# Stress, Strain and Deformation
# **Bending Stresses and Strains**

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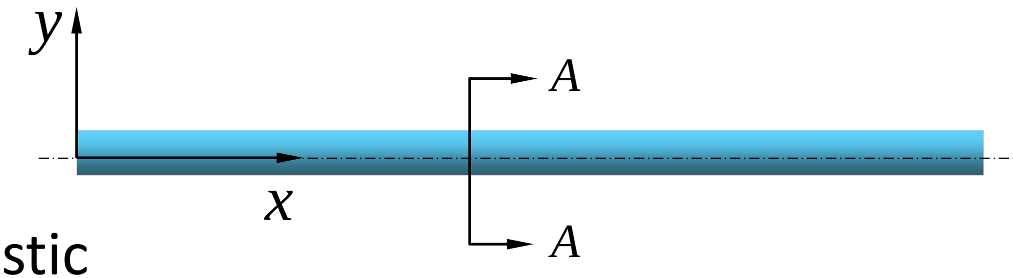
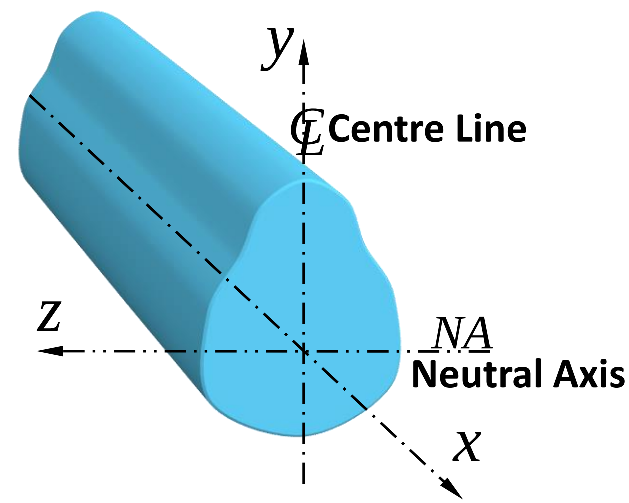
# Assumptions

- Beams are straight
- Cross-sections are constant
- Material response is linear elastic
- Material properties are 'constant':
  - Isotropic
  - Homogeneous
- Loading in the plane of symmetry
  - *i.e.* always in the plane $x$-$y$
  - Also called symmetric bending



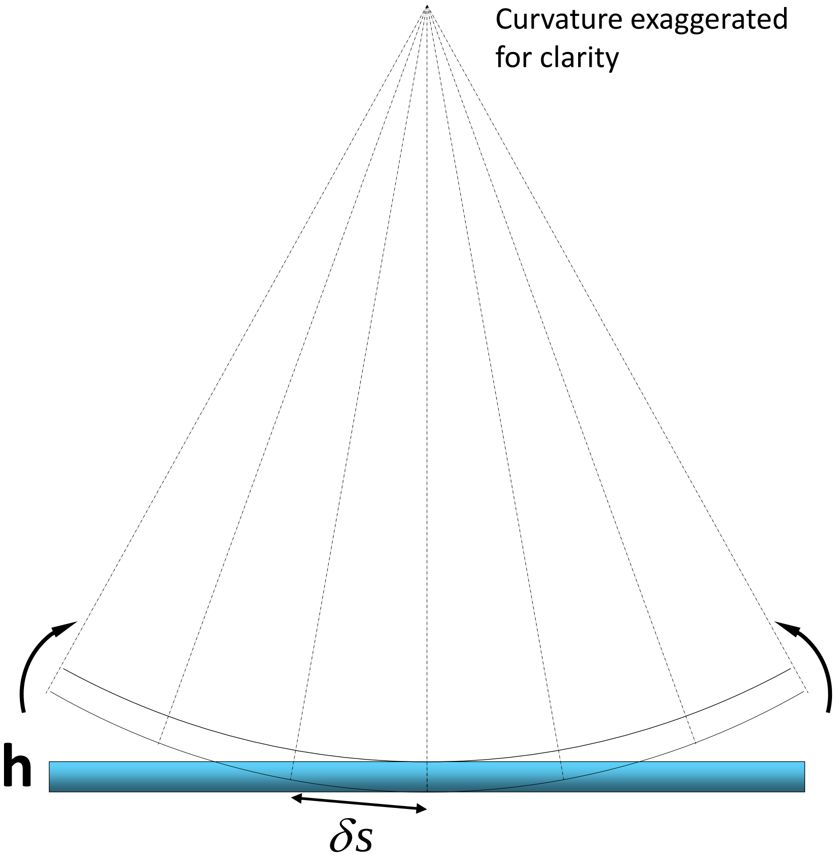




Section $A$-$A$

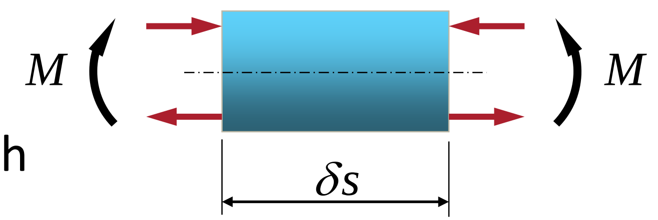
Note: choice of axes may be different in different textbooks !

