Vibrations 2, Lecture 12
Free vibration characteristics

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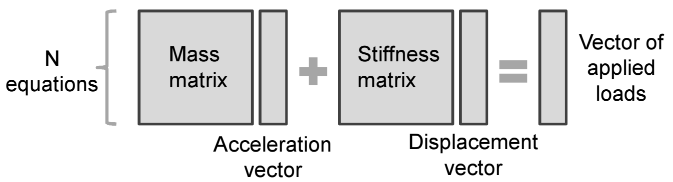
Lecture 11 review

Concept revision:

- Newton method (dynamic equilibrium)
- Free Body Diagram
- Degree of Freedom
- Multi degree-of-freedom system
- Equations of Motion

Equation of motion for NDOF system (no damping):

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t)$$



Lecture 12

- Free response
- Eigenvalue problem
  - Eigenvalues and eigenvectors
  - Natural frequencies and mode shapes
- Characteristic equation
- Numerical solutions using Matlab

Free vibrations

In this lecture, we will study *the characteristics of free vibration response* and the methods of their calculation. Consider a linear forced undamped system:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t)$$

We will study free vibrations of this system, $\mathbf{f}(t)=0$:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

We *assume* that this system can vibrate freely at a *single* frequency:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \sin(\omega t + \varphi) \\ a_2 \sin(\omega t + \varphi) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(\omega t + \varphi) = \mathbf{a}\sin(\omega t + \varphi)$$

x is the vector of displacements, **a** is the vector of displacement amplitudes, ω is the frequency of vibration, φ is the phase angle.

Free vibration response

We use the *assumed motion function* to find the free response of the system:

$$\mathbf{x} = \mathbf{a}\sin(\omega t + \varphi)$$
$$\dot{\mathbf{x}} = \omega\mathbf{a}\cos(\omega t + \varphi)$$
$$\ddot{\mathbf{x}} = -\omega^2\mathbf{a}\sin(\omega t + \varphi)$$

We substitute the vector of displacements and accelerations to the EOM:

$$-\omega^2\mathbf{M}\mathbf{a}\sin(\omega t + \varphi) + \mathbf{K}\mathbf{a}\sin(\omega t + \varphi) = \mathbf{0}$$

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a}\sin(\omega t + \varphi) = \mathbf{0}$$

This equation is valid for *all* time instants, then:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = \mathbf{0}$$

We obtained a linear (matrix) equation with the *unknown* vector of amplitudes **a** *and unknown* frequency ω. This problem is called ***eigenvalue problem***.

Eigenvalue problem

The eigenvalue problem:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = \mathbf{0}$$

For 2DOF systems, this problem has two solutions: ω_1^2, $\mathbf{a}_1$ and ω_2^2, $\mathbf{a}_2$. The values ω_i^2 are called *eigenvalues* and the vectors $\mathbf{a}_i$ are called *eigenvectors* or *mode shapes*. Eigenvalues are squares of the undamped *natural frequencies* ω_i.

We will consider *two* different ways of solving the eigenvalue problem. For small systems (for example 2DOF) we can use the *analytical* method, while for larger problems, or where repeated calculations are needed, we use the *numerical* method which is now available in Matlab.

Note: this problem is equivalent to the eigenvalue problem which was studied in Mathematics 1. Multiply the above equation by the inverse of the mass matrix:

$$\left.\begin{aligned} &\mathbf{M}^{-1}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = \mathbf{M}^{-1}\mathbf{0} \\ &(\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{M}^{-1}\mathbf{M})\mathbf{a} = \mathbf{0} \\ &(\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{I})\mathbf{a} = \mathbf{0} \end{aligned}\right\} \quad \begin{aligned} &(\mathbf{A} - \lambda\mathbf{I})\mathbf{a} = \mathbf{0} \\ &\mathbf{A} = \mathbf{M}^{-1}\mathbf{K},\ \lambda = \omega^2 \end{aligned}$$

Solving eigenvalue problem

The eigenvalue problem: $$(\mathbf{K}-\omega^2\mathbf{M})\mathbf{a}=\mathbf{0}$$

This system of linear equations has the two types of solutions:

- *One trivial* solution, where **a**=0 (zero amplitudes of vibration = no vibration)
- *Non-trivial* solutions, where **a**≠0 (non-zero vibration amplitudes)

Analytical solution: We are interested in non-trivial solutions. Assuming **a**≠0, the condition for the non-trivial solutions is that the determinant of the system matrix is zero. This condition is called *the frequency or characteristic equation*:

$$\det(\mathbf{K}-\omega^2\mathbf{M})=0$$

Numerical solution: An alternative approach to solving this problem uses *numerical methods*. Matlab can determine all eigenvalues and eigenvectors of the standard problem:

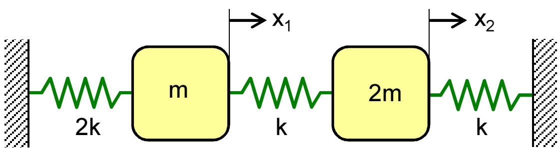
$$\mathbf{K}\,\mathbf{a}_i=\lambda_i\mathbf{M}\,\mathbf{a}_i$$

$$\lambda_i=\omega_i^2$$

```
» K= ...          % stiffness matrix
» M= ...          % mass matrix
» [Evec,Eval]=eig(K,M);
```


Example

Find the natural frequencies and mode shapes of the following system:



$$\mathbf{M}\,\ddot{\mathbf{x}}+\mathbf{K}\,\mathbf{x}=\mathbf{0}$$

$$\mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \qquad \mathbf{K}=\begin{bmatrix} 2k+k & -k \\ -k & k+k \end{bmatrix}$$

2DOF system is described by the stiffness and mass matrices of size 2×2. The determinant det(**K**-ω^2**M**) represents a polynomial of order 2 in ω^2 (or order 4 in ω). The quadratic polynomial (in ω^2) has two solutions – two eigenvalues. (This approach can be extended to systems with arbitrary number of DOFs).

Example

The characteristic equation:

$$\det(\mathbf{K}-\omega^2\mathbf{M}) = \det\left(\begin{bmatrix} 3k-\omega^2 m & -k \\ -k & 2(k-\omega^2 m) \end{bmatrix}\right) = 0$$

$$a(\omega^2)^2 + b(\omega^2) + c = 0; \; a = 2m^2, b = -8mk, c = 5k^2$$

$$\omega_{1,2}^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{(8\pm\sqrt{24})}{4}\frac{k}{m}$$

For example, if m=1 kg and k=600 N/m: $\omega_1 = 21.6\,\text{rad/s}, \; \omega_2 = 44.0\,\text{rad/s}$

From Matlab:

```
» m=1; k=600;
» K=[3*k,-k;-k,2*k];
» M=[m,0;0,2*m];
» Eval=eig(K,M);
» sqrt(Eval)
```

⇨

```
» sqrt(Eval)

ans =
   21.5674
   43.9869
```


Example

To determine the mode shapes (eigenvectors), we will make use of the original eigenvalue problem with previously calculated natural frequencies ω_1 and ω_2.

$$(\mathbf{K}-\omega_i^2\mathbf{M})\mathbf{a}_i = \mathbf{0} \Rightarrow \mathbf{a}_i = \ldots$$

Mode shape 1:

$$\begin{bmatrix} 3k-\omega_1^2 m & -k \\ -k & 2(k-\omega_1^2 m) \end{bmatrix}\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Mode shape 2:

$$\begin{bmatrix} 3k-\omega_2^2 m & -k \\ -k & 2(k-\omega_2^2 m) \end{bmatrix}\begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The above equations do not allow calculation of *unique* mode shapes $\mathbf{a}_1$ and $\mathbf{a}_2$ (the system matrices are singular). If we *choose* one equation in each case and *define* one vibration amplitude as 1 then the other amplitude can be calculated:

$$(3k-\omega_i^2 m)\,a_{1,i} - k\,a_{2,i} = 0 \;\Rightarrow\; a_{1,i} = \frac{k}{3k-\omega_i^2 m}a_{2,i}$$

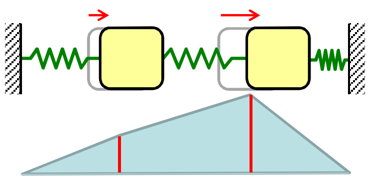
If we assume that $a_{2,i}$=1:

$$a_{1,i} = \frac{k}{3k-\omega_i^2 m} \;\Rightarrow\; \mathbf{a}_i = \begin{bmatrix} k/(3k-\omega_i^2 m) \\ 1 \end{bmatrix}$$

Example

Mode shape 1:

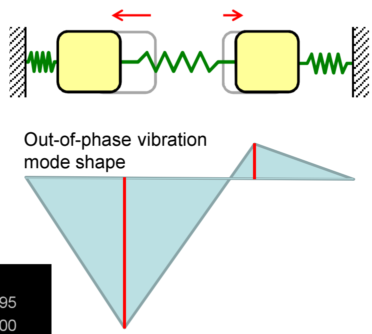
$$\mathbf{a}_1 = \begin{bmatrix} k/(3k - \omega_1^2 m) \\ 1 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 1 \end{bmatrix}$$



In-phase vibration mode shape

Mode shape 2:

$$\mathbf{a}_2 = \begin{bmatrix} k/(3k - \omega_2^2 m) \\ 1 \end{bmatrix} = \begin{bmatrix} -4.45 \\ 1 \end{bmatrix}$$



From Matlab:

```
» m=1; k=600;
» K=[3*k,-k;-k,2*k];
» M=[m,0;0,2*m];
» [Evec,Eval]=eig(K,M);
» Evec(:,1)/Evec(2,1), Evec(:,2)/Evec(2,2)
```

```
ans =
   0.4495
   1.0000
ans =
  -4.4495
   1.0000
```


Summary

- Structural systems can vibrate freely (naturally) at their natural frequencies with their vibration shapes proportional to the corresponding mode shapes
- MDOF systems have M natural frequencies and M mode shape vectors of size M
- Natural frequencies and mode shapes are determined by solving the eigenvalue problems
- Analytical methods use the characteristic equations to solve for the eigenvalues, the eigenvectors are *then* solved from the original eigenvalue problems