StM3 – Composite Laminate Analysis

Lecture 3 :
- ply failure criteria
- micromechanics of uni-directional ply

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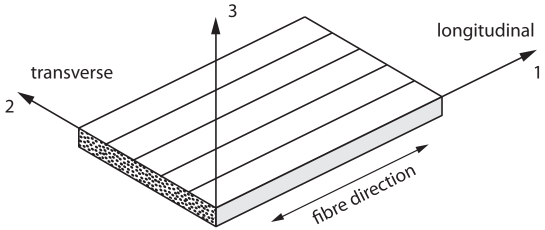
Summary – I

- uniaxial fibre-reinforced composite lamina/ply



- specially orthotropic material in plane stress (σ_{11}, σ_{22}, τ_{12})

Summary – II

compliance matrix $\boldsymbol{S}$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix}$$

$$S_{11} = \frac{1}{E_{11}} \qquad S_{22} = \frac{1}{E_{22}}$$

$$S_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \qquad S_{66} = \frac{1}{G_{12}}$$

note:

$$\frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}}$$

Summary – III

reduced stiffness matrix $\boldsymbol{Q} = \boldsymbol{S}^{-1}$

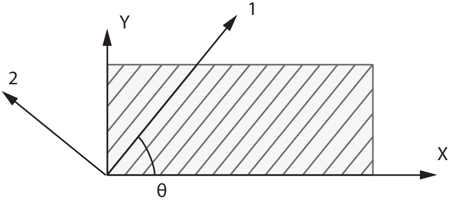
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21} E_{11}}{1 - \nu_{12}\nu_{21}} \qquad Q_{66} = G_{12}$$

Summary – IV

- generally orthotropic material



transform stress and strain between
structural (xyz) and material (123) coordinate systems

Summary – V

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

$$\bar{\boldsymbol{Q}} = \boldsymbol{T}^{-1} \boldsymbol{Q} \boldsymbol{R} \boldsymbol{T} \boldsymbol{R}^{-1}$$

$$\bar{Q}_{11} = Q_{11} \cos^4 \theta + 2 \left(Q_{12} + 2Q_{66}\right) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta$$

$$\bar{Q}_{22} = Q_{11} \sin^4 \theta + 2 \left(Q_{12} + 2Q_{66}\right) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta$$

$$\bar{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right) \sin^2 \theta \cos^2 \theta + Q_{12} \left(\sin^4 \theta + \cos^4 \theta\right)$$

$$\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right) \sin^2 \theta \cos^2 \theta + Q_{66} \left(\sin^4 \theta + \cos^4 \theta\right)$$

$$\bar{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right) \sin \theta \cos^3 \theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right) \cos \theta \sin^3 \theta$$

$$\bar{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right) \cos \theta \sin^3 \theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right) \sin \theta \cos^3 \theta$$

Summary – VI

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

$$\bar{\boldsymbol{S}} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}\boldsymbol{S}\boldsymbol{T}$$

$$\bar{S}_{11} = S_{11}\cos^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta + S_{22}\sin^4\theta$$

$$\bar{S}_{22} = S_{11}\sin^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta + S_{22}\cos^4\theta$$

$$\bar{S}_{12} = (S_{11} + S_{22} - S_{66})\sin^2\theta\cos^2\theta + S_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

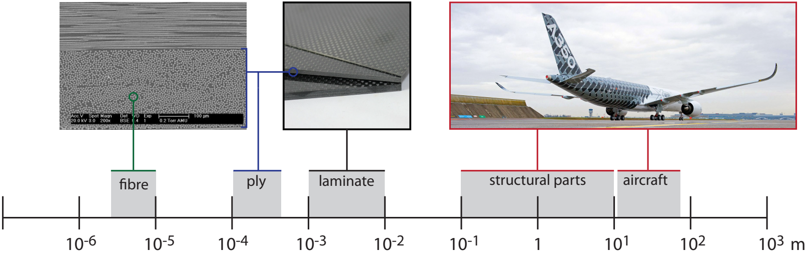
$$\bar{S}_{66} = 2\,(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\sin^2\theta\cos^2\theta + S_{66}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})\sin\theta\cos^3\theta - (2S_{22} - 2S_{12} - S_{66})\cos\theta\sin^3\theta$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})\cos\theta\sin^3\theta - (2S_{22} - 2S_{12} - S_{66})\sin\theta\cos^3\theta$$

