Vibrations 2, Lecture 17
Lagrange's equations

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Lecture 16 review

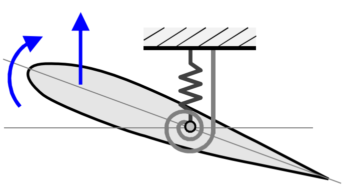
Forced linear *damped* MDOF system: $\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t)$

Steady-state harmonic response: $\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}_0 e^{i\omega t}$

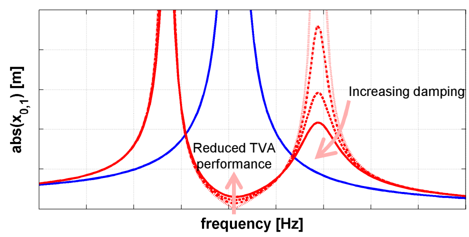
$$\mathbf{x}(t) = \left((-\omega^2\mathbf{M} + i\omega\,\mathbf{C} + \mathbf{K})^{-1}\,\mathbf{f}_0 \right) e^{i\omega t}$$

Aero examples/applications:

Elastically suspended rigid wing with aero loads:



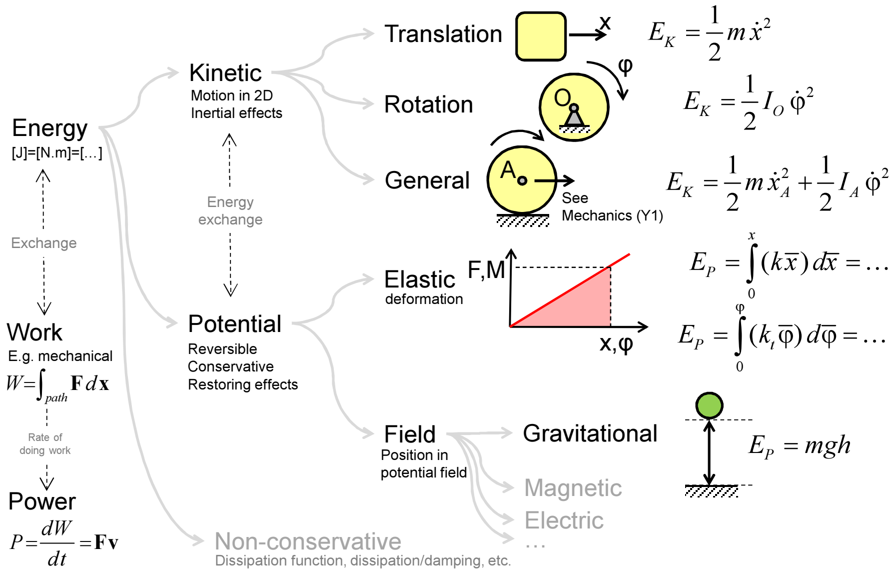
Damped TVA:



Lecture 17

- Overview of energy related concepts
- Lagrange's equations
- Example: 2DOF wing example
- Example: 2 coupled pendulums

Energy and work revision



Lagrange's equations

Lagrange's equations for NDOF system:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \qquad j=1,2,\ldots,N$$

where:

$$\mathcal{L} = E_K - E_P$$

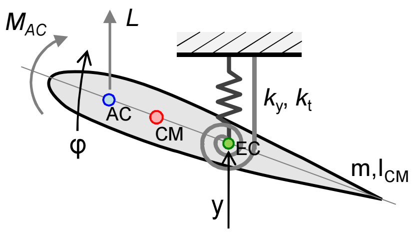
- $\mathcal{L}$ is the Lagrangian function
- E_K is the total kinetic energy (energy stored in motion)
- E_P is the total potential energy (energy stored in deformation, position, …)
- q_j are the generalized coordinates (angle, deformation, deflection, …)
- Q_j are the generalized loads (forces or moments)

Properties and advantages:

- Using scalar (energy) functions instead of vectors = "safer" derivations
- Consistent approach when working with rectilinear and angular DOFs and loads
- Minimum number of equations (EOMs) for multi-body and multi-DOF systems
- Programmable

2DOF wing example (Lagrange)

Consider a simplified 2DOF model of a rigid wing suspended on the linear and torsional *springs* (these springs represent elasticity of real wings). Derive the matrix EOM for this system using Lagrange's equations.



