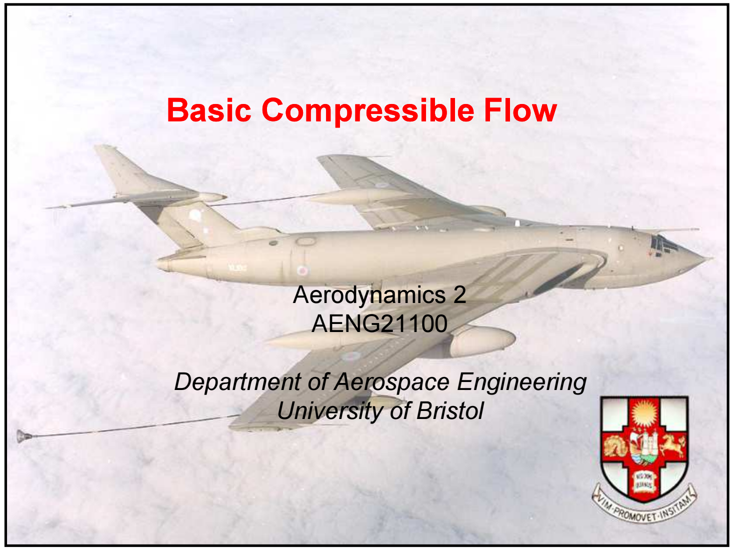


Basic Compressible Flow

Aerodynamics 2
AENG21100

Department of Aerospace Engineering
University of Bristol

There's just too much!

- Part of university learning is distilling and understanding a large amount of material yourself. Everyone's mind will do this differently. I can't throw information in to your brain.
- Since everyone does this differently, you have to take responsibility for your own learning.
- **You need to know what you don't know, then you need to make sure you know it. Don't have unknown unknowns.**

- I can't know what people don't know. So, you each have to tell me. Then I can help you to learn it.
- Normal English resumes

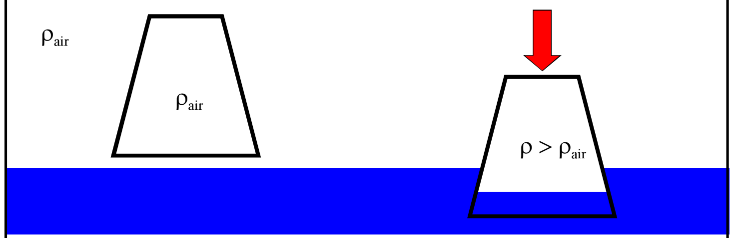
What do I need to know?

- **Well, what would you like to know?**
- There are some things you simply must know - for example, definitions of Cp, or using deflection angles to compute an expansion fan. These occur throughout the tutorial sheets you have been solving, so you will be familiar with them by the end of the course.
- In comparison, I would not expect you to derive the normal shock equations off the top of your head. However, I would expect you to understand where they come from (mass/momentum/energy conservation) and what they mean.
- Put another way, if you were given a line in their derivation, it would be reasonable to expect you to know the next line, but I would not expect you to know the entire working.

Basic Compressible Flow

- review equations of motion of a fluid
  - conservation of energy
  - need for thermodynamics in compressible flow
- review basic thermodynamic concepts
  - energy, enthalpy and entropy …
- speed of sound
  - propagation of information
  - Mach Number
- 1D compressible flow
  - 'compressible Bernoulli'
- isentropic duct flows with varying area
  - critical conditions

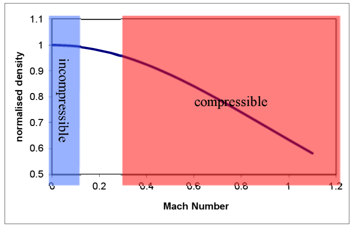
Density is a function of pressure



Density is a function of temperature

Density is a function of energy

Density is a function of flight speed



Equations of Motion (1)

- 5 unknowns — For 1-D flow. For 3-D flows we have 7 unknowns as **V** is a vector
  - pressure p, density ρ, velocity vector $\mathbf{V}$
  - internal energy e, temperature T
- therefore 5 equations required
  1. *conservation of mass* – 'continuity'

     $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$ 1 equation +

  2. *conservation of linear momentum* – Newton's 2nd law

     Force=rate of change of momentum

     $$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{f} + \mathbf{F}_{\text{visc}}$$ *(VECTOR EQUATION)* 3 equations +

  3. *conservation of energy + two equations of state …* 1 equation + 2 equations =7 equations

Equations of motion (2): Conservation of Energy

- 'energy cannot either be created or destroyed, merely changed in form', therefore need to balance

  The idea of production balancing dissipation

- ***fluid energy***
  - internal energy — thermal
  - kinetic energy
  - potential energy

  $\left.\begin{array}{l}\text{kinetic energy}\\ \text{potential energy}\end{array}\right\}$ mechanical

- ***heat transfer and work done***
  - work done by body forces
  - work done by pressure forces
  - heat transfer
  - viscous dissipation

- which necessitates 2 'equations of state' (for specific internal energy e and temperature T). Typical equations are $p=\rho RT$ and $e=c_v T$. These allow simplified equations

Equations of motion(3): Thermal Energy Equation

- previously, energy equation split into *mechanical* and *thermal* energy components
- mechanical energy derived from momentum equation
  - accounts for kinetic & potential energy, body forces, work done by pressure gradient
  - can therefore be subtracted from total energy equation to give

$$\rho \frac{De}{Dt} = -p\nabla \cdot \mathbf{V} + \rho \dot{q} + \dot{Q}_{visc}$$

See Navier Stokes equations in yr1

steady Euler equations in yr1

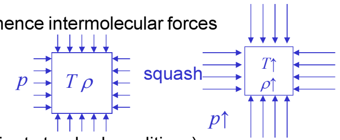
- for adiabatic, inviscid flow this becomes $\rho \frac{De}{Dt} = -p\nabla \cdot \mathbf{V}$ — continuity: conservation of mass — 0

  Adiabatic flow → no heat lost or gained
- for incompressible flow $\nabla \cdot \mathbf{V} = 0$ and hence internal energy e is constant $\frac{De}{Dt} = 0 \rightarrow e = \text{const}$
- thermodynamics irrelevant to *incompressible* fluid flow

Basic Thermodynamics (1)

- equation of state
  - Consider ***thermally*** perfect gas (hence intermolecular forces negligible)

$$\boxed{p \ = \ \rho R T}$$



  - R = gas constant (287 *J/kgK* for air at standard conditions)
- specific internal energy e *(ie energy per unit mass)*
  - sum of translational, rotational, vibrational and electronic energies
  - thermodynamic state variable = function of temperature only

$$de \ = \ c_v dT$$ therefore $e\uparrow \rightarrow T\uparrow$. note generally $de \ = \ c_v(T)dT$

  - for ***calorically*** perfect gas c_v = constant, hence

$$\boxed{e \ = \ c_v T}$$ $T\!=\!0$ $e\!=\!0$:remember T in ^{0}K

  - c_v = specific heat at constant volume (717 *J/kgK* for air at standard conditions)

c_v :how much energy needed to raise the temperature of 1kg by 1^0K, with volume kept constant

Enthalpy – its usefulness will become clear…

$Fds = pAds = pdV = mpdv$ Work associated with expansion

$pv = RT$

$pdv + vdp = RdT$ dp=0 – constant pressure

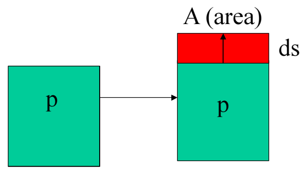
$pdv = RdT$

$mdh = mde + mpdv$ work in per unit mass (specific) quantities

$dh = de + pdv = de + RdT = C_v dT + RdT = C_p dT$

$h = C_p dT$

$R = C_p - C_v$



Basic Thermodynamics (2)

- specific enthalpy h
  - defined as

$$\boxed{h \ = \ e + \frac{p}{\rho} \ = \ e + RT}$$ using $p\!=\!\rho RT$

  - second term can be thought of as 'pressure energy'
  - as for e, a thermodynamic state variable – hence **for a perfect gas**

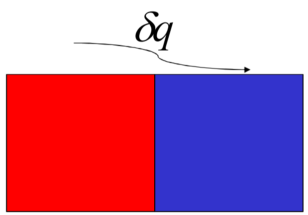
$$\boxed{h \ = \ c_p T}$$ we could use as an alternative state equation

  - c_p = specific heat at constant pressure (1004 *J/kgK* for air at standard conditions)
- from definition of e and h

$$\boxed{R \ = \ c_p - c_v}$$

$$h \ = \ e + RT = c_v T + RT = c_p T \Rightarrow RT = c_p T - c_v T$$

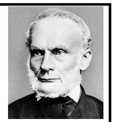
Entropy



$$\frac{-\delta q}{T_{hot}} + \frac{\delta q}{T_{cold}} \geq 0$$

(could only be = if both temperatures almost the same…)

Basic Thermodynamics (3)



Rudolf Clausius

- ratio of specific heats γ

$$\boxed{\gamma = \frac{c_p}{c_v}}$$

useful shorthand.

may not always be the same value but we can usually consider it constant.

  - $\gamma \approx$ 1.4 for air at standard conditions (more accurately, 1.403)

- entropy s represents degree of disorder   actual value of s not important.
  - determines ***direction*** of thermodynamic process
  - defined as

$$ds = \frac{\delta q_{rev}}{T} = \frac{\delta q}{T} + ds_{irrev}$$

2^{nd} law of thermodynamics.

  - δq = amount of heat added to system at temperature T
  - ds_{irrev} is entropy ***increase*** due to dissipative phenomena (viscosity, thermal conductivity and mass diffusion) occurring within the system – always positive

IREVERSIBLE PROCESS *"non-isentropic"* $ds>0$.
REVERSIBLE PROCESS *"isentropic"* $ds=0$.
this is an idealised situation

Rearranging

1st Law

$$de = \delta q - \delta w$$

$$\delta q = Tds \quad 2^{nd}\text{ Law}$$

Ideal gas law $pv = RT$

Differentiate $pdv + vdp = RdT$

$$pdv = RdT - vdp$$

$$Tds = de + \delta w = C_v dT + pdv$$

$$Tds = C_v dT + RdT - vdp = C_p dT - vdp$$

$$R = C_p - C_v$$

$$\frac{v}{T} = \frac{R}{p}$$

$$\boxed{ds = C_p \frac{dT}{T} - \frac{R}{p} dp}$$

Basic Thermodynamics(4): Isentropic Processes and entropy

- change in specific entropy is   for a flow from position 1 to 2.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

  - derived from integration of 1st law in terms of entropy
- for a reversible process $s_2 - s_1 = 0$ $\rightarrow$ ***'Isentropic'***
- with some algebra, this gives

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}} \Rightarrow p = \text{const.}\rho^{\gamma} \Rightarrow \boxed{\frac{dp}{d\rho} = \frac{\gamma p}{\rho}}$$

- relates pressure, temperature & density for an isentropic process
  - representative of many practical compressible flow problems

Basic Thermodynamics(4): Isentropic Processes and entropy

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$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

  - derived from integration of 1st law in terms of entropy

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p} \quad =0 \quad \textit{as flow is assumed isentropic}$$

$$\frac{c_p}{R}\frac{dT}{T} = \frac{dp}{p} \rightarrow \textit{Integrate from 1 to 2} \rightarrow \ln\left(\left(\frac{T_2}{T_1}\right)^{C_p/R}\right) = \ln\left(\frac{p_2}{p_1}\right)$$

$$\left(\frac{T_2}{T_1}\right)^{C_p/R} = \left(\frac{p_2}{p_1}\right) \quad \textit{from previous} \quad \frac{c_p}{R} = \frac{c_p}{c_p - c_v} = \frac{\gamma}{\gamma - 1} \rightarrow \left(\frac{p_2}{p_1}\right) = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$\textit{using} \quad p = \rho RT \qquad \left(\frac{p_2}{p_1}\right) = \left(\frac{R\rho_1 p_2}{R\rho_2 p_1}\right)^{\gamma/(\gamma-1)} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma/(\gamma-1)}\left(\frac{p_2}{p_1}\right)^{\gamma/(\gamma-1)}$$

$$\left(\frac{p_2}{p_1}\right)^{1-\left(\gamma/(\gamma-1)\right)} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma/(\gamma-1)} \qquad \left(\frac{p_2}{p_1}\right) = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

Basic Thermodynamics(4):
Isentropic Processes and entropy

- change in specific entropy is   for a flow from position 1 to 2.

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  – derived from integration of 1st law in terms of entropy

- for a reversible process $s_2 - s_1 = 0$ → ***'Isentropic'***

differentiating *w.r.t* ρ

$$\frac{dp}{d\rho} = \text{const} \times \gamma\rho^{\gamma-1} = (\text{const} \times \rho^{\gamma})\gamma\rho^{-1}$$

- with some algebra, this gives

$$\boxed{\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}} \Rightarrow p = \text{const.}\rho^{\gamma} \Rightarrow \boxed{\frac{dp}{d\rho} = \frac{\gamma p}{\rho}}$$

- relates pressure, temperature & density for an isentropic process
  – representative of many practical compressible flow problems

Basic Thermodynamics(5):
Total Temperature (1)

- T_0 - also known as 'stagnation temperature'   and reservoir temperature
  – similar concept to total pressure p_0 in potential flow
- derive from 'full' conservation of energy equation, making the assumptions
  – body forces negligible
  – adiabatic – no heat addition
  – inviscid – no ***external*** viscous losses   still have internal dissipation. No isentropic assumption.
  – steady flow
- in terms of enthalpy, gives   will take more manipulation

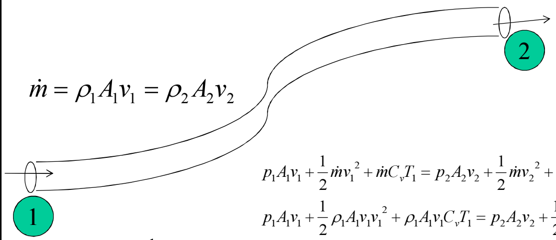
$$\frac{D\left(h + V^2/2\right)}{Dt} = 0 \Rightarrow \boxed{h + \frac{V^2}{2} = \text{constant} = h_0}$$

enthalpy of a flow brought to rest adiabatically

  – **along a streamline** (compare with Bernoulli's Equation)

Substantial derivative is the time rate of change of a fluid element moving with the flow ie along a streamline.

Streamtube



$$\dot{m} = \rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

$$p_1A_1v_1 + \frac{1}{2}\dot{m}v_1^2 + \dot{m}C_vT_1 = p_2A_2v_2 + \frac{1}{2}\dot{m}v_2^2 + \dot{m}C_vT_2$$

$$p_1A_1v_1 + \frac{1}{2}\rho_1A_1v_1v_1^2 + \rho_1A_1v_1C_vT_1 = p_2A_2v_2 + \frac{1}{2}\rho_2A_2v_2v_2^2 + \rho_2A_2v_2C_vT_2$$

$$\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 + C_vT_1 = \frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 + C_vT_2$$

Rate KE entering $\frac{1}{2}\dot{m}v^2$

Rate work done by pressure forces $Fv = pAv$

Rate internal energy entering $\dot{m}C_vT$

Ignore internal energy and assume const density - Bernoulli

$$p + \frac{1}{2}\rho v^2 = const = p_0$$

Very rarely used in this part of the course!

$$h + \frac{1}{2}v^2 = const = h_0$$

'Compressible' Bernoulli

Used all the time!

$$\boxed{C_pT + \frac{1}{2}v^2 = const = h_0 = C_pT_0}$$

Stagnation Temperature

$$\boxed{C_pT + \frac{1}{2}v^2 = const = h_0 = C_pT_0}$$

Velocity? Urgh! Much prefer Mach number!

$$C_pT_0 = C_pT + \frac{1}{2}M^2\gamma RT$$

Remember $v = Ma$, $a^2 = \gamma RT$

$$T_0 = T + \frac{1}{2}\frac{M^2\gamma RT}{C_p}$$

$$\frac{\gamma R}{C_p} = \frac{\gamma(C_p - C_v)}{C_p} = \gamma\left(1 - \frac{1}{\gamma}\right) = \gamma - 1$$

$$T_0 = T\left(1 + \frac{(\gamma-1)M^2}{2}\right)$$

Surely such simple, humble equation can be of little use?

This is **NOT** heating due to friction – inviscid flow! It is analogous to heating as a result of compression in a bike pump

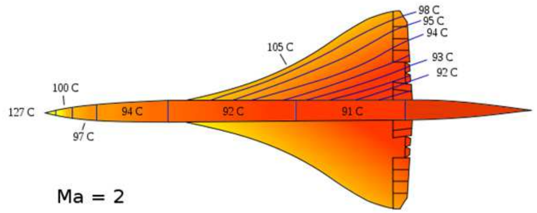
Planes

M=0.8, T=219K, T0=247K=-26C ✔

M=2.02, T=217K, T0=394K=121C ~ max allowed for aluminium 127C – so max Mach effectively determined by T0 equation ✔

M=3, T=222K, T0=622K=349C (Titanium used) ✔

M=25, T=180K?, T0=22680K=22407C? ✘
Air no longer a continuum, gamma no longer the same (or even constant), no longer adiabatic (air radiates energy) so this is **not** accurate. Actual max for shuttle (for any M) ~ 1600C



Ma = 2

Basic Thermodynamics (6):
Total Temperature (2)

- for a calorically perfect gas $h=c_pT$ hence *see* BC1.7

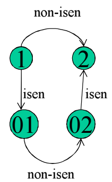
$$\boxed{c_pT+\frac{V^2}{2} = c_pT_0} \qquad c_pT_0 = h_0$$

  - where T_0 is the 'total' or 'stagnation' or 'reservoir' temperature
  - an alternative form of the energy equation along a streamline
- temperature of fluid element brought to rest *adiabatically*
- no assumption about *entropy* made in derivation
- in an *isentropic* process T_0, p_0 and ρ_0 are constant
  - eg a sound wave
- in a ***non****isentropic* (but adiabatic) process **only** T_0 is constant
  - eg a shock wave

T_o = *"total energy"*
P_o = *"total usable energy"*

Compressible flow
⇓ *assume no heat loss*
Adiabatic flow
⇓ *assume reversible*
Isentropic flow

Basic Thermodynamics (7):
Stagnation pressure and entropy



- Consider an adiabatic process from condition 1 to 2

$p_1T_1V_1s_1 \longrightarrow \quad p_2T_2V_2s_2 \longrightarrow$

- Derive "stagnation" values corresponding to 1 and 2 using an **isentropic** deceleration.
- Entropy change between 1 and 2 equals entropy change in stagnation values.

$$s_2 - s_1 = c_p\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = c_p\ln\left(\frac{T_{o_2}}{T_{o_1}}\right) - R\ln\left(\frac{p_{o_2}}{p_{o_1}}\right)$$ *see* BC1.12

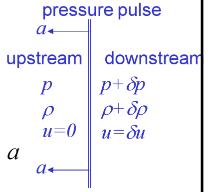
- For **adiabatic** flow $T_{01} = T_{02} \rightarrow \ln\left(\frac{T_{02}}{T_{01}}\right) = 0 \rightarrow -\frac{s_2 - s_1}{R} = \ln\left(\frac{p_{o_2}}{p_{o_1}}\right)$

$$\frac{p_{o_2}}{p_{o_1}} = e^{-\left(\frac{s_2-s_1}{R}\right)}$$

- non-isentropic – total pressure "lost" $\quad \frac{p_{o_2}}{p_{o_1}} < 1 \rightarrow p_{o_2} < p_{o_1}$

  Isentropic – total pressure conserved $\quad \frac{p_{o_2}}{p_{o_1}} = 1 \rightarrow p_{o_2} = p_{o_1}$

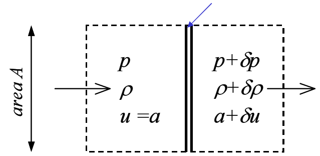
Speed of Sound (1)



- rate of propagation of pressure ‘information’
  - instantaneous in incompressible fluid
  - finite velocity a in compressible flow
- start with plane pressure pulse moving at velocity a
- assume pulse is of infinitesimal strength, hence
  - changes in fluid properties p, ρ and u are ‘small’ later assume products are negligible
  - adiabatic process
  - isentropic process
- use control volume moving with the wave front Galilean Transformation
  - equivalent to superimposing freestream velocity a so pulse becomes stationary
- apply momentum & continuity equations to flow through control volume next slide

Speed of Sound (2)

stationary pulse
flow moving at speed a



- apply simple continuity and momentum equations
  - neglect 2nd and 3rd order terms
  - gives Newton’s result

$$\boxed{\frac{dp}{d\rho} = a^2} \sim \frac{1}{'compressibility'}$$

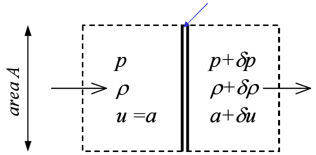
- now apply *isentropic* relation, followed by equation of state

$$\Rightarrow \boxed{a^2 = \frac{\gamma p}{\rho}} \Rightarrow \boxed{a = \sqrt{\gamma RT}}$$

- speed of sound is a function of static temperature T only

Speed of Sound (2)

stationary pulse
flow moving at speed a



- apply simple continuity and momentum equations
  - neglect 2nd and 3rd order terms
  - gives Newton’s result

continuity: mass flow conserved $\dot m = \rho a A = (\rho+\delta\rho)(a+\delta u)A$

neglecting products of small terms $\rho\delta u + a\delta\rho = 0$

momentum: force= rate of change of momentum = change in momentum rate

net force $= pA-(p+\delta p)A = (\rho+\delta\rho)(a+\delta u)^2 A - \rho a^2 A$

neglecting products of small terms $-\delta p = 2a\rho\delta u + a^2\delta\rho$

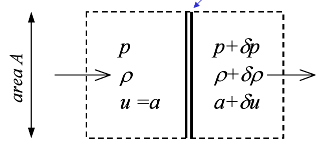
from continuity, sub in term for $\rho\delta u$ *gives*

$$-\delta p = -2a^2\delta\rho + a^2\delta\rho \rightarrow \frac{\delta p}{\delta\rho} = a^2$$

so in the limit of infinitely small disturbances $\boxed{\frac{dp}{d\rho} = a^2}$

Speed of Sound (2)

stationary pulse
flow moving at speed a



- apply simple continuity and momentum equations
  - neglect 2nd and 3rd order terms
  - gives Newton’s result

$$\boxed{\frac{dp}{d\rho} = a^2} \sim \frac{1}{'compressibility'} \qquad \frac{d\rho}{dp} = \text{compressibility}$$

- now apply *isentropic* relation, followed by equation of state from *BC1.9* $\frac{dp}{d\rho} = \frac{\gamma p}{\rho} \rightarrow a^2 = \frac{\gamma p}{\rho}$

$$\Rightarrow \boxed{a^2 = \frac{\gamma p}{\rho}} \Rightarrow \boxed{a = \sqrt{\gamma RT}}$$

using the equation of state $p = \rho RT$

$a^2 = \gamma RT$

- speed of sound is a function of static temperature T only

For air at standard temperature and pressure (STP: p=1atm γ=1.403 R=287.1 T=288°K)

a= 340.6

Speed of Sound (3) : Mach Number

Ernst Mach

- *local* Mach Number defined as

$$\boxed{M = \frac{V}{a}} \sim \frac{directed\ KE}{random\ thermal\ energy} \qquad M^2 = \frac{V^2}{a^2} = \frac{V^2}{\gamma RT} = \frac{V^2}{\gamma R}\frac{c_v}{e} = \left(\frac{c_v}{\gamma R}\right)\frac{V^2}{e}$$

- dynamic pressure q can be given in terms of M

$$\boxed{q = \frac{1}{2}\rho V^2 = \frac{1}{2}p\gamma M^2} \qquad q = \frac{1}{2}\rho V^2 = \frac{1}{2}\rho M^2 a^2 = \frac{1}{2}\rho M^2 \frac{\gamma p}{\rho}$$

- from which it follows that the pressure coefficient C_p is

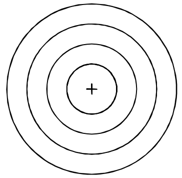
note: use pressure ratio rather than difference

$$\boxed{C_p = \underbrace{\frac{p-p_\infty}{q_\infty}}_{definition\ of\ c_p} = \frac{(p/p_\infty - 1)}{\frac{\gamma}{2}M_\infty^2}} \qquad \frac{p-p_\infty}{\frac{1}{2}\rho_\infty v_\infty^2} = \frac{p-p_\infty}{\frac{1}{2}\rho_\infty M_\infty^2 \frac{\gamma p_\infty}{\rho_\infty}} = \frac{p-p_\infty}{\frac{1}{2}M_\infty^2 \gamma p_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

Speed of Sound (4) : Mach Cone (1)

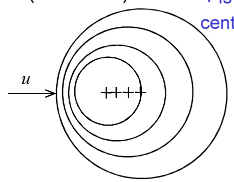
note: a=maximum velocity of pressure information for **small** disturbances. Strong pressure pulses such as shocks can move faster than sound, eg from explosions

- point sound source in 'still air'
  - pressure wave radiated in all directions
  - *spherical* wave front of radius at
  - all of fluid eventually disturbed



- now add freestream velocity $u < a$ (subsonic)
  - spherical wave fronts displaced downstream by distance ut
  - all of fluid still eventually disturbed

+ is the apparent center of disturbance

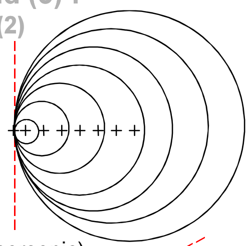


- 'ripples on a pond'
  - close analogy between shallow water waves and sound waves

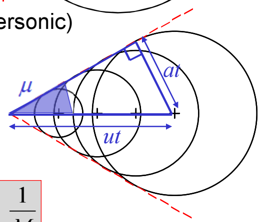
different values of γ

Speed of Sound (5) : Mach Cone (2)

- increase freestream to $u = a$ (sonic)
  - no wavefronts propagated upstream
  - 'zone of silence' ahead of source
  - no 'warning' of downstream disturbances
  - pressure rise at front no longer infinitesimal



- finally, increase freestream to $u > a$ (supersonic)
  - spherical wavefronts swept downstream to form a *'Mach Cone'*
  - defines lateral extent of source influence



- half-angle μ of cone = *'Mach Angle'*
- surface of cone = *'Mach Wave'* $\boxed{\sin\mu = \frac{1}{M}}$

$$\sin\mu = \frac{at}{ut} = \frac{a}{u} = \frac{1}{M}$$

1D Compressible Flow (1): Energy Equation Revisited

- for an ***adiabatic*** process from BC1.11

$$\boxed{c_pT + \frac{V^2}{2} = c_pT_0}$$

  - where T_0 is the 'total' or 'stagnation' or 'reservoir' temperature
- more useful in terms of Mach Number M and constants γ R *etc*, but not other state variables such as: p, ρ *etc.*
- so, substitute

BC2.2 $M^2 = \frac{V^2}{\gamma RT}$, $c_p = \frac{\gamma R}{\gamma - 1}$ BC1.8

to obtain

$$\boxed{T_0 = T\left\{1 + \frac{\gamma - 1}{2}M^2\right\}}$$

ADIABATIC fundamental energy equation in terms of M & T

- no assumption made about entropy (yet) so equation valid through shock waves

1D Compressible Flow (2): Local Speed of Sound

- Mach Number based on *local* speed of sound a
- but a varies with *local* temperature T ..

$$a = \sqrt{\gamma RT} \quad \Rightarrow \quad a_0/a = \sqrt{T_0/T}$$

$a_0 = \sqrt{\gamma R T_0}$ $\quad$ a_o = reservoir speed of sound

and hence

substitute into energy relationship in previous slide BC2.6

$$\boxed{a_0 = a\left\{1+\frac{\gamma-1}{2}M^2\right\}^{0.5}}$$ ADIABATIC

- 'stagnation' or 'reservoir' speed a_0 is constant in an adiabatic flow
- ***'critical'*** value a^* is speed of sound for M = 1
  - common reference value in duct flows
  - $a^* = 0.913 a_0$ for air at STP

$$\boxed{a^* = a_0\sqrt{\frac{2}{\gamma+1}}}$$

later we will see values of p^*, ρ^* *etc.*

Map – lectures 1-3

Today

Adiabatic – no heat lost or gained – total enthalpy constant along a streamline

Energy equation

$$\frac{T_0}{T} = 1+\frac{\gamma-1}{2}M^2$$

Valid through shocks!