# Assumptions – Bending Deformation

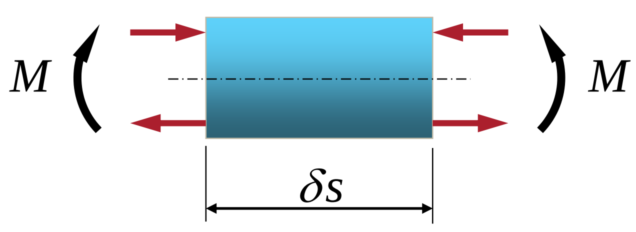
- Small deformations
- Pure bending
  - Neglecting shear deformation
  - Deformation in shape of circular arc
- Cross-sections remain plane
  - and rotate about the neutral plane (*i.e.* no shear deformation)

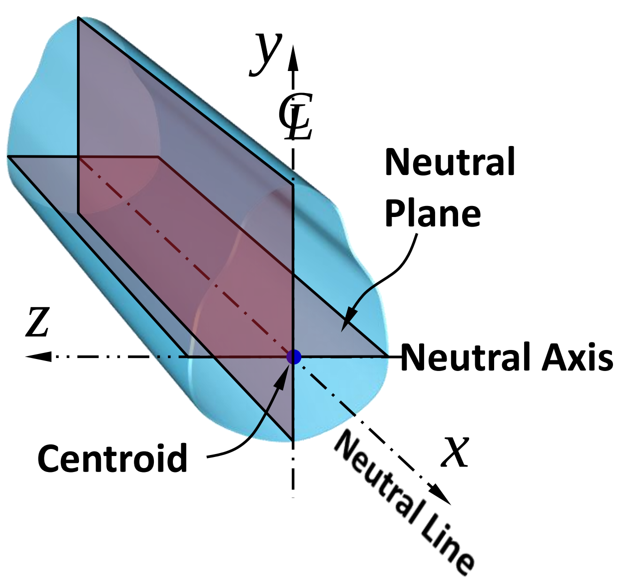



- Consider a small element of **arc-length** $\delta s$ under a **bending moment** $M$
  - The moment is equivalent to compression on one 'side' and tension on the other, *e.g.* a '**couple**'
  - So the length decreases on one side and increases on the other forming a **curvature**, with radius $R$ and curvature $1/R = \kappa$ → (kappa)



- Somewhere in the middle the length will be unchanged; here the normal stress will be zero - this is the **neutral plane**
- The intersection of the neutral plane with the loading plane is called the **neutral line**
- The intersection of the neutral plane with the cross section plane is called the **neutral axis**

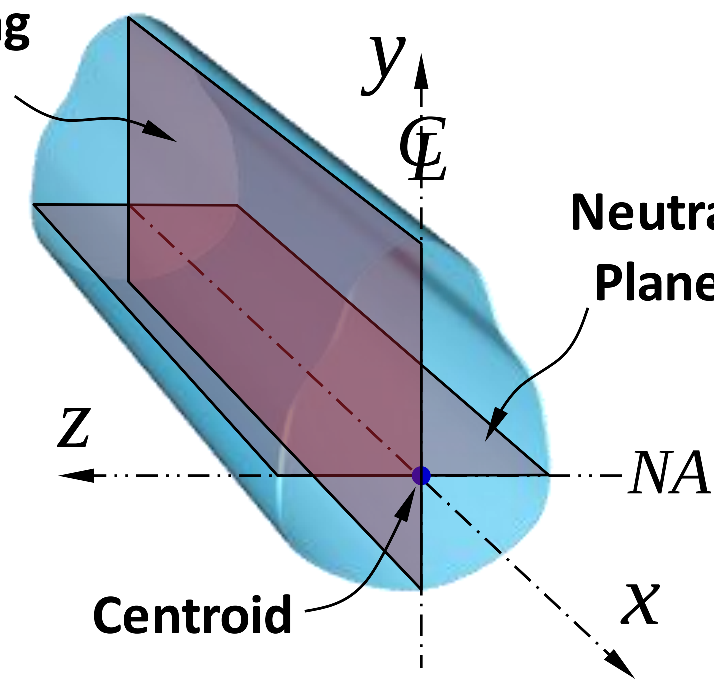


# Assumptions – Symmetric Bending



- Bending acts in a section-symmetric plane
- Neutral axis perpendicular to symmetric bending plane and through section centroid
- (This is not the case for non-symmetric bending as we will see in StM2)



# Derivation of Bending Strains (1/2)

- Consider the beam element ds at a distance y from the neutral line, where $R$ = **radius of curvature**

- After bending, assuming small displacements:

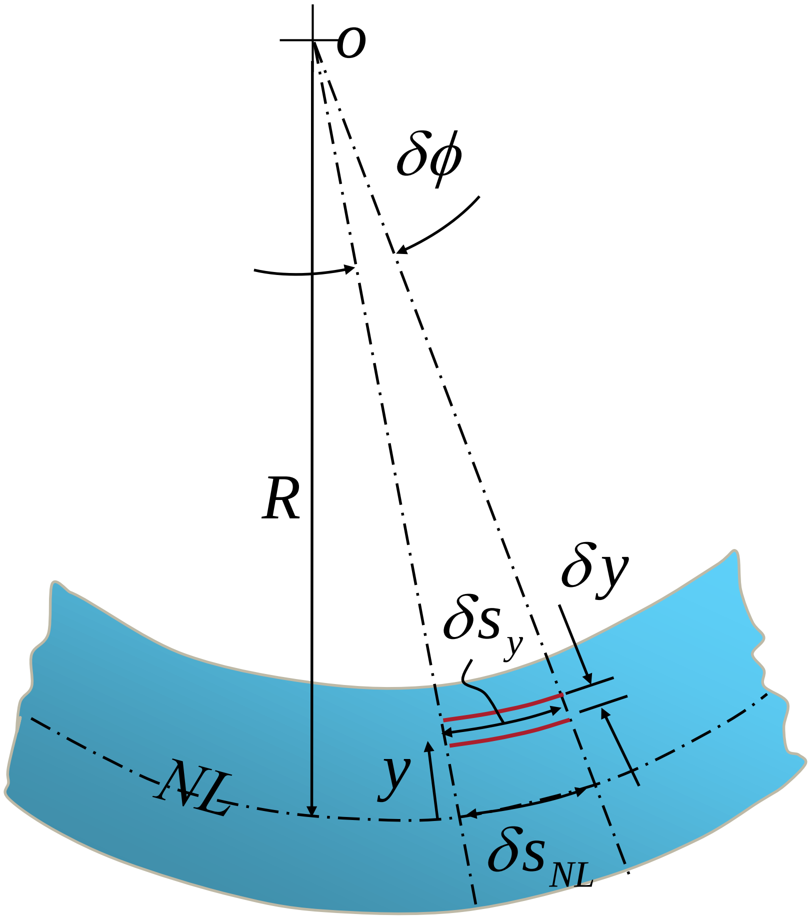
$$\delta\phi \cong \tan\delta\phi \cong \frac{\delta s_{NL}}{R}$$

$$\boxed{\delta s_{NL} = R\,\delta\phi}$$

- And also:

$$\delta\phi \cong \tan\delta\phi \cong \frac{\delta s_y}{R-y}$$

$$\boxed{\delta s_y = (R-y)\delta\phi}$$



# Derivation of Bending Strains (2/2)



- But before bending (*i.e.* when straight)

$$\delta s_y = \delta s_{NL}$$

- So the change in length of the 'element' at $y$ is:

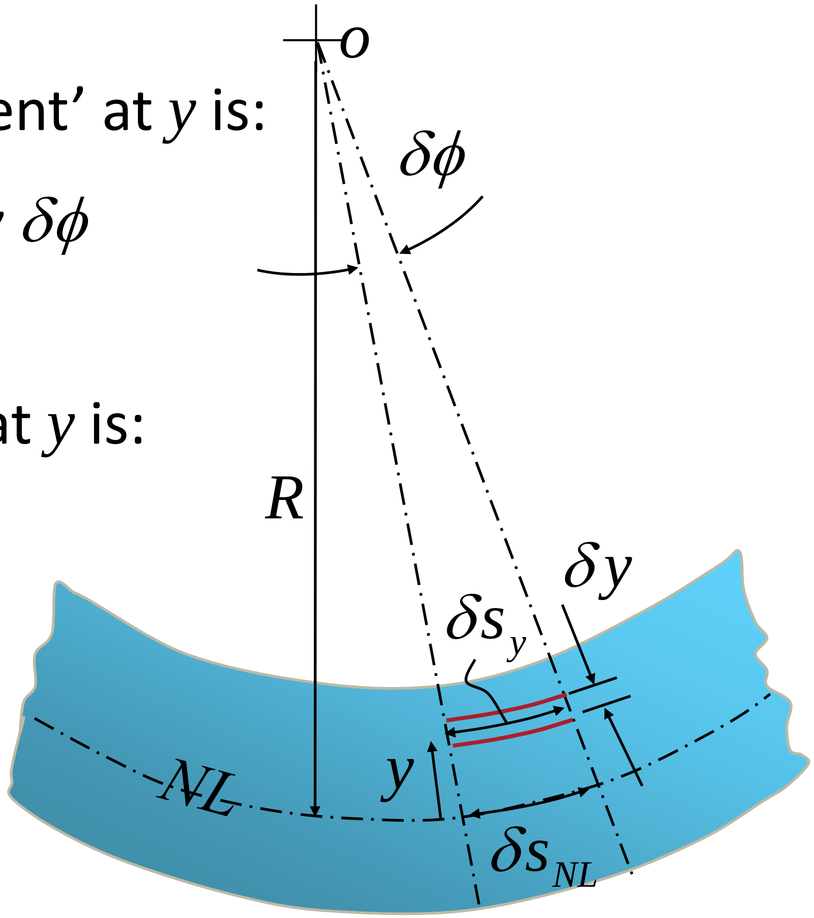
$$\left(\delta s_y - \delta s_{NL}\right) = \left(R - y\right)\delta\phi - R\,\delta\phi = -y\,\delta\phi$$

