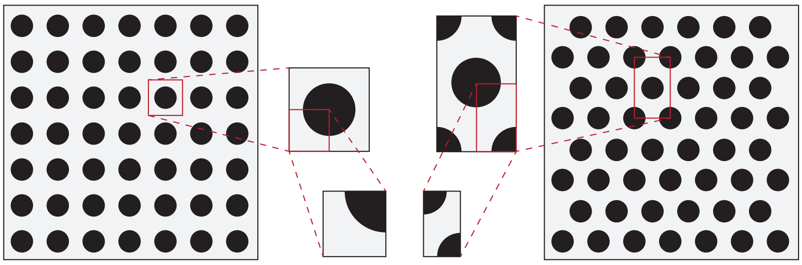
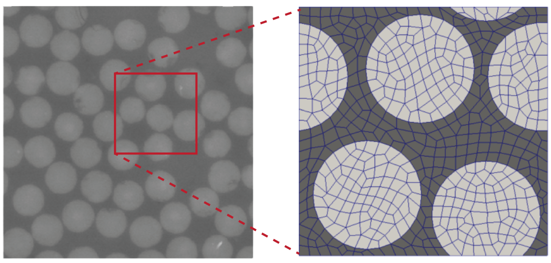
Handout 2 – Micromechanics of Uni-Directional Lamina

The structural analysis of composite structures considers material anisotropy, but regards the properties to be macroscopically homogeneous across a ply or laminate. However, at the *micro-mechanical* level the stresses and strains will vary between the constituent components, *i.e.* fibres and matrix. The objective is to determine the macroscopic composite material properties as a function of its constituents.

Representative Volume Element At the micromechanical level a composite material is heterogeneous, and the interaction between the constituents determines the macro-mechanical properties. An important concept in micromechanics is the *Representative Volume Element* (RVE). This is the smallest region of material over which the mechanical properties can be considered representative for the macroscopic composite as a whole, allowing the homogenisation of the micro-mechanical properties. For homogeneous isotropic materials this could be any element of material, but for composites the choice of RVE is informed by the microstructure and will affect the predicted macroscopic properties.

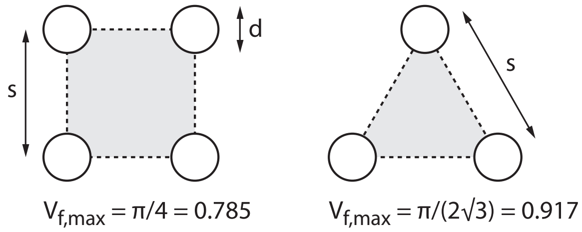




Once an RVE has been established, the micromechanics of a composite can be modelled using different approaches. In the *mechanics of materials* approach simplifying assumptions regarding the stress/strain distributions enable the formulation of straightforward analytical models. More detailed and rigorous models rely on *elasticity* methods (often using Finite Element Modelling), and the *semi-empirical* models attempt to fit curves to experimental results. This handout will primarily take a mechanics of materials approach, but includes other models where these provide more insight or more accurate solutions.

Note that the actual properties of a composite will also depend on processing variables, such as fibre volume fraction, residual stresses, voids, fibre misalignment, *etc*. This means that micromechanical predictions will necessarily be approximate.

The **fibre volume fraction** V_f is an important parameter in the micro-mechanical modelling of composites. Assuming no voids, the matrix volume fraction $V_m = (1 - V_f)$. The upper bound for V_f is determined by the packing geometry of the fibres, with diameter d and spacing s.

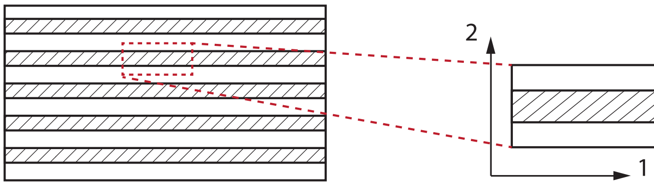


The fibre volume fraction usually falls within the following (approximate) range:

$$0.3 < V_f < 0.8$$

2.1 Micromechanics of Continuous Fibre Composites

Most fibre-reinforced composites make use of continuous fibres, which are assumed to be uniformly distributed and perfectly aligned. Therefore, the stress distribution may be assumed constant along their length, resulting in a simple 2D plane stress representative volume element.



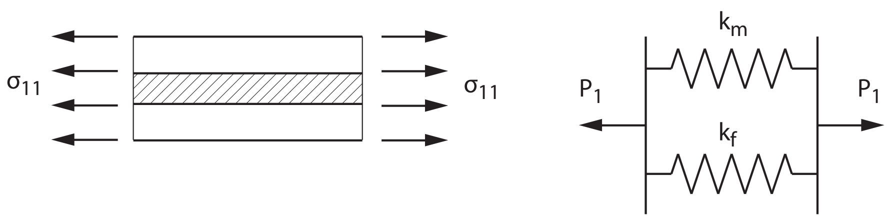
The homogenised composite mechanical properties are derived assuming that: (i) the fibres and matrix are both linear-elastic, isotropic; and (ii) the fibres and matrix are perfectly bonded.