Entropy Constant

$p = const \times \rho^\gamma$

+ideal gas

$p = \rho RT$

Speed of sound

$a^2 = \gamma RT$

1[st] and 2[nd] Laws of Thermo.

Isentropic flow relations

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

$$\frac{p_0}{p} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

Not valid through shocks! Still valid in between

Cons. of mass

Mach-area Relation + Con-di nozzles + Mach-rate of area change relation

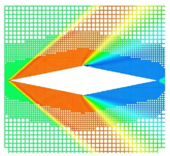
NB we have not yet done shocks!

Problems we shall solve

- Working out pressure/density/temperature from Mach number. Usually done using tables.
- Working out Mach number from area ratio for nozzles, then using Mach number for pressure, density and temperature.

ONCE WE HAVE COVERED SHOCKS (next lecture)…

- Cp on supersonic wedge/bicon aerofoils – allows us to find Cl, Cd and Cm.
- Supersonic pitot probes
- Nozzles with simple shock patterns



1D Compressible Flow (3): 'Compressible Bernoulli'

- make the additional assumption of an ***isentropic*** process

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^\gamma = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1}$$ from BC1.9

- substitute in total temperature equation to give

$$\boxed{p_0 = p\left\{1+\frac{\gamma-1}{2}M^2\right\}^{\frac{\gamma}{\gamma-1}} \qquad \rho_0 = \rho\left\{1+\frac{\gamma-1}{2}M^2\right\}^{\frac{1}{\gamma-1}}}$$

$$\left(\frac{p_0}{p}\right) = \left(\frac{T_0}{T}\right)^{\gamma/\gamma-1} = \left\{1+\frac{\gamma-1}{2}M^2\right\}^{\gamma/\gamma-1}$$

ISENTROPIC

$$\left(\frac{\rho_0}{\rho}\right)^\gamma = \left(\frac{p_0}{p}\right) = \left\{1+\frac{\gamma-1}{2}M^2\right\}^{\gamma/\gamma-1}$$

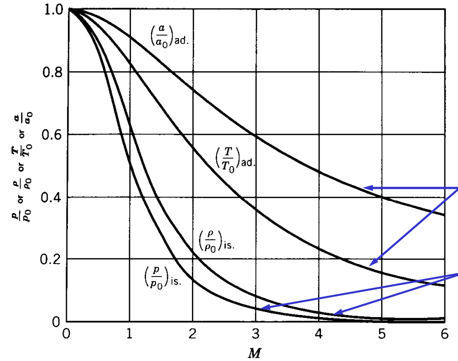
- compressible equivalent of Bernoulli's Equation $\quad$ p_o constant in an isentropic flow
  - relationship between pressure, density and velocity
- ratios T_0/T, p_0/p and ρ_0/ρ given in compressible flow tables

Tables

- A wide variety of tables will be used in the course. The isentropic flow relations are conveniently tabulated as a function of Mach number. Be prepared to interpolate linearly in Mach number.
- Other tables describe shocks and expansions – these will be covered in the next few lectures

M	p_o/p	ρ_o/ρ	T_o/T	A/A^*
0.50	1.187	1.130	1.050	1.340
0.52	1.203	1.141	1.054	1.303
0.54	1.220	1.152	1.059	1.270
0.56	1.238	1.164	1.063	1.240
0.58	1.257	1.177	1.068	1.213

1D Compressible Flow (4): Compressible Flow Relations



Flow parameters versus Mach number of adiabatic (ad.) and isentropic (is.) flows.

tabulated data: must interpolate between data points

valid through strong pressure waves such as shocks *ie* non-isentropic flows

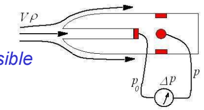
only valid for isentropic flow

(NB ratios inverted to keep magnitude < 1)

1D Compressible Flow (5): Pitot-Static Probe in Compressible Flow

example of compressible Bernoulli

- incompressible pitot-static equation



assume flow decelerated isentropically so $p_{pitot} = p_o$

Bernoulli $V_{measured} = \sqrt{\dfrac{2(p_0 - p)}{\rho}}$ *for compressible ρ variable*

- expanding the pressure ratio p_0/p equation (for air) from BC2.8

$$\frac{p_0}{p} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1} = 1+\frac{\gamma M^2}{2}\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\dots\right]$$

use binomial theorem with $\gamma = 1.4$

$$(1+x)^n = \left(1+nx+\frac{n(n-1)}{2}x^2+\cdots\right)$$

and hence using $q = \gamma M^2 p/2$ from BC2.3

$$p_0 - p = \frac{1}{2}\rho V^2\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\dots\right]$$

$$p_0 = p + \frac{\gamma M^2 p}{2}\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\dots\right]$$

- term in [ ] is basic compressibility correction $p_0 - p = \dfrac{\rho V^2_{measured}}{2}$

$$\boxed{V = V_{measured}\Big/\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\dots\right]^{0.5}}$$

$$V_{measured} = \sqrt{2\frac{p_0-p}{\rho}} = \sqrt{V^2\left[1+\frac{M^2}{4}+\frac{M^4}{40}+\cdots\right]}$$

1D Compressible Flow (6): Airspeed Corrections

for info-not in exam

- ASIR *'Airspeed Indicator Reading'* — basic calibration (compressible flow at SL)

  \+ Instrument Error Correction
- IAS *'Indicated Airspeed'* V_i

  \+ Pressure Error Correction (position error in static reading)
- CAS *'Calibrated Airspeed'* V_c

  \+ Compressibility Correction (altitude effect on M)
- EAS *'Equivalent Airspeed'* V_e

  \+ Density Correction (altitude effect on density)
- TAS *'True Airspeed'* V

Towards 1D Euler equation

- For the steady Euler equations the momentum equations can be written out in full as (cons. of mass has been substituted in)

$$\boxed{\begin{aligned} \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} \\ \rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} \\ \rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} \end{aligned}}$$

- These three momentum equations can, in certain circumstances, be reduced to one equation which links pressure to velocity and density.

Irrotational Flow

Assume irrotational flow i.e. vorticity (or elemental angular or rotational velocity) is zero.

$$\nabla\times\mathbf{V} = \mathbf{i}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\mathbf{j}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\mathbf{k}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0 \Rightarrow \frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}$$

$$\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0 \Rightarrow \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$$

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0 \Rightarrow \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$$

$$\boxed{\begin{aligned} \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} \\ \rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} \\ \rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} \end{aligned}} \begin{aligned} &\times dx \\ &+ \\ &\times dy \\ &+ \\ &\times dz \end{aligned}$$

Substitute for highlighted terms

$$u = u(x,y,z)$$

$$du = \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$ definition of derivative $$udu = \frac{1}{2}d\left(u^2\right)$$

$$\frac{1}{2}d\left(u^2+v^2+w^2\right) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right)$$

Alternative approach – if you're happy with 1D

$$\frac{\partial(\rho u^2+p)}{\partial x}=0$$ ← Momentum eq'n in 1D

$$\rho u\frac{\partial u}{\partial x}+u\frac{\partial(\rho u)}{\partial x}+\frac{\partial p}{\partial x}=0$$ ← Expanding

$$\frac{\partial(\rho u)}{\partial x}=0$$ ← Mass conservation in 1D

$$\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0$$ ← Substitute mass conservation into momentum eq'n

If flow 1D, then we only have x direction, so partial notation can be dropped.

Most compact approach

$$\rho\mathbf{V}\cdot\nabla\mathbf{V} = -\nabla p$$

Vector calculus identity

$$\boxed{\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) = \mathbf{V}\cdot\nabla\mathbf{V}+\mathbf{V}\times\nabla\times\mathbf{V}}$$

Zero if irrotational

Option (a)

Taking the dot product gives

$$\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V})\cdot\mathbf{ds} = (\mathbf{V}\cdot\nabla\mathbf{V})\cdot\mathbf{ds}+\underline{(\mathbf{V}\times\nabla\times\mathbf{V})\cdot\mathbf{ds}}$$ Option (b)

Zero – bracket term is perpendicular to streamline by definition

We can now take

$$(\rho\mathbf{V}\cdot\nabla\mathbf{V})\cdot\mathbf{ds} = -\nabla p\cdot\mathbf{ds}$$

to arrive at

$$\frac{\rho}{2}\nabla(\mathbf{V}\cdot\mathbf{V})\cdot\mathbf{ds} = -\nabla p\cdot\mathbf{ds}$$

Valid everywhere if flow irrotational, or only along a streamline otherwise

- Since $V^2 = (u^2 + v^2 + w^2)$ this is equivalent to

$$\boxed{dp = -\rho V\,dV}$$

1D Euler equation

This relationship still applies to **compressible & Incompressible flows**

As an aside, we can integrate this to get Bernoulli's equation if we assume the density is constant! Or in the compressible case we can substitute with $p = k\rho^\gamma$

$$\frac{1}{\rho} = \left(\frac{p}{k}\right)^{\frac{-1}{\gamma}} \longrightarrow \int \left(\frac{p}{k}\right)^{\frac{-1}{\gamma}} dp = \int -V dV$$

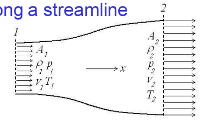
Integrate + set constant of integration so that p=p0 at stagnation

$$\left(\frac{1}{k}\right)^{\frac{-1}{\gamma}} p^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{\gamma-1} = -\frac{V^2}{2} + \left(\frac{1}{k}\right)^{\frac{-1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{\gamma-1} \longrightarrow \left(1 + \frac{M^2(\gamma-1)}{2}\right)^{\frac{\gamma}{\gamma-1}} = \frac{p_0}{p}$$

i.e. the isentropic flow relations again! Perhaps this explains the origins of `compressible Bernoulli'

Back to work and …1D Isentropic Duct Flow

and flow along a streamline



- flow in a duct with *slowly* changing area A
  - angle between streamlines is 'small'
  - impact of boundary layer small (for turbulent flow)
- flow is therefore effectively one-dimensional

"convergent divergent nozzles"

- assume inviscid, steady flow with negligible height variation
- momentum equation is the **1D Euler Equation**

$$dp = -\rho V\,dV \Rightarrow \boxed{\frac{dp}{\rho} + V dV = 0}$$

$$dp = a^2 d\rho \qquad \frac{a^2 d\rho}{\rho} + V dV = 0$$

  - from previous work

from handout 5.12

- continuity equation given by derivative of mass flow

$$\frac{dV}{V} + \frac{dA}{A} - \frac{V^2}{a^2}\frac{dV}{V} = 0$$

$$\dot{m} = \rho A V = \text{const} \Rightarrow \boxed{\frac{dV}{V} + \frac{dA}{A} + \frac{d\rho}{\rho} = 0}$$

$d\dot{m} = \rho A dV + \rho V dA + AV d\rho = d(\text{const}) = 0$ then divide by $\rho A V$

Area Velocity Variation (Adiabatic Flow)

- using Newton's result for a then gives

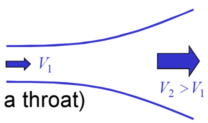
from BC2.2 $\frac{dp}{d\rho} = a^2$ sub $dp = a^2 d\rho$ into 1D Euler $\Rightarrow$ $\boxed{\frac{dA}{A} = (M^2 - 1)\frac{dV}{V}}$ *ADIABATIC*

  - a direct relation between area variation and velocity variation
- now introduce the element of length along the duct dx

$$\boxed{(M^2 - 1)\frac{1}{V}\frac{dV}{dx} = \frac{1}{A}\frac{dA}{dx}}$$

or flow streamline

- consider an *accelerating* flow $dV/dx > 0$
  - $M < 1 \to dA/dx < 0 \to$ *converging* duct
  - $M > 1 \to dA/dx > 0 \to$ *diverging* duct (!)
  - $M = 1 \to dA/dx = 0 \to$ *minimum* area (ie a throat)



Area Ratio (Isentropic Flow)

- sonic (M = 1) duct flow can **only** occur at a throat – ie where the area A_t is a local minimum
- since M =1 here, this area is also the 'critical' area A^*
- A^* is often used as a reference area in compressible flow
- use isentropic equations to relate local area A andMach Number M to A^*

$$\frac{\rho^*}{\rho} = \left(\frac{1 + \frac{\gamma-1}{2}.1^2}{1 + \frac{\gamma-1}{2}M^2}\right)^{\frac{-1}{\gamma-1}} \quad and \quad \frac{a^*}{a} = \left(\frac{1 + \frac{\gamma-1}{2}.1^2}{1 + \frac{\gamma-1}{2}M^2}\right)^{\frac{-1}{2}}$$

mass conservation $\rho A V = \rho^* A^* V^*$

$V = Ma \quad V^* = a^* \quad \Rightarrow \quad \rho A M a = \rho^* A^* a^*$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{a}\frac{1}{M} \quad \text{then using}$$

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho} \qquad \frac{a^*}{a} = \frac{a^*}{a_0}\frac{a_0}{a}$$

$$\boxed{\frac{A}{A^*} = \frac{1}{M}\left\{\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}}$$

using isentropic density/speed of sound relationships

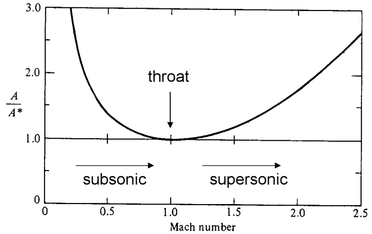
2 solutions: $M>1$, $M<1$
  - given in standard tables

Area Ratio (Isentropic Flow)

- sonic (M = 1) duct flow can **only** occur at a throat – ie where the area A_t is a local minimum
- since M =1 here, this area is also the 'critical' area A^*
- A^* is often used as a reference area in compressible flow
  - use isentropic equations to relate local area A and Mach Number M to A^*
  - given in standard tables

$$\frac{A}{A^*} = \frac{1}{M}\left\{\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right\}^{\frac{\gamma+1}{2(\gamma-1)}}$$

ISENTROPIC



M=0.38: area reduction to accelerate to sonic flow (M=1) is 1/1.658

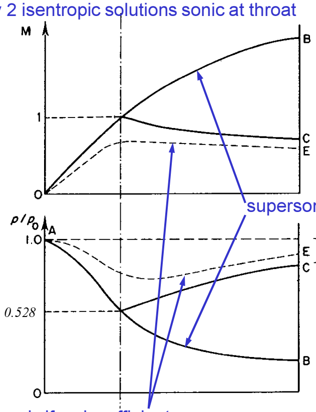
M=1.98: area reduction to decelerate to sonic flow (M=1) is 1/1.658

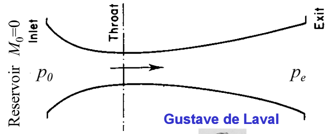
Laval Nozzle

19^{th} Century steam turbine designed using isentropic area ratio and ..

Infinitely many solutions depending on p_e but only 2 isentropic solutions sonic at throat

Compressible Bernoulli $p_0 = p\left\{1+\frac{\gamma-1}{2}M^2\right\}^{\frac{\gamma}{\gamma-1}}$





supersonic if p_e sufficient

Gustave de Laval

Subsonic throughout

isentropic flow

isentropic flow not possible

Shock system in the nozzle

Ideally expanded

isentropic flow (in nozzle)

but shock system outside the nozzle

subsonic if p_e insufficient

'Choking'

- once M=1 reached at the throat further reductions in p_e have **no** effect on subsonic flow upstream
  - no pressure 'information' can propagate past the throat
- therefore mass flow through duct also unaffected
  - sonic throat → duct is 'choked'
- mass flow can be written in non-dimensional form

$$\frac{\dot{m}\sqrt{RT_0}}{Ap_0} = \sqrt{\gamma}\frac{p/p_0}{\sqrt{T/T_0}}M = fn(\gamma, M)\ only$$

$m = \rho AV \quad M = \frac{V}{a}$

$a^2 = \gamma RT \quad p = \rho RT$

and rearrange

  - with maximum value of 0.686 (for air) at M = 1
- maximum (choked) mass flow is then

$$\dot{m}_{max} = 0.686\frac{A_t p_0}{\sqrt{RT_0}} = 0.0404\frac{A_t p_0}{\sqrt{T_0}}$$

'Choking'

$$\dot{m} = \rho AV = \frac{p}{RT}AMa = \frac{p}{RT}AM\sqrt{\gamma RT}$$

$$\dot{m} = \sqrt{\frac{\gamma}{RT}}\frac{p}{p_0}p_0AM = \sqrt{\frac{\gamma}{R\frac{T}{T_0}T_0}}\frac{p}{p_0}p_0AM$$

For M=1 T/T0=0.833
p/p0=0.528

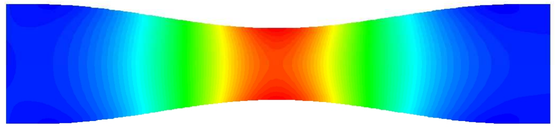
$$\frac{\dot{m}\sqrt{RT_0}}{Ap_0} = \sqrt{\frac{\gamma}{T/T_0}}\frac{p}{p_0}M = \sqrt{\frac{1.4}{0.833}}0.528\times 1 = 0.685$$

Basic Compressible Flow Review

- review equations of motion of a fluid
  - conservation of energy – fundamental!
  - 1D Euler equation
  - need for thermodynamics in compressible flow – provides additional equations to close or simplify the problem
- review basic thermodynamic concepts
  - energy, enthalpy and entropy … you should know these from your 1st year
- speed of sound
  - propagation of information
- 1D compressible flow
  - 'compressible Bernoulli' – builds on energy equation with the isentropic assumption
- isentropic duct flows with varying area
  - critical conditions – possibility of choking

Fully subsonic
(flood plot shows Mach number)

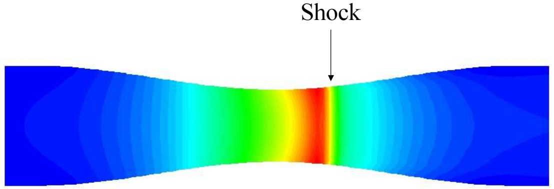
Here Athroat/Aexit=0.6
Remember Athroat is only = to A* if the nozzle is choked



Isentropic (and symmetric)

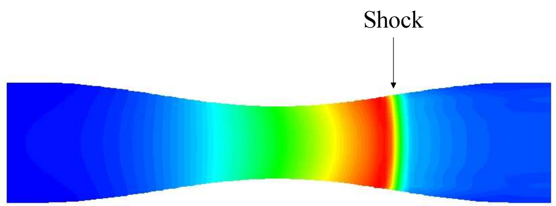
NB these are 2D CFD images, so not completely uniform across nozzle

Reduce downstream pressure – now choked



Non-isentropic (and non-symmetric – mirroring does not produce a valid solution)

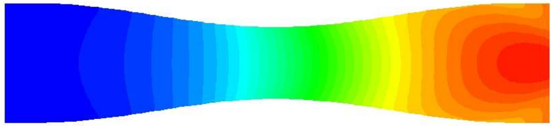
Reduce downstream pressure further – now shock moves further downstream



Non-isentropic (and non-symmetric – mirroring does not produce a valid solution)

Subsequent pressure drop downstream produces
fully supersonic flow

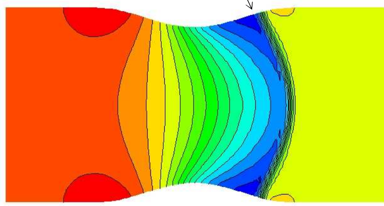
Shock system might exist at
nozzle exit



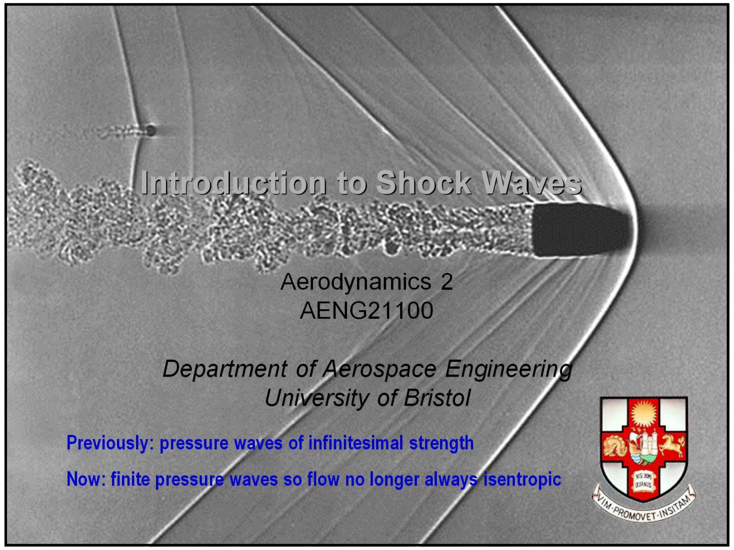
Isentropic
(but may or may not remain isentropic downstream)

This case is non-symmetric, but a left-right mirror is also a valid solution, so a link between entropy and symmetry still holds

Next 3 lectures will focus on shockwaves



- You should now be able to attempt the 1st tutorial sheet!
- Tutorial session – in 2 weeks time?
- Make sure you attempt all questions before you come – this will clarify in your own mind what you know and what you need to practice.



Map so far

Isentropic flow along streamlines and nozzles → Non isentropic and isentropic flow meeting corners → Applications to aerofoils

Done | Next 3 lectures | Future

Next 3 lectures

- Normal shocks ← Today
- Oblique shocks
- Expansion fans

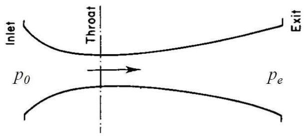
What is a shockwave?

- It is a discontinuity in velocity, temperature, pressure, total pressure, density and total density
- But not in total temperature – why?
- It takes the flow irreversibly from > Mach 1 to < Mach 1
- It is the line along which downstream perturbations coalesce when they meet a supersonic upstream zone
- It is ~ 60 molecular mean free paths (MFP) wide so <<< thinner than anything else. We do not need to use models on this scale – the Euler equations are still satisfactory
- Similar to a breaking wave on a beach - what we are interested in is the pressure behind the shock (height of water) related to its Mach number (speed of wave hitting beach). Not interested in the frothing white region near the front (since it is ~ 60MFP only)

Aerodynamics 2 : *Slide SW.4*

Hydraulic analogy

- Surface water waves travel at a set speed – the Froude number (Fr) is the ratio between the speed of the water and the speed of these waves. It is analogous to the Mach number
- So if we get some water moving at Fr>1 we should see behaviour similar to shockwaves

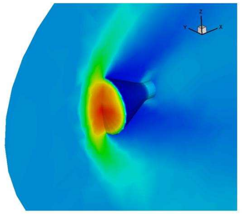


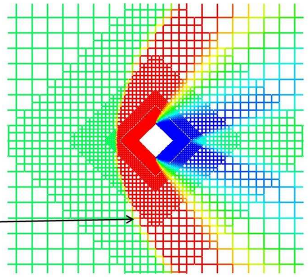
Aerodynamics 2 : *Slide SW.5*

Other examples

Aerodynamics 2 : *Slide SW.6*

Last year - Chelyabinsk meteor 'Mach 60' - ?





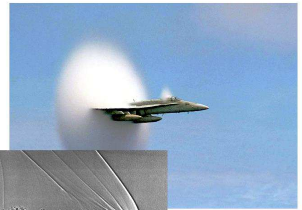
Pressure jump through shock breaks windows…

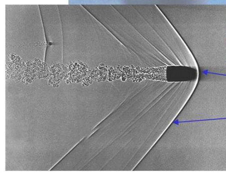
Sonic boom rocks large part of Britain
Daily Mail 13 April 2012
'A sonic boom is the sound associated with the shockwaves created when an object travels through the air and breaks the sound barrier…'
'The noise contains large amounts of sound energy, meaning sonic booms are often mistaken for explosions…'

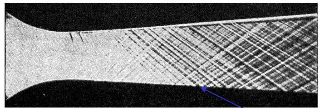
Aerodynamics 2 : *Slide SW.7*

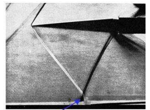
Shock Waves etc

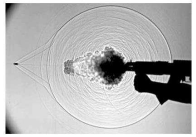
- normal shock waves
  - extension of sound wave analysis
  - change in properties through shock
- 2D supersonic flow
  - Mach waves
  - Prandtl-Meyer expansion
  - oblique shock waves
- shocks in ducts
  - intakes & nozzles
  - supersonic pitot-static











Shock "reflections"

Aerodynamics 2 : *Slide SW.8*

Shock Spotting

- Wait for clear, bright sunlight
- Best from directly above, so close to midday
- Keep looking. Depending on aircraft weight and Mach number, the shock will move in and out of the light (forwards with lower weight or lower Mach number)
- Turbulence makes them more obvious as they move back and forth a few cm. The incidence is changing in the gusts, altering lift coefficient and shock position
- If you get a good photo, send it to me!

