Signals part 1.5 –Fourier and Laplace, Correlation

Fourier

- Fourier is credited with the idea that time domain waveforms can be represented by summations of sinusoids (his basis functions).
  - *But the maths uses basis function that are analytic.*

- A signal made up of summed sinusoids is called a Fourier series.

- Since sinusoids have a unique frequency component hence the concept reveals the mapping between time and frequency domains.

- This works well for periodic signals that continue indefinitely or signals which we can define as constant over some time.

- Fourier series are very useful for signals

Fourier transform

$$\hat{X}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

- This is the definition of the Fourier transform, and the inverse transform below.

$$x(t) = \int_{-\infty}^{\infty} \hat{X}(f)\, e^{j2\pi f t}\, df$$

- It requires the unit of the signal and its transform to be the reciprocal of each other: i.e. if we have seconds in the time domain, the transform is in cycles per second (or frequency in Hertz).

Fourier transform

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- It is common to use ω in the frequency domain, but still use seconds in the time domain. In which case we must compensate by dividing the inverse transform by 2π, thus restoring the units.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}(\omega)\, e^{j\omega t}\, d\omega$$

- The Inverse Fourier transform can be seen as the sum of the basis functions

Interpreting the inverse transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}(\omega)\, e^{j\omega t}\, d\omega$$

This is the size of the component at 'ω'

Unity magnitude basis function at 'ω'

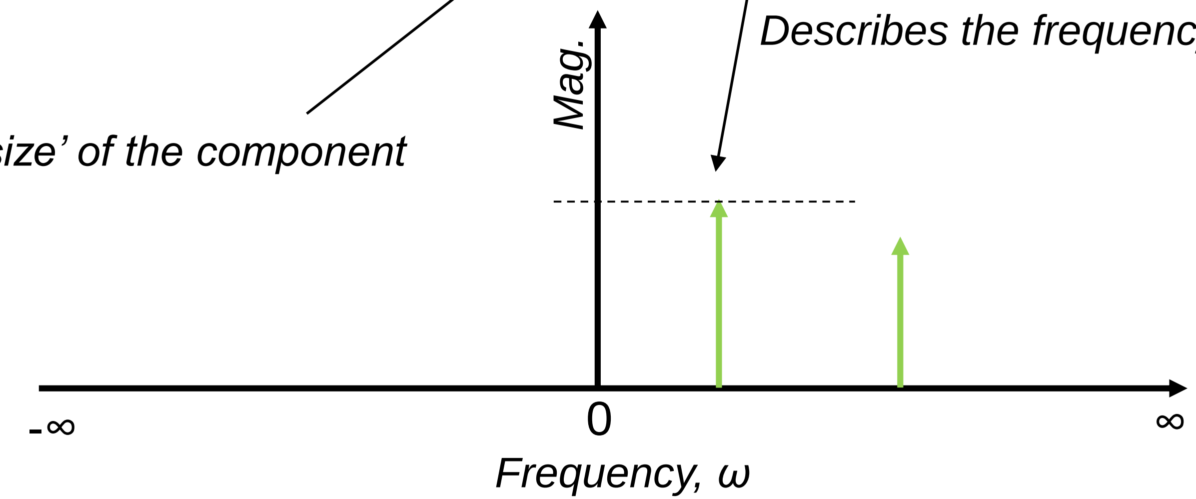
- The Inverse Fourier transform can be seen as representing the time series as the sum of weighted frequency components

Interpreting the inverse transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}(\omega)\, e^{j\omega t}\, d\omega$$

