle between r and v.
$\hat{\boldsymbol{n}}$ disappears, as it is the magnitude of vector h that we seek.

Magnitude of h ...2

Question: What is the magnitude of $\vec{\boldsymbol{h}}$?

Tangential component $\vec{\boldsymbol{v}}_{\boldsymbol{\theta}}$
Radial component $\vec{\boldsymbol{v}}_{\boldsymbol{r}}$



Answer:

$$h = |\boldsymbol{r}||\boldsymbol{v}|\sin\alpha \; = |\boldsymbol{r}||\boldsymbol{v}|\cos\gamma \qquad \text{(4-6)}$$

γ is the flight path angle, tan γ=$\vec{\boldsymbol{v}}_{\boldsymbol{r}}$/$\vec{\boldsymbol{v}}_{\boldsymbol{\theta}}$ and $v = \sqrt{{v_\theta}^2 + v_r^2}$

The tangential component in the theta direction is called $\vec{\boldsymbol{v}}_{\boldsymbol{\theta}} = v\cos(\gamma)$. The component in the radial direction is called $\vec{\boldsymbol{v}}_{\boldsymbol{r}} = v\sin(\gamma)$. From the definition of the cross product h=rvsinα which is a constant. Angle γ (gamma) is defined as the 'flight path angle'.
tan γ=$\vec{\boldsymbol{v}}_{\boldsymbol{r}}$/$\vec{\boldsymbol{v}}_{\boldsymbol{\theta}}$.

Magnitude of h …3

- Conservation of momentum means that for any point on an orbit:

$$r_1 v_1 cos\gamma_1 = r_2 v_2 cos\gamma_2 \quad \text{(4-6a)}$$

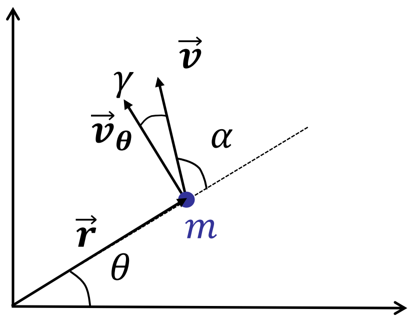
- For which points is γ=0?

The flight path angle is 0 for periapse and apoapse. Draw an ellipse with r from a focal point if you want to see this.

So $r_a v_a = r_p v_p$

Magnitude of h …4

Question: What is the magnitude of $\vec{\boldsymbol{h}}$?



We know from circular motion:

$$v_\theta = r\omega = r\frac{d\theta}{dt} \qquad (4\text{-}7)$$

Also, as $v_{\boldsymbol{\theta}} = v\cos(\gamma)$, then: (4-8)

$$h = rv cos\gamma = rv_\theta = \mathrm{r}\left(r\frac{d\theta}{dt}\right) = r^2\dot{\theta} \qquad (4\text{-}9)$$

$$\boxed{h = r^2\dot{\theta}} \qquad (4\text{-}10)$$

For motion in a circle of radius r, the angular rate of rotation, also known as angular velocity, is $r\omega$.
Let's write $\omega = \frac{d\theta}{dt}$=$\dot{\theta}$ to make things easier later.

Numerical Example

What are the components of velocity, v_θ and v_r , for a spacecraft with γ =10°, altitude 20000km, h=70000km^2/s?

$h = rv_\theta$, so we have:

$$v_\theta = \frac{h}{r} = \frac{70000}{20000 + 6378} = 2.65 km/s$$

tan $\gamma = v_r / v_\theta$ so $\tan\left(\frac{10*\pi}{180}\right) = \frac{v_r}{2650}$

so $\boldsymbol{v_r} = 467$m/s

Steps in Solving the Kepler Problem

1. **Use conservation of angular momentum to show that the motion is planar**
2. **Find the magnitude of $\boldsymbol{h}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{\theta}$**
3. Derive the equation of motion of the orbiting body (in polar coordinates)
4. Resolve eq. of motion tangentially to prove K2
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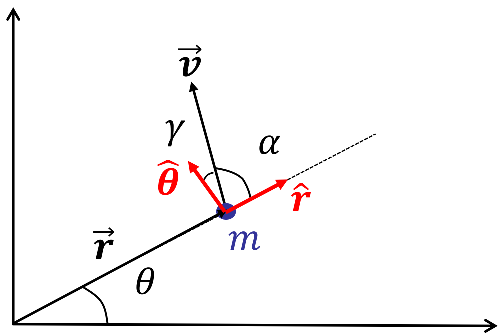
6. Use conservation of energy to solve for the constants of integration in 5.