Aerodynamics 2 : *Slide SW.9*

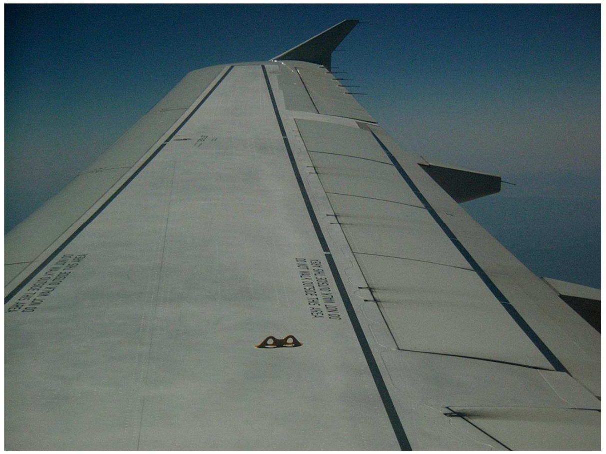
Shocks

Aerodynamics 2 : *Slide SW.10*

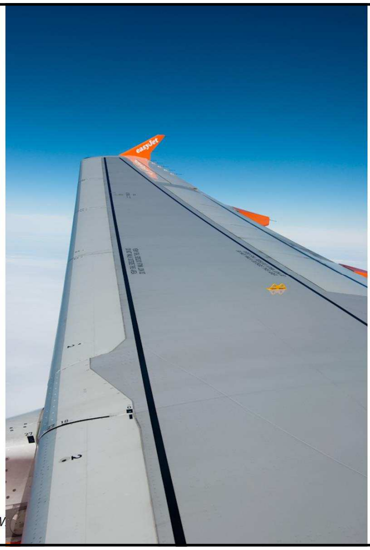
Shocks

Aerodynamics 2 : *Slide SW.11*

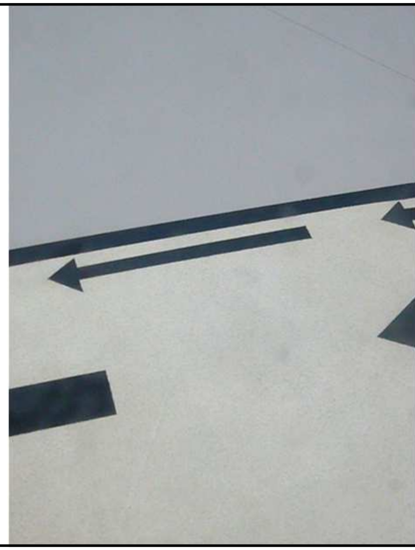
Shocks



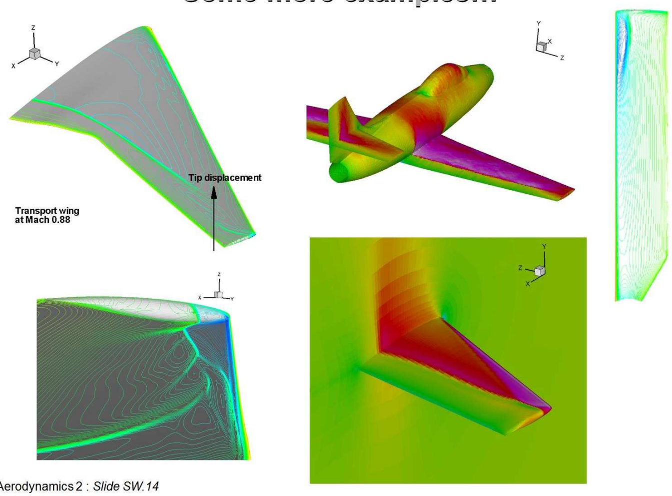
Aerodynamics 2 : *Slide SW.12*

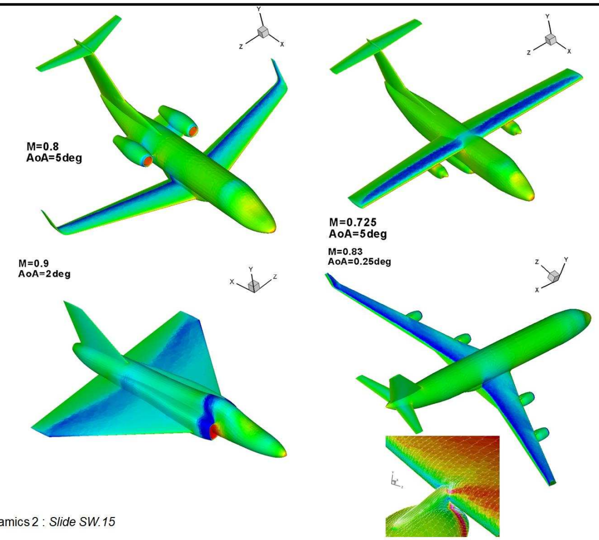


Aerodynamics 2 : *Slide SW*



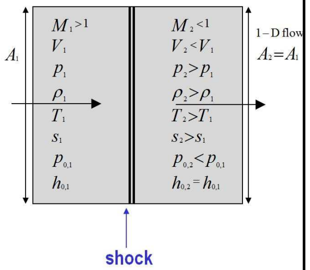
Some more examples…





Normal Shock Waves (1)

- discontinuity in supersonic flow
  - due to 'large' disturbance
  - Compression & deceleration
  - width ~ 6x10^{-8}m
  - Subsonic flow allows upstream propagation: flow adjusts, no shocks
- similar approach to sound wave
  - control volume moving with the shock
  - 'station numbers': 1 & 2 up-/down-stream
- shock has finite strength, hence
  - adiabatic process- changes too rapid for heat transfer
  - ***nonisentropic process*** - but still inviscid (externally)
  - changes in fluid properties p, ρ and u are 'large'
- apply momentum, continuity and energy equations to flow through control volume
  - rewrite in terms of Mach Number M and speed of sound a



Normal Shock Waves (2)
Mach Number Variation Through Shock

- Continuity $\dot{m}=\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \rightarrow \rho_1 a_1 M_1 = \rho_2 a_2 M_2$
- Momentum **See previous analysis of infinitesimal (sound) waves**

pressure force=rate of change of momentum

$$p_1 A_1 - p_2 A_2 = \rho_1 A_1 V_1 \left(V_2 - V_1\right)$$

apply continuity and eqn. of state $a^2=\gamma RT=\frac{\gamma p}{\rho}$ $\quad p=\frac{\rho a^2}{\gamma}$

$$p_1 - p_2 = \rho_1 V_1 (V_2 - V_1) \;\Rightarrow\; \frac{\rho_1 a_1^2}{\gamma} - \frac{\rho_2 a_2^2}{\gamma} = \rho_1 a_1 M_1 (a_2 M_2 - a_1 M_1)$$

rearrange $\rho_1 a_1^2\left(1+\gamma M_1^2\right) = \rho_2 a_2^2\left(1+\gamma M_2^2\right)$

- Energy – adiabatic flow so use temperature energy eqn. $\frac{T_0}{T}=1+\frac{\gamma-1}{2}M^2$ **BC18**

$$\left(\frac{T_1}{T_2}\right)=\left(\frac{T_0}{T_2}\frac{T_1}{T_0}\right)=\left(1+\frac{\gamma-1}{2}M_2^2\right)\Big/\left(1+\frac{\gamma-1}{2}M_1^2\right)=\left(\frac{a_1^2}{\gamma R}\right)\Big/\left(\frac{a_2^2}{\gamma R}\right)=\left(\frac{a_1^2}{a_2^2}\right) \quad \text{using} \quad a=\sqrt{\gamma RT}$$

$$a_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right) = a_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

Algebraic Interlude (not examinable)

$$\rho_1 a_1 M_1 = \rho_2 a_2 M_2$$

$$\rho_1 a_1^2\left(1+\gamma M_1^2\right) = \rho_2 a_2^2\left(1+\gamma M_2^2\right)$$

$$\frac{a_1}{M_1}\left(1+\gamma M_1^2\right) = \frac{a_2}{M_2}\left(1+\gamma M_2^2\right)$$

$$\left(\frac{a_1}{a_2}\right)^2 = \left(\frac{M_1}{M_2}\frac{\left(1+\gamma M_2^2\right)}{\left(1+\gamma M_1^2\right)}\right)^2$$

$$\frac{\left(2+(\gamma-1)M_2^2\right)}{\left(2+(\gamma-1)M_1^2\right)} = \left(\frac{M_1}{M_2}\frac{\left(1+\gamma M_2^2\right)}{\left(1+\gamma M_1^2\right)}\right)^2$$

$$a_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right) = a_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

$$\frac{a_1^2}{a_2^2} = \frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)} = \frac{\left(2+(\gamma-1)M_2^2\right)}{\left(2+(\gamma-1)M_1^2\right)}$$

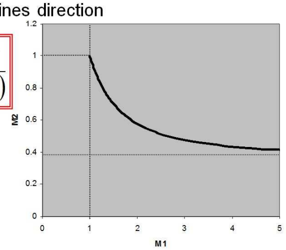
Normal Shock Waves (3)
Mach Number Variation Through Shock

- Eliminate ρ and a from previous continuity momentum and energy equations

$$\left(M_1^2-M_2^2\right)\left\{2+(\gamma-1)\left(M_1^2+M_2^2\right)-2\gamma M_1^2M_2^2\right\}=0$$

- Solutions:
  - $M_1 = M_2$ no shock
  - symmetric in M – but entropy change determines direction

$$\boxed{M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}} \quad \boxed{M_1^2 = \frac{2+(\gamma-1)M_2^2}{2\gamma M_2^2-(\gamma-1)}}$$



- upstream $M_1 \geq 1 \rightarrow$ downstream $M_2 \leq 1$
  - only occurs in *supersonic* flow
  - downstream flow *subsonic* (but $M_2 > 0.378$)

Normal Shock Waves (4)
Flow Properties: Variation Through Shock

- pressure ratio
  - ***compression*** process
- temperature ratio
  - Temperature rise
- density ratio

compression process

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left(2+(\gamma-1)M_1^2\right)\left(2\gamma M_1^2-(\gamma-1)\right)}{(\gamma+1)^2M_1^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2} \rightarrow 6 \text{ as } M_1 \rightarrow \infty$$

see standard texts for derivation

>1
M_1>1

- total pressure ratio
  - NB some authors use p_t for p_0
  - p_0 ***reduces*** across shock
  - direct link between entropy gain and loss in total pressure
  - in practice the ratio p_{02}/p_1 is more useful
  - $(p_{02}/p_1)=(p_{02}/p_{01})(p_{01}/p_1)$ where (p_{01}/p_1) from compressible Bernoulli

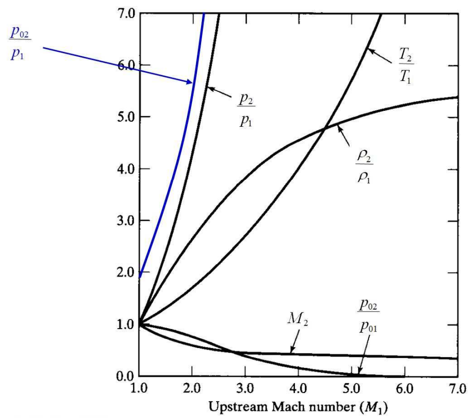
$$\boxed{-R\ln\left(\frac{p_{02}}{p_{01}}\right) = s_2-s_1} = fn(M_1)$$

from BC1.12 $\quad s_2-s_1 = c_p\ln\left(\frac{T_2}{T_1}\right)-R\ln\left(\frac{p_2}{p_1}\right) = fn(M_1) - fn(M_1)$

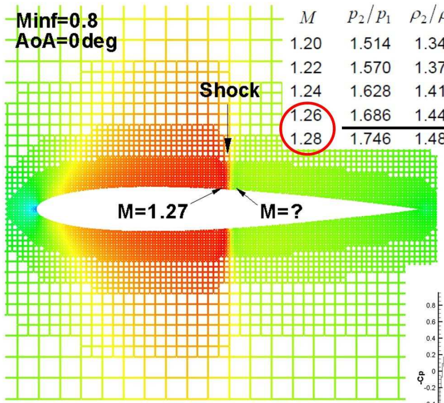
- ratios p_2/p_1, T_2/T_1, ρ_2/ρ_1 and p_{02}/p_1 tabulated vs M_1

formulae too complex for routine use so use tables when γ=1.403

Normal Shock Relations

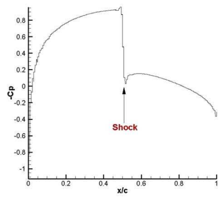


Example 1



M	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2	p_{02}/p_1
1.20	1.514	1.341	1.129	0.842	2.410
1.22	1.570	1.376	1.141	0.830	2.469
1.24	1.628	1.410	1.154	0.818	2.529
1.26	1.686	1.445	1.167	0.807	2.591
1.28	1.746	1.480	1.179	0.796	2.653

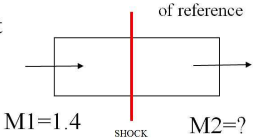
Linear interp. gives M=0.802

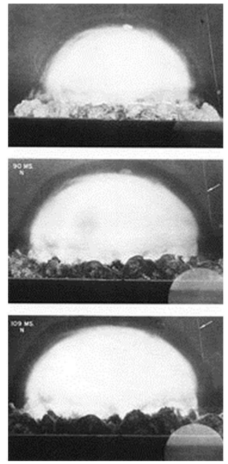


Example 2

Mach number consistent with average value 10s after 1MT explosion

Shock moving at M=1.4 into air at STP





From tables M2=0.74, T2/T1=1.256

So air exits at 0.74*340*sqrt(1.256)=282m/s (in the moving frame)

Frame of reference moving at 1.4*340=476m/s

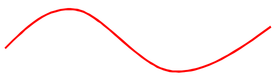
So air behind shock has velocity of 476-282=194m/s=698km/h=433mph

T2 =1.256*288=362K=89C

P2=2.121atm=16PSI above atmospheric

Sound wave

Shockwave

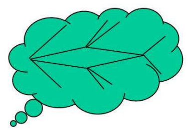


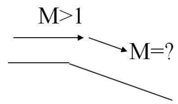
Speed of air behind wave ~0mph

Speed of air behind wave ~100s mph

Today

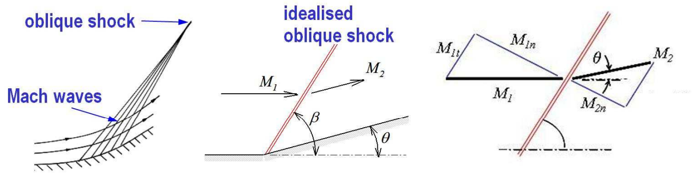
- Oblique shocks – normal shocks at an angle, with some trigonometry, and some unexpected complications
- Detached shocks – a cross between oblique and normal shocks (for these you only need to consider the normal part, anything else is beyond this course)
- Next week – expansions!





2D Supersonic Flow
Oblique Shock

- for 'large' negative deflection angle Mach waves coalesce to form an *oblique shock wave*
  - a nonisentropic compression wave

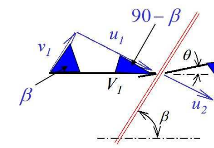


- use normal shock relations for normal velocity component
  - no change in tangential velocities

$$M_{1n} = M_1 \sin\beta, \quad M_{2n} = M_2 \sin(\beta-\theta)$$

- M_{2n} and ratios p_2/p_1, T_2/T_1, ρ_2/ρ_1 tabulated vs M_{1n}
- β still unknown

2D Supersonic Flow
Oblique Shock angle



$$\tan(\beta) = \frac{u_1}{v_1}$$

$$\tan(\beta-\theta) = \frac{u_2}{v_2}$$

consider 1D flow normal to the shock

- From continuity

$$\rho_1 u_1 = \rho_2 u_2 \quad \rightarrow \quad \frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} \qquad v_1 = v_2 \qquad \frac{u_1/v_1}{u_2/v_2} = \frac{\rho_2}{\rho_1}$$

$$\frac{\tan(\beta)}{\tan(\beta-\theta)} = \frac{\rho_2}{\rho_1}$$

- use normal shock relations and some algebra to give

$$\tan(\theta) = \frac{\cot(\beta)\left\{(M_1\sin\beta)^2 - 1\right\}}{\left(\gamma+1/2\right)M_1^2 - \left\{(M_1\sin\beta)^2 - 1\right\}}$$

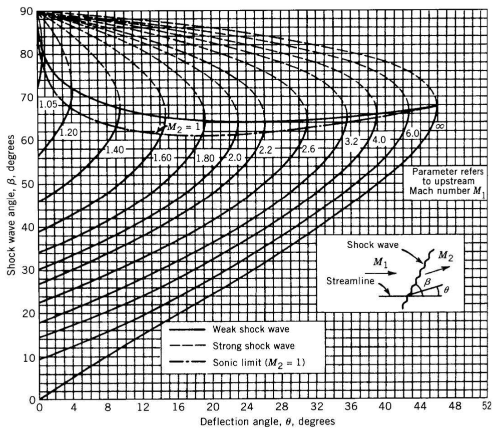
$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{\left(2+(\gamma-1)M_1^2\right)\left(2\gamma M_1^2 - (\gamma-1)\right)}{(\gamma+1)^2 M_1^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}$$

- Graphical solution: variation of β with θ for upstream Mach No

2D Supersonic Flow
Oblique Shock Angle



Finding the shock angle

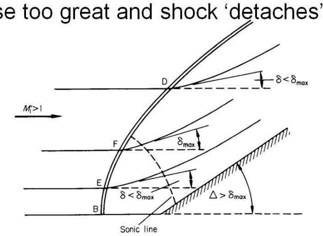
- Given the shock angle, the wedge angle may be found explicitly
- However, finding the shock angle from the wedge angle is not easy (though it can be done through solution of a cubic with much algebra – see NASA Report 187173 'Exact and Approximate Solutions to the Oblique Shock Equations for Real-Time Applications', 1991)
- It is often done using an iterative procedure
  (or graphically if desired)
- Could use bisection, direct iteration (below) or a Newton-Raphson procedure
- Similar ideas can be used to give the maximum shock angle
- Rearrange the method below and start from ~90degrees to get the strong shock

```
(start beta at some small value, give gamma, M1 and theta)
do i=1,100 ! Usually ~ 10 iterations would be enough
  bot=0.5*(gam+1)*M1**2-((M1*sin(beta))**2-1)
  top=asin((sqrt(tan(theta)*bot*tan(beta)+1))/M1)
  beta=top
enddo
```

For M1=2, theta=10,
beta_initial=0.1deg
30.02954
37.60822
39.03094
39.27807
39.32045
39.32770
39.32894
39.32916
39.32919
39.32920
39.32920
39.32920(12)

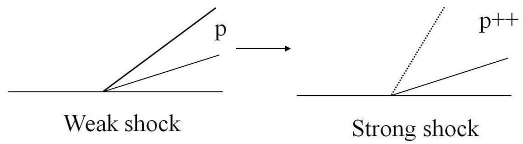
Strong, Weak & Detached Shocks

- for given wedge angle θ and onset Mach Number M_1 there are two solutions to the equations
  - 'weak' shock occurs in external flows ($\beta_{weak} < \beta_{strong}$)
  - 'strong' shock can occur in internal flows with high back pressure
- M_2 supersonic for weak shocks except for small region near θ_{max}
- for $\theta > \theta_{max}$ pressure rise too great and shock 'detaches'
  - curved shock 'stands off' ahead of wedge
  - effectively 'normal' near wedge – subsonic region behind shock
  - curved shock generates vorticity

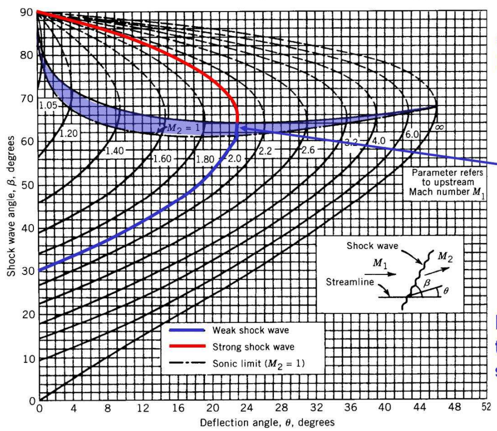


Weak/Strong shock

- If a weak shock exists, and then the downstream pressure is increased (perhaps by a gauze or some other external mechanism), the solution can switch to the strong shock
- Unfortunately some abuse of terminology takes place – weak shocks are sometimes called 'strong' if they have a big change in pressure across them



2D Supersonic Flow
Oblique Shock Angle

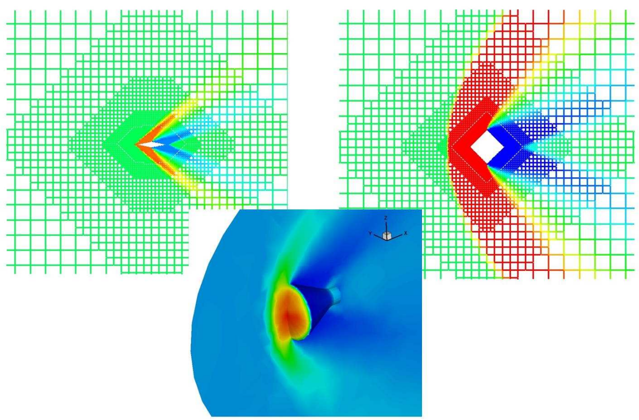


consider Mach 2 line M_1=2

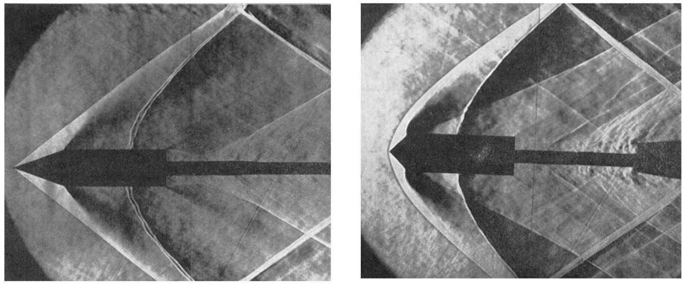
Subsonic flow downstream of oblique shock If θ greater than 22.45°

For M_1=2: if θ greater than 22.93° oblique shock detaches

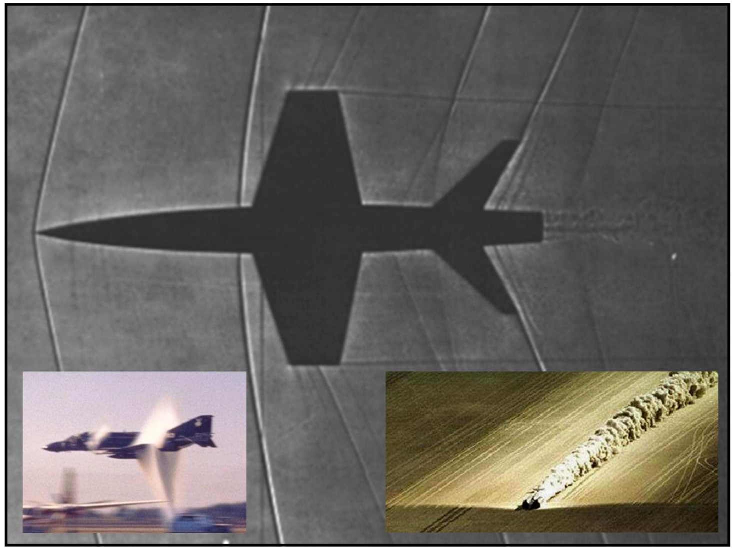
Oblique shock and a detached bow shock for M=2



Schlieren Images of Cone Shock System



Aerodynamics 2 : *Slide SW.33*



Oblique shock example

- Mach number usually known to be one of the values in the tables (increment of 0.1)
- Deflections given in increments of 2 degrees up to the maximum
- Usual to interpolate in terms of deflection, and possibly Mach number too – although unlikely in practice
- Try finding M2 for M1=1.87, deflection=14.7 degrees

For 1.8 have 1.288+(14.7-14)*(1.194-1.288)/(16-14)=1.2551

For 1.9 have 1.39+(14.7-14)*(1.304-1.39)/(16-14)=1.3599

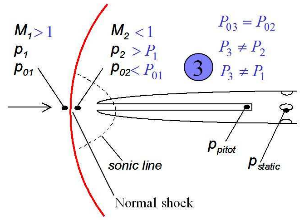
Then for 1.87 have 1.2551+(1.87-1.8)*(1.3599-1.2551)/(1.9-1.8)=1.33

Aerodynamics 2 : *Slide SW.35*

Detached Shock Example: Supersonic Pitot-Static

- detached shock ahead of probe
  - $p_{static} \neq p_2$
  - $p_{pitot} = p_{02}$
- can relate p_{02} to p_1 and M_1 only using: previous derivations, some algebra &

$$\frac{p_{02}}{p_1} = \left(\frac{p_{02}}{p_2}\right)\left(\frac{p_2}{p_1}\right)$$

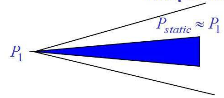


- Rayleigh's Supersonic Pitot Equation

$$\boxed{\frac{p_{02}}{p_1} = \frac{\left(\frac{\gamma+1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}}$$

tabulated in normal shock tables

- given in compressible flow tables
- How do we calculate upstream static, p_1 ?



Aerodynamics 2 : *Slide SW.36*

The double wedge aerofoil (M=2) – part 1

$$\theta = \tan^{-1}\left(\frac{0.1}{0.5}\right) = 11.310\deg$$

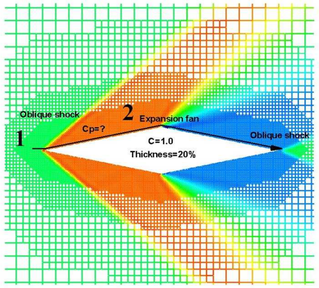
Get angle – note this is the 'half angle'

$$\frac{p_2}{p_1} = 1.708 + \frac{1.891 - 1.708}{12 - 10}(11.310 - 10) = 1.828$$

Interpolate linearly to find pressure ratio across oblique shock

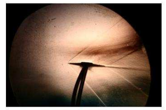
$$C_p = \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p_2}{p_1} - 1\right) = \frac{2}{1.4 \times 2^2}(1.828 - 1) = 0.296$$

p1=p_inf for this case (only true here because this is a shock into the freestream flow)

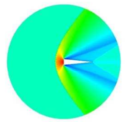


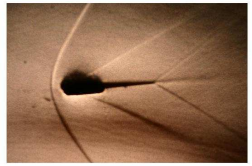
Aerodynamics 2 : *Slide SW.37*

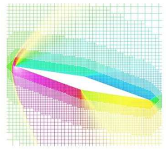
Compressible Flow Lab (next term)

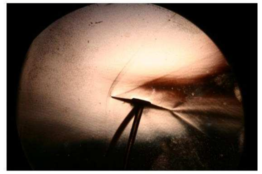


- Opportunity to see oblique shocks and detached bow shocks in practice









Aerodynamics 2 : *Slide S*

Today

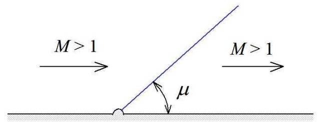
- Expansion fans and the Prandtl-Meyer function
- Expansion example
- Wave reflections

Aerodynamics 2 : *Slide SW.39*

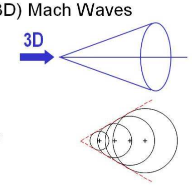
2D Supersonic Flow

Mach Waves building blocks for larger disturbances

- start with supersonic flow along a *straight* surface
- isolated perturbations on surface generate disturbances in flow
  - eg scratches, rivets, panel edges etc
  - straight surface hence upstream and downstream conditions equal
  - infinitesimal pressure wave propagating at Mach Angle μ
  - point perturbations generate conical (3D) Mach Waves



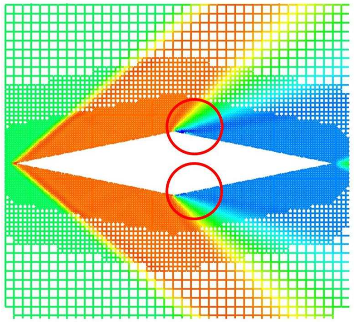
$$\sin\mu = \frac{1}{M}$$



- slightly different situation if perturbation is a 2D corner …

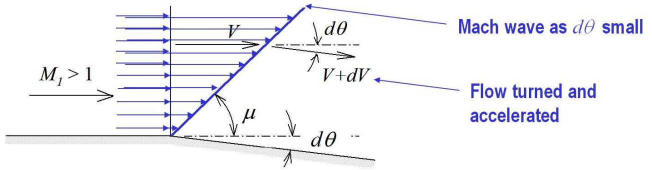
Aerodynamics 2 : *Slide SW.40* larger disturbance

The double wedge aerofoil – part 2



2D Supersonic Flow
'Small' Flow Deflection (1)

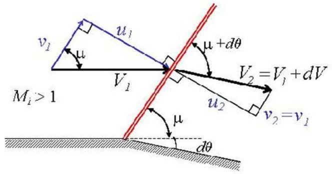
- surface (and hence flow) deflected by small angle $d\theta$



- no effect of deflection felt upstream of Mach wave
- 2D flow → flow properties ***constant*** along the wave
  - therefore no pressure gradient parallel (tangential) to the wave
- no change in ***tangential*** velocity component across the wave
  - only need to deal with changes in ***normal*** velocity component
  - also applicable to oblique shock wave analysis

2D Supersonic Flow
'Small' Flow Deflection (2)

- equate upstream and downstream tangential velocities
  - make small angle approximations & ignore 2nd order terms



$v_1 = V_1\cos(\mu) \qquad v_2 = (V_1+dV)\cos(\mu+d\theta)$

$u_1 = V_1\sin(\mu) \qquad u_2 = (V_1+dV)\sin(\mu+d\theta)$

tangential velocity unchanged

use cos(A+B)=cos(A)cos(B)-sin(A)sin(B)

$v_1 = v_2$

$$V_1\cos(\mu) = (V_1+dV)\begin{pmatrix}\cos(\mu)\cos(d\theta) - \\ \sin(\mu)\sin(d\theta)\end{pmatrix}$$

$d\theta$ **small** $\Rightarrow \cos(d\theta)\approx 1 \quad \sin(d\theta)\approx d\theta$

$V_1\cos(\mu) = (V_1+dV)(\cos(\mu)-\sin(\mu)d\theta)$

$dVd\theta \approx 0$

$\cancel{V_1\cos(\mu)} = \cancel{V_1\cos(\mu)} - V_1\sin(\mu)d\theta + dV\cos(\mu)$

$\sin(\mu) = \frac{1}{M} \;\therefore\; \cos(\mu) = \sqrt{1-\sin^2(\mu)} = \frac{\sqrt{M^2-1}}{M}$

$$\boxed{\frac{dV}{d\theta} = \frac{V}{\sqrt{M^2-1}}}$$

- positive for $M > 1$ → *expansion = acceleration*

2D Supersonic Flow
'Small' Flow Deflection (3)

- apply 1D Euler Equation + Newton's speed of sound equation

$dp = -\rho V dV \qquad a^2 = \gamma RT = \frac{\gamma p}{\rho}$

$dp = -\frac{\gamma p}{a^2}VdV \quad = -\frac{\gamma p V^2}{a^2}\frac{dV}{V} \quad = -\gamma p M^2\frac{dV}{V}$

$\frac{dV}{d\theta} = \frac{V}{\sqrt{M^2-1}} \Rightarrow \frac{dV}{V} = \frac{d\theta}{\sqrt{M^2-1}} \Rightarrow dp = \frac{-\gamma M^2 d\theta\, p}{\sqrt{M^2-1}}$

$$\boxed{\frac{dp}{d\theta} = \frac{-\gamma M^2}{\sqrt{M^2-1}}p}$$

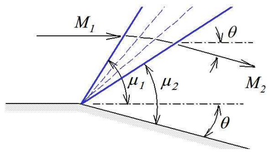
- negative for $M > 1$ → *expansion = pressure drop*
- need to integrate for finite deflection angle (Prandtl-Meyer expansion)

The integration

- For your information – no need to memorise or reproduce, but you should understand the final line

$$\frac{dV}{d\theta} = \frac{V}{\sqrt{M^2 - 1}} \quad \text{so} \quad \int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{V} dV$$

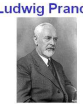
Fortunately Messrs. Prandtl and Meyer have done this for you – so no integration heroics required. In return, we humbly refer to the integral of the RHS as the Prandtl-Meyer function ν



$$\boxed{\theta_2 - \theta_1 = \theta = \nu(M_2) - \nu(M_1)}$$

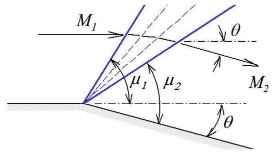
2D Supersonic Flow
Prandtl-Meyer Expansion

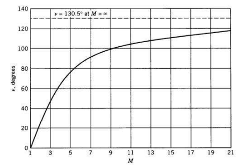
Ludwig Prandtl



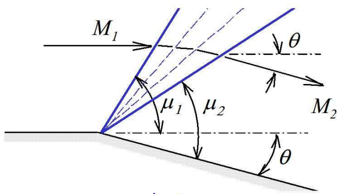
Theodor Meyer

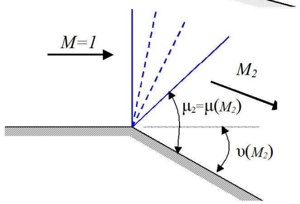
- consider 'large' positive deflection angle θ
  - flow expands through series of infinitesimal expansions $d\theta$
  - isentropic *'expansion fan'*





- integral starts from M_1 =1 to give *Prandtl-Meyer angle* ν as a function of downstream Mach Number M_2 only
  - ie $\nu(M) \equiv \theta(M_1=1, M_2=M)$
  - a 'fictitious' angle through which sonic flow (ie at a Mach Number of 1) would have to be turned to give downstream Mach Number M
- For $M_1>1$, $\theta(M_1,M_2)$ *is difference of expansion angles for* M_1 *and* M_2





- For $M_1>1$, $\theta(M_1,M_2)$ *is difference of expansion angles for* M_1 *and* M_2
- $\theta = \nu(M_2) - \nu(M_1)$

What is that function?