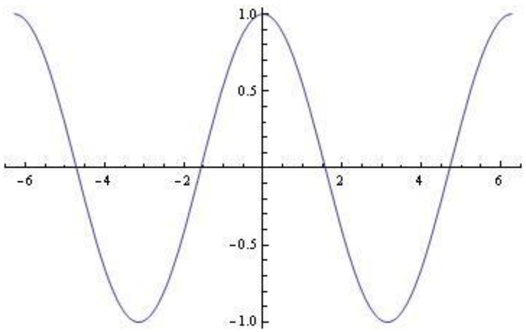
Describes the 'size' of the component

Describes the frequency of the component

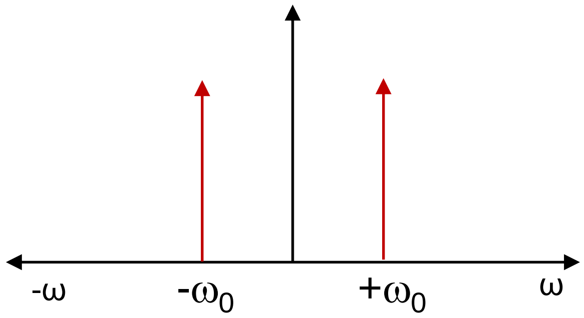


Integral is over all ω

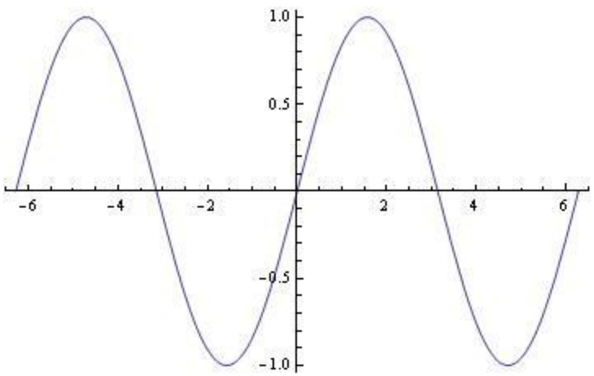
Some illustrative Fourier transforms



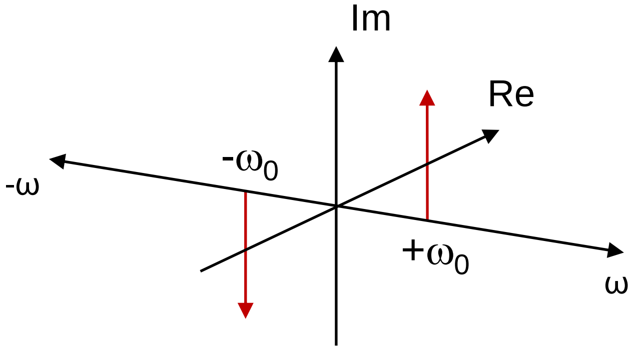
$\cos(\omega_0 t)$



$$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

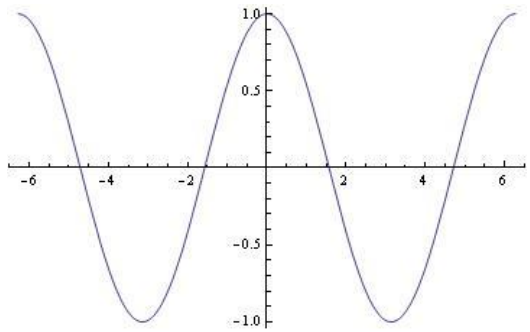


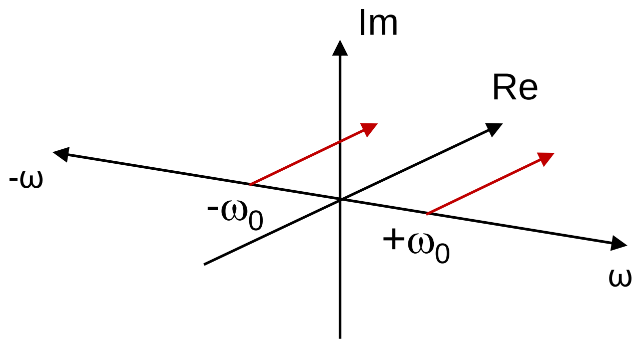
$\sin(\omega_0 t)$



$$\frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

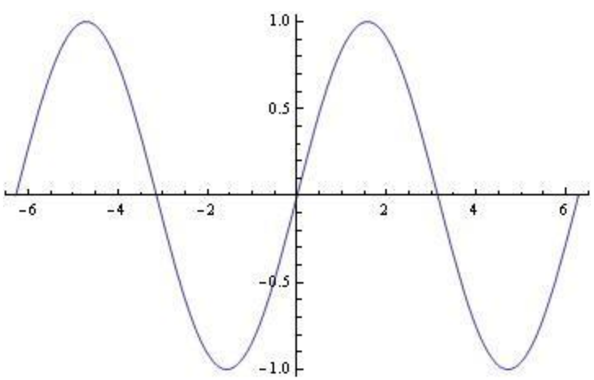
Some illustrative Fourier transforms

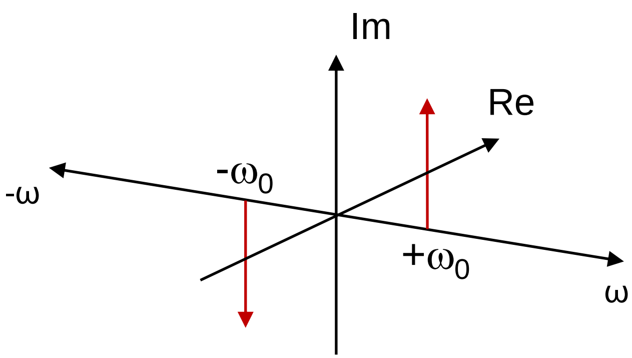




$$\cos(\omega_0 t)$$

$$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$





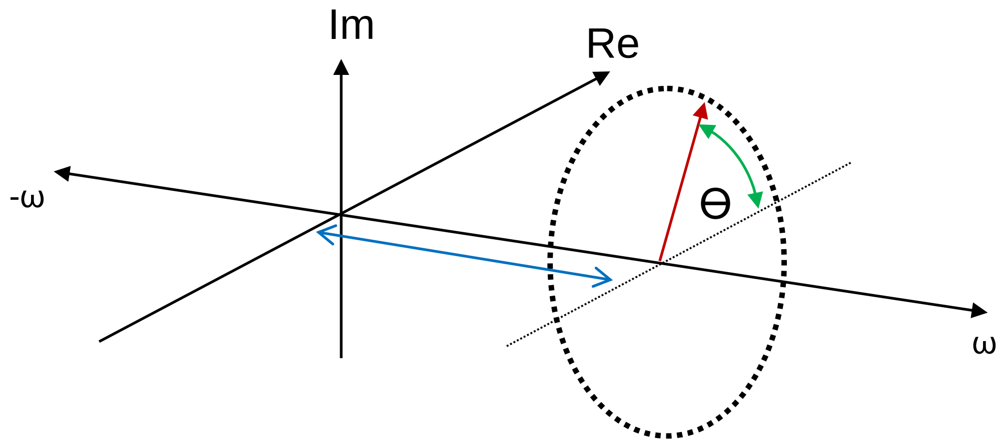
$$\sin(\omega_0 t)$$

$$\frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Some illustrative Fourier transforms

- The impulse function has infinite magnitude
  - the Fourier transform integrates over all time – hence for signals that extend for all time the magnitude is infinite. *This is not a very practical quality for us*.
  - You can follow the maths to find this – if the units of the signal are in 'v', the units of the transform (y axis) are in 'v.s' – we are integrating a voltage over a time.
- Each frequency component is actually expressed as a complex number
  - The phase of a frequency component is captured this way, e.g. sine is cosine shifted 90°

Visualising the Fourier domain



Representing the Fourier domain in this way we capture the frequency, the magnitude and the phase (relative to a cosine) of each component.

Note: as time does not exist, the component is 'stationary' on the phase plane.

Laplace

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$Where:\ s = r + j\omega$$

$$f(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\gamma - jT}^{\gamma + jT} F\,(s)\, e^{-st} ds$$

- Laplace went one stage further than Fourier and represented his time domain functions with sinusoidal basis functions which could grow or shrink.
- The physical significance of this is that the S domain can model systems that gain or lose energy over time – e.g. the free response of a mass/spring/damper or the forced response of that system.
- This idea is central to the control theory you will study next term – stable systems

The definition of the inverse Laplace transform appears more complex – the limits of integration are such that growing (or decaying) exponentials are captured and we are now transforming between ‘t’ and ‘s’, hence the 1/2πj unit correction.