Polar form 1

Polar form ($|\mathbf{r}|$, θ) provides a convenient representation:

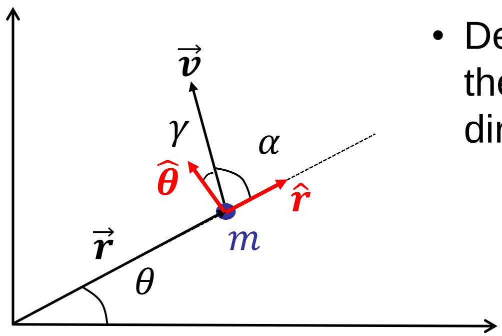


A polar vector is a vector in two dimensions specified as a magnitude (or length) 'r' and a direction (or angle) from the horizontal 'θ'.

In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a fixed point and an angle from a fixed direction.

Polar form 2

Polar form ($|\mathbf{r}|$, θ) provides a convenient representation:



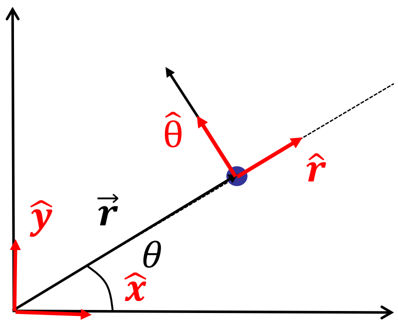
- Define $\hat{r}$ and $\hat{\theta}$ as unit vectors in the radial and tangential directions respectively

We will show that:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \hat{\boldsymbol{\theta}}\dot{\theta}, \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\hat{\mathbf{r}}\dot{\theta} \qquad (4\text{-}11)$$

(proof on next slides)

Derivatives of Vectors in Polar Coordinates



$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\cos\theta + \hat{\boldsymbol{y}}\sin\theta \qquad (4\text{-}12)$$
$$\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{x}}\sin\theta + \hat{\boldsymbol{y}}\cos\theta$$

$$\frac{d\hat{\boldsymbol{r}}}{d\theta} = \dot{\hat{\boldsymbol{x}}}\cos\theta - \hat{\boldsymbol{x}}\sin\theta + \dot{\hat{\boldsymbol{y}}}\sin\theta + \hat{\boldsymbol{y}}\cos\theta \qquad (4\text{-}13)$$
$$\frac{\boldsymbol{d}\hat{\boldsymbol{\theta}}}{d\theta} = -\dot{\hat{\boldsymbol{x}}}\sin\theta - \hat{\boldsymbol{x}}\cos\theta + \dot{\hat{\boldsymbol{y}}}\cos\theta - \hat{\boldsymbol{y}}\sin\theta$$

$$\boxed{\dot{\hat{\boldsymbol{x}}} = \dot{\hat{\boldsymbol{y}}} = \boldsymbol{0}} \qquad (4\text{-}14)$$

$$\frac{d\hat{\boldsymbol{r}}}{d\theta} = -\hat{\boldsymbol{x}}\sin\theta + \hat{\boldsymbol{y}}\cos\theta = \hat{\boldsymbol{\theta}} \qquad (4\text{-}15)$$
$$\frac{\boldsymbol{d}\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\boldsymbol{x}}\cos\theta - \hat{\boldsymbol{y}}\sin\theta = -\hat{\boldsymbol{r}}$$

Remember: x = rcosθ and y = rsinθ and equally: r = √(x2 + y2) and θ = tan-1 (y/x)
But remember that when solving for θ, we must use the Quadrant rules.
Quadrant rules: I: x; II: π-x; III π+x; ; IV: 2π-x .
From the definition of unit vectors: $\dot{\hat{\boldsymbol{x}}} = \dot{\hat{\boldsymbol{y}}} = \boldsymbol{0}$. They are “fixed unit vectors”.

