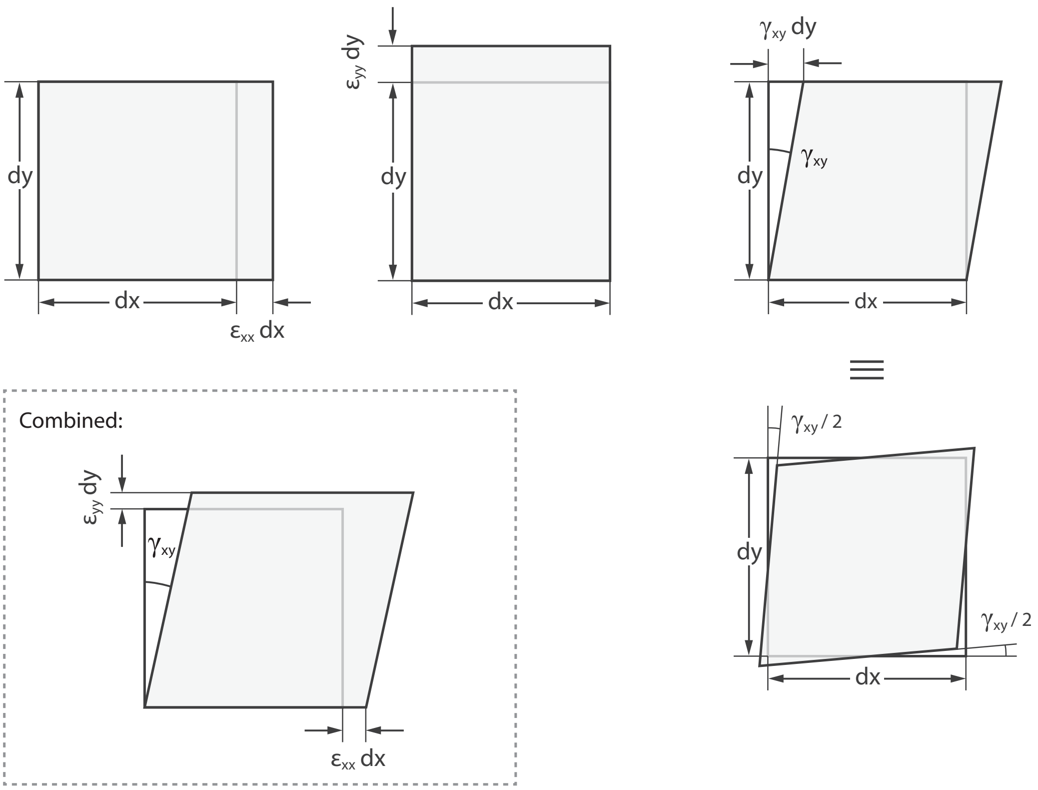
Handout 2 – 2D Strain and Strain Measurements

This handout introduces the properties of two-dimensional strain. Deriving the strain transformation equations leads to a Mohr's circle of strain. Experimental strain measurement techniques will be described, focusing on the use of strain gauges.

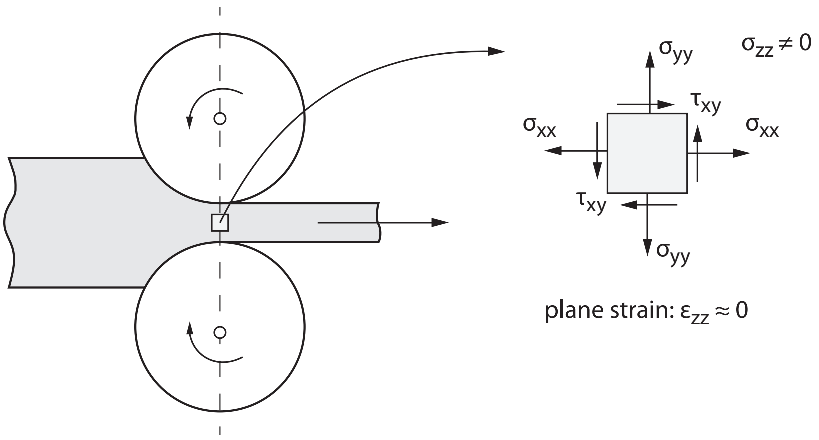
2.1 Plane Strain and Plane Stress

Two-dimensional deformations are composed of direct strains ε_{xx}, ε_{yy} and a shear strain γ_{xy}. The shear strain represents the change in angle between two orthogonal directions; note that a shear deformation is volume preserving. If the only deformations are those in the XY plane, $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$, the element is considered to be in *plane strain*.



The similarity between the definitions of *plane strain* and *plane stress* (where σ_{zz}, τ_{xz} and τ_{yz} are zero) invites the misunderstanding these can occur simultaneously. In fact, for an element in plane stress the through-thickness strain ε_{zz} will in general be non-zero due to Poisson's ratio effects, and for an element in plane strain the out-of-plane stress will generally be non-zero to enforce the plane strain condition.

These two assumptions are used for very different applications: while plane stress is used for thin-walled structures, plane strain is used to describe the stress deep inside an elastic body (where z-dimensions are much greater than x and y). One application of plane strain would be in forming processes, such as rolling, drawing and forging, where flow in a particular direction is constrained by the geometry of the machinery.



Despite the fundamental differences in applications, the stress and strain transformations for plane stress and plane strain are identical, as the out-of-plane direct stresses and strains do not affect the in-plane transformation equations.