- It is this

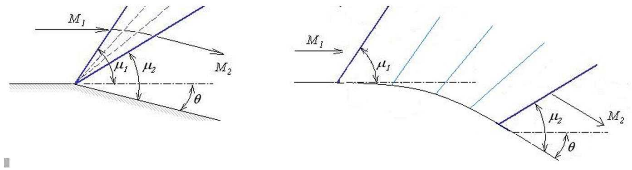
$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right)$$

…and it is unwieldy and tricky to solve for M given ν

…so we don't. Instead use tables and trusty linear interpolation!

2D Supersonic Flow
Prandtl-Meyer Expansion (2)

- Given the upstream Mach number M_1 and the expansion angle, how do we find the downstream Mach number M_2
  - $\theta = \nu(M_2) - \nu(M_1)$
  - We know θ and can calculate $\nu(M_1)$
  - Therefore must find M_2 such that $\nu(M_2) = \theta + \nu(M_1)$
- As long as flow remains isentropic expansion path can be ignored



Prandtl-Meyer expansion angle tabulated in supersonic isentropic tables – hurrah!

Example for trailing part of the wedge

- M1=1.59 for which the PM function=14.535deg
- Deflection=22.6 degrees from the geometry

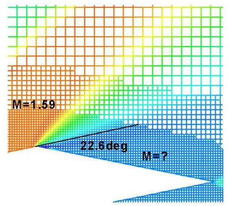
We find the second PM value…

$$\nu(M_2) = 22.6 + \nu(M_1) = 22.6 + 14.535 = 37.135$$

Then look up the Mach number for that value of the PM function, to get…

$$M_2 = 2.42$$

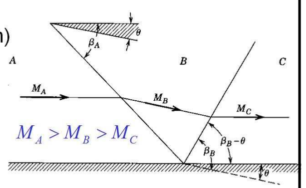
Same for pressure ratios etc.

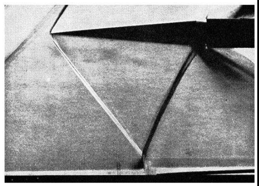


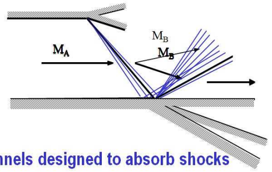
Everything in degrees, not radians!

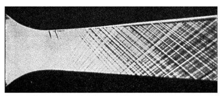
Wave Reflections

- Mach Waves (expansion & compression) and oblique shock waves are “reflected” at flow boundaries
  - flow tangency for solid boundaries
  - pressure equilibrium for jet boundaries
- for solid walls the type of reflection depends on the wall inclination. eg oblique shock can be reflected as a shock, as an expansion fan or be absorbed







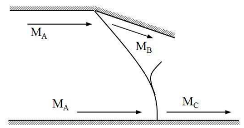


wind tunnels designed to absorb shocks

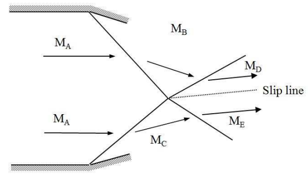
Wave Reflections (2)

Solid surface reflections

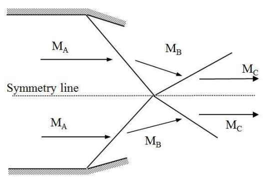
Mach wave reflection



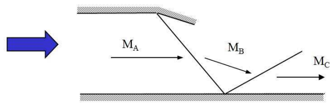
If M_c insufficient for 2nd Oblique shock

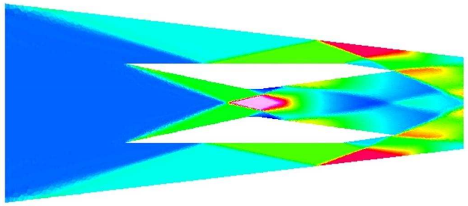


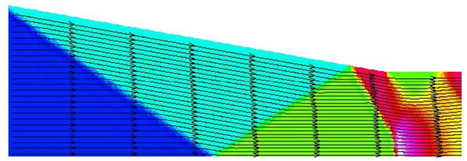
Slip line : direction defined by equality of pressure $P_D = P_E$

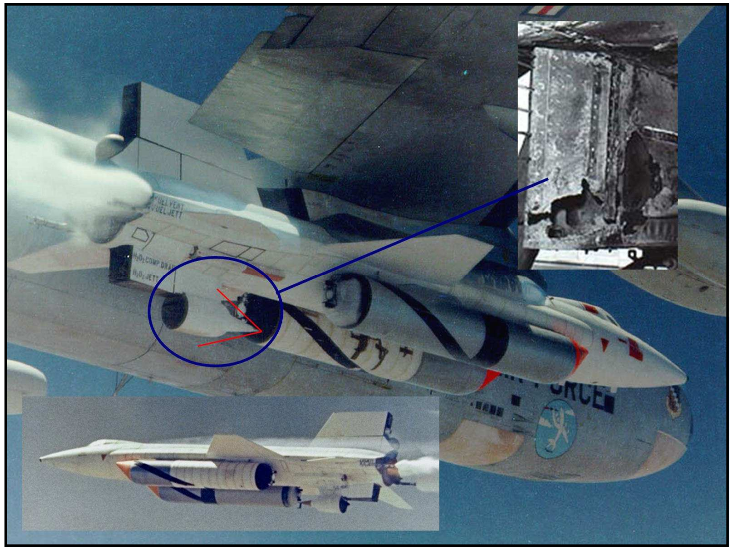


Equality of symmetry plane and solid wall





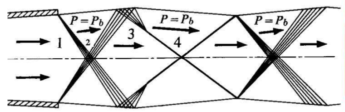




Wave Reflections (3)

Fluid boundary reflections

- for fluid boundary (ie edge of jet) pressure equilibrium gives alternating compression and expansion waves
  - shock diamonds
  - viscosity smears out effect



Underexpanded supersonic nozzle

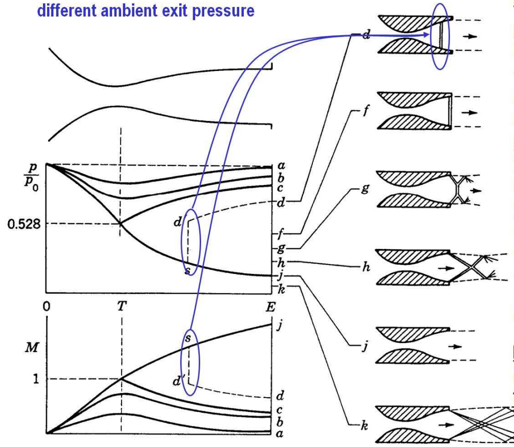
- Shock or expansion at exit dependant on pressure

Under-expanded nozzle has $P_1 > P_b$, Over-expanded nozzle has $P_1 < P_b$, what will this look like?

Low Mach numbers & viscosity can greatly complicate the picture.

Shocks in Convergent-Divergent Nozzles

each case, a-k, corresponds to a different ambient exit pressure



a-c, subsonic isentropic

c-j, exit pressure > pressure in divergent nozzle (over--expanded). Compression process needed to raise pressure

c-f, normal shock in nozzle increases pressure in divergent duct

f-j, Compression in jet outside duct. Can be complex mix of normal & oblique shocks.

j, Supersonic isentropic flow. (ideally expanded)

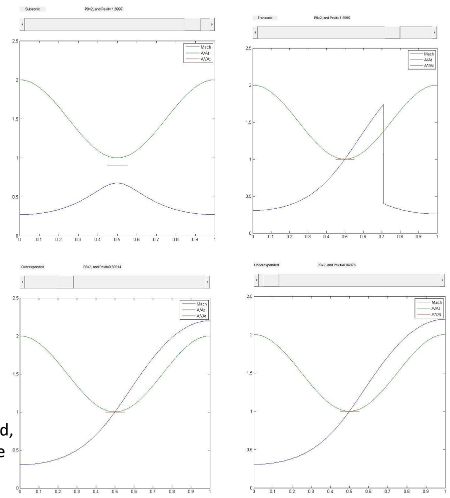
k, underexpanded jet. Expansion process needed to drop jet pressure at exit.

$$\frac{A}{A_t} = 2 - \sin^2(\pi x)$$

P0_start=2b
Pexit varies with slider

Subsonic

Transonic



Supersonic – overexpanded, shock outside nozzle

Supersonic – underexpanded, expansion outside nozzle

Case 1 – subsonic, Pe=1.9b

Subsonic

A/At	A/A*	M
2.000000	2.227262	0.271405
1.975480	2.199955	0.275103
1.904324	2.120714	0.286461
1.793511	1.997309	0.306276
1.653910	1.841846	0.335861
1.499214	1.669571	0.376924
1.344595	1.497382	0.431105
1.205218	1.342168	0.498667
1.094753	1.219151	0.575202
1.024036	1.140399	0.644801
1.000002	1.113634	0.674986
1.025009	1.141482	0.643668
1.096603	1.221212	0.573659
1.207764	1.345003	0.497200
1.347587	1.500714	0.429886
1.502359	1.673073	0.375979
1.656899	1.845174	0.335163
1.796051	2.000138	0.305789
1.906166	2.122765	0.286154
1.976443	2.201028	0.274955
2.000000	2.227262	0.271405

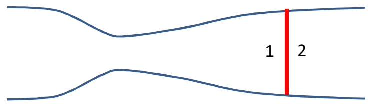
$$\frac{p_0}{p_e} = \frac{2}{1.9} = 1.0526$$

Mach number from pressure ratio

$$M = 0.26 + 0.02 \times \frac{1.0526 - 1.048}{1.056 - 1.048} = 0.2715$$

Important – the static pressure of the air at exit is equal to the downstream static pressure. This is only true for a subsonic jet discharging

Nomenclature – note the shock is not present for this calculation



Note p02=p0e

Case 2 – transonic, Pe=1.5b

Choked - transonic

Shock location	A/At at shock	A*_beforeshock/A*_aftershock	P0_aftershock/P_beforeshock
0.717860	1.399713	0.828832	4.512082

M_before	M_after	P0/P_before	P0/P_after	P0/P_exit
1.763973	0.470818	5.443907	1.164310	1.105108

(1/(P0/P_exit))*(P0_aftershock/P_beforeshock)*(1/(P0/P_before))= 1.500001

A/At	A/A*	M
2.000000	2.000000	0.305813
1.975480	1.975480	0.310090
1.904324	1.904324	0.323260
1.793511	1.793511	0.346367
1.653910	1.653910	0.381217
1.499214	1.499214	0.430434
1.344595	1.344595	0.497411
1.205218	1.205218	0.585953
1.094753	1.094753	0.699361
1.024036	1.024036	0.838842
1.000002	1.000002	1.005006
1.025009	1.025009	1.180775
1.096603	1.096603	1.365608
1.207764	1.207764	1.545454
1.347587	1.347587	1.711442
1.502359	1.245202	0.556472
1.656899	1.373290	0.483123
1.796051	1.488624	0.434346
1.906166	1.579890	0.403086
1.976443	1.638138	0.385651
2.000000	1.657663	0.380178

Before we can answer this problem, we need to know the downstream pressure for two other cases…

1. Choked, **but with** subsonic flow after throat (done on last slide)
2. Ideally expanded
3. Choked, with supersonic flow after throat **followed by a normal shock exactly at the nozzle exit**

These results will tell us what's happening in the nozzle.

Option 1 – choked/subsonic

For this case we know At=A*. So pressure at exit for this would be

$$\frac{p_0}{p} = 1.065 + (2 - 2.035)\frac{1.074 - 1.065}{1.921 - 2.035} = 1.0678$$ Pressure ratio from A/A*

$$p = \frac{2}{1.0678} = 1.8731$$

Lower than this, and there will certainly be a supersonic region in the nozzle, **but we don't know about the shock position yet**

Options 2 and 3

Ideally expanded

Pressure from area

$$\frac{p_0}{p} = 10.366 + (2 - 1.966)\frac{10.695 - 10.366}{2.001 - 1.966} = 10.6856$$

$$p = \frac{2}{10.6856} = 0.1872$$

This is pretty low. Below this the nozzle would be underexpanded (expansions outside nozzle), and above this it is overexpanded (shocks at exit or outside nozzle).

Exit Mach for ideal case is $M = 2.18 + 0.02 \times \frac{2 - 1.966}{2.001 - 1.966} = 2.199$

An important case is "ideal+normal shock". So compute a normal shock from here

$$\frac{p_{02}}{p_1} = 6.614 + (2.199 - 2.18)\frac{6.726 - 6.614}{0.02} = 6.7204$$ Using p02/p1 column

$$M = 0.55 + (2.199 - 2.18)\frac{0.548 - 0.55}{0.02} = 0.5481$$ Downstream (exit) M

$$\frac{p_0}{p} = 1.22 + (0.5481 - 0.54)\frac{1.238 - 1.22}{0.02} = 1.2273$$ Downstream (exit) total pressure ratio

$$\frac{p_e}{p_{01}} = \frac{p_e}{p_{0e}}\frac{p_{0e}}{p_1}\frac{p_1}{p_{01}} = \frac{1}{1.2273}6.7204 \times \frac{1}{10.6856} = 0.5124 \rightarrow \boxed{p_e = 2 \times 0.5124 = 1.0248}$$

Transonic summary

- If Pe below 1.0248, then shocks/expansions are outside nozzle, and nozzle fully supersonic
- If Pe above 1.0248, then shock is **inside** divergent part of nozzle - transonic
- If Pe above 1.8731, then fully subsonic

Case 2 – transonic, Pe=1.5b

We now know there must be a shock in the divergent part.

It is actually quite hard to work out (analytically) exactly where. However, we know everything we need to from the relationship

$$p_e = p_{01}\frac{p_e}{p_{0e}}\frac{p_{0e}}{p_1}\frac{p_1}{p_{01}}$$

Note p02=p0e

We can easily iterate (bisect) on the shock position, until the calculated exit pressure matches the specified exit pressure. This is how the matlab tool works.

Alternatively, it is quite easy to find the exact exit pressure for a specified shock location, by using the same procedure as on the previous slide, but with the shock at any other location. This is the type of question you should be able to solve – see tutorial sheet for more examples.

Ie. You would not be expected to find the shock position from the exit pressure, but you should be able to find the exit pressure given the shock position.

Case 3 – supersonic, Pe=1b

Choked - supersonic

A/At	A/A*	M
2.000000	2.000000	0.305813
1.975480	1.975480	0.310090
1.904324	1.904324	0.323260
1.793511	1.793511	0.346367
1.653910	1.653910	0.381217
1.499214	1.499214	0.430434
1.344595	1.344595	0.497411
1.205218	1.205218	0.585953
1.094753	1.094753	0.699361
1.024036	1.024036	0.838842
1.000002	1.000002	1.005006
1.025009	1.025009	1.180775
1.096603	1.096603	1.365608
1.207764	1.207764	1.545454
1.347587	1.347587	1.711442
1.502359	1.502359	1.857397
1.656899	1.656899	1.979508
1.796051	1.796051	2.075629
1.906166	1.906166	2.144632
1.976443	1.976443	2.185966
2.000000	2.000000	2.199399

$$\cancel{\frac{p_0}{p_e} = \frac{2}{1} = 2}$$

Not true, because we know this is supersonic discharge

Just find M from A/A*. We know At=A*, so

$$M = 2.18 + 0.02 \times \frac{2 - 1.966}{2.001 - 1.966} = 2.199$$ Supersonic

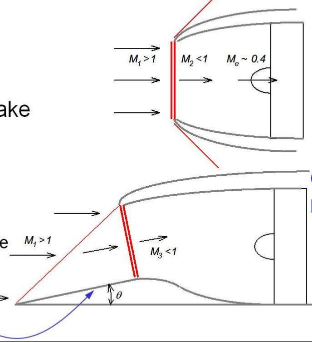
$$M = 0.3 + 0.02 \times \frac{2 - 2.035}{1.921 - 2.035} = 0.3061$$ Subsonic

Supersonic Intakes

- conventional engines require subsonic flow at fan face (<0.4M)
  - implies shock system at supersonic speeds
  - shock provides compression – but at a price
- basic Pitot intake
  - normal shock at lip
  - 2-stage compression
  - inefficient above 1.5M
- external compression intake
  - external oblique shock
  - normal shock at lip
  - 3-stage compression
  - reduced pressure loss
  - variable geometry desirable

Shock means:compression, pressure rise but total pressure drop.

P_0 ↓=↓ available mechanical energy.
=↓ thrust



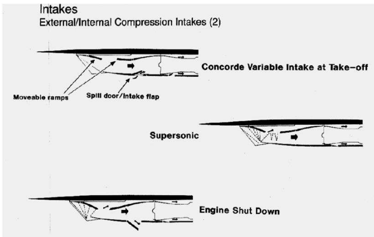
Intake design: minimise drop In P_0

Oblique shock has less P_0 ↓for same M↓

Normal shock losses decrease with M

Aerodynamics 2 : *Slide SW.58*

Concorde



B-58…22.4% lost in accidents!

Aerodynamics 2 : *Slide SW.59*

Argh! I hate summations!

Force coefficients

$$\mathbf{F} = \int p\mathbf{n}ds$$ **n** is inward normal

$$p = C_p q_\infty + p_\infty$$

$$\mathbf{F} = \int C_p q_\infty \mathbf{n} ds$$

$$\mathbf{C}_F = \frac{\mathbf{F}}{q_\infty c} = \int C_p \mathbf{n} d\left(\frac{s}{c}\right)$$

$$C_X = \int C_p \mathbf{n}\cdot\mathbf{i} d\left(\frac{s}{c}\right)$$

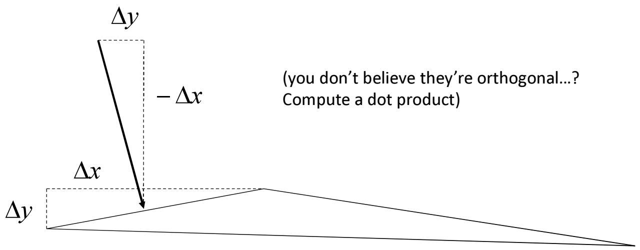
$$C_Y = \int C_p \mathbf{n}\cdot\mathbf{j} d\left(\frac{s}{c}\right)$$

$$\mathbf{n}' = \begin{pmatrix} n_x \\ n_y \end{pmatrix}\frac{s}{c} = \begin{pmatrix} \Delta y \\ -\Delta x \end{pmatrix}$$

$$C_X = \sum_{i=1}^{i=N} C_{p_i} \mathbf{n}'_{x_i} = \sum_{i=1}^{i=N} C_{p_i} \Delta y_i$$

$$C_Y = \sum_{i=1}^{i=N} C_{p_i} \mathbf{n}'_{y_i} = -\sum_{i=1}^{i=N} C_{p_i} \Delta x_i$$

(inward!) Normals

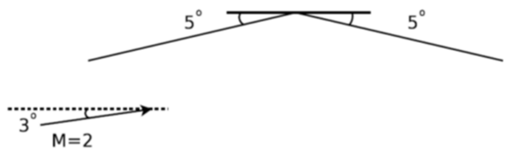


(you don't believe they're orthogonal…?
Compute a dot product)

Best approach is to work out the signs on each contribution by **common sense**

Or, go for the doctrinaire approach of memorising a system for doing it, but wave goodbye to an element of your sanity.

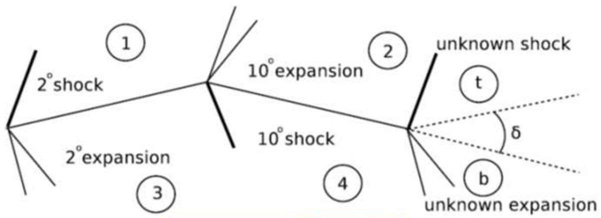
Full aerofoil example

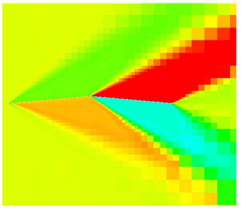


Picture
Strategise
Calculate
Integrate

Pedantic students cash in

Picture





Strategise

Surface 1 requires only $\frac{p_1}{p_\infty}$

Surface 2 requires $\frac{p_2}{p_\infty} = \frac{p_2}{p_{02}} \frac{p_{01}}{p_1} \frac{p_1}{p_\infty}$

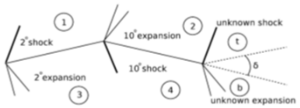
Surface 3 requires only $\frac{p_3}{p_\infty} = \frac{p_3}{p_{03}} \frac{p_{0\infty}}{p_\infty}$

Surface 4 requires $\frac{p_4}{p_\infty} = \frac{p_4}{p_3} \frac{p_3}{p_{03}} \frac{p_{0\infty}}{p_\infty}$

Surface 1

$$C_{p_1} = \frac{2}{\gamma M_\infty^2}(1.118 - 1) = 0.0421$$

Remember to use M_freestream for finding Cp values, NOT M_local



Surface 2 $\frac{p2}{p_\infty} = \frac{p2}{p02}\frac{p01}{p1}\frac{p1}{p_\infty}$

For surface 2, we need ν for the flow from surface 1 (where M=1.928 downstream of the shock). also need the total pressure ratio for surface 1

$$\mathcal{L}(6.916, 7.134, 1.92, 1.94, 1.928) = 7.0032$$

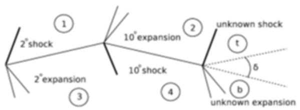
now ν for surface 1

$$\nu(1.928) = \mathcal{L}(24.090, 24.650, 1.92, 1.94, 1.928) = 24.314$$

What we actually want is the total pressure ratio, not the Mach number, so

$$\nu^{-1}(34.314) = \mathcal{L}(12.504, 12.901, 36.690, 34.314) = 12.6025$$

$$C_{p_2} = \frac{2}{\gamma M_\infty^2}\left(1.118\frac{7.0032}{12.6025} - 1\right) = -0.1350$$



Surface 3 $\frac{p3}{p_\infty} = \frac{p3}{p03}\frac{p0_\infty}{p_\infty}$

Moving to surface 3, which is an expansion of 2°

$$\nu(2) = 26.32$$

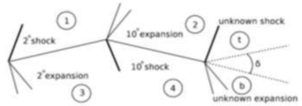
$$\nu^{-1}(28.32) = \mathcal{L}(8.596, 8.868, 27.95, 28.49, 28.32) = 8.782$$

and we already know $\frac{p0_\infty}{p_\infty} = 7.830$ so

$$C_{p_3} = \frac{2}{\gamma M_\infty^2}\left(\frac{7.830}{8.782} - 1\right) = -0.0386$$

We now need the Mach number on surface 3, which is

$$\nu^{-1}(26.320 + 2) = \mathcal{L}(2.06, 2.08, 27.95, 28.49, 28.32) = 2.0737$$



Surface 4 $\frac{p4}{p_\infty} = \frac{p4}{p3}\frac{p3}{p03}\frac{p0_\infty}{p_\infty}$

For surface 4 there is a shock through 10°. Interpolating in terms of Mach number for the shock for pressure ratio

$$\mathcal{L}(1.708, 1.736, 2.0, 2.1, 2.0737) = 1.7286 \quad (101)$$

and for downstream Mach

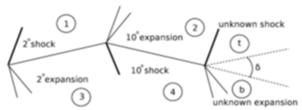
$$\mathcal{L}(1.64, 1.731, 2.0, 2.1, 2.0737) = 1.7071 \quad (102)$$

We will need the total pressure ratio here later for the slip line, so

$$\mathcal{L}(4.941, 5.093, 1.70, 1.72, 1.7071) = 4.9950 \quad (103)$$

Finally

$$C_{p_2} = \frac{2}{\gamma M_\infty^2}\left(1.7286\frac{7.830}{8.782} - 1\right) = 0.1929 \quad (104)$$



Integrate

$$C_X = 0.5\tan(5)(0.0421 + 0.0386 + 0.1350 + 0.1929) = 0.01787$$

$$C_Y = 0.5(-0.0421 - 0.0386 + 0.1350 + 0.1929) = 0.1236$$

$$C_L = C_Y \cos(3) - C_X \sin(3) = 0.1225$$

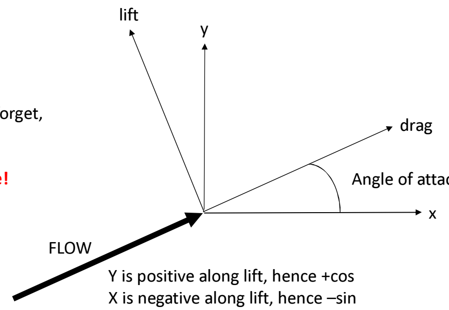
$$C_D = C_Y \sin(3) + C_X \cos(3) = 0.0243$$

...huh?

$$C_M = 0.5(0.0421 \times 0.25 + 0.0386 \times 0.25 - 0.1350 \times 0.75 - 0.1929 \times 0.75) + 0.5^3 \tan^2(5)(0.0421 + 0.0386 + 0.1350 - 0.1929) = -0.1129$$

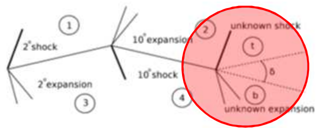
Rotation

If you forget, **draw the picture!**



Y is positive along lift, hence +cos
X is negative along lift, hence –sin
Y is positive along drag, hence +sin
X is positive along drag, hence +cos

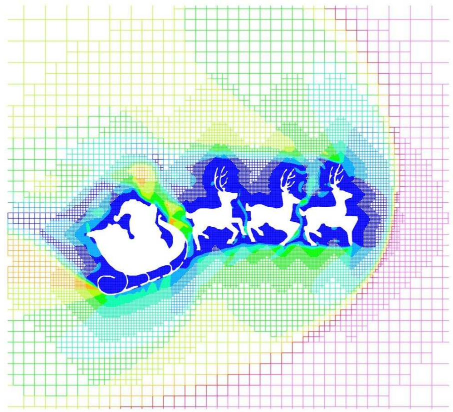
Slip line



Angle will be nearly aligned with freestream.