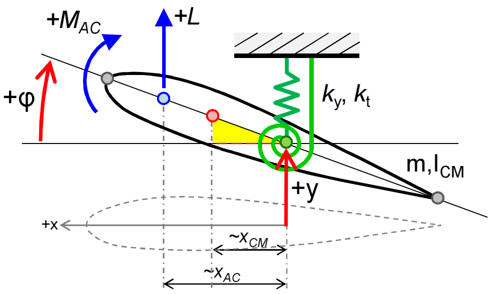
AC … aerodynamic centre
CM … centre of mass
EC … elastic centre

DOFs: 2 (y,φ)
Small angle φ: $\cos\varphi=1$, $\sin\varphi=\varphi$

m … wing mass
I_{CM} … moment of inertia about CM

2DOF wing example (Lagrange)

DOFs, generalized coordinates and motion (kinematics):



Number of DOFs: 2 = heaving (up-down) + pitching (rotation about EC)

Generalized coordinates: q_1 = y, q_2 = φ; use positive orientations / deflections

Motion (important points): EC, CM, AC

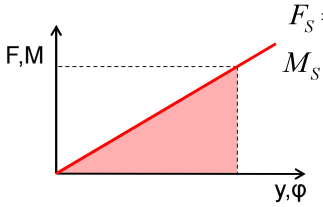
$x_{EC} \approx 0 \qquad x_{CM} \approx 0 \qquad\qquad x_{AC} \approx 0$

$y_{EC} \approx y \qquad y_{CM} \approx y + \varphi x_{CM} \Rightarrow \dot{y}_{CM} \approx \dot{y} + \dot{\varphi} x_{CM} \qquad y_{AC} \approx y + \varphi x_{AC} \Rightarrow \dot{y}_{AC} = \ldots$

2DOF wing example (Lagrange)

Energies: kinetic and potential (no damping)

Potential energy:



$$F_S = k_y y$$

$$M_S = k_\varphi \varphi$$

$$E_{P,1} = \int_0^y F_S \, d\bar{y} = \ldots \qquad E_{P,2} = \int_0^\varphi M_S \, d\bar{\varphi} = \ldots$$

The potential energy stored in any deformed elastic member is equal to the area under the curve (line) in the load-deflection characteristics of this member.

Potential energy in our case is equal to the area enclosed by the *triangle*:

$$E_{P,1} = \frac{1}{2}(k_y y)y = \frac{1}{2}k_y y^2 \qquad E_{P,2} = \frac{1}{2}(k_\varphi \varphi)\varphi = \frac{1}{2}k_\varphi \varphi^2$$

The total potential energy: $E_P = \frac{1}{2}k_y y^2 + \frac{1}{2}k_\varphi \varphi^2$

2DOF wing example (Lagrange)

Kinetic energy:

- General motion in 2D
  - vertical CM motion characterized by y_{CM} $\quad E_{K,1} = \frac{1}{2} m(\dot{y} + \dot{\varphi} x_{CM})^2$
  - rotation about CM characterized by φ $\quad E_{K,2} = \frac{1}{2} I_{CM} \dot{\varphi}^2$

The total potential energy: $E_K = \frac{1}{2} m(\dot{y}^2 + 2\dot{y}\dot{\varphi} x_{CM} + \dot{\varphi}^2 x_{CM}^2) + \frac{1}{2} I_{CM} \dot{\varphi}^2$

Lagrangian using generalized coordinates: $\mathcal{L} = E_K - E_P$

$$\mathcal{L} = E_K - E_P = \frac{1}{2} m(\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 x_{CM} + \dot{q}_2^2 x_{CM}^2) + \frac{1}{2} I_{CM} \dot{q}_2^2 - \frac{1}{2} k_y q_1^2 - \frac{1}{2} k_\varphi q_2^2$$

$\mathcal{L}$ is the function of the generalized coordinates (displacements), velocities and all system parameters.

$$\mathcal{L} = \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2; m, I_{CM}, x_{CM}, k_y, k_\varphi)$$

2DOF wing example (Lagrange)

Lagrange's equations: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \qquad j=1,2$

