Aerofoils in Compressible Flow (1)

Aerodynamics 2
AENG21100

Department of Aerospace Engineering
University of Bristol

So far

- Subsonic/supersonic compressible isentropic flow
- Supersonic flow on to wedges and round corners (normal/oblique shocks and expansion fans) – wedge aerofoils
- …next – a little of both applied to aerofoils and wings, and some practical calculations!

Today

- To derive the linearised potential equation
- To describe how the linearised potential equation can be used to derive the following theories
  - subsonic flows– Prandtl-Glauert
  - supersonic - Ackeret

The Full Potential Equation

- Consider inviscid compressible flow over a body in a uniform stream which is isentropic and irrotational. Irrotational flow means the velocity field in terms of velocity potential ϕ

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

$$\mathbf{u} = \nabla \phi$$

- Then to obtain an equation to solve for ϕ, use continuity, momentum (1D Euler) & isentropic speed of sound (**$dp=a^2d\rho$**).

Full potential derivation (not examinable)

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p$$

$$\nabla\cdot\rho\mathbf{u} = 0$$

$$p = k\rho^{\gamma}$$

$$\mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

$$\nabla p = \gamma k\rho^{\gamma-1}\nabla\rho = \frac{\gamma p}{\rho}\nabla\rho = a^2\nabla\rho$$

$$\mathbf{u}\cdot\frac{\nabla\rho}{\rho} + \nabla\cdot\mathbf{u} = 0$$

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -a^2\nabla\rho$$

$$\nabla\phi\cdot\frac{\nabla\rho}{\rho} + \nabla\cdot\nabla\phi = 0$$

$$\frac{\nabla\rho}{\rho} = -\frac{1}{a^2}\mathbf{u}\cdot\nabla\mathbf{u}$$

$$\nabla\cdot\nabla\phi - \frac{1}{a^2}\nabla\phi\cdot((\nabla\nabla\phi)\nabla\phi) = 0$$

Full potential derivation (not examinable)

$$\nabla\nabla\phi = \begin{pmatrix} \frac{\partial^2 \phi}{\partial x^2} & \frac{\partial^2 \phi}{\partial x \partial y} \\ \frac{\partial^2 \phi}{\partial y \partial x} & \frac{\partial^2 \phi}{\partial y^2} \end{pmatrix}$$

Decompose to constant+perturbation

$$\nabla \cdot (\mathbf{V}_\infty + \nabla\phi') - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla\phi') \cdot ((\nabla(\mathbf{V}_\infty + \nabla\phi'))(\mathbf{V}_\infty + \nabla\phi')) = 0$$

Remove gradients of constants

$$\nabla \cdot \nabla\phi' - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla\phi') \cdot ((\nabla\nabla\phi')(\mathbf{V}_\infty + \nabla\phi')) = 0$$

Expand

$$\nabla \cdot \nabla\phi' - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla\phi') \cdot ((\nabla\nabla\phi')\mathbf{V}_\infty + (\nabla\nabla\phi')\nabla\phi') = 0$$

Ignore products of derivatives

$$\nabla \cdot \nabla\phi' - \frac{1}{a^2}\mathbf{V}_\infty^T(\nabla\nabla\phi')\mathbf{V}_\infty = 0$$

$$\mathbf{V}_\infty^T\,(\nabla\nabla\phi')\,\mathbf{V}_\infty = \begin{pmatrix} u_\infty & v_\infty \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi'}{\partial x^2} & \frac{\partial^2 \phi'}{\partial x \partial y} \\ \frac{\partial^2 \phi'}{\partial y \partial x} & \frac{\partial^2 \phi'}{\partial y^2} \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix}$$

Keep only u part

$$\mathbf{V}_\infty^T\,(\nabla\nabla\phi')\,\mathbf{V}_\infty = u_\infty^2 \frac{\partial^2 \phi'}{\partial x^2}$$

Full potential derivation (not examinable)

$$\left(a^2 - u_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2} + a^2\frac{\partial^2\phi'}{\partial y^2} + a^2\frac{\partial^2\phi'}{\partial z^2} = 0$$

Energy equation

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left((\mathbf{V}_\infty + \nabla\phi')\cdot(\mathbf{V}_\infty + \nabla\phi')\right)$$

$$\frac{T_0}{T} = \frac{a_0^{\ 2}}{a^2} = 1 + \frac{\gamma - 1}{2}M^2$$

$$a_0^{\ 2} = a^2 + \frac{\gamma - 1}{2}V^2$$

Ignore gradient products

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}2u_\infty\frac{\partial\phi'}{\partial x}$$

$$a_\infty^2 = a_0^2 - \frac{\gamma-1}{2}(u_\infty^2 + u_\infty^2)$$

Cancel through by a_inf

$$\left(1 - M_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} + \frac{\partial^2\phi'}{\partial z^2} = 0$$

The Full Potential Equation

Unknowns are speed of sound and velocity potential – need a 2nd eq'n

- The equation to solve is given by

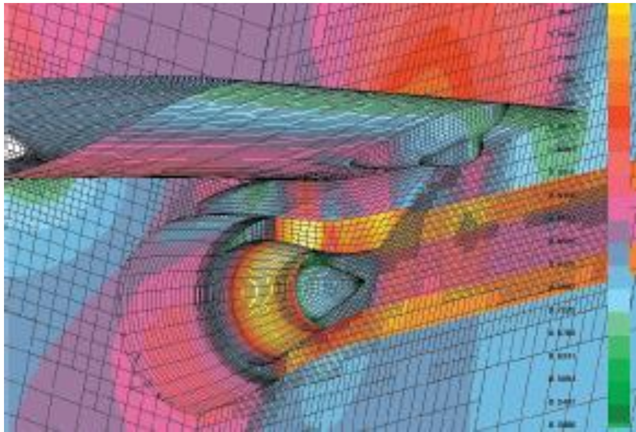
$$\left[a^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-2\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0$$

do not memorise!

- This is a closed form in ϕ *as* a given by

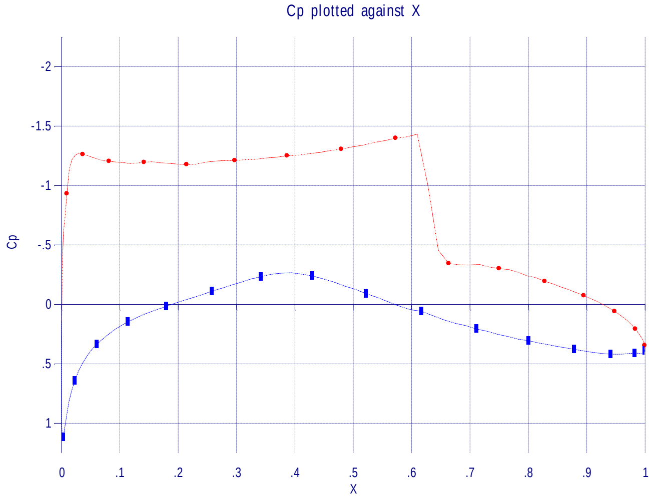
$$a^2=a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

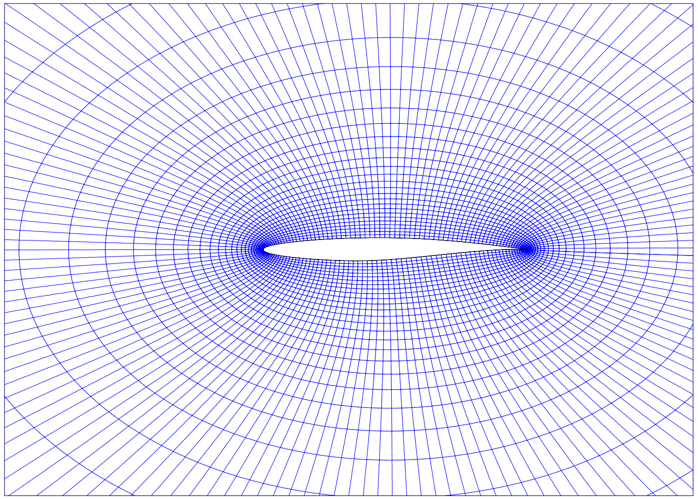
- Where a_0 is a known property of the flow.



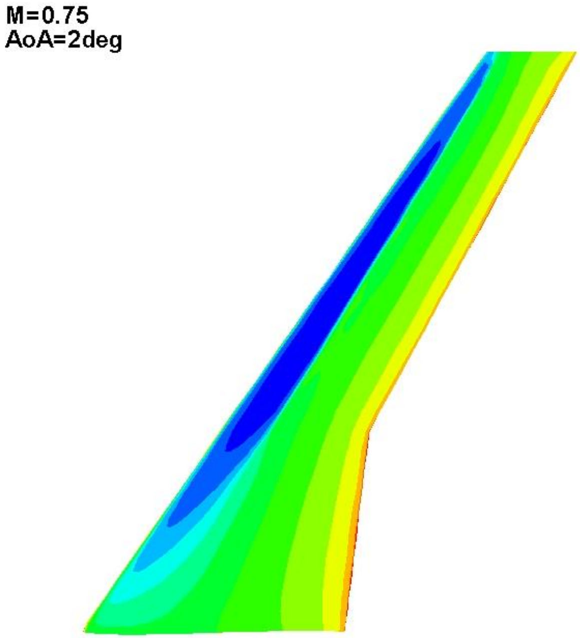
Solving the Full Potential Equation

2D - VGK





3D - FP



VGK and FP available through ESDU

The Full Potential Equation

- Compare this equation to solve for the compressible flow potential ϕ

$$\left[a^2 - \left(\frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} - 2\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

NONLINEAR

- With the equation to solve for the incompressible flow potential ϕ

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

LINEAR

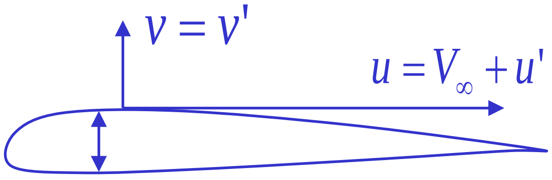
Nonlinear so numerical solutions only and no superposition of solutions. As an alternative to numerical solutions look for algebraic solutions by considering small perturbations to free stream velocity and linearising.

Linearising the Full Potential Equation

- Now assume u and v are **'small'** perturbations to free stream V_∞

$$u = V_\infty + u' = V_\infty + \frac{\partial \phi'}{\partial x} \qquad \phi = V_\infty x + \phi'$$

$$v = v' = \frac{\partial \phi'}{\partial y}$$



- A very good approximation for thin bodies at low α
- Substitute perturbation velocity potential, $\hat{\phi}$ into "velocity potential equation" to give "perturbation velocity potential equation"

$$\left[a^2 - \left(V_\infty + \frac{\partial \phi'}{\partial x}\right)^2\right]\frac{\partial^2 \phi'}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi'}{\partial y}\right)^2\right]\frac{\partial^2 \phi'}{\partial y^2} - 2\left(V_\infty + \frac{\partial \phi'}{\partial x}\right)\left(\frac{\partial \phi'}{\partial y}\right)\frac{\partial^2 \phi'}{\partial x \partial y} = 0$$

Linearising the Full Potential Equation

- For a freestream Mach Number $\boldsymbol{M_\infty}$ **< 0.8** or 1.2<$\boldsymbol{M_\infty}$ **< 5** ($i.e.$ outside transonic region) nonlinear terms negligible. This gives

$$\boxed{\left(1-M_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} \approx 0}$$

Linearised velocity potential eqn.

Note that around M=1, shock waves arise and the flow is no longer isentropic, an assumption made in the derivation of this equation. However can sometimes give reasonable approximations in this flow regime providing shocks not too strong.

  - This equation can be transformed to equivalent incompressible Laplace Equation, for which analytic solutions are possible
  - The form of the required transformation depends on whether M_∞ is subsonic (Prandtl-Glauert) or supersonic (Ackeret)

Linearised Pressure Coefficient

- From the definition of the pressure coefficient we can make approximations similar to those used to derive the linear velocity potential equation (see handout for a reference to derivation). This is needed in subsequent theory.

$$C_p \equiv \frac{p-p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = -\frac{2u'}{V_\infty} + \frac{u'^2+v'^2}{V_\infty^2} + H.O.T.$$

$$\frac{\hat{u}}{V_\infty} < 1,\ \frac{\hat{v}}{V_\infty} < 1$$

$$\frac{\hat{u}^2}{V_\infty^2} << 1,\ \frac{\hat{v}^2}{V_\infty^2} << 1$$

$$C_p = 1 - \frac{v^2}{V_\infty^2} = 1 - \frac{(v'^2 + (V_\infty + u')^2)}{V_\infty^2} \approx -2\frac{u'}{V_\infty}$$

$$\boxed{C_p \approx -\frac{2u'}{V_\infty}}$$

(approach not strictly true for compressible case but result is valid; see next slide for full approach)

Linearising Cp (not examinable)

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

$$T_\infty\left(1 + \frac{\gamma-1}{2}M_\infty^2\right) = T\left(1 + \frac{\gamma-1}{2}M^2\right)$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma-1}{2}\left(M_\infty^2 - \frac{TM^2}{T_\infty}\right) = \frac{\gamma-1}{2a_\infty^2}\left(V_\infty^2 - V^2\right)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2a_\infty^2}\left(V^2 - V_\infty^2\right)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2a_\infty^2}\left((V_\infty + V')^2 - V_\infty^2\right) = 1 - \frac{\gamma-1}{2a_\infty^2}\left(2V_\infty V' + V'^2\right)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2}M_\infty^2\left(2\frac{V'}{V_\infty} + \frac{V'^2}{V_\infty^2}\right) = 1 - \epsilon$$

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p}{p_\infty} = (1-\epsilon)^{\frac{\gamma}{\gamma-1}} \approx 1 - \frac{\gamma}{\gamma-1}\epsilon$$

$$C_p \approx \frac{2}{\gamma M_\infty^2}\left(1 - \frac{\gamma-1}{2}\frac{\gamma}{\gamma-1}M_\infty^2\left(2\frac{V'}{V_\infty} + \frac{V'^2}{V_\infty^2}\right) - 1\right)$$

$$C_p \approx \frac{2}{\gamma M_\infty^2}\left(\frac{1}{2}\frac{\gamma}{1}M_\infty^2\left(2\frac{V'}{V_\infty} + \frac{V'^2}{V_\infty^2}\right)\right)$$

$$\boxed{C_p \approx -2\frac{V'}{V_\infty}}$$

Subsonic Prandtl-Glauert Correction (1)

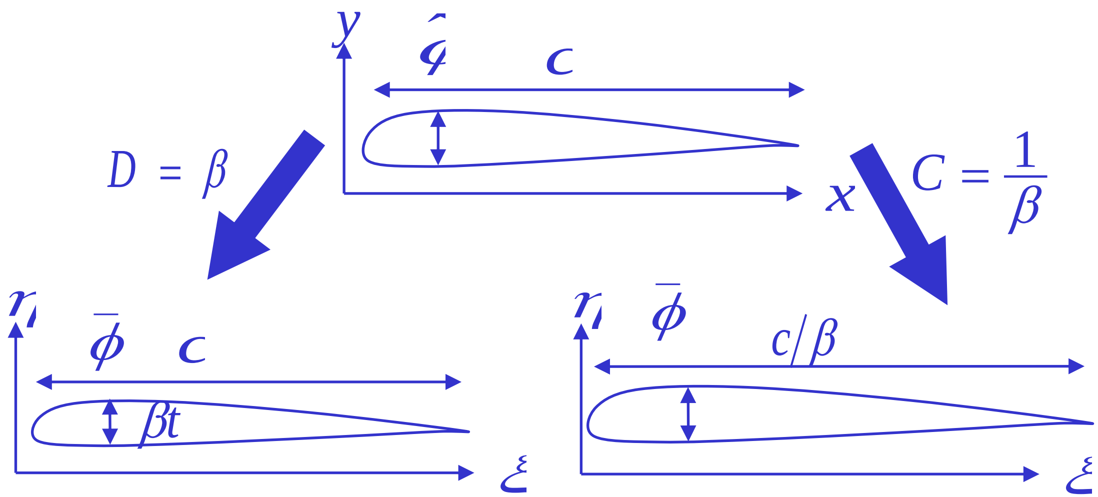
- Consider linearised velocity potential equation and apply coordinate transformation in terms of Glauert Factor β

$$x_{ic} = Cx \qquad \bar{\phi}(x_{ic}, y_{ic}) = \beta\,\phi'(x, y)$$

$$y_{ic} = Dy \qquad \beta = \sqrt{1 - M_\infty^2}$$

- Then there are two possibilities: C=1, D=β *and* C=1/β, D=1 to transform to a Laplace equation (equivalent incompressible flow solution) which is easier to solve

$$\boxed{\left(1 - M_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} \approx 0}$$



$$\frac{\partial \phi}{\partial x_c} = \frac{\partial \phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial x_c} + \frac{\partial \phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial x_c} + \frac{\partial \phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial x_c} = \frac{\partial \phi}{\partial x_{ic}}$$

Choose

$$x_c = x_{ic}$$

$$\beta y_c = y_{ic}$$

$$\beta z_c = z_{ic}$$

$$\frac{\partial \phi}{\partial y_c} = \frac{\partial \phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial y_c} + \frac{\partial \phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial y_c} + \frac{\partial \phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial y_c} = \frac{\partial \phi}{\partial y_{ic}}\beta$$

$$\frac{\partial \phi}{\partial z_c} = \frac{\partial \phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial z_c} + \frac{\partial \phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial z_c} + \frac{\partial \phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial \phi}{\partial z_{ic}}\beta$$

$$\frac{\partial \phi_{x_c}}{\partial x_c} = \frac{\partial \phi_{x_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial x_c} + \frac{\partial \phi_{x_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial x_c} + \frac{\partial \phi_{x_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial x_c} = \frac{\partial^2 \phi}{\partial x_{ic}^2}$$

$$\frac{\partial \phi_{y_c}}{\partial y_c} = \frac{\partial \phi_{y_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial y_c} + \frac{\partial \phi_{y_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial y_c} + \frac{\partial \phi_{y_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial \phi_{y_c}}{\partial y_{ic}}\beta = \frac{\partial^2 \phi}{\partial y_{ic}^2}\beta^2$$

$$\frac{\partial \phi_{z_c}}{\partial z_c} = \frac{\partial \phi_{z_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial z_c} + \frac{\partial \phi_{z_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial z_c} + \frac{\partial \phi_{z_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial \phi_{z_c}}{\partial z_{ic}}\beta = \frac{\partial^2 \phi}{\partial z_{ic}^2}\beta^2$$

Slide not examinable!

$$\boxed{\begin{gathered}\beta^2\frac{\partial^2 \phi}{\partial x_{ic}^2} + \beta^2\frac{\partial^2 \phi}{\partial y_{ic}^2} + \beta^2\frac{\partial^2 \phi}{\partial z_{ic}^2} = 0 \\ \frac{\partial^2 \phi}{\partial x_{ic}^2} + \frac{\partial^2 \phi}{\partial y_{ic}^2} + \frac{\partial^2 \phi}{\partial z_{ic}^2} = 0\end{gathered}}$$

Subsonic Prandtl-Glauert Correction (1)

- In the new coordinate system the new equation to solve is a Laplace equation

$$\frac{\partial^2 \overline{\phi}}{\partial x_{ic}{}^2} + \frac{\partial^2 \overline{\phi}}{\partial y_{ic}{}^2} = 0$$

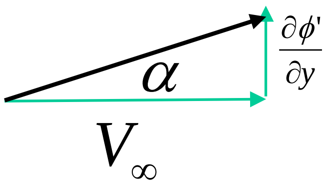
- The pressure coefficient is given by

Remember…

$$u' = \frac{\partial \phi'}{\partial x} = \frac{\partial \left(\frac{\overline{\phi}}{\beta} \right)}{\partial x}$$

$$C_p \approx -\frac{2u'}{V_\infty} = \frac{1}{\beta}\left(-\frac{2\overline{u}}{V_\infty} \right)$$

- Where $\overline{u} = \partial \phi / \partial x_{ic}$ is the solution of Laplace's equation in transformed space i.e. it is equivalent to an incompressible solution



Why did we scale the potential?

Consider the angle of any streamline…

$$\alpha \approx \frac{\frac{\partial \phi'}{\partial y}}{V_\infty} = \beta \frac{\frac{\partial(\bar{\phi}/\beta)}{\partial y_{ic}}}{V_\infty} = \frac{\frac{\partial \bar{\phi}}{\partial y_{ic}}}{V_\infty}$$

Remember…

$$\beta\phi' = \bar{\phi}$$

$$x_{ic} = x$$

$$y_{ic} = \beta y$$

The y-scaling reduces the angle, hence beta here

The potential scaling cancels it again here

This means the angle of attack is the same in both cases, and therefore the shape is the same in both cases (because a streamline runs along the surface).

Note that this is not **quite how it is done in 3D**, although the working is nearly the same. **See year 3**.