2.1.1 Longitudinal Stiffness (E_{11})

In the fibre direction, it may be assumed that the fibres and and matrix experience the same strain:

$$\varepsilon_{11} = \varepsilon_f = \varepsilon_m$$

This *iso-strain* or *Voigt* model can be represented by two *parallel* springs.



where the load is split between the fibres and matrix:

$$P_1 = P_f + P_m \qquad \rightarrow \qquad \sigma_{11} = \sigma_f \frac{A_f}{A_c} + \sigma_m \frac{A_m}{A_c}$$
$$= \sigma_f V_f + \sigma_m V_m$$

Dividing by ε_{11} gives the homogenised elastic modulus:

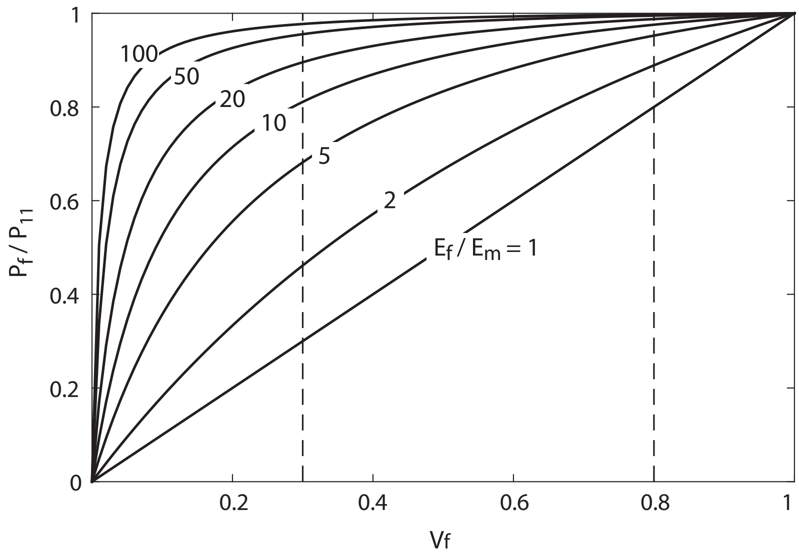
$$E_{11} = E_f V_f + E_m V_m = \sum_{i=1}^{n} E_i V_i \tag{2.1}$$

The resulting **rule of mixtures** is a simple linear relationship, but gives good agreement with experiments in tension, although some deviation may be found in compression.

The fraction of load carried by the fibres:

$$\frac{P_f}{P_1} = \frac{E_f/E_m}{E_f/E_m + V_m/V_f}$$

shows a non-linear relationship with the fibre-volume fraction, and depends on the ratio of E_f/E_m. Common material combinations are glass/epoxy and carbon/epoxy with E_f/E_m of approximately 20 and 100 respectively. Thus, the composite longitudinal modulus is a fibre-dominated property.



Example 2.1 – Equivalent Modulus

Consider a carbon/epoxy composite, using standard modulus fibres (Toray T700G) with $E_f = 240$ GPa, and an epoxy resin (Hexcel RTM6) $E_m = 2.9$ GPa. To achieve an equivalent elastic modulus to aerospace Aluminium, $E = 70$ GPa, we use the rule of mixtures

$$\frac{E_{11}}{E_f} = V_f + \frac{E_m}{E_f}(1 - V_f)$$

to find the required fibre volume fraction $V_f = 0.283$.

Taking represenative fibre and matrix densities $\rho_f = 1.8 \cdot 10^3$ kg/m^3 and $\rho_m = 1.14 \cdot 10^3$ kg/m^3, the resulting composite density is derived using the rule of mixtures:

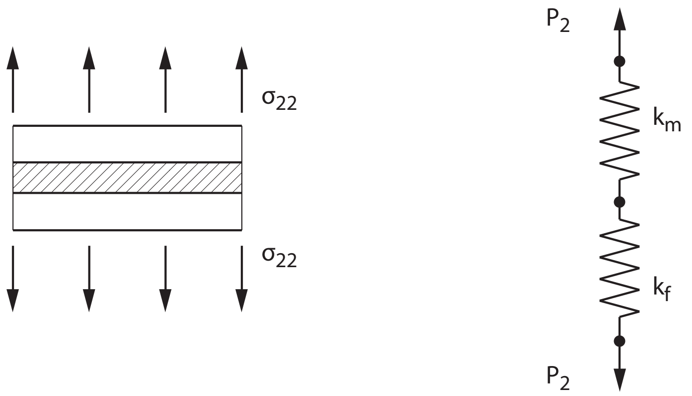
$$\rho_c = \rho_f V_f + \rho_m (1 - V_f) = 1.29 \cdot 10^3 \text{ kg/m}^3$$

Therefore, the specific modulus of the carbon/epoxy ply is therefore approximately double that of Aluminium, which has a density of $\rho = 2.7 \cdot 10^3$ kg/m^3.

