Handout 2 – Statics: Friction, Work & Virtual Work

Meriam & Kraige, Statics: 6/2, 6/3, 6/8, 7/2 – 7/4

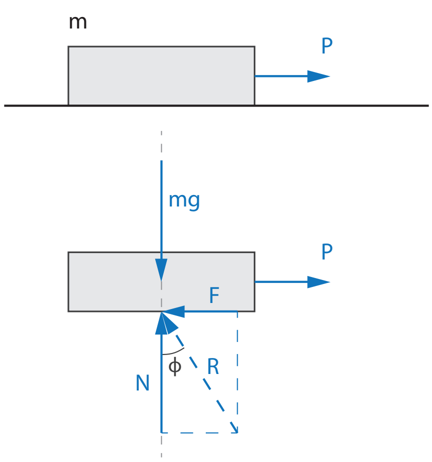
Friction is often considered undesirable, and engineers go to great lengths to minimize the friction in many mechanical systems. However, it is also a critical force in many applications: it is necessary to stand a ladder against a wall, enables a car to accelerate and brake, holds screws and nuts in place, *etc.*

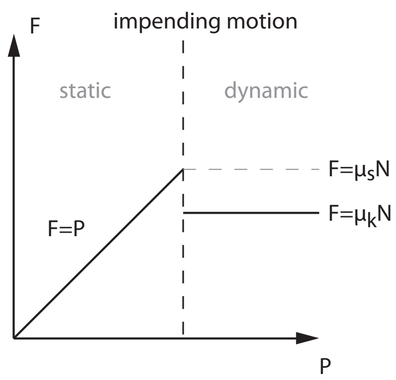
A distinction is made between *dry friction* and *fluid friction*. Dry friction, also known as Coulomb friction, is encountered between unlubricated solids. The friction force can be defined as the component of the contact force that is tangent to the contact plane. It resists the relative motion (or tendency towards motion) between the two bodies. Fluid friction develops between layers in a fluid and not only depends on the velocity gradient, but also viscosity of the fluid – it is therefore the subject of fluid mechanics.

The dry friction is a result of surface roughness at a microscopic level. Contact between such surfaces only takes place at a number of small, irregular asperities. Friction is the result of the interaction between the asperities of the mating surfaces, and is the load required to overcome adhesive bonding and produce the required inelastic deformations.

In most engineering applications, however, the modelling of friction is highly simplified and based on results from experimentation. This has shown that friction is: (i) proportional to the normal force, and (ii) independent of the contact area.

2.1 Static vs Dynamic Friction





The friction is found to change once an object starts to move. The **static** friction occurs up to the point of slippage, and the static friction force is given as:

$$F_s \leq \mu_s N \tag{2.1}$$

where μ_s is the coefficient of static friction. Note that the exact value of F_s follows from static equilibrium considerations.

Once slippage occurs, the **dynamic** (or kinetic) friction coefficient μ_k is used

$$F_k = \mu_k N \tag{2.2}$$

where, in general, $\mu_k < \mu_s$. In other words, once an object starts to slip, less force is required to keep it moving. The dynamic coefficient of friction may decrease slightly as the velocity increases, but is nominally considered to be constant.

Surface	μ_s	μ_k
steel on steel	0.74	0.57
glass on glass	0.94	0.40
metal on metal (lubricated)	0.15	0.06
ice on ice	0.10	0.03
teflon on teflon	0.04	0.04
tire on concrete	1.00	0.80
tire on wet road	0.60	0.40
tire on snow	0.30	0.20

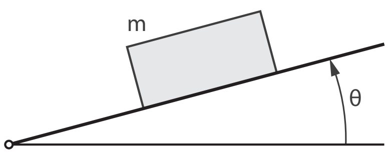
Friction Angle The friction can also be described by the friction angle ϕ

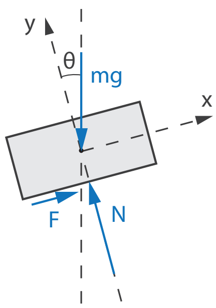
$$\tan\phi = \frac{F}{N} = \mu \tag{2.3}$$

which can be a helpful interpretation for certain problems, by placing a single resultant force R at an angle ϕ to the contact normal.