Subsonic Prandtl-Glauert Correction (1)

$$C_p \approx -\frac{2u'}{V_\infty} = \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$$

- The term in brackets in the Cp equation is the linearised incompressible pressure coefficient

$$C_p = 1 - \frac{v^2}{V_\infty^2} = 1 - \frac{(v'^2 + (V_\infty + u')^2)}{V_\infty^2} = 1 - \frac{(v'^2 + V_\infty{}^2 + 2V_\infty u' + u'^2)}{V_\infty^2} \approx -2\frac{u'}{V_\infty}$$

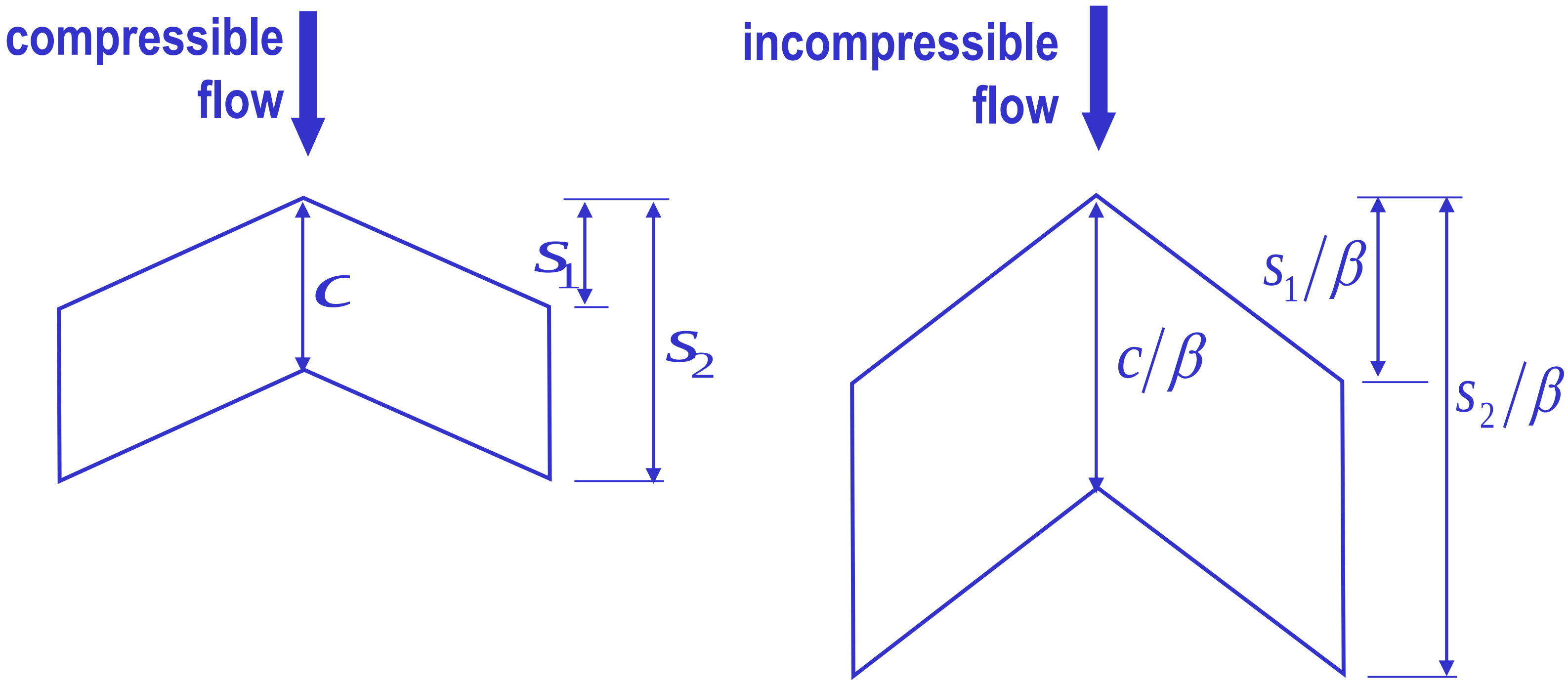
Call this incompressible pressure coefficient $\overline{C}_{p,0}$

- Hence

$$\boxed{C_p \approx = \frac{1}{\beta}\overline{C}_{p,0}}$$

Subsonic Prandtl-Glauert Correction (2)

- Thus a compressible flow over an aerofoil of thickness ratio t/c has the <u>same scaled **pressure distribution**</u> as incompressible flow over an aerofoil with thickness ratio $(t/c)_0$ where $\boxed{(t/c) = (t/c)_0/\beta}$



same scaled pressure distribution

$$\boxed{C_p = fn\left(\frac{(t/c)}{\beta}\right)}$$

Subsonic Prandtl-Glauert Correction (3)

- An alternative interpretation is to relate pressure (and hence lift and pitching moment) to incompressible values for *same* aerofoil **geometry**

$$\boxed{C_p = \frac{C_{p0}}{\beta}}$$

$$\boxed{c_l = \frac{c_{l0}}{\beta}, \quad c_m = \frac{c_{m0}}{\beta}}$$

C_p=compressible
C_{p0}=incompressible
about same geometry

- This indicates that incompressible data can be corrected and used to give a good first guess in design.

Note again – it is only possible to relate compressible/incompressible on the same geometry for any Mach number in 2D. This no longer holds in 3D.

Subsonic Prandtl-Glauert Correction (3)

- Due to the link between Incompressible analysis and Prandtl-Glauert correction, they have common features
  - Predicted drag (=0) (D'Alembert's paradox)
  - aerodynamic centre (at 0.25c) unaffected

- Prandtl-Glauert not the only correction factor

  e.g. Karmen-Tsien

$$C_p = \frac{C_{p0}}{\beta + \frac{C_{p0} M_\infty^2}{2(1+\beta)}}$$

This is better for unsteady

Note that P-G & K-T corrections are good for external aero. Other corrections are used for internal flows

Supersonic Linearised Ackeret Theory (1)

- Consider again the linearised velocity potential equation, but now for $M_\infty > 1$ no upstream influence. Remember thickness and α must be small.
- Then in this case the pressure coefficient can be related directly to local surface inclination θ in radians (positive 'into flow') by

$$\frac{dp}{d\theta} = \frac{\gamma M^2}{\sqrt{M^2-1}}\,p$$

Assume p=p_inf, then integrate wrt angle

$$p - p_\infty = \frac{\gamma M^2\theta}{\sqrt{M^2-1}}\,p_\infty$$

$$\frac{p}{p_\infty} - 1 = \frac{\gamma M^2\theta}{\sqrt{M^2-1}}$$

Multiply by 2/(gam*M^2)

$$\longrightarrow \boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2-1}}}$$

(-ve sign dropped from last lecture; angle positive 'into' the flow)

Example

- Shock, M=2, deflection =4 degrees

Oblique shock theory

$$C_p = \frac{2}{\gamma M_\infty{}^2}\left(\frac{p_2}{p_1}\frac{p_1}{p_\infty} - 1\right) = \frac{2}{1.4\times 2^2}\left(1.247\times 1 - 1\right) = 0.0882$$

Ackeret's Method

$$C_p \approx \frac{2\times 4\times \dfrac{\pi}{180}}{\sqrt{2^2 - 1}} = 0.0806$$

ANGLE IN RADIANS!

8.7% error - fairly small, especially at low angles!

Supersonic Linearised Ackeret Theory (2)

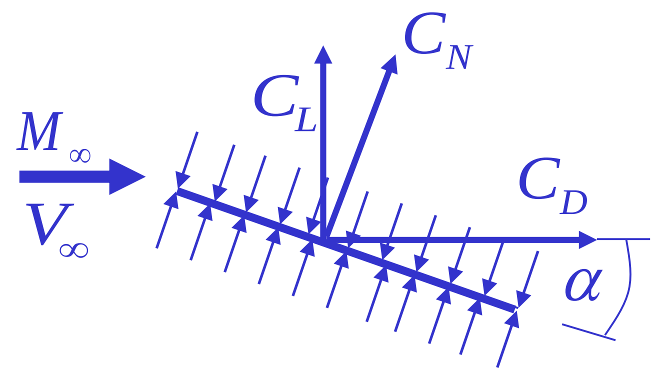
- Applying this to a *flat plate* 2D aerofoil at angle of attack α, this gives a *constant* pressure distribution

$$C_{p_{\text{lower}}} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}, \quad C_{p_{\text{upper}}} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}}, \quad c_N = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

- The lift and lift-dependent drag are

$$c_l = c_N \cos(\alpha) \approx c_N = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$c_d = c_N \sin(\alpha) \approx c_N \alpha = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$$



- Supersonic flow Drag ≠0 ("wave drag")
- Gives centre of pressure at 0.5c (0.25c for incompressible)

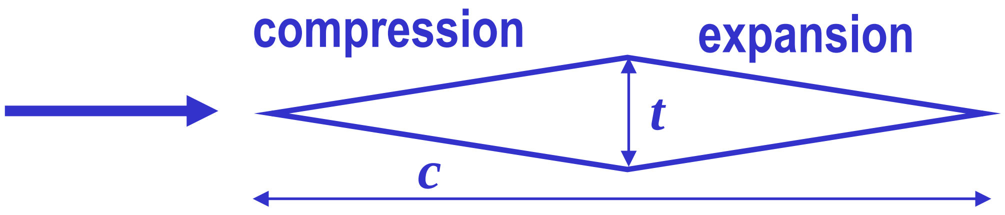
$$\boxed{c_d = \frac{\sqrt{M_\infty^2 - 1}}{4} c_l^2}$$

General approximate result for wave drag, we can use for most aerofoils at low α

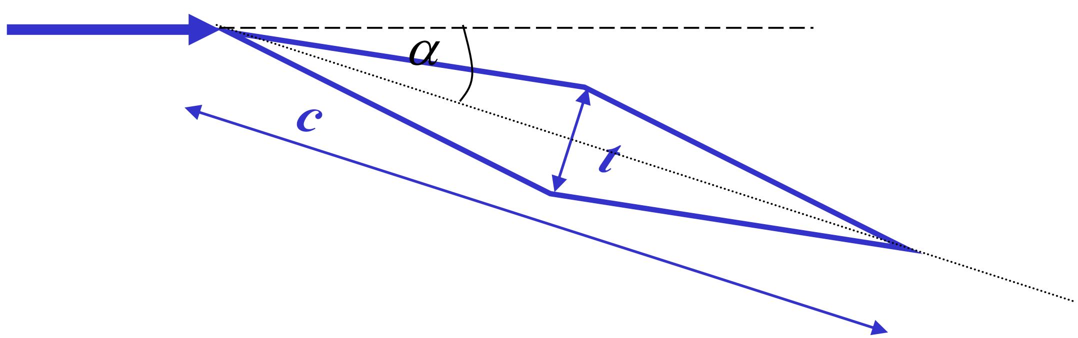
Supersonic Linearised Ackeret Theory (3)

- For a thin 'double wedge' high speed section, linearised Ackeret theory gives

$$\boxed{c_d = 4\frac{(t/c)^2}{\sqrt{M_\infty^2 - 1}}}$$



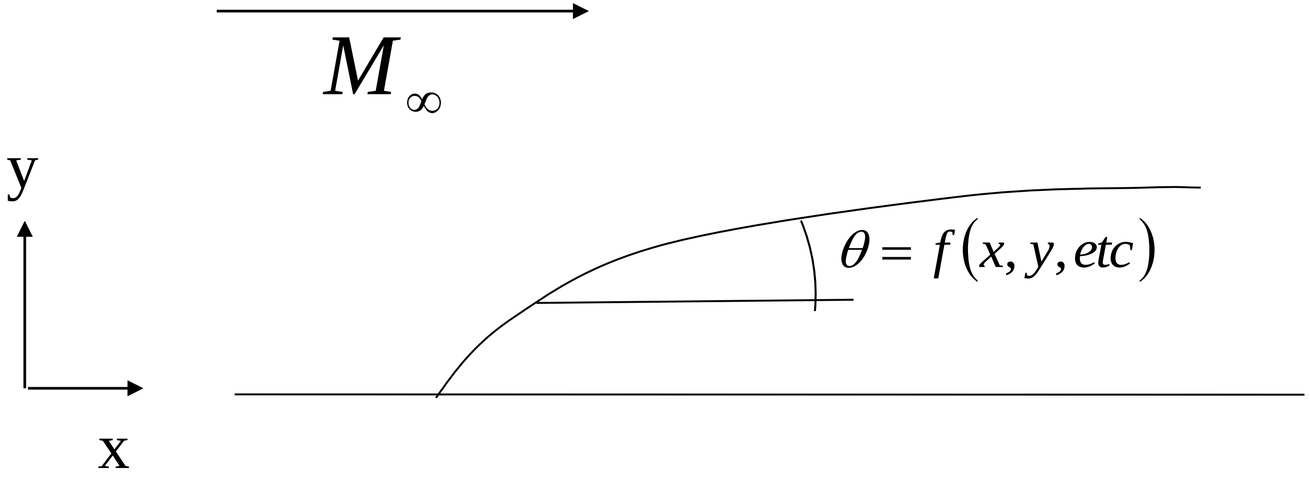
$$\boxed{c_d = 4\frac{\alpha^2 + (t/c)^2}{\sqrt{M_\infty^2 - 1}}}$$



- Note that there are now two components to the wave drag, one due to incidence and one due to thickness

Why use Ackeret?

- Convenient simplification for curved surfaces, especially where the angle may be a function of other variables
- In this case an integral is needed for Cl or Cd (taking appropriate force component for each one)

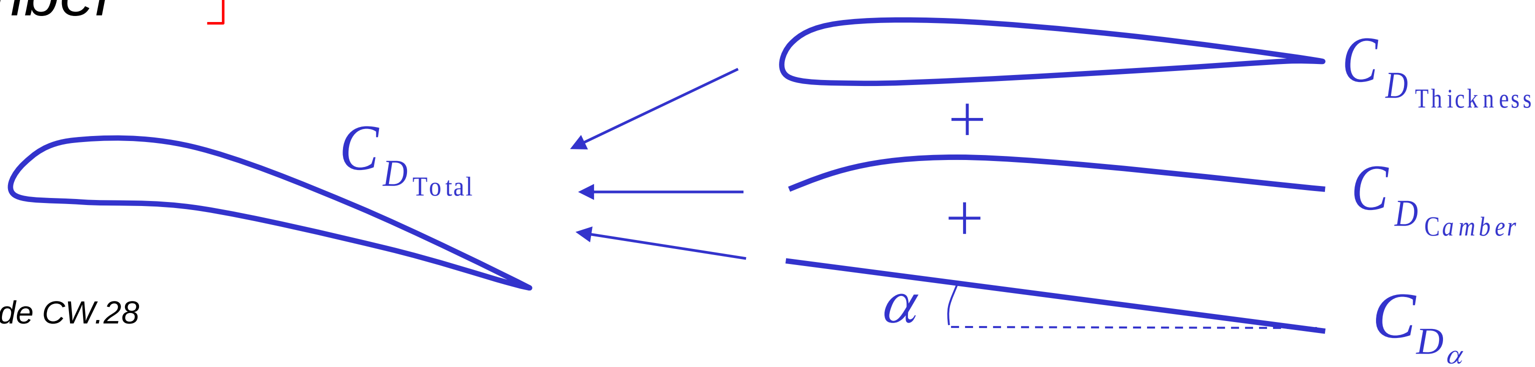


Wave Drag

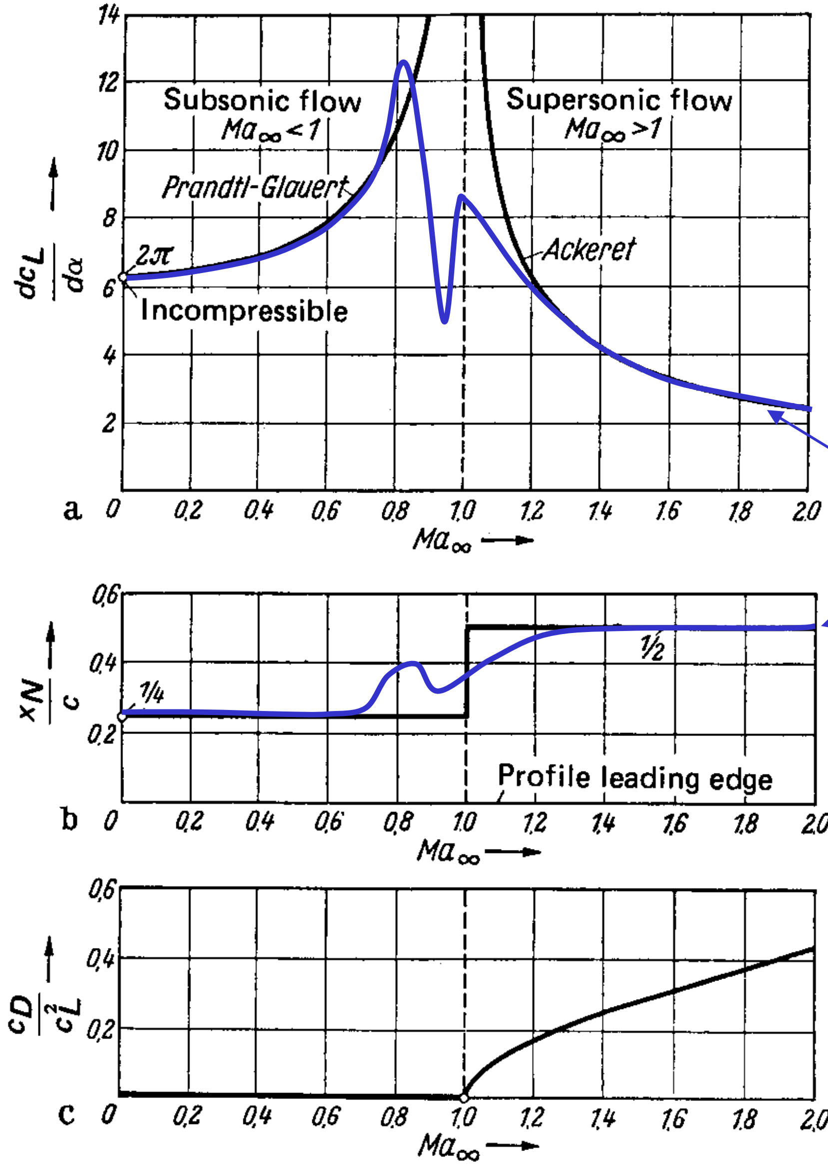
- Energy is lost to wave system shed by aerofoil
  - hence ‘wave drag’
  - occurs even in isentropic flow
- Previous two examples of the application of Ackeret theory show components due to incidence (which is lift dependent) and thickness. In general, wave drag has three components, due to:
  - *lift*
  - *thickness*
  - *camber*

present at zero lift conditions

build up drag value by components



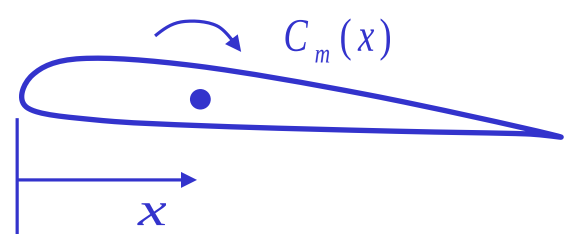
Linearised 2D Aerofoil Characteristics



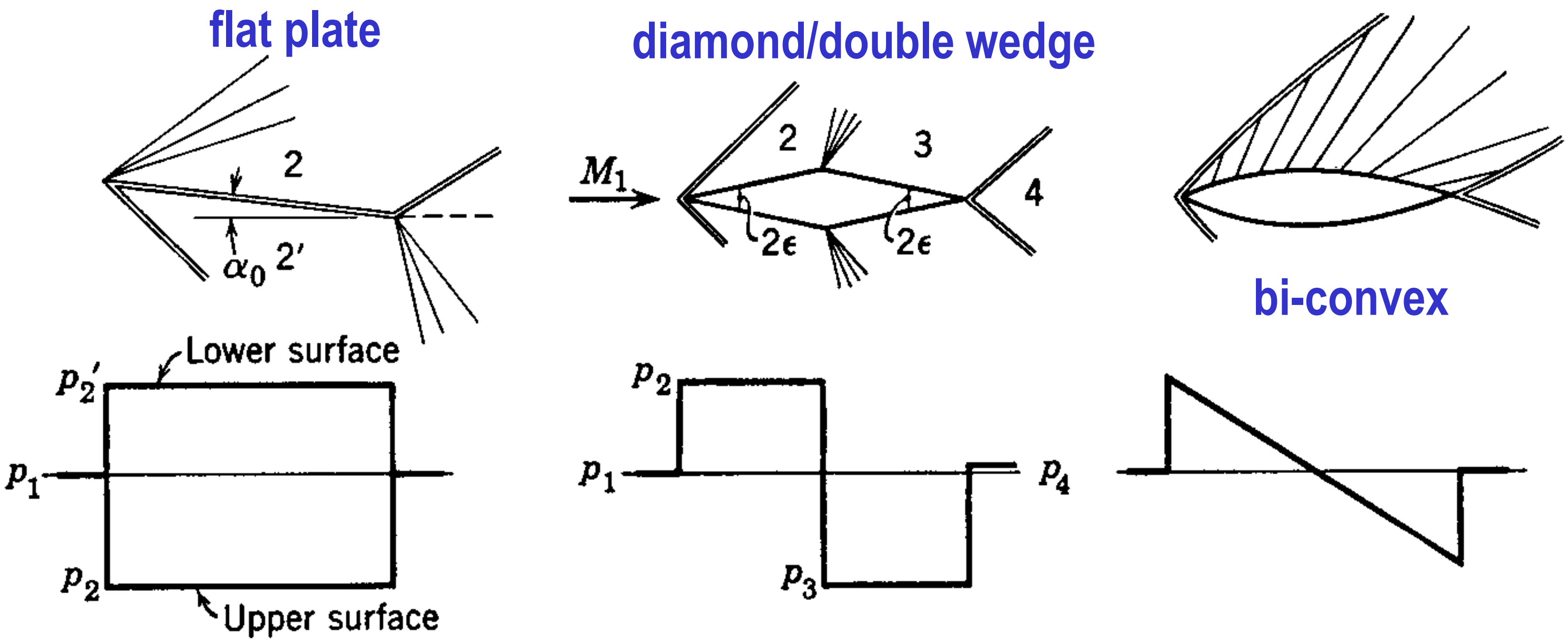
typical values

x_N=aerodynamic center

$$\frac{dC_m(x)}{d\alpha} = 0 \quad @\ x = x_N$$



‘Real’ Supersonic Aerofoil Flows



- NB modelling shock waves is not theoretically possible using linearised small perturbation theory…
  - but using the analysis for flows with shocks shows rather small errors at low α
  - wave drag is mostly due to energy ‘lost’ in expansion and compression waves (as opposed to total pressure loss through oblique shocks)

Revision Objectives

You should be able to

- Explain the assumptions made to derive the linearised velocity potential equation for compressible flow and state that equation
- Explain the link between compressible and incompressible flow for subsonic Prandtl-Glauert
- Apply the Prandtl-Glauert correction to model problems
- To quote the pressure coefficient for supersonic Ackeret theory
- State the three components of wave drag
- Sketch typical flow patterns on real supersonic aerofoils

Aerofoils in Compressible Flow (2)

Aerodynamics 2
AENG21100

Department of Aerospace Engineering
University of Bristol

Today

- To explain transonic 2D aerofoil behaviour
- To introduce the concept of critical Mach Number and how to calculate it

Shocks

Shocks

Shocks


easyJet

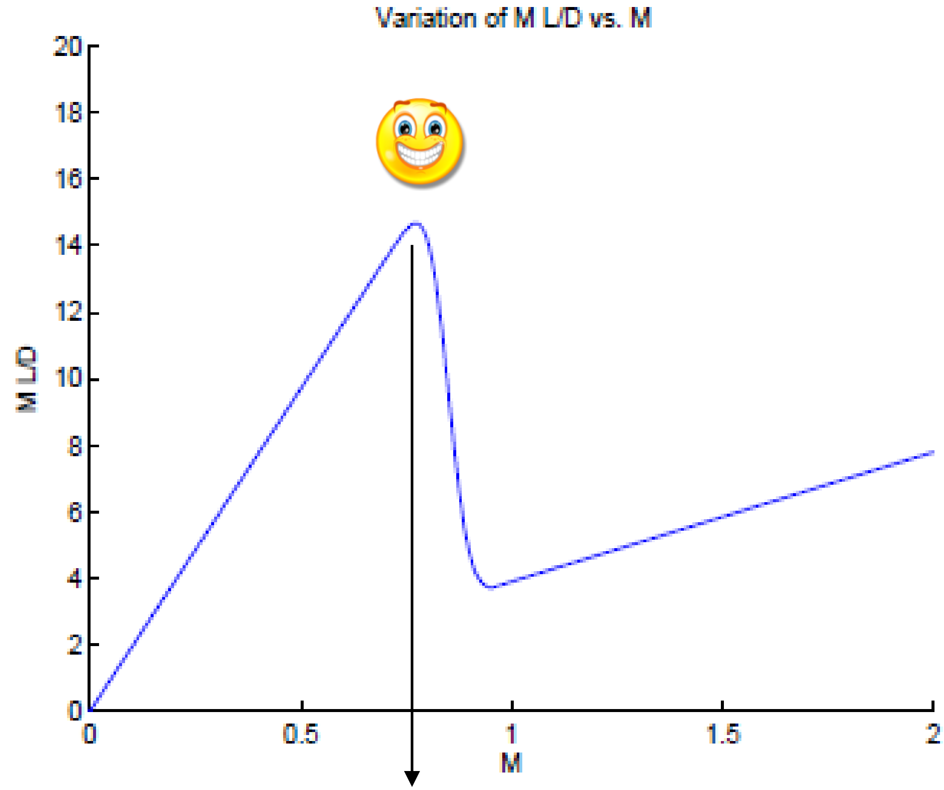
Shock Spotting

- Wait for clear, bright sunlight
- Best from directly above, so close to midday
- Keep looking. Depending on aircraft weight and Mach number, the shock will move in and out of the light (forwards with lower weight or lower Mach number)
- Turbulence makes them more obvious as they move back and forth a few cm. The incidence is changing in the gusts, altering lift coefficient and shock position
- If you get a good photo, send it to me!