A much higher modulus can be achieved for a minimal increase in weight, by increasing the fibre volume fraction. However, keep in mind that the transverse stiffness will be significantly lower than Aluminium!

2.1.2 Transverse Stiffness (E_{22})

In the transverse fibre direction it may be assumed that the matrix and fibre are both subjected to the same stress, with different resulting strains. This *iso-stress* or *Reuss* model can be represented by two elastic components in series.



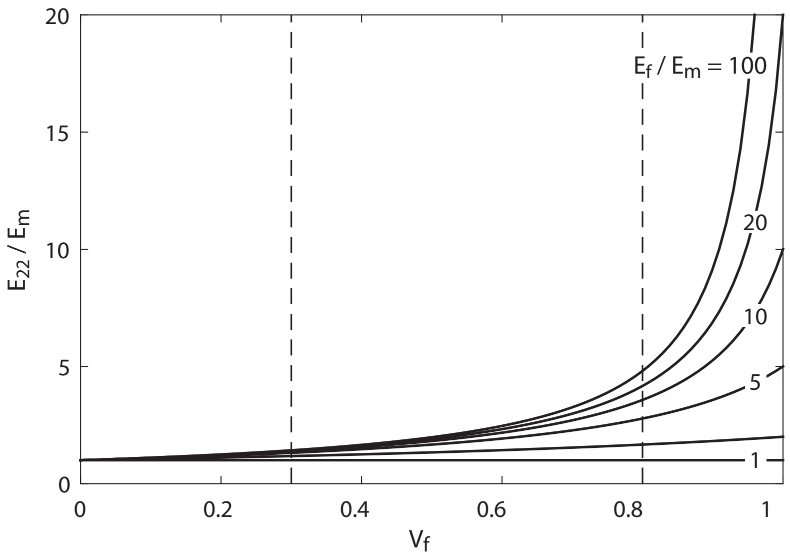
The total composite strain is found as:

$$\varepsilon_{22} = \varepsilon_f V_f + \varepsilon_m V_m \qquad \rightarrow \qquad \frac{\sigma_{22}}{E_{22}} = \frac{\sigma_f}{E_f} V_f + \frac{\sigma_m}{E_m} V_m$$

and with $\sigma_{22} = \sigma_f = \sigma_m$, this results in the **inverse rule of mixtures**:

$$\frac{1}{E_{22}} = \frac{V_f}{E_f} + \frac{V_m}{E_m} \tag{2.2}$$

As illustrated by plotting E_{22}/E_m, the transverse modulus is dominated by the matrix properties.



The assumptions for the homogenisation of the transverse modulus do not hold up to closer scrutiny, and the predictions of the inverse rule of mixtures therefore do not closely match experimental results. For instance, the stress distribution around the fibres will be three-dimensional, and the stiffness disparity will result in strain mismatches at the fibre/matrix interface.

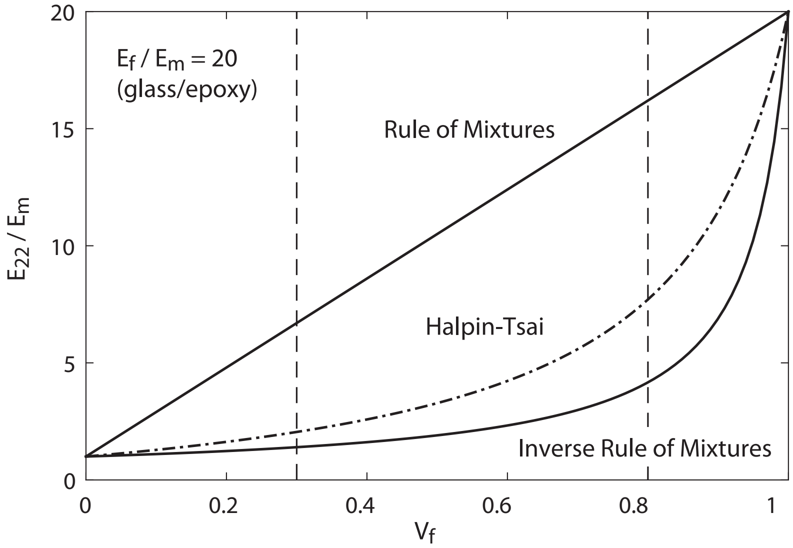
The **Halpin-Tsai** semi-empirical model offers a better fit for the transverse elastic modulus.