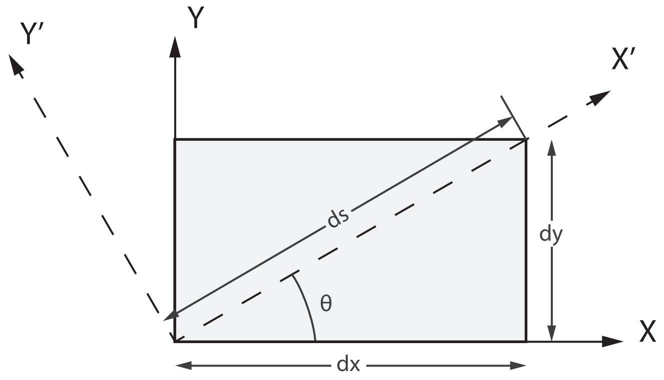
Strain Tensor As with stresses, we can define a strain tensor $\bar{\varepsilon}$:

$$\bar{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

Here we use the *mathematical* shear strain $\varepsilon_{xy} = \gamma_{xy}/2$ rather than the *engineering* shear strain γ_{xy}. This allows us to exploit the parallels with the Cauchy stress tensor.

2.2 Strain Transformation – Engineering Approach

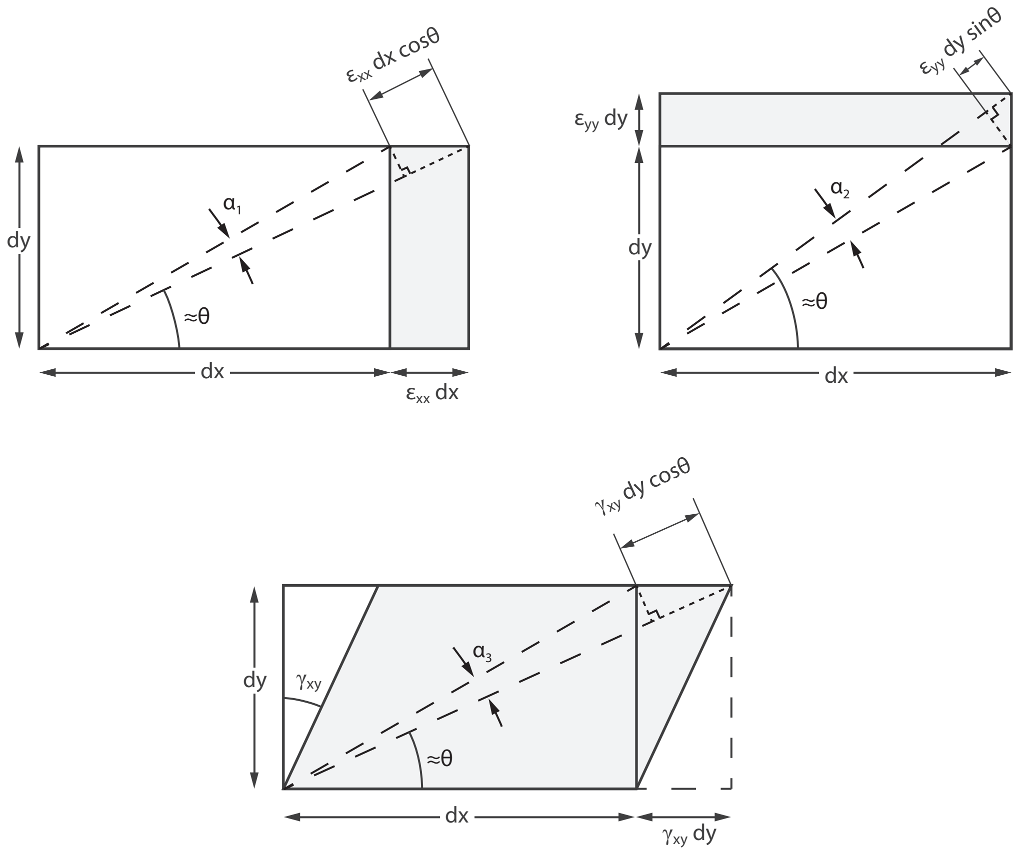
As with the stress calculations, we are interested in strains in directions other than the XY axes. Whereas the stress transformation equations followed from *equilibrium* considerations, for strain we shall derive these from *compatibility* of the deformed configuration. Let us consider an infinitesimal element with dimensions $dx \times dy$, with its diagonal conveniently aligned with the X' axis.



By superimposing three plane strain deformations (ε_{xx}, ε_{yy} and γ_{xy}) we can formulate expressions for the direct strain $\varepsilon_{x'x'}$ (*i.e.* change in length of the diagonal) and shear strain $\gamma_{x'y'}$ (*i.e.* change in orientation of diagonal).

2.2.1 Strain Transformation: Direct Strain

First we consider the direct strain $\varepsilon_{x'x'}$ at an angle θ to the XY axes; this is found by calculating the change in length of the diagonal of our infinitesimal element. Throughout the analysis we assume that the change in orientation of the diagonal, α_i, is very small and therefore $\theta + \alpha_i \approx \theta$.



The total increase in length of the diagonal is given as:

$$\Delta d = \varepsilon_{xx} dx \cos\theta + \varepsilon_{yy} dy \sin\theta + \gamma_{xy} dy \cos\theta$$

and the direct strain of the diagonal is therefore:

$$\begin{aligned} \varepsilon_{x'x'} = \frac{\Delta d}{ds} &= \varepsilon_{xx} \frac{dx}{ds} \cos\theta + \varepsilon_{yy} \frac{dy}{ds} \sin\theta + \gamma_{xy} \frac{dy}{ds} \cos\theta \\ &= \varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \end{aligned} \tag{2.1}$$

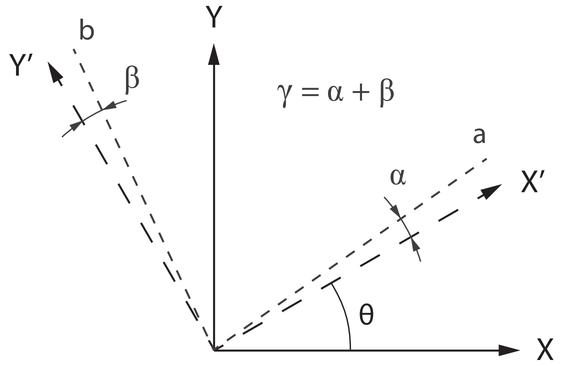
where $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$ (see undeformed geometry).