Example 2.1 – Block on Slope

A block with mass m is placed on a slope, whose angle θ can be changed. The coefficient of static friction between the slope and the block is μ_s. At what angle does the block start sliding down the slope?





The equilibrium parallel ($\|$) and perpendicular ($\perp$) to the plane are written as:

$$\begin{aligned} \sum F_x & \qquad F - mg\sin\theta = 0 \\ \sum F_y & \qquad N - mg\cos\theta = 0 \end{aligned}$$

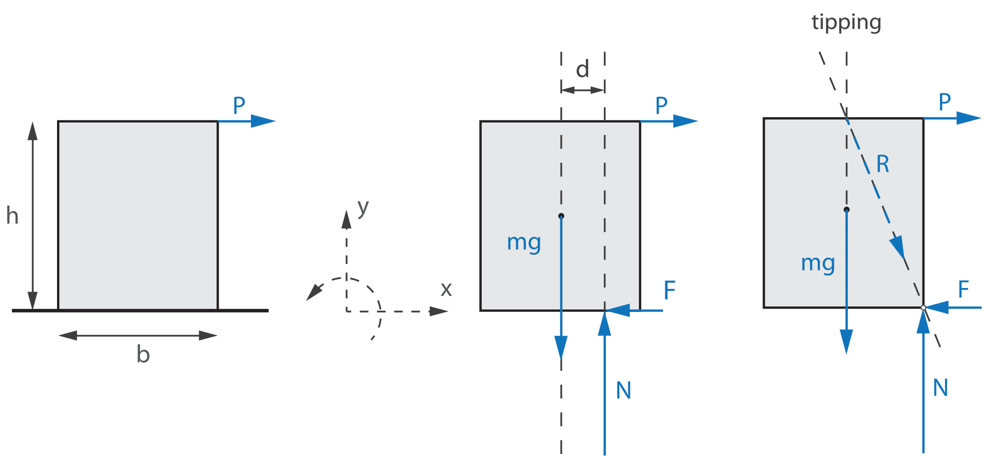
at the onset of sliding, we find

$$\frac{F}{N} = \tan\theta = \mu_s = \tan\phi_s$$

Therefore the block starts sliding when θ reaches the static friction angle ϕ_s. This set-up can thus be used to measure the coefficient of friction.

Example 2.2 – Tipping or Slipping

A block with height h, width b, and mass m, is pulled at its top edge with force P.



Under what conditions will the block slip or tip? Intuitively, how would you expect the ratio h/b and the friction coefficient to play a role?

From the FBD:

$$\sum F_x : \qquad P - F = 0$$

$$\sum F_y : \qquad N - mg = 0$$

$$\sum M_G : \qquad Nd - \frac{h}{2}(P + F) = 0$$

which gives:

$$F = P$$

$$N = mg$$

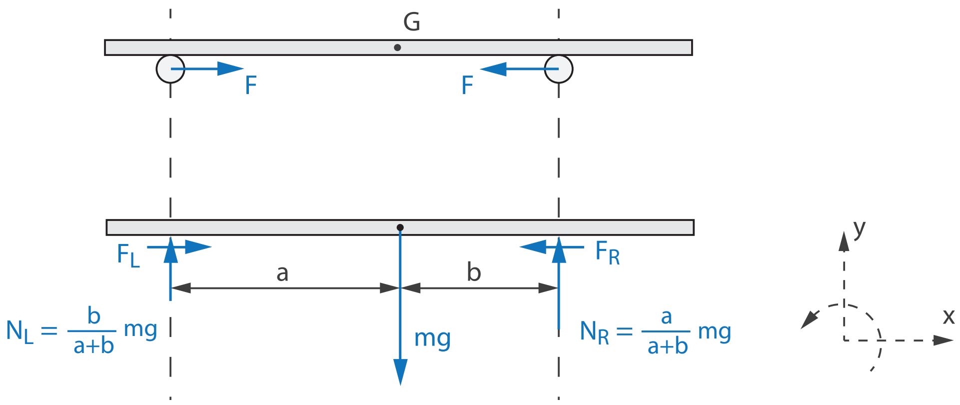
$$Nd = hP$$

As the force P is increased, it either exceeds the static friction force ($\frac{P}{N} > \mu$), and the block slips, or the distance d increases until the block tips around the corner point ($d = b/2$). This corresponds to the direction of the resultant force R (of P and mg) passing the corner ($\frac{P}{N} > \frac{b/2}{h}$).