Visualising the Laplace domain

- …is much harder!
  - We already used up all of our 3-D space representing the Fourier domain…...

Practical usage

- We have outlined the theory of Laplace and Fourier, but practically we may have take a few shortcuts.

- There is some fascinating maths to explore if you are that way inclined – you have probably already come across functions that have Laplace transforms but not Fourier transforms.

- With real signals we almost always have time domain signals that are real valued (not analytic) and are periodic over some finite time. We normally use Fourier type transforms on signals.

- For LTI systems we use the Laplace transform and then typically set $s = j\omega$ to extract the 'sinusoidal steady state' behaviour.

Correlation

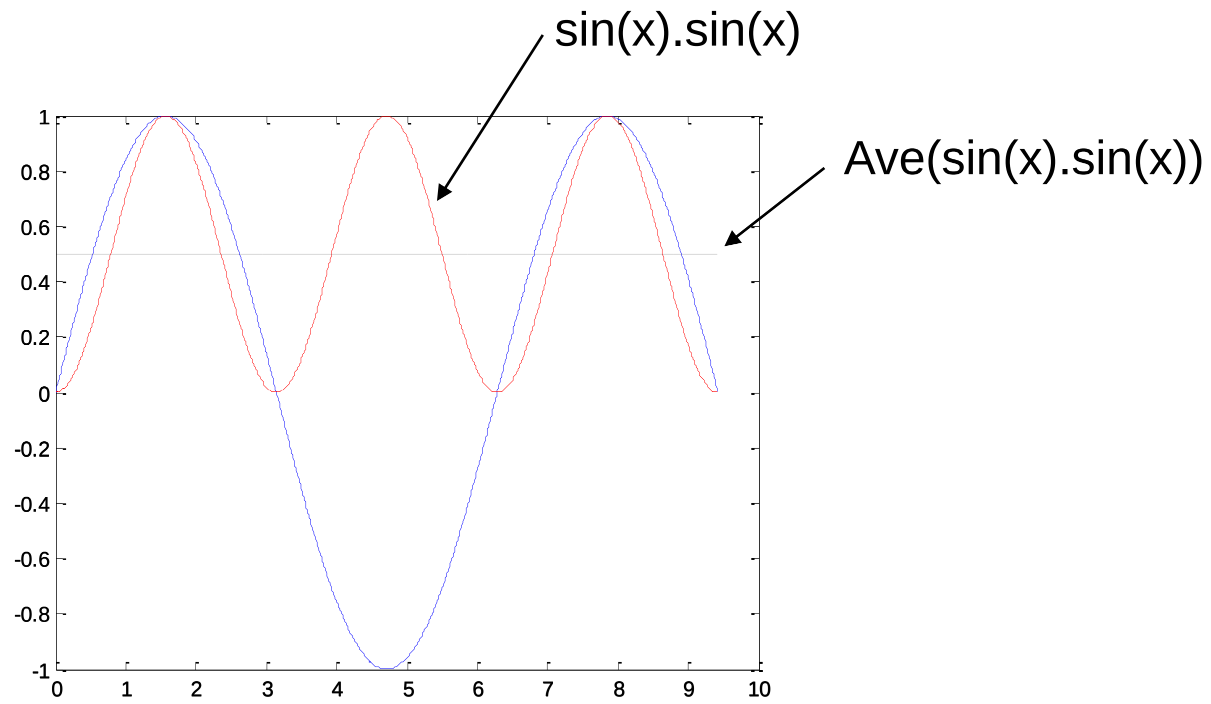
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

Value in frequency domain at ‘ω’ = integral of : (the time domain waveform * the rotating vector at ‘ω’)