$$\frac{E_{22}}{E_m} = \frac{1 + \xi\eta V_f}{1 - \eta V_f} \tag{2.3}$$

with

$$\eta = \frac{(E_f/E_m) - 1}{(E_f/E_m) + \xi} \tag{2.4}$$

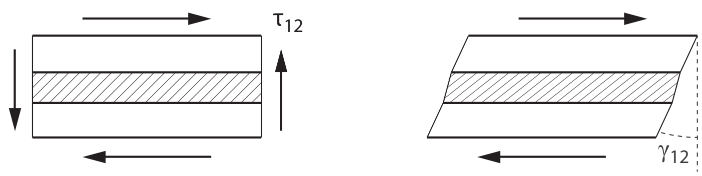
where ξ is obtained from curve fitting; for instance, for circular fibres in a square array $\xi = 2$. Note that $\xi = 0$ returns the inverse rule of mixtures, which provides a lower bound on the transverse unidirectional stiffness.



2.1.3 Shear Modulus (G_{12})

The shear modulus follows the constant stress model ($\tau = \tau_f = \tau_m$), where:

$$\gamma_{12} = \gamma_f V_f + \gamma_m V_m$$



This leads to the inverse rule of mixtures

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

Similarly to the transverse modulus, the inverse rule of mixtures for the shear modulus is not sufficiently accurate for design purposes. The Halpin-Tsai relationship can be used for improved prediction of the shear modulus, using $\xi = 1$.

2.1.4 Poisson's Ratio (ν_{12})

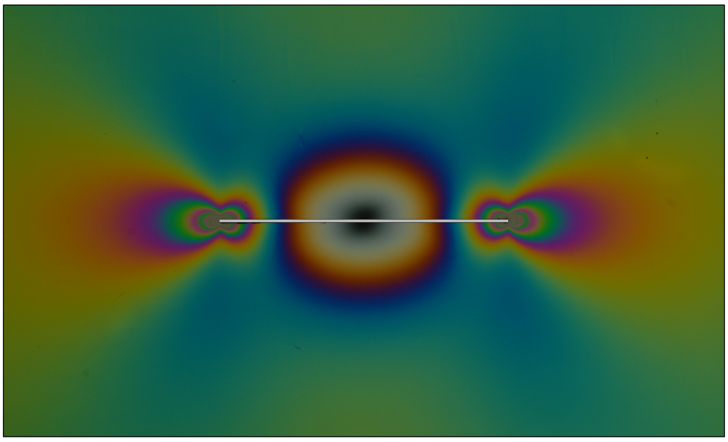
The homogenisation of Poisson's ratio ν_{12} follows the rule of mixtures:

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

where ν_{21} is calculated using $\nu_{21} = (E_{22}/E_{11})\nu_{12}$.

2.2 Micromechanics of Short-Fibre Composites

In the derivation of the homogenised properties of continuous fibre-reinforced composites, it was assumed that the stresses and strains did not vary along the fibre length. For short-fibre composites, this assumption must be relaxed and the representative volume element now considers a finite-length *stiff* fibre embedded in a *compliant* matrix. The resulting stress distribution is nonuniform along the length of the fibre with high shear stresses at the ends, as illustrated in the photoelastic experimental result (contours indicate constant shear stress).

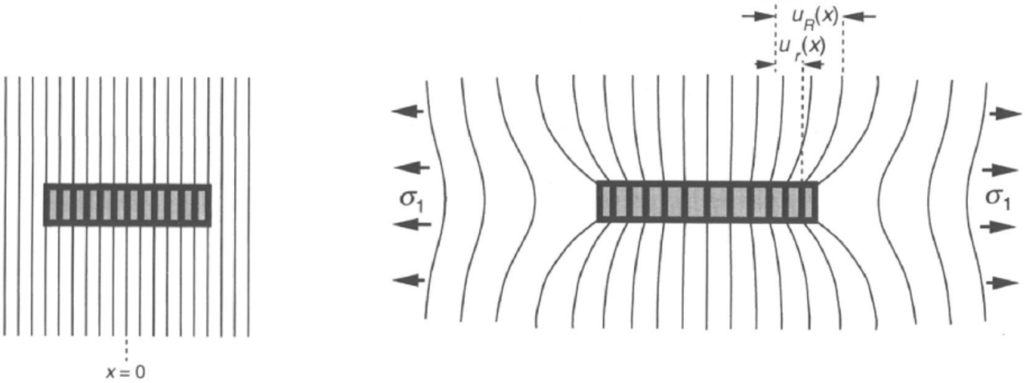


The following analysis focuses on the mechanics of short-fibre composites, but also offers valuable insight into the strength of continuous fibre composites after fibre fracture.