Compute by making pressure at top equal to pressure at bottom of slip line.

Full slip line calculation is in the tips and tricks document...



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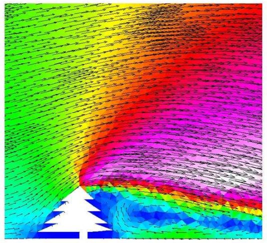
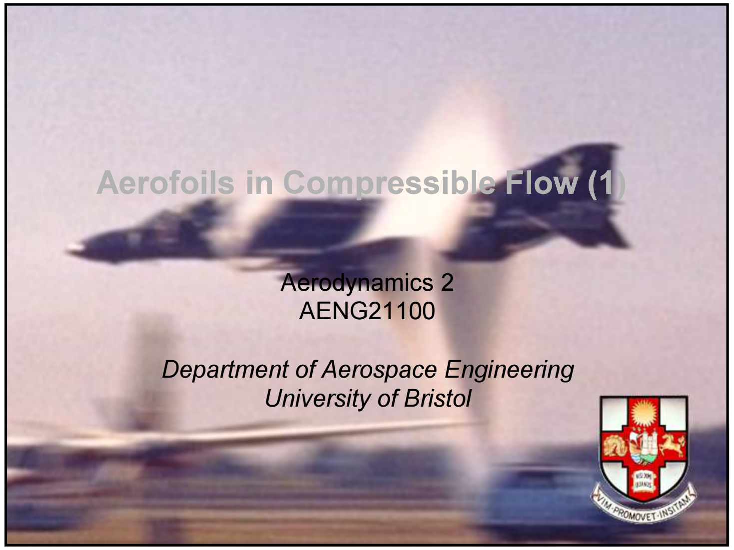


Aerodynamics 2 : *Slide SW.61*



Aerofoils in Compressible Flow (1)

Aerodynamics 2
AENG21100

Department of Aerospace Engineering
University of Bristol

So far

- Subsonic/supersonic compressible isentropic flow
- Supersonic flow on to wedges and round corners (normal/oblique shocks and expansion fans) – wedge aerofoils
- …next – a little of both applied to aerofoils and wings, and some practical calculations!

Today

- To derive the linearised potential equation
- To describe how the linearised potential equation can be used to derive the following theories
  - subsonic flows– Prandtl-Glauert
  - supersonic - Ackeret

The Full Potential Equation

- Consider inviscid compressible flow over a body in a uniform stream which is isentropic and irrotational. Irrotational flow means the velocity field in terms of velocity potential ϕ

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

$$\mathbf{u} = \nabla \phi$$

- Then to obtain an equation to solve for ϕ, use continuity, momentum (1D Euler) & isentropic speed of sound ($dp=a^2 d\rho$).

Full potential derivation (not examinable)

$$\nabla \cdot \rho \mathbf{u} = 0$$

$$\mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u} \cdot \frac{\nabla \rho}{\rho} + \nabla \cdot \mathbf{u} = 0$$

$$\nabla \phi \cdot \frac{\nabla \rho}{\rho} + \nabla \cdot \nabla \phi = 0$$

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p$$

$$p = k\rho^{\gamma}$$

$$\nabla p = \gamma k \rho^{\gamma-1} \nabla \rho = \frac{\gamma p}{\rho} \nabla \rho = a^2 \nabla \rho$$

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -a^2 \nabla \rho$$

$$\frac{\nabla \rho}{\rho} = -\frac{1}{a^2} \mathbf{u} \cdot \nabla \mathbf{u}$$

$$\nabla \cdot \nabla \phi - \frac{1}{a^2} \nabla \phi \cdot ((\nabla \nabla \phi) \nabla \phi) = 0$$

Full potential derivation (not examinable)

$$\nabla \nabla \phi = \begin{pmatrix} \frac{\partial^2 \phi}{\partial x^2} & \frac{\partial^2 \phi}{\partial x \partial y} \\ \frac{\partial^2 \phi}{\partial y \partial x} & \frac{\partial^2 \phi}{\partial y^2} \end{pmatrix}$$

Decompose to constant+perturbation

$$\nabla \cdot (\mathbf{V}_\infty + \nabla \phi') - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla \phi') \cdot ((\nabla(\mathbf{V}_\infty + \nabla \phi'))(\mathbf{V}_\infty + \nabla \phi')) = 0$$

Remove gradients of constants

$$\nabla \cdot \nabla \phi' - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla \phi') \cdot ((\nabla \nabla \phi')(\mathbf{V}_\infty + \nabla \phi')) = 0$$

Expand

$$\nabla \cdot \nabla \phi' - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla \phi') \cdot ((\nabla \nabla \phi')\mathbf{V}_\infty + (\nabla \nabla \phi')\nabla \phi') = 0$$

Ignore products of derivatives

$$\nabla \cdot \nabla \phi' - \frac{1}{a^2}\mathbf{V}_\infty^T (\nabla \nabla \phi') \mathbf{V}_\infty = 0$$

$$\mathbf{V}_\infty^T (\nabla \nabla \phi') \mathbf{V}_\infty = \begin{pmatrix} u_\infty & v_\infty \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \phi'}{\partial x^2} & \frac{\partial^2 \phi'}{\partial x \partial y} \\ \frac{\partial^2 \phi'}{\partial y \partial x} & \frac{\partial^2 \phi'}{\partial y^2} \end{pmatrix} \begin{pmatrix} u_\infty \\ v_\infty \end{pmatrix}$$

Keep only u part

$$\mathbf{V}_\infty^T (\nabla \nabla \phi') \mathbf{V}_\infty = u_\infty^2 \frac{\partial^2 \phi'}{\partial x^2}$$

Full potential derivation (not examinable)

$$(a^2 - u_\infty^2)\frac{\partial^2 \phi'}{\partial x^2} + a^2 \frac{\partial^2 \phi'}{\partial y^2} + a^2 \frac{\partial^2 \phi'}{\partial z^2} = 0$$

Energy equation

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}((\mathbf{V}_\infty + \nabla \phi') \cdot (\mathbf{V}_\infty + \nabla \phi'))$$

$$\frac{T_0}{T} = \frac{{a_0}^2}{a^2} = 1 + \frac{\gamma - 1}{2} M^2$$

$${a_0}^2 = a^2 + \frac{\gamma - 1}{2} V^2$$

Ignore gradient products

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2} 2u_\infty \frac{\partial \phi'}{\partial x}$$

$$a_\infty^2 = a_0^2 - \frac{\gamma - 1}{2}(u_\infty^2 + u_\infty^2)$$

Cancel through by a_inf

$$(1 - M_\infty^2)\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{\partial^2 \phi'}{\partial z^2} = 0$$

The Full Potential Equation

- The equation to solve is given by

Unknowns are speed of sound and velocity potential – need a 2nd eq'n

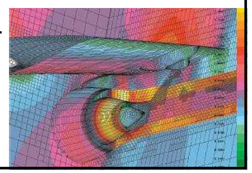
$$\left[a^2 - \left(\frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} - 2\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

do not memorise!

- This is a closed form in ϕ as a given by

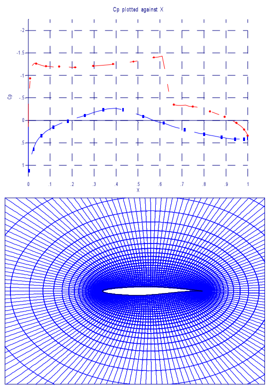
$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]$$

- Where a_0 is a known property of the flow.

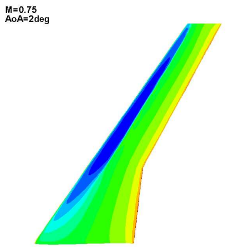


Solving the Full Potential Equation

2D - VGK



3D - FP



VGK and FP available through ESDU

The Full Potential Equation

- Compare this equation to solve for the compressible flow potential ϕ

$$\left[a^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-2\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0$$

NONLINEAR

- With the equation to solve for the incompressible flow potential ϕ

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

LINEAR

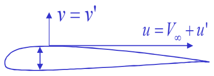
Nonlinear so numerical solutions only and no superposition of solutions. As an alternative to numerical solutions look for algebraic solutions by considering small perturbations to free stream velocity and linearising.

Linearising the Full Potential Equation

- Now assume u and v are **'small'** perturbations to free stream V_∞

$$u=V_\infty+u'=V_\infty+\frac{\partial\phi'}{\partial x}\qquad \phi=V_\infty x+\phi'$$

$$v=v'=\frac{\partial\phi'}{\partial y}$$



- A very good approximation for thin bodies at low α
- Substitute perturbation velocity potential, $\hat{\phi}$ into "velocity potential equation" to give "perturbation velocity potential equation"

$$\left[a^2-\left(V_\infty+\frac{\partial\phi'}{\partial x}\right)^2\right]\frac{\partial^2\phi'}{\partial x^2}+\left[a^2-\left(\frac{\partial\phi'}{\partial y}\right)^2\right]\frac{\partial^2\phi'}{\partial y^2}-2\left(V_\infty+\frac{\partial\phi'}{\partial x}\right)\left(\frac{\partial\phi'}{\partial y}\right)\frac{\partial^2\phi'}{\partial x\partial y}=0$$

Linearising the Full Potential Equation

- For a freestream Mach Number M_∞ **< 0.8** or 1.2<M_∞ **< 5** (*i.e.* outside transonic region) nonlinear terms negligible. This gives

$$\boxed{\left(1-M_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2}\approx 0}$$

Linearised velocity potential eqn.

Note that around M=1, shock waves arise and the flow is no longer isentropic, an assumption made in the derivation of this equation. However can sometimes give reasonable approximations in this flow regime providing shocks not too strong.

- This equation can be transformed to equivalent incompressible Laplace Equation, for which analytic solutions are possible
- The form of the required transformation depends on whether M_∞ is subsonic (Prandtl-Glauert) or supersonic (Ackeret)

Linearised Pressure Coefficient

- From the definition of the pressure coefficient we can make approximations similar to those used to derive the linear velocity potential equation (see handout for a reference to derivation). This is needed in subsequent theory.

$$C_p \equiv \frac{p-p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = -\frac{2u'}{V_\infty} + \frac{u'^2+v'^2}{V_\infty^2} + H.O.T.$$

$$\frac{\hat{u}}{V_\infty} < 1, \quad \frac{\hat{v}}{V_\infty} < 1$$

$$\frac{\hat{u}^2}{V_\infty^2} << 1, \quad \frac{\hat{v}^2}{V_\infty^2} << 1$$

$$C_p = 1 - \frac{v^2}{V_\infty^2} = 1 - \frac{(v'^2 + (V_\infty + u')^2)}{V_\infty^2} \approx -2\frac{u'}{V_\infty}$$

$$\boxed{C_p \approx -\frac{2u'}{V_\infty}}$$

(approach not strictly true for compressible case but result is valid; see next slide for full approach)

Linearising Cp (not examinable)

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

$$T_\infty\left(1 + \frac{\gamma-1}{2}M_\infty^2\right) = T\left(1 + \frac{\gamma-1}{2}M^2\right)$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma-1}{2}\left(M_\infty^2 - \frac{TM^2}{T_\infty}\right) = \frac{\gamma-1}{2a_\infty^2}(V_\infty^2 - V^2)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2a_\infty^2}(V^2 - V_\infty^2)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2a_\infty^2}((V_\infty + V')^2 - V_\infty^2) = 1 - \frac{\gamma-1}{2a_\infty^2}(2V_\infty V' + V'^2)$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma-1}{2}M_\infty^2\left(2\frac{V'}{V_\infty} + \frac{V'^2}{V_\infty^2}\right) = 1 - \epsilon$$

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p}{p_\infty} = (1-\epsilon)^{\frac{\gamma}{\gamma-1}} \approx 1 - \frac{\gamma}{\gamma-1}\epsilon$$

$$C_p \approx \frac{2}{\gamma M_\infty^2}\left(1 - \frac{\gamma-1}{2}\frac{\gamma}{\gamma-1}M_\infty^2\left(2\frac{V'}{V_\infty} + \frac{V'^2}{V_\infty^2}\right) - 1\right)$$

$$C_p \approx \frac{2}{\gamma M_\infty^2}\left(\frac{1}{2}\frac{\gamma}{1}M_\infty^2\left(2\frac{V'}{V_\infty} + \frac{V'^2}{V_\infty^2}\right)\right)$$

$$\boxed{C_p \approx -2\frac{V'}{V_\infty}}$$

Subsonic Prandtl-Glauert Correction (1)

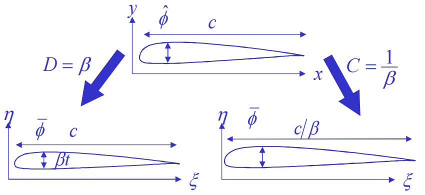
- Consider linearised velocity potential equation and apply coordinate transformation in terms of Glauert Factor β

$$x_{ic} = Cx \qquad \bar{\phi}(x_{ic}, y_{ic}) = \beta\phi'(x,y)$$

$$y_{ic} = Dy \qquad \beta = \sqrt{1 - M_\infty^2}$$

- Then there are two possibilities: C=1, D=β *and* C=1/β, D=1 to transform to a Laplace equation (equivalent incompressible flow solution) which is easier to solve

$$\boxed{(1 - M_\infty^2)\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} \approx 0}$$



$$\frac{\partial\phi}{\partial x_c} = \frac{\partial\phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial x_c} + \frac{\partial\phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial x_c} + \frac{\partial\phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial x_c} = \frac{\partial\phi}{\partial x_{ic}}$$

$$\frac{\partial\phi}{\partial y_c} = \frac{\partial\phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial y_c} + \frac{\partial\phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial y_c} + \frac{\partial\phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial y_c} = \frac{\partial\phi}{\partial y_{ic}}\beta$$

$$\frac{\partial\phi}{\partial z_c} = \frac{\partial\phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial z_c} + \frac{\partial\phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial z_c} + \frac{\partial\phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial\phi}{\partial z_{ic}}\beta$$

Choose
$$x_c = x_{ic}$$
$$\beta y_c = y_{ic}$$
$$\beta z_c = z_{ic}$$

$$\frac{\partial\phi_{x_c}}{\partial x_c} = \frac{\partial\phi_{x_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial x_c} + \frac{\partial\phi_{x_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial x_c} + \frac{\partial\phi_{x_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial x_c} = \frac{\partial^2\phi}{\partial x_{ic}^2}$$

$$\frac{\partial\phi_{y_c}}{\partial y_c} = \frac{\partial\phi_{y_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial y_c} + \frac{\partial\phi_{y_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial y_c} + \frac{\partial\phi_{y_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial\phi_{y_c}}{\partial y_{ic}}\beta = \frac{\partial^2\phi}{\partial y_{ic}^2}\beta^2$$

$$\frac{\partial\phi_{z_c}}{\partial z_c} = \frac{\partial\phi_{z_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial z_c} + \frac{\partial\phi_{z_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial z_c} + \frac{\partial\phi_{z_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial\phi_{z_c}}{\partial z_{ic}}\beta = \frac{\partial^2\phi}{\partial z_{ic}^2}\beta^2$$

Slide not examinable!

$$\boxed{\begin{aligned} \beta^2\frac{\partial^2\phi}{\partial x_{ic}^2} + \beta^2\frac{\partial^2\phi}{\partial y_{ic}^2} + \beta^2\frac{\partial^2\phi}{\partial z_{ic}^2} &= 0 \\ \frac{\partial^2\phi}{\partial x_{ic}^2} + \frac{\partial^2\phi}{\partial y_{ic}^2} + \frac{\partial^2\phi}{\partial z_{ic}^2} &= 0 \end{aligned}}$$

Subsonic Prandtl-Glauert Correction (1)

- In the new coordinate system the new equation to solve is a Laplace equation

$$\frac{\partial^2\bar{\phi}}{\partial x_{ic}{}^2} + \frac{\partial^2\bar{\phi}}{\partial y_{ic}{}^2} = 0$$

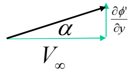
- The pressure coefficient is given by

$$C_p \approx -\frac{2u'}{V_\infty} = \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$$

Remember…

$$u' = \frac{\partial\phi'}{\partial x} = \frac{\partial\left(\frac{\bar{\phi}}{\beta}\right)}{\partial x}$$

- Where $\bar{u} = \partial\bar{\phi}/\partial x_{ic}$ is the solution of Laplace's equation in transformed space i.e. it is equivalent to an incompressible solution



Why did we scale the potential?

Consider the angle of any streamline…

$$\alpha \approx \frac{\frac{\partial\phi'}{\partial y}}{V_\infty} = \beta\frac{\frac{\partial(\bar{\phi}/\beta)}{\partial y_{ic}}}{V_\infty} = \frac{\frac{\partial\bar{\phi}}{\partial y_{ic}}}{V_\infty}$$

Remember…

$\beta\phi' = \bar{\phi}$

$x_{ic} = x$

$y_{ic} = \beta y$

The y-scaling reduces the angle, hence beta here

The potential scaling cancels it again here

This means the angle of attack is the same in both cases, and therefore the shape is the same in both cases (because a streamline runs along the surface).
Note that this is not **quite how it is done in 3D**, although the working is nearly the same. **See year 3**.

Subsonic Prandtl-Glauert Correction (1)

$$C_p \approx -\frac{2u'}{V_\infty} = \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$$

- The term in brackets in the Cp equation is the linearised incompressible pressure coefficient

$$C_p = 1 - \frac{v^2}{V_\infty^2} = 1 - \frac{(v'^2 + (V_\infty + u')^2)}{V_\infty^2} = 1 - \frac{(v'^2 + V_\infty{}^2 + 2V_\infty u' + u'^2)}{V_\infty^2} \approx -2\frac{u'}{V_\infty}$$

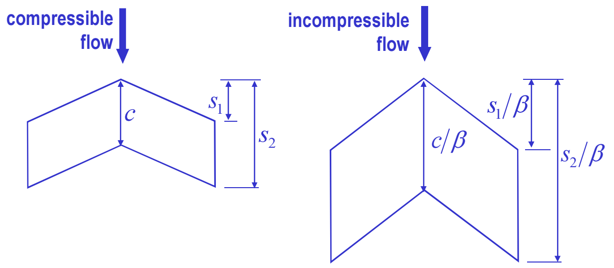
Call this incompressible pressure coefficient $\bar{C}_{p,0}$

- Hence

$$\boxed{C_p \approx = \frac{1}{\beta}\bar{C}_{p,0}}$$

Subsonic Prandtl-Glauert Correction (2)

- Thus a compressible flow over an aerofoil of thickness ratio *t/c* has the <u>same scaled **pressure distribution**</u> as incompressible flow over an aerofoil with thickness ratio $(t/c)_0$ where $\boxed{(t/c) = (t/c)_0/\beta}$



same scaled pressure distribution

$$\boxed{C_p = fn\left(\frac{(t/c)}{\beta}\right)}$$

Subsonic Prandtl-Glauert Correction (3)

- An alternative interpretation is to relate pressure (and hence lift and pitching moment) to incompressible values for *same* aerofoil **geometry**

$$\boxed{C_p = \frac{C_{p0}}{\beta}} \qquad \boxed{c_l = \frac{c_{l0}}{\beta},\quad c_m = \frac{c_{m0}}{\beta}}$$

C_p=compressible
C_{p0}=incompressible
about same geometry

- This indicates that incompressible data can be corrected and used to give a good first guess in design.

Note again – it is only possible to relate compressible/incompressible on the same geometry for any Mach number in 2D. This no longer holds in 3D.

Subsonic Prandtl-Glauert Correction (3)

- Due to the link between Incompressible analysis and Prandtl-Glauert correction, they have common features
  - Predicted drag (=0) (D'Alembert's paradox)
  - aerodynamic centre (at 0.25c) unaffected
- Prandtl-Glauert not the only correction factor

e.g. Karmen-Tsien

$$C_p = \frac{C_{p0}}{\beta + \dfrac{C_{p0}M_\infty^2}{2(1+\beta)}}$$

This is better for unsteady

Note that P-G & K-T corrections are good for external aero. Other corrections are used for internal flows

Supersonic Linearised Ackeret Theory (1)

- Consider again the linearised velocity potential equation, but now for $M_\infty > 1$ no upstream influence. Remember thickness and α must be small.
- Then in this case the pressure coefficient can be related directly to local surface inclination θ in radians (positive 'into flow') by

$$\frac{dp}{d\theta} = \frac{\gamma M^2}{\sqrt{M^2-1}}\, p$$

Assume p=p_inf, then integrate wrt angle

$$p - p_\infty = \frac{\gamma M^2 \theta}{\sqrt{M^2-1}}\, p_\infty$$

$$\frac{p}{p_\infty} - 1 = \frac{\gamma M^2 \theta}{\sqrt{M^2-1}}$$

Multiply by 2/(gam*M^2)

$$\longrightarrow \boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}}$$

(-ve sign dropped from last lecture; angle positive 'into' the flow)

Example

- Shock, M=2, deflection =4 degrees

Oblique shock theory

$$C_p = \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p_2}{p_1}\frac{p_1}{p_\infty} - 1\right) = \frac{2}{1.4\times 2^2}(1.247\times 1 - 1) = 0.0882$$

Ackeret's Method

$$C_p \approx \frac{2\times 4\times \dfrac{\pi}{180}}{\sqrt{2^2-1}} = 0.0806$$

ANGLE IN RADIANS!

8.7% error - fairly small, especially at low angles!

Supersonic Linearised Ackeret Theory (2)

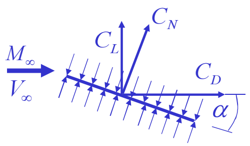
- Applying this to a *flat plate* 2D aerofoil at angle of attack α, this gives a *constant* pressure distribution

$$C_{P\text{lower}} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}, \quad C_{P\text{upper}} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}}, \quad c_N = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

- The lift and lift-dependent drag are

$$c_l = c_N \cos(\alpha) \approx c_N = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

$$c_d = c_N \sin(\alpha) \approx c_N \alpha = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$$



- Supersonic flow Drag ≠0 ("wave drag")
- Gives centre of pressure at 0.5c (0.25c for incompressible)

$$\boxed{c_d = \frac{\sqrt{M_\infty^2 - 1}}{4} c_l^2}$$

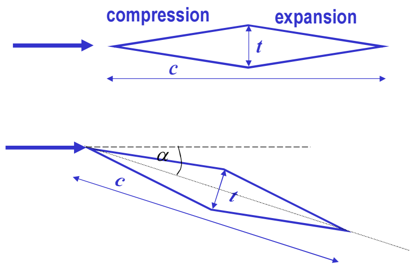
General approximate result for wave drag, we can use for most aerofoils at low α

Supersonic Linearised Ackeret Theory (3)

- For a thin 'double wedge' high speed section, linearised Ackeret theory gives

$$\boxed{c_d = 4\frac{(t/c)^2}{\sqrt{M_\infty^2 - 1}}}$$

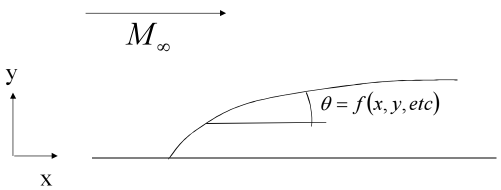
$$\boxed{c_d = 4\frac{\alpha^2 + (t/c)^2}{\sqrt{M_\infty^2 - 1}}}$$



- Note that there are now two components to the wave drag, one due to incidence and one due to thickness

Why use Ackeret?

- Convenient simplification for curved surfaces, especially where the angle may be a function of other variables
- In this case an integral is needed for Cl or Cd (taking appropriate force component for each one)

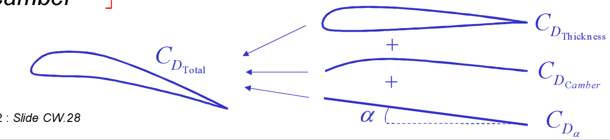


Wave Drag

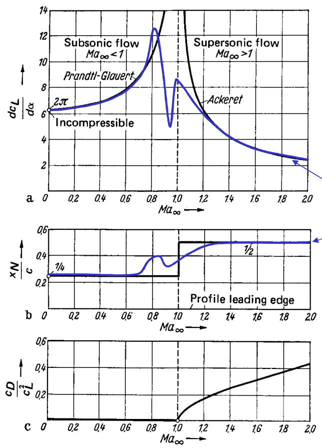
- Energy is lost to wave system shed by aerofoil
  - hence 'wave drag'
  - occurs even in isentropic flow
- Previous two examples of the application of Ackeret theory show components due to incidence (which is lift dependent) and thickness. In general, wave drag has three components, due to:
  - *lift*
  - *thickness*
  - *camber*

present at zero lift conditions

build up drag value by components



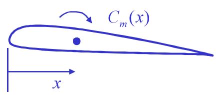
Linearised 2D Aerofoil Characteristics



typical values

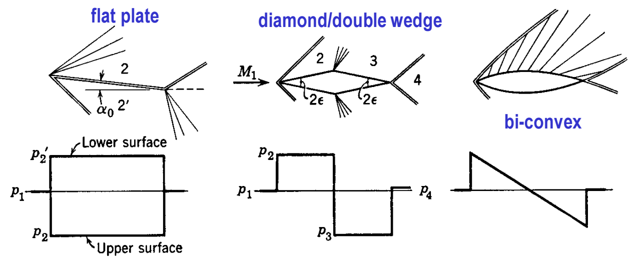
x_N=aerodynamic center

$$\frac{dC_m(x)}{d\alpha}=0 \quad @x=x_N$$



Aerodynamics 2 : *Slide CW.29*

'Real' Supersonic Aerofoil Flows



- NB modelling shock waves is not theoretically possible using linearised small perturbation theory…
  - <u>but</u> using the analysis for flows with shocks shows rather small errors at low α
  - wave drag is mostly due to energy 'lost' in expansion and compression waves (as opposed to total pressure loss through oblique shocks)

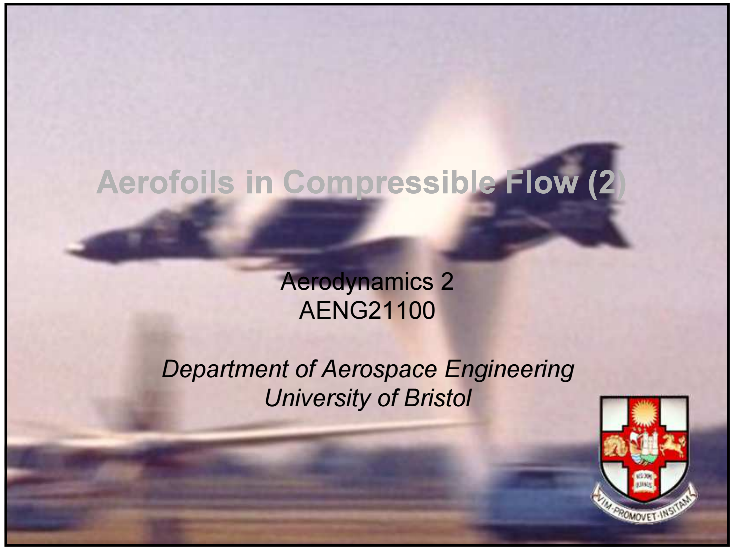
Aerodynamics 2 : *Slide CW.30*

Revision Objectives

You should be able to

- Explain the assumptions made to derive the linearised velocity potential equation for compressible flow and state that equation
- Explain the link between compressible and incompressible flow for subsonic Prandtl-Glauert
- Apply the Prandtl-Glauert correction to model problems
- To quote the pressure coefficient for supersonic Ackeret theory
- State the three components of wave drag
- Sketch typical flow patterns on real supersonic aerofoils

Aerodynamics 2 : *Slide CW.31*



Today

- To explain transonic 2D aerofoil behaviour
- To introduce the concept of critical Mach Number and how to calculate it

Aerodynamics 2 : *Slide CW.33*

Shocks

Aerodynamics 2 : *Slide CW.34*

Shocks

Aerodynamics 2 : *Slide CW.35*

Shocks

Aerodynamics 2 : *Slide CW.36*

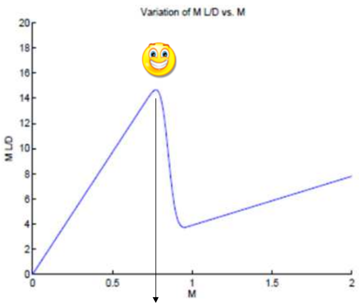
Shock Spotting

- Wait for clear, bright sunlight
- Best from directly above, so close to midday
- Keep looking. Depending on aircraft weight and Mach number, the shock will move in and out of the light (forwards with lower weight or lower Mach number)
- Turbulence makes them more obvious as they move back and forth a few cm. The incidence is changing in the gusts, altering lift coefficient and shock position
- If you get a good photo, send it to me!