Correlation of $f(t)$ and $g(t) = \int_{-\infty}^{\infty} f(t)g(t)dt$

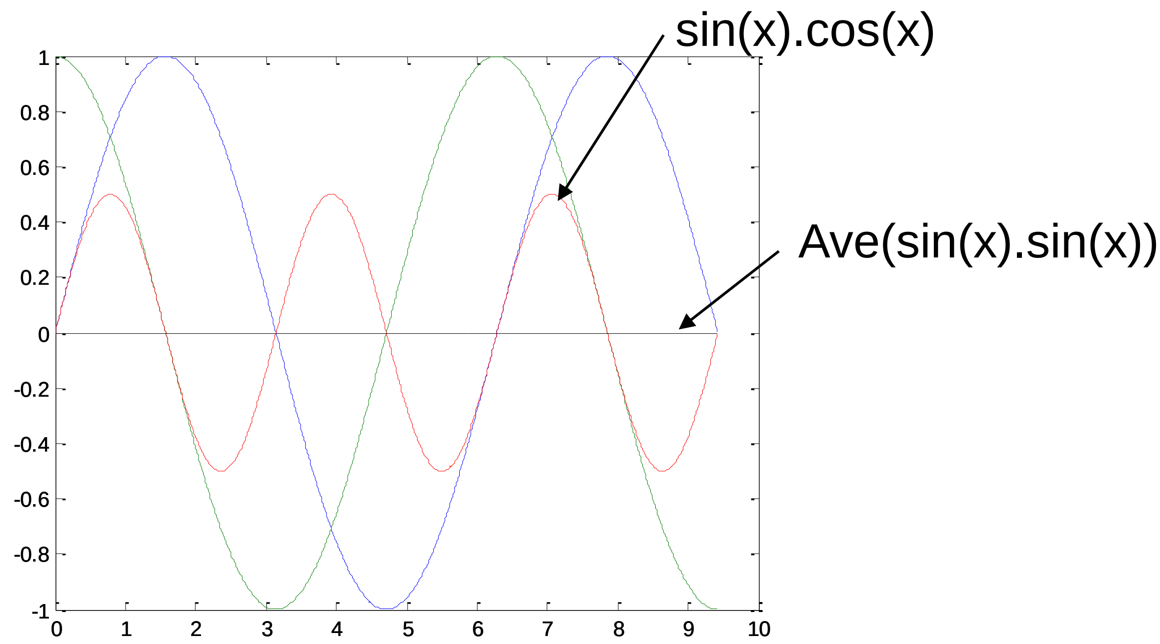
- The Fourier (and Laplace) transforms are examples of a correlation integral.
- The integrating (or averaging) the product of two signals gives a measure of similarity.

Correlation – simple example



- To correlate sin(x) with itself, multiply sin(x) and sin(x) and average the result over several cycles
- Correlating sin(x) with itself gives a value of 1/2

Correlation – simple example



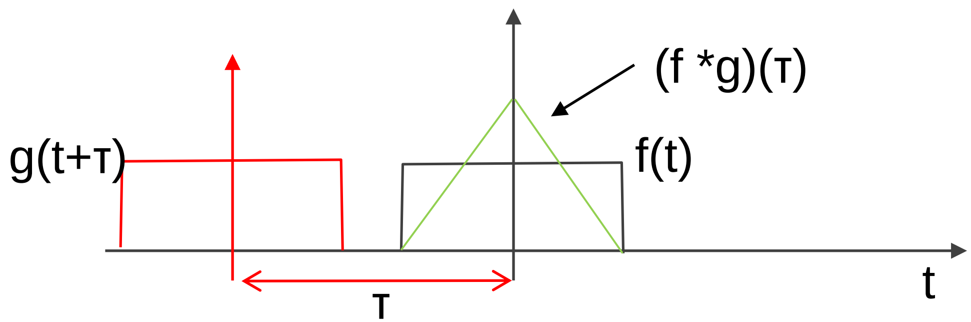
- Correlating sin(x) with cos(x) gives a value of 0