Equation of Motion for One Body System 1

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \hat{\boldsymbol{\theta}}\dot{\theta}, \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\hat{\mathbf{r}}\dot{\theta} \qquad \text{(4-11)}$$

We want a 2nd order differential equation, 'M', for the position 'r' in terms of 'θ', we need to find:

$$M[r(\theta),\dot{r}(\theta),\ddot{r}(\theta),\theta]$$

So let's start with what we know, which is that:

$$\boldsymbol{r} = r\hat{\boldsymbol{r}} \qquad \text{(4-16)}$$

Now if we differentiate to get $\dot{\boldsymbol{r}}$ ...

Newtonian mechanics: an equation of motion M takes the general form of a second order ordinary differential equation (ODE) in the position r wrt time of the object, but our intermediate step is to find it in terms of theta.

Equation of Motion for One Body System 2

Given that $\mathbf{r} = r\hat{\mathbf{r}}$ and using: $\dfrac{d\hat{\mathbf{r}}}{dt} = \hat{\boldsymbol{\theta}}\dot{\theta}, \quad \dfrac{d\hat{\boldsymbol{\theta}}}{dt} = -\hat{\mathbf{r}}\dot{\theta}$ (4-11)

1. $\dot{\mathbf{r}} = \dfrac{d\hat{\mathbf{r}}}{dt} r + \hat{\mathbf{r}}\dot{r} = \hat{\boldsymbol{\theta}}\dot{\theta} r + \hat{\mathbf{r}}\dot{r}$ (differentiate) (4-16)

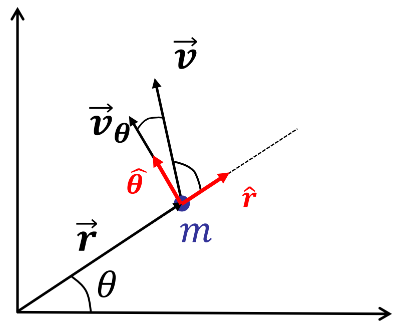
2. $\ddot{\mathbf{r}} = \left[\dfrac{d\hat{\boldsymbol{\theta}}}{dt}\dot{\theta} r + \hat{\boldsymbol{\theta}}\ddot{\theta} r + \hat{\boldsymbol{\theta}}\dot{\theta}\dot{r}\right] + \left[\dfrac{d\hat{\mathbf{r}}}{dt}\dot{r} + \hat{\mathbf{r}}.\ddot{r}\right]$ (differentiate) (4-17)

3. $\ddot{\mathbf{r}} = \hat{\mathbf{r}}\ddot{r} - \hat{\mathbf{r}} r\dot{\theta}^2 + \underbrace{\hat{\boldsymbol{\theta}}\dot{\theta}\dot{r} + \hat{\boldsymbol{\theta}}\ddot{\theta} r + \hat{\boldsymbol{\theta}}\dot{\theta}\dot{r}}$ (substitution) (4-18)

4. $\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}}$ (collect terms) (4-19)

1. First we differentiate using the product rule, then substitute in for dr/dt using (4-11)
2. Then we differentiate again
3. Then we substitute in again for dθ/dt and dr/dt using (4-11)
4. Then we group for r hat and theta hat

Equation of Motion for One Body System 3



- From the previous assumption that $M_1 \gg m_2$, we can treat m_2 as a particle, hence, $\mu = GM_1$
- Remember (3-7): $\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}$

$$\boxed{\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}} \qquad (4\text{-}20)$$

Why are (3-12) and (4-20) equivalent?
(3-12) is an expression for the relative motion of M and m; here we are assuming that m is moving under a central force and so the relative motion is that of m, i.e. the expression we have just derived.
The acceleration has two components: one radial and one tangential.