- Therefore the strain of the element at $y$ is:

$$\varepsilon = \frac{\delta s_y - \delta s_{NL}}{\delta s_{NL}} = \frac{-y\,\delta\phi}{R\,\delta\phi} = -\frac{1}{R}y$$

$$\boxed{\varepsilon = -\frac{1}{R}y}$$

**KEY POINT**



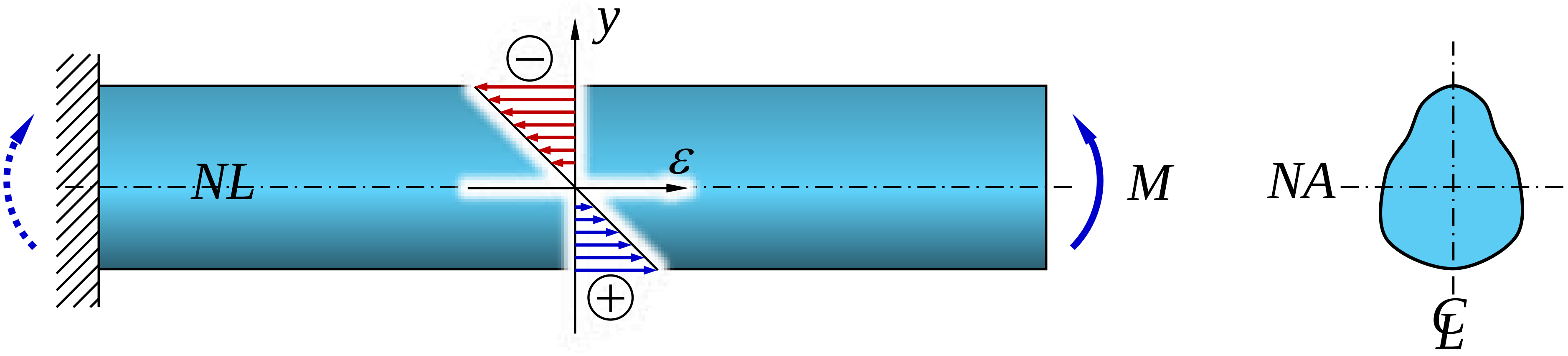
# Bending Strains



- The derived strain distribution is:

$$\boxed{\varepsilon = -\frac{1}{R} y}$$

- Note the **linear variation of strains** with distance from the neutral axis $NA$:



# Bending Stresses

- For linear elastic stress-strain behaviour (*i.e.* applying Hooke's law) the direct stress (normal to section) is:

$$\varepsilon = \frac{-y}{R} \quad , \quad \sigma = E\,\varepsilon \qquad \therefore \qquad \boxed{\sigma = -\frac{E}{R}\,y}$$

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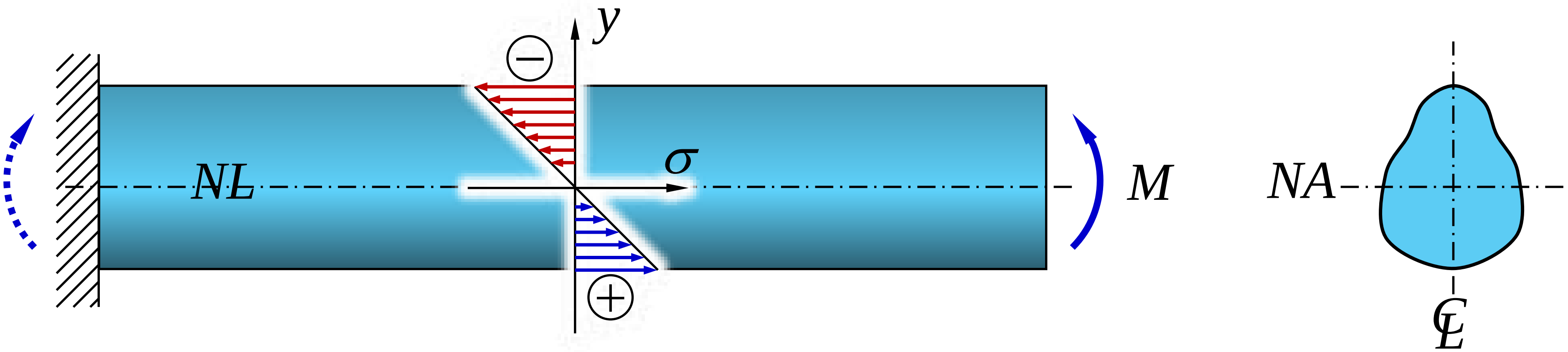