Why Transonic?

Aircraft tend to be designed to maximise range, which depends on speed*L/D (from the Breguet range equation)

$$R = \frac{V}{sfc}\frac{L}{D}log(\frac{W_o + W_f}{W_o})$$



Transonic

2D Transonic Aerofoils-critical Mach Number

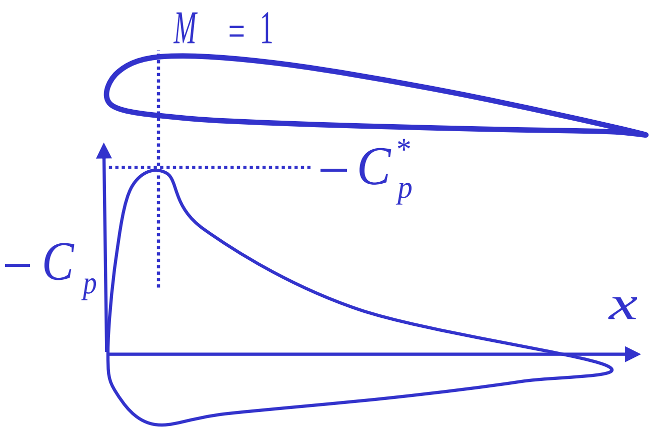
- Transonic behaviour begins when sonic flow first appears on the aerofoil surface
- Identification of sonic point important in aerofoil design
- Assuming flow is isentropic (P_0 constant) up to the sonic point
  - Use compressible C_p equation – isentropic flow relation for pressure
  - Set local $M=1$ to get critical pressure coefficient C_p*
- $M_\infty = M_{cr}$ when minimum pressure coefficient $C_{pmin} = C_p*$
- Still need to identify C_{pmin} for each aerofoil…experiment or a panel method might be used

2D Transonic Aerofoils

Critical Mach Number

Mach 0.8 to 1.2
mixed subsonic & supersonic flows

$$C_p = \left(\frac{p/p_\infty - 1}{\tfrac{1}{2}\gamma M_\infty^2} \right) \qquad \frac{p}{p_\infty} = \left(\frac{p}{p_{0\infty}} \right) \bigg/ \left(\frac{p_\infty}{p_{0\infty}} \right)$$



$$\frac{p}{p_\infty} = \frac{\left(1 + \frac{\gamma - 1}{2} 1^2\right)^{\frac{-\gamma}{\gamma-1}}}{\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{-\gamma}{\gamma-1}}} = \frac{\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma}{\gamma-1}}}$$

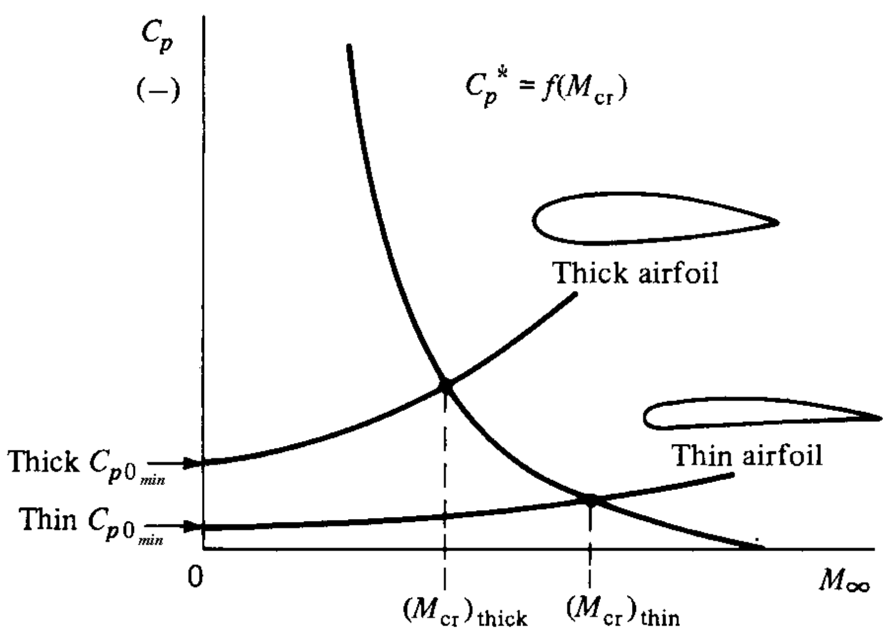
$$C_p^* = \frac{2}{\gamma M_\infty^2} \left\{ \left(\frac{1 + 0.5(\gamma - 1) M_\infty^2}{1 + 0.5(\gamma - 1)} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right\} = \mathrm{f}(M_\infty)$$

Some kind of corrected Cp – Prandtl-Glauert!

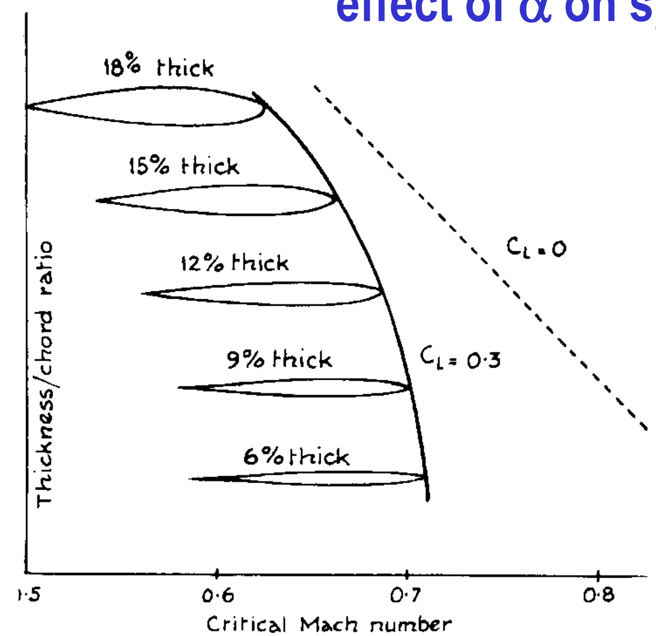
Don’t try to solve analytically!

2D Transonic Aerofoils
Critical Mach Number

- Highest local velocities correspond to minimum C_p (suction peak)
  - depends on section and angle of attack
  - generally scales directly with t/c
  - but does not necessarily occur at maximum thickness
- Approximate C_p in terms of incompressible pressure distribution C_{p0} scaled with M_∞ by Prandtl-Glauert correction



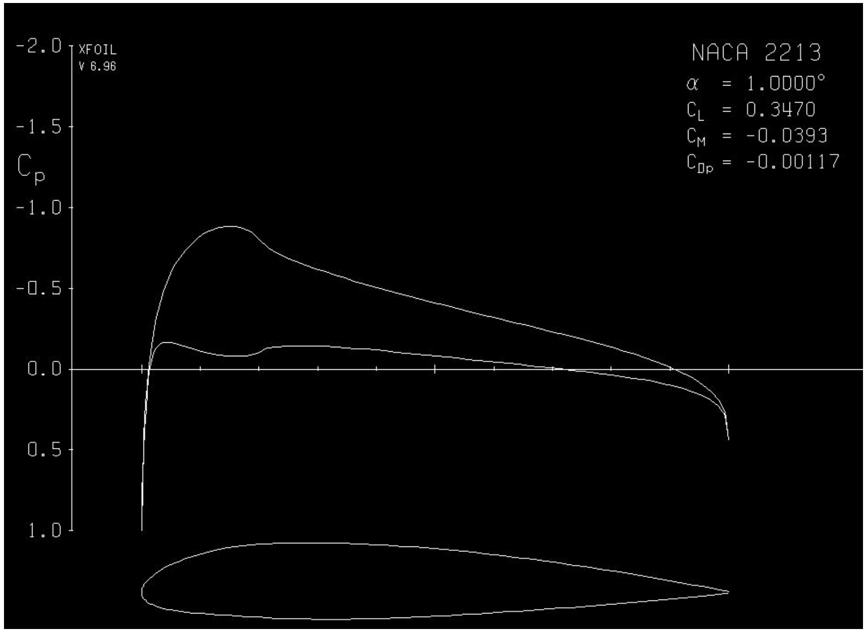
effect of α on symmetric aerofoil

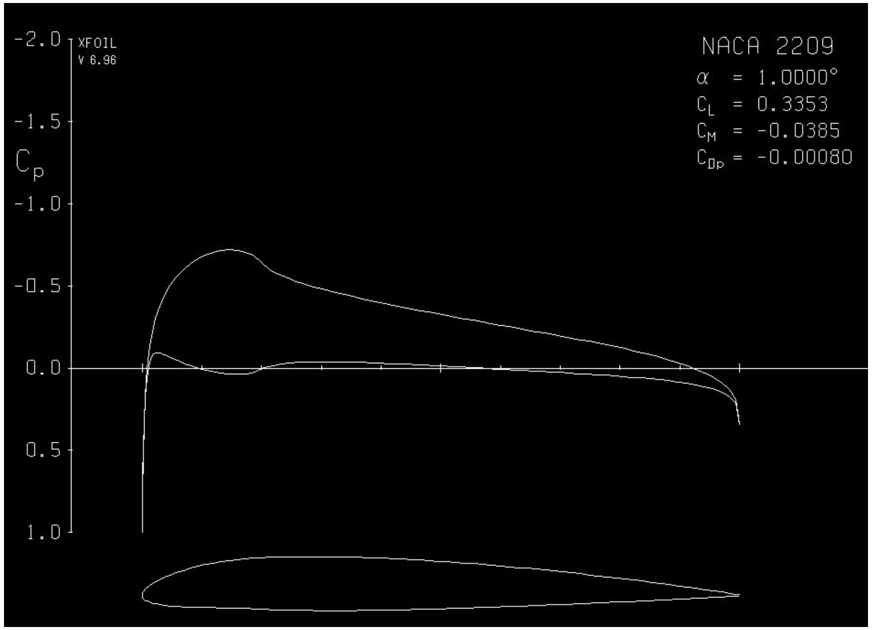


as α increases suction peak ($-C_p$) increases, so critical M reduces. General relationship in terms of C_l

Example – Spitfire Mcrit

Root - NACA 2213, tip NACA 2209





(Xfoil output)

Max value of U/Uinf=1.37, so Cpmin=-0.88 for AoA=1deg

At any AoA, most +ve Cpmin=-0.55

So…

How to…

- Having seen the equations, this is the quick way using tables…

Look up this ratio from tables

$$C_p^{\ *} = \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p}{p_{0\infty}}\frac{p_{0\infty}}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p}{p_{0\infty}}\frac{p_{0\infty}}{p_\infty} - 1\right) = \boxed{\frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{1}{1.895}\frac{p_{0\infty}}{p_\infty} - 1\right)}$$

$$\boxed{C_p^{\ PG} = \frac{C_{p\,\min}}{\sqrt{1 - M_\infty^{\ 2}}}}$$

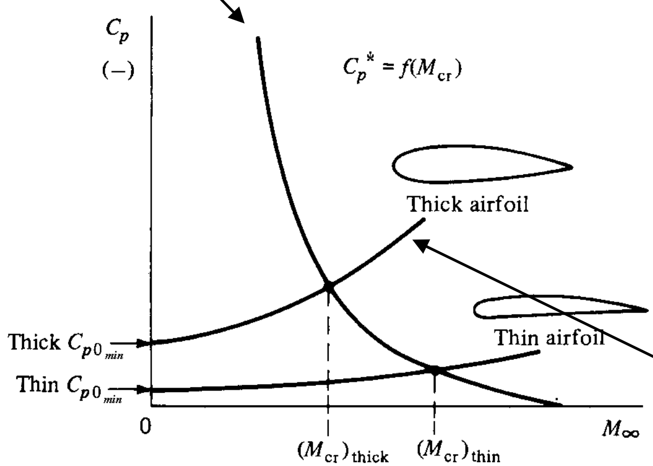
Both are easy to work out, so now we just need a method to solve iteratively – bisection is an easy option

At M=0.7, Cp*=(2/(1.4*0.7^2))*(1.388/1.895-1)=-0.78
CpPG=-0.88/sqrt(1-0.7^2)=-1.23

Some bisection for Cpmin=-0.88

M lower	M upper	M middle	Cp PG	Cp*	
0.6000	0.8000	0.7000	-1.2322	-0.7782	Make upper
0.6000	0.7000	0.6500	-1.1580	-1.0074	Make upper
0.6000	0.6500	0.6250	-1.1273	-1.1420	Make lower
0.6250	0.6500	0.6375	-1.1422	-1.0728	Make upper
0.6250	0.6375	**0.6313**	-1.1346	-1.1069	

Cp*



Mcrit

CpPG

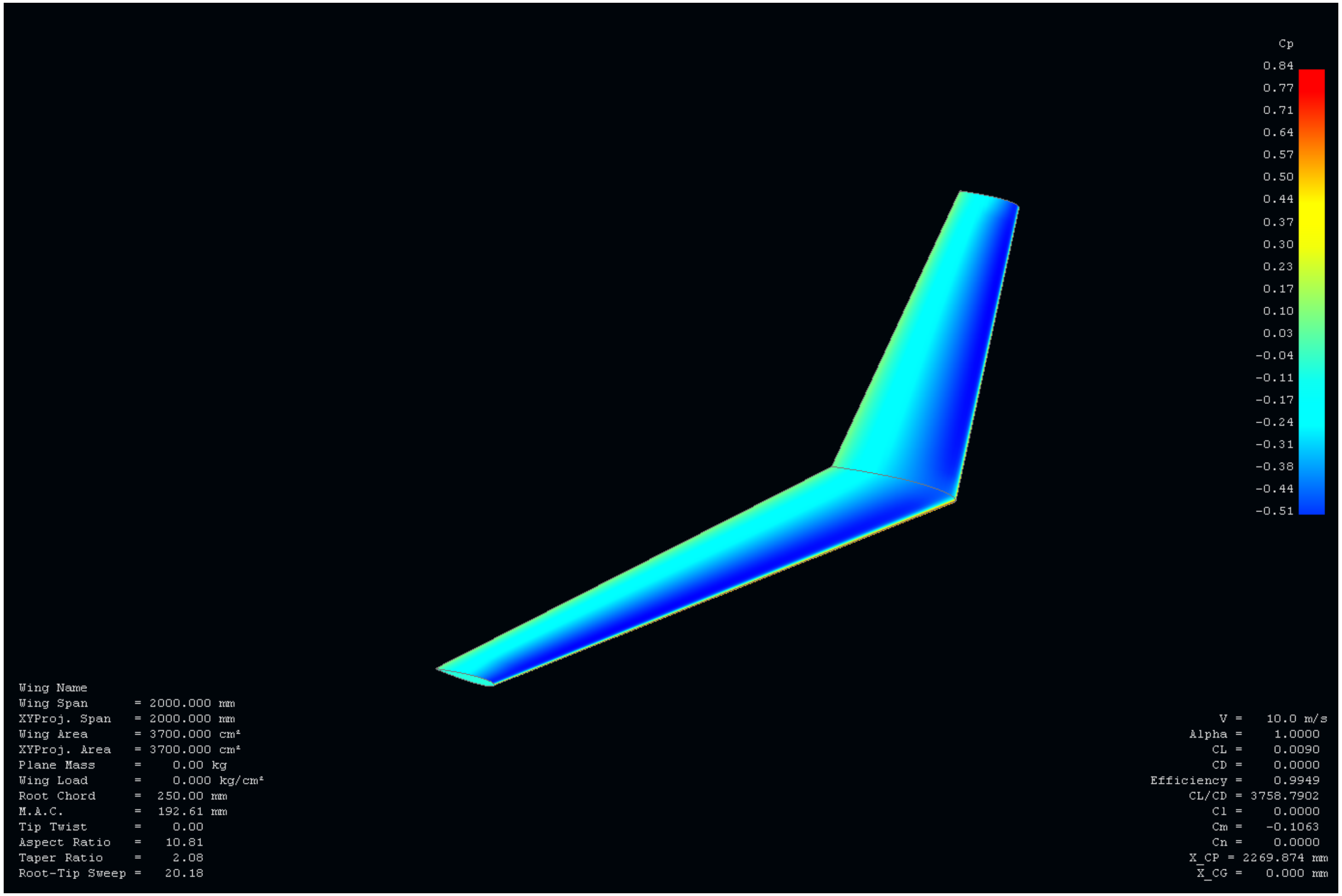
Increases to M=0.702 in the best case (Cpmin=-0.55)

Is it accurate?

- No 3D effects here, but would be included if Cpmin came from a 3D set of data
- The Cp here was from a 2D incompressible result, the 3D one would probably be less –ve, thereby raising Mcrit. Highest ever recorded Mach no. for the Spitfire was 0.89, 27% over our Mcrit
- Depends on incidence – I chose 1deg to run in Xfoil. If the pilot pulls harder Mcrit drops slightly. Highest possible Mcrit value occurs when the minimum Cp anywhere on the wing is maximised (made more positive)

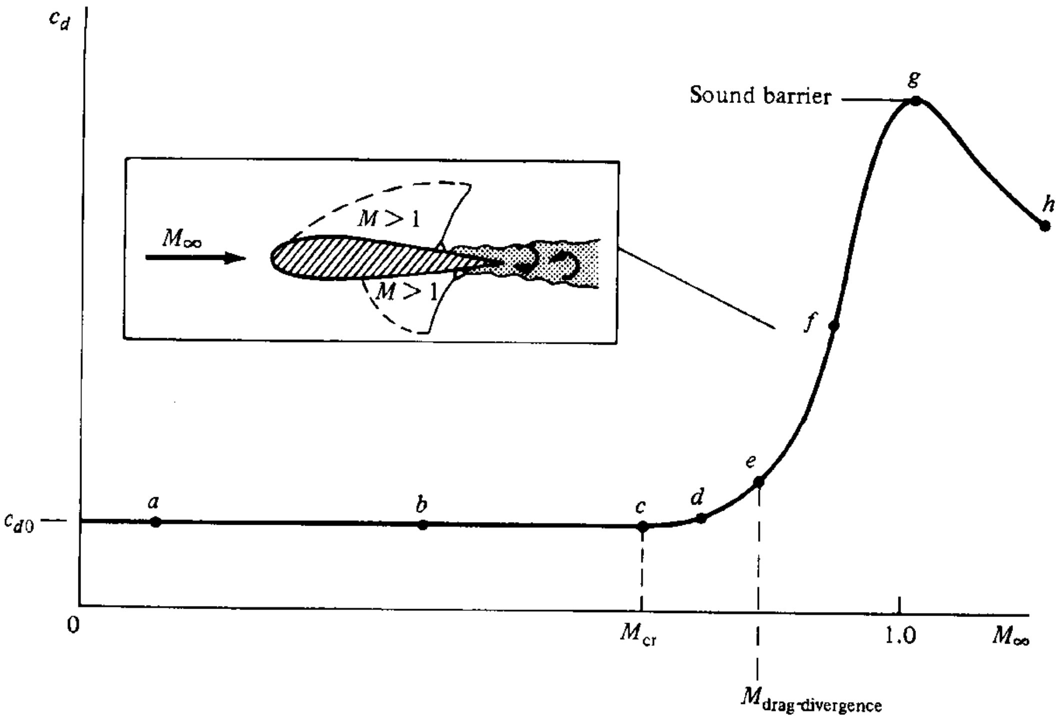
3D

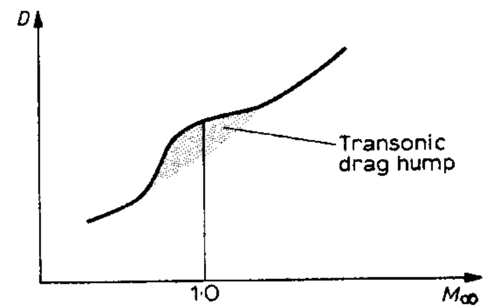
AoA=1deg, Cpmin=-0.51, Mcrit=0.713

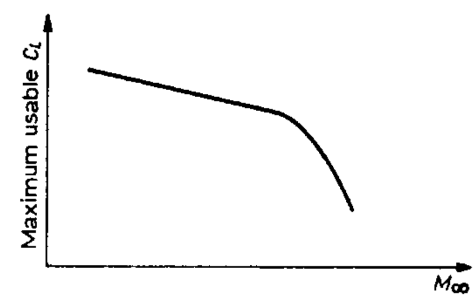


Above M_{cr} OR why is M_{CR} so important

- 'drag divergence': rapid increase in drag just above M_{cr}
- 'shock stall' : M_{cr} decreases with α, above M_{cr} shocks form and separation induced at shock attachment point







