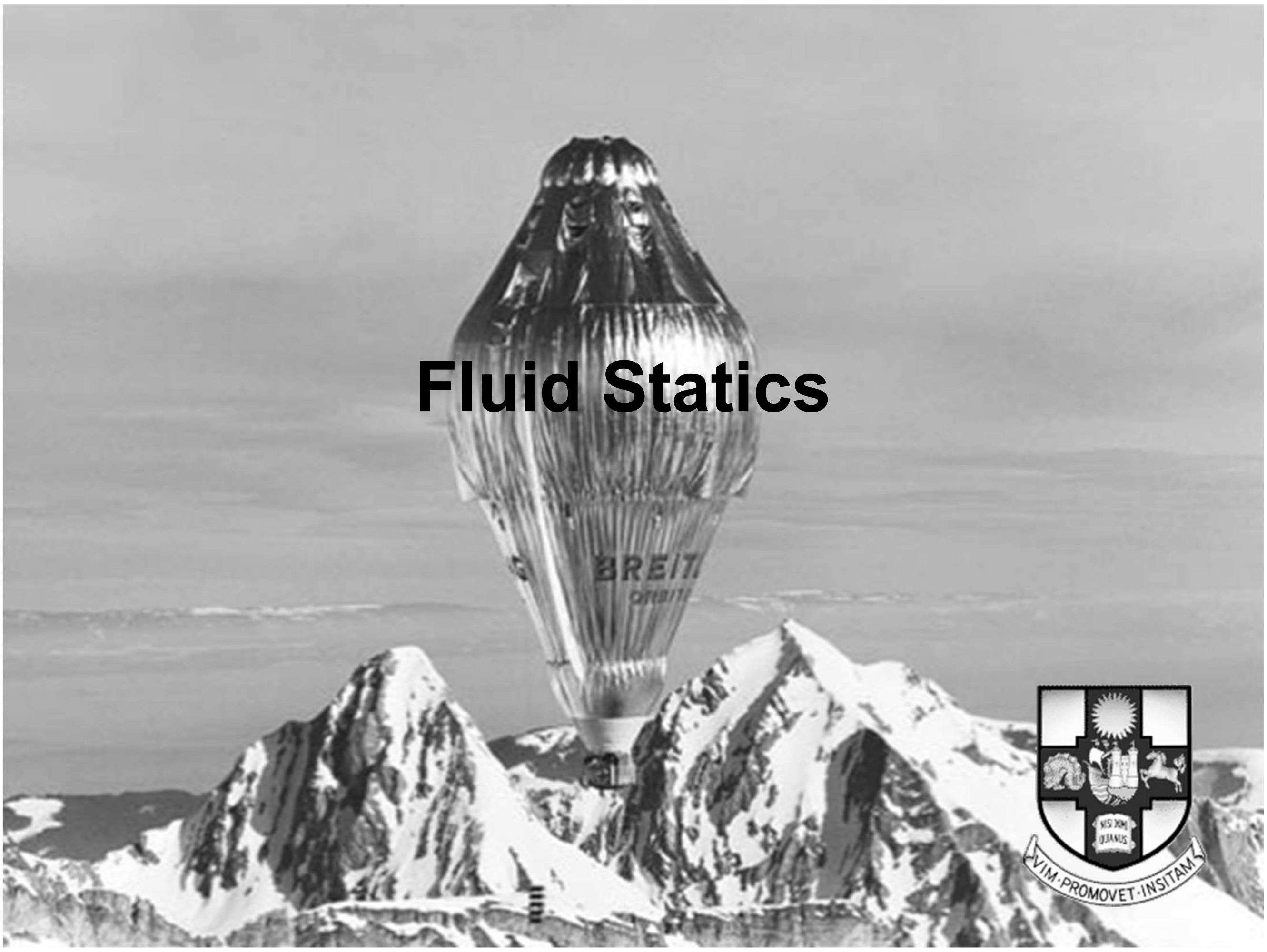
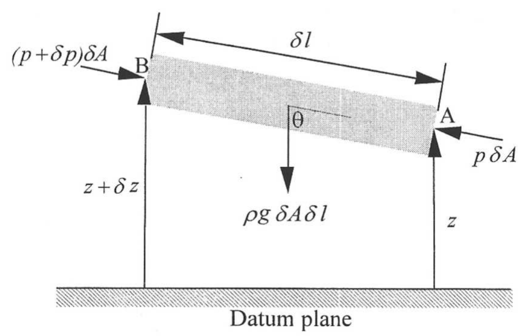



# Fluid Statics

## Pressure Variation in a Static Fluid

- Consider cylindrical fluid element A-B
- Area of ends, $\delta A$, is small so pressure can be considered constant on each end face
- $\delta p$ is difference in pressure between A & B
- Fluid at rest so only gravity and pressure
- Pressure acts normal to the element surface
- No net force in any direction
- Resolve forces in direction BA
- First use of Control volume analysis in Fluids1
- Simplify and take the limit as $\delta z \rightarrow 0$
  Gives "hydrostatic equation"

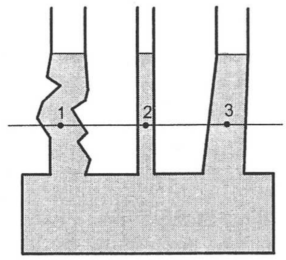
$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$

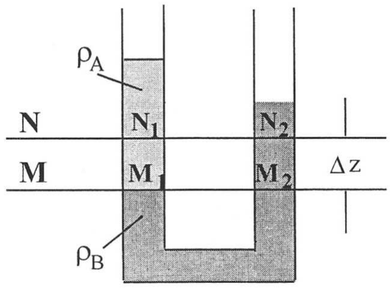



- *The pressure at any two points on a horizontal plane is the same provided they are connected by a line <u>wholly</u> in the fluid*

# Equilibrium of Constant Density Fluid

- Constant density fluid:
- Integrate previous equation for $z$ from a datum ($z=0$) to $h$, to give "constant density hydrostatic equation" $\boxed{p+\rho g h = \text{constant}}$
- Liquid under atmospheric pressure, $p_a$
  - Datum at surface where $p(z=0)=p_a$
  - $h$ increases downward $z=-h$ , $p=p(-h)$
  - "Gauge pressure" defined as $p(z=-h)-p_a$
  - Container shape irrelevant
  - What if we have two fluids?

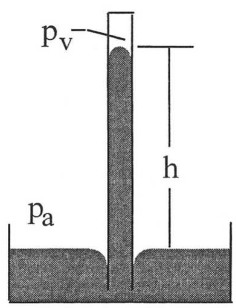


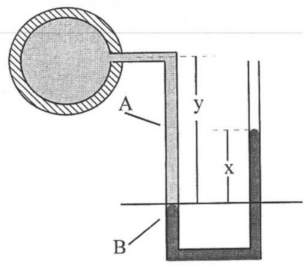



# Pressure Measurement

- Units: pressure is a measure of force per unit area but can often be defined with different units
  - N/m$^2$ "Newtons per metre squared"
  - Pa "Pascals" 1 Pa=1 N/m$^2$
  - bar 1 bar =100,000 Pa
  - atm "standard atmosphere" 1 atm=101,295 Pa ≈101,300 Pa
  - psi "pounds per square inch" 1 psi =6892.7 Pa
  - mmHg "mm of mercury" 1 mmHg =1 tor =133.28 Pa
- Pressure measured relative to another pressure, usually atmospheric, rather than a vacuum. Pressure transducers still require calibration.
- Barometer: $p_a = p_v + \rho_{Hg} g h$
  - Mercury vapour pressure $p_v$ small relative to $p_a$
  - $p_v$ =0.16N/m$^2$ $p_a$ ≈10$^5$
  - $\rho_{Hg}$ =13560 Kg/m$^3$ so column height is short



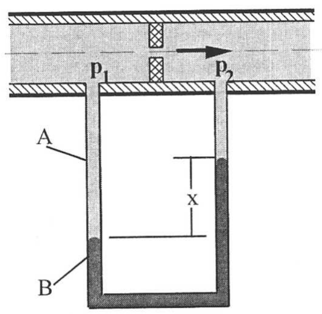
# Pressure Measurement (2)




- U-tube manometer
  - Measures the difference in pressure
  - Fluid B (liquid) is immiscible with fluid A
  - Density of B greater than A

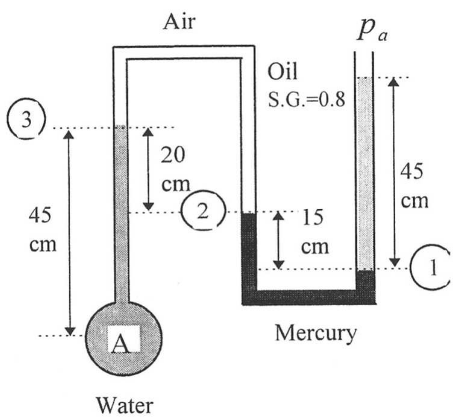
$$p + \rho_A g y = p_a + \rho_B g x$$

  - Interface of fluid A &B <u>or</u> B & atmosphere usually has increased area so y <u>or</u> x stays approximately fixed
- Example of use in flowing fluid
  - Measure the difference either side of a restriction
  - Attachments are perpendicular to flow
  - For small differences adjust the density of B
  - and/or Incline the measuring sections of the tube






# Example: Determine Gauge pressure at A




Applying the constant density hydrostatic equation at the interfaces is the best technique for this type of problem

$$p + \rho g h = \text{constant}$$

$$\begin{aligned} p_1 &= p_a + 800 \; x \; 9.81 \; x \; 0.45 \;\; N/m^2 \\ &= p_a + 3531.6 \;\; N/m^2 \end{aligned}$$

$$\begin{aligned} p_2 &= p_1 - 13560 \; x \; 9.81 \; x \; 0.15 \;\; N/m^2 \\ &= p_1 - 19953.6 \;\; N/m^2 \end{aligned}$$

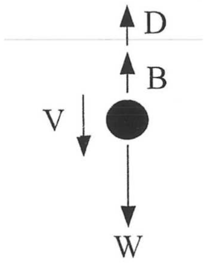
$$\begin{aligned} p_3 &= p_2 - 1.2 \; x \; 9.81 \; x \; 0.2 \;\; N/m^2 \\ &= p_2 - 2.35 \;\; N/m^2 \end{aligned}$$

$$\begin{aligned} p_A &= p_3 + 1000 \; x \; 9.81 \; x \; 0.45 \;\; N/m^2 \\ &= p_3 + 4414.5 \;\; N/m^2 \end{aligned}$$

$$p_A = 4414.5 - 2.35 - 19953.6 + 3531.6 + p_a \quad \rightarrow \quad p_A - p_a = -12009.9 N/m^2$$

Remember to be consistent with the direction (h measured upwards)

# Example: Terminal velocity of a rain drop




Assume that the drop is a sphere of radius 1mm, the air and water have densities of 1.2 kg/m$^3$ and 1000 kg/m$^3$ .
The drag on the spherical droplet is given by the formulae
$D = C_D \tfrac{1}{2}\rho_a V^2 A_\times \qquad C_D = 0.45$
From Archimedes' principle, the buoyancy force is given by the weight of displaced fluid,
$B = \tfrac{4}{3}\pi r^3 \rho_a g$

This is not a true static problem, but we shall use it to demonstrate buoyancy. We are replacing the relative motion of air and water by a simple force relation. At terminal velocity no acceleration so all forces in balance, then

$$D = W - B$$

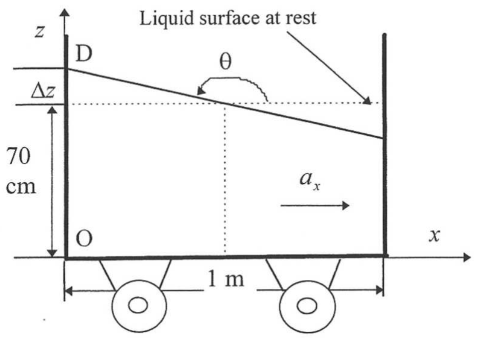
$$C_D \tfrac{1}{2}\rho_a \pi r^2 V^2 = \tfrac{4}{3}\pi r^3 g(\rho_w - \rho_a)$$

$$V^2 = \frac{8rg}{3C_D\rho_a}(\rho_w - \rho_a) \;\rightarrow\; V = \sqrt{\frac{8\times(1\times10^{-3})\times9.81\times(1000-1.2)}{3\times0.45\times1.2}} = 6.96\text{m/s}$$

Note: Buoyancy problems often include the use of the equation of state *i.e.* $p = \rho RT$ or for a fixed temperature $pV = $ const



# Example: Rigid body translation of a liquid




Consider the adjacent trolley filled with oil (S.G 0.8) accelerating at a constant 5m/s$^2$. Calculate the slope of the surface (θ) and the gauge pressure at O.
This type of problem (unlike the previous) can be treated as static because there is no relative motion between the fluid and the container.
Generally we must consider accelerations in x and z given in this case by $a_x$=5m/s and $a_z$=0.

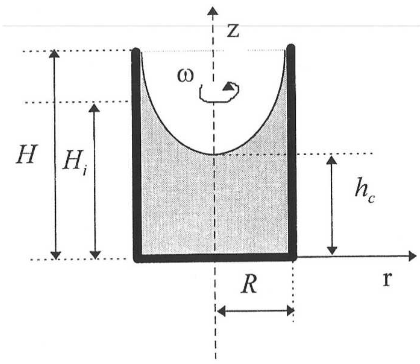
The fluid behaves as though under the influence of an effective gravity vector:
($g_x$,$g_z$)=(-5, 9.81)
The fluid forms a surface perpendicular to this vector

$$\tan\theta = \frac{dz}{dx} = -\frac{a_x}{g + a_z} \qquad \rightarrow \quad \theta = \tan^{-1}(-5/9.81) = 153^o$$

$\Delta z = 0.5\tan(27^o) = 0.2548m$
Two choices for gauge pressure, find h perpendicular to the surface and use $\sqrt{a_x^2 + (g + a_z)^2}$ or use g and the vertical distance from D. Can you think of another way?

$$p_O - p_a = \rho g(0.7 + \Delta z) \quad \rightarrow \quad 0.8\times1000\times9.81\times(0.7 + 0.2548) = 7493.27 N/m^2$$

# Example: Rigid body rotation of a liquid




The container is initially filled to height $H_1$. Find angular speed, the shape of the free surface and the pressure at the bottom corner of the container if the liquid surface just touches the lip at height $H$.

Rotational symmetry, so consider $r$ and $z$ only.
Centripetal acceleration provides a pressure gradient in $r$

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \qquad \frac{\partial p}{\partial z} = -\rho g$$

Integrating the centripetal terms $p = \frac{1}{2}\rho\omega^2 r^2 + f(z)$

Taking the partial differential w.r.t. $z$ ($r$ fixed) gives

$$\frac{\partial p}{\partial z} = \frac{\partial f}{\partial z} = \frac{df}{dz} \quad \rightarrow \quad \frac{df}{dz} = -\rho g \quad \rightarrow \quad f(z) = -\rho g z + C \rightarrow \quad p = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + C$$

Complete centripetal equation for $p$ and evaluating the constant at the centre $p(r=0, z=h_c) = p_a$

$$p = \frac{1}{2}\rho\omega^2 r^2 + p_a - \rho g(z - h_c)$$

At the lip with maximum angular speed ω=Ω and pressure given by $p(r=R, z=H) = p_a$

$$\frac{1}{2}\rho\Omega^2 R^2 = \rho g(H - h_c)$$

The same result for all the surface points shows that the local height above the centreline increases with $r^2$, forming a parabaloid

$$(z - h_c) = \left(\frac{\Omega^2}{2g}\right) r^2$$



# Example: Rigid body rotation of a liquid (2)

The area under the parabaloid is exactly half the base times the height so the original height of the liquid is half way between the minimum and maximum fluid heights

$$h_c = 2H_i - H \qquad \leftarrow \qquad H_i = \frac{H + h_c}{2}$$

Substituting into the equation for angular speed

$$\frac{1}{2}\rho\Omega^2 R^2 = \rho g(H - h_c) \quad \rightarrow \quad \Omega^2 = \frac{2g(H - h_c)}{R^2} \quad \rightarrow \quad \Omega^2 = \frac{2g(H - (2H_i - H))}{R^2}$$

$$\Omega = \sqrt{\frac{4g(H - H_i)}{R^2}}$$

The equation for the free surface therefore becomes

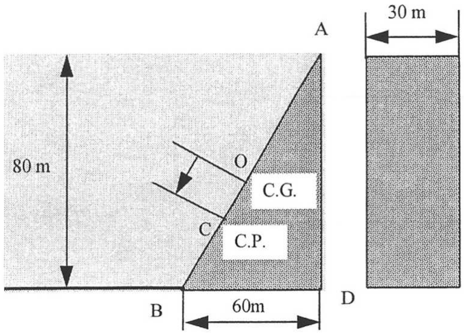
$$\frac{1}{2}\rho\Omega^2 r^2 = \rho g(z - h_c) \quad \rightarrow \quad z = \frac{\Omega^2 r^2}{2g} + h_c \quad \rightarrow \quad z = \frac{4g(H - H_i)}{2gR^2} r^2 + (2H_i - H)$$

$$z = \left[2\tfrac{(H - H_i)}{R^2}\right] r^2 + (2H_i - H)$$

And the pressure at the bottom corner

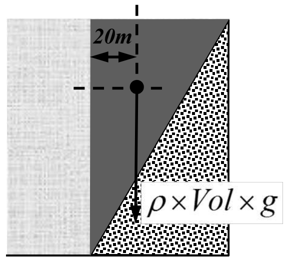
$$p = p_a + 2\rho g(H - H_i) + \rho g(2H_i - H) = p_a + \rho g H$$

# Example: Hydrostatic Thrust on Submerged Surface



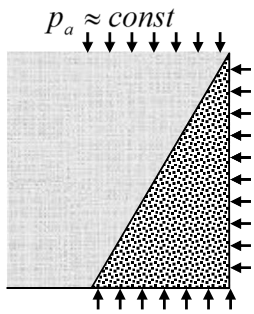

Find the magnitude and point of action of the hydrostatic force exerted on the dam.

Let us consider the vertical and horizontal components of the force ($F_v$ & $F_H$) separately (we will also look at the problem for a general submerged surface as we go)




Vertical Force: Can be calculated as the integral of the vertical component of pressure force over the area. Gives a simple relation for planar surfaces but complex otherwise.
As the fluid is static so the forces balance. Hence, $F_v$ equals the **effective weight of fluid above**.

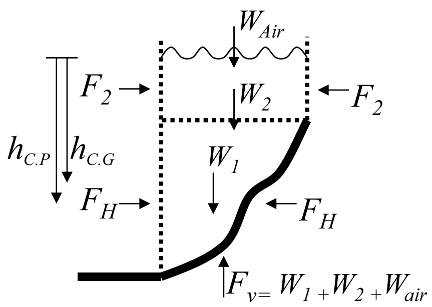
For the dam in our case $F_v = 1000 \times \frac{60}{2} \times 80 \times 30 \times 9.81 = 7.0632 \times 10^8 N$



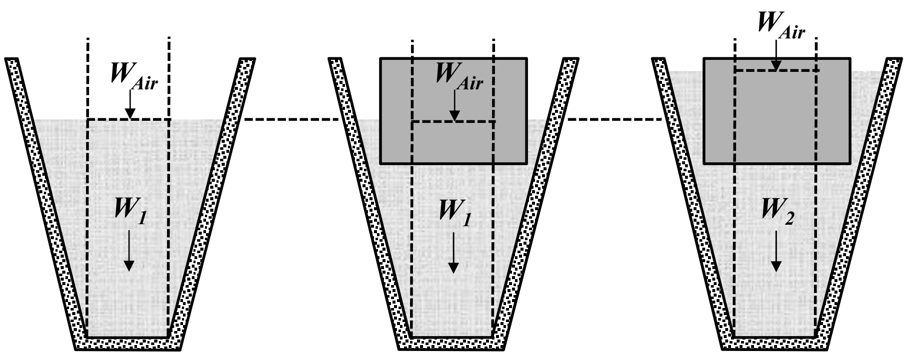

Consider atmospheric pressure to act all around the outer boundary and that it is approximately constant over the height.



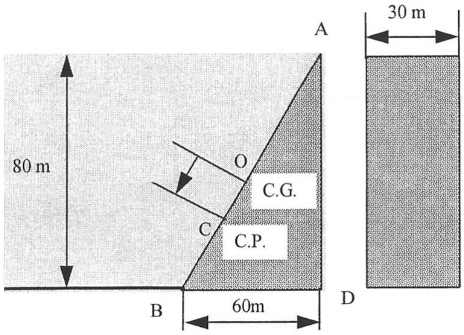
# Example: Hydrostatic Thrust on Submerged Surface (2)
# Aside:Vertical force on general submerged surfaces




Note the definition “effective weight of fluid above”.
Is there any difference in vertical force at the bottom of a water tank for the cases below?



## Example: Hydrostatic Thrust on Submerged Surface (3)




The horizontal force on the submerged surface $F_H$ is the same as the force on a plane formed by projecting the surface horizontally. Atmospheric pressure is ignored as it acts on either side of the dam.

$$F_H = \iint_A p dA = \rho g \iint_A h dA = \rho g h_{C.G} A$$

The horizontal force on a submerged surface is therefore equal to the **pressure at the centre of gravity of the projected shape** times the **area of that projected shape**. Hence for the Dam, the projected shape is a rectangle of 80m by 30m and so

$$F_H = 1000 \times 9.81 \times 40 \times (80 \times 30) = 9.4176 \times 10^8 N$$

The total force $F$ is then given by

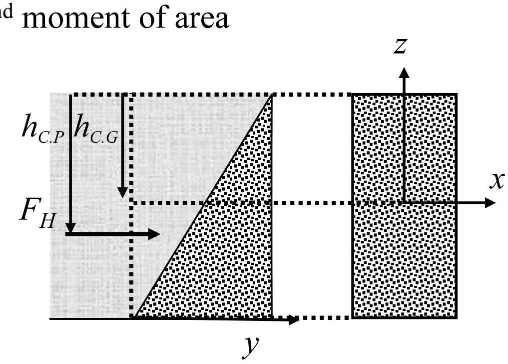
$$F = \sqrt{(F_H)^2 + (F_v)^2} = 1.1772 \times 10^9 N$$



## Example: Hydrostatic Thrust on Submerged Surface (4)

However the horizontal force acts through the centre of pressure (C.P.) and not the centre of gravity. The centre of pressure is found by considering moments about the centre of gravity of the projected area:

$$F_H (h_{C.G} - h_{C.P}) = \iint_A z p dA = -\rho g I_{xx}$$ Where $I_{xx}$ is the 2nd moment of area




using our previous formulae for $F_H$ and rearranging

$$h_{C.P} - h_{C.G} = \frac{I_{xx}}{h_{C.G} A}$$

For a rectangle of height $H$ and base $b$, $I_{xx} = bH^3/12$

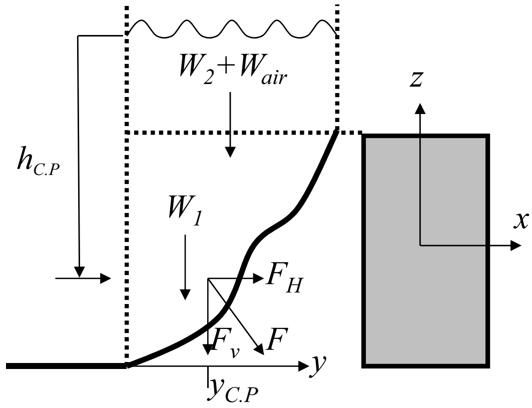
Hence the line of action for the horizontal force is $h_{C.P} - h_{C.G} = \dfrac{30 \times 80^3}{12 \times 40 \times 30 \times 80} = 13.333 m$

So the horizontal force acts a depth of 53.333m (as does the total force)

For the current dam case, or any plane surface, this is all we need to know. However for the general case we need a little more work..

## Example: Hydrostatic Thrust on Submerged Surface (5)
## Aside:Horizontal force on general submerged surfaces

The line of action of the vertical force ($y_{CP}$) is also found as the centre of mass of the column of fluid above the surface.




The true point of action of the total thrust F is found as the intersection of the vector F (with origin defined by $h_{CP}$ and $y_{CP}$) and the surface.

Note that in the planar case (*ie* the dam problem just completed) the origin defined by $h_{CP}$ and $y_{CP}$ lies on the surface.



## Learning Outcomes:
## "What you should have learnt so far"

- The hydrostatic equation and the constant density hydrostatic equation
- How these relations apply to simple pressure measurement devices and the definition of gauge pressure.
- The definition of standard pressure units
- Buoyancy problems
- How to apply the constant density hydrostatic equation to solve relative steady problems with constant accelerations.
- How to solve thrust problems on simple submerged surfaces