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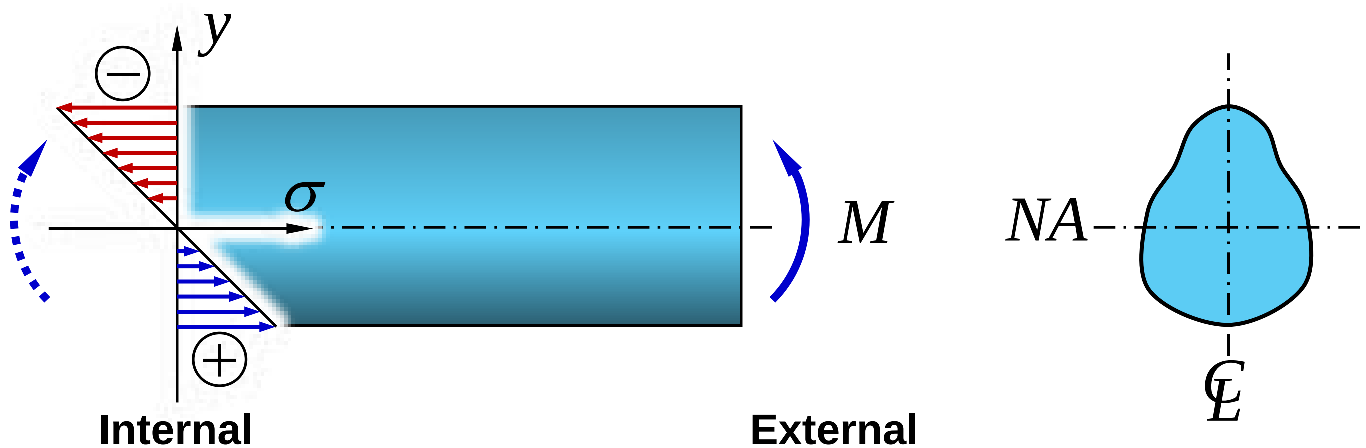


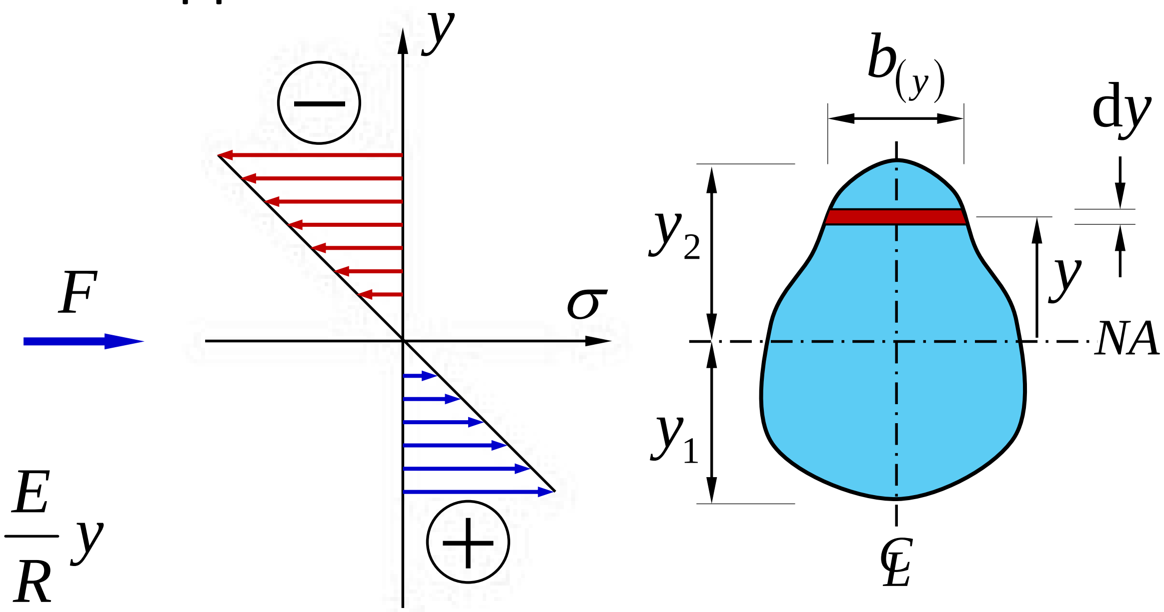

- **External** forces and moments are reacted by **internal** forces and moments which are the result of **internal stresses**

- For **pure bending** the total **internal resultant force** must be **zero**, since no external axial force is applied

$$\int_{y_1}^{y_2} \mathrm{d}F = 0$$

$$\mathrm{d}F = \sigma_{(y)}\, b_{(y)}\, \mathrm{d}y$$

$$\mathrm{d}F = \sigma_{(y)}\, \mathrm{d}A$$




$$\int_{y_1}^{y_2} \sigma_{(y)}\, \mathrm{d}A = 0 \qquad \sigma_{(y)} = -\frac{E}{R}\, y$$

$$\int_{y_1}^{y_2} -\frac{E}{R}\, y\, \mathrm{d}A = 0$$

However $E$ and $R$ are constant with respect to $y$, therefore:

$$\boxed{\int_{y_1}^{y_2} y\, \mathrm{d}A = 0}$$

→ The **1st moment of area** is **zero** about $NA$

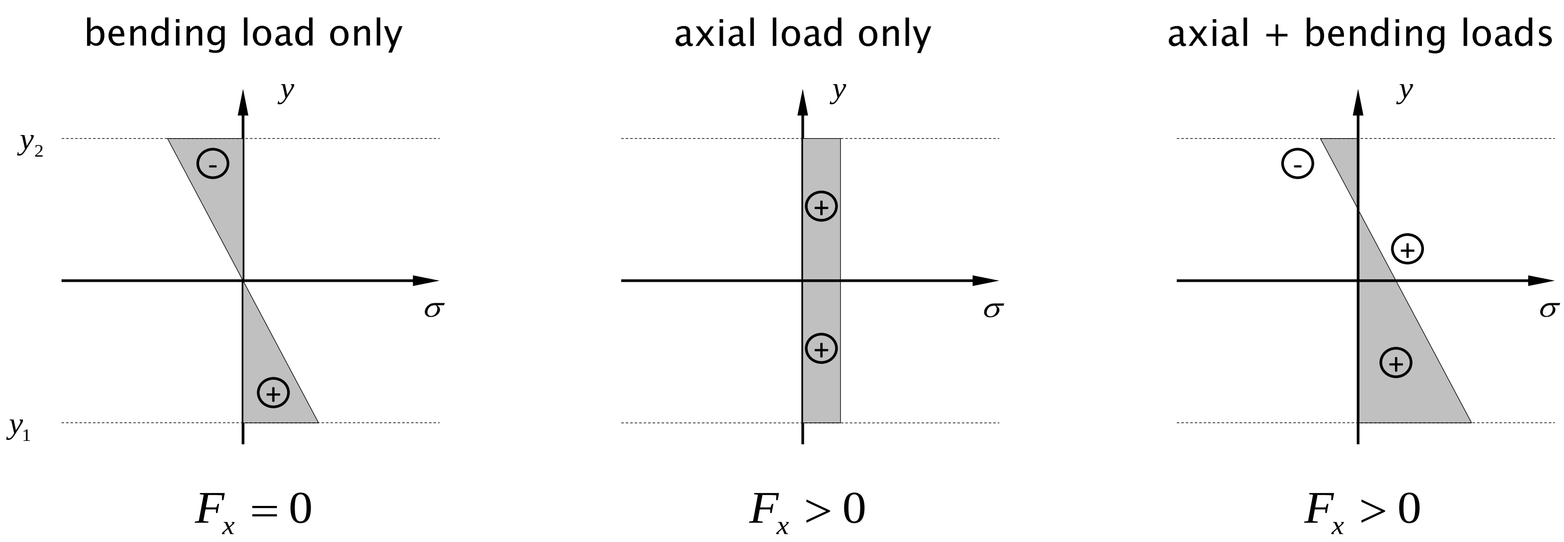
→ $NA$ must pass through the **centroid** of the cross-section

# Internal Axial Forces

- For a constant width $b$, the resultant internal axial force $F_x$ can be interpreted as the 'summation' of the 'areas' under the curves of stress distribution through the thickness

$$\boxed{F_x = b\int_{y_1}^{y_2} \sigma_{(y)}\, \mathrm{d}y}$$

- (Note that these areas can have different 'signs')



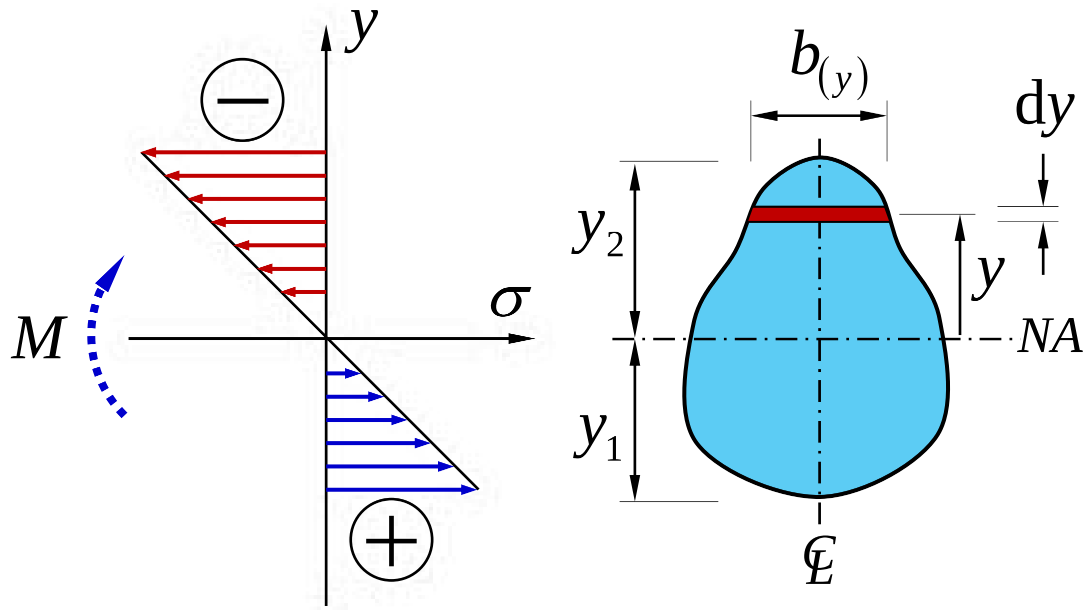
- For **pure bending** the **internal resultant moment** must be **nonzero**, to balance out the external moment $M$