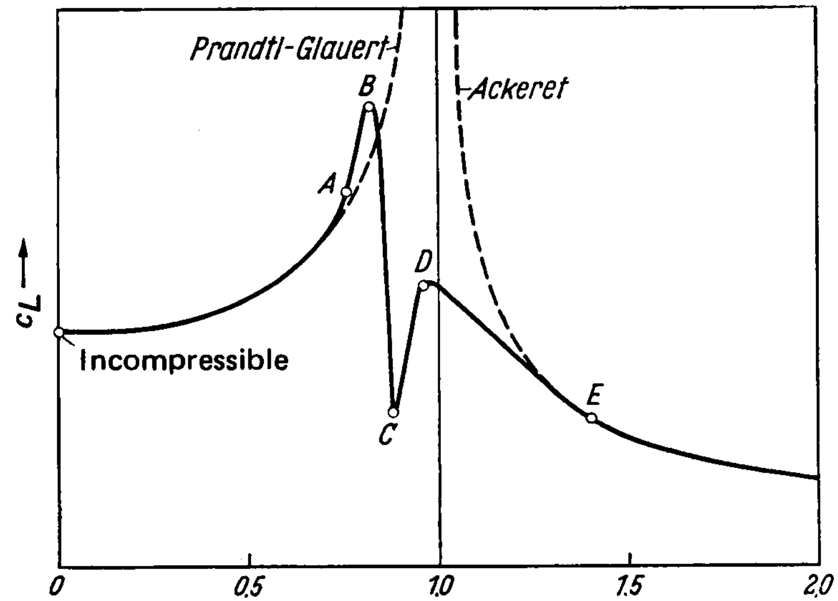
- Aerodynamic centre moves aft: increased stability – but potentially a strong nose down pitching moment

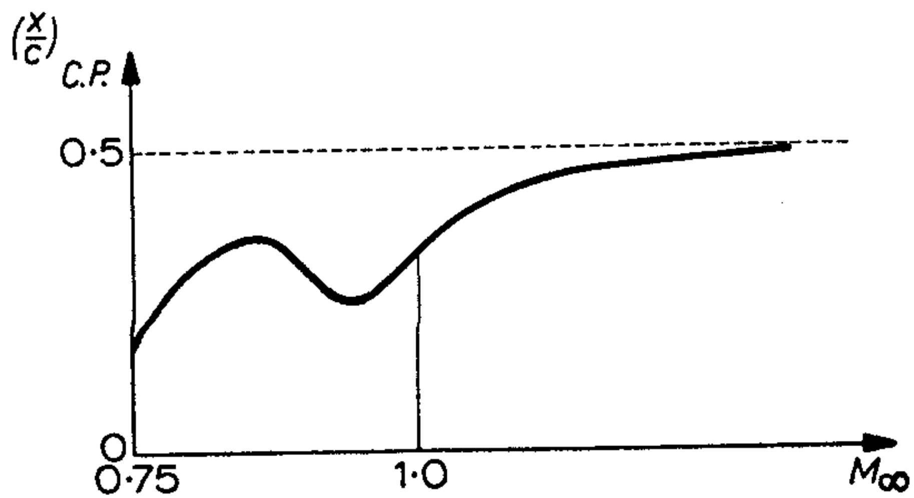
Transonic 2D Aerodynamic Characteristics

Transonic Vs Linearised

- complex lift behaviour
  coefficient
- drag peak near M=1
- centre of pressure and

aerodynamic centre shift

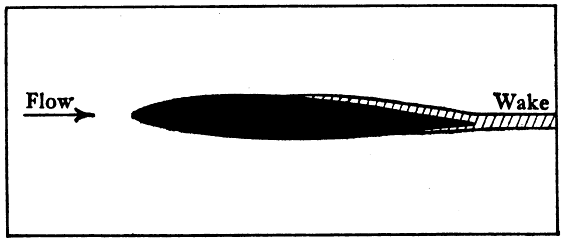


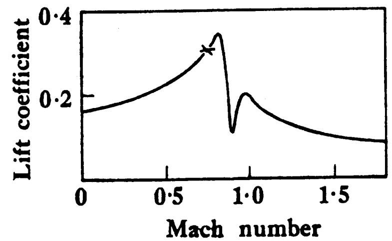


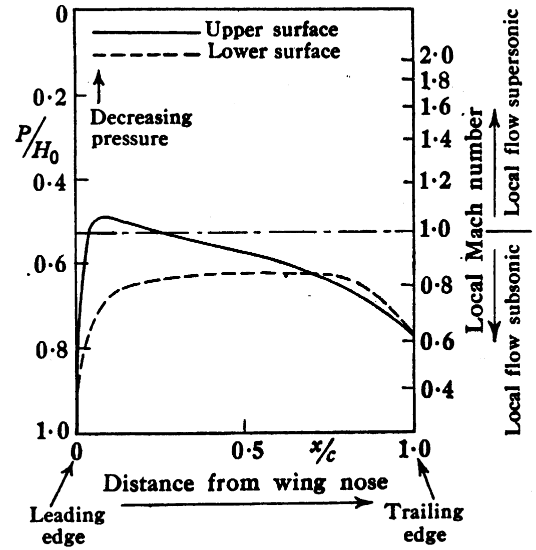
Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Subsonic

Typical behaviour, of aerofoil as Mach No increases.

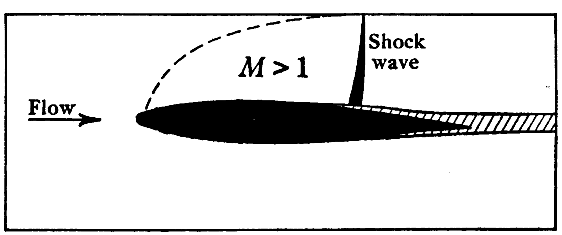
$M_1 = 0\cdot75$

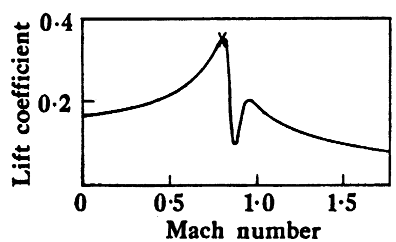


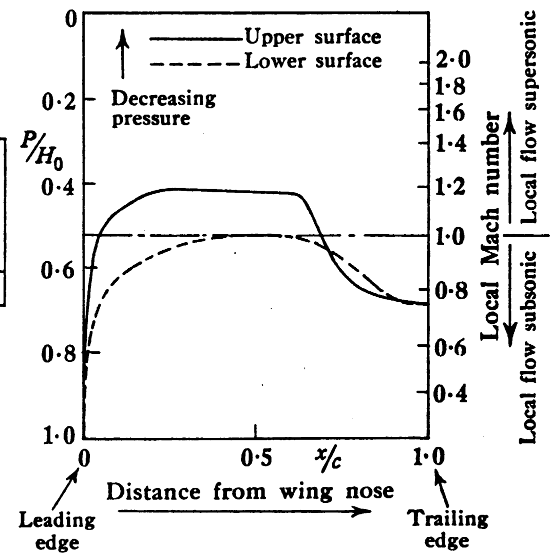




$M_1 = 0\cdot81$

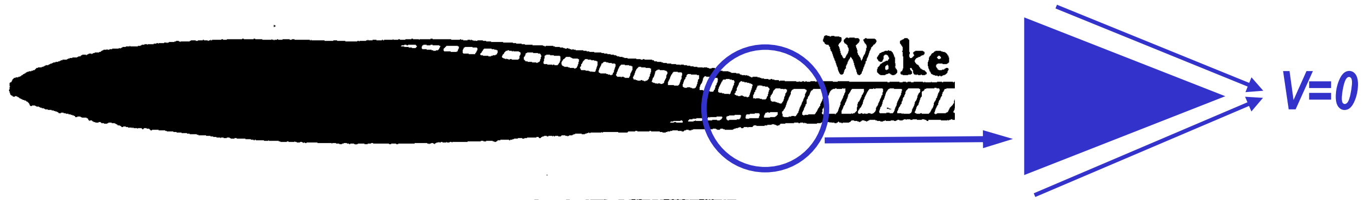




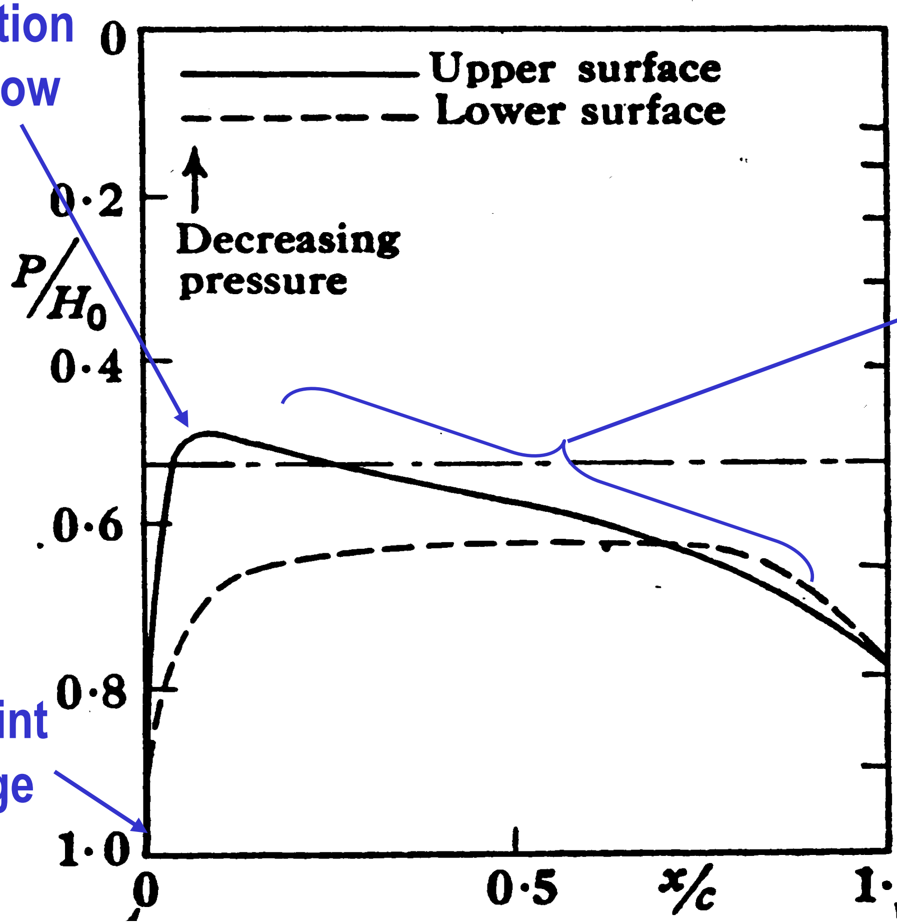


Transonic 2D Aerodynamic Characteristics

Transonic Aerofoil Flows - Subsonic



Significant suction & supersonic flow



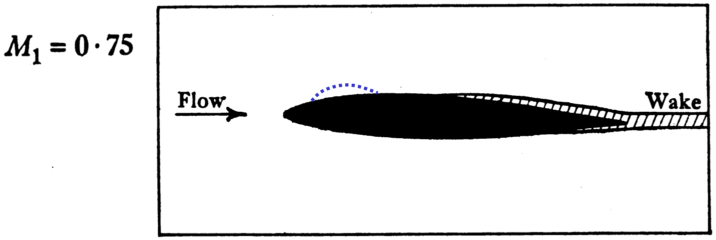
Stagnation point at leading edge

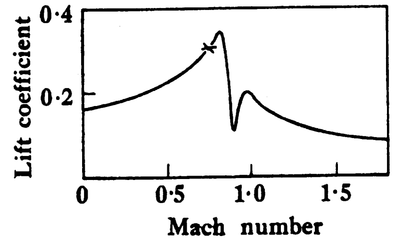
As peak suction increases, large compression required, leads to shock formation

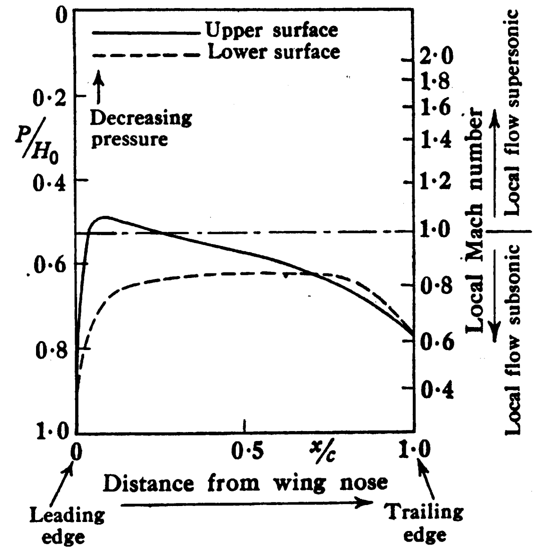
Kutta condition implies zero velocity at trailing edge (stagnation point) for incompressible.

Compressible trailing edge has high static pressure & subsonic velocities in the wake. Upper & lower surface pressures equal.

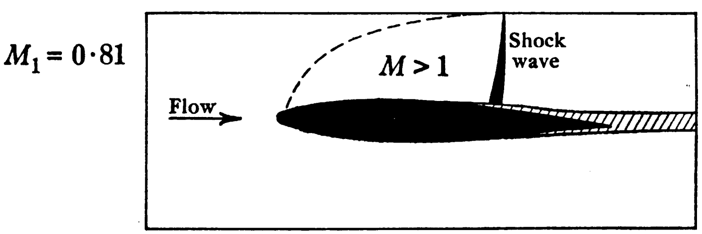
Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Subsonic

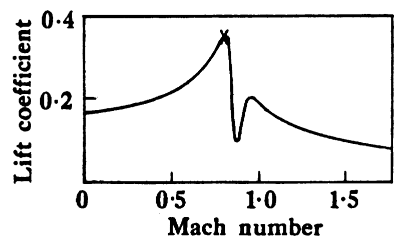


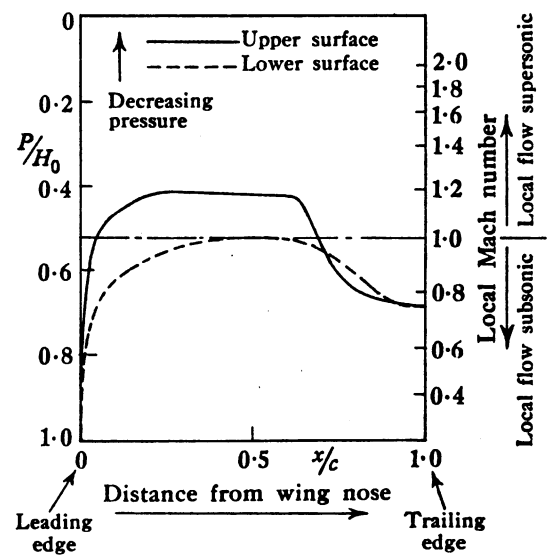




$M > M_{CR}$ →small region of supersonic flow.
Flow is still isentropic but starts to diverge from
P-G solutions





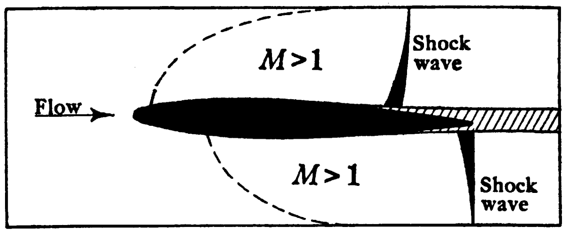


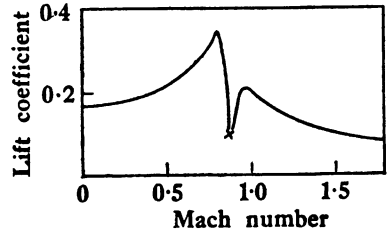
i) supersonic region terminates with a normal shock.
ii) Aero centre moves aft
iii) Drag rise due to losses in P_o through shock

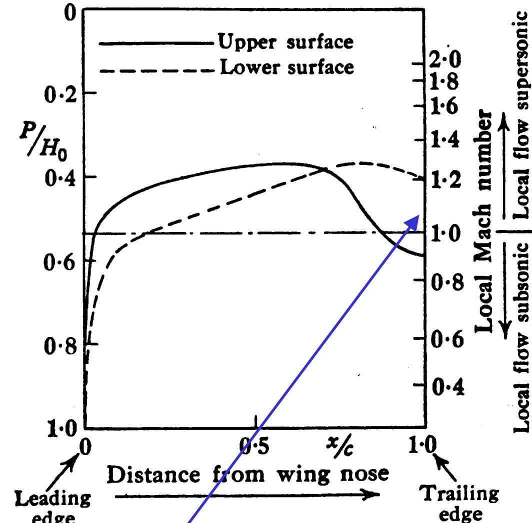
Transonic 2D Aerodynamic Characteristics

Transonic Aerofoil Flows - Transonic

$M_1 = 0{\cdot}89$







i) Supersonic flow on lower surface. Extent of supersonic region increases rapidly (flat surface).

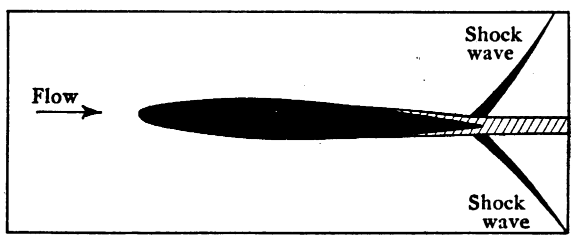
ii) Drag continues to rise.

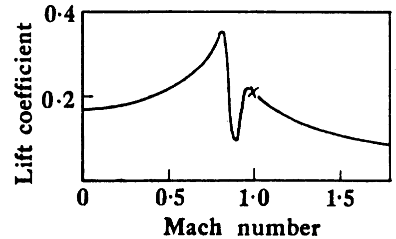
iii) Large lift loss

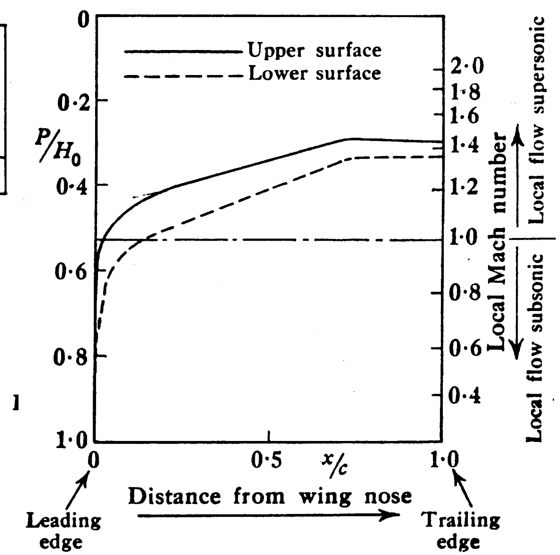
iv) Aero centre moves forward

Pressure adjusts in the wake

$M_1 = 0{\cdot}98$







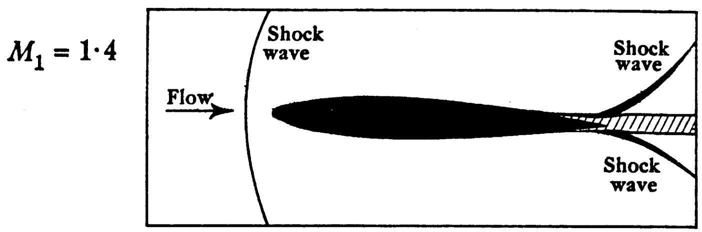
i) Shocks reach trailing edge (terminating shock system).

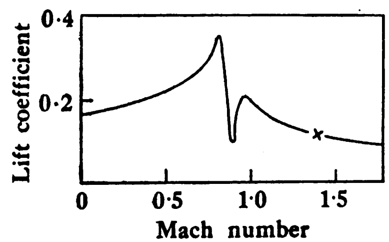
ii) Aero centre moves aft again

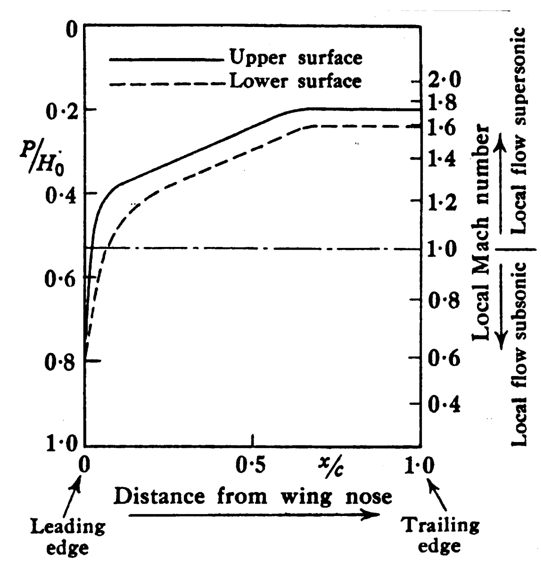
iii) Drag continues to rise.