Therefore, if $\frac{b/2}{h} < \mu$, and the block is small and slender, it will tip rather than slip under the applied load.

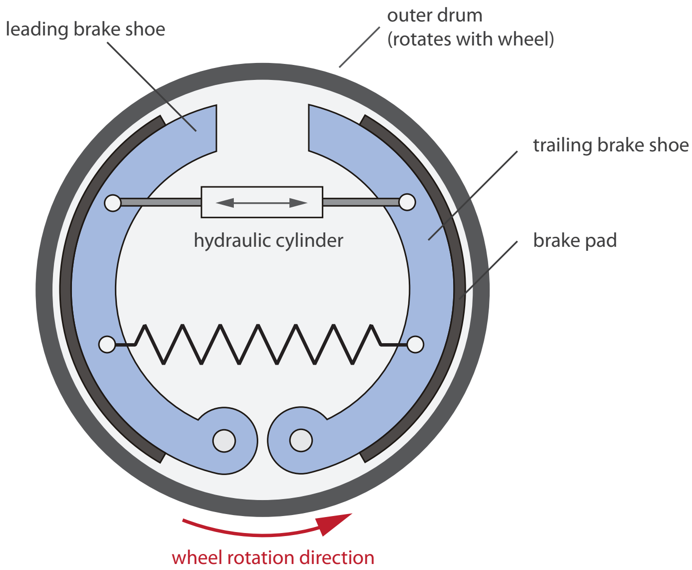
Example 2.3 – Balanced Ruler

A classic example to illustrate the difference in static and dynamic friction is that of the balanced ruler. A ruler is supported on two fingers, which are gradually (quasi-statically) moved towards each other. Can you explain what happens?



Example 2.4 – Drum Brakes

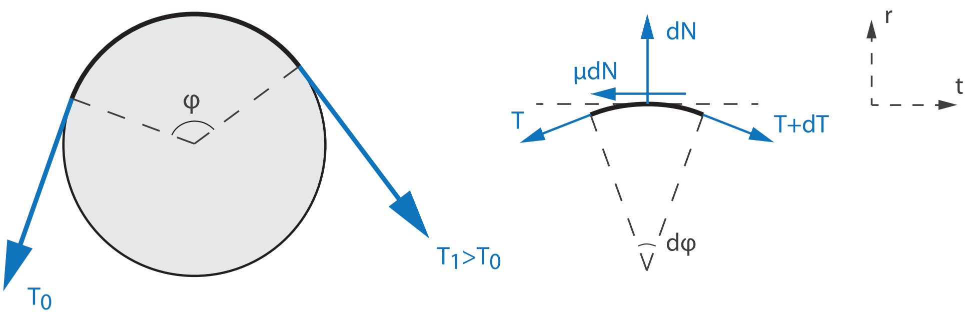
A drum brake is an elegant example of using friction to its best advantage. While it has generally been superseded by disc brakes in cars, it can stil be found in the hand brake, or in some bicycle wheels.



A hydraulic cylinder pushes out the shoes, which are pressed against the inside of the wheel hub and thereby brake the wheel. In the 'leading/trailing'configuration shown, the rotation of the drum will drag the leading shoe tighter onto the friction surface, increasing the braking torque wihout additional mechanical effort (*i.e.* it is self-locking). The flip side is that it makes it harder to modulate the sensitivity of the brakes, which may lock up completely. The trailing shoe is much less effective for the rotation direction shown, as it will detach from the drum as a result of the friction.

2.2 Capstan Equation

The friction force can also vary along the contact area. A classic problem is the capstan, where a rope is wrapped around a cylindrical post over an angle φ. The tension in both ends of the rope differs due to the effect of friction, μ, with $T_1 > T_0$.