Cross-correlation

- In the previous examples the waveforms where ‘static’ and the correlation gave a single value.
- We can introduce a additional time delay into one waveform, τ, resulting in the operation ‘cross-correlation’

$$(f * g)(\tau) = \int_{-\infty}^{\infty} f^*(t) g(t+\tau) dt$$

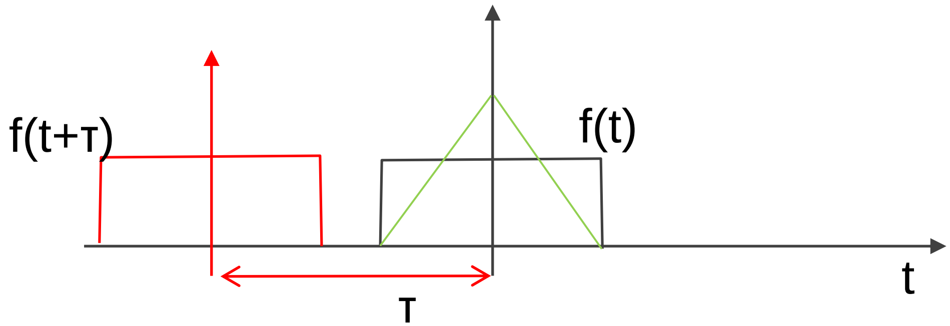
- Cross-correlation ‘slides’ one waveform over another, giving a correlation at each relative position (i.e. as a function of τ)



Auto-correlation

- The example on the previous slide is the cross correlation of a waveform with itself – this is known as auto-correlation.

$$(f * f)(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau) dt$$



- *Why f^* (complex conjugate)? Because if the signal is analytical, the imaginary part would produce a negative contribution to the integral, hence we use the conjugate*

Correlation

- Correlation is a powerful tool;
  - Can find the time differences between similar signals

    Performing a cross-correlation and looking for the value of offset that produces the peak value, indicates the time difference.
  - Can find particular components in a signal

    As used in the Fourier transform correlation can find components within a signal.
  - Can extract signals from noise

    A signal that is hidden by unwanted noise can be extracted by correlation – e,g, radio signals from space.

    Also GPS satellites all transmit of the same frequency. The messages are ‘overlaid’ when received, but are separated but correlation with ‘gold’ codes – orthogonal binary sequences.

Fourier series

The Fourier series is a particular implementation of the ideas of Fourier and especially useful for practical, engineering applications.

$$f(t) = A_0 + \sum_{n=1}^{\infty} a_n \sin n\omega t + \sum_{n=1}^{\infty} b_n \cos n\omega t$$

The Fourier series expresses the time series as the sum of a constant, sine and cosine harmonic components, normally over a fixed period of time.

It is very useful for period functions that are the sum of harmonic series

Fourier series coefficients

The Fourier coefficients can be found by evaluating the three expressions;

$$A_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t)\delta t$$

Mean of the function

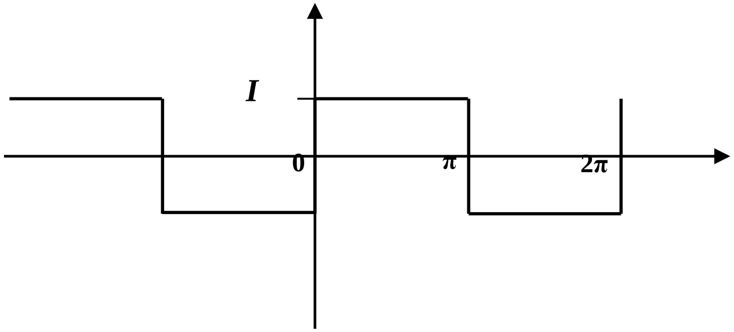
$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\delta t$$

Correlation of the function with sine n^{th} harmonic

$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\delta t$$

Correlation of the function with cosine n^{th} harmonic

Example



To simplify things we will normalise frequency

By inspection of the current waveform we see that;

•The average value is zero, hence $A_0=0$.