Equation of Motion for One Body System 4

$$\ddot{\mathbf{r}} = \left[\underset{1}{\ddot{r}} - \underset{2}{r\dot{\theta}^2}\right]\hat{\mathbf{r}} + \left[\underset{3}{2\dot{r}\dot{\theta}} + \underset{4}{r\ddot{\theta}}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}} \qquad (4\text{-}20)$$

1. Due to acceleration along **r**
2. Centripetal acceleration

3. Coriolis acceleration due to change in radius
4. Euler acceleration due to angular acceleration

When an object simultaneously rotates about a point and moves relative to that point, an acceleration results from this. This acceleration is called Coriolis acceleration. It acts in the circumferential direction. Euler acceleration is due to the tangential force.

Steps in Solving the Kepler Problem

1. **Use conservation of angular momentum to show that the motion is planar**
2. **Find the magnitude of h in terms of r and θ**
3. **Derive the equation of motion of the orbiting body (in polar coordinates)**
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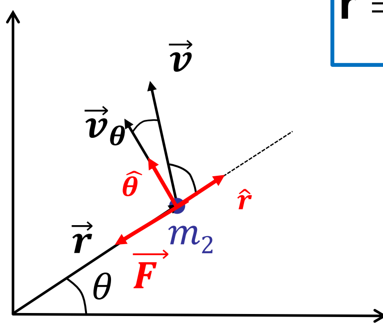
5. Resolve radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.

So what? We have now derived an alternative equation of motion in polar coordinates. We can use this expression to:

derive some useful relationships;
prove Kepler's Laws, and
derive an expression for r in terms of θ

Tangential Component 1 ($\hat{\theta}$)

$$\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}} \qquad \text{(4-20)}$$



$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \qquad \text{(4-21)}$$

- Recall: $h = r^2\dot{\theta}$ (4-10)
- Thus: $\dot{h} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$ (4-22)
- CoAM $\frac{\dot{h}}{r} = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$ (4-23)

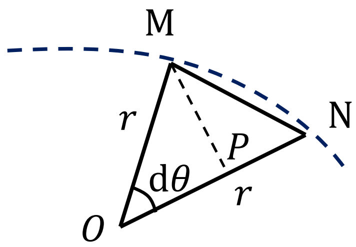
$$h = r^2\dot{\theta} \equiv const \qquad \text{(4-24)}$$

This agrees with our earlier result in (4-10) and (4-6)

If we compare LHS and RHS of 4-20 we can see that there is no theta/tangential component so the tangential component is =0.
If our object is increasing its speed or slowing down, there is a non-zero tangential acceleration in direction of motion.
Differentiating r^2 gives 2rdr/dt=2r(rdot)
From CoAM=Conservation of Angular Momentum, we know that hdot=0.

We can use 4-21 to prove Kepler 2nd law in the next few slides.

Tangential Component 2



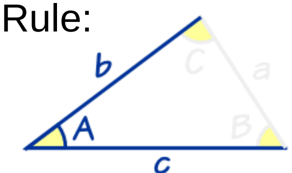
- We can also now consider the area, $\mathrm{d}A$, swept out by $\boldsymbol{r}$ in a time dt.
- First let us approximate the area of the arc segment, as the area of the triangle MON. From the rule:

$$dA = \frac{1}{2} r r \sin(d\theta) \qquad (4\text{-}25)$$

- For small angles $\sin\theta \approx \theta$ thus:

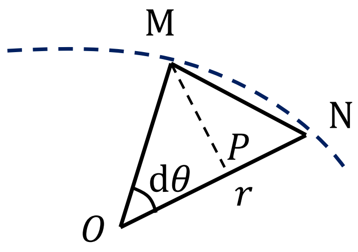
$$dA = \frac{1}{2} r^2 d\theta \qquad (4\text{-}26)$$

Rule:



Area = $\frac{1}{2}$ bc sin A

Tangential Component 3



- We can also now consider the area, $\mathrm{d}A$, swept out by $\boldsymbol{r}$ in a time dt.
- First let us approximate the area of the arc segment as the area of the triangle MON:

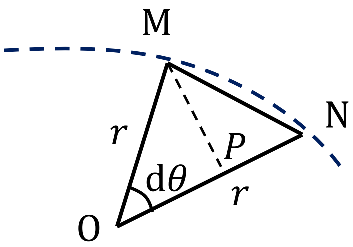
$$dA = \frac{1}{2} r r \sin(d\theta)$$

- For small angles $\sin\theta \approx \theta$ thus:

$$dA = \frac{1}{2} r^2 d\theta$$

Also note that as $d\theta \to 0$, the error in our approximation to the area swept out by $\boldsymbol{r}$ also tends to zero

Tangential Component 4



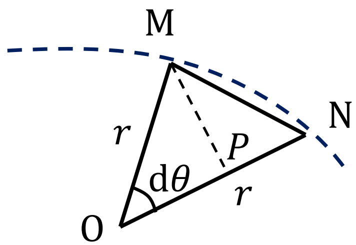
- We can also now consider the area, dA, swept out by $\boldsymbol{r}$ in a time dt:

$$dA = \frac{1}{2} r^2 d\theta \qquad (4\text{-}26)$$

- Differentiating to find the rate of change of area gives:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} \qquad (4\text{-}27)$$

Tangential Component 5



- We have shown that:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \qquad \text{(4-27)}$$

- Substitute using $h = r^2\dot{\theta}$: (4-10)

$$\frac{dA}{dt} = \frac{h}{2} \equiv const! \qquad \text{(4-28)}$$

Thus we have proved Kepler's second law:
"the line joining a planet to the Sun sweeps out equal areas in equal intervals of time".

Steps in Solving the Kepler Problem

1. **Use conservation of angular momentum to show that the motion is planar**
2. **Find the magnitude of $\boldsymbol{h}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{\theta}$**
3. **Derive the equation of motion of the orbiting body (in polar coordinates)**
4. **Resolve eq. of motion tangentially to prove K2**
5. Resolve radially to derive an expression for the position of the orbiting body, r, as a function of its position in the orbit, θ: the 'orbit equation'.
6. Use conservation of energy to solve for the constants of integration in 5.

Next we will resolve radially…

Summary

1. An analytical solution for the position of the body or spacecraft is useful to perform calculations without computers.
2. The *Kepler problem* consisted of determining the radial and angular coordinates, of an object in a Keplerian orbit about the Sun as a function of time.
3. Conservation of sp. angular momentum gives: $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$
4. Flight path angle $'\gamma'$ is the angle from velocity vector $'\boldsymbol{v}'$

to the tangential component of the velocity vector $'v_\theta'$ and is 0 at peri and apoapse.

5. If we resolve the eq. of motion tangentially we can prove K2.

Useful equations

1. $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$
2. $h = r v_\theta$
3. $v = \sqrt{{v_\theta}^2 + v_r^2}$
4. $\tan \gamma = v_r / v_\theta$
5. $h = r^2 \dot{\theta}$
6. $\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}$
7. $r_a v_a = r_p v_p$

Test Yourself!

1. Is a 'barycentre' exactly the same as the centre of mass?
2. What assumption allows us to use 1 body central force model for the motion of Mercury wrt the Sun?
3. Why do we use polar coordinates for planetary motion?
4. Use a diagram and simple equation to show which vectors determine the plane of an orbit.
5. Given that $\mathbf{r} = r\hat{\boldsymbol{r}}$ and using: $\dfrac{d\hat{\mathbf{r}}}{dt} = \hat{\boldsymbol{\theta}}\dot{\theta}$, $\dfrac{d\hat{\boldsymbol{\theta}}}{dt} = -\hat{\mathbf{r}}\dot{\theta}$

   prove that:

$$\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{\mathbf{r}} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{\theta}}$$