Lagrangian: $\mathcal{L} = \frac{1}{2} m(\dot{q}_1^2 + 2\dot{q}_1 \dot{q}_2 x_{CM} + \dot{q}_2^2 x_{CM}^2) + \frac{1}{2} I_{CM} \dot{q}_2^2 - \frac{1}{2} k_y q_1^2 - \frac{1}{2} k_\varphi q_2^2$

Partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m\dot{q}_1 + \dot{q}_2 m x_{CM} \qquad \frac{\partial \mathcal{L}}{\partial q_1} = -k_y q_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \dot{q}_1 m x_{CM} + \dot{q}_2 m x_{CM}^2 + I_{CM} \dot{q}_2 \qquad \frac{\partial \mathcal{L}}{\partial q_2} = -k_\varphi q_2$$

Substituting all relationships into Lagrange's equations:

$$\frac{d}{dt}\left(m\dot{q}_1 + m x_{CM} \dot{q}_2\right) + k_y q_1 = Q_1 \qquad \Rightarrow m\ddot{q}_1 + m x_{CM} \ddot{q}_2 + k_y q_1 = Q_1$$

$$\frac{d}{dt}\left(m x_{CM} \dot{q}_1 + (I_{CM} + m x_{CM}^2)\dot{q}_2\right) + k_\varphi q_2 = Q_2 \Rightarrow m x_{CM} \ddot{q}_1 + (I_{CM} + m x_{CM}^2)\ddot{q}_2 + k_\varphi q_2 = Q_2$$

2DOF wing example (Lagrange)

Generalized loads are determined using the *equivalency* between the total power generated by the applied loads (moments, forces) and the total power produced by the generalized loads and their corresponding generalized velocities.

$$P = Q_1\,\dot{q}_1 + Q_2\,\dot{q}_2$$

$$P = L\,\dot{y}_{AC} + M\dot{\varphi} = L(\dot{y} + \dot{\varphi}\,x_{AC}) + M\dot{\varphi} = L\dot{y} + (M + Lx_{AC})\dot{\varphi} = \underline{L}\,\dot{q}_1 + \underline{(M + Lx_{AC})}\,\dot{q}_2$$

From this: $m\ddot{q}_1 + m\,x_{CM}\ddot{q}_2 + k_y q_1 = L$

$$m\,x_{CM}\ddot{q}_1 + (I_{CM} + m\,x_{CM}^2)\ddot{q}_2 + k_\varphi q_2 = M + Lx_{AC}$$

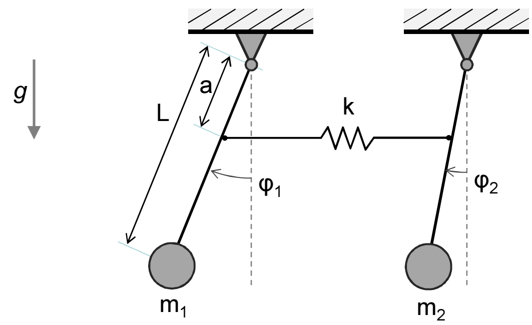
Matrix form of the EOM:

$$\begin{bmatrix} m & mx_{CM} \\ mx_{CM} & I_{CM} + mx_{CM}^2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} k_y & 0 \\ 0 & k_t \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} L \\ M_{AC} + Lx_{AC} \end{bmatrix}$$

After this step (*modelling*), one would typically perform *analysis* of the EOMs, e.g. calculate **M** and **K** and perform eigenvalue or harmonic response analysis (see L12-14), performance and design analyses (see L15,16) etc.

2 coupled pendulums (Lagrange)

Consider the two coupled pendulums. Derive EOMs for this 2DOF system.