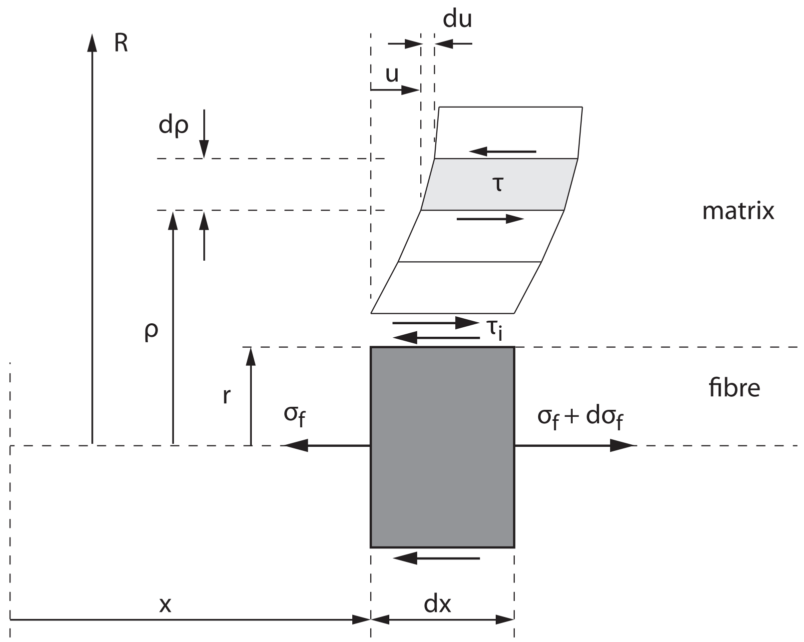
2.2.1 Cox's Shear Lag Model (not examinable)

The analysis of the axial fibre and interfacial shear stress distribution follows **Cox's shear lag model**, as described in Hull and Clyne (1996). Although more rigorous models exist, this model sufficiently captures key features while maintaining simplicity of derivation.

The shear lag model assumes that: (i) the fibres are cylindrical (length L and radius r); (ii) the fibres and matrix behave as linear-elastic, isotropic materials; (iii) there is no slip at the matrix-fibre interface; and (iv) the fibre axial stress vanishes at the fibre ends and there is no interaction between adjacent fibres.



As a result of a longitudinal load σ_1, the matrix and fibre strain non-uniformly. The load transfer from the matrix to the fibre is achieved through interfacial shear stress τ_i, which results in a non-uniform stresses in the fibre: $\sigma_f(x)$. The aim is to capture this fibre stress and interfacial shear stress distribution.



Consider the axial force balance for an element dx of the fibres:

$$-2\pi r\,dx\,\tau_i = \pi r^2\,d\sigma_f$$

$$\therefore \qquad \frac{d\sigma_f}{dx} = -\frac{2\tau_i}{r} \tag{2.5}$$

to relate the fibre stress σ_f to the interfacial shear stress τ_i at the matrix-fibre interface. The position x along the fibre is measured from the fibre midpoint (total length L).

The radial shear stress distribution in the matrix is obtained from the equilibrium of shear forces on concentric annuli at radius ρ around the fibre, to find τ relative to the interfacial shear stress τ_i:

$$2\pi\rho\,dx\,\tau = 2\pi r\,dx\,\tau_i \qquad \rightarrow \qquad \tau = \tau_i\left(\frac{r}{\rho}\right)$$

As the interface boundary condition prescribes no slip between the fibre and matrix, expressions for the matrix shear strains γ and axial displacement u are required. Assuming no radial displacement,

$$\frac{\partial u}{\partial \rho} = \gamma = \frac{\tau}{G_m} = \frac{\tau_i}{G_m}\left(\frac{r}{\rho}\right)$$

with matrix shear modulus G_m. This can be integrated, at a given location x, to find the displacement u_R at a distance R, relative to displacement u_r at the matrix-fibre interface:

$$\int_{u_r}^{u_R} du = \frac{\tau_i r}{G_m}\int_r^R \frac{1}{\rho}d\rho \quad \rightarrow \quad u_R - u_r = \frac{\tau_i r}{G_m}\ln\left(\frac{R}{r}\right) \tag{2.6}$$

It is assumed that the matrix strain is uniform at a far-field radius R from the fibre. The ratio R/r depends on the fibre packing arrangement and fibre volume fraction. However, the results are weakly sensitive to these details as R/r appears in the logarithmic term. For a hexagonal fibre arrangement (fibre spacing $2R$):

$$V_f \approx \frac{\pi r^2}{(2R)\,(R\sqrt{3})} \qquad \rightarrow \qquad \left(\frac{R}{r}\right)^2 = \frac{\pi}{2\sqrt{3}V_f} \approx \frac{1}{V_f}$$

Combining Equation 2.5, Equation 2.6 and the no-slip boundary condition, $u_r = u_f$, provides:

$$\frac{d\sigma_f}{dx} = -\frac{4G_m\left(u_R - u_f\right)}{r^2\ln(1/V_f)}$$

Finally, consider the fibre strain ε_f

$$\frac{du_f}{dx} = \varepsilon_f = \frac{\sigma_f}{E_f}$$

and far-field matrix strain ε_m

$$\frac{du_R}{dx} \approx \varepsilon_m \sim \varepsilon_1$$

which can be approximated as the overall composite strain ε_1.