Why Transonic?

Aircraft tend to be designed to maximise range, which depends on speed*L/D (from the Breguet range equation)

$$R = \frac{V}{sfc}\frac{L}{D}log(\frac{W_o + W_f}{W_o})$$



Transonic

2D Transonic Aerofoils-critical Mach Number

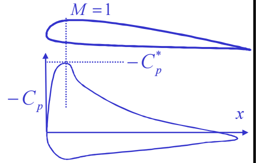
- Transonic behaviour begins when sonic flow first appears on the aerofoil surface
- Identification of sonic point important in aerofoil design
- Assuming flow is isentropic (P_0 constant) up to the sonic point
  - Use compressible C_p equation – isentropic flow relation for pressure
  - Set local $M=1$ to get critical pressure coefficient C_p*
- $M_\infty = M_{cr}$ when minimum pressure coefficient C_{pmin} = C_p*
- Still need to identify C_{pmin} for each aerofoil…experiment or a panel method might be used

2D Transonic Aerofoils
Critical Mach Number

Mach 0.8 to 1.2
mixed subsonic & supersonic flows

$$C_p = \left(\frac{p/p_\infty - 1}{\frac{1}{2}\gamma M_\infty^2} \right) \qquad \frac{p}{p_\infty} = \left(\frac{p}{p_{0\infty}}\right) \Big/ \left(\frac{p_\infty}{p_{0\infty}}\right)$$

$$\frac{p}{p_\infty} = \frac{\left(1+\frac{\gamma-1}{2}1^2\right)^{\frac{-\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{-\gamma}{\gamma-1}}} = \frac{\left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}}}$$



$$C_p^* = \frac{2}{\gamma M_\infty^2}\left\{ \left(\frac{1+0.5(\gamma-1)M_\infty^2}{1+0.5(\gamma-1)} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right\} = f(M_\infty)$$

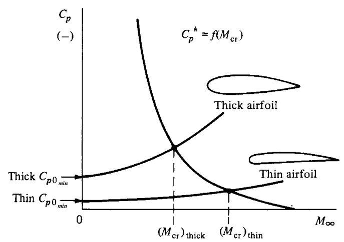
Some kind of corrected Cp – Prandtl-Glauert!

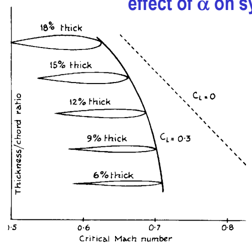
Don't try to solve analytically!

2D Transonic Aerofoils
Critical Mach Number

- Highest local velocities correspond to minimum C_p (suction peak)
  - depends on section and angle of attack
  - generally scales directly with t/c
  - but does not necessarily occur at maximum thickness
- Approximate C_p in terms of incompressible pressure distribution C_{p0} scaled with M_∞ by Prandtl-Glauert correction

effect of α on symmetric aerofoil

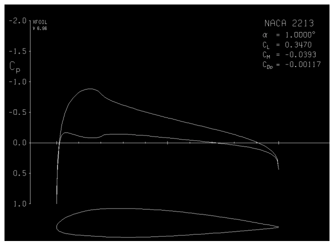


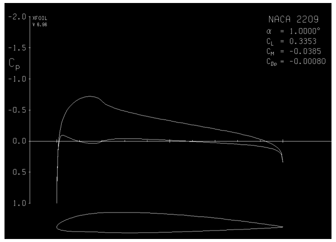


as α increases suction peak ($-C_p$) increases, so critical M reduces. General relationship in terms of C_l

Example – Spitfire Mcrit

Root - NACA 2213, tip NACA 2209





(Xfoil output)

Max value of U/Uinf=1.37, so Cpmin=-0.88 for AoA=1deg

At any AoA, most +ve Cpmin=-0.55

So…

How to…

- Having seen the equations, this is the quick way using tables…

Look up this ratio from tables

$$C_p^* = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_{0\infty}}\frac{p_{0\infty}}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_{0\infty}}\frac{p_{0\infty}}{p_\infty} - 1\right) = \boxed{\frac{2}{\gamma M_\infty^2}\left(\frac{1}{1.895}\frac{p_{0\infty}}{p_\infty} - 1\right)}$$

$$\boxed{C_p^{PG} = \frac{C_{p\min}}{\sqrt{1-M_\infty^2}}}$$

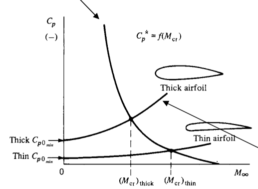
Both are easy to work out, so now we just need a method to solve iteratively – bisection is an easy option

At M=0.7, Cp*=(2/(1.4*0.7^2))*(1.388/1.895-1)=-0.78
CpPG=-0.88/sqrt(1-0.7^2)=-1.23

Some bisection for Cpmin=-0.88

M lower	M upper	M middle	Cp PG	Cp*	
0.6000	0.8000	0.7000	-1.2322	-0.7782	Make upper
0.6000	0.7000	0.6500	-1.1580	-1.0074	Make upper
0.6000	0.6500	0.6250	-1.1273	-1.1420	Make lower
0.6250	0.6500	0.6375	-1.1422	-1.0728	Make upper
0.6250	0.6375	0.6313	-1.1346	-1.1069	

Cp*



Mcrit

CpPG

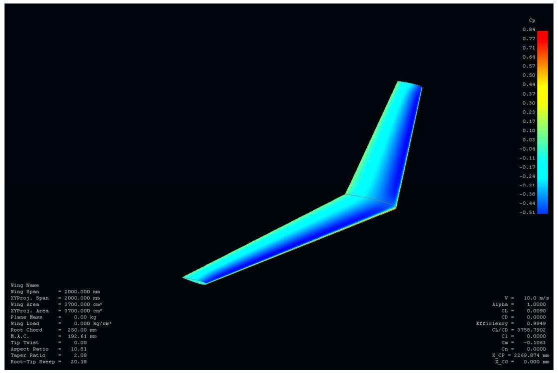
Increases to M=0.702 in the best case (Cpmin=-0.55)

Is it accurate?

- No 3D effects here, but would be included if Cpmin came from a 3D set of data
- The Cp here was from a 2D incompressible result, the 3D one would probably be less –ve, thereby raising Mcrit. Highest ever recorded Mach no. for the Spitfire was 0.89, 27% over our Mcrit
- Depends on incidence – I chose 1deg to run in Xfoil. If the pilot pulls harder Mcrit drops slightly. Highest possible Mcrit value occurs when the minimum Cp anywhere on the wing is maximised (made more positive)

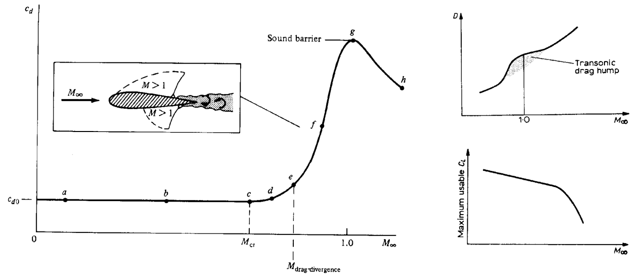
3D

AoA=1deg, Cpmin=-0.51, Mcrit=0.713



Above M_{cr} OR why is M_{CR} so important

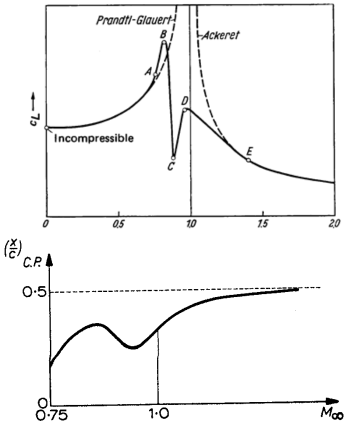
- ‘drag divergence’: rapid increase in drag just above M_{cr}
- ‘shock stall’ : M_{cr} decreases with α, above M_{cr} shocks form and separation induced at shock attachment point



- Aerodynamic centre moves aft: increased stability – but potentially a strong nose down pitching moment

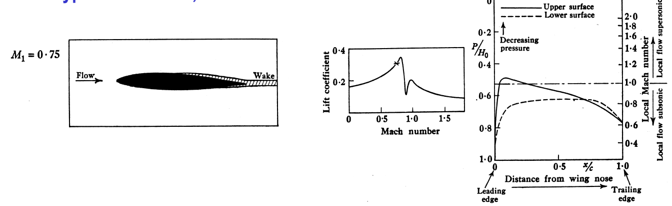
Transonic 2D Aerodynamic Characteristics
Transonic Vs Linearised

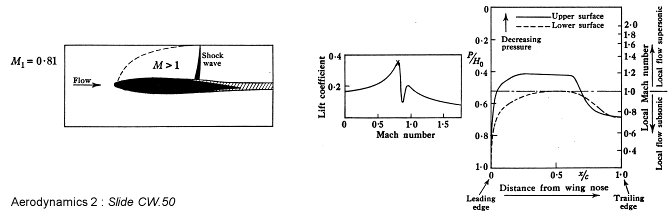
- complex lift behaviour
- drag peak near M=1 coefficient
- centre of pressure and aerodynamic centre shift



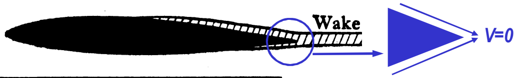
Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Subsonic

Typical behaviour, of aerofoil as Mach No increases.

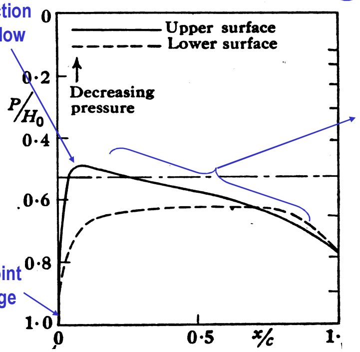




Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Subsonic



Significant suction & supersonic flow



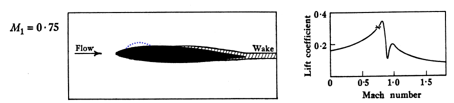
Stagnation point at leading edge

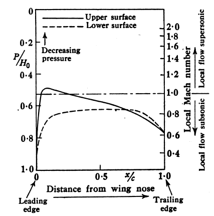
As peak suction increases, large compression required, leads to shock formation

Kutta condition implies zero velocity at trailing edge (stagnation point) for incompressible.

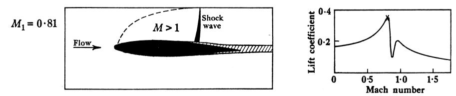
Compressible trailing edge has high static pressure & subsonic velocities in the wake. Upper & lower surface pressures equal.

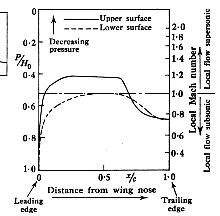
Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Subsonic





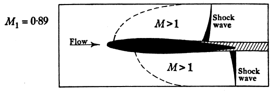
$M > M_{CR}$ →small region of supersonic flow.
Flow is still isentropic but starts to diverge from P-G solutions

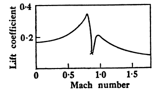


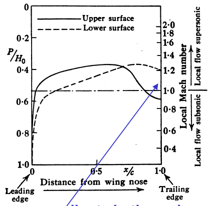


i) supersonic region terminates with a normal shock.
ii) Aero centre moves aft
iii) Drag rise due to losses in P_o through shock

Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Transonic

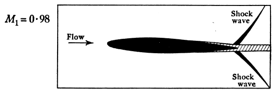


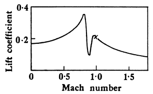


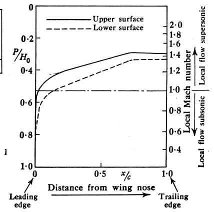


i) Supersonic flow on lower surface. Extent of supersonic region increases rapidly (flat surface).
ii) Drag continues to rise.
iii) Large lift loss
iv) Aero centre moves forward

Pressure adjusts in the wake

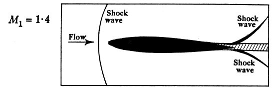


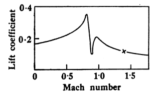


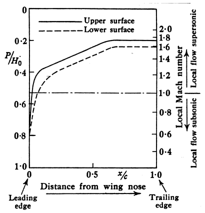


i) Shocks reach trailing edge (terminating shock system).
ii) Aero centre moves aft again
iii) Drag continues to rise.
iii) lift recovers

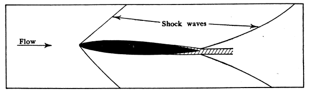
Transonic 2D Aerodynamic Characteristics
Transonic Aerofoil Flows - Supersonic



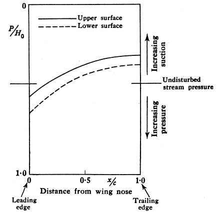


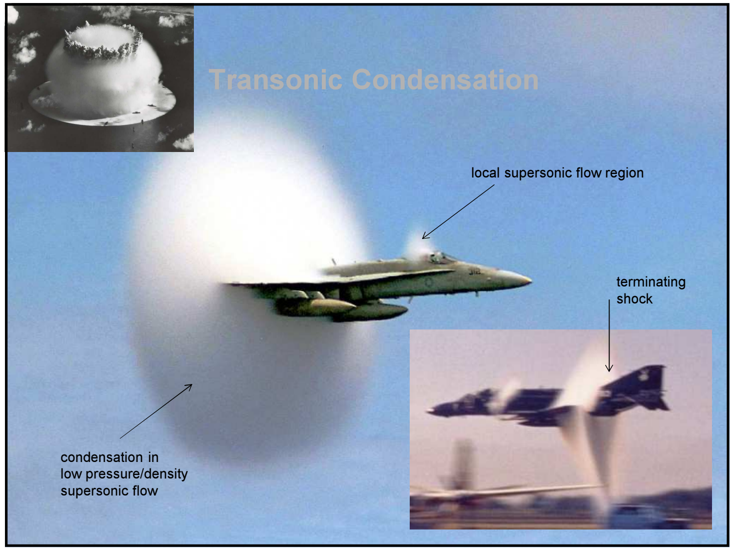


i) Bow shock forms
ii) possible terminating shocks at trailing edges
iii) Aero centre fixed at approx 0.5C

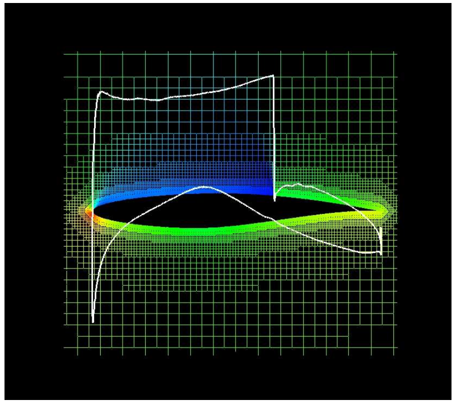


Bow shock reattaches as oblique shocks --
Mach number depends on leading edge shape.

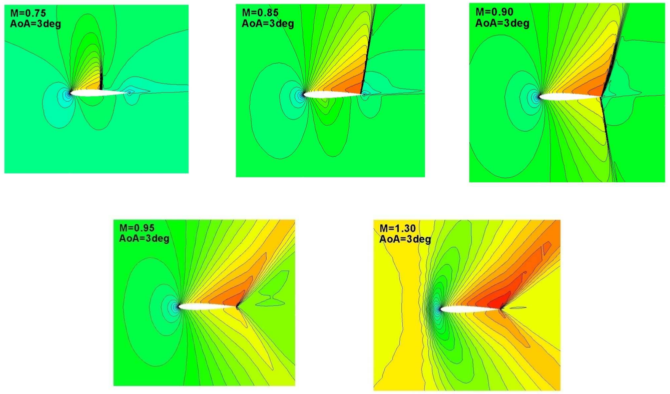




Transonic Aerofoil (RAE2822) - inviscid



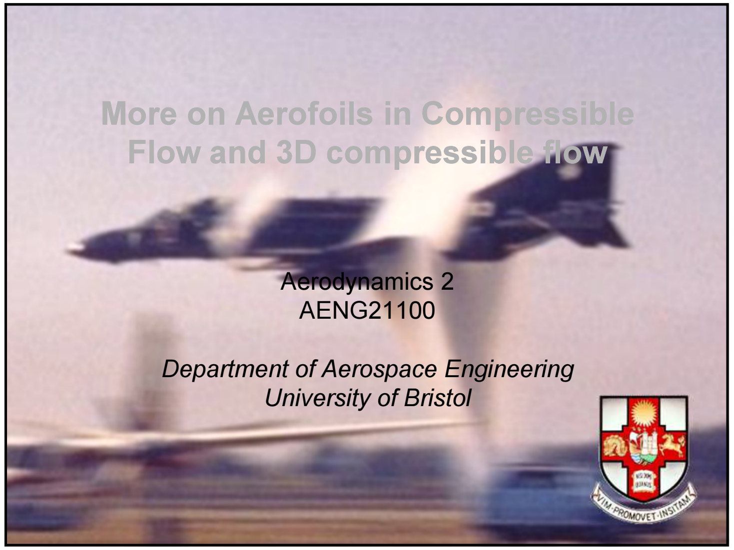
Transonic-supersonic NACA0012



Revision Objectives

You should be able to

- Describe transonic aerofoil behaviour using words, pictures and graphs, in terms of Cl, Cd, Cm and the aerodynamic centre
- Calculate the critical Mach number of an aerofoil given Cpmin



More on Aerofoils in Compressible Flow and 3D compressible flow

Aerodynamics 2
AENG21100

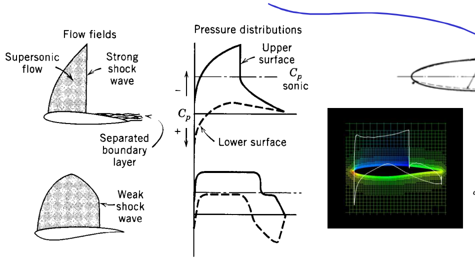
Department of Aerospace Engineering
University of Bristol

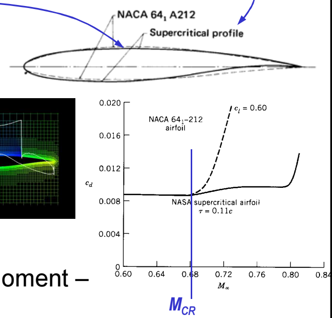
Aims for this Lecture

- To introduce supercritical aerofoils for transonic flow
- To look at 3D compressible flows. In particular at
  - The effect of wing sweep
  - Wing characteristics
  - The area rule

Supercritical Aerofoil

- Delay drag divergence by weakening upper surface shock
  - 'aft-loaded' section
  - flat upper surface with strongly cambered aft region
  - also gives improved structure - more volume for metal and fuel

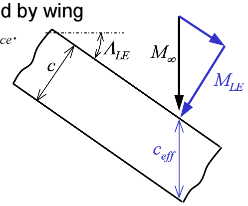


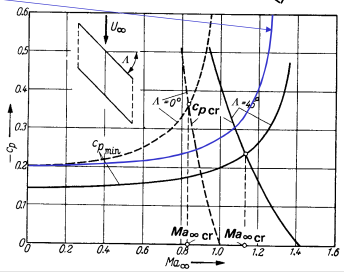


… but watch out for the pitching moment – gains may be lost if this increases

Wing Sweep – Basic Effect

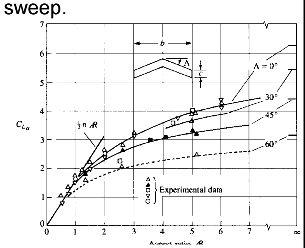
- Transonic & supersonic design characterised by wing sweep. Increase M_{crit} and hence $M_{drag\text{-}divergence}$. Sweep can lower supersonic drag by a factor of 2 to 3
- Two Effects
  - reduced effective Mach Number M_{LE}
  - reduced effective thickness t/c
- Prandtl-Glauert correction and critical pressure coefficient curves, scale with M_{LE} not M_∞
- Reduced pressure increase with Λ and hence reduced incompressible suction peak C_{pmin}
- Result is M_{crit} increased

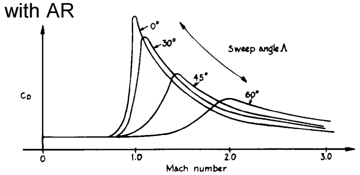


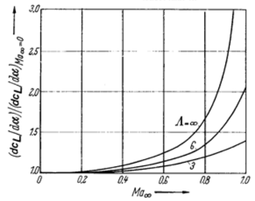


Effect of Sweep on Subsonic Wings

- For transonic flow, drag rise delayed and reduced by sweep. But when M_{LE} supersonic, wave drag increased by sweep.
- Consider Prandtl-Glauert correction as a transformation to an 'equivalent' incompressible flow. Aspect ratio reduced and sweep increased with increasing M
  - lift curve slope reduced
  - no effect on induced drag factor k
- Prandtl-Glauert scaling effects reduce with AR

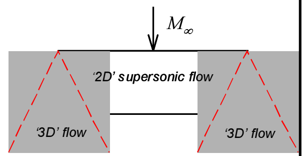


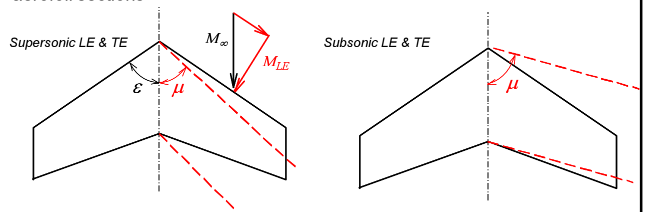




Supersonic Swept Wing Considerations

- Interaction of aircraft components limited by Mach cone, very important for control
- Leading-edge sweep very important
  - "Supersonic" leading edges generate bow shocks ($M_{LE} > 1$)
  - "Subsonic" leading edge gives lower wave drag and can use subsonic aerofoil sections



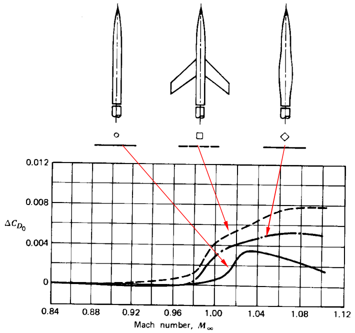


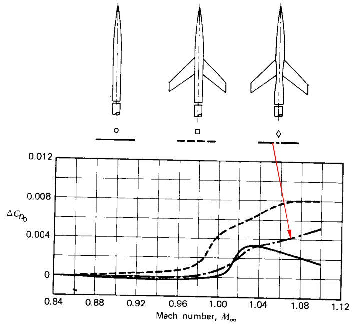
F-15 Clipped tips

Aerodynamics 2 : *Slide CW.65*

Zero-Lift Drag

- At zero lift the drag is made up from: skin friction on surface, wave drag due to thickness distribution and "base drag" due to afterbody separation. Below are some experimental results



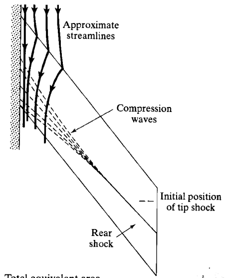


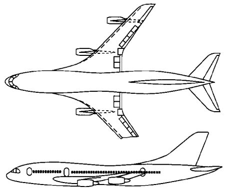
- fuselage + bulge
- fuselage + waist

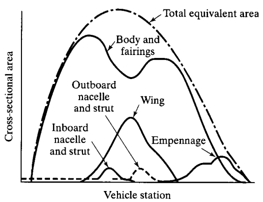
Aerodynamics 2 : *Slide CW.66*

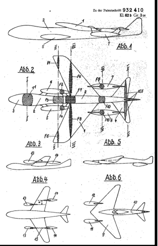
The Transonic 'Area Rule'

- Drag rise is a function of the rate of change of the longitudinal distribution of cross-sectional area (think how Ackeret's Cp is dependent on the gradient of the surface in 2D)
- Aim to match distribution of 'minimum drag' Sears-Haack body



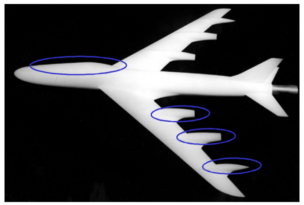


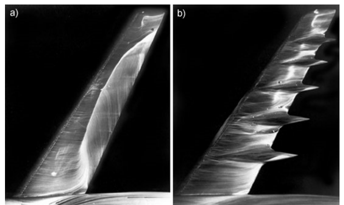




Aerodynamics 2 : *Slide CW.67*

The Transonic 'Area Rule'

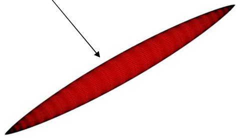




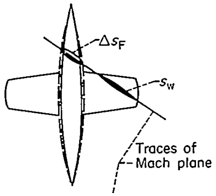
Rear shock

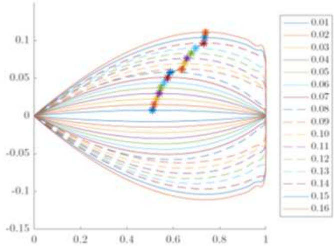
Aerodynamics 2 : *Slide CW.68*

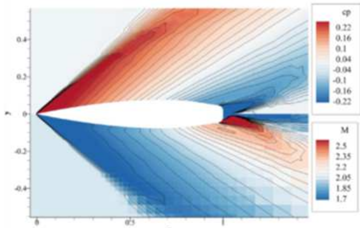
Sears-Haack body - influence of the idea



In supersonic flow, it is the area distribution along the mach cone that matters, not the flow axis





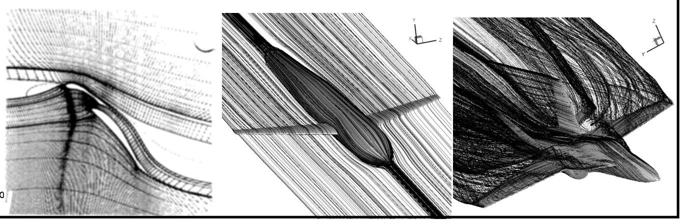


Sad but historically significant; approx. 1km increase in range over earlier designs. Idea originally French/German, dating from ~1890s.