Consider the equilibrium of an infinitesimal element of the rope, subtending angle $d\varphi$, in radial direction

$$\sum F_r : \qquad dN - T\sin\frac{d\varphi}{2} - (T+dT)\sin\frac{d\varphi}{2} = 0$$

and tangential direction

$$\sum F_t : \qquad (T+dT)\cos\frac{d\varphi}{2} - \mu dN - T\cos\frac{d\varphi}{2} = 0$$

With small-angle approximations and ignoring higher-order terms, we find:

$$T d\varphi = dN$$
$$dT = \mu dN$$

which are combined and integrated to give

$$\begin{aligned} \frac{dT}{T} &= \mu d\varphi \\ \int_{T_0}^{T_1} \frac{1}{T} dT &= \int_0^{\varphi} \mu d\varphi \\ \ln\left(\frac{T_1}{T_0}\right) &= \mu\varphi \\ \frac{T_1}{T_0} &= e^{\mu\varphi} \end{aligned} \tag{2.4}$$

The consequences of this equation are very powerful. If a rope with $\mu = 0.3$ is wrapped a full turn around the capstan ($\varphi = 2\pi$) the amplification between $T_1/T_0 \approx 6.5$! If the capstan is actively driven by a motor, only a small additional force would be required to haul large weights.

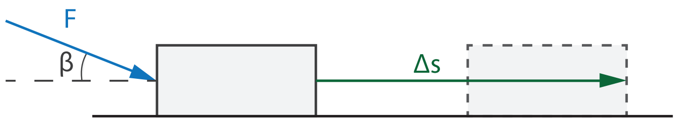
2.3 Work & Energy

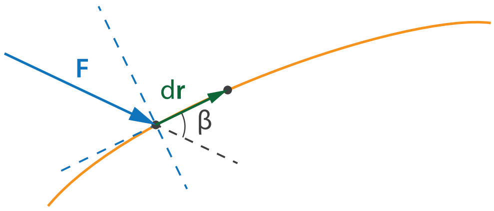
Up to now the examples of friction have generally shown benefits, for example, allowing objects to stay in place on a slope, or enabling cars to brake. In practise, however, friction is often undesirable, as **energy** is dissipated/lost as a result. In other words, **work** is required to overcome the friction.

The work U done by a force F is equal to the product of the force times the displacement Δs. Specifically, it is the product of the component of the force in the direction of the displacement:

$$U = F\cos\beta\Delta s$$

where β is the angle between the force and displacement vectors. Any component of the force perpendicular to the displacement does not contribute to the work.





For the case where both direction and magnitude of the force and displacement are variable, the work dU exerted by the force $\boldsymbol{F}$ over a displacement $d\boldsymbol{r}$ is equal to:

$$dU = \boldsymbol{F}\cdot d\boldsymbol{r} = F\cos\beta\, ds \tag{2.5}$$

Note that the dot product projects the force onto the displacement vector. The total work is then given as

$$U = \int_0^s F\cos\beta\, ds \tag{2.6}$$

Similarly, the work of a couple is equal to

$$U = \int_0^\theta M d\theta$$

where $d\theta$ is the infinitesimal angle of rotation.

Work is a *scalar* and can either be positive or negative. For instance, friction acts opposite to the direction of motion and will therefore exert negative work, and dissipate energy.

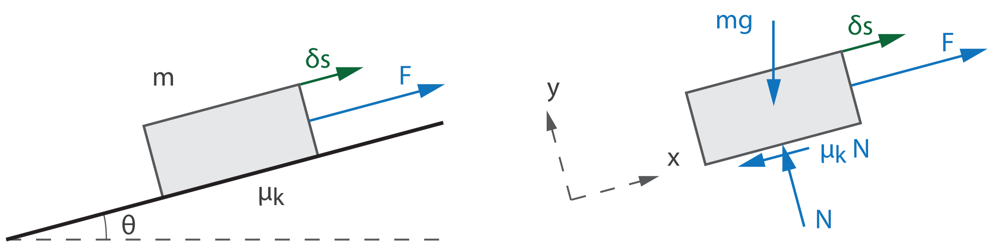
The concept of work and energy will become particularly powerful in the study of dynamics. Note that in this part of the unit the bodies are assumed to move quasi-statically (*i.e.* no accelerations, and thus no velocity and no kinetic energy).