iii) lift recovers

Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Supersonic



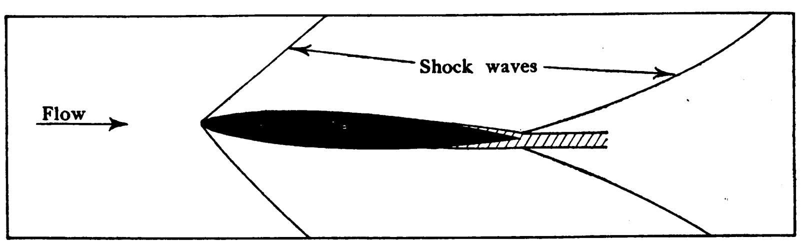




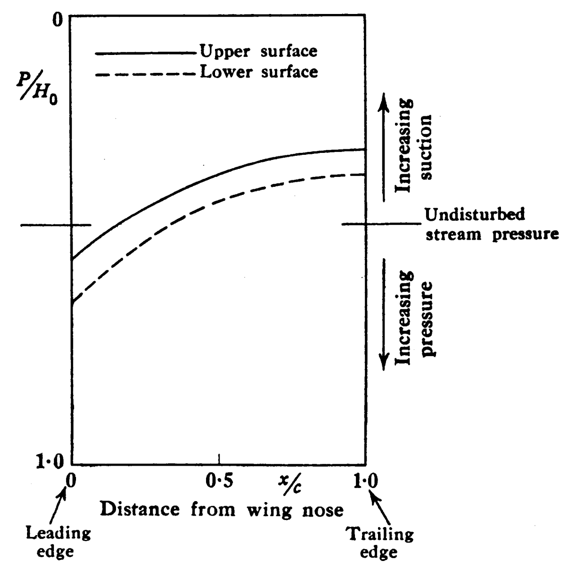
i) Bow shock forms

ii) possible terminating shocks at trailing edges

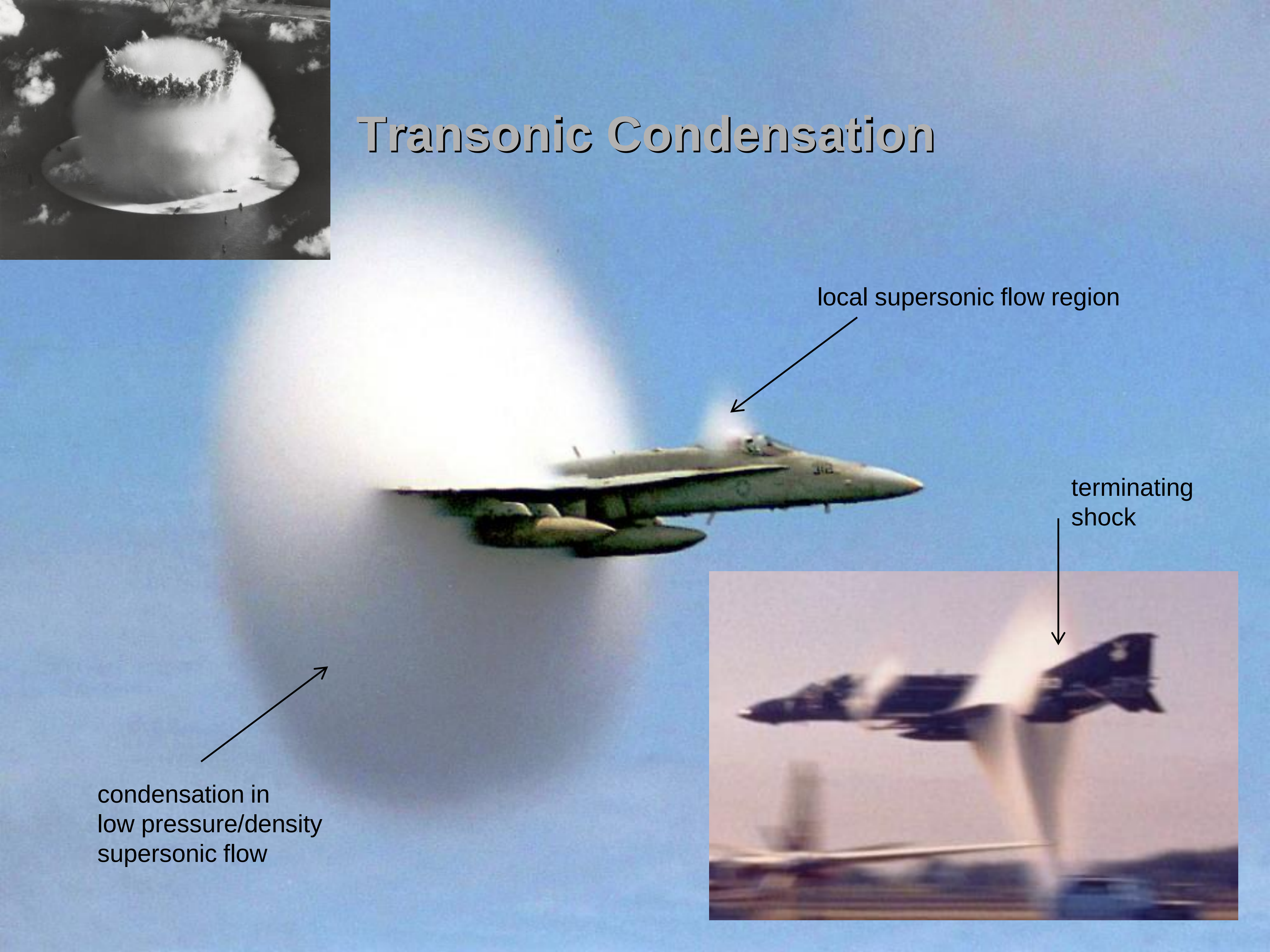
iii) Aero centre fixed at approx 0.5C



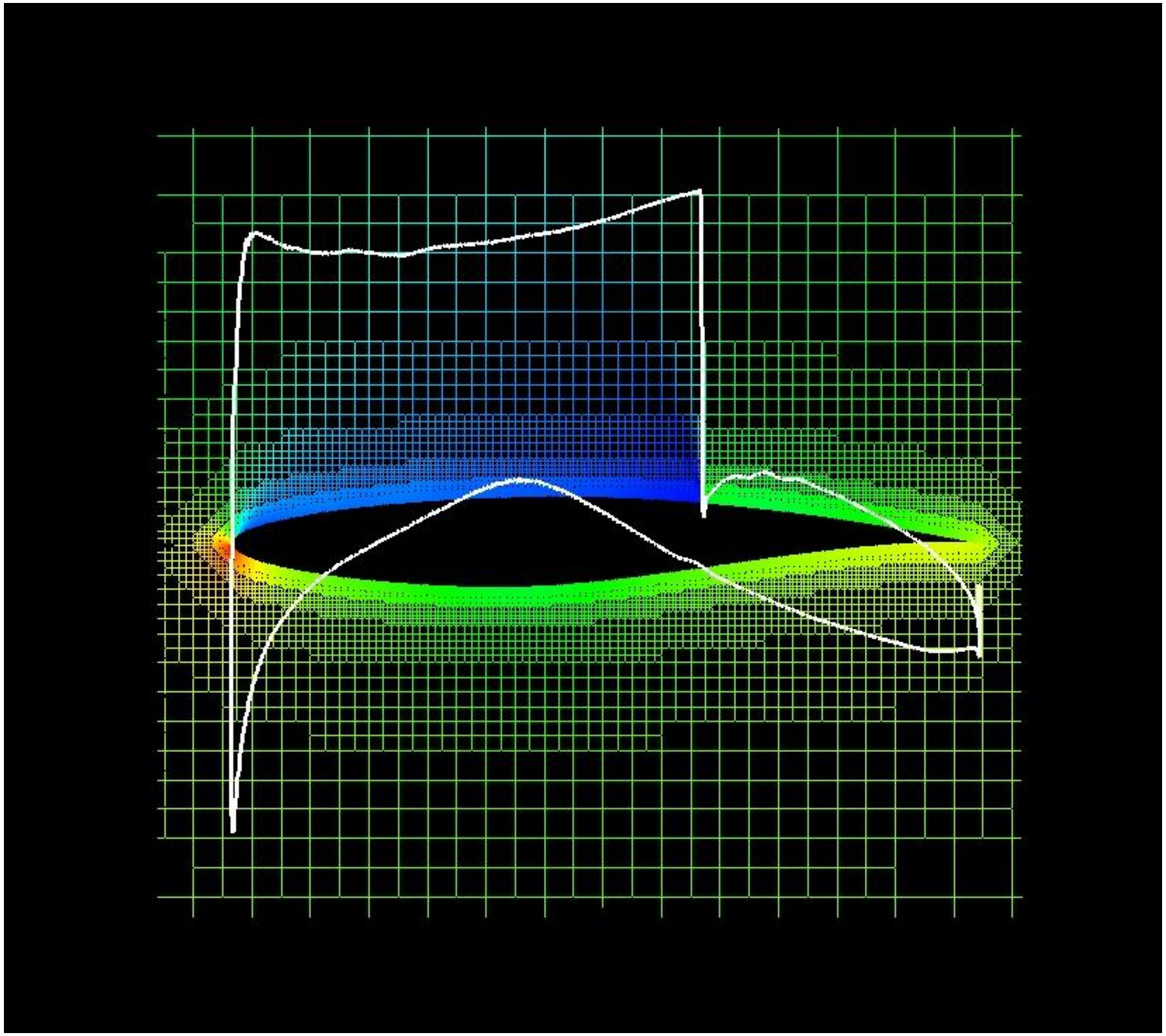
Bow shock reattaches as oblique shocks --
Mach number depends on leading edge shape.



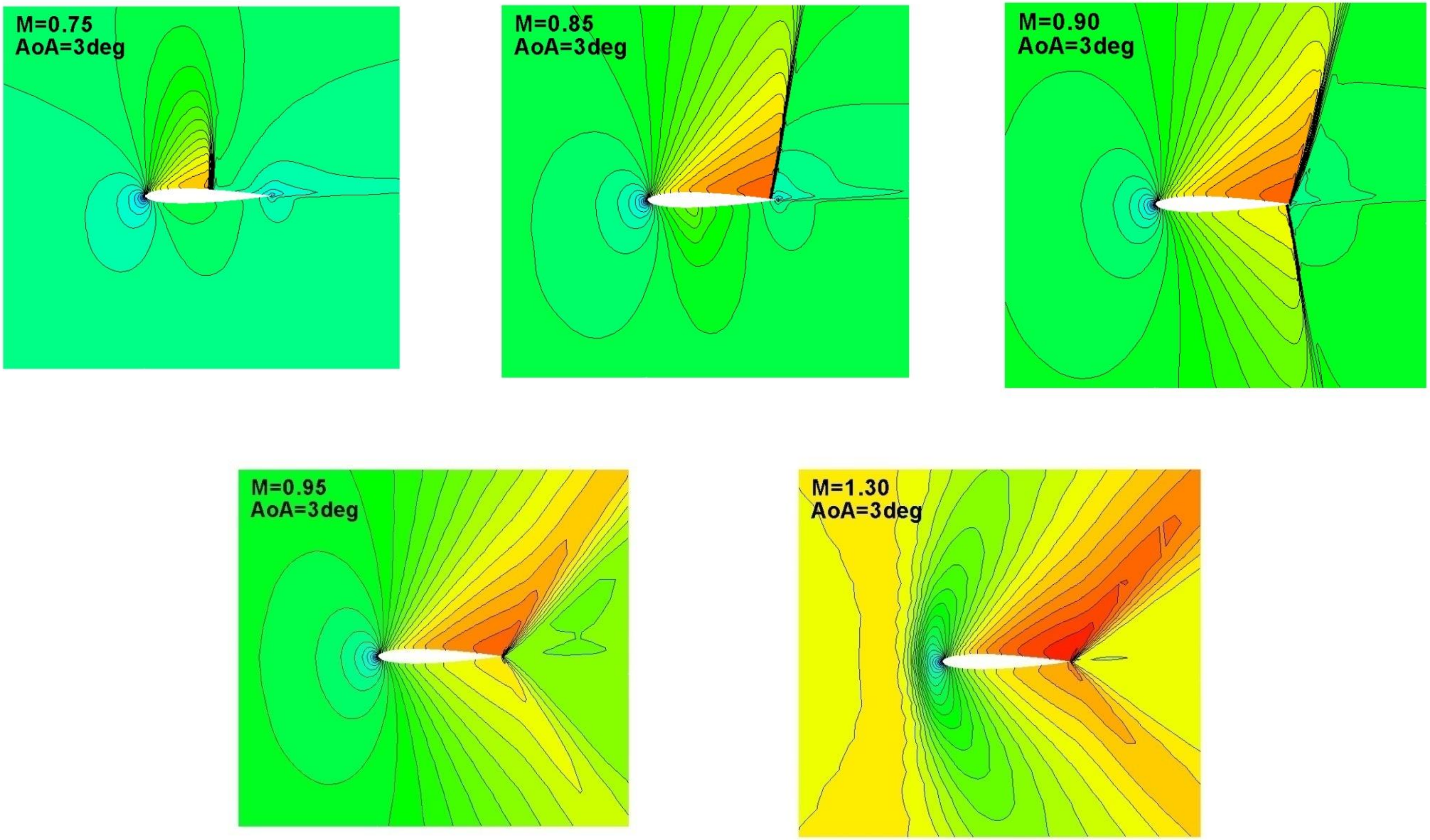
Transonic Condensation



Transonic Aerofoil (RAE2822) - inviscid



Transonic-supersonic NACA0012



Revision Objectives

You should be able to

- Describe transonic aerofoil behaviour using words, pictures and graphs, in terms of Cl, Cd, Cm and the aerodynamic centre
- Calculate the critical Mach number of an aerofoil given Cpmin

More on Aerofoils in Compressible Flow and 3D compressible flow

Aerodynamics 2
AENG21100

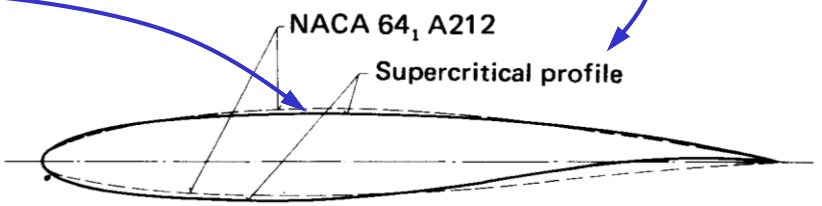
Department of Aerospace Engineering
University of Bristol

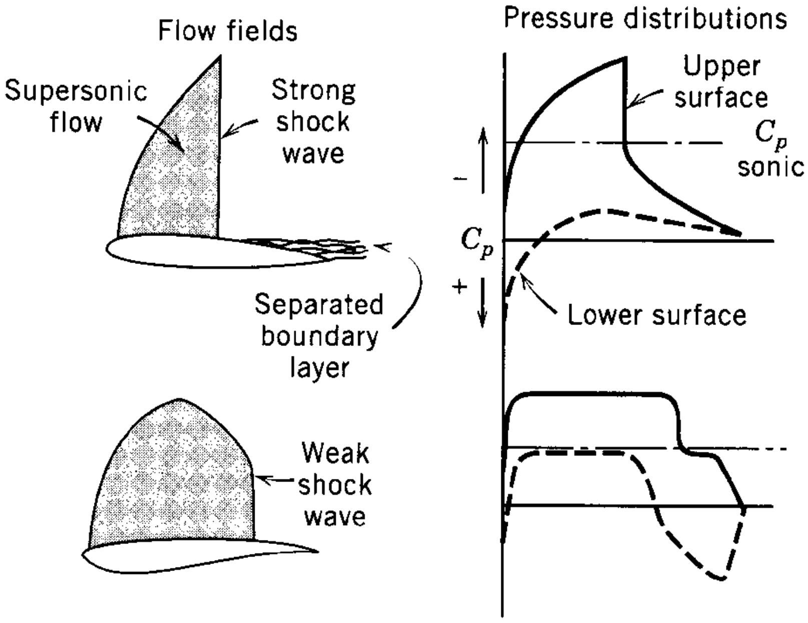
Aims for this Lecture

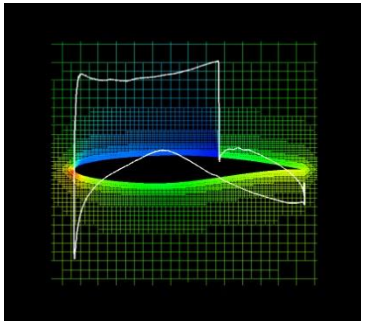
- To introduce supercritical aerofoils for transonic flow
- To look at 3D compressible flows. In particular at
  - The effect of wing sweep
  - Wing characteristics
  - The area rule

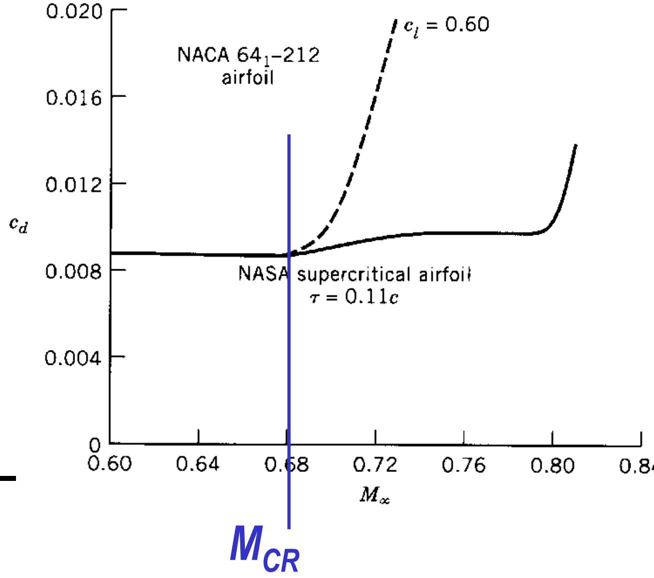
Supercritical Aerofoil

- Delay drag divergence by weakening upper surface shock
  - ‘aft-loaded’ section
  - flat upper surface with strongly cambered aft region
  - also gives improved structure - more volume for metal and fuel





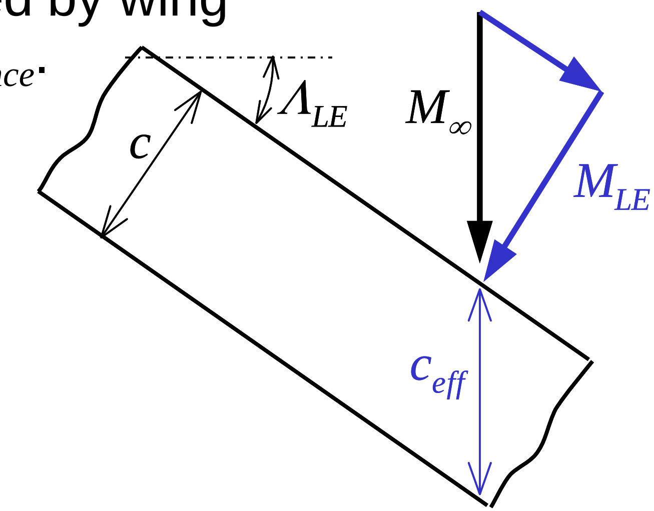


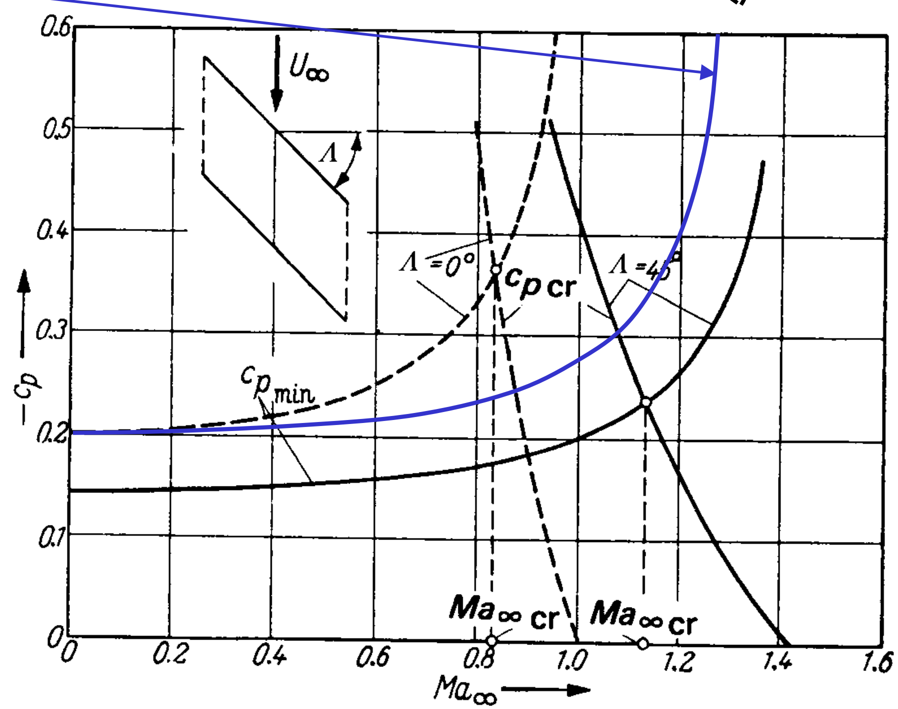


… but watch out for the pitching moment – gains may be lost if this increases

Wing Sweep – Basic Effect

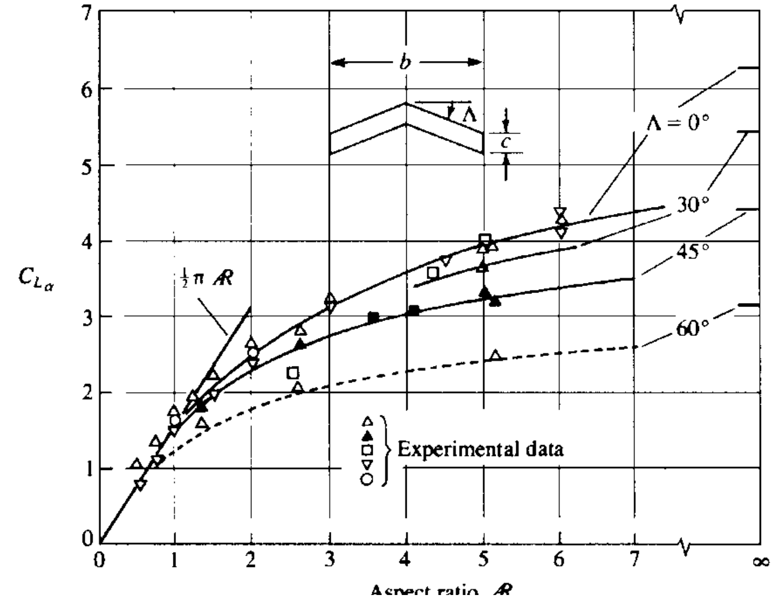
- Transonic & supersonic design characterised by wing sweep. Increase M_{crit} and hence $M_{drag\text{-}divergence}$. Sweep can lower supersonic drag by a factor of 2 to 3
- Two Effects
  - reduced effective Mach Number M_{LE}
  - reduced effective thickness t/c
- Prandtl-Glauert correction and critical pressure coefficient curves, scale with M_{LE} not M_∞
- Reduced pressure increase with Λ and hence reduced incompressible suction peak C_{pmin}
- Result is M_{crit} increased

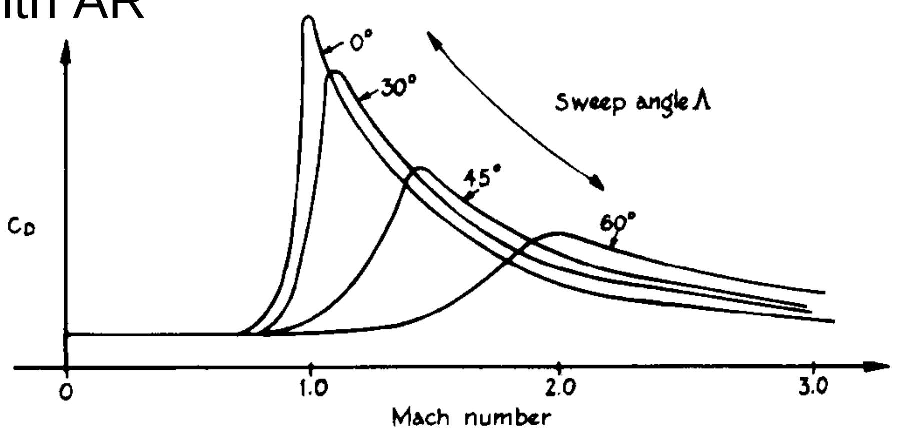


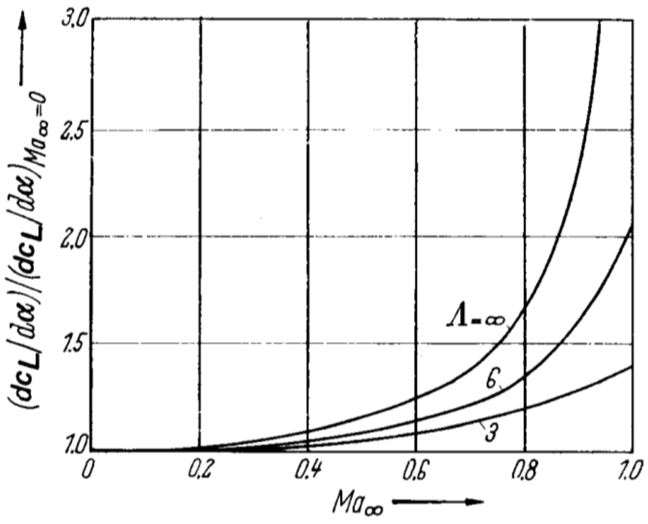


Effect of Sweep on Subsonic Wings

- For transonic flow, drag rise delayed and reduced by sweep. But when M_{LE} supersonic, wave drag increased by sweep.
- Consider Prandtl-Glauert correction as a transformation to an 'equivalent' incompressible flow. Aspect ratio reduced and sweep increased with increasing M
  - lift curve slope reduced
  - no effect on induced drag factor *k*
- Prandtl-Glauert scaling effects reduce with AR