DOFs: 2 (φ_1,φ_2)
Small angle assumption: cosφ=1, sinφ=φ

2 coupled pendulums (Lagrange)

Number of DOFs: 2 = pendulum 1 rotation + pendulum 2 rotation

Generalized coordinates: $q_1 = \varphi_1$, $q_2 = \varphi_2$; use positive orientations

Potential energy due to deformed spring, assume $\varphi_1 > \varphi_2$:

$$E_{P,1} = \frac{1}{2}k\Delta^2 = \frac{1}{2}k(a\varphi_1 - a\varphi_2)^2 = \frac{1}{2}ka^2(\varphi_1^2 - 2\varphi_1\varphi_2 + \varphi_2^2) = \frac{1}{2}ka^2(q_1^2 - 2q_1q_2 + q_2^2)$$

Potential energy due to gravity <u>increases with increasing</u> φ and has the form: (mass × g × *height*), where *height* represents the height of the mass from the reference position (e.g. equilibrium position):

$$E_{P,2} = m_1 g\Delta h_1 = m_1 g(L - L\cos\varphi_1) = m_1 gL(1 - \cos q_1)$$

$$E_{P,3} = m_2 g\Delta h_2 = m_2 g(L - L\cos\varphi_2) = m_2 gL(1 - \cos q_2)$$

The total potential energy:

$$E_P = \frac{1}{2}ka^2(q_1^2 - 2q_1q_2 + q_2^2) + m_1 gL(1 - \cos q_1) + m_2 gL(1 - \cos q_2)$$

2 coupled pendulums (Lagrange)

Kinetic energy due to moving masses m_1 and m_2:

Magnitude of the velocity = radius of the circular trajectory × angular speed

$$|\mathbf{v}_1| = L\dot{\varphi}_1, \quad |\mathbf{v}_2| = L\dot{\varphi}_2$$

The total kinetic energy:

$$E_K = E_{K,1} + E_{K,2} = \frac{1}{2}m_1|\mathbf{v}_1|^2 + \frac{1}{2}m_2|\mathbf{v}_2|^2 = \frac{1}{2}m_1L^2\dot{\varphi}_1^2 + \frac{1}{2}m_2L^2\dot{\varphi}_2^2 = \frac{1}{2}m_1L^2\dot{q}_1^2 + \frac{1}{2}m_2L^2\dot{q}_2^2$$

Lagrangian: $\mathcal{L} = E_K - E_P$

$$\mathcal{L} = \frac{1}{2}m_1L^2\dot{q}_1^2 + \frac{1}{2}m_2L^2\dot{q}_2^2 - \frac{1}{2}ka^2(q_1^2 - 2q_1q_2 + q_2^2) - m_1gL(1 - \cos q_1) - m_2gL(1 - \cos q_2)$$

Partial derivatives (assume small angles φ_i):

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_1} = m_1L^2\dot{q}_1 \qquad \frac{\partial\mathcal{L}}{\partial q_1} = -ka^2(q_1 - q_2) - m_1gL\sin q_1 \approx -ka^2(q_1 - q_2) - m_1gL\,q_1$$

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_2} = m_2L^2\dot{q}_2 \qquad \frac{\partial\mathcal{L}}{\partial q_2} = -ka^2(-q_1 + q_2) - m_2gL\sin q_2 \approx -ka^2(-q_1 + q_2) - m_2gL\,q_2$$

2 coupled pendulums (Lagrange)

Substituting all relationships into Lagrange's equations:

$$\frac{d}{dt}\left(m_1 L^2 \dot{q}_1\right) + ka^2(q_1 - q_2) + m_1 g L q_1 = Q_1 = 0$$

$$\frac{d}{dt}\left(m_2 L^2 \dot{q}_2\right) + ka^2(-q_1 + q_2) + m_2 g L\, q_2 = Q_2 = 0$$

Note zero generalized loads Q_i, no external loads are applied, therefore free vibrations problem

Matrix form of the EOM:

$$\begin{bmatrix} m_1 L^2 & 0 \\ 0 & m_2 L^2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} ka^2 + m_1 g L & -ka^2 \\ -ka^2 & ka^2 + m_2 g L \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Usual other problem solving tasks:
- Substitute numerical values and determine **M** and **K**
- Eigenvalue analysis (e.g. via characteristic equation / determinant)
- Eigenvalues / natural frequencies
- Eigenvalue problem, use computed eigenvalues and determine eigenvectors
- …

Summary

- Modelling using Lagrange's equations and further analysis:
  - DOFs, generalized coordinates
  - kinetic and potential energy, Lagrangian
  - partial and time derivatives, generalized loads
  - EOMs, matrix format (if linear problem), **K** and **M**
  - Eigenvalues, eigenvectors
  - Free, harmonic or general time response
  - Aero-elastic analyses (divergence, reversal, gust, flutter, loads, etc.)