Mechanical Efficiency In a system with friction, some of the work done by external forces is dissipated in the form of heat. The output work of a machine is therefore always less than the input work. The ratio of the two amounts is the mechanical efficiency, e, of the system:

$$e = \frac{U_{\text{out}}}{U_{\text{in}}} \tag{2.7}$$

Example 2.5 – Block dragged up Slope

Consider a block with mass m being pulled up a slope at angle θ, with a coefficient of dynamic friction μ_k, by a force F.



The *input* work is over an infinitesimal distance δs is:

$$U_{\text{in}} = F\delta s = mg\,(\sin\theta + \mu_k\cos\theta)\,\delta s$$

The *output* work is due to the vertical displacement of centre of mass:

$$U_{\text{out}} = mg\sin\theta\,\delta s$$

Therefore, the efficiency e is

$$e = \frac{mg\sin\theta\,\delta s}{mg\,(\sin\theta + \mu_k\cos\theta)\,\delta s} = \frac{1}{1+\mu_k\frac{1}{\tan\theta}}$$

For the case of $\mu_k = \mu_s$, a self-locking slope (the block will not slide down under gravity: $\theta \leq \phi_s$) the efficiency can thus be no greater than 50%.

2.4 Potential Energy

The work done by a force can be stored as *potential energy*. Here we briefly describe elastic and gravitational potential energy, and we shall return to the subject in more detail when discussing dynamics.

Elastic Potential Energy The elastic spring force F_s is proportional to the elongation u and the spring stiffness k:

$$F_s = k\,u$$

The energy stored in a spring, V_s can therefore be found by integrating over the elongation

$$V_s = \int_0^u F du = \int_0^u k\,u\,du = \frac{1}{2}ku^2 \tag{2.8}$$

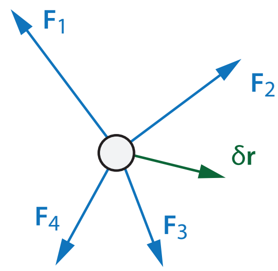
Gravitational Potential Energy The gravitational potential energy V_g due to a mass m displaced over a height h is given by:

$$V_g = \int_0^h mg\,dh = mgh \tag{2.9}$$

Note that the potential energy is calculated with respect to an arbitrary datum, or reference height.

2.5 Virtual Work and Equilibrium

There exists a class of problems where the bodies are composed of members that can move relative to each other, *i.e.* mechanisms, and therefore numerous equilibrium configurations are possible. Such problems can be solved by isolating each of the invidual components and successively formulating and solving the equilibrium equations. However, this can be a time consuming excercise, and the **principle of virtual work** offers a more elegant approach.



Consider a *particle in equilibrium* under multiple forces $\boldsymbol{F}_i$. Let the particle be displaced by an arbitrary but infinitesimally small **virtual displacement** $\delta \boldsymbol{r}$. The **virtual work** done by a force $\boldsymbol{F}_i$ acting on the particle during the virtual displacement is:

$$\delta U_i = \boldsymbol{F}_i \cdot \delta \boldsymbol{r} = F_i \cos\beta \delta s$$

Note that the force $\boldsymbol{F}_i$ is assumed to remain constant during any infinitesimal displacement. The total virtual work done by the forces on the particle is found as:

$$\begin{aligned} \delta U &= \boldsymbol{F}_1 \cdot \delta \boldsymbol{r} + \boldsymbol{F}_2 \cdot \delta \boldsymbol{r} \ldots = \sum \boldsymbol{F} \cdot \delta \boldsymbol{r} \\ &= \sum F_x \delta x + \sum F_y \delta y + \sum F_z \delta z = 0 \end{aligned}$$

which must evidently be zero since $\sum \boldsymbol{F} = 0$. This result may appear trivial, but the consequences of this formulation are powerful when extended to (systems of) rigid bodies. If a *rigid body in equilibrium* is subjected to a virtual displacement, forces applied at different points on the body may have different virtual displacements, but the total virtual work will still be zero.