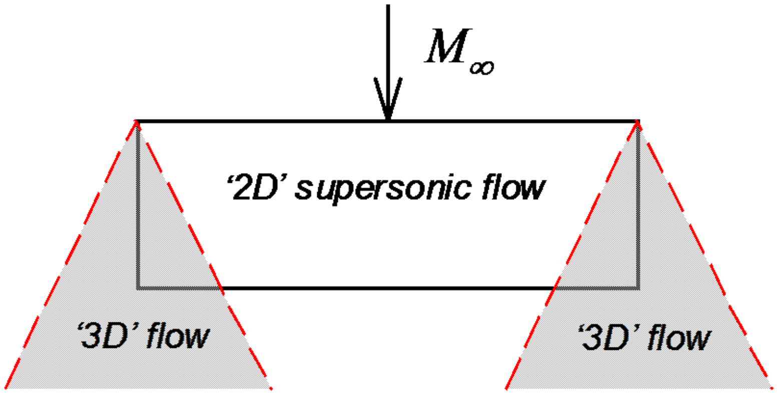


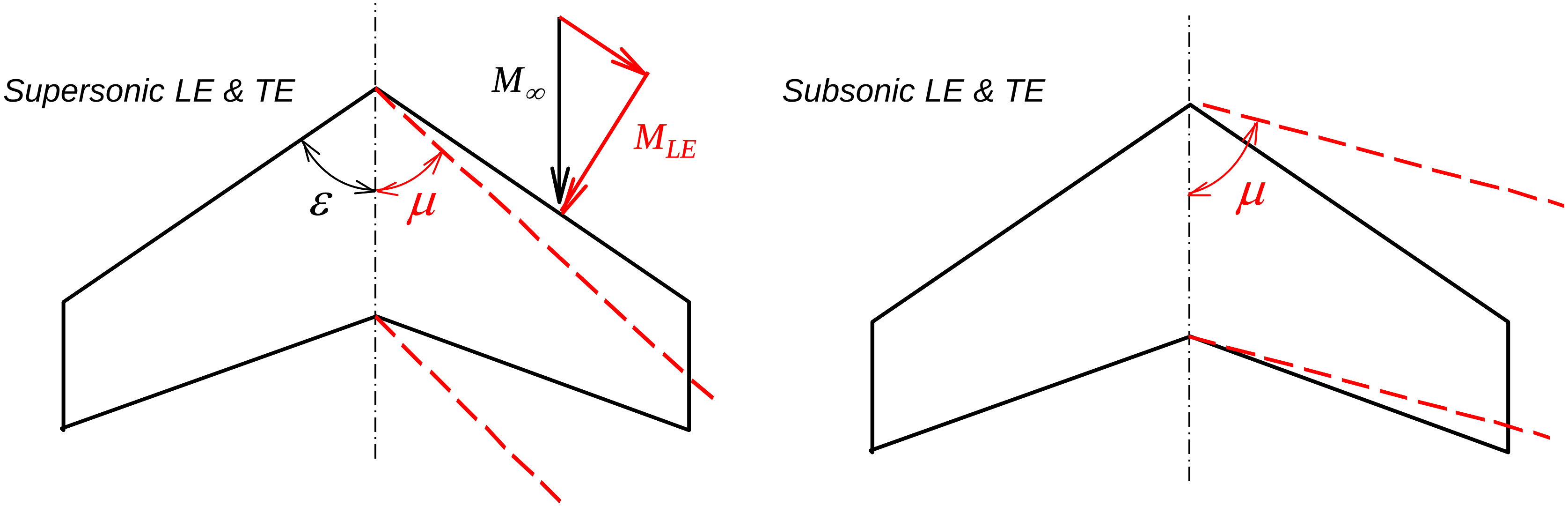




Supersonic Swept Wing Considerations

- Interaction of aircraft components limited by Mach cone, very important for control
- Leading-edge sweep very important
  - "Supersonic" leading edges generate bow shocks ($M_{LE} > 1$)
  - "Subsonic" leading edge gives lower wave drag and can use subsonic aerofoil sections

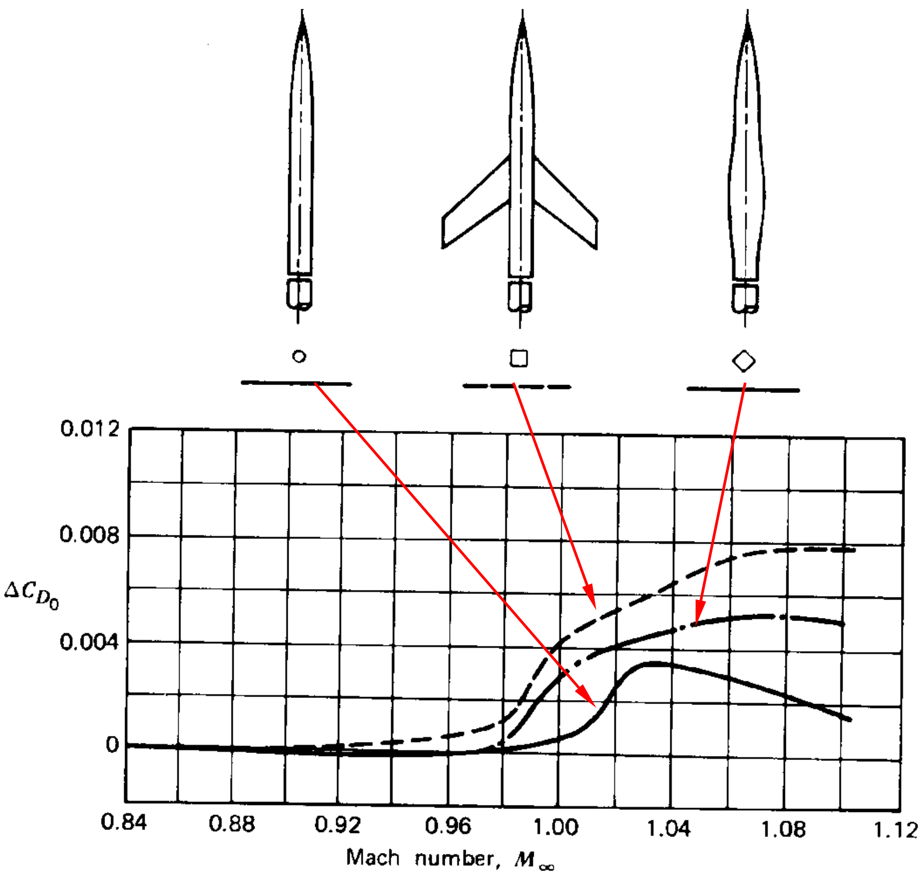


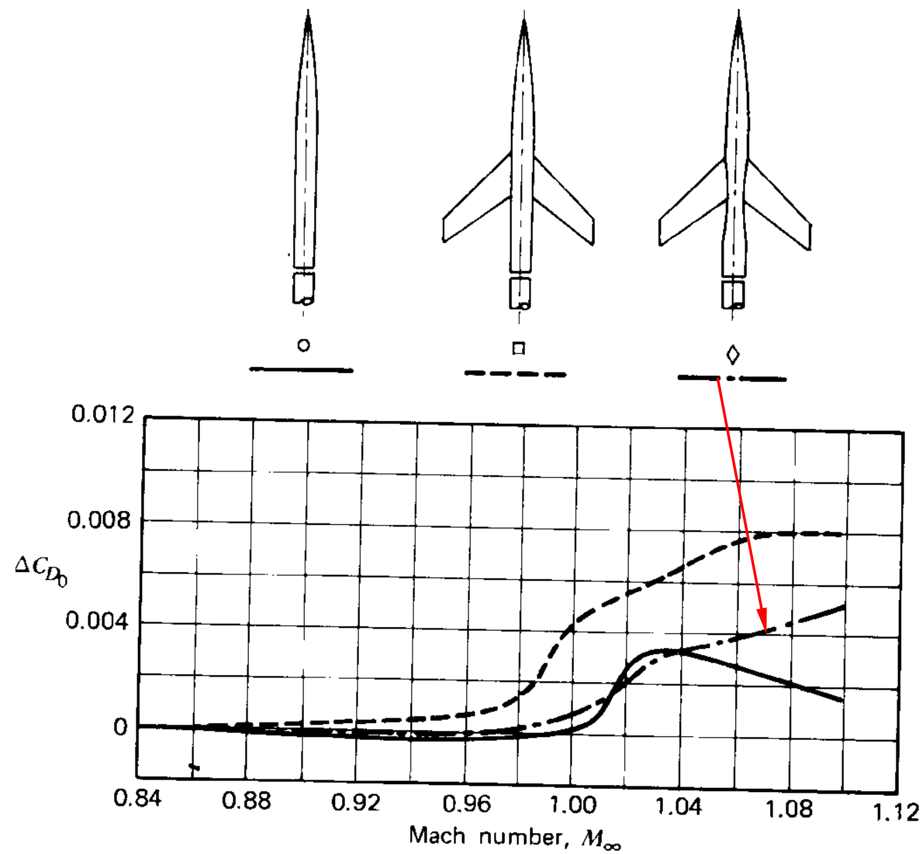


F-15 Clipped tips

Zero-Lift Drag

- At zero lift the drag is made up from: skin friction on surface, wave drag due to thickness distribution and "base drag" due to afterbody separation. Below are some experimental results

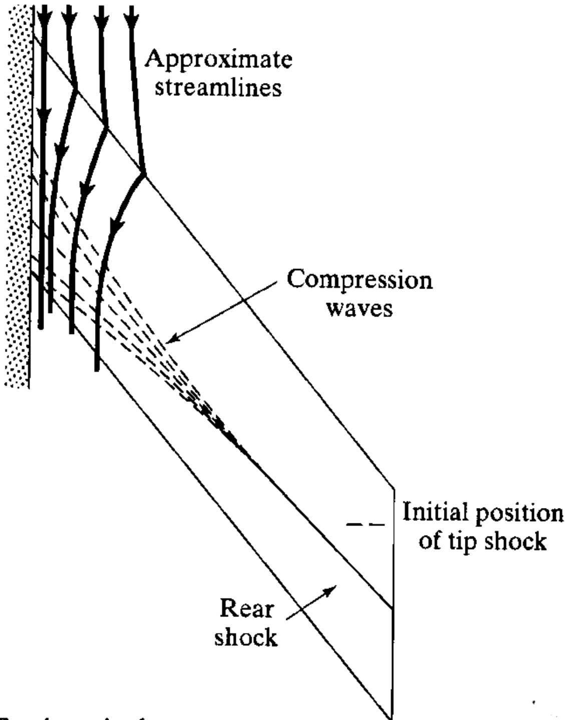


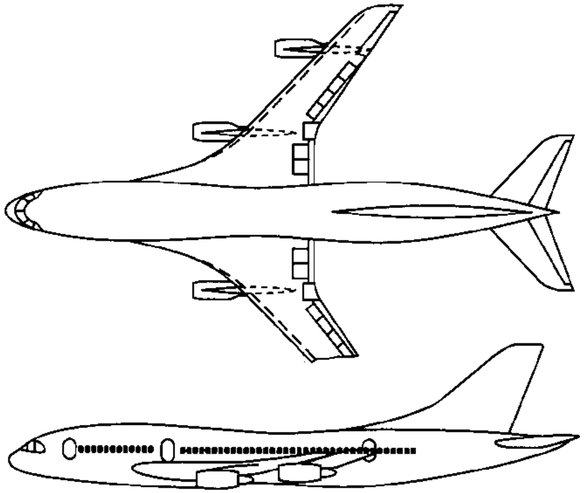


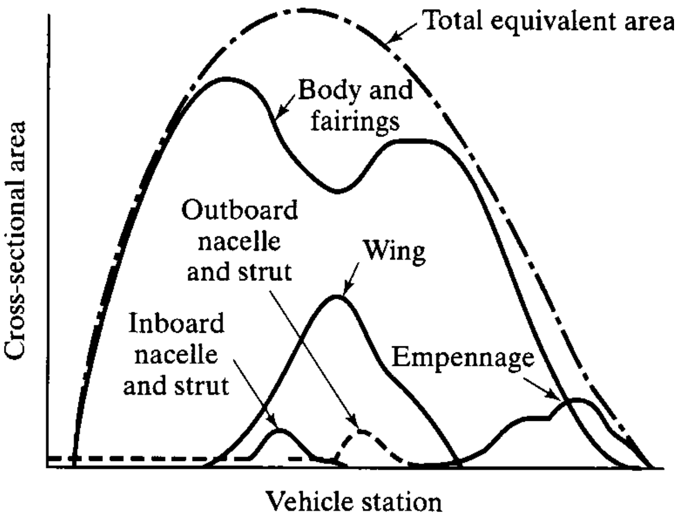
- fuselage + bulge
- fuselage + waist

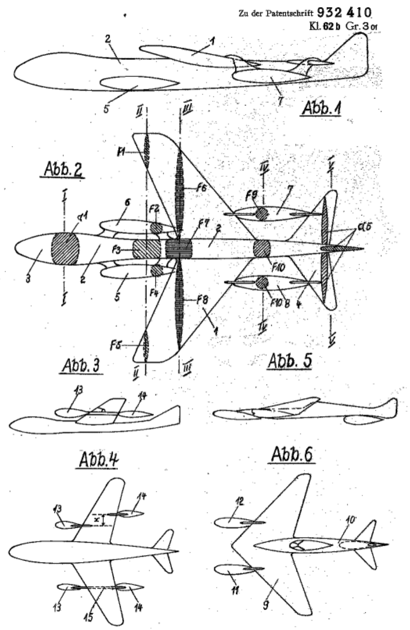
The Transonic ‘Area Rule’

- Drag rise is a function of the rate of change of the longitudinal distribution of cross-sectional area (think how Ackeret’s Cp is dependent on the gradient of the surface in 2D)
- Aim to match distribution of ‘minimum drag’
  Sears-Haack body

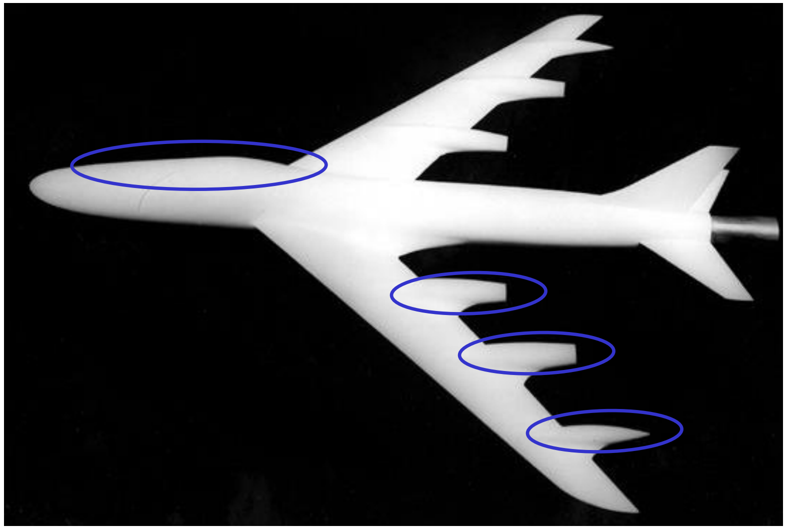


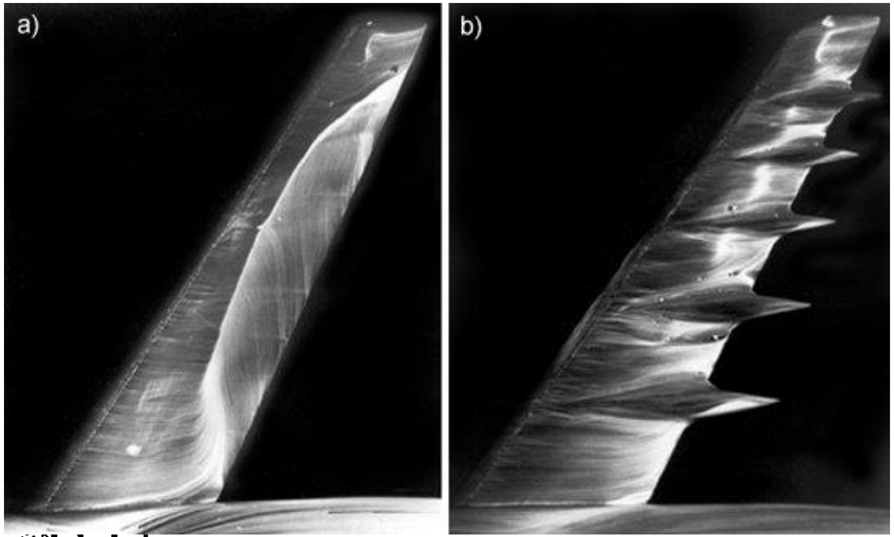




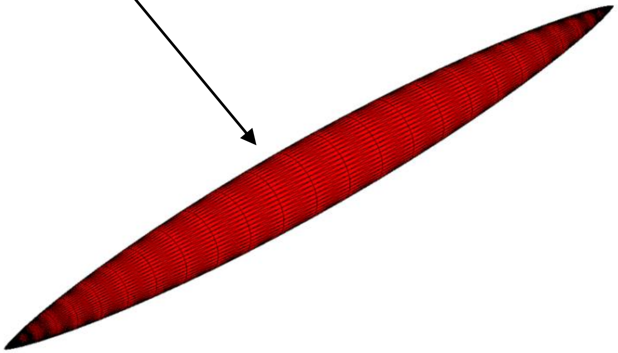


The Transonic 'Area Rule'

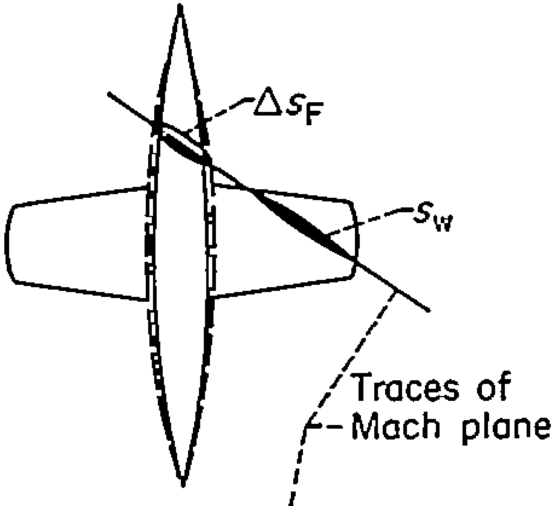


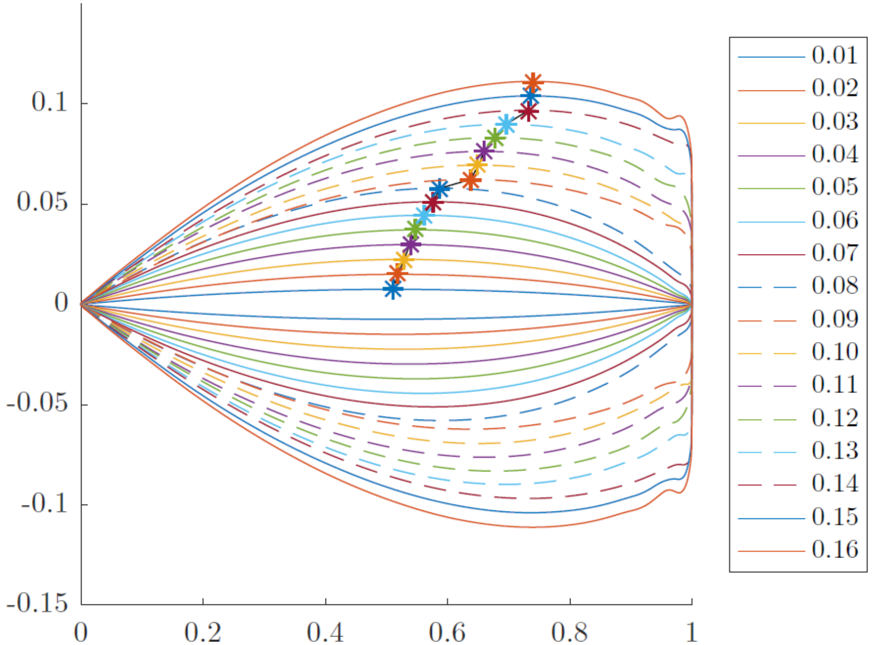


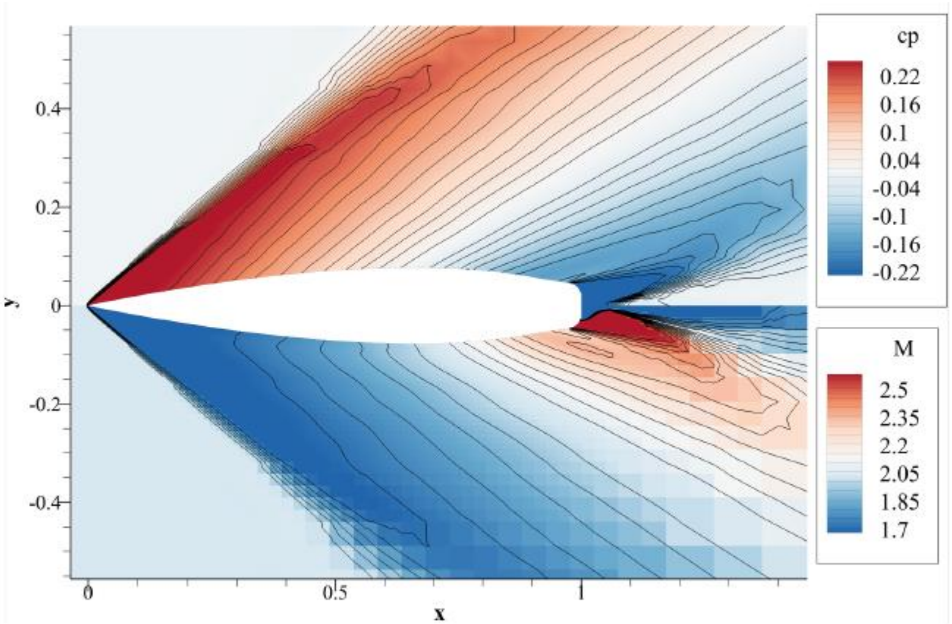
Sears-Haack body - influence of the idea



In supersonic flow, it is the area distribution along the mach cone that matters, not the flow axis





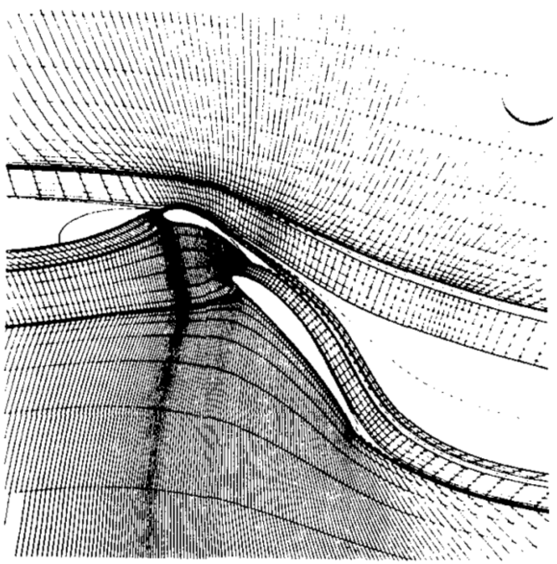


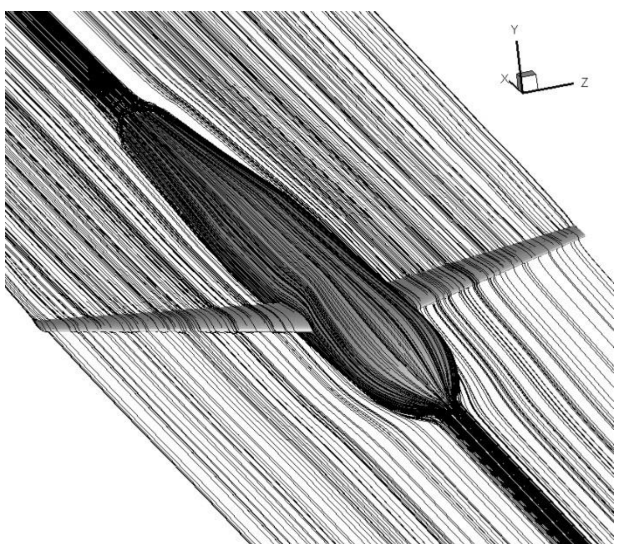
Sad but historically significant; approx. 1km increase in range over earlier designs. Idea originally French/German, dating from ~1890s.