Applying these boundary conditions shows that the fibre stress distribution is governed by:

$$\frac{d^2\sigma_f}{dx^2} = \frac{n^2}{r^2}\left(\sigma_f - E_f\varepsilon_1\right)$$

with

$$n = \sqrt{\left[\frac{4G_m}{E_f\ln\left(1/V_f\right)}\right]} \tag{2.7}$$

The second-order ODE has a standard solution of the form:

$$\sigma_f = E_f\varepsilon_1 + B\sinh\left(\frac{nx}{r}\right) + D\cosh\left(\frac{nx}{r}\right)$$

which in combination with the loading boundary conditions

$$\sigma_f = 0 \qquad \text{at} \qquad x = \pm L/2$$

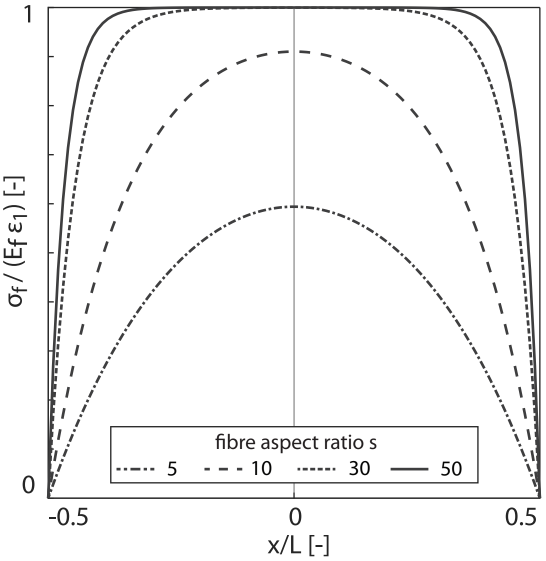
provides expressions for fibre stress and interfacial shear stress:

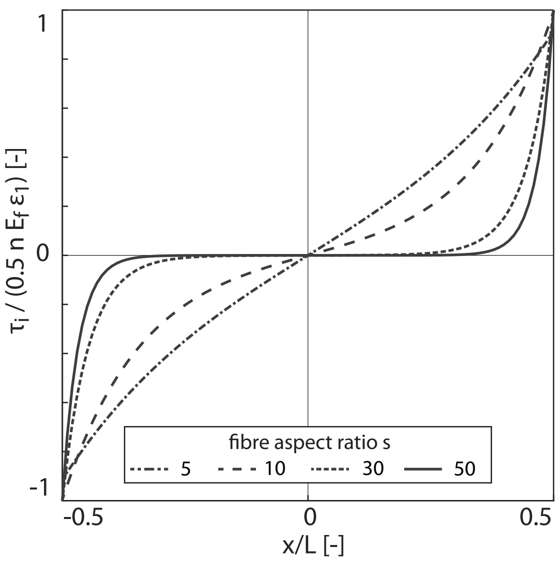
$$\sigma_f = E_f\varepsilon_1\left[1 - \frac{\cosh\left(\frac{nx}{r}\right)}{\cosh\left(ns\right)}\right] \tag{2.8}$$

$$\tau_i = \frac{n}{2}E_f\varepsilon_1\frac{\sinh\left(\frac{nx}{r}\right)}{\cosh\left(ns\right)} \tag{2.9}$$

with fibre aspect ratio $s = L/d$, and where L is fibre length and d the fibre diameter. The derivation has been non-rigorous and neglected stress transfer at the end of the fibres, but nonetheless provides useful insights.

Stress Transfer Length The axial stress σ_f increases from the fibre ends, and for sufficiently high fibre aspect ratios s will reach a plateau; here the fibre and matrix strains will be equal. Conversely, the interfacial shear stress τ_i decreases from the ends to reach zero along a plateau. Crucially, for continuous fibre composites there are no end-effects, and the axial stress will be constant and interfacial shear stress zero along the entire fibre length.