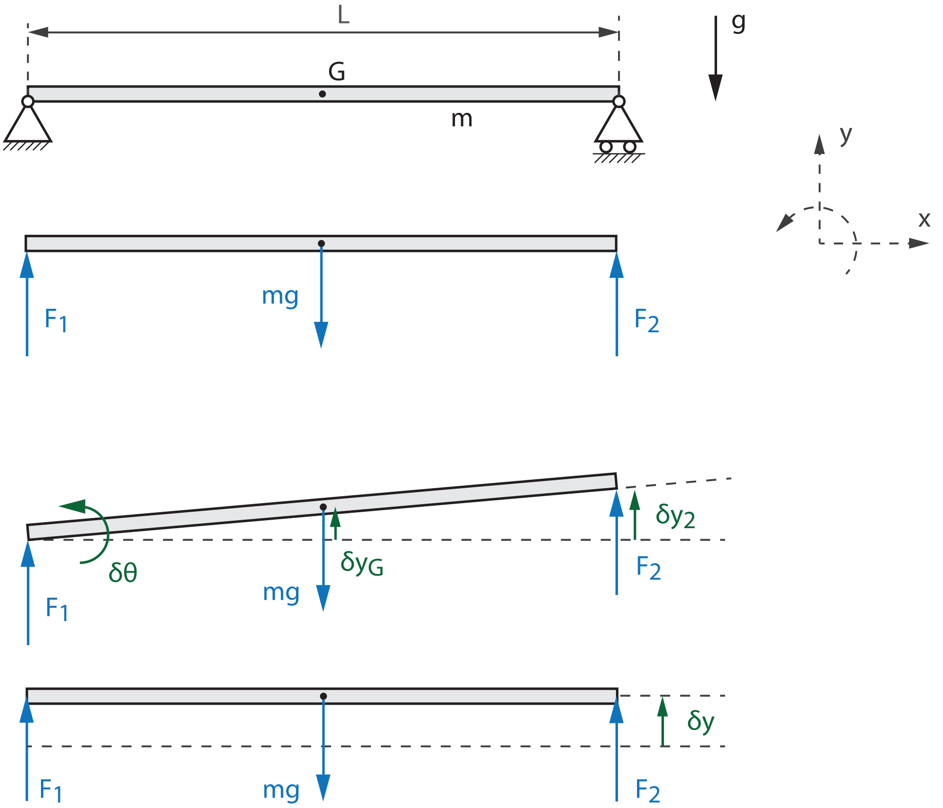
The power of the principle of virtual work becomes evident for *systems of rigid bodies*. Any internal forces will not contribute to the virtual work; and if the virtual displacements are compatible with the kinematic constraints, they will be orthogonal to the reaction forces and therefore also not contribute to the total virtual work. This enables us to find the desired external forces or equilibrium configuration directly, without needing to isolate each individual member, or without reference to any of the reaction forces.

The **principle of virtual work** can now be stated as follows: **the sum of the virtual work done by all external forces on system of rigid bodies in equilibrium is zero for any and all virtual displacements consistent with the constraints**.

One of the applications is in the design of mechanisms, and you will use this in your AVDASI 2 project to design slat and flap mechanisms for a small-scale wing.

Example 2.6 – Virtual Work on Rigid Body

Consider a simply supported beam of length L and mass m. The support reaction forces can be found using equilibrium equations, but can also be found using the principle of virtual work.



After drawing the FBD, apply a virtual rotation $\delta\theta$ at the left-hand support. This results in virtual displacements for the applied loads and reactions, and the total virtual work is given as:

$$\begin{aligned}\delta U &= -mg\,\delta y_G + F_2\,\delta y_2 \\ &= -mg\frac{L}{2}\delta\theta + F_2 L\delta\theta = \left(F_2 - \frac{mg}{2}\right) L\delta\theta = 0\end{aligned}$$

where $\delta y_G = \frac{L}{2}\delta\theta$ and $\delta y_2 = L\delta\theta$ follow from kinematics. This gives the expected result of $F_2 = mg/2$.

