of scalar fields

# 5. Integration over areas and volumes

*How do we integrate when there is more than one variable $\int\int dxdy$ (a double integral)? Does the order matter? What do such integrals represent physically? Can changing co-ordinates help? Is case of three variables $\int\int\int dxdydz$ really any harder than two variables? Conceptually no, but examples show that the limits can be more tricky.*

Integration so far has represented the area under a curve as a single variable $x$ (or $t$) varies along a line (or curve).

This enables one to calculate, e.g.

- the velocity of a particle, given its acceleration $\boldsymbol{a}(t)$
- the mass of a thin rod whose density/per unit length $\rho(x)$ varies as a function of the length
- the charge on a wire, given its charge density $f(x)$

which are all functions of a single variable.

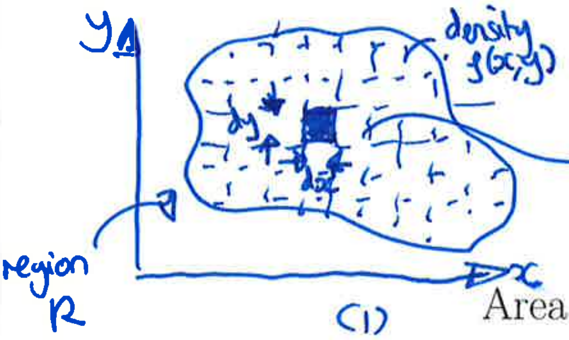
However, in many engineering applications we want to integrate over several variables, e.g. to find centre of gravity, moment of inertia, volume.

In this chapter we shall exclusively deal with **scalar fields** $f(\boldsymbol{r})$.

We start with the 2D case...

## 5.1 Double integrals

E.g. calculate the mass of a lamina occupying a region $R$ in the $(x, y)$-plane, whose density variation is $\rho(x, y)$




mass of a little square bit of lamina $= \rho(x,y)\,dx\,dy$

mass of entire lamina $= \sum_{y}\sum_{x} \rho(x,y)\,dx\,dy \longrightarrow \iint_R \rho(x,y)\,dx\,dy$

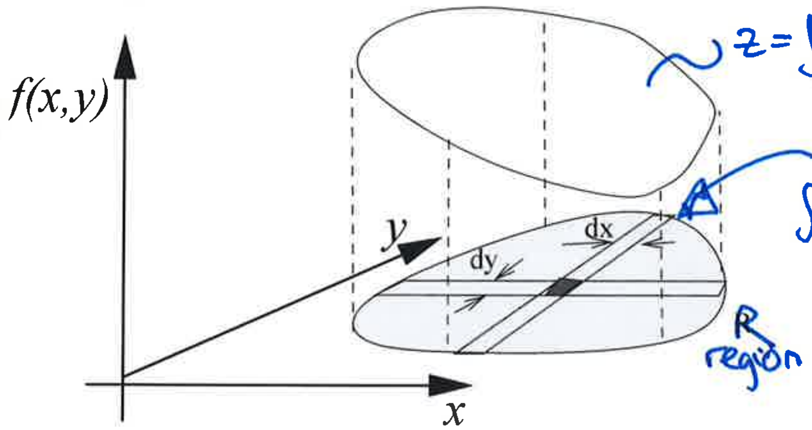
(I) Area of $R = \int\int_R dA = \int\int_R dxdy. = \iint_R 1.\,dxdy$

(II) Therefore, total mass $= \int\int_R \rho(x, y)dxdy.$

This is an example of a double integral, which we write in Cartesian co-ordinates as

$$\int\int_R f(x, y)dxdy,$$

and interpret as representing the volume under the 'height function' $f(x, y)$ above the region $R$.



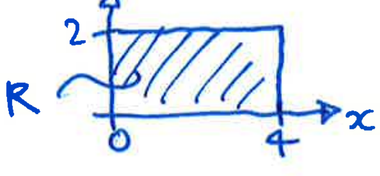

Such integrals are calculated by first integrating with respect to one variable, then the other. NOTATION: perform the inner integral first

$$\boxed{\int\int f(x, y)dxdy = \int \left[\int f(x, y)dx\right] dy} \tag{4.1}$$

**Worked example 5.1** *Calculate the integral*

$$\int\int_R (x^2 + y^2)dxdy = \int_{y=0}^{2}\int_{x=0}^{4} (x^2+y^2)\,dx\,dy$$

*where $R$ is the rectangle* $0 \leq x \leq 4$, $0 \leq y \leq 2$.



In the above example, the region $R$ was a rectangle, so the limits on $x$ and $y$ were obvious. But what are the limits on $x$ and $y$ for a general region $R$? Here it matters in which order we perform the integration.

Suppose we perform the integration with respect to $x$ first, they as in (4.1). Then we should express the region $R$ in the form

integrate second — integrate first

$$R : c \leq y \leq d \quad p(y) \leq x \leq q(y)$$

conventional numbers — limits are functions of y

Then

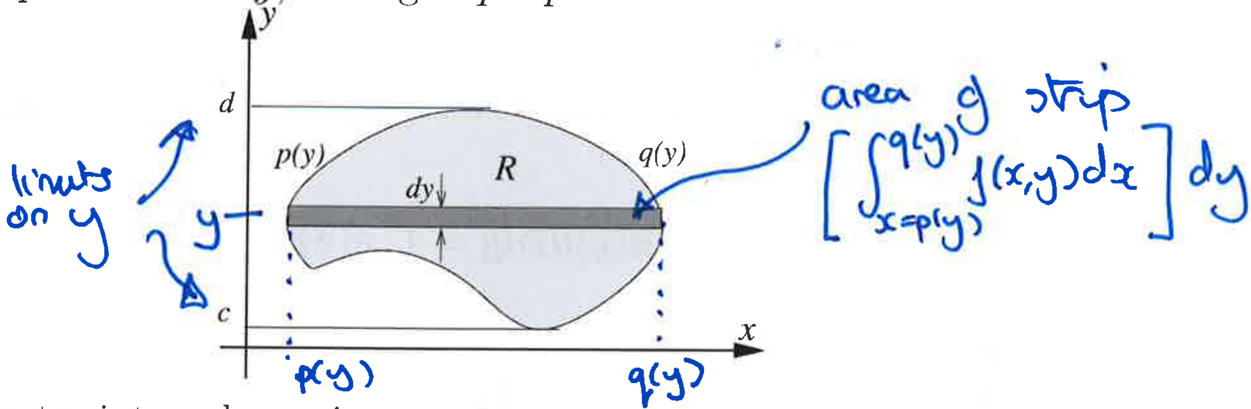
$$\int\int_R f(x,y)dxdy = \int_{y=c}^{d} \left[\int_{x=p(y)}^{q(y)} f(x,y)dx\right] dy$$

and the inner integral is

$$\int_{x=p(y)}^{q(y)} f(x,y)dx := P(y)$$

for which $y$ is just a constant, has limits ~~which~~ may be (and in general are) functions of $y$.