Led to tactical change in WW1, especially long rang machine gun use. Improved accuracy at fixed range – bullet would fall less during flight, due to shorter flight time.

An understanding of the area rule

- In 2D streamlines form the edges of con-di nozzles, or a con-di-con-di… nozzle. We hope that none of these nozzles chokes, otherwise a shock will occur (transonic case). A good area distribution achieves this
- In 3D we have 'streamtubes', but the same argument holds. Remember tubes are bounded by the 'rest' of the air – which is what makes things complicated! How much the tubes are 'squeezed' depends on the x-sectional area of the body



Why?

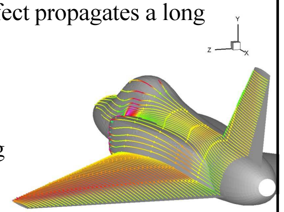
Recall from first lectures

$$\left(M^2-1\right)\frac{1}{V}\frac{dV}{dx} = \frac{1}{A}\frac{dA}{dx}$$

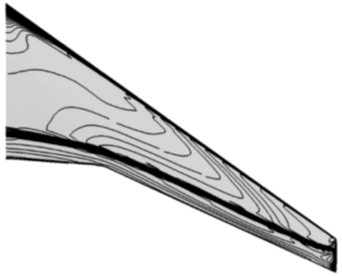
So if M=1 the area **cannot be changing for a streamtube**

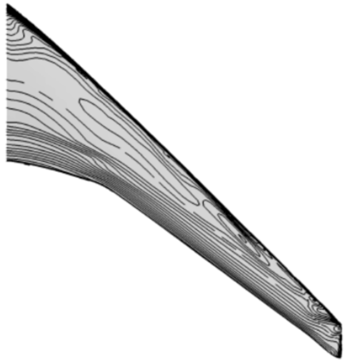
This means streamtubes around the aircraft interact rather strongly, which means if one is displaced outwards, by a small change in cross sectional area of the aircraft, this effect propagates a long way, and has quite a large influence

Richard Whitcomb, originator of the supercritical aerofoil, termed M=1 flow 'pipefitter's flow', as it is similar to fitting rigid pipes around the aircraft



Wing sweep may be thought of as
an application of the area rule!

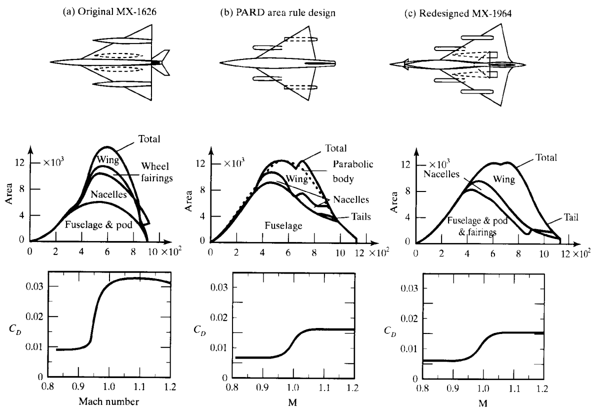




Domain-Element Method for Aerodynamic Shape
Optimization Applied to a Modern Transport Wing
AIAA JOURNAL
Vol. 47, No. 7, July 2009

Best sweep configuration not a
simple problem!

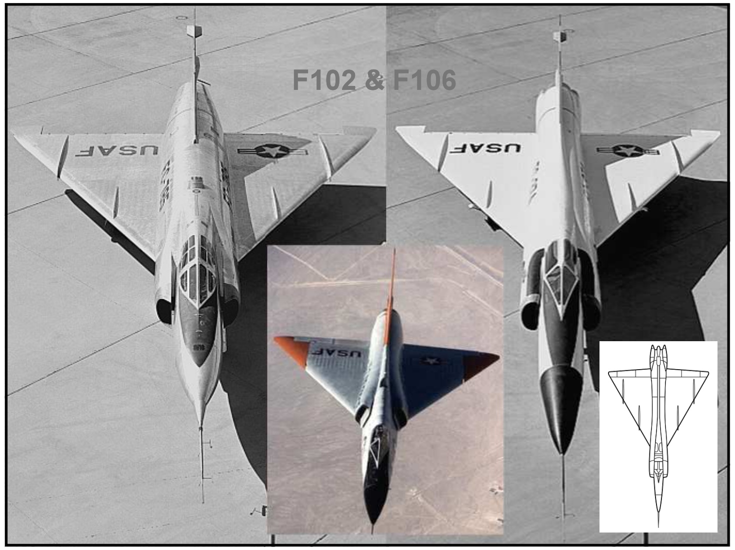
B58 'Area Rule'



F-102

- Oft quoted case, where initial aircraft was limited to M=0.98 in level flight
- After area ruling, structural and engine improvements, M=1.22 achieved.

F102 & F106



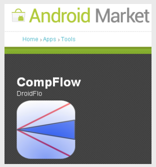
Blackburn Buccaneer

T-38

Revision Objectives

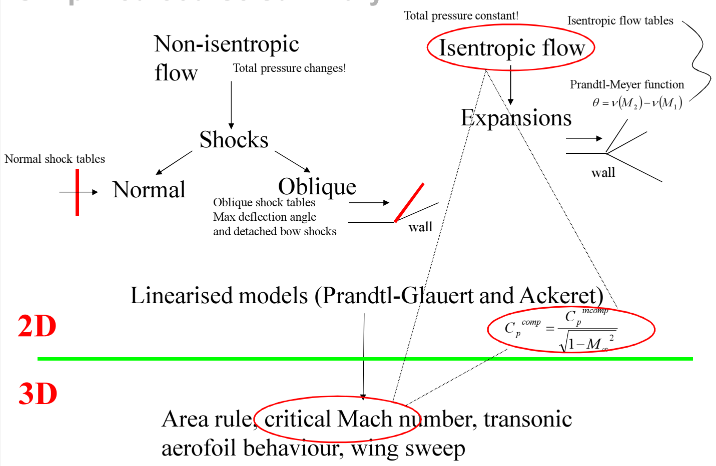
You should be able to

- Explain sweep effects
- Explain the area rule and its effect on aircraft design

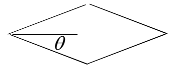


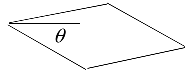
(free!)

Simplified course summary



Tutorial sheet 3





- Q1 - shock/expansion theory followed by a comparison to linear theory.

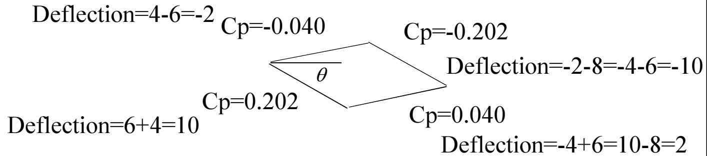
$$L=\int_S p\cos(\theta)ds \quad \text{and} \quad C_p=\frac{2}{\gamma M_\infty^{\;2}}\left(\frac{p}{p_\infty}-1\right)=\frac{p-p_\infty}{q_\infty}$$

$$L=\int_S \left(C_p q_\infty + p_\infty\right)\cos(\theta)ds$$

but a constant pressure cancels in a loop integral, so...

$$C_L=\frac{L}{q_\infty c}=\int_S C_p\cos(\theta)d\frac{s}{c} \qquad C_D=\frac{D}{q_\infty c}=\int_S C_p\sin(\theta)d\frac{s}{c}$$

Q1 – example to get Cl and Cd



Ackeret's linear Cp gives (remember radians)...

Cl=(0.5/cos(4))*[0.04*cos(2)+0.202*cos(10)+ (unit chord)
0.202*cos(10)+0.04*cos(2)]=0.24
Cd=(0.5/cos(4))*[2*0.04*sin(2)+2*0.202*sin(10)]=0.037
Cl/Cd=6.5

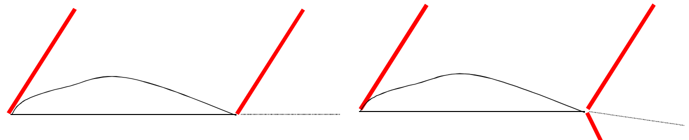
$$\boxed{C_N=\frac{4\alpha}{\sqrt{M_\infty^2-1}}\approx C_L} \longrightarrow 0.24$$

$$\boxed{c_d = 4\frac{\alpha^2+(t/c)^2}{\sqrt{M_\infty^2-1}}} \longrightarrow 0.037$$

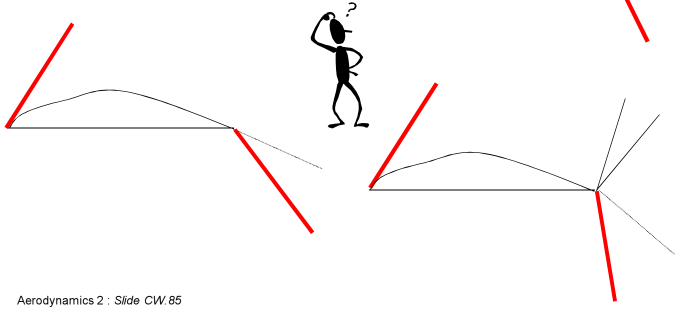
Q3

- The calculation of the flow angle at the trailing edge is an example of an equal pressure condition needing to be satisfied – we cannot have a pressure discontinuity in a wake (otherwise the wake would move!)
- In general there will be a slip line discontinuity, across which velocity changes
- Not particularly straightforward to find the angle of a slip line. Trickier if there's more than one!

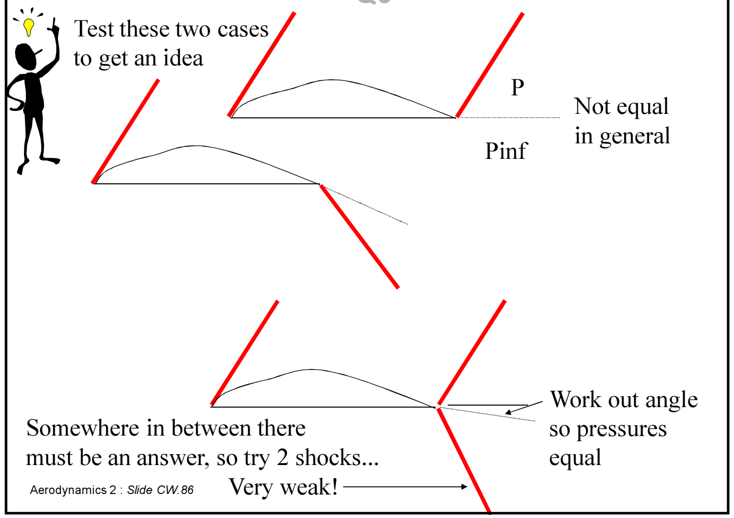
Q3



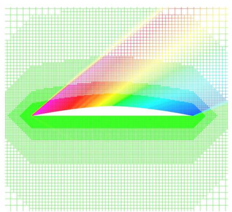
If aerofoil not symmetric, you can expect a slip line

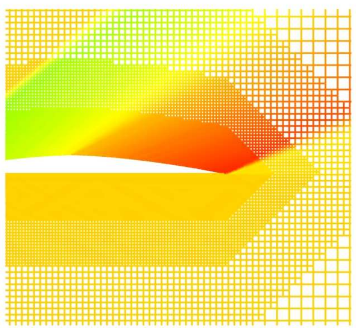


Q3

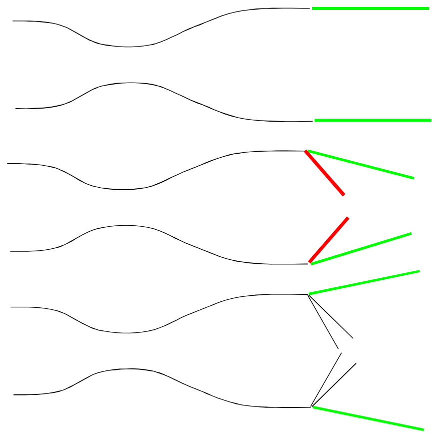


Q3





Q4 - easy after Q3!



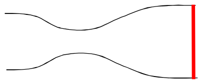
Ideally expanded - Pexit=isentropic pressure at nozzle exit

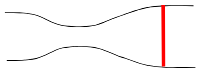
Over expanded - Pexit>isentropic pressure at nozzle exit

Under expanded - Pexit<isentropic pressure at nozzle exit

Q4

- The internal normal shock cases are simpler. The point is that given the shock position it is easy to work out the exit pressure that produces that shock position
- It is not so easy to find the shock position given the exit pressure. This would be similar to the slip line problem - we would be trying to match a given exit pressure, and anything where a pressure condition is matched is harder to do
- …but it could be done with a fairly simple iterative procedure (bisection?) that kept moving the shock until the exit pressure was correct





Q5/6 - Mcrit

- These are as in the lecture, with a minor variation. Start by working out the compressible Cpmin on the aerofoil, then use PG to get the incompressible Cpmin.
- Then solve as usual using bisection - remember what the graphs look like so you know which way to move
- Practice this procedure (or any other if you prefer, so long as you are clear about what you are doing, and it works)

AERODYNAMICS 2

Examples Sheet 1a – One-Dimensional Isentropic Flow

For all cases assume that the flow is one-dimensional and isentropic.

For air, assume $\gamma = 1.4$ and $R = 287$ J/kgK. Take 0°C as 273 K, and remember that 1 bar = 10^5 Pa.

Use subsonic & supersonic isentropic flow tables from your lecture notes – for these example questions you will need to use linear interpolation.

1. An air flow of Mach 2 is produced in a duct of rectangular cross-section 25cm x 20cm. If the air is supplied from a reservoir at a pressure of 6 bar and a temperature of 20°C, calculate the pressure, temperature, velocity and mass flow of the air in the duct.

   Ans: 0.766 bar, -111°C , 510.5 m/s, 42.0 kg/s

2. At a certain section along a duct in which air is flowing the Mach Number is 0.4, the pressure is 2 bar and the temperature is 350K. Calculate the pressure and temperature at a section further downstream where the Mach Number is 0.8. What is the percentage change in density between the two sections? If the duct cross-sectional area at the upstream section is 100cm^2, determine the cross-sectional area at the downstream section.

   Ans: 1.46 bar, 320 K , -20%, 65.3cm^2

3. (a) An aeroplane is flying at a velocity of 240m/s at low level, where the atmospheric pressure and temperature can be taken to have their standard sea level values of 1.013 bar and 288 K. What Mach Number is the aircraft flying at? What pressure would be measured by a pitot probe mounted on the aircraft nose? A static pressure measurement is made on the upper surface of the wing at a point where the local flow velocity is sonic (ie M = 1.0). What is the pressure at this point, and what is the local velocity?

   (b) A model of the aeroplane is tested in a simple wind tunnel in which the air flow is produced by drawing in air from the atmosphere (again assume standard sea level conditions) through a smooth contraction into the working section and then into a low pressure system. If the freestream Mach Number in the tunnel matches the flight value of part (a), what is the corresponding static pressure and freestream velocity in the working section? What pressure would be measured by a nose-mounted pitot probe?

   Ans: (a) 0.706, 1.41 bar, 0.746 bar, 326 m/s
   (b) 0.726 bar, 229 m/s, 1.013 bar

4. (a) At a certain section of a two-dimensional channel the width is 4 cm and the Mach Number is 0.5. What are the two values of Mach Number possible at a downstream section where the width is 6 cm? For each of the two Mach Numbers, sketch how the channel width must vary between the two sections.

   (b) If the reservoir pressure is 1.02 bar, calculate the pressure at the downstream section needed to maintain each type of flow.

   Ans: (a) 0.304, 2.205
   (b) 0.956 bar, 0.0946 bar

5. (a) An axisymmetric convergent-divergent nozzle supplied with air from a reservoir produces a flow of Mach Number 2.2 and static pressure 0.5 bar. The exit diameter is 50 cm. What is the diameter of the throat and the value of the static pressure there? What are the Mach Number and static pressure of the flow at the section in the converging part of the nozzle where the diameter is 45 cm?

   (b) If a different flow is set up in the nozzle such that the Mach Number at the exit is 0.2, what are the Mach Numbers at the throat and at the section in the converging part where the diameter is 45 cm?

   Ans: (a) 35.34 cm, 2.82 bar; 0.391, 4.81 bar
   (b) 0.438, 0.251

AERODYNAMICS 2

Examples Sheet 2-Shocks and Expansions

For air, assume $\gamma = 1.4$ and $R = 287$ J/kgK. Take 0°C as 273 K, and remember that 1 bar = 10^5 Pa.

1. Consider the flow of air through a normal shock wave. Upstream of the shock wave the Mach Number is 2.5, the pressure is 0.2 bar and the temperature is –57°C (conditions in the stratosphere). Calculate the total temperature upstream and downstream of the shock wave, and the corresponding total pressures. What is the downstream Mach Number? What would the total temperature downstream of the shock wave be if the upstream Mach Number were 5? (Use formula in handouts for $M = 5$ case)

   Ans: 215°C , 215°C , 3.42 bar, 1.71 bar, 0.514, 1023°C

2. There is supersonic flow in a duct. At a certain place, the static pressure is 0.4 bar and a pitot tube in the flow registers a pressure of 1.9 bar. Calculate the Mach Number at this place. What is the percentage error in the measured pitot pressure compared to the true total pressure here? At another place somewhat further downstream, the Mach Number is 10% greater. What pressure would a pitot tube register at this second station?

   Assume isentropic flow between the two stations, and that the pitot tubes are very small compared with the dimensions of the flow. As a result, the local disturbance produced by the upstream pitot tube will have a negligible effect on the rest of the flow.

   Ans: 1.816, -19%, 1.70 bar

3. A uniform air stream of Mach Number 2.1 is expanded two-dimensionally over a sharp corner for which the angle of deflection of the surface is 6°. What is the Mach Number of the flow after the expansion? If the upstream total pressure is 3 bar, what are the static pressures before and after the expansion?

   Ans: 2.333, 0.328 bar, 0.228 bar

4. An air stream at Mach Number 2.2 is deflected through an angle of 20° by a two-dimensional wedge. What is the angle of the attached oblique shock wave, and the Mach Number behind the shock?

   What would be the pressures indicated by pitot tubes aligned in the local stream direction and placed (a) in front of the shock and (b) behind the shock, if the total pressure of the flow approaching the wedge is 10 bar?

   If the Mach Number of the air stream is reduced, at what value of that Mach Number will the shock wave become detached?

   Ans: 48°, 1.40, 6.29 bar, 8.45 bar, 1.84

AERODYNAMICS 2

Examples Sheet 3

For air, assume $\gamma = 1.4$ and R = 287 J/kgK. Take 0°C as 273 K, and remember that 1 bar = 10^5 Pa.

1. The wing of a guided missile has a cross-section in the form of a symmetrical double wedge with a total included angle at the leading-edge of 8°. Calculate the pressure coefficients for each of the four surfaces when it is at an angle of incidence of 6° at a Mach Number of 2. What would be the error in these pressures if linear small disturbance theory were used to calculate them? ($C_p = 2\theta / \sqrt{M_\infty^2 - 1}$, where θ is positive 'into' the flow)

   Ans: -0.039, -0.162, +0.253, +0.043
   +4%, +24%, -20%, -6%

2. The first part of the air intake for a jet engine of a supersonic aircraft is a wedge which deflects the flow through an angle of 12° , effectively two-dimensionally, at cruise conditions at a Mach number of 1.8. This produces an attached oblique shock wave which is followed in the intake by a normal shock wave. There is then a subsonic diffuser which reduces the air velocity to a Mach number of 0.4 at the engine face. Assuming one-dimensional isentropic flow in the diffuser, calculate the ratio of the static pressure at the engine face to the ambient atmospheric pressure

   Ans: 4.85

3. The wing of a stealthy supersonic aircraft has an airfoil section described by a flat lower surface and a purely convex upper surface. The total included angle at the leading-edge is 16° and the total included angle at the trailing edge is 4°. Given that the aircraft is flying at Mach 2 with the lower surface horizontal, calculate the Mach number just ahead of the trailing edge on the upper and lower surfaces. Find the angle at which the flow leaves the trailing edge.
   You may assume that a normal shock does **not** form on the leading edge of the lower surface.

   Ans: 2.1003, 2.0, 0.121°

4. A two-dimensional convergent divergent nozzle takes in air from the atmosphere (pressure=1bar) and discharges into a large chamber whose pressure can be varied. The height of the throat is 13mm and the height of the exit plane of the nozzle is 35mm.
   a. Calculate the value of the chamber pressure for which a parallel supersonic jet will be obtained.
   b. If the chamber pressure is (i) 10% less than this value, or (ii) twice this value, calculate the angle of inclination of each jet on leaving the nozzle exit plane, assuming that the jet is also effectively two-dimensional.
   c. As the chamber pressure increases above this latter value, the shock waves forming from the lips of the exit plane of the nozzle become stronger. Assuming that the flow within the nozzle remains one-dimensional and inviscid, estimate the value of the chamber pressure at which a normal shock would occur across the exit plane of the nozzle.

d. For even higher values of chamber pressure, a normal shock would occur somewhere in the divergent part of the nozzle. If the normal shock occurs where the height of the nozzle is 25mm, calculate the value of the chamber pressure and the value of the pitot pressure measured at the exit plane of the nozzle. Note: In practise the real flow will be considerably different, due to interaction of the viscous boundary layer and the shock waves. See A2 lab.

Ans:
a. 0.0563 bar
b. 1.54^{o} outwards, 11.08^{o} inwards
c. 0.410 bar
d. 0.595 bar, 0.649 bar
e. 0.967 bar, 0.975 bar

5. Wind tunnel data is taken for a wing at an onset Mach number of 0.3. It is found that a maximum Mach number of 0.4 is reached on the surface. What is the critical freestream Mach number M_{crit}? You may assume isentropic flow from the free stream to the point at which the Mach number just reaches 1.

Ans: 0.6637

6. Consider a wing in a wind tunnel where the reservoir pressure is $1atm$ and the onset Mach number is 0.3 . The minimum pressure on the aerofoil is found to be $0.9atm$. What is the critical Mach number of the aerofoil under these conditions? What is the critical Mach number of the aerofoil in free flight where the ambient pressure is $0.2bar$? You may assume isentropic flow from the free stream to the point at which the Mach number just reaches 1.

Ans: 0.6819

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Compressible Flow Tables for a Perfect Gas with γ=1.403

Subsonic isentropic Relationships

M	p_o/p	ρ_o/ρ	T_o/T	A/A^*
0.00	1.000	1.000	1.000	0.000
0.02	1.000	1.000	1.000	28.933
0.04	1.001	1.001	1.000	14.477
0.06	1.003	1.002	1.001	9.663
0.08	1.004	1.003	1.001	7.259
0.10	1.007	1.005	1.002	5.820
0.12	1.010	1.007	1.003	4.863
0.14	1.014	1.010	1.004	4.181
0.16	1.018	1.013	1.005	3.672
0.18	1.023	1.016	1.007	3.277
0.20	1.028	1.020	1.008	2.963
0.22	1.034	1.024	1.010	2.707
0.24	1.041	1.029	1.012	2.495
0.26	1.048	1.034	1.014	2.317
0.28	1.056	1.040	1.016	2.165
0.30	1.065	1.046	1.018	2.035
0.32	1.074	1.052	1.021	1.921
0.34	1.083	1.059	1.023	1.822
0.36	1.094	1.066	1.026	1.735
0.38	1.105	1.074	1.029	1.658
0.40	1.117	1.082	1.032	1.590
0.42	1.129	1.091	1.036	1.529
0.44	1.143	1.100	1.039	1.474
0.46	1.156	1.109	1.043	1.424
0.48	1.171	1.119	1.046	1.380

M	p_o/p	ρ_o/ρ	T_o/T	A/A^*
0.50	1.187	1.130	1.050	1.340
0.52	1.203	1.141	1.054	1.303
0.54	1.220	1.152	1.059	1.270
0.56	1.238	1.164	1.063	1.240
0.58	1.257	1.177	1.068	1.213
0.60	1.276	1.190	1.073	1.188
0.62	1.297	1.203	1.077	1.166
0.64	1.318	1.217	1.083	1.145
0.66	1.340	1.232	1.088	1.126
0.68	1.364	1.247	1.093	1.110
0.70	1.388	1.263	1.099	1.094
0.72	1.413	1.280	1.104	1.080
0.74	1.440	1.297	1.110	1.068
0.76	1.467	1.314	1.116	1.057
0.78	1.496	1.332	1.123	1.047
0.80	1.525	1.351	1.129	1.038
0.82	1.556	1.371	1.135	1.030
0.84	1.589	1.391	1.142	1.024
0.86	1.622	1.412	1.149	1.018
0.88	1.657	1.433	1.156	1.013
0.90	1.693	1.455	1.163	1.009
0.92	1.730	1.478	1.171	1.006
0.94	1.769	1.502	1.178	1.003
0.96	1.809	1.526	1.186	1.001
0.98	1.851	1.551	1.194	1.000
1.00	1.895	1.577	1.201	1.000

Supersonic isentropic Relationships

M	p_o/p	ρ_o/ρ	T_o/T	A/A^*	ν
1.00	1.895	1.577	1.201	1.000	0.000
1.02	1.940	1.604	1.210	1.000	0.130
1.04	1.987	1.631	1.218	1.001	0.350
1.06	2.035	1.659	1.226	1.003	0.640
1.08	2.085	1.688	1.235	1.005	0.970
1.10	2.137	1.718	1.244	1.008	1.330
1.12	2.191	1.749	1.253	1.011	1.730
1.14	2.247	1.781	1.262	1.015	2.160
1.16	2.305	1.814	1.271	1.020	2.600
1.18	2.365	1.847	1.281	1.025	3.070
1.20	2.428	1.882	1.290	1.030	3.550
1.22	2.492	1.917	1.300	1.037	4.050
1.24	2.559	1.954	1.310	1.043	4.560
1.26	2.628	1.991	1.320	1.050	5.090
1.28	2.700	2.030	1.330	1.058	5.620
1.30	2.774	2.069	1.341	1.066	6.160
1.32	2.851	2.110	1.351	1.075	6.710
1.34	2.930	2.152	1.362	1.084	7.270
1.36	3.013	2.195	1.373	1.094	7.830
1.38	3.098	2.239	1.384	1.104	8.400
1.40	3.186	2.284	1.395	1.115	8.970
1.42	3.277	2.330	1.406	1.126	9.550
1.44	3.372	2.378	1.418	1.138	10.130
1.46	3.470	2.427	1.430	1.150	10.710
1.48	3.571	2.477	1.441	1.163	11.300
1.50	3.675	2.529	1.453	1.176	11.880
1.52	3.784	2.582	1.466	1.190	12.470
1.54	3.896	2.636	1.478	1.204	13.060
1.56	4.011	2.692	1.490	1.219	13.650
1.58	4.131	2.749	1.503	1.234	14.240
1.60	4.255	2.807	1.516	1.250	14.830
1.62	4.383	2.867	1.529	1.266	15.420
1.64	4.516	2.929	1.542	1.283	16.010
1.66	4.653	2.992	1.555	1.300	16.600
1.68	4.795	3.056	1.569	1.318	17.190
1.70	4.941	3.123	1.582	1.337	17.770
1.72	5.093	3.191	1.596	1.356	18.360
1.74	5.249	3.260	1.610	1.375	18.940
1.76	5.411	3.332	1.624	1.396	19.520
1.78	5.578	3.405	1.638	1.416	20.100
1.80	5.751	3.480	1.653	1.438	20.680
1.82	5.930	3.556	1.667	1.460	21.260
1.84	6.115	3.635	1.682	1.482	21.830
1.86	6.305	3.715	1.697	1.505	22.400
1.88	6.502	3.798	1.712	1.529	22.970