Lecture Outline

- composite lamina failure criteria
- composite micromechanics
  - continuous fibre composites
  - short-fibre composites



Composite Lamina Strength – I

macromechanics of uni-directional lamina:

stiffness: four elastic constants (E_{11}, E_{22}, G_{12}, ν_{12})

strength: (at least) five independent parameters

$$
\begin{aligned}
X_t &= \text{longitudinal tensile strength} \\
X_c &= \text{longitudinal compressive strength} \\
Y_t &= \text{transverse tensile strength} \\
Y_c &= \text{transverse compressive strength} \\
S &= \text{in-plane intralaminar shear strength}
\end{aligned}
$$

Composite Lamina Strength – II

strength properties of unidirectional prepreg materials

Material	X_t	X_c	Y_t	Y_c	S	
HS carbon/epoxy	1500	1200	50	250	70	MPa
HM carbon/epoxy	1000	850	40	200	60	MPa
E-glass/epoxy	1000	600	30	110	40	MPa
Kevlar/epoxy	1300	280	30	140	60	MPa

note: strength parameters defined in material axes

Composite Lamina Strength – III

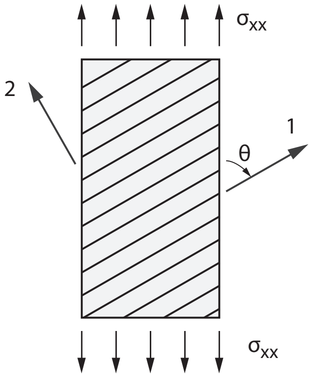
isotropic materials: both stiffness and strength are isotropic; failure criteria expressed in terms of principal stresses

composite materials: both stiffness and strength are highly anisotropic, complicating formulation failure criteria

- anisotropy: principal stresses less insightful
- composite strength depends on direction of loading
- failure *mechanisms* depend on the direction of loading (*e.g.* fibre fracture or matrix shear failure)

Experimental Test Case – I

test case: off-axis loading of unidirectionally reinforced lamina



$$\sigma_{11} = \sigma_{xx} \cos^2 \theta$$

$$\sigma_{22} = \sigma_{xx} \sin^2 \theta$$

$$\tau_{12} = \sigma_{xx} \sin \theta \cos \theta$$

note: test set-up cannot generate combined tension and compression

Experimental Test Case – II

example plots use material properties for Glass-Epoxy laminate:

E_{11}	$=$	54 GPa	X_t	$=$	1035 MPa
E_{22}	$=$	18 GPa	X_c	$=$	1035 MPa
ν_{12}	$=$	0.25	Y_t	$=$	28 MPa
G_{12}	$=$	9 GPa	Y_c	$=$	138 MPa
			S	$=$	55 MPa

Maximum Stress Criterion – I

maximum stress: failure if one of the stresses in the natural axes exceeds the corresponding allowable stress

$$-X_c < \sigma_{11} < X_t \qquad\qquad - Y_c < \sigma_{22} < Y_t$$

$$|\tau_{12}| < S$$

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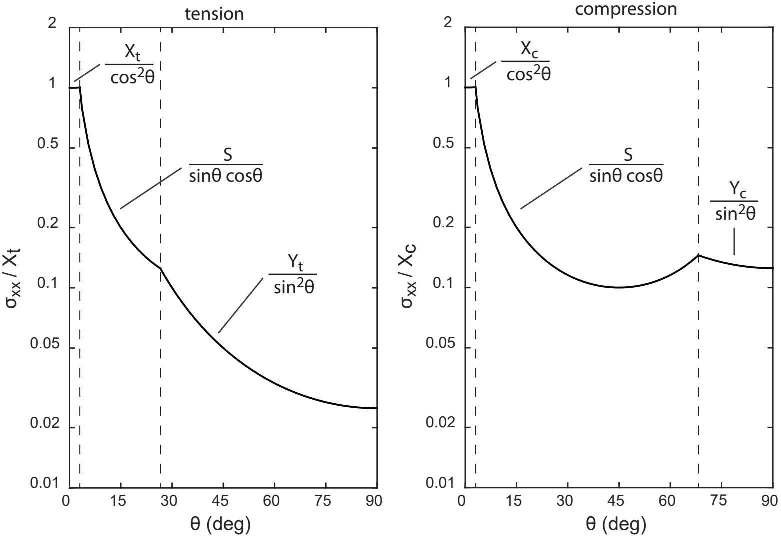
substitute transformed stresses for test case:

$$-\frac{X_c}{\cos^2\theta} < \sigma_{xx} < \frac{X_t}{\cos^2\theta} \qquad\qquad -\frac{Y_c}{\sin^2\theta} < \sigma_{xx} < \frac{Y_t}{\sin^2\theta}$$

$$|\sigma_{xx}| < \left|\frac{S}{\sin\theta\cos\theta}\right|$$

Maximum Stress Criterion – II



Tsai-Hill Failure Criterion – I

Tsai-Hill: extension of Von Mises criterion to specially orthotropic materials; includes *interaction between failure modes*

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 - \frac{\sigma_{11}}{X}\frac{\sigma_{22}}{X} \leq 1$$

$X = X_t$ or X_c and $Y = Y_t$ or Y_c depending on sign of σ_{11} and σ_{22}!

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numerous failure criteria exist, often requiring more experimental data (*e.g.* Tsai-Wu, Puck); composite strength remains an active area of research (see The World Wide Failure Exercise II).

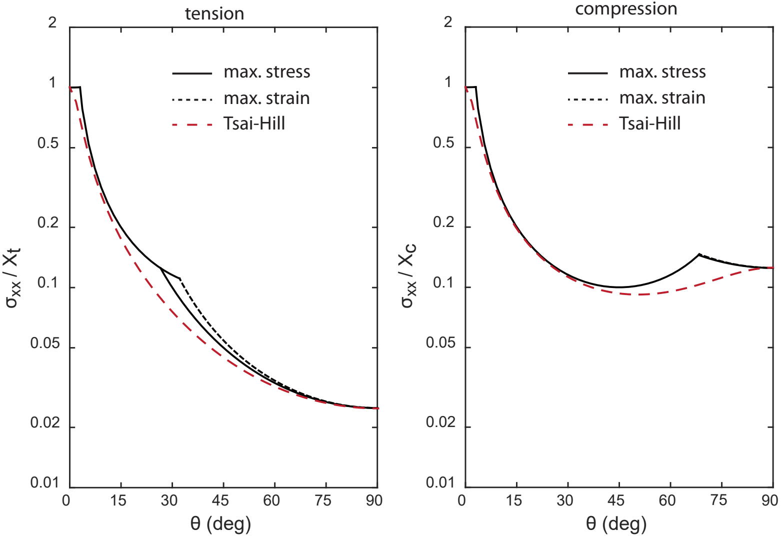
Tsai-Hill Failure Criterion – II

substitute transformed stresses for test case:

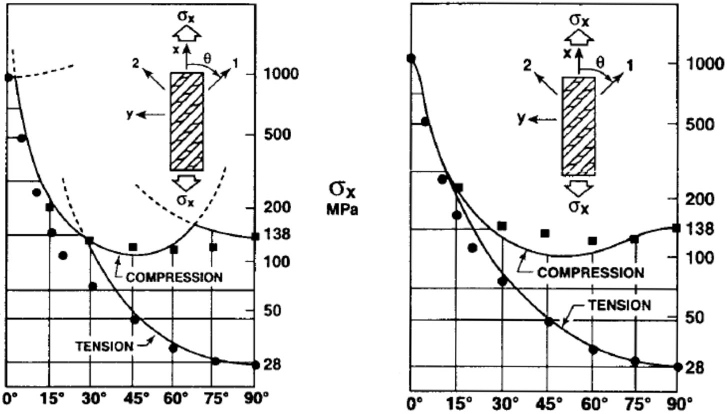
$$\frac{\cos^4\theta}{X^2} + \left[\frac{1}{S^2} - \frac{1}{X^2}\right]\cos^2\theta\sin^2\theta + \frac{\sin^4\theta}{Y^2} = \frac{1}{\sigma_{xx}^2}$$