Physically we are summing the area under the curve along thin vertical strips of width $dy$, of length $q-p$




The outer integral now is

$$\int_{y=c}^{d} P(y)dy$$

is not a function of $x$ or $y$ so its limits are constants. This is the sum of all the horizontal strips between $y=c$ and $y=d$.

**Worked example 5.2** *Calculate the area of the triangle with vertices at* $(x,y) =$ A $(1,3)$, B $(3,3)$ C *and* $(3,7)$.

The key to these problems is to DRAW A PICTURE OF THE REGION OF INTEGRATION and carefully work out the limits on the integrals.

EXERCISE

**Worked example 5.3** *Consider the integral*

$$\int_{y=0}^{4}\int_{x=\sqrt{y}}^{2}(y+xy)dxdy.$$

Separable
$y+xy = y(x+1)$
but
doesn't help!

*Sketch the region of integration.*

## Properties of double integrals

The following are more or less obvious from thinking of the integral as the volume under the height function $f(x,y)$.

**Linear** If $\alpha$ and $\beta$ are constant scalars

$$\int\int_R(\alpha f(x,y)+\beta g(x,y))dxdy = \alpha\int\int_R f(x,y)dxdy+\beta\int\int_R g(x,y)dxdy$$

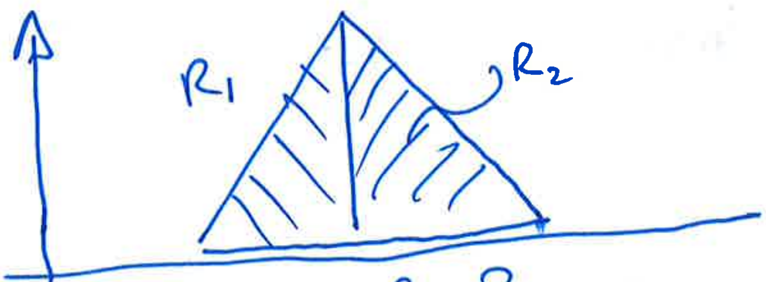
**Decomposable** If $R = R_1 + R_2$

$$\int\int_{R} f(x,y)dxdy = \int\int_{R_1} f(x,y)dxdy + \int\int_{R_2} f(x,y)dxdy$$

$=R_1+R_2$

**Separable** If limits are constant (if the region $R$ is a rectangle) and $f(x,y) = g(x)h(y)$ then

$$\int_a^b\int_c^d g(x)h(y)dxdy = \int_c^d g(x)dx\int_a^b h(y)dy.$$




$R=R_1+R_2$

$g(x)=x$
$h(y)=y$

separable $f(x,y) = xy$

non-separable $f(x,y) = xy+1$

## Applications of double integrals

2D Laminar

Given a distribution of mass in a region $R$ of the $(x, y)$-plane:

1. **area** $A$, total **mass** $M$

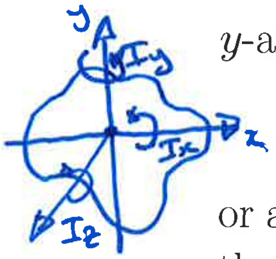
$$A = \int\int_R dxdy, \qquad M = \int\int_R \rho(x, y)dxdy$$

where $\rho(x, y)$ is the density.

2. The **centre of gravity** of the mass in $R$ has co-ordinates $\overline{x}$, $\overline{y}$, where

$$\overline{x} = \frac{1}{M}\int\int_R x\rho(x, y)dxdy, \qquad \overline{y} = \frac{1}{M}\int\int_R y\rho(x, y)dxdy.$$

3. The **moment of inertia** of the mass in $R$ about the $x$- and $y$-axes respectively




$$I_x = \int\int_R y^2\rho(x, y)dxdy, \qquad I_y = \int\int_R x^2\rho(x, y)dxdy,$$

or about an axis perpendicular to the $(x, y)$-plane passing through the point $(a, b)$

$$I_z = \int\int_R \left[(x-a)^2 + (y-b)^2\right]\rho(x, y)dxdy.$$

[linear motion : Force = mass × accⁿ
rotational motion : Torque = moment inertia × angular accⁿ]

$a = b = 0$ origin

$$I_z = \iint_R (x^2+y^2)\rho(x,y)\,dxdy$$

**Worked example 5.4** *Let $\rho(x, y) = 1$ be the density in the region*

$$R : 0 \le x \le \sqrt{1-y^2}, \quad 0 \le y \le 1.$$

*Find the center of gravity, the moments of inertia $I_x$, $I_y$ about the $x$ and $y$ axes, and the polar moment of inertia $I_z$ about an axis through the origin.*