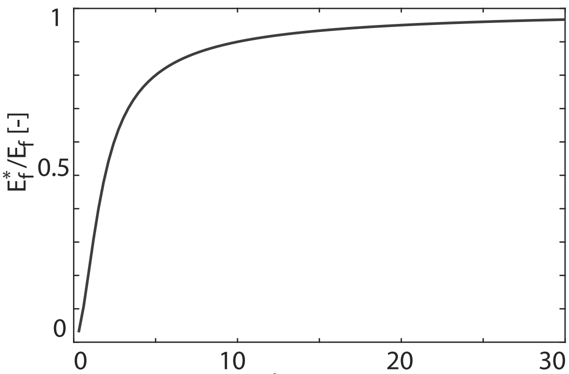
The transfer of the applied stress to the fibre is governed by the shear-lag parameter $\lambda = ns$. As λ increases, the fibre stress builds up faster to achieve a longer plateau. The shear-lag parameter increases with: (i) fibre aspect ratio s; (ii) matrix/fibre stiffness ratio (G_m/E_f); and (iii) fibre volume fraction V_f. (The graphs used representative values $G_m/E_f = 1/40$ and $V_f = 0.5$.)

Effective Modulus From the previous results it may be observed that short fibres offer inefficient reinforcement, if they do not allow the maximum achievable stress to be built up along their length. A side-effect of the axial stress build-up is that the effective fibre modulus is reduced. The averaged fibre stress is found as:

$$\bar{\sigma}_f = \frac{1}{L}\int_{-L/2}^{L/2} \sigma_f dx = E_f \varepsilon_1 \left[1 - \frac{\tanh(ns)}{ns}\right]$$

to give the effective fibre modulus

$$E_f^* = E_f \left[1 - \frac{\tanh(ns)}{ns}\right]$$



2.3 Micromechanics of Composite Strength

The micromechanics of composite strength and failure is less well-established. Moreover, the mechanics of materials models seldom provide accurate predictions, and engineering applications will rely on experimental results. This handout will primarily focus on qualitative descriptions of the failure modes under different loading conditions, and will also not consider failure of composite laminates through delamination (separation of plies).

2.3.1 Longitudinal Strength (X_t, X_c)

Tensile Strength As a first approximation, the composite tensile strength is found using a rule of mixtures:

$$X_t = \sigma_f V_f + (\sigma_m)_{\varepsilon_f} (1 - V_f) \tag{2.10}$$

where σ_f is the tensile fibre strength and $(\sigma_m)_{\varepsilon_f}$ is the matrix stress at the maximum tensile strain in the fibres. This analysis assumes that: (i) the composite failure is driven by fibre fracture; (ii) material is linear-elastic up to the point of failure; and (iii) all fibres have equal strength and will thus fail simultaneously. Although the observed tensile failure of composites is sudden, the actual failure mechanism is more intricate.

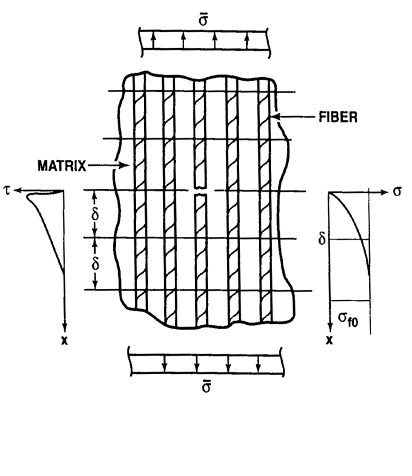
For brittle fibres, the strength of **individual fibres** is driven by defects. This means that the length of the fibre will affect its strength: longer fibres are weaker, as the probability of significant defects increases. The strength of individual fibres is captured using a *Weibull* distribution; the probability of failure at stress σ is given by:

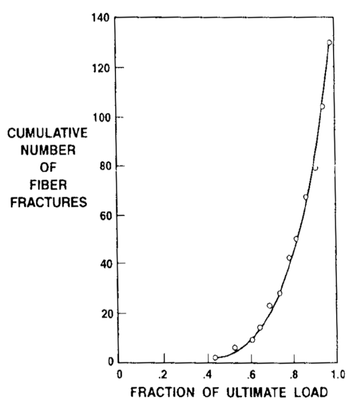
$$p(\sigma) = 1 - \exp\left[-\left(\frac{L}{L_0}\right)\left(\frac{\sigma}{\sigma_0}\right)^m\right]$$

with fibre length L and characteristic fibre strength σ_0 for length L_0. The Weibull modulus m is determined from the mean fibre strength and standard deviation, with typical values $m = 2 - 10$. A low value for m introduces more uncertainty for the failure stress.