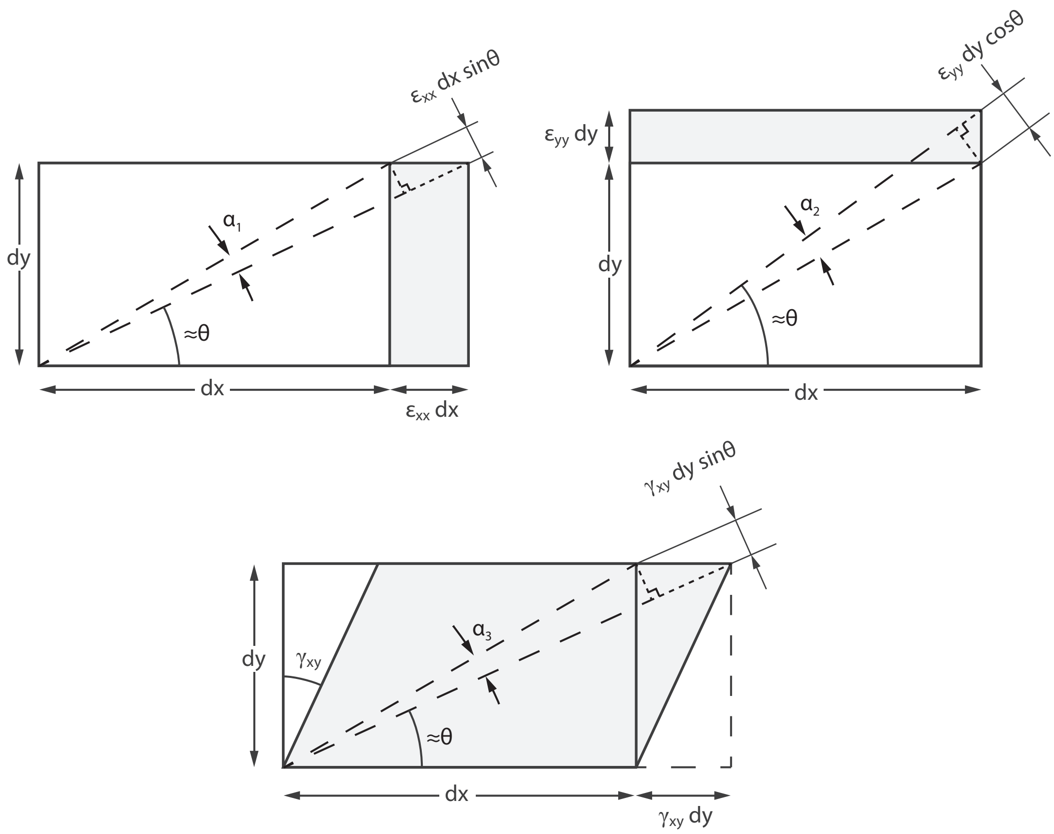
For the strain in Y' direction, we subsitute $\theta_{Y'} = \theta + \pi/2$ to find:

$$\varepsilon_{y'y'} = \varepsilon_{xx} \sin^2\theta + \varepsilon_{yy} \cos^2\theta - \gamma_{xy} \sin\theta\cos\theta \tag{2.2}$$

2.2.2 Strain Transformation: Shear Strain

Next, we consider the change in orientation of the diagonal of the infinitesimal element. The shear strain $\gamma_{x'y'}$ is equal to the change in angle between lines that were initially along the X' and Y' axes.





In our analysis we assume that the changes in angle α_i are small, and therefore $\tan\alpha_i \simeq \alpha_i$. This gives the total change in angle α as:

$$
\begin{aligned}
\alpha &= -\alpha_1 + \alpha_2 - \alpha_3 \\
&= -\frac{\varepsilon_{xx} dx \sin\theta}{ds} + \frac{\varepsilon_{yy} dy \cos\theta}{ds} - \frac{\gamma_{xy} dy \sin\theta}{ds} \\
&= -\varepsilon_{xx} \cos\theta \sin\theta + \varepsilon_{yy} \sin\theta \cos\theta - \gamma_{xy} \sin^2\theta
\end{aligned}
$$

again using $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$ from the undeformed geometry. The rotation β of the Y' axis is then found by substituting $\theta + \pi/2$, and correcting for the counter-clockwise rotation (β is taken to be positive in clockwise direction) to give:

$$\beta = -\varepsilon_{xx}\cos\theta\sin\theta + \varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}\cos^2\theta$$

and with $\gamma_{x'y'} = \alpha + \beta$ we find:

$$\gamma_{x'y'} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \tag{2.3}$$

2.2.3 Combined Strain Transformation

We can now combine our strain transformation equations into a single transformation matrix $\boldsymbol{T}$:

$$\begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \gamma_{x'y'}/2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy}/2 \end{bmatrix} \tag{2.4}$$

where we have used $\gamma_{xy}/2$. Note that this is the same[4] transformation matrix as derived for plane stress even though they have been derived using completely different methods, *i.e.* geometry and equilibrium. It is also worth emphasising that these transformation equations are independent of the material properties!