results in single failure criterion

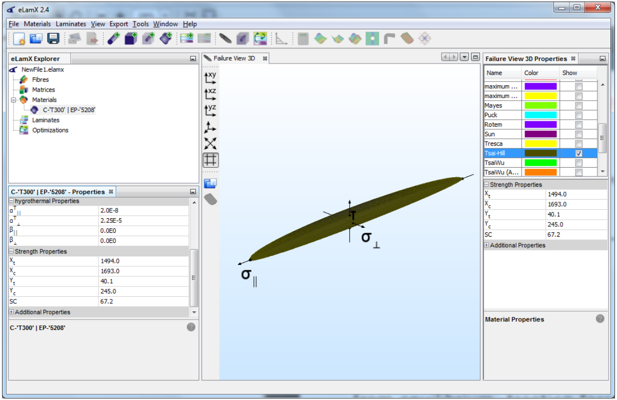
Tsai-Hill Failure Criterion – III



Tsai-Hill Failure Criterion – IV



eLamX^2 demo



task: visually compare failure envelopes (max. stress, Tsai-Hill)

Summary

- macromechanical strength (X_t, X_c, Y_t, Y_c, S)
- failure criteria: maximum stress, maximum strain, Tsai-Hill
- composite strength more complicated than isotropic!

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example problems: Q1, Q2, Q3

Revision Objectives

Revision Objectives:

- recall the five strength parameters for specially orthotropic laminate (X_t, X_c, Y_t, Y_c, S);
- appreciate different failure mechanisms as function of ply angle in angled ply;
- calculate strength of a lamina, using Tsai-Hill or maximum stress failure criteria;

Micro-Mechanics – I

macro-mechanics of composite lamina: *homogenised* properties

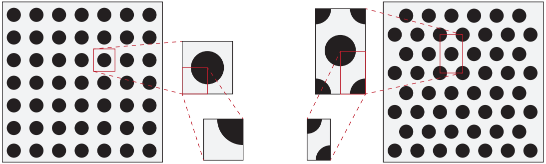
micro-mechanical approach: predict the macroscopic composite material properties as function of constituents (fibres + matrix)

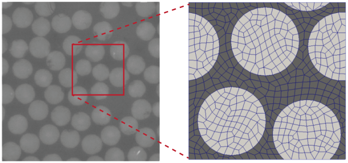
at micromechanical level material is *heterogeneous*

Representative Volume Element: smallest region of material over which mechanical properties are considered representative for macroscopic composite, enabling homogenisation

Micro-Mechanics – II

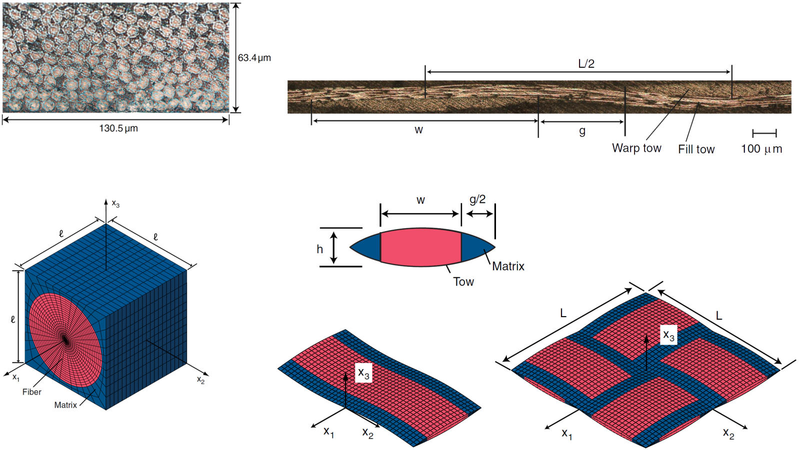
choice of RVE: informed by composite microstructure, and affects predicted macroscopic properties





