EMAT10100 Engineering Maths I

Lecture 12: Determinants and the matrix inverse

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Looking back looking forward

Last time:

- Square $n \times n$ matrices as transformations on n-vectors
  - linear: maps straight lines to straight lines
  - special case of $n = 2$: transformations of the plane scaling, rotation, reflexion and shear
- Determinants give area (or volume) scale factor
  - formula for two by two case
- Singular matrices (determinant equals zero)
- Identity and diagonal matrices

This time:

- Determinant