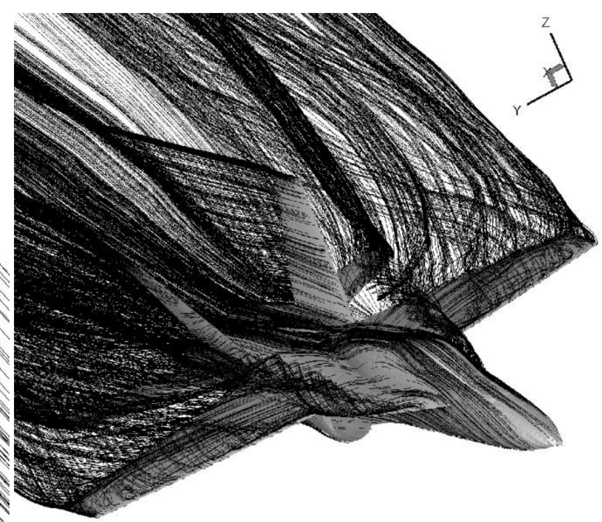
Led to tactical change in WW1, especially long rang machine gun use. Improved accuracy at fixed range – bullet would fall less during flight, due to shorter flight time.

An understanding of the area rule

- In 2D streamlines form the edges of con-di nozzles, or a con-di-con-di… nozzle. We hope that none of these nozzles chokes, otherwise a shock will occur (transonic case). A good area distribution achieves this
- In 3D we have ‘streamtubes’, but the same argument holds. Remember tubes are bounded by the ‘rest’ of the air – which is what makes things complicated! How much the tubes are ‘squeezed’ depends on the x-sectional area of the body







Why?

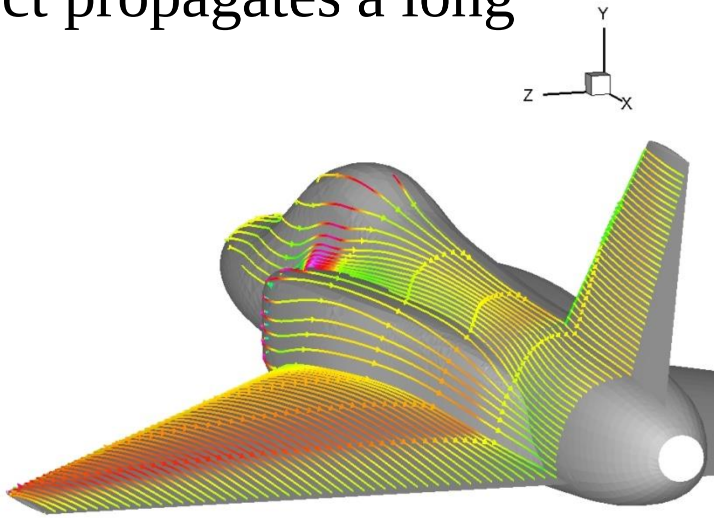
Recall from first lectures

$$\boxed{\left(M^2 - 1\right)\frac{1}{V}\frac{dV}{dx} = \frac{1}{A}\frac{dA}{dx}}$$

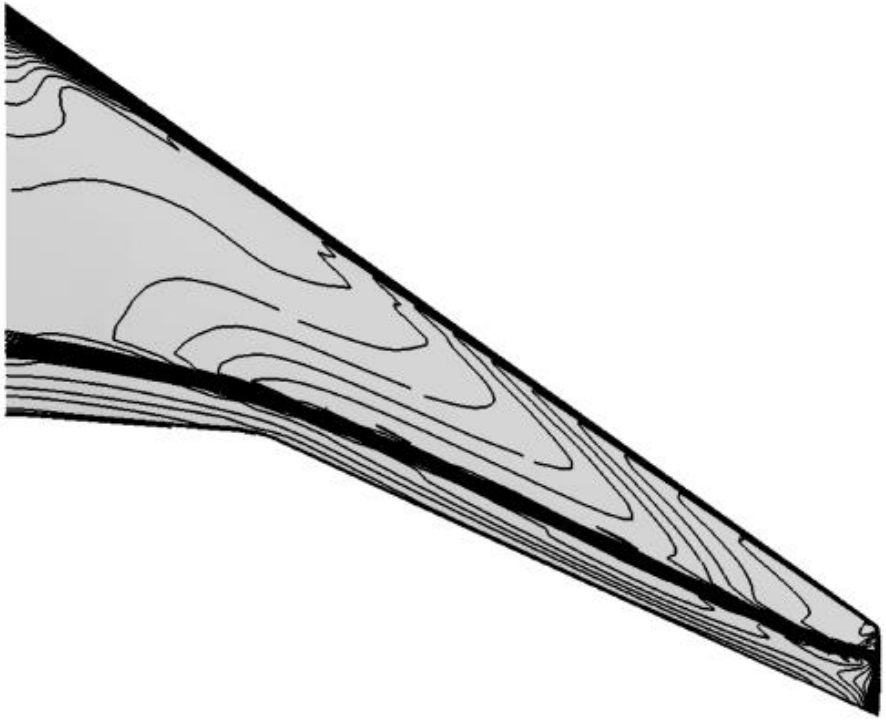
So if M=1 the area **cannot be changing for a streamtube**

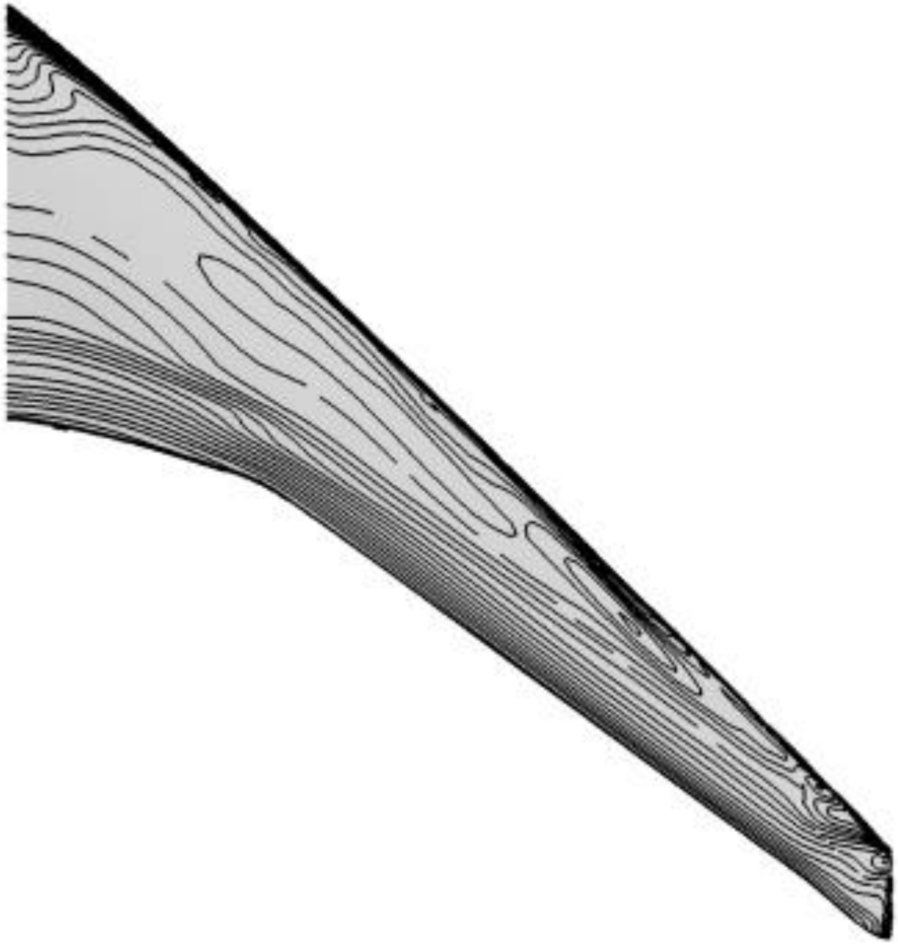
This means streamtubes around the aircraft interact rather strongly, which means if one is displaced outwards, by a small change in cross sectional area of the aircraft, this effect propagates a long way, and has quite a large influence

Richard Whitcomb, originator of the supercitical aerofoil, termed M=1 flow ‘pipefitter’s flow’, as it is similar to fitting rigid pipes around the aircraft



Wing sweep may be thought of as an application of the area rule!

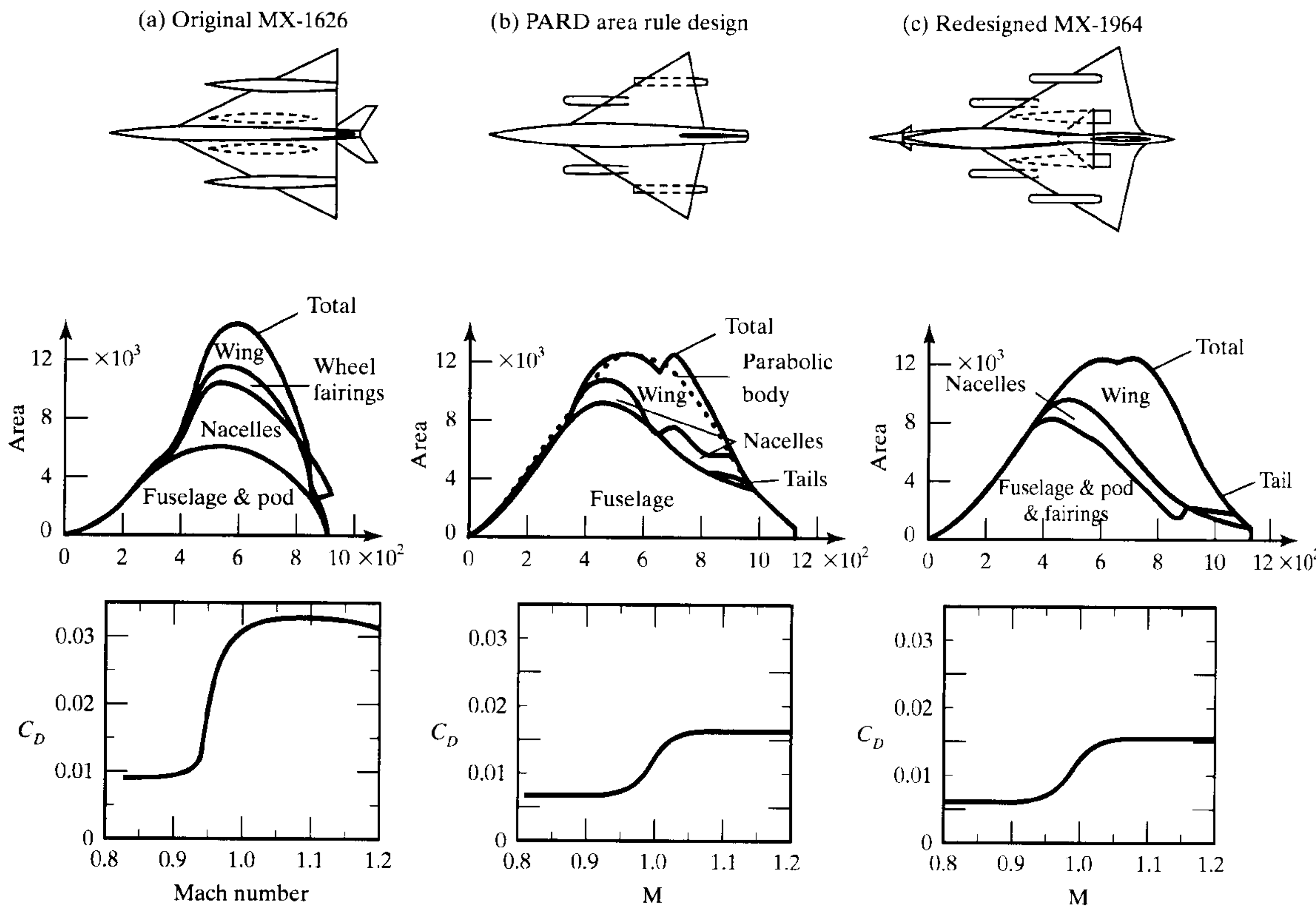




Domain-Element Method for Aerodynamic Shape Optimization Applied to a Modern Transport Wing
AIAA JOURNAL
Vol. 47, No. 7, July 2009

Best sweep configuration not a simple problem!

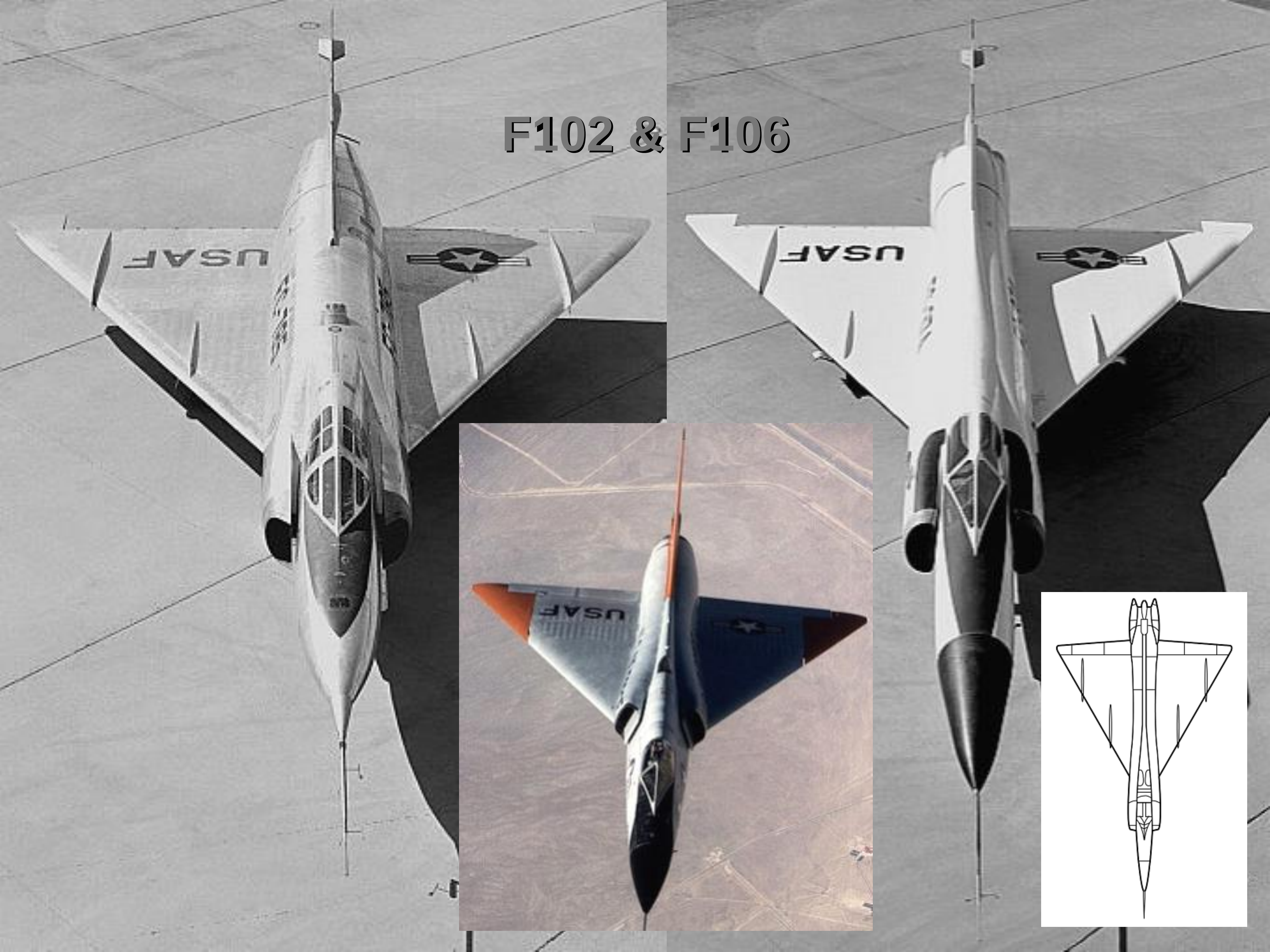
B58 'Area Rule'



F-102

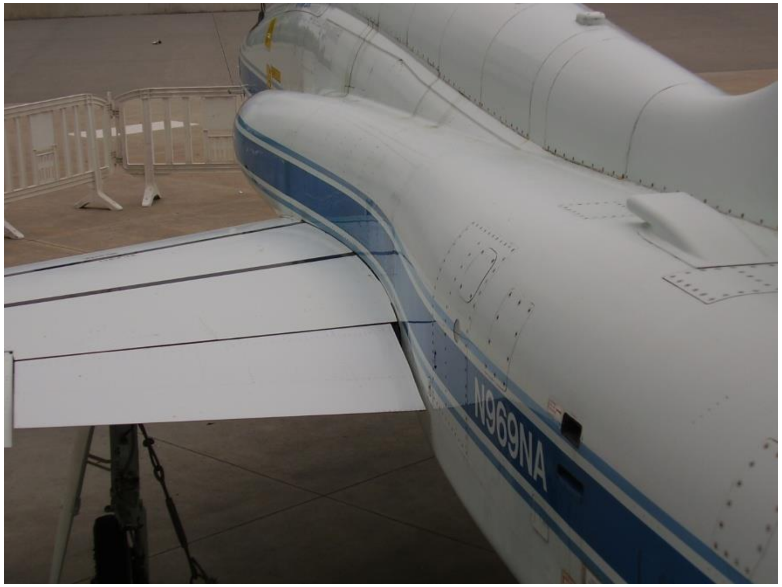
- Oft quoted case, where initial aircraft was limited to M=0.98 in level flight
- After area ruling, structural and engine improvements, M=1.22 achieved.

F102 & F106



Blackburn Buccaneer

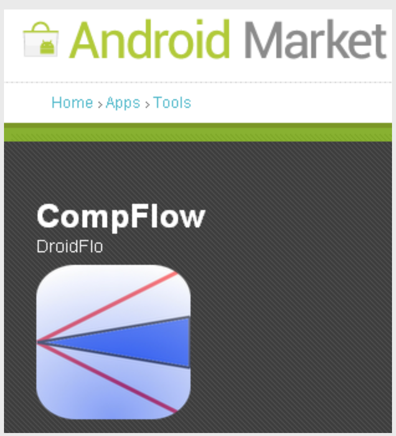
T-38



Revision Objectives

You should be able to

- Explain sweep effects
- Explain the area rule and its effect on aircraft design



(free!)

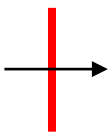
Simplified course summary

Non-isentropic flow

Total pressure changes!

Shocks

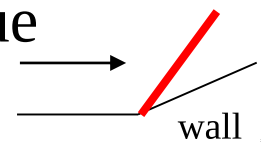
Normal shock tables



Normal

Oblique

Oblique shock tables
Max deflection angle
and detached bow shocks



Total pressure constant!

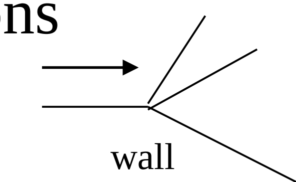
Isentropic flow

Isentropic flow tables

Prandtl-Meyer function

$$\theta = \nu(M_2) - \nu(M_1)$$

Expansions



Linearised models (Prandtl-Glauert and Ackeret)

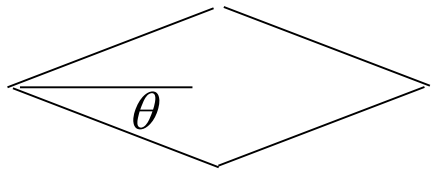
$$C_p^{\,comp} = \frac{C_p^{\,incomp}}{\sqrt{1 - M_\infty^{\,2}}}$$

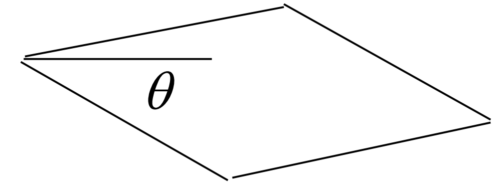
2D

3D

Area rule, critical Mach number, transonic aerofoil behaviour, wing sweep

Tutorial sheet 3





- Q1 - shock/expansion theory followed by a comparison to linear theory.

$$L = \int_S p\cos(\theta)\,ds \qquad \text{and} \qquad C_p = \frac{2}{\gamma M_\infty{}^2}\left(\frac{p}{p_\infty} - 1\right) = \frac{p - p_\infty}{q_\infty}$$

$$L = \int_S \left(C_p q_\infty + p_\infty\right)\cos(\theta)\,ds$$

but a constant pressure cancels in a loop integral, so...