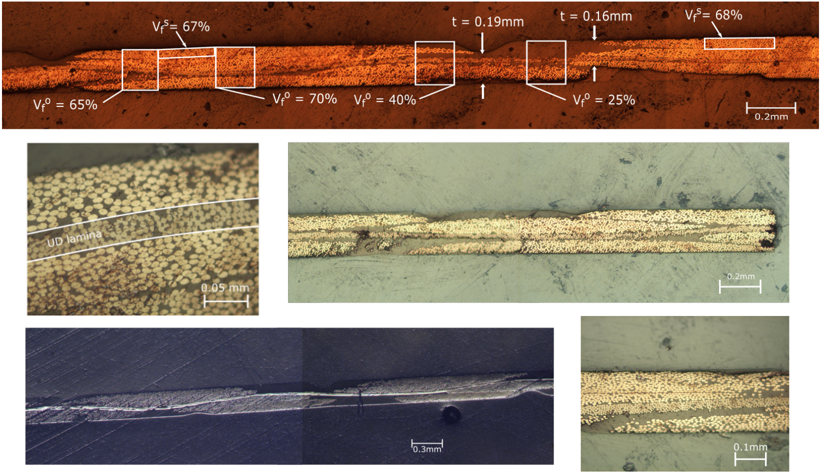
modelling: *mechanics of materials*, *elasticity*, *semi-empirical*

Micro-Mechanics – III



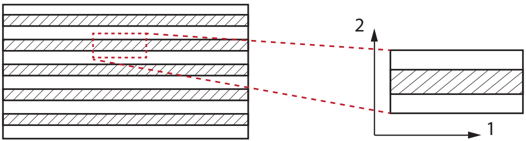
K. Kwok and S. Pellegrino (2017), "*Micromechanics Models for Viscoelastic Plain-Weave Composite Tape Springs*", AIAA Journal 55(1):309-321, doi:10.2514/1.J055041

Micro-Mechanics – IV



Reece Lincoln (2017), "*Characterisation of the in-plane properties of an ultra-thin CFRP woven laminate*", IXP Report, University of Bristol

Continuous Fibre Composites



micromechanics of **continuous fibre-reinforced composites**

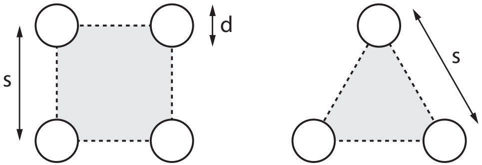
- fibres are uniformly distributed and perfectly aligned
- fibres and matrix are linear-elastic and isotropic
- perfect bonding between fibres and matrix

stresses may be assumed constant along length

Fibre Volume Fraction

key composite parameter: **fibre volume fraction** V_f

upper bound defined by fibre packing geometry:



$V_{f,max} = \pi/4 = 0.785$ $\qquad$ $V_{f,max} = \pi/(2\sqrt{3}) = 0.917$

usually within (approximate) range:

$$0.3 < V_f < 0.8$$

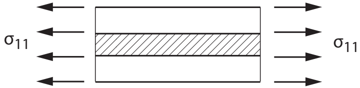
no voids: matrix volume fraction $V_m = (1 - V_f)$

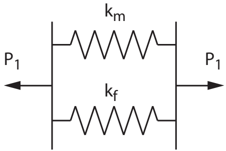
Longitudinal Stiffness (E_{11}) – I

longitudinal: assume fibres and and matrix experience same strain

$$\varepsilon_{11} = \varepsilon_f = \varepsilon_m$$

modelled as two *parallel* springs





Longitudinal Stiffness (E_{11}) – II

load split between fibres and matrix:

$$P_1 = P_f + P_m \qquad \rightarrow \qquad \sigma_{11} = \sigma_f \frac{A_f}{A_c} + \sigma_m \frac{A_m}{A_c}$$
$$= \sigma_f V_f + \sigma_m V_m$$