•It is an odd function hence will only contain sine terms. $(b_n=0)$

•Because the waveform has quarter-wave symmetry there are no even harmonics. $(a_n=0$, *for* $n=2,4,6$, *etc..)*

We therefore only need to solve the integral

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} I \sin nx\,\delta x \qquad \text{for } n=1,3,5 \text{ etc}$$

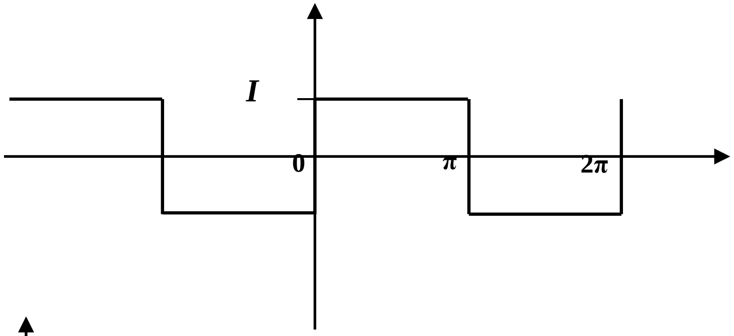
Example

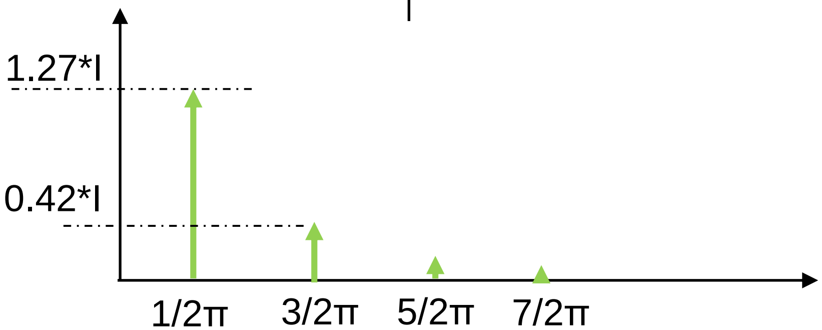
This gives us the result;

$$a_n = \frac{4I}{\pi n} \qquad (n = 1,3,5,7, etc)$$

The harmonic content of the AC signal is then;

Harmonic	Magnitude (rel. I)
1st	1.27
3rd	0.42
5th	0.25
7th	0.18





Polar form of Fourier series

$$f(t) = A_0 + \sum_{n=1}^{\infty} a_n \sin n\omega t + \sum_{n=1}^{\infty} b_n \cos n\omega t$$

$$A_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} f(t)\,dt \qquad a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t)\sin(n\omega t)\,dt \qquad b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t)\cos(n\omega t)\,dt$$

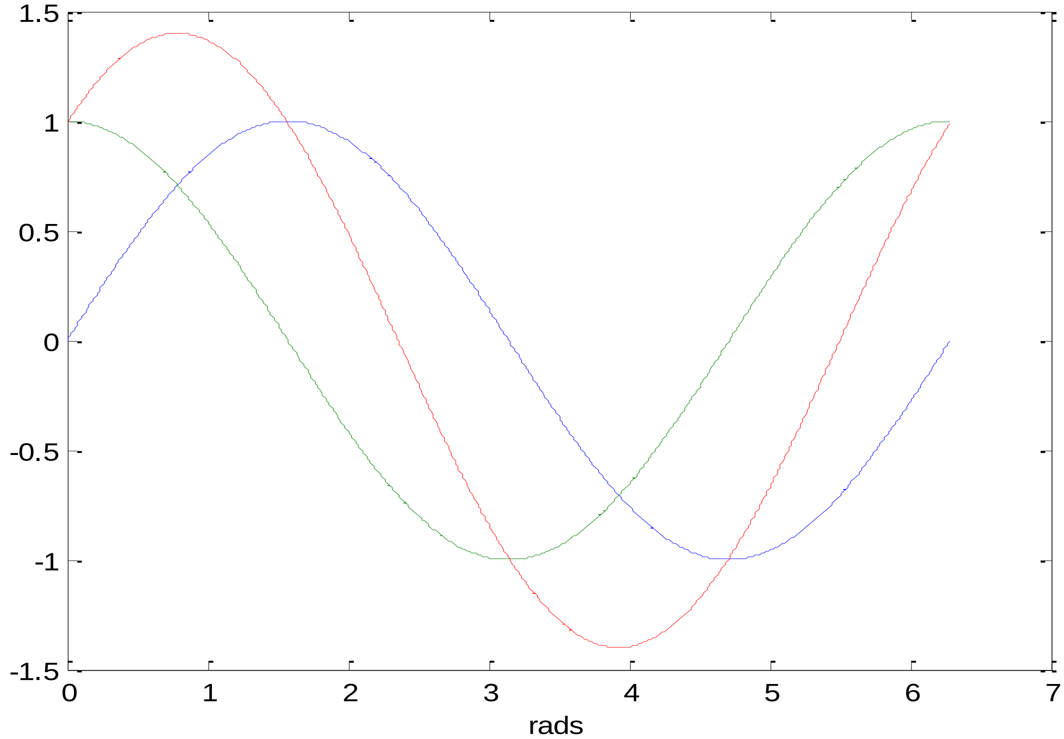
The Fourier series above is expressed in rectangular form. This resolves any frequency component with a phase offset into sine and cosine components, however it may be more intuitive to describe the Fourier series in a polar form;