$$\int_{y_1}^{y_2} y \, \mathrm{d}F = -M$$

$$\int_{y_1}^{y_2} y \, \sigma_{(y)} \, \mathrm{d}A = -M$$

$$\sigma_{(y)} = -\frac{E}{R} y$$

$$\int_{y_1}^{y_2} \frac{E}{R} y^2 \, \mathrm{d}A = M$$




$$\frac{E}{R} \int_{y_1}^{y_2} y^2 \, \mathrm{d}A = M$$

$$\boxed{\int_{y_1}^{y_2} y^2 \, \mathrm{d}A = I}$$

→ The **2nd moment of area** of the cross-section is called $I_{(\text{ref. axis})}$

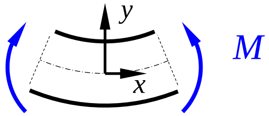
$$\boxed{\frac{1}{R} EI = \kappa \, EI = M}$$

→ The product $EI$ is called the '**flexural modulus**' of the beam

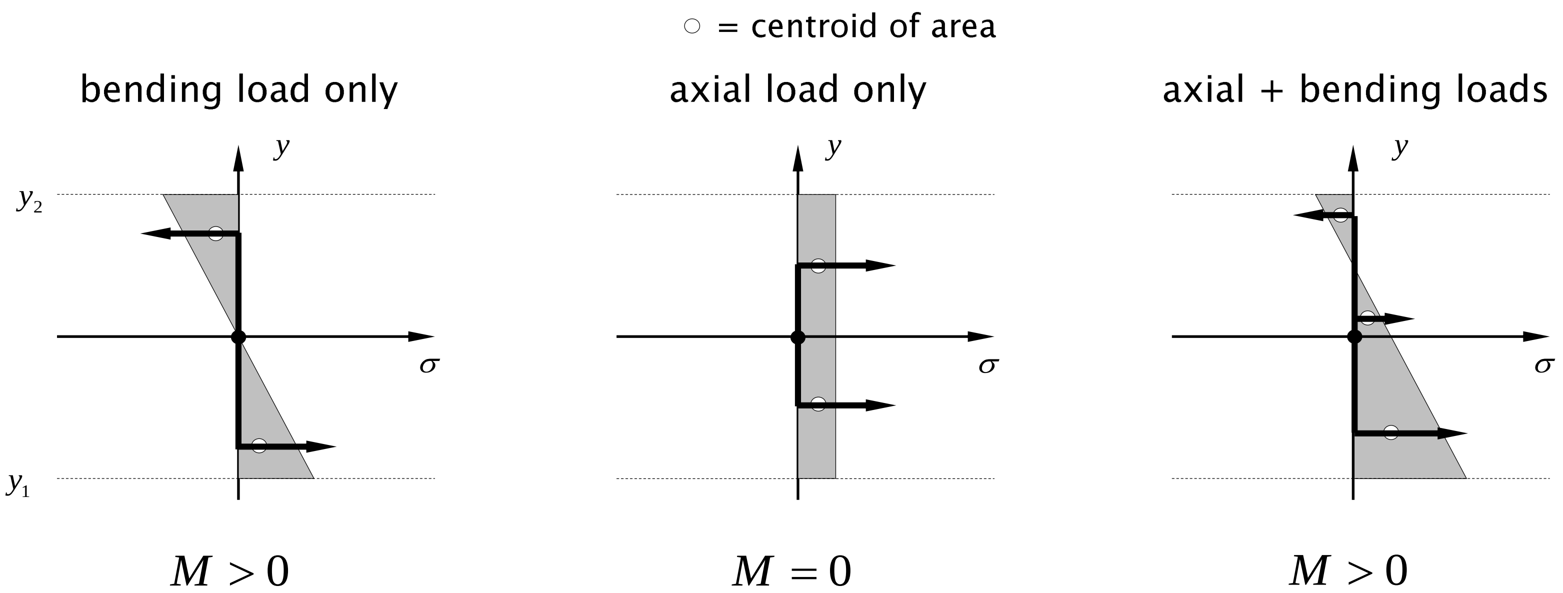
# Bending Moment

- The internal bending moment is given by the integral:

$$\boxed{M = -b\int_{y_1}^{y_2} \sigma_{(y)}\, y\, \mathrm{d}y}$$




- This can be interpreted as the '**moment of area**' of the stress distribution:



KEY POINTS



- We can now re-arrange the equations:

$$\frac{EI}{R} = M \quad \therefore \quad \frac{M}{I} = \frac{E}{R}$$

$$\sigma = -\frac{E}{R}y \quad \therefore \quad -\frac{\sigma}{y} = \frac{E}{R}$$

KEY POINT

$$\boxed{-\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

# Stress, Strain and Deformation
# **Second Moment of Areas**

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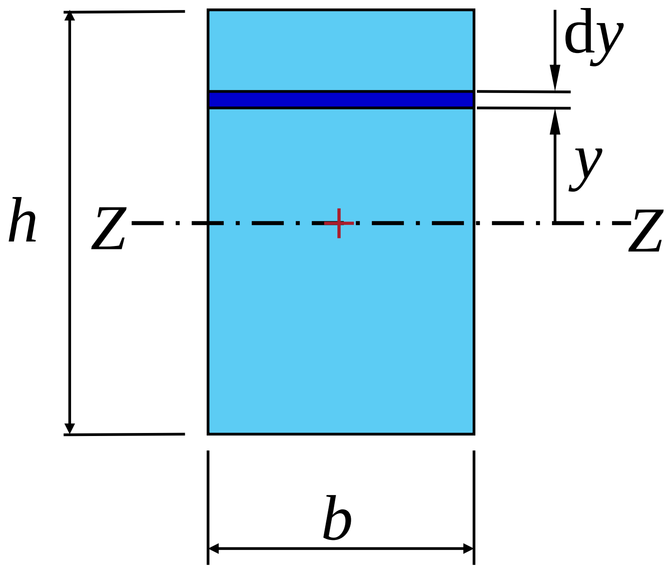
# Rectangular and Square Cross-Sections

KEY POINTS

- Consider a **rectangular cross-section** of height $h$ and width (or breadth) $b$
- The second moment of area about a centroidal $Z$-$Z$ axis is:




$$I = \int_{-h/2}^{h/2} y^2 \, \mathrm{d}A$$

$$I = \int_{-h/2}^{h/2} y^2 \, b \, \mathrm{d}y$$

$$I = \left[ \frac{y^3}{3} b \right]_{-h/2}^{h/2}$$

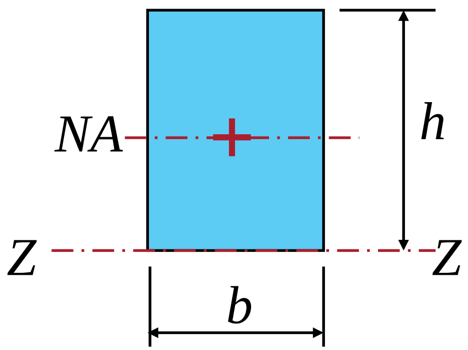
$$\boxed{I = \frac{b\,h^3}{12}}$$

- For **square cross-sections** $b = h$, therefore $\boxed{I = \frac{h^4}{12}}$

# Expressions of $I$ for standard plane areas

KEY POINTS

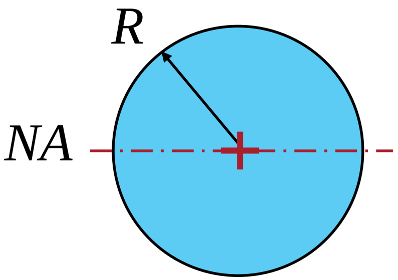
Rectangular




$$I_{NA} = \frac{b\,h^3}{12}$$

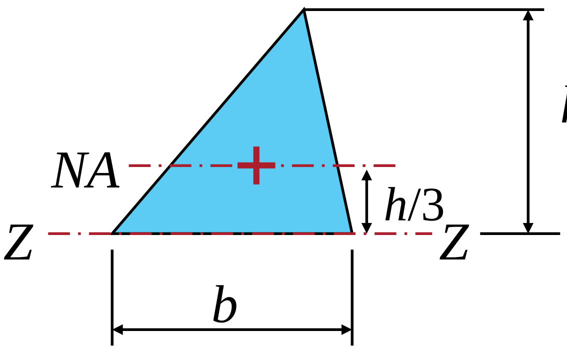
$$I_{ZZ} = \frac{b\,h^3}{3}$$

Circular




$$I_{NA} = \frac{\pi\,R^4}{4}$$

Triangular

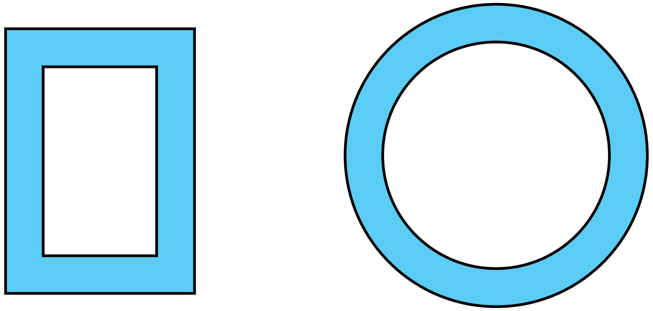



$$I_{NA} = \frac{b\,h^3}{36}$$

$$I_{ZZ} = \frac{b\,h^3}{12}$$

# $I$ of concentric hollow cross-sections

- For concentric hollow cross-sections, *e.g.*



- The second moment of area is simply:

$$\boxed{I = I_{\text{outer}} - I_{\text{inner}}}$$