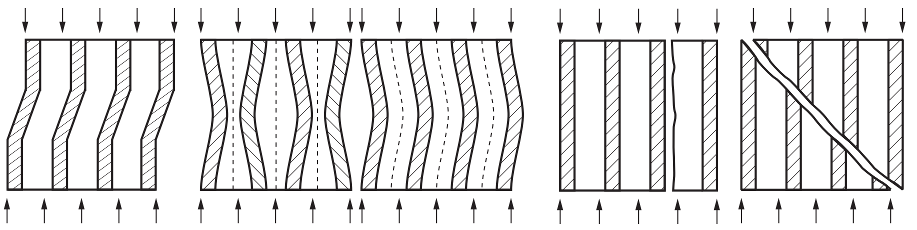
For a **bundle of loose fibres**, as individual fibres break, the load is redistributed to the remaining fibres until the bundle fails as a whole. The resulting bundle strength is found to be *less* than individual fibre strength, and will also decrease with length, although it is independent of the number of fibres.

In **fibre-reinforced composites**, the matrix plays a key role in redistributing the load after fibre fracture. In effect, the tensile load is transferred from one side of the fracture to the other, through high shear stress in the surrounding matrix material; this is analogous to the load transfer mechanism discussed for short-fibre composites. Therefore, at a sufficiently far distance away from the fracture, approximately 10 - 100 fibre diameters, the fibre will again carry the same load as the other fibres. For a tough matrix, the damage can accumulate until macroscopic material fracture occurs; this failure mode will involve fibre pull-out. Due to the redistribution of load through the matrix, the composite strength is *greater* than the mean fibre strength.



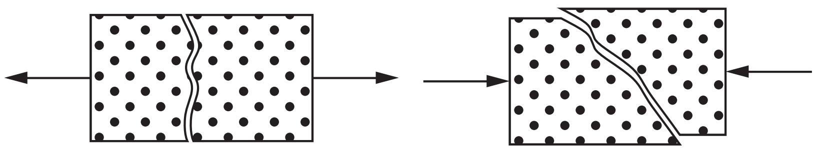


Compressive Strength In longitudinal compressive loading, various failure modes may be observed: shear instability followed by flexural fibre fracture, fibre micro-buckling, interface debonding due to Poisson's ratio mismatch, and shear failure without buckling. Accurate measurement of the intrinsic compressive strength has proved to be difficult, and test results typically depend on specimen geometry and test method. Fibre misalignment and/or waviness may also significantly affect experimental results.



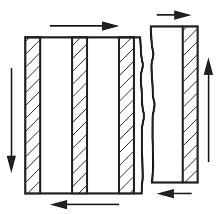
2.3.2 Transverse Strength (Y_t, Y_c)

In the transverse direction, the composite failure is dominated by the matrix strength. Under *tensile* loading, the high stiffness fibres constrain the deformation of the matrix, and the resulting strain concentrations lead to interface failure. The individual micro-cracks subsequently coalesce, and propagate through the matrix. As a result, the composite transverse failure occurs at much lower strain than the unrestrained matrix material. The strain concentration factor increases with V_f and E_f. Thus, the improvement in longitudinal stiffness/strength comes at the expense of lower composite transverse strength. Under transverse *compressive* loading, the composite failure is due to a combination of matrix shear failure and debonding.



2.3.3 Shear Strength (S)

The shear failure of a composite is a combination of matrix shear failure and constituent debonding.



2.4 Summary

In *micro-mechanical* modelling the aim is to predict macroscopic composite material properties as a function of its constituents components (fibres + matrix). An important step is the selection of a representative volume element (RVE), the smallest region of material over which the mechanical properties can be considered representative for the macroscopic composite.

A mechanics of materials approach was taken, where simplifying structural assumptions are made to express the interplay between the constituents. For continuous fibre-reinforced composites, the macroscopic elastic properties E_{11} and ν_{12} can be predicted using the rule of mixtures. For the E_{22} and G_{12} the inverse rule of mixtures is not sufficiently accurate, and the semi-empirical Halpin-Tsai relationship is used.

For short-fibre composites, the load transfer between the matrix and fibre was described using a shear lag model, which showed high interfacial shear stresses at the ends, and a build-up of axial fibre stress along the length. For sufficiently high aspect ratio fibres the axial fibre stress will reach a plateau, and the shear stress vanishes; this is the limiting case of continuous fibres.

The micromechanics of composite strength was discussed qualitatively, to highlight the importance of the matrix in composite strength, and the variety of failure modes, which make the analysis of composite strength particularly challenging.

Revision Objectives Handout 2:

- describe the role of a Representative Volume Element (RVE) in micro-mechanical modelling;
- recall the rule of mixtures for E_{11} and ν_{12};
- recall the inverse rule of mixtures for E_{22} and G_{12};
- discuss the assumptions in the derivation of the (inverse) rule of mixtures;
- apply the Halpin-Tsai semi-empirical relationship for E_{22} and G_{12};
- describe load transfer between fibre and matrix in short-fibre composites;
- describe the different micromechanical composite failure modes;
- calculate longitudinal tensile strength X_t using rule of mixtures;