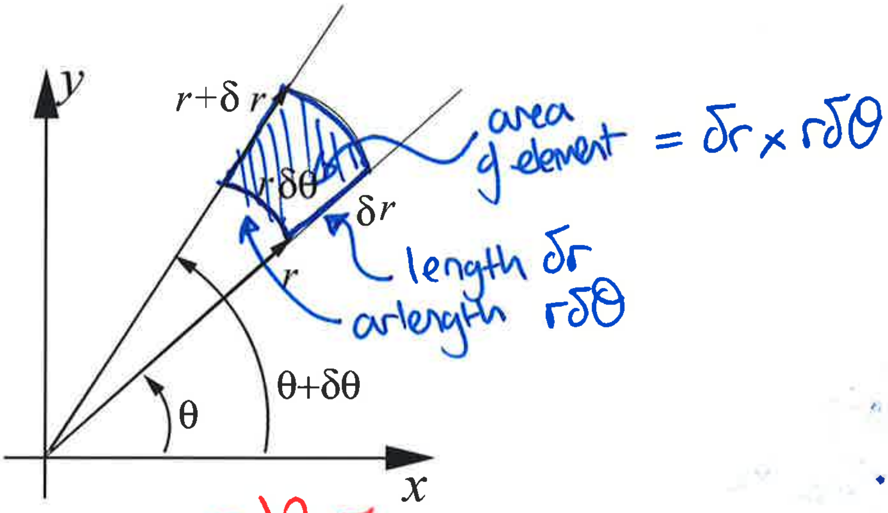
## 5.2 Changing co-ordinates in double integrals

The previous example would have been much easier to calculate had we used **cylindrical polar co-ordinates**:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

In order to do that we need to know what an infinitesimal piece of area $dA$ is in polar co-ordinates. Note it is NOT $drd\theta$...

$dA = dxdy$


... instead it is $rdrd\theta$. That is

$dA$

$$\boxed{\int\int f(x,y)\,dxdy = \int\int f(r\cos\theta, r\sin\theta)\,rdrd\theta.}$$

Note these co-ordinates are most useful when the region $R$ is composed of segments of circles and straight lines. Of course, we have to calculate the new limits on the $r$ and $\theta$ integrals carefully.

**Worked example 5.5** *Repeat the integrations in Worked example 5.4 using cylindrical polar co-ordinates*

More generally, if we change co-ordinates to some other co-ordinate system $(x,y) \to (u,v)$

$$x = x(u,v), \quad y = y(u,v)$$

defines the change in coordinates

then it can be shown that the infinitesimal piece of area

$$\boxed{dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,}$$

where the **area scale factor**

JACOBIAN = area scale factor

$$\boxed{|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

is the determinant of the so called **Jacobian matrix** $J$.

$\left[\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\right]$

## Cylindrical polar coordinates

The coordinate change is then $(x,y) = r(\cos\theta, \sin\theta)$, so

$x = r\cos\theta$
$y = r\sin\theta$

$$dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} drd\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} drd\theta = rdrd\theta$$

$r\cos^2\theta + r\sin^2\theta = r$

*i.e.* $dxdy = rdrd\theta$
(as derived earlier from first principles)

ie VOLUME

## 5.3 Triple integrals

In principle triple integrals are the same as double integrals, with an extra unknown. Triple integrals are also known as **Volume integrals**.

Consider a solid body occupying a volume $V$ in 3D. Let its density by $\rho(x, y, z)$. The mass of a small element $\Delta x\Delta y\Delta z$ is $\rho(x, y, z)\Delta x\Delta y\Delta z$.

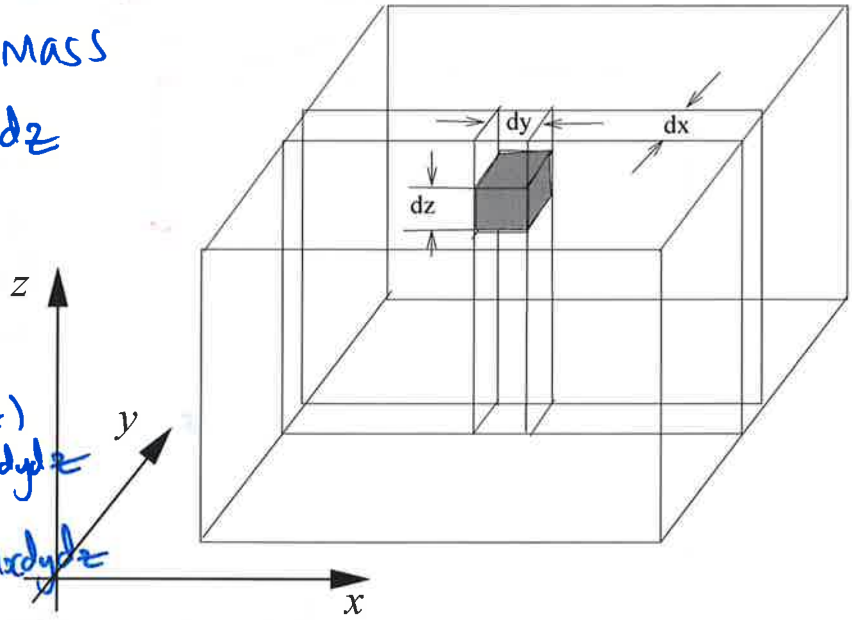
little volume, mass

$\rho(x,y,z)dxdydz$

total mass of volume

$= \sum_x \sum_y \sum_z \rho(x,y,z) dxdydz$

$\rightarrow \int\int\int \rho(x,y,z)dxdydz$ ✓




Hence the total mass of the solid is

$$\boxed{M = \int\int\int_V \rho(x, y, z)dxdydz}$$

The limits on the integration depend on the shape of the body, and are best illustrated by examples.

**Worked example 5.6** *Find the volume* $V = \int\int\int dxdydz$ *bounded by the planes* $x = 0$, $y = 0$ $z = 0$ *and* $x + y + z = a$.

Note that (assuming we integrate with respect to $x$, then $y$, then $z$) the innermost limits may depend on the other two variables ($y$ and $z$), the middle limits may depend on the outer variable ($z$), whereas the outer limits are constants.

$$\boxed{\int\int\int_V f(x,y,z)dxdydz = \int_{z=a}^{z=b}\left[\int_{y=g(z)}^{h(z)}\left(\int_{x=p(y,z)}^{q(y,z)} f(x,y,z)dx\right)dy\right]dz}$$

where the volume has been parametrised as

$$V : a \leq z \leq b, \quad g(z) \leq y \leq h(z), \quad p(y,z) \leq x \leq q(y,z).$$

## Properties of triple integrals

Triple integrals satisfy the same three properties of being **linear**, **decomposable** and, in the case of constant limits **separable**.