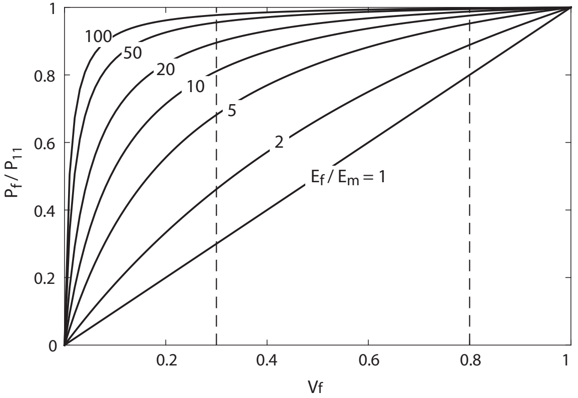
divided by ε to give the elastic moduli:

$$E_{11} = E_f V_f + E_m V_m = \sum_{i=1}^{n} E_i V_i$$

this is **rule of mixtures** or *Voigt* iso-strain model

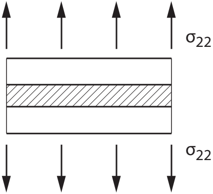
Longitudinal Stiffness (E_{11}) – III

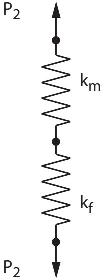
$$\frac{P_f}{P_1} = \frac{E_f/E_m}{E_f/E_m + E_m V_m}$$



Transverse Stiffness (E_{22}) – I

<u>transverse</u>: assume fibres and and matrix experience same stress





results in **inverse rule of mixtures** or *Reuss* iso-stress model:

$$\frac{1}{E_{22}} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

Transverse Stiffness (E_{22}) – II

assumptions for transverse modulus do not hold up to closer scrutiny, and predictions do not match experimental validation

Halpin-Tsai semi-empirical model offers better fit:

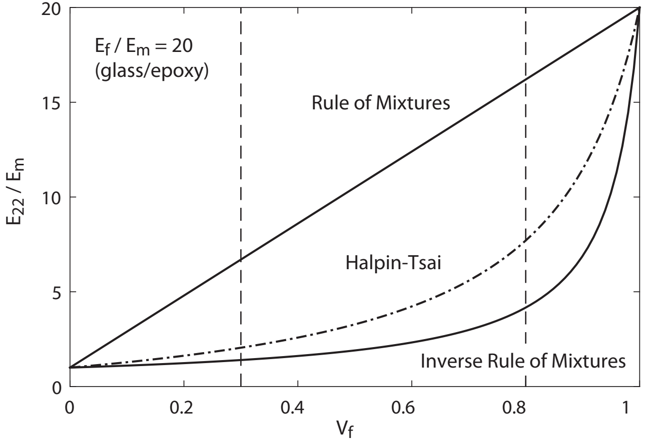
$$\frac{E_{22}}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}$$

with

$$\eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi}$$

where ξ is obtained from curve fitting; for circular fibres $\xi = 2$

Transverse Stiffness (E_{22}) – III



Shear Modulus (G_{12}) & Poisson's ratio (ν_{12})

shear modulus: inverse rule of mixtures

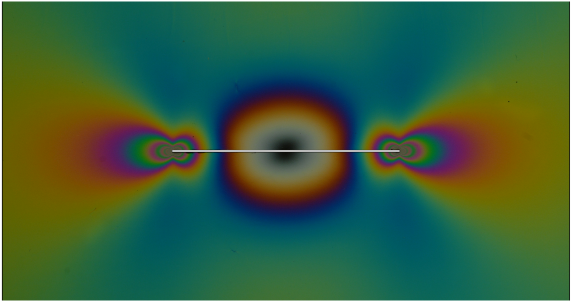
$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

not sufficiently accurate: Halpin-Tsai ($\xi = 1$)