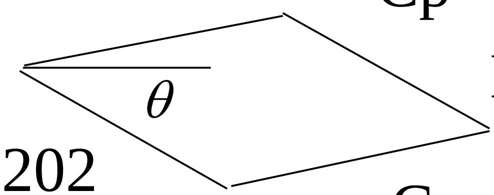
$$C_L = \frac{L}{q_\infty c} = \int_S C_p \cos(\theta)\,d\frac{s}{c} \qquad C_D = \frac{D}{q_\infty c} = \int_S C_p \sin(\theta)\,d\frac{s}{c}$$

Q1 – example to get Cl and Cd

Deflection=4-6=-2

Cp=-0.040

Cp=-0.202



Deflection=-2-8=-4-6=-10

Cp=0.202

Deflection=6+4=10

Cp=0.040

Deflection=-4+6=10-8=2

Ackeret’s linear Cp gives (remember radians)...

Cl=(0.5/cos(4))*[0.04*cos(2)+0.202*cos(10)+ 0.202*cos(10)+0.04*cos(2)]=0.24 (unit chord)

Cd=(0.5/cos(4))*[2*0.04*sin(2)+2*0.202*sin(10)]=0.037

Cl/Cd=6.5

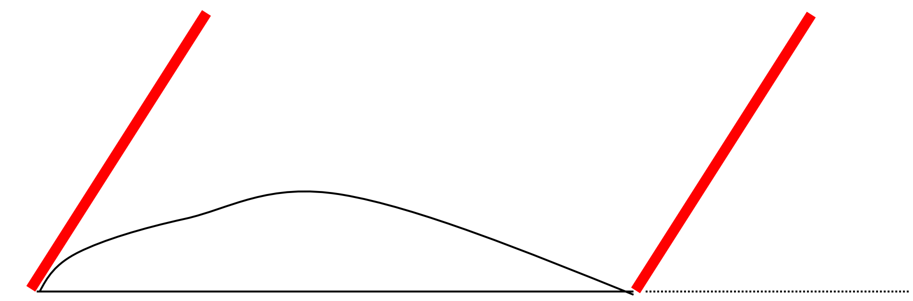
$$\boxed{C_N = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \approx C_L} \longrightarrow 0.24$$

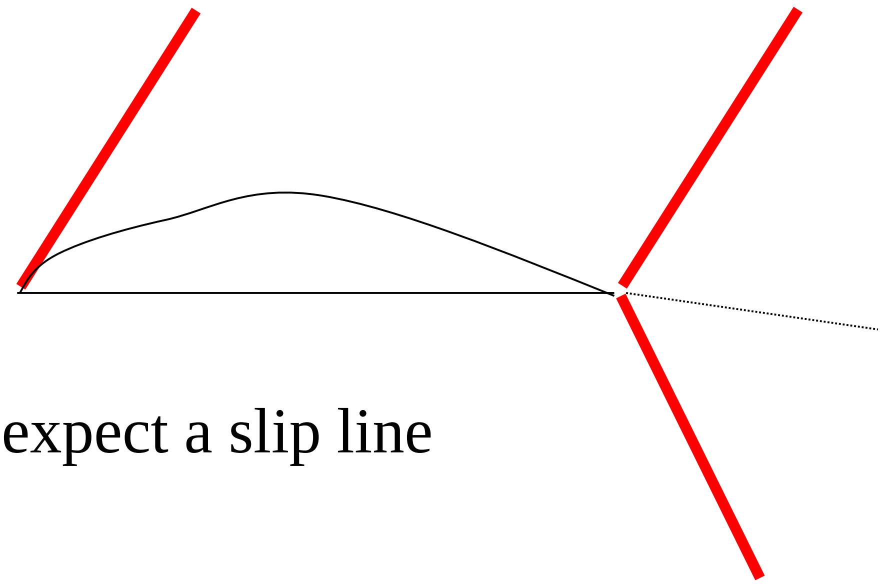
$$\boxed{c_d = 4\frac{\alpha^2 + (t/c)^2}{\sqrt{M_\infty^2 - 1}}} \longrightarrow 0.037$$

Q3

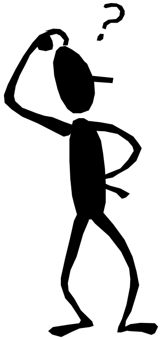
- The calculation of the flow angle at the trailing edge is an example of an equal pressure condition needing to be satisfied – we cannot have a pressure discontinuity in a wake (otherwise the wake would move!)
- In general there will be a slip line discontinuity, across which velocity changes
- Not particularly straightforward to find the angle of a slip line. Trickier if there’s more than one!

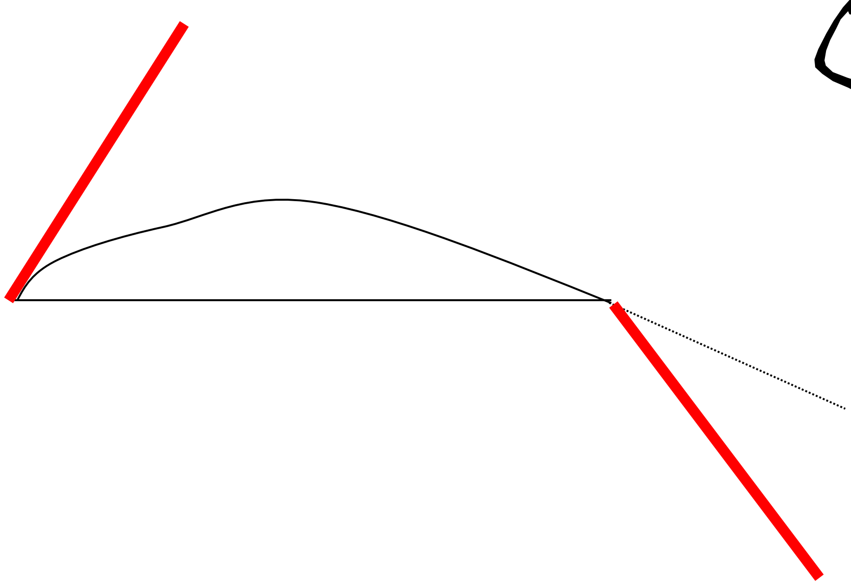
Q3

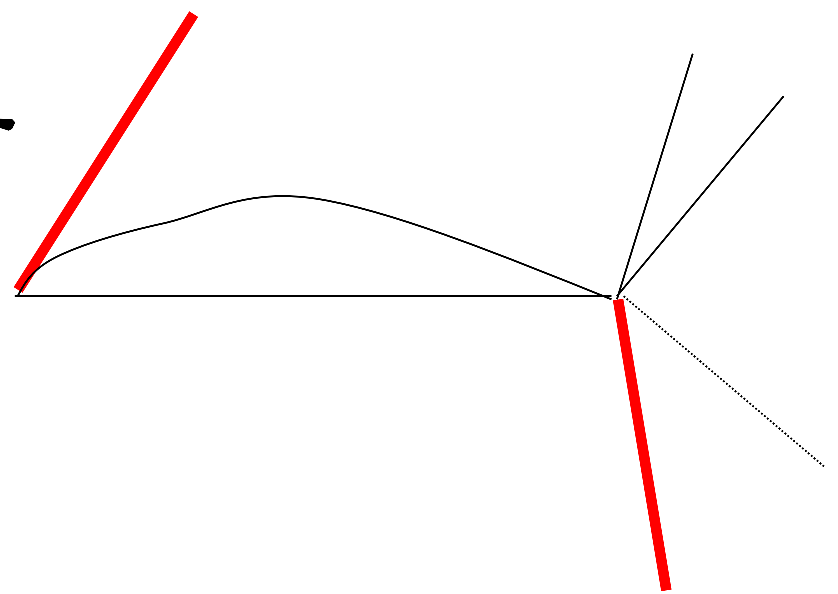




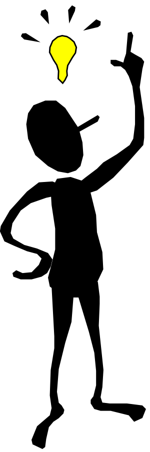
If aerofoil not symmetric, you can expect a slip line



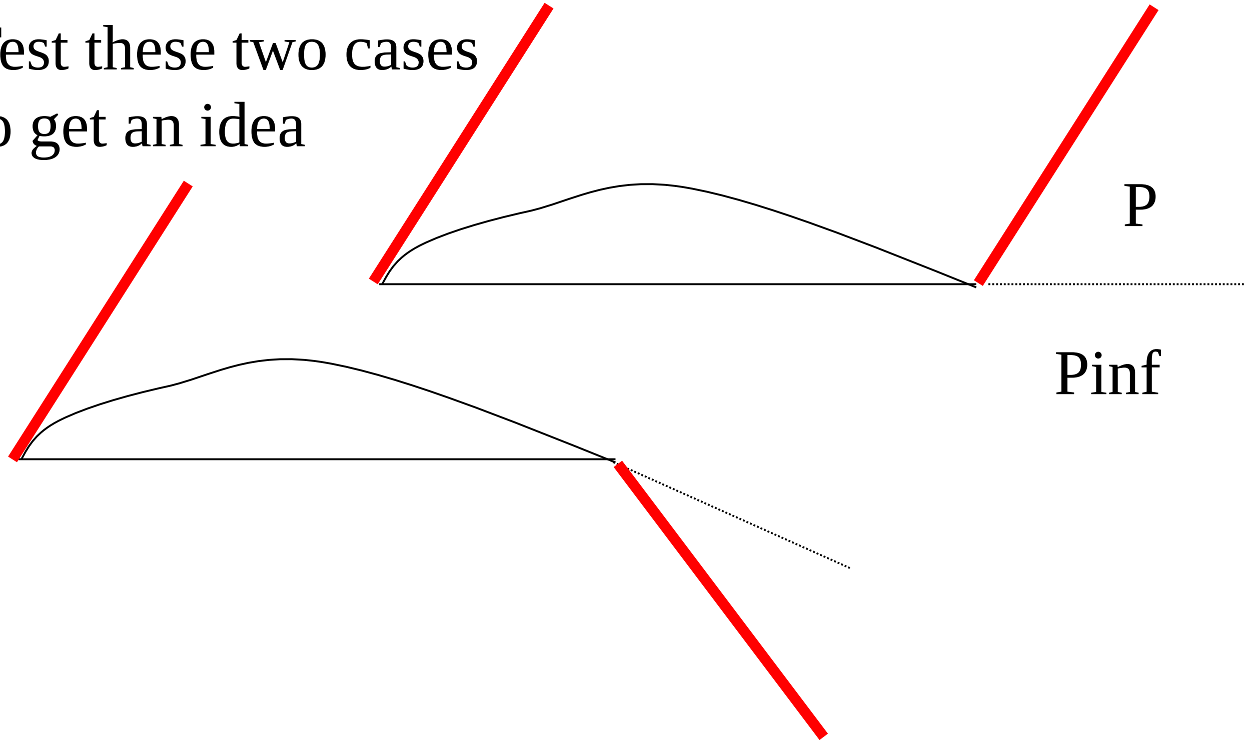




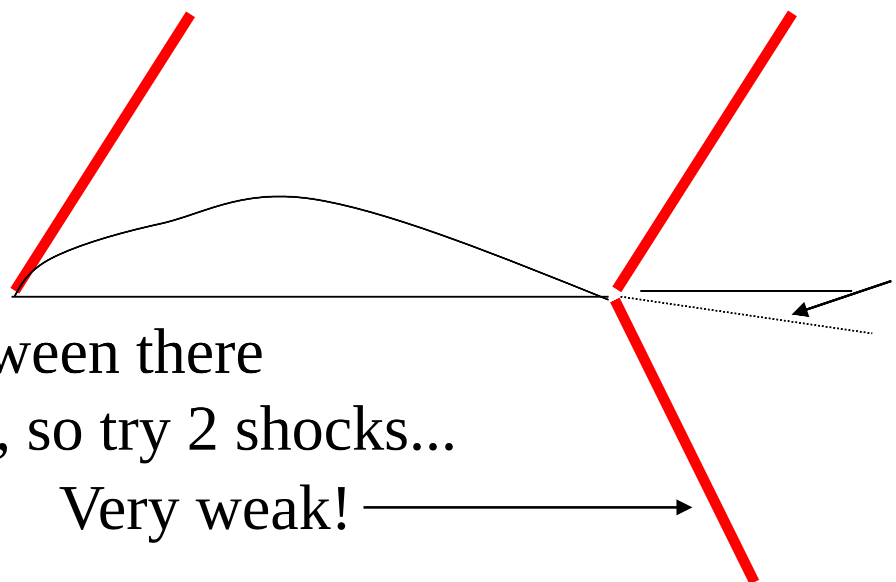
Q3



Test these two cases
to get an idea



Not equal
in general

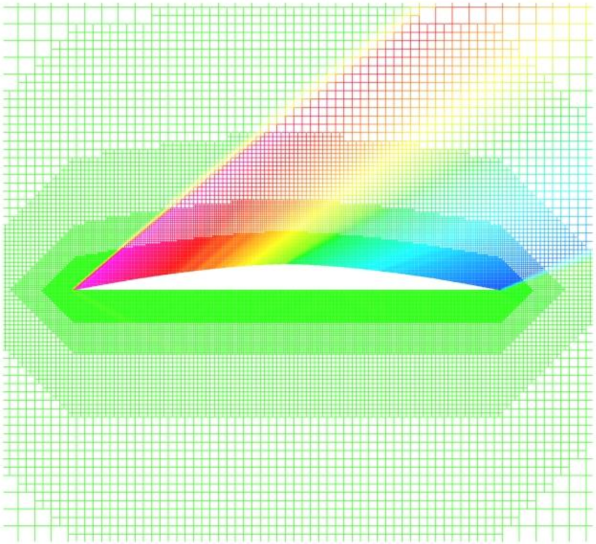


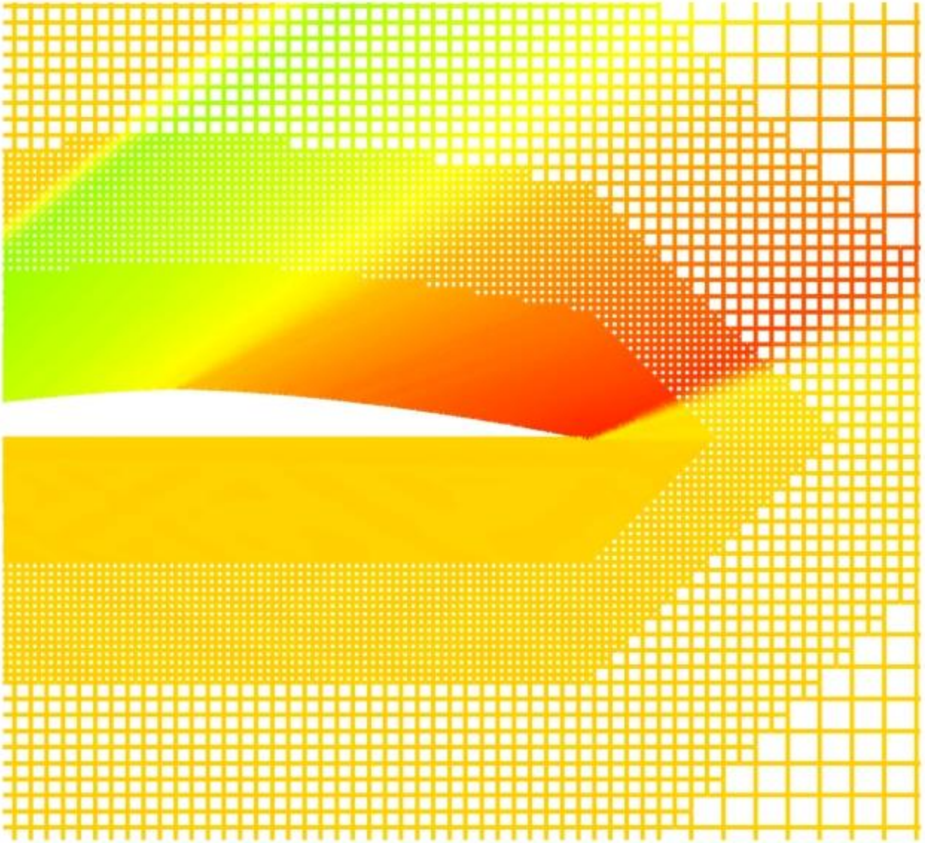
Work out angle
so pressures
equal

Somewhere in between there
must be an answer, so try 2 shocks...

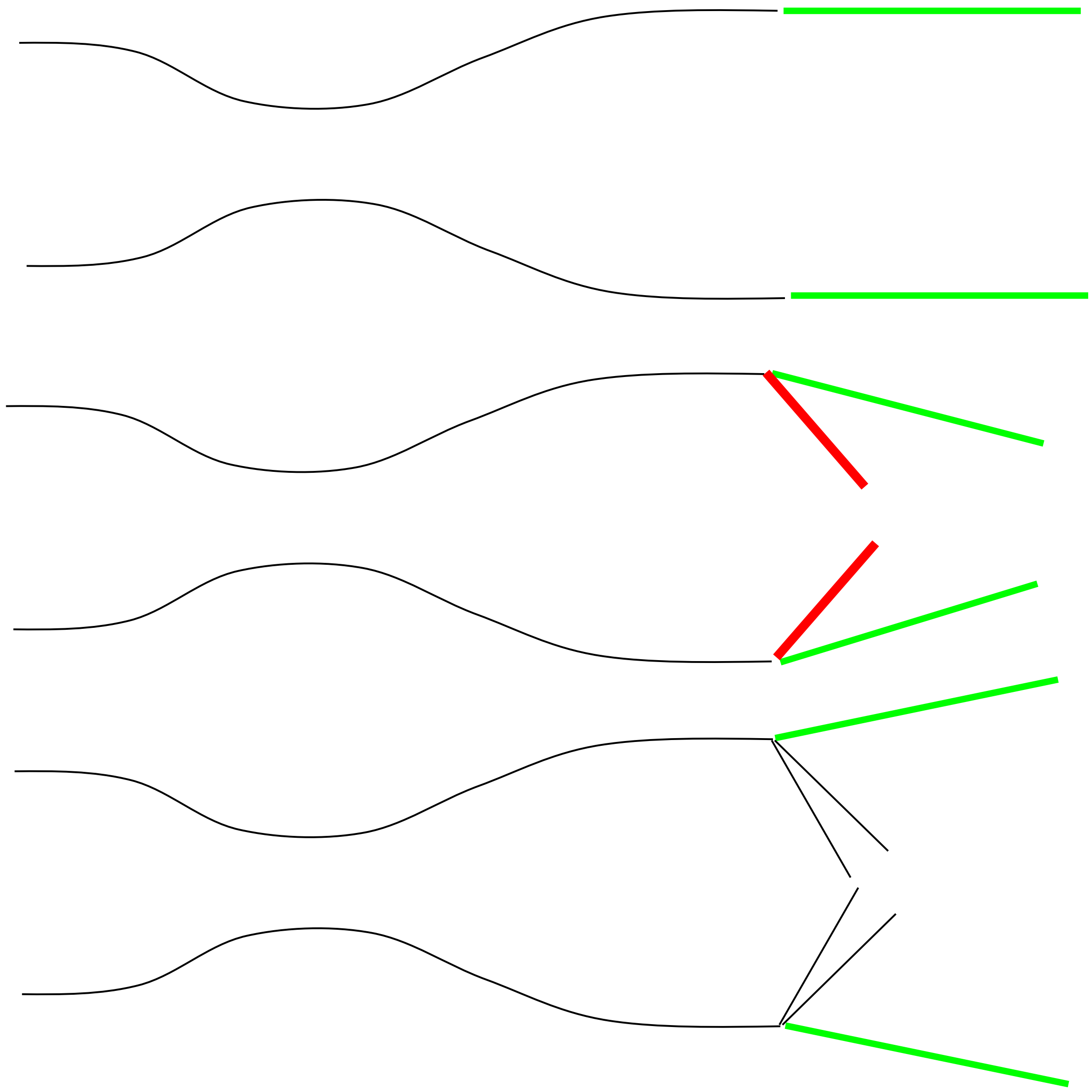
Very weak!

Q3





Q4 - easy after Q3!



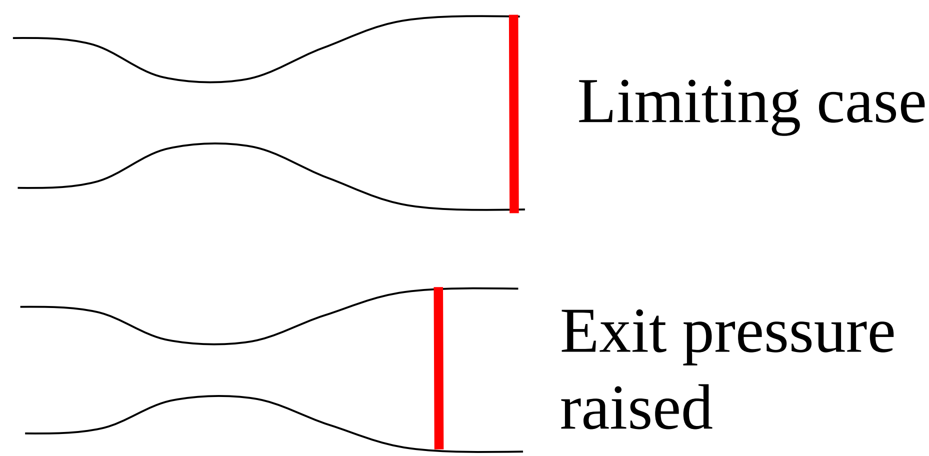
Ideally expanded - Pexit=isentropic pressure at nozzle exit

Over expanded - Pexit>isentropic pressure at nozzle exit

Under expanded - Pexit<isentropic pressure at nozzle exit

Q4

- The internal normal shock cases are simpler. The point is that given the shock position it is easy to work out the exit pressure that produces that shock position
- It is not so easy to find the shock position given the exit pressure. This would be similar to the slip line problem - we would be trying to match a given exit pressure, and anything where a pressure condition is matched is harder to do
- …but it could be done with a fairly simple iterative procedure (bisection?) that kept moving the shock until the exit pressure was correct



Q5/6 - Mcrit

- These are as in the lecture, with a minor variation. Start by working out the compressible Cpmin on the aerofoil, then use PG to get the incompressible Cpmin.
- Then solve as usual using bisection - remember what the graphs look like so you know which way to move
- Practice this procedure (or any other if you prefer, so long as you are clear about what you are doing, and it works)