E.g. separable, with $\rho(x,y,z) = f(x)g(y)h(z)$:

$$\int_a^b \int_g^h \int_p^q f(x)g(y)h(z)dxdydz = \int_p^q f(x)dx \int_g^h g(y)dy \int_a^b h(z)dz.$$

in this case the volume $V$ is a cuboid.

## Applications of triple integrals

We can also use triple integrals to define centres of gravity and moments of inertia for 3D bodies in analogy with the 2D cases.

1. **Volume** $V$, total Mass $M$

$$V = \int\int\int_V dxdydz, \qquad M = \int\int\int_V \rho(x,y,z)dxdydz$$

where $\rho(x,y,z)$ is the density (mass per unit volume).

2. **Centre of gravity** $(\overline{x}, \overline{y}, \overline{z})$,

$$\text{where} \quad \overline{x} = \frac{1}{M}\int\int\int_V x\rho(x,y,z)dxdydz, \quad \text{similarly } \overline{y}, \overline{z}$$

3. **Moments of inertia**

$x \Leftrightarrow (y^2+z^2)$

$y \Leftrightarrow (x^2+z^2)$

$z \Leftrightarrow (x^2+y^2)$

$$I_x = \int\int\int_V (y^2+z^2)\rho(x,y,z)dxdydz$$

$$I_y = \int\int\int_V (x^2+z^2)\rho(x,y,z)dxdydz$$

$$I_z = \int\int\int_V (x^2+y^2)\rho(x,y,z)dxdydz$$

same as 2D case

**Worked example 5.7** *Find the moment of inertia about the z-axis of the ~~region R~~ volume V bounded by the parabolic ~~cylinder~~ prism $z = 4 - x^2$, and the planes $x = 0$, $y = 0$, $z = 0$, $y = 6$, assuming the density to be a constant $\rho$.*

As with double integrals, triple integrals can sometimes be easier to evaluate in different co-ordinate systems.

$$dV = dxdydz = |J|\,dudvdw$$

## 5.3 Changing co-ordinates in triple integrals

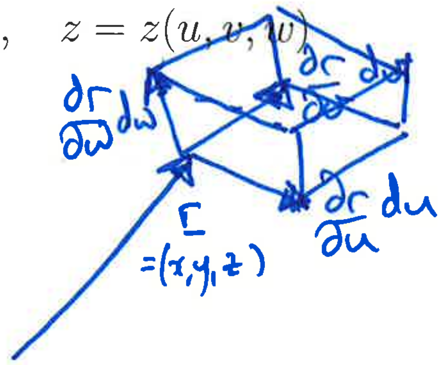
A general co-ordinate change $(x, y, z) \to (u, v, w)$ can be written in the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

$$\underline{r} = \underline{r}(u, v, w) = (x, y, z)$$

$$\underline{r} + d\underline{r} = \underline{r}(u+du, v+dv, w+dw)$$

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u}du + \frac{\partial \underline{r}}{\partial v}dv + \frac{\partial \underline{r}}{\partial w}dw$$




NOT EXAMINABLE.

Now from the diagram, we see that the infinitesimal piece of volume $dV$ is the volume of the parallelepiped spanned by the vectors

$$\frac{\partial \boldsymbol{r}}{\partial u}du, \quad \frac{\partial \boldsymbol{r}}{\partial v}dv, \quad \frac{\partial \boldsymbol{r}}{\partial w}dw.$$

From elementary vector algebra, this volume is given by a triple scalar product

triple scalar product

$$dV = \left(\frac{\partial \boldsymbol{r}}{\partial u}du\right) \cdot \left[\left(\frac{\partial \boldsymbol{r}}{\partial v}dv\right) \times \left(\frac{\partial \boldsymbol{r}}{\partial w}dw\right)\right] = \left[\frac{\partial \boldsymbol{r}}{\partial u} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial v}\right) \times \left(\frac{\partial \boldsymbol{r}}{\partial w}\right)\right] dudvdw$$

But the triple scalar product is equal to the determinant of all three vectors

IS EXAMINABLE

$$dxdydz = \boxed{dV = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} dudvdw := |J|dudvdw}$$

where $J$ is the **Jacobian matrix** of the transformation $(u, v, w) \mapsto (x, y, z)$, sometimes written as

$$J := \frac{\partial(x, y, z)}{\partial(u, v, w)} := \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

## Examples of 3-dimensional co-ordinate changes

- Linear changes of co-ordinates (rotations and shear) + displacement

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

3x3 matrix

Then $|J| = |A|$

And $\boxed{dV = dxdydz = |A|dudvdw}$

- Cylindrical polar co-ordinates.

In 2D we already saw how to change to polar co-ordinates. Cylindrical co-ordinates just adds the $z$ direction

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Then $|J| = r$ (derivation given earlier in notes)

And $\boxed{dV = dxdydz = rdrd\theta dz}$

- Spherical polar co-ordinates

θ - azimuthal
φ - axial

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$|J| = r^2\sin\theta$

And $\boxed{dV = dxdydz = r^2\sin\theta\, drd\theta d\phi}$

Proved from:

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -1 & 0 \end{vmatrix}$$

$$= r^2\sin\theta(\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi)$$

$$= r^2\sin\theta$$

EXERCISE
work through

### Application to triple integrals

So for any co-ordinate transformation, the scale factor we need to evaluate triple integrals is the determinant $|J|$ of the Jacobian $J$ of the co-ordinate transformation, that is

$$\boxed{\begin{aligned}&\int\int\int_V f(x,y,z)dxdydz\\ &= \int\int\int_V f(x(u,v,w),y(u,v,w),z(u,v,w))|J|dudvdw\end{aligned}}$$

Of course, we have to carefully consider the new limits. This is best illustrated by examples

scalar field over $u, v, w$

Jacobian

**Worked example 5.8** *Find the volume of the region above the $(x,y)$-plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.*

EXTRA EXAMPLE 5.9 Calculate the $z$-coordinate of the centre of gravity of a hemisphere $x^2+y^2+z^2=a^2$, $z \geq 0$.

## Summary

- Integration of scalar functions over an area $\Rightarrow$ Double Integrals
- All the work is in evaluating the $x$ and $y$ limits. Sometimes easier to change the order of integration

$$\int\int_R f(x,y)dxdy = \int\int_R f(x,y)dydx$$

but then the limits change.

- changing variables is easy if you remember $x = x(u,v)$, $y = y(u,v)$ implies

$$\int\int_R f(x,y)dxdy = \int\int_R f(x(u,v),y(u,v)) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dudv$$

- triple integrals are conceptually similar

$$\int\int\int_V f(x,y,z)dxdydz$$

NON - EXAMINABLE

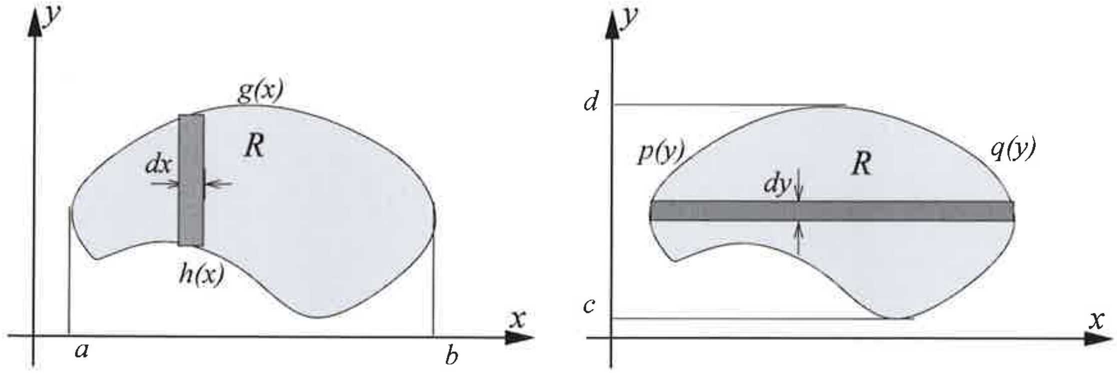
### Extra: Reversing order of integration in area integrals

It is also possible to integrate with respect to $y$ first and then with respect to $x$.

$$\int\int_R f(x,y)dxdy = \int\int_R f(x,y)dydx$$

But note that in general the limits change, since we must now express the region $R$ in terms of inequalities

$$a \leq x \leq b \quad g(x) \leq y \leq h(x)$$




So that

$$\int\int_R f(x,y)dydx = \int_{x=a}^{b}\left[\int_{y=h(x)}^{g(x)} f(x,y)dy\right]dx$$

**Worked example 5.9** *Calculate the area of the triangle in worked example 5.2, by integrating with respect to y first and then with respect to x.*

**Worked example 5.10** *Consider the integral*

$$\int_{y=0}^{4}\int_{x=\sqrt{y}}^{2}(y+xy)dxdy.$$

*Sketch the region of integration. Calculate the integral directly and by changing the order. Which is easier?*

## Example 5.1

find $\displaystyle\int_{y=0}^{2}\left[\int_{x=0}^{4}(x^2+y^2)\,dx\right]dy$

integrate from inside to out

ie (i) w.r.t. $x$

(ii) w.r.t. $y$

$$\int_{x=0}^{4}(x^2+y^2)\,dx = \left[\frac{x^3}{3} + xy^2\right]_0^4$$

$$= \frac{64}{3} + 4y^2$$

just a function of $y$

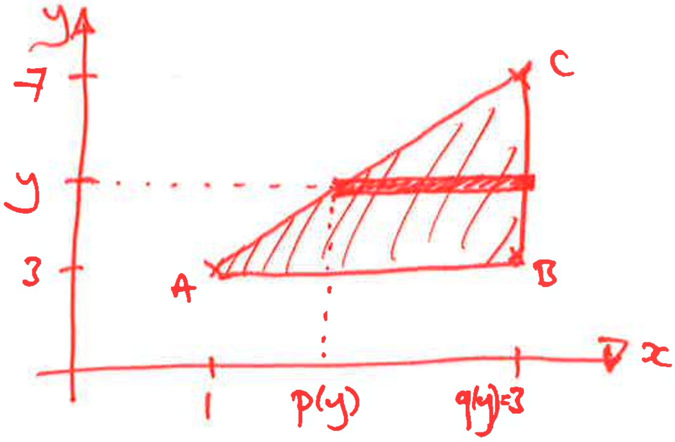
$$\therefore \int_{y=0}^{2}\int_{x=0}^{4}(x^2+y^2)\,dx\,dy = \int_{y=0}^{2}\left(\frac{64}{3}+4y^2\right)dy$$

$$= \left[\frac{64y}{3} + \frac{4y^3}{3}\right]_0^2$$

$$= \frac{128}{3} + \frac{32}{3} = \frac{160}{3}$$

note: doesn't matter if integrated $y$ first then $x$

Example 5.2




write integral of form

$$\int_{y=3}^{7}\int_{x=p(y)}^{q(y)} 1 \, dx \, dy$$

identify $q(y) = 3$

find $p(y)$ : value of $x$ on line AC at y-ordinate $y$

AC : $y = mx + c$

at A : $3 = m + c$ — ①

at B : $7 = 3m + c$ — ②

② − ① $4 = 2m$ $\therefore m = 2$

sub ① $3 = 2 + c$ $\therefore c = 1$

ie $y = 2x + 1$ ; $x = p$ @ $y = y$

$\therefore$ $y = 2p + 1$ $\Rightarrow$ $p = (y-1)/2$

x-integral first : $\displaystyle\int_{x=p(y)}^{q(y)} 1 \, dx = \int_{x=\frac{y-1}{2}}^{3} dx = \Big[x\Big]_{\frac{y-1}{2}}^{3}$

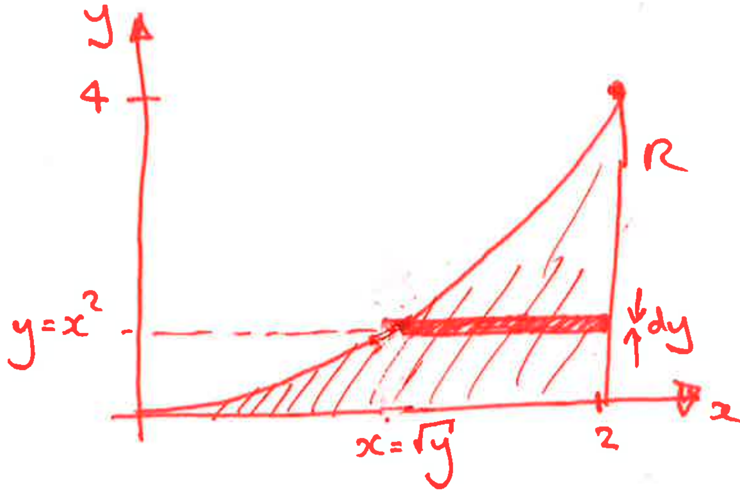
$= 3 - \left(\frac{y-1}{2}\right) = 7/2 - y/2$

y-integral next : $\displaystyle\int_{y=3}^{7} \left(7/2 - y/2\right) dy = \left[\frac{7y}{2} - \frac{y^2}{4}\right]_{y=3}^{7}$

$= \frac{49}{2} - \frac{49}{4} - \frac{21}{2} + \frac{9}{4}$

$= 4$

## EXAMPLE 5.3




(i) order in question

$$\int_{y=0}^{4}\int_{x=\sqrt{y}}^{2} (y+xy)\,dx\,dy$$

$$= \int_{y=0}^{4}\left[yx + \frac{x^2y}{2}\right]_{x=\sqrt{y}}^{2} dy$$

$$= \int_{y=0}^{4}\left(2y + 2y - y^{3/2} - y^2/2\right) dy$$

$$= \left[4y^2/2 - \tfrac{2}{5}y^{5/2} - y^3/6\right]_0^4 = 32 - \frac{64}{5} - \frac{64}{6} = \underline{\underline{\frac{128}{15}}}$$

EXTRA

(ii) reverse order of integration

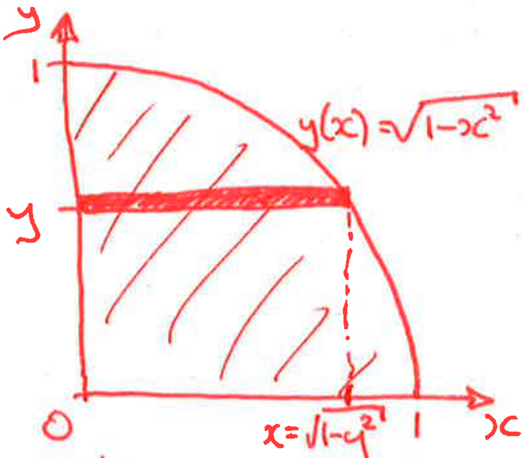
$$\int_{x=0}^{2}\int_{y=0}^{x^2} (y+xy)\,dy\,dx$$

$$= \int_{x=0}^{2}\left[y^2/2 + \frac{xy^2}{2}\right]_0^{x^2} dx$$

$$= \int_{x=0}^{2}\left(x^4/2 + x^5/2\right) dx$$

$$= \left[x^5/10 + x^6/12\right]_0^2 = 32/10 + 64/12 = \underline{\underline{128/15}}$$

## Example 5.4




$R: 0 \le x \le \sqrt{1-y^2}, \quad 0 \le y \le 1$

$$M = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} 1.\, dx\, dy$$

$$= \int_{y=0}^{1}\left[x\right]_{x=0}^{\sqrt{1-y^2}} dy = \int_{y=0}^{1}\sqrt{1-y^2}\, dy = \int_{\theta=0}^{\pi/2}\sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/2}\cos^2\theta\, d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}(1+\cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \qquad \begin{array}{l} y = \sin\theta \\ dy = \cos\theta\, d\theta \end{array}$$

$\therefore \underline{\underline{M = \pi/4}}$ (Note: obvious from geometry of circle)

$$\bar{x} = \frac{1}{M}\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} x\, dx\, dy = \frac{1}{M}\int_{y=0}^{1}\left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{2M}\int_{y=0}^{1}(1-y^2)\, dy = \frac{1}{2M}\left[y - \frac{y^3}{3}\right]_{y=0}^{1} = \frac{1}{3M}$$

$\therefore \underline{\underline{\bar{x} = 4/3\pi}}$

$$\bar{y} = \frac{1}{M}\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y\, dx\, dy = \frac{1}{M}\int_{y=0}^{1}\left[yx\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{M}\int_{y=0}^{1} y\sqrt{1-y^2}\, dy = \frac{1}{M}\left[\frac{(1-y^2)^{3/2}}{3/2}\times -\frac{1}{2}\right]_{y=0}^{1} = \frac{1}{3M}$$

$\therefore \underline{\underline{\bar{y} = 4/3\pi}}$ (Note: could spot $\bar{x} = \bar{y}$ from symmetry of shape under $\bar{x} \leftrightarrow \bar{y}$)

## Example 5.4 (cont)

$$I_x = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2 \, dx\, dy$$

$$= \int_{y=0}^{1}\left[y^2 x\right]_{x=0}^{\sqrt{1-y^2}} dy = \int_{y=0}^{1} y^2\sqrt{1-y^2}\, dy$$

$y=\sin\theta$
$dy=\cos\theta\, d\theta$

$$= \int_{\theta=0}^{\pi/2} \sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \int_{\theta=0}^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{4}\sin^2 2\theta\, d\theta = \frac{1}{8}\int_{\theta=0}^{\pi/2}(1-\cos 4\theta)\,d\theta = \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_{\theta=0}^{\pi/2}$$

$$\therefore \underline{\underline{I_x = \pi/16}}$$

$$I_y = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} x^2\, dx\, dy$$

$$= \int_{y=0}^{1}\left[x^3/3\right]_{x=0}^{\sqrt{1-y^2}} dy = \frac{1}{3}\int_{y=0}^{1}(1-y^2)^{3/2}\, dy$$

$y=\sin\theta$
$dy=\cos\theta\, d\theta$

$$= \frac{1}{3}\int_{\theta=0}^{\pi/2}\cos^4\theta\, d\theta \qquad \cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right)$$

$$= \frac{1}{4}\left(1+2\cos 2\theta+\frac{1}{2}+\frac{\cos 4\theta}{2}\right)$$

$$= \frac{1}{12}\int_{\theta=0}^{\pi/2}\left(3/2 + 2\cos 2\theta + \cos 4\theta/2\right)d\theta$$

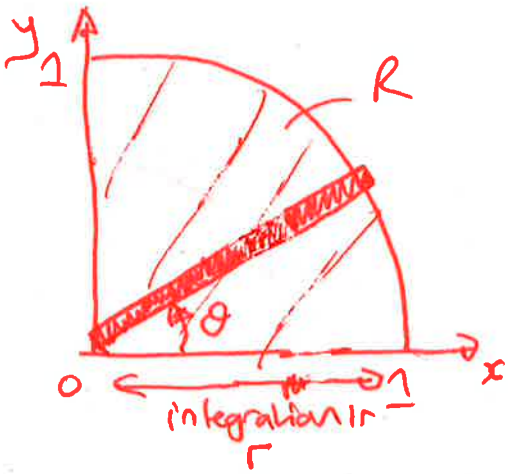
$$= \frac{1}{12}\left[3\theta/2 + \underbrace{\sin 2\theta}_{0} + \underbrace{\sin 4\theta}_{0}/8\right]_{\theta=0}^{\pi/2}$$

$$\therefore \underline{\underline{I_y = \pi/16}}$$

(Note: again could have used symmetry $x \leftrightarrow y$)

$$I_z = I_x + I_y = \underline{\underline{\pi/8}}$$

# EXAMPLE 5.5




change variables $x = r\cos\theta$
$y = r\sin\theta$

limits on R: $0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}$

area element: $dx\,dy = r\,dr\,d\theta$

$$M = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\,dr\,d\theta = \int_{\theta=0}^{\pi/2} d\theta \int_{r=0}^{1} r\,dr = \frac{\pi}{2}\cdot\frac{1}{2} = \underline{\underline{\frac{\pi}{4}}}$$

$$\bar{x} = \frac{1}{M}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} x\,r\,dr\,d\theta = \frac{1}{M}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\cos\theta\,dr\,d\theta$$

$$= \frac{1}{M}\int_{\theta=0}^{\pi/2}\cos\theta\,d\theta\int_{r=0}^{1} r^2\,dr = \frac{1}{M}\,1\cdot\frac{1}{3} = \underline{\underline{\frac{4}{3\pi}}}$$

$$\bar{y} = \frac{1}{M}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} y\,r\,dr\,d\theta = \frac{1}{M}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\sin\theta\,dr\,d\theta$$

$$= \frac{1}{M}\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\int_{r=0}^{1} r^2\,dr = \frac{1}{M}\,1\cdot\frac{1}{3} = \underline{\underline{\frac{4}{3\pi}}}$$

$$I_x = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} y^2\,r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\sin^2\theta\,d\theta\int_{r=0}^{1} r^3\,dr = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\pi/2} \times \frac{1}{4} = \underline{\underline{\frac{\pi}{16}}}$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$I_y = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} x^2\,r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^3\cos^2\theta\,dr\,d\theta$$

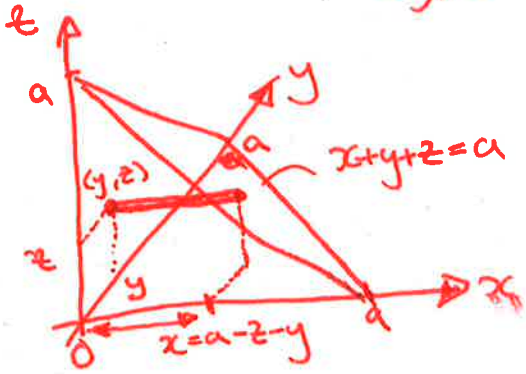
$$= \int_{\theta=0}^{\pi/2}\cos^2\theta\,d\theta\int_{r=0}^{1} r^3\,dr = \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\pi/2} \times \frac{1}{4} = \underline{\underline{\frac{\pi}{16}}}$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

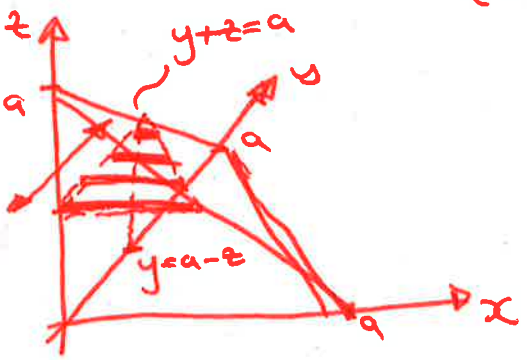
$$I_z = I_x + I_y = \underline{\underline{\pi/8}}$$

EXAMPLE 5.6

(i) line integration (y,z)




(ii) triangular section (z)




(i) <u>x inequality</u> : line (y,z)

for a (y,z) position, integrate from $x=0$ to plane $x+y+z=a$

ie $0 \le x \le a-y-z$

(ii) <u>y inequality</u> : triangular section (z)

for a z-position, integrate from $y=0$ to line $y+z=a$

ie $0 \le y \le a-z$

(iii) z <u>inequality</u> : z-extent of object

ie $0 \le z \le a$.

therefore

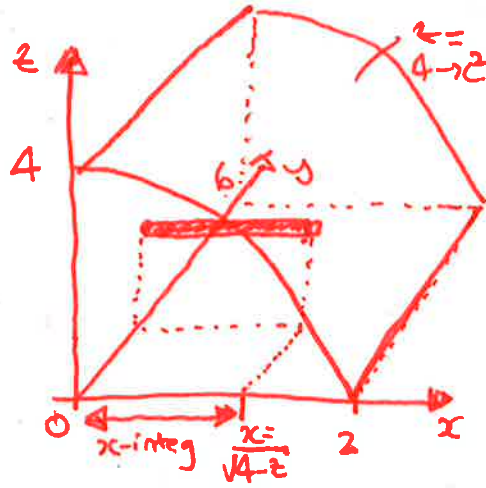
$$V = \int_{z=0}^{a}\int_{y=0}^{a-z}\int_{x=0}^{a-y-z} dx\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a-z}(a-y-z)\,dy\,dz = \int_{z=0}^{a}\left[ay-\frac{y^2}{2}-zy\right]_{y=0}^{a-z} dz$$

$$= \int_{z=0}^{a}\left(a(a-z)-\frac{(a-z)^2}{2}-z(a-z)\right)dz$$

$$= \int_{z=0}^{a}\left(\frac{a^2}{2}-az+\frac{z^2}{2}\right)dz = \left[\frac{a^2z}{2}-\frac{az^2}{2}+\frac{z^3}{6}\right]_{z=0}^{a} = \frac{a^3}{6}$$

## EXAMPLE 5.7




limits :

$$0 \le x \le \sqrt{4-z}$$

$$0 \le y \le 6$$

$$0 \le z \le 4$$

$$I_z = \int_{z=0}^{4} \int_{y=0}^{6} \int_{x=0}^{\sqrt{4-z}} \rho x (x^2 + y^2)\, dx\, dy\, dz$$

$$= \rho \int_{z=0}^{4} \int_{y=0}^{6} \left[ \frac{x^3}{3} + x y^2 \right]_{x=0}^{\sqrt{4-z}} dy\, dz$$

$$= \rho \int_{z=0}^{4} \int_{y=0}^{6} \left( \frac{1}{3}(4-z)^{3/2} + y^2 (4-z)^{1/2} \right) dy\, dz$$

$$= \rho \int_{z=0}^{4} \left[ \frac{y}{3}(4-z)^{3/2} + \frac{y^3}{3}(4-z)^{1/2} \right]_{y=0}^{6} dy\, dz$$

$$= \frac{6\rho}{3} \int_{z=0}^{4} (4-z)^{3/2} + 36 (4-z)^{1/2}\, dz$$

$$= 2\rho \left[ -\frac{(4-z)^{5/2}}{5/2} - 36 \frac{(4-z)^{3/2}}{3/2} \right]_{z=0}^{4}$$

$$= 4\rho \left( \frac{4^{5/2}}{5} + \frac{36 \times 4^{3/2}}{3} \right) = \underline{\underline{4\rho \left( \frac{32}{5} + 96 \right)}}$$

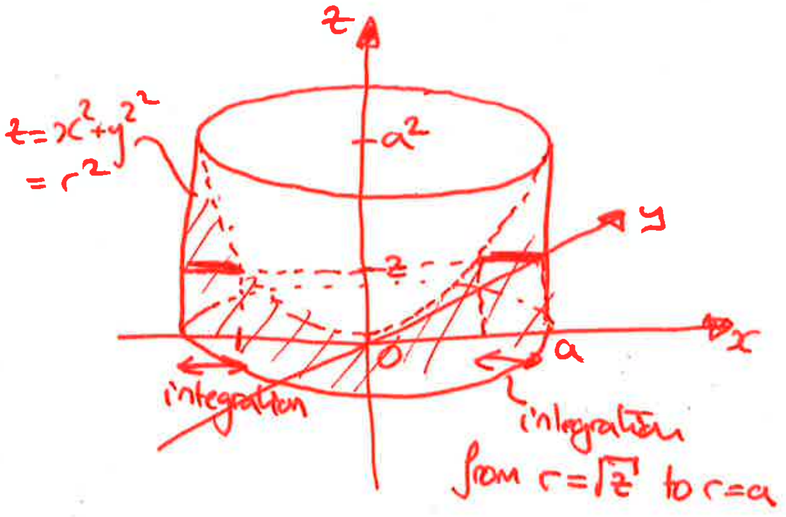
Example 8.8

volume $V$ :

$$\sqrt{z} \le r \le a$$

$$0 \le \theta \le 2\pi$$

$$0 \le z \le a^2$$




so
$$V = \int_{z=0}^{a^2}\int_{\theta=0}^{2\pi}\int_{r=\sqrt{z}}^{a} r\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{a^2}\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_{r=\sqrt{z}}^{a} d\theta\,dz$$

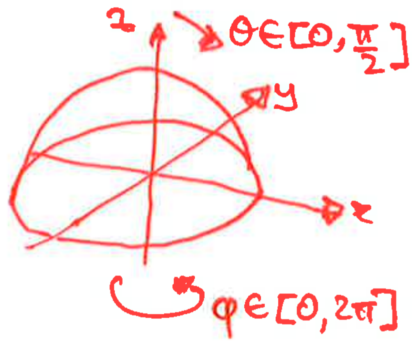
$$= \int_{z=0}^{a^2}\int_{\theta=0}^{2\pi}\frac{a^2-z}{2}\,d\theta\,dz$$

$$= \int_{z=0}^{a^2}\pi(a^2-z)\,dz$$

$$= \left[\pi\left(a^2 z - \frac{z^2}{2}\right)\right]_{z=0}^{a^2}$$

$$= \pi\left(a^4 - \frac{a^4}{2}\right) = \underline{\underline{\frac{\pi a^4}{2}}}$$

Example 5.9 : volume: $0<r<a$, $0<\theta<\frac{\pi}{2}$, $0<\varphi<2\pi$




$$\bar{z} = \frac{1}{M}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta \cdot r^2\sin\theta \, dr\, d\theta\, d\varphi$$

$$= \frac{1}{M}\int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\, d\theta \int_{r=0}^{a} r^3\, dr$$

$$= \frac{1}{M} \times 2\pi \times \frac{1}{2} \times \frac{a^4}{4} = \frac{\pi a^4}{4M}$$

$$M = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2\sin\theta\, dr\, d\theta\, d\varphi = \frac{2\pi a^3}{3}$$

½ vol of sphere

$$\therefore \quad \bar{z} = \frac{\pi a^4}{4M} \Big/ \frac{2\pi a^3}{3} = \underline{\underline{\frac{3a}{8}}}$$

extra

$$\bar{x} = \frac{1}{M}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta\cos\varphi \; r^2\sin\theta\, dr\, d\theta\, d\varphi = 0$$

$$\int_{\varphi=0}^{2\pi} \cos\varphi\, d\varphi = 0$$

Similarly $\bar{y} = 0$

(can expect $\bar{x} = \bar{y} = 0$ by cylindrical symmetry of hemisphere)