Next, apply a virtual displacement δy to the entire beam, which results in the following virtual work:

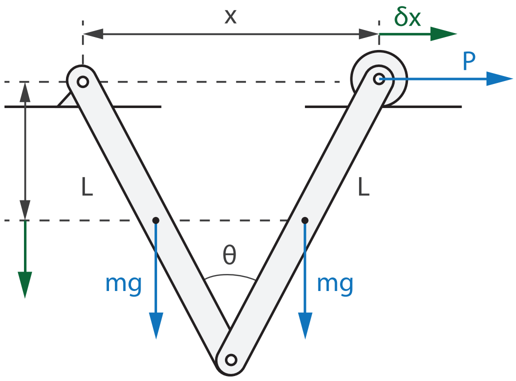
$$\delta U = F_1\delta y - mg\,\delta y + F_2\,\delta y = (F_1 + F_2 - mg)\,\delta\theta = 0$$

which gives sufficient information to find $F_1 = mg/2$.

Clearly, these results could also have been found from force and moment equilibrium. The principle of virtual work becomes more powerful for systems of rigid bodies, as shown in the next example.

Example 2.7 – Equilibrium of Linkage

Consider the following linkage. Each of the two uniform hinged bars has a mass m and a length L. For a given force P determine the angle θ for equilibrium.



For a virtual displacement δx and δh the virtual work becomes:

$$\delta U = P\delta x + 2mg\delta h = 0$$

where both virtual displacements δx and δh are expressed in terms of θ.

$$x = 2L\sin\frac{\theta}{2} \quad \rightarrow \quad \delta x = \frac{dx}{d\theta}\delta\theta = L\cos\frac{\theta}{2}\delta\theta$$

$$h = \frac{L}{2}\cos\frac{\theta}{2} \quad \rightarrow \quad \delta h = \frac{dh}{d\theta}\delta\theta = -\frac{L}{4}\sin\frac{\theta}{2}\delta\theta$$

Substituting into the virtual work equation gives:

$$\delta U = PL\cos\frac{\theta}{2}\delta\theta - 2mg\frac{L}{4}\sin\frac{\theta}{2}\delta\theta = \left(PL\cos\frac{\theta}{2} - 2mg\frac{L}{4}\sin\frac{\theta}{2}\right)\delta\theta = 0$$

resulting in:

$$\theta = 2\arctan\frac{2P}{mg}$$

This solution could also have been found by drawing the FBD for each of the two bars, and applying equilibrium equations. The principle of virtual work, however, is faster and more elegant.

2.6 Minimum Potential Energy

The principle of virtual work can be extended, by explicitly including energy stored and released from elastic and gravitational potential energy (V_s and V_g):

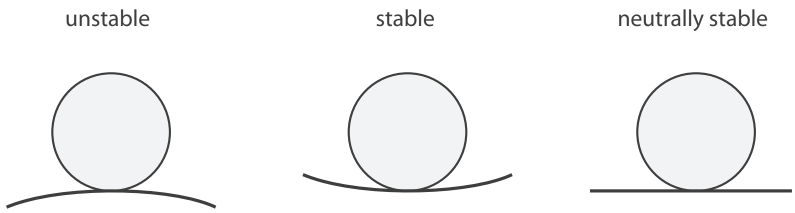
$$\delta U - (\delta V_s + \delta V_g) = 0$$

where there δU now only includes the effect of *active* forces.

Consider a special case where movement is a result of changes in elastic and gravitational potential energy, and where external forces are zero. This implies that the equilibrium configuration lies at a point where $\delta V = 0$, and the total potential energy therefore has a stationary value. If the movement of the system can be described in a single variable λ, the equilibrium conditions is found as:

$$\frac{dV}{d\lambda} = 0$$

This equilibrium point may, however, be a local maximum or minimum of the potential energy function, and should therefore be checked for *stability*. The classic analog is that of a ball on a surface.