Poisson's ratio: rule of mixtures

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

Short-Fibre Composites – I

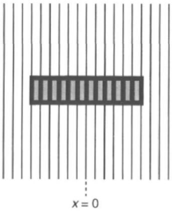


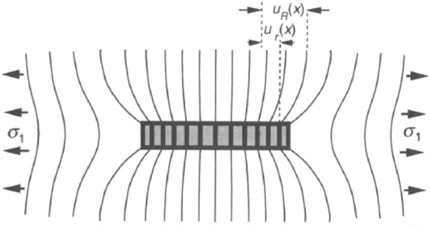
micromechanics of **short-fibre composites**

- stress distribution nonuniform along fibre length
- fibre/matrix load transfer through interfacial shear stress
- high interfacial shear stress at ends

Short-Fibre Composites – II

finite-length *stiff* fibre embedded in a *compliant* matrix





modelled using Cox' **shear-lag** model (non-examinable)

Short-Fibre Composites – III

fibre stress and interfacial shear stress at macroscopic strain ε_1

$$\sigma_f = E_f \varepsilon_1 \left[1 - \frac{\cosh\left(\frac{nx}{r}\right)}{\cosh\left(ns\right)} \right]$$

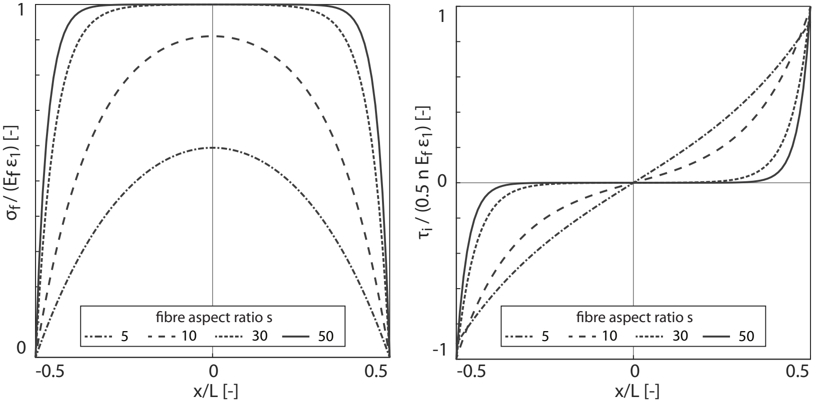
$$\tau_i = \frac{n}{2} E_f \varepsilon_1 \frac{\sinh\left(\frac{nx}{r}\right)}{\cosh\left(ns\right)}$$

where fibre aspect ratio $s = L/d$ and shear-lag parameter $\lambda = ns$

$$n = \sqrt{\left[\frac{4 G_m}{E_f \ln\left(1/V_f\right)} \right]}$$

with fibre modulus E_f, matrix shear modulus G_m

Short-Fibre Composites – IV

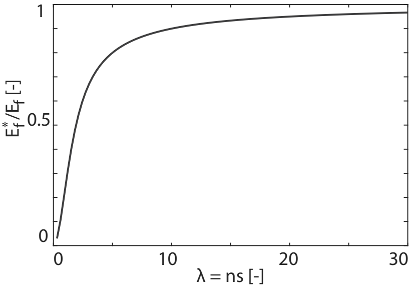


stress transfer length: fibre stress σ_f reaches plateau with $\tau_i \approx 0$ for sufficiently high shear-lag parameter $\lambda = ns$

Short-Fibre Composites – V

short fibres are inefficient reinforcement; effective fibre modulus

$$E_f^* = E_f \left[1 - \frac{\tanh(ns)}{ns} \right]$$



Summary

- micromechanics: interaction of composite constituents
- homogenisation: Representative Volume Element (RVE)
- key parameter: fibre volume fraction V_f
- continuous fibre-reinforced composites

  derived (inverse) rule of mixtures for macroscopic properties
- short-fibre composites

  nonuniform stresses along fibre length; stress transfer length

Revision Objectives

Revision Objectives:

- describe role of a Representative Volume Element (RVE) in micro-mechanical modelling;
- recall rule of mixtures for E_{11} and ν_{12};
- recall inverse rule of mixtures for E_{22} and G_{12};
- discuss assumptions in derivation of (inverse) rule of mixtures;
- apply Halpin-Tsai semi-empirical relationship for E_{22} and G_{12};
- describe load transfer between fibre and matrix in short-fibre composites;