$$f(t) = A_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega t + \theta_n)$$

$$C_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$$

$$\theta_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$

Polar form of Fourier series



For example the red waveform can be resolved into sine and cosine components *sin(θ) + cos(θ)* (rectangular form) but it may be more intuitive to express in polar notation *1.4sin(θ-π/4).*

Coefficients of the Fourier series

You will notice some formulations of the inverse Fourier transform/series multiply the correlation integral by some factor divided by the period over which the integral is taken, i.e.

$$A_0 = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t)\,dt \qquad\qquad a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin(n\omega t)\,dt$$

A_n it follows the definition of mean, i.e. *1/(period of integration)* but for a_n and b_n the factor is different. Why is this? And how do you work it out?

First remember the correlation is telling us " how much of the component *sin(nωt)* is in *f(t) ",* and if the signal continues for all time, the integral becomes larger as the period over which it is taken increases. If a signal is infinite in time, then all components will tend to infinity – and this is not really what we want to know.

It is easy (but somewhat longwinded) to work out what we must multiply the integral by, from the simple consideration that if f(t) = sin(ωt) and n =1, we require $a_n = 1$.

Units of the Fourier transform

Following on the previous slide, consider the units of the Fourier transform;

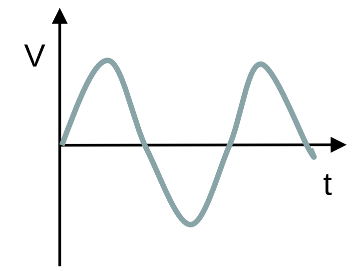
$$\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = F(f)$$

*Units of the signal * time*

Units of the signal

dimensionless

Adds units of time



If the time domain signal is in volts…

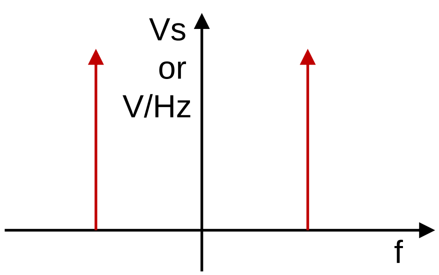
$$\int_{-\infty}^{\infty} \hat{F}(f)\, e^{j2\pi f t}\, df = x(t)$$

Units of the signal

*Units of the signal * time*

dimensionless

Adds units of frequency i.e 1/time



…the frequency domain 'signal' is in volts*seconds

Power and energy signals

Signals that are finite, such as exponential decay are described as ‘energy signals’ because the total energy in the signal is finite i.e.

$$E(t) = \int_{t_s}^{t_s+T} |f(t)|^2 \, dt \qquad\qquad 0 < \lim_{T\to\infty} E(t) < \infty$$

Signals that go on indefinitely, sine, unit step, etc. are sometimes described as ‘power signals’ that is because we need to work with the energy of the signal per unit time. The average power and definition of a power signal:

$$P(t) = \frac{1}{T} \int_{0}^{T} |f(t)|^2 \, dt \qquad\qquad 0 < \lim_{T\to\infty} P(t) < \infty$$

These definitions are useful as they guides the approach required to analyse a particular signal