An equilibrium is stable if it is at a local energy minimum:

$$\frac{dV^2}{d\lambda^2} > 0$$

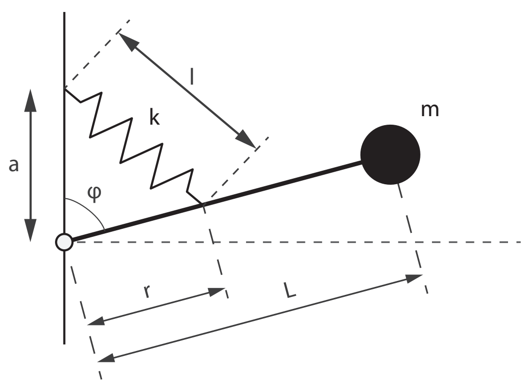
or unstable if it is at a local energy maximum:

$$\frac{dV^2}{d\lambda^2} < 0$$

The special case where the second-order derivative is zero will be illustrated next.

Example 2.8 – Static Balancer

In a simple robot arm, a payload with mass m is attached to the end of a massless bar with length L, and is supported by a spring with stiffness k.



The length l of the spring is calculated as:

$$l^2 = a^2 + r^2 - 2ar\cos\varphi$$

The spring energy and potential energy can then be calculated as:

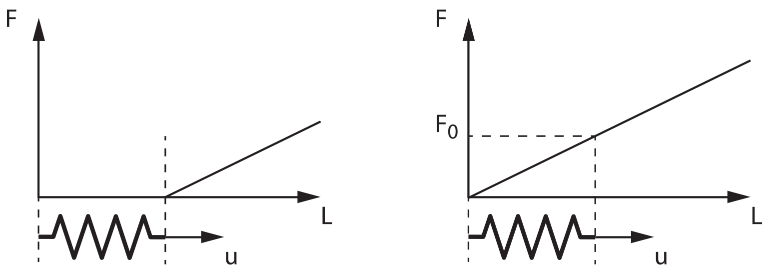
$$V_s = \frac{1}{2}k\left(l - l_0\right)^2$$
$$V_g = mgL\cos\varphi$$

where l_0 is the initial length of the spring. The equilibrium position of the robot arm could then be found by looking at the local energy minimum:

$$\frac{dV}{d\varphi} = \frac{d(V_s + V_g)}{d\varphi} = 0$$

and solving for φ. In general this will result in a single, stable equilibrium configuration for the robot arm.

However, let us consider a special case where $l_0 = 0$. This represents a so-called zero-free-length spring, where the spring force is proportional to the *length* of the spring. This can be achieved by prestressing the spring, so that the coils are very tightly wound and a force F_0 is required to separate them.



In that case the total energy is given by:

$$V = V_p + V_g = mgL\cos\varphi + \frac{1}{2}k\left(a^2 + r^2 - 2ar\cos\varphi\right)$$

and energy minimum is found as:

$$\frac{dV}{d\varphi} = -mgL\sin\varphi + akr\sin\varphi = 0$$
$$= \sin\varphi\left(akr - mgL\right) = 0$$

which is satisfied for *any* angle φ as long as the following condition is satisfied:

$$k = \frac{mgL}{ar}$$

This means that the robot arm will be in equilibrium for *any* position, as all positions will have the same total amount of energy (spring and gravitational). This also means that changing configuration requires no work, and the mass is effectively 'weightless'!

This is also the working principle of an Anglepoise desk lamp, which can effortlessly be repositioned (well, the 1930s models at least!)

In robotics the same principle can be used to counterbalance the mass of the robot arm, allowing the engineer to use less powerful motors to move the manipulator.

Revision Objectives Handout 2:

Friction

- recognise the difference between static and kinematic friction
- express friction using friction coefficient (μ) and angle ($\tan\phi = \mu$)
- recall and apply the capstan equation ($T_1/T_0 = e^{\mu\varphi}$)
- calculate the efficiency e of a mechanical system involving friction
- solve statics problems that involve friction

Work and Energy - Statics

- calculate the work done by a force and moment ($U = \int F \cos\beta ds$, $U = \int M d\theta$)
- recognise that work and energy are scalar quantities
- apply principle of virtual work to determine equilibrium force or position of a mechanism ($\delta U = 0$)
- appreciate the elegant working principle behind an Anglepoise lamp