M	p_o/p	ρ_o/ρ	T_o/T	A/A^*	ν
1.90	6.706	3.882	1.727	1.554	23.530
1.92	6.916	3.969	1.743	1.579	24.090
1.94	7.134	4.057	1.758	1.604	24.650
1.96	7.358	4.148	1.774	1.631	25.210
1.98	7.590	4.240	1.790	1.658	25.770
2.00	7.830	4.335	1.806	1.685	26.320
2.02	8.077	4.432	1.822	1.714	26.860
2.04	8.332	4.532	1.839	1.743	27.410
2.06	8.596	4.634	1.855	1.772	27.950
2.08	8.868	4.738	1.872	1.803	28.490
2.10	9.149	4.844	1.889	1.834	29.020
2.12	9.439	4.953	1.906	1.866	29.560
2.14	9.738	5.065	1.923	1.898	30.090
2.16	10.047	5.179	1.940	1.932	30.610
2.18	10.366	5.295	1.958	1.966	31.130
2.20	10.695	5.415	1.975	2.001	31.650
2.22	11.035	5.536	1.993	2.037	32.170
2.24	11.385	5.661	2.011	2.073	32.680
2.26	11.746	5.789	2.029	2.111	33.190
2.28	12.119	5.919	2.047	2.149	33.690
2.30	12.504	6.052	2.066	2.188	34.190
2.32	12.901	6.189	2.085	2.228	34.690
2.34	13.310	6.328	2.103	2.269	35.180
2.36	13.732	6.470	2.122	2.311	35.670
2.38	14.167	6.616	2.141	2.353	36.160
2.40	14.615	6.764	2.161	2.397	36.650
2.42	15.078	6.916	2.180	2.441	37.130
2.44	15.555	7.072	2.200	2.487	37.600
2.46	16.046	7.230	2.219	2.533	38.080
2.48	16.553	7.392	2.239	2.580	38.550
2.50	17.075	7.558	2.259	2.629	39.010
2.52	17.614	7.727	2.280	2.678	39.480
2.54	18.168	7.899	2.300	2.729	39.940
2.56	18.740	8.076	2.321	2.780	40.390
2.58	19.329	8.256	2.341	2.832	40.840
2.60	19.935	8.440	2.362	2.886	41.290

Flow Through a Normal Shock Wave

M	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2	p_{02}/p_1
1.00	1.000	1.000	1.000	1.000	1.895
1.02	1.047	1.033	1.013	0.981	1.940
1.04	1.095	1.067	1.027	0.962	1.986
1.06	1.144	1.101	1.040	0.945	2.035
1.08	1.194	1.135	1.053	0.928	2.084
1.10	1.245	1.169	1.065	0.912	2.135
1.12	1.297	1.203	1.078	0.897	2.188
1.14	1.350	1.237	1.091	0.882	2.241
1.16	1.404	1.272	1.104	0.868	2.296
1.18	1.458	1.306	1.116	0.855	2.353
1.20	1.514	1.341	1.129	0.842	2.410
1.22	1.570	1.376	1.141	0.830	2.469
1.24	1.628	1.410	1.154	0.818	2.529
1.26	1.686	1.445	1.167	0.807	2.591
1.28	1.746	1.480	1.179	0.796	2.653
1.30	1.806	1.515	1.192	0.786	2.717
1.32	1.867	1.550	1.205	0.776	2.782
1.34	1.929	1.584	1.218	0.767	2.848
1.36	1.992	1.619	1.231	0.757	2.915
1.38	2.056	1.654	1.243	0.748	2.984
1.40	2.121	1.688	1.256	0.740	3.053
1.42	2.187	1.723	1.269	0.732	3.124
1.44	2.254	1.757	1.283	0.724	3.196
1.46	2.321	1.792	1.296	0.716	3.269
1.48	2.390	1.826	1.309	0.709	3.343
1.50	2.460	1.860	1.322	0.701	3.418
1.52	2.530	1.894	1.336	0.694	3.494
1.54	2.602	1.928	1.349	0.688	3.571
1.56	2.674	1.962	1.363	0.681	3.650
1.58	2.747	1.996	1.377	0.675	3.729
1.60	2.822	2.029	1.391	0.669	3.810
1.62	2.897	2.063	1.405	0.663	3.892
1.64	2.973	2.096	1.419	0.657	3.975
1.66	3.050	2.129	1.433	0.652	4.059
1.68	3.128	2.162	1.447	0.646	4.144
1.70	3.207	2.194	1.461	0.641	4.230
1.72	3.287	2.227	1.476	0.636	4.317
1.74	3.368	2.259	1.491	0.631	4.405
1.76	3.449	2.292	1.505	0.626	4.494
1.78	3.532	2.324	1.520	0.621	4.585

M	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2	p_{02}/p_1
1.80	3.616	2.355	1.535	0.617	4.676
1.82	3.700	2.387	1.550	0.612	4.769
1.84	3.786	2.418	1.566	0.608	4.862
1.86	3.872	2.449	1.581	0.604	4.957
1.88	3.959	2.480	1.596	0.600	5.052
1.90	4.048	2.511	1.612	0.596	5.149
1.92	4.137	2.541	1.628	0.592	5.247
1.94	4.227	2.572	1.644	0.588	5.346
1.96	4.318	2.602	1.660	0.585	5.446
1.98	4.410	2.632	1.676	0.581	5.547
2.00	4.503	2.661	1.692	0.578	5.649
2.02	4.597	2.691	1.709	0.574	5.752
2.04	4.692	2.720	1.725	0.571	5.856
2.06	4.788	2.749	1.742	0.568	5.961
2.08	4.884	2.777	1.759	0.565	6.067
2.10	4.982	2.806	1.776	0.562	6.174
2.12	5.080	2.834	1.793	0.559	6.283
2.14	5.180	2.862	1.810	0.556	6.392
2.16	5.280	2.889	1.828	0.553	6.502
2.18	5.382	2.917	1.845	0.550	6.614
2.20	5.484	2.944	1.863	0.548	6.726
2.22	5.587	2.971	1.881	0.545	6.840
2.24	5.691	2.998	1.899	0.542	6.954
2.26	5.797	3.024	1.917	0.540	7.070
2.28	5.903	3.051	1.935	0.537	7.187
2.30	6.010	3.077	1.953	0.535	7.304
2.32	6.117	3.102	1.972	0.533	7.423
2.34	6.226	3.128	1.991	0.530	7.543
2.36	6.336	3.153	2.009	0.528	7.664
2.38	6.447	3.178	2.028	0.526	7.786
2.40	6.558	3.203	2.048	0.524	7.909
2.42	6.671	3.228	2.067	0.522	8.033
2.44	6.784	3.252	2.086	0.520	8.158
2.46	6.899	3.276	2.106	0.518	8.284
2.48	7.014	3.300	2.126	0.516	8.411
2.50	7.131	3.324	2.145	0.514	8.539
2.52	7.248	3.347	2.165	0.512	8.668
2.54	7.366	3.370	2.186	0.510	8.798
2.56	7.485	3.393	2.206	0.508	8.930
2.58	7.605	3.416	2.226	0.506	9.062
2.60	7.726	3.439	2.247	0.504	9.195

Flow Through a Plane Oblique Shock Wave

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
1.1	0	65.38	1.000	1.000	1.000	1.100
1.1	1.51	76.30	1.166	1.116	1.045	0.971
1.1	0	90.00	1.245	1.169	1.065	0.912
1.2	0	56.44	1.000	1.000	1.000	1.200
1.2	2	61.06	1.120	1.084	1.033	1.111
1.2	3.94	71.98	1.353	1.239	1.092	0.950
1.2	2	83.85	1.495	1.329	1.125	0.855
1.2	0	90.00	1.514	1.341	1.129	0.842
1.3	0	50.28	1.000	1.000	1.000	1.300
1.3	2	53.48	1.107	1.075	1.03	1.224
1.3	4	57.46	1.234	1.161	1.063	1.140
1.3	6	63.50	1.413	1.278	1.106	1.027
1.3	6.65	69.39	1.561	1.370	1.139	0.936
1.3	6	75.32	1.679	1.441	1.165	0.864
1.3	4	81.63	1.764	1.491	1.183	0.812
1.3	2	86.05	1.796	1.509	1.190	0.792
1.3	0	90.00	1.806	1.515	1.192	0.786
1.4	0	45.58	1.000	1.000	1.000	1.400
1.4	2	48.18	1.103	1.073	1.029	1.329
1.4	4	51.13	1.220	1.152	1.059	1.255
1.4	6	54.65	1.355	1.241	1.092	1.173
1.4	8	59.40	1.528	1.350	1.132	1.074
1.4	9.41	67.71	1.792	1.507	1.189	0.927
1.4	8	75.85	1.984	1.615	1.229	0.819
1.4	6	80.46	2.058	1.655	1.244	0.777
1.4	4	83.97	2.096	1.675	1.251	0.755
1.4	2	87.07	2.115	1.685	1.255	0.743
1.4	0	90.00	2.121	1.688	1.256	0.740
1.5	0	41.81	1.000	1.000	1.000	1.500
1.5	2	44.07	1.103	1.073	1.029	1.432
1.5	4	46.55	1.217	1.150	1.058	1.361
1.5	6	49.34	1.344	1.234	1.090	1.288
1.5	8	52.59	1.490	1.326	1.123	1.207
1.5	10	56.72	1.668	1.435	1.163	1.113
1.5	12	64.58	1.976	1.610	1.227	0.957
1.5	12.09	66.58	2.045	1.647	1.241	0.921
1.5	12	68.56	2.109	1.682	1.254	0.889
1.5	10	75.95	2.305	1.783	1.293	0.786
1.5	8	79.68	2.375	1.819	1.306	0.748
1.5	6	82.64	2.417	1.839	1.314	0.725
1.5	4	85.24	2.442	1.851	1.319	0.711
1.5	2	87.66	2.455	1.858	1.322	0.704
1.5	0	90.00	2.460	1.860	1.322	0.701

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
1.6	0	38.68	1.000	1.000	1.000	1.600
1.6	2	40.73	1.105	1.074	1.029	1.532
1.6	4	42.94	1.219	1.152	1.059	1.464
1.6	6	45.35	1.345	1.235	1.090	1.393
1.6	8	48.05	1.486	1.324	1.122	1.319
1.6	10	51.14	1.645	1.421	1.158	1.239
1.6	12	54.93	1.835	1.531	1.198	1.147
1.6	14	60.64	2.103	1.679	1.253	1.021
1.6	14.62	65.82	2.320	1.791	1.295	0.919
1.6	14	70.78	2.498	1.879	1.330	0.834
1.6	12	75.85	2.643	1.948	1.357	0.762
1.6	10	79.07	2.714	1.980	1.371	0.726
1.6	8	81.67	2.759	2.001	1.379	0.702
1.6	6	83.95	2.788	2.014	1.384	0.687
1.6	4	86.05	2.807	2.023	1.388	0.676
1.6	2	88.05	2.818	2.028	1.390	0.671
1.6	0	90.00	2.822	2.029	1.391	0.669
1.7	0	36.03	1.000	1.000	1.000	1.700
1.7	2	37.93	1.107	1.075	1.030	1.632
1.7	4	39.96	1.224	1.155	1.060	1.564
1.7	6	42.15	1.352	1.239	1.091	1.494
1.7	8	44.54	1.493	1.328	1.124	1.423
1.7	10	47.19	1.648	1.423	1.159	1.347
1.7	12	50.20	1.824	1.525	1.196	1.266
1.7	14	53.82	2.031	1.640	1.238	1.174
1.7	16	58.89	2.306	1.784	1.293	1.055
1.7	16.98	65.31	2.618	1.936	1.352	0.919
1.7	16	71.32	2.861	2.047	1.398	0.809
1.7	14	75.62	2.999	2.107	1.423	0.745
1.7	12	78.52	3.073	2.139	1.437	0.709
1.7	10	80.88	3.122	2.159	1.446	0.684
1.7	8	82.95	3.156	2.173	1.452	0.667
1.7	6	84.83	3.180	2.183	1.456	0.655
1.7	4	86.61	3.195	2.190	1.459	0.647
1.7	2	88.32	3.204	2.193	1.461	0.642
1.7	0	90.00	3.207	2.194	1.461	0.641

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
1.8	0	33.75	1.000	1.000	1.000	1.800
1.8	2	35.54	1.111	1.078	1.031	1.731
1.8	4	37.45	1.231	1.159	1.062	1.662
1.8	6	39.49	1.362	1.246	1.094	1.593
1.8	8	41.69	1.506	1.336	1.127	1.522
1.8	10	44.08	1.663	1.431	1.162	1.449
1.8	12	46.71	1.837	1.533	1.199	1.371
1.8	14	49.70	2.033	1.641	1.239	1.288
1.8	16	53.25	2.261	1.761	1.284	1.194
1.8	18	58.09	2.559	1.908	1.341	1.074
1.8	19.15	64.97	2.939	2.081	1.412	0.920
1.8	18	71.31	3.227	2.203	1.465	0.798
1.8	16	75.26	3.371	2.261	1.491	0.734
1.8	14	77.98	3.452	2.292	1.506	0.697
1.8	12	80.18	3.506	2.313	1.515	0.671
1.8	10	82.10	3.544	2.328	1.522	0.652
1.8	8	83.85	3.572	2.339	1.527	0.639
1.8	6	85.47	3.592	2.346	1.531	0.629
1.8	4	87.02	3.605	2.351	1.533	0.622
1.8	2	88.52	3.613	2.354	1.535	0.618
1.8	0	90.00	3.616	2.355	1.535	0.617
1.9	0	31.76	1.000	1.000	1.000	1.900
1.9	2	33.47	1.114	1.080	1.032	1.830
1.9	4	35.28	1.239	1.165	1.064	1.760
1.9	6	37.22	1.374	1.253	1.097	1.689
1.9	8	39.28	1.522	1.346	1.131	1.618
1.9	10	41.51	1.684	1.444	1.166	1.545
1.9	12	43.92	1.861	1.546	1.204	1.470
1.9	14	46.58	2.056	1.654	1.243	1.390
1.9	16	49.59	2.276	1.769	1.287	1.304
1.9	18	53.16	2.532	1.895	1.336	1.206
1.9	20	58.01	2.865	2.048	1.399	1.080
1.9	21.12	64.77	3.282	2.225	1.475	0.922
1.9	20	70.93	3.598	2.349	1.532	0.796
1.9	18	74.79	3.758	2.408	1.561	0.729
1.9	16	77.41	3.848	2.441	1.577	0.689
1.9	14	79.53	3.908	2.462	1.587	0.662
1.9	12	81.35	3.952	2.478	1.595	0.641
1.9	10	83.00	3.985	2.489	1.601	0.626
1.9	8	84.52	4.009	2.498	1.605	0.615
1.9	6	85.95	4.027	2.504	1.608	0.606
1.9	4	87.33	4.039	2.508	1.610	0.600
1.9	2	88.67	4.046	2.510	1.612	0.597
1.9	0	90.00	4.048	2.511	1.612	0.596

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
2.0	0	30.00	1.000	1.000	1.000	2.000
2.0	2	31.65	1.118	1.083	1.033	1.928
2.0	4	33.40	1.247	1.170	1.066	1.856
2.0	6	35.25	1.388	1.262	1.100	1.785
2.0	8	37.22	1.541	1.358	1.135	1.713
2.0	10	39.33	1.708	1.458	1.172	1.640
2.0	12	41.60	1.891	1.563	1.210	1.564
2.0	14	44.06	2.091	1.672	1.250	1.485
2.0	16	46.77	2.312	1.787	1.294	1.402
2.0	18	49.83	2.560	1.908	1.341	1.311
2.0	20	53.49	2.850	2.042	1.396	1.208
2.0	22	58.60	3.235	2.206	1.467	1.072
2.0	22.93	64.65	3.647	2.367	1.541	0.924
2.0	22	70.18	3.966	2.483	1.598	0.805
2.0	20	74.19	4.156	2.548	1.631	0.729
2.0	18	76.80	4.260	2.583	1.649	0.686
2.0	16	78.88	4.329	2.605	1.662	0.657
2.0	14	80.65	4.380	2.622	1.671	0.634
2.0	12	82.23	4.418	2.634	1.677	0.617
2.0	10	83.68	4.447	2.643	1.682	0.604
2.0	8	85.04	4.468	2.650	1.686	0.594
2.0	6	86.33	4.484	2.655	1.689	0.587
2.0	4	87.57	4.495	2.659	1.691	0.582
2.0	2	88.79	4.501	2.661	1.692	0.579
2.0	0	90.00	4.503	2.661	1.692	0.578
2.1	0	28.44	1.000	1.000	1.000	2.100
2.1	2	30.04	1.122	1.086	1.034	2.026
2.1	4	31.73	1.256	1.176	1.068	1.953
2.1	6	33.52	1.403	1.271	1.103	1.879
2.1	8	35.42	1.562	1.371	1.140	1.806
2.1	10	37.45	1.736	1.475	1.177	1.731
2.1	12	39.61	1.926	1.582	1.217	1.655
2.1	14	41.94	2.132	1.694	1.259	1.576
2.1	16	44.46	2.359	1.810	1.303	1.494
2.1	18	47.25	2.609	1.932	1.351	1.406
2.1	20	50.42	2.891	2.060	1.404	1.310
2.1	22	54.25	3.224	2.202	1.465	1.199
2.1	24	59.97	3.692	2.384	1.549	1.044
2.1	24.56	64.60	4.034	2.506	1.610	0.927
2.1	24	68.90	4.314	2.601	1.659	0.829
2.1	22	73.43	4.563	2.680	1.703	0.736
2.1	20	76.12	4.686	2.718	1.724	0.688
2.1	18	78.21	4.767	2.742	1.738	0.655
2.1	16	79.96	4.825	2.760	1.749	0.631
2.1	14	81.51	4.870	2.773	1.756	0.612
2.1	12	82.91	4.904	2.783	1.762	0.597
2.1	10	84.22	4.930	2.790	1.767	0.585
2.1	8	85.45	4.949	2.796	1.770	0.577
2.1	6	86.63	4.964	2.800	1.773	0.570
2.1	4	87.77	4.974	2.803	1.774	0.565
2.1	2	88.89	4.980	2.805	1.775	0.563
2.1	0	90.00	4.982	2.806	1.776	0.562

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
2.2	0	27.04	1.000	1.000	1.000	2.200
2.2	2	28.59	1.127	1.089	1.035	2.123
2.2	4	30.24	1.266	1.183	1.071	2.048
2.2	6	31.99	1.418	1.281	1.107	1.973
2.2	8	33.84	1.585	1.385	1.145	1.898
2.2	10	35.80	1.766	1.492	1.184	1.822
2.2	12	37.89	1.964	1.603	1.225	1.744
2.2	14	40.12	2.179	1.719	1.268	1.664
2.2	16	42.52	2.414	1.838	1.314	1.581
2.2	18	45.13	2.671	1.961	1.362	1.494
2.2	20	48.03	2.956	2.088	1.415	1.401
2.2	22	51.35	3.279	2.224	1.475	1.298
2.2	24	55.46	3.667	2.375	1.544	1.177
2.2	26	63.32	4.345	2.610	1.664	0.963
2.2	26.05	64.60	4.444	2.642	1.682	0.931
2.2	26	65.84	4.537	2.672	1.698	0.900
2.2	24	72.45	4.970	2.802	1.774	0.751
2.2	22	75.34	5.122	2.846	1.800	0.695
2.2	20	77.49	5.219	2.873	1.817	0.658
2.2	18	79.26	5.288	2.891	1.829	0.630
2.2	16	80.80	5.340	2.906	1.838	0.609
2.2	14	82.18	5.380	2.916	1.845	0.593
2.2	12	83.46	5.411	2.925	1.850	0.580
2.2	10	84.65	5.435	2.931	1.854	0.569
2.2	8	85.78	5.453	2.936	1.858	0.561
2.2	6	86.87	5.467	2.940	1.860	0.555
2.2	4	87.93	5.477	2.942	1.862	0.551
2.2	2	88.97	5.482	2.944	1.862	0.548
2.2	0	90.00	5.484	2.944	1.863	0.548
2.3	0	25.77	1.000	1.000	1.000	2.300
2.3	2	27.30	1.131	1.092	1.036	2.221
2.3	4	28.91	1.276	1.189	1.073	2.143
2.3	6	30.62	1.435	1.292	1.111	2.066
2.3	8	32.43	1.608	1.399	1.150	1.989
2.3	10	34.34	1.798	1.510	1.191	1.910
2.3	12	36.37	2.005	1.626	1.233	1.831
2.3	14	38.53	2.230	1.745	1.278	1.750
2.3	16	40.84	2.474	1.867	1.325	1.666
2.3	18	43.33	2.741	1.993	1.376	1.578
2.3	20	46.05	3.034	2.122	1.430	1.486
2.3	22	49.09	3.360	2.256	1.489	1.387
2.3	24	52.62	3.733	2.399	1.556	1.275
2.3	26	57.22	4.198	2.562	1.639	1.138
2.3	27.39	64.63	4.875	2.775	1.757	0.934
2.3	26	71.12	5.362	2.912	1.842	0.777
2.3	24	74.42	5.564	2.965	1.877	0.708
2.3	22	76.70	5.683	2.996	1.897	0.665
2.3	20	78.53	5.765	3.016	1.911	0.634
2.3	18	80.09	5.827	3.032	1.922	0.610
2.3	16	81.47	5.874	3.043	1.930	0.591
2.3	14	82.73	5.911	3.053	1.936	0.576
2.3	12	83.90	5.940	3.060	1.941	0.564
2.3	10	85.01	5.963	3.065	1.945	0.555
2.3	8	86.06	5.980	3.070	1.948	0.547
2.3	6	87.07	5.993	3.073	1.951	0.542
2.3	4	88.06	6.002	3.075	1.952	0.538
2.3	2	89.04	6.008	3.076	1.953	0.536
2.3	0	90.00	6.010	3.077	1.953	0.535

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
2.4	0	24.62	1.000	1.000	1.000	2.400
2.4	2	26.12	1.136	1.095	1.037	2.318
2.4	4	27.71	1.286	1.196	1.076	2.238
2.4	6	29.38	1.452	1.302	1.115	2.158
2.4	8	31.16	1.633	1.414	1.155	2.078
2.4	10	33.04	1.831	1.529	1.198	1.998
2.4	12	35.02	2.048	1.649	1.242	1.916
2.4	14	37.13	2.283	1.772	1.288	1.833
2.4	16	39.38	2.540	1.899	1.338	1.748
2.4	18	41.78	2.818	2.028	1.39	1.659
2.4	20	44.38	3.122	2.159	1.446	1.566
2.4	22	47.23	3.457	2.294	1.507	1.468
2.4	24	50.44	3.830	2.434	1.574	1.361
2.4	26	54.29	4.267	2.585	1.651	1.238
2.4	28	59.88	4.865	2.771	1.755	1.071
2.4	28.62	64.68	5.328	2.902	1.836	0.937
2.4	28	69.06	5.699	3.000	1.900	0.825
2.4	26	73.29	6.002	3.075	1.952	0.728
2.4	24	75.81	6.154	3.111	1.978	0.677
2.4	22	77.74	6.255	3.135	1.996	0.641
2.4	20	79.35	6.329	3.152	2.008	0.614
2.4	18	80.76	6.385	3.164	2.018	0.593
2.4	16	82.03	6.429	3.174	2.025	0.576
2.4	14	83.19	6.464	3.182	2.031	0.562
2.4	12	84.28	6.491	3.188	2.036	0.551
2.4	10	85.31	6.513	3.193	2.04	0.542
2.4	8	86.29	6.530	3.197	2.043	0.535
2.4	6	87.24	6.543	3.200	2.045	0.530
2.4	4	88.18	6.552	3.202	2.046	0.527
2.4	2	89.09	6.557	3.203	2.047	0.524
2.4	0	90.00	6.558	3.203	2.048	0.524

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
2.5	0	23.58	1.000	1.000	1.000	2.500
2.5	2	25.05	1.141	1.098	1.039	2.415
2.5	4	26.61	1.297	1.203	1.078	2.332
2.5	6	28.27	1.469	1.313	1.119	2.250
2.5	8	30.02	1.659	1.429	1.161	2.167
2.5	10	31.86	1.866	1.549	1.205	2.084
2.5	12	33.82	2.093	1.673	1.251	2.001
2.5	14	35.89	2.340	1.801	1.299	1.915
2.5	16	38.08	2.609	1.932	1.351	1.827
2.5	18	40.42	2.901	2.064	1.405	1.737
2.5	20	42.93	3.218	2.199	1.463	1.643
2.5	22	45.65	3.564	2.336	1.526	1.544
2.5	24	48.66	3.947	2.476	1.594	1.439
2.5	26	52.12	4.379	2.622	1.671	1.323
2.5	28	56.47	4.904	2.783	1.762	1.183
2.5	29.73	64.75	5.803	3.026	1.918	0.940
2.5	28	71.80	6.419	3.172	2.024	0.760
2.5	26	74.76	6.626	3.218	2.059	0.695
2.5	24	76.87	6.754	3.246	2.081	0.652
2.5	22	78.57	6.844	3.265	2.096	0.621
2.5	20	80.02	6.911	3.279	2.108	0.597
2.5	18	81.31	6.964	3.29	2.117	0.578
2.5	16	82.48	7.006	3.298	2.124	0.562
2.5	14	83.57	7.039	3.305	2.13	0.550
2.5	12	84.59	7.066	3.311	2.134	0.539
2.5	10	85.56	7.087	3.315	2.138	0.531
2.5	8	86.49	7.103	3.318	2.141	0.525
2.5	6	87.39	7.115	3.321	2.143	0.520
2.5	4	88.27	7.124	3.322	2.144	0.516
2.5	2	89.14	7.129	3.323	2.145	0.514
2.5	0	90.00	7.131	3.324	2.145	0.514

M	δ	β	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2
2.6	0	22.62	1.000	1.000	1.000	2.600
2.6	2	24.07	1.146	1.102	1.040	2.512
2.6	4	25.62	1.308	1.210	1.081	2.426
2.6	6	27.25	1.487	1.324	1.123	2.341
2.6	8	28.98	1.685	1.444	1.166	2.256
2.6	10	30.80	1.902	1.570	1.212	2.170
2.6	12	32.73	2.140	1.698	1.260	2.083
2.6	14	34.77	2.399	1.831	1.311	1.995
2.6	16	36.93	2.682	1.965	1.364	1.905
2.6	18	39.22	2.988	2.102	1.421	1.813
2.6	20	41.66	3.32	2.240	1.482	1.717
2.6	22	44.29	3.681	2.380	1.547	1.617
2.6	24	47.16	4.077	2.521	1.617	1.512
2.6	26	50.38	4.516	2.665	1.695	1.398
2.6	28	54.20	5.025	2.818	1.783	1.269
2.6	30	59.58	5.702	3.000	1.900	1.098
2.6	30.74	64.84	6.299	3.145	2.003	0.943
2.6	30	69.55	6.762	3.247	2.082	0.817
2.6	28	73.47	7.087	3.315	2.138	0.721
2.6	26	75.87	7.256	3.349	2.167	0.669
2.6	24	77.71	7.368	3.371	2.186	0.632
2.6	22	79.24	7.451	3.387	2.200	0.605
2.6	20	80.58	7.515	3.399	2.211	0.583
2.6	18	81.78	7.565	3.408	2.219	0.565
2.6	16	82.87	7.605	3.416	2.226	0.550
2.6	14	83.89	7.637	3.422	2.232	0.539
2.6	12	84.86	7.663	3.427	2.236	0.529
2.6	10	85.77	7.683	3.431	2.240	0.521
2.6	8	86.66	7.699	3.434	2.242	0.515
2.6	6	87.51	7.711	3.436	2.244	0.510
2.6	4	88.35	7.720	3.437	2.246	0.507
2.6	2	89.18	7.724	3.438	2.247	0.505
2.6	0	90.00	7.726	3.439	2.247	0.504