2.2.4 Mohr's Circle of Strain

The analogy between the transformation matrices for stress and strain naturally leads to corresponding results, and we can find principal strains, maximum shear strains, and formulate a Mohr's circle for strain.

Principal directions:

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{2.5}$$

Principal strains:

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{2.6}$$

Maximum shear strains:

$$\gamma_{\text{max,min}} = \pm\sqrt{\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2 + \gamma_{xy}^2} = \varepsilon_1 - \varepsilon_2 \tag{2.7}$$

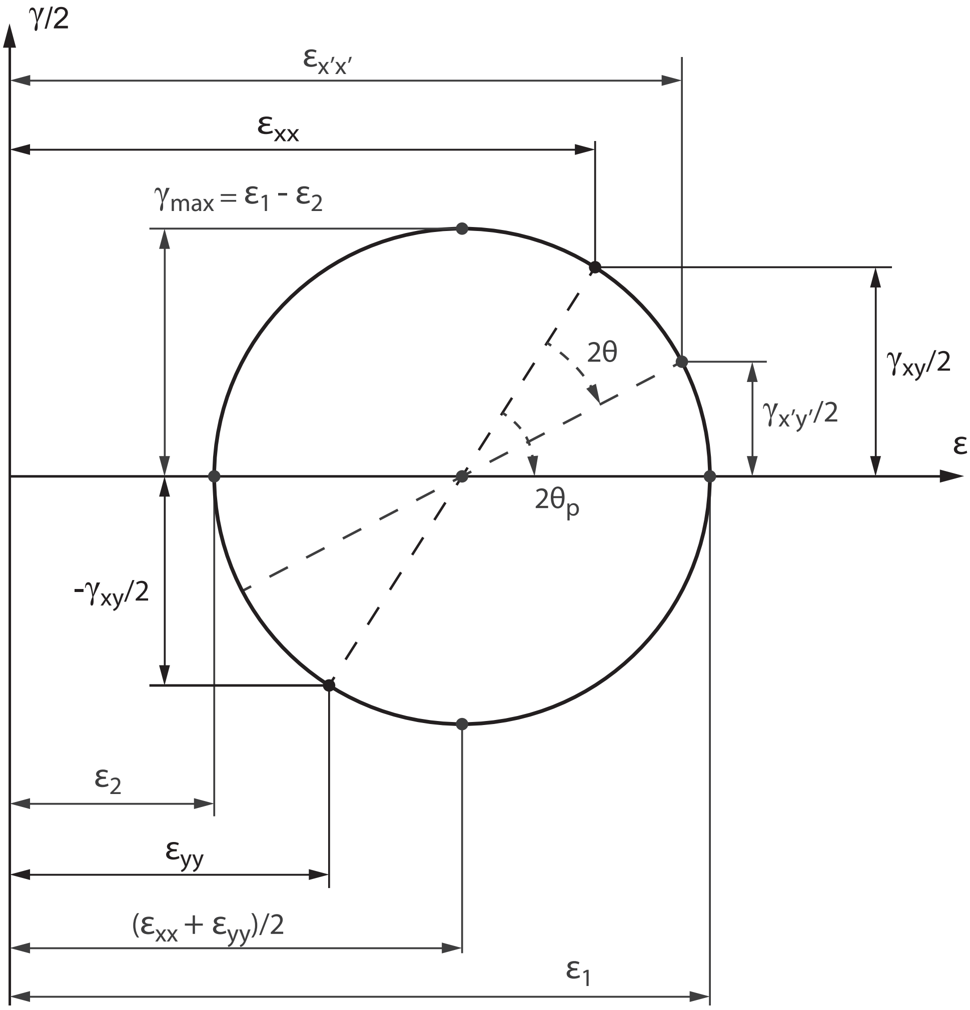
The above equations were derived by substituting ε_{xx}, ε_{yy} and $\gamma_{xy}/2$ for σ_{xx}, σ_{yy} and τ_{xy}, respectively, in Equations 1.4, 1.5 and 1.7.

[4] Looking back to your notes on beam deflections, you will find that this is the same coordinate transformation equation as for the second moments of area I_{xx}, I_{yy} and I_{xy}:

$$\begin{bmatrix} I_{x'x'} \\ I_{y'y'} \\ -I_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} I_{xx} \\ I_{yy} \\ -I_{xy} \end{bmatrix}$$

It turns out that stress, strain and second moment of area are all examples of second-rank tensors, and therefore share the same properties such as principal directions. Subtleties in sign conventions exist, so take care when revising from other sources, such as textbooks.

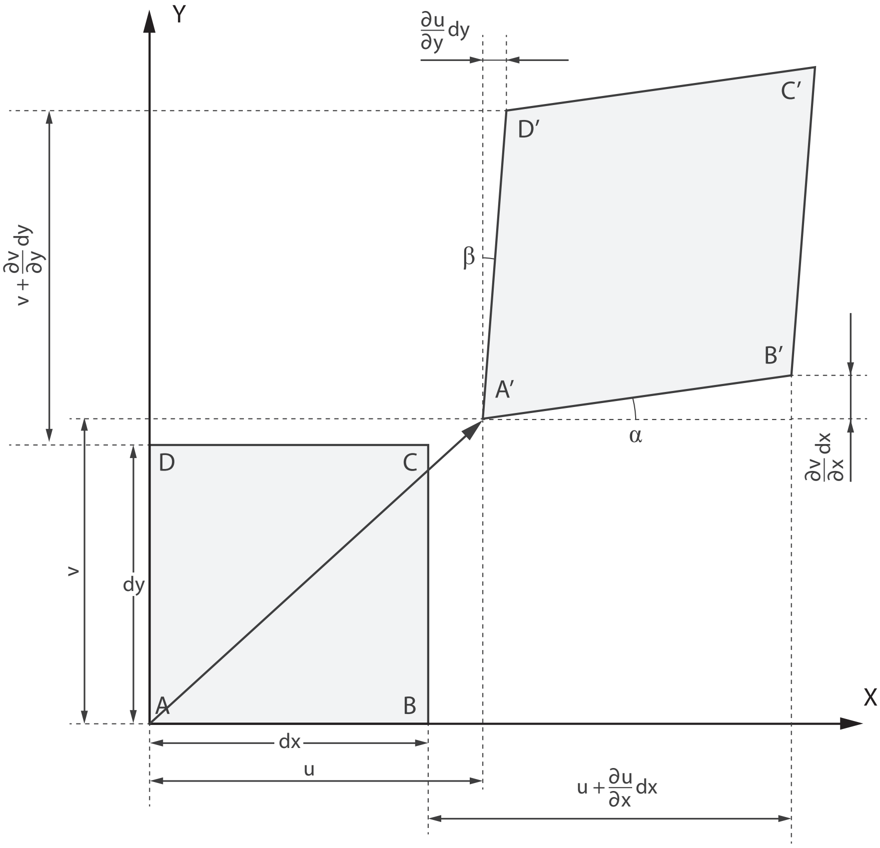
Mohr's circle for strain:



Please note we have maintained the sign convention we used for Mohr's circle for stress, *i.e.* a CCW rotation of the coordinate system is represented by a CW rotation on Mohr's circle. Some textbooks use a convention where $\gamma_{xy}/2$ is plotted positive downward, to ensure all rotations are CCW.

2.3 Strain Transformation – Mathematical Approach

An alternative method to deriving the strain transformation equations, is to approach strain from a mathematical point of view. Let $u(x, y)$ and $v(x, y)$ describe the displacement field of points (x, y) in the deformed body. The deformation of an infinitesimal element ABCD can then be described in terms of u and v.



The displaced position of corner A is given by (u, v), and the position of the other points can then be described using a Taylor expansion. For example, for point B:

$$u_B = u + \frac{\partial u}{\partial x}dx + \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(dx)^2 + \dots$$

Taking only the linear terms, the direct strains are then found as:

$$\varepsilon_{xx} = \frac{\left(u + \frac{\partial u}{\partial x}dx\right) - u}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\left(v + \frac{\partial v}{\partial y}dy\right) - v}{dy} = \frac{\partial v}{\partial y}$$

The shear strain $\gamma_{xy} = \alpha + \beta$, which can be found as:

$$\alpha = \frac{\partial v}{\partial x} \qquad \beta = \frac{\partial u}{\partial y} \qquad \therefore \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

where use is made of small-strain approximations ($\tan\alpha \approx \alpha$, $\partial u/\partial x \ll 1$, and $\partial v/\partial y \ll 1$).

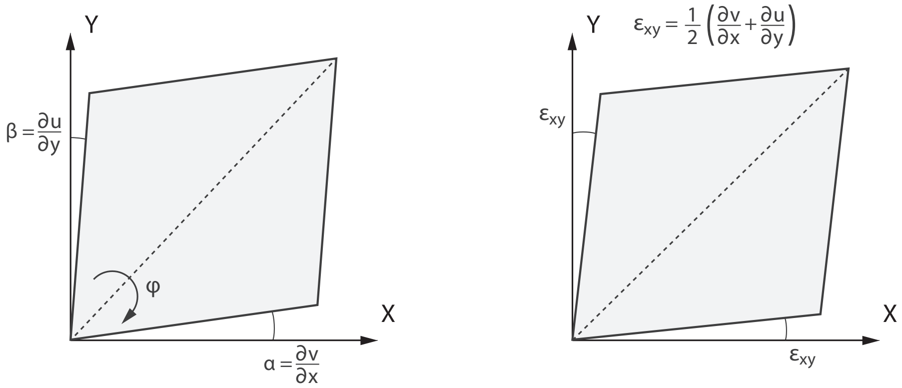
The three strain components are now given as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{2.8}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{2.9}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2.10}$$

Note that the infinitesimal element undergoes a small rigid-body rotation, φ, in addition to a (pure) shear deformation, ε_{xy}.



The mathematical shear strain is

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

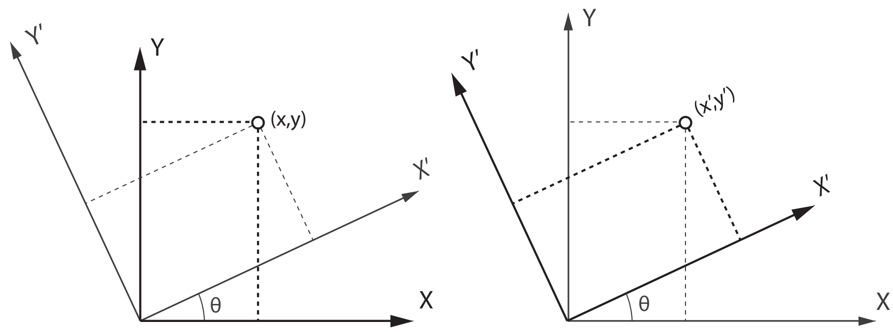
and the rigid-body rotation φ is therefore:

$$\varphi = \alpha - \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

2.3.1 Coordinate Transformation *(not examinable)*

Now consider the coordinate transformation from XY to $X'Y'$ in the form of rotation matrices, to find

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$



Substituting into mathematical formulations for strain in transformed coordinates $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$ and $\gamma_{x'y'}$ gives:

$$\begin{aligned}
\varepsilon_{x'x'} = \frac{\partial u'}{\partial x'} &= \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} \\
&= \left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial v}{\partial x}\right)\cos\theta + \left(\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial v}{\partial y}\right)\sin\theta \\
&= \cos^2\theta\frac{\partial u}{\partial x} + \sin^2\theta\frac{\partial v}{\partial y} + 2\sin\theta\cos\theta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + 2\gamma_{xy}\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} &= \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} \\
&= \left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial v}{\partial x}\right)\sin\theta + \left(-\sin\theta\frac{\partial u}{\partial y} + \cos\theta\frac{\partial v}{\partial y}\right)\cos\theta \\
&= \sin^2\theta\frac{\partial u}{\partial x} + \cos^2\theta\frac{\partial v}{\partial y} - 2\sin\theta\cos\theta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - 2\gamma_{xy}\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
\gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} &= \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} \\
&= \left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial v}{\partial x}\right)\cos\theta + \left(-\sin\theta\frac{\partial u}{\partial y} + \cos\theta\frac{\partial v}{\partial y}\right)\sin\theta \ldots \\
&\qquad \ldots - \left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial v}{\partial x}\right)\sin\theta + \left(\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial v}{\partial y}\right)\cos\theta \\
&= -2\sin\theta\cos\theta\frac{\partial u}{\partial x} + 2\sin\theta\cos\theta\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\cos^2\theta - \sin^2\theta\right) \\
&= -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right)
\end{aligned}$$

which is identical to the strain transformation equations derived previously. While the engineering approach provides a more intuitive understanding for plane strain, the mathematical approach can straightforwardly be extended to three-dimensional states of strain.

NB: this derivation is *not* examinable!

2.4 Stress/Strain Measurements

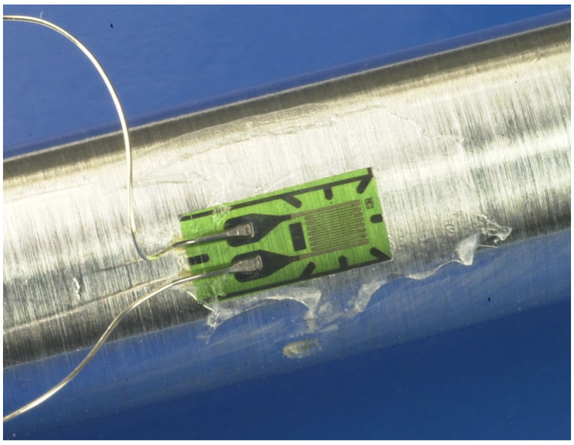
Many classic problems in structural mechanics have been solved analytically, and the validity of those solutions has been painstakingly verified using experiments. Modern structural analysis often makes use of Finite Element Analysis (FEA) to calculate the stress and strain, which may give engineers a false sense of certainty regarding the correctness of the results. In cases where material properties are not well established, or where the loading conditions are not precisely known *a priori* (*e.g.* fluid structure interaction) methods are still needed to measure the stress and strain experimentally.

A state of plane stress is determined by measuring σ_{xx}, σ_{yy} and τ_{xy} at a point. Measuring those stresses directly, however, turns out to be rather challenging. Instead, often the strains on the surface of the body are measured, allowing us to then reconstruct the state of stress. Keep in mind that this relies on knowing the material model that relates stress and strain — the topic of the next handout!

In *full field* measurements, the strains are measured across the surface of the structure simultaneously. This provides valuable insight into the global stress distributions, before focusing in on highly stressed areas. Such techniques include **photoelasticity** and **digital image correlation (DIC)**. More commonly, however, strains are measured at discrete points on a surface, using **strain gauges**. These provide accurate results and do not require intensive post-processing (such as is the case for DIC).

2.4.1 Strain Gauges

A strain gauge consists of a long thin wire, looping back and forth, which is attached to the surface of the structure being measured.

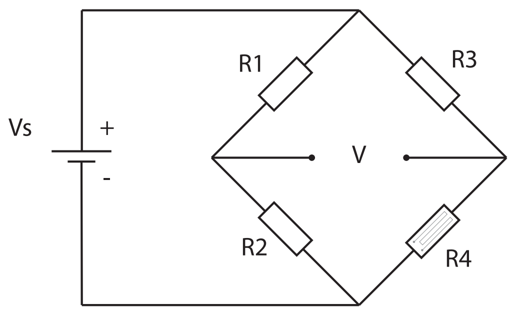


The electrical resistance R of the wire varies with change of length, *i.e.* strain; the change in resistance is further amplified by the change in cross-section due to its Poisson's ratio. As the strain gauge is bonded to the surface, it will experience the same strain as the structure. Note the strain gauge can only measure strains in the direction it is oriented, as transverse and shear strains will not change the length of the wire.

The change in resistance ΔR of the strain gauge is amalgamated into a Gauge Factor κ, which depends on the wire material used.

$$\frac{\Delta R}{R} = \kappa\,\varepsilon$$

Using strain gauges, the direct stress in a specific direction can be determined by measuring its change in resistance using a Wheatstone bridge.



The measured voltage V is given as:

$$V/V_s = \frac{R_4}{R_3 + R_4} - \frac{R_2}{R_1 + R_2}$$

When $R_1/R_2 = R_3/R_4$ the Wheatstone bridge is balanced, and $V = 0$. For the Quarter-Bridge configuration shown, where $R_4 = R + \Delta R$ is the strain gauge, and assuming $R_1 = R_2$ and $R_3 = R$, the measured voltage as a function of the strain is:

$$\frac{V}{V_s} = \frac{\Delta R}{4R + 2\Delta R} \approx \frac{\kappa}{4}\varepsilon$$

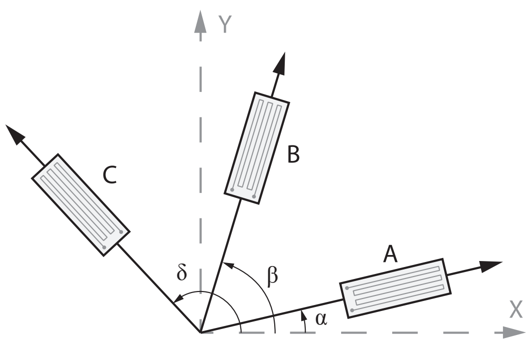
The measured strains will be in the order of micro-strains ($\mu\varepsilon = 1 \cdot 10^{-6}$), and the signal will therefore need to be further amplified. There is an art and skill to using strain gauges, for example, to compensate for temperature variations — this is beyond the scope of this course, but will be covered again in 3rd year Signals, Sensors and Controls.

NB: the details of the Wheatstone bridge are non-examinable.

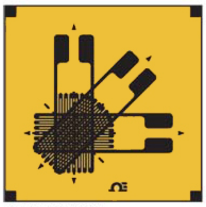
Strain Gauge Rosettes To fully reconstruct a state of stress, three independent strain measurements in three different directions are required. Calculating the strains in a desired coordinate system can then be achieved using the strain transformation equations:

$$\begin{aligned}
\varepsilon_A &= \varepsilon_{xx}\cos^2\alpha + \varepsilon_{yy}\sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha \\
\varepsilon_B &= \varepsilon_{xx}\cos^2\beta + \varepsilon_{yy}\sin^2\beta + \gamma_{xy}\sin\beta\cos\beta \\
\varepsilon_C &= \varepsilon_{xx}\cos^2\delta + \varepsilon_{yy}\sin^2\delta + \gamma_{xy}\sin\delta\cos\delta
\end{aligned} \tag{2.11}$$

where α, β and δ are the angles of the three strain gauges with respect to the axes of interest. Solving the three equations simultaneously will give the strains ε_{xx}, ε_{yy} and γ_{xy}, which then enable the stresses and principal stresses and strains to be found.



A variety of standard strain gauge configurations is available, with strain gauges oriented at different angles, and are often referred to as *rosettes*.



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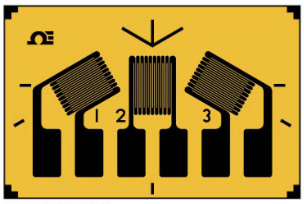

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Tee : 0/90° rosette Only applicable if the principal strain directions are known *a priori* through other considerations, such as cylindrical pressure vessels or shafts in torsion. See for example Example 3.3.

Rectangular: 45° rosette A common strain gauge configuration is the 45°rosette, where $\alpha = 0$, $\beta = 45°$ and $\delta = 90°$ and therefore:

$$
\begin{aligned}
\varepsilon_0 &= \varepsilon_{xx} \\
\varepsilon_{90} &= \varepsilon_{yy} \\
\varepsilon_{45} &= \frac{\varepsilon_0 + \varepsilon_{90} + \gamma_{xy}}{2}
\end{aligned}
$$

and thus

$$\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_0 - \varepsilon_{90}$$

After some re-arranging, it may be shown that the principal stresses are given by:

$$\sigma_{1,2} = \frac{E}{2}\left[\frac{\varepsilon_0 + \varepsilon_{90}}{1-\nu} \pm \frac{1}{1+\nu}\sqrt{2\left(\varepsilon_0 - \varepsilon_{45}\right)^2 + 2\left(\varepsilon_{45} - \varepsilon_{90}\right)^2}\right]$$

Delta : 60° Another common configuration is the Delta rosette, where the strain gauges are set at 60°angles, and the measured strains are therefore ε_0, ε_{60} and ε_{120}. If we align the x-axis with the 0°direction, we obtain:

$$
\begin{aligned}
\varepsilon_0 &= \varepsilon_{xx} \\
\varepsilon_{60} &= \frac{\varepsilon_{xx} + 3\varepsilon_{yy} + \sqrt{3}\gamma_{xy}}{4} \\
\varepsilon_{120} &= \frac{\varepsilon_{xx} + 3\varepsilon_{yy} - \sqrt{3}\gamma_{xy}}{4}
\end{aligned}
$$

Adding and subtracting ε_{60} and ε_{120} respectively gives the following relationships:

$$
\begin{aligned}
\varepsilon_{yy} &= \frac{-\varepsilon_0 + 2\varepsilon_{60} + 2\varepsilon_{120}}{3} \\
\gamma_{xy} &= \frac{2\varepsilon_{60} - 2\varepsilon_{120}}{\sqrt{3}}
\end{aligned}
$$

Finding an expression for the principal stresses of the Delta rosette is then a matter of some mildly tedious algebra, and ultimately yields:

$$\sigma_{1,2} = \frac{1}{3}\left[\frac{E}{1-\nu}\left(\varepsilon_0 + \varepsilon_{60} + \varepsilon_{120}\right) \pm \frac{E}{1+\nu}\sqrt{2\left(\varepsilon_0 - \varepsilon_{60}\right)^2 + 2\left(\varepsilon_{60} - \varepsilon_{120}\right)^2 + 2\left(\varepsilon_{120} - \varepsilon_0\right)^2}\right]$$

NB: do not memorise equations for specific rosettes, but use Equation 2.11

Example 2.1 – Strain Gauge Measurements

Q: A 45° rosette is used to measure the strains on the surface of a thin-walled aircraft fuselage. Derive the shear strain γ_{xy} from the strain gauge measurements.

A: As strain gauges A and C are aligned with the XY axes, the direct strains are known directly:

$$\varepsilon_{xx} = \varepsilon_A$$

$$\varepsilon_{yy} = \varepsilon_C$$

For the shear strain, use the strain transformation equation:

$$\varepsilon_B = \frac{1}{2}\left[(\varepsilon_{xx} + \varepsilon_{yy}) + (\varepsilon_{xx} - \varepsilon_{yy})\cos 2\beta + \gamma_{xy}\sin 2\beta\right]$$

where for $\beta = \pi/4$ we find:

$$\varepsilon_B = \frac{1}{2}\left[(\varepsilon_A + \varepsilon_C) + \gamma_{xy}\right]$$

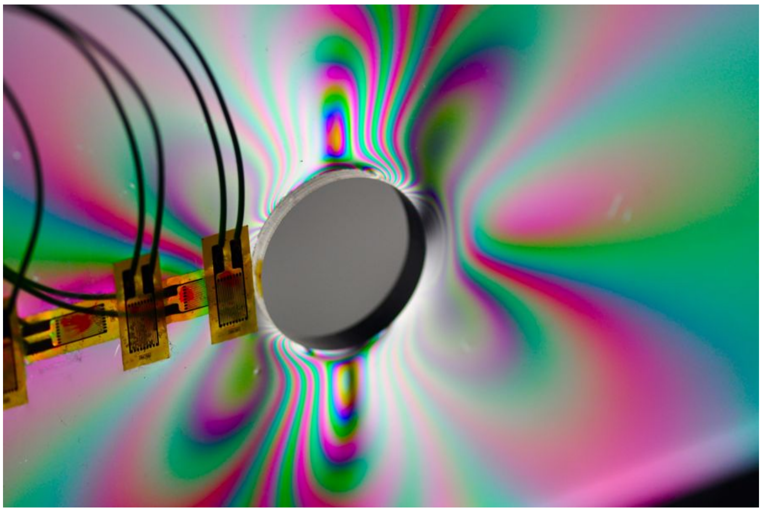
which provides:

$$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C$$

Once ε_{xx}, ε_{yy} and γ_{xy} are derived, the standard equations for principal strains and directions can be employed to further analyse the strain state.

2.4.2 Photoelasticity

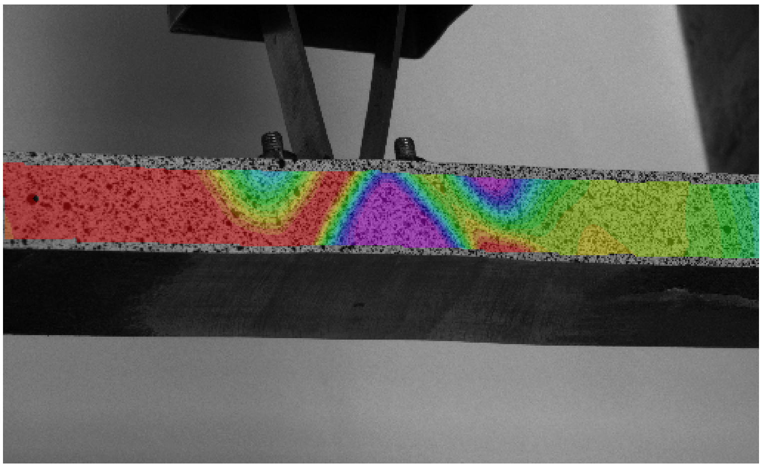
Photoelasticity is a classic full-field stress measurement technique. It relies on a photoelastic effect, where the material refractive index changes with stress. A photoelastic coating is applied to the structure, and the magnitude and direction of the stresses at any point can then be determined by examining the fringe patterns produced by a polarised light source.



This allows imaging of iso-static (where the *difference in principal stresses* is equal) and iso-clinic (where either of the principal directions coincides with the axis of polarisation) fringes. Although photo-elasticity gives a good insight into the stress distribution, it only visualises the *difference* in principal stresses, not the magnitude of the stresses.

2.4.3 Digital Image Correlation (DIC)

Another full-field measurement technique is Digital Image Correlation (DIC). A fine speckle pattern is applied to the structure, and the relative displacement of the speckles is tracked with a camera during the loading of the structure, enabling the strains to be calculated; see Section 2.3. Using stochastic cross-correlation methods, subpixel accuracy can be achieved.



The computationally intensive post-processing of high-resolution images precluded the practical use of DIC until relatively recently. What is more, the measurements need to be done under controlled circumstances (*e.g.* constant lighting conditions) so will not replace strain gauges in all applications. Nonetheless, its ability to perform contactless, full-field strain measurements is powerful!

2.5 Summary

In this handout we introduced the concept of plane strain and its relationship to plane stress. The strain transformation equations were derived from *compatibility* considerations, and were found to be identical to the stress transformation equations. This analogy allowed us to formulate concepts such as principal strains and directions, and establish a Mohr's circle for strain.

The strain transformation equations enabled us to reconstruct a state of stress from experimental strain measurements using strain gauges. Two alternative, full-field stress/strain measurement techniques, photo-elasticity and Digital Image Correlation, were also briefly introduced.

Revision Objectives Handout 2:

- explain the differences between plane stress and plane strain;
- derive strain transformation equations from first principles (geometrically only);
- appreciate and make use of parallels between strain and stress in terms of principal axes, transformation equations and Mohr's circle;
- calculate principal strains and directions, maximum/minimum shear strains;
- express strains in terms of a displacement field u, v;
- transform strains from experimental strain gauge readings (given transformation equations);
- describe different experimental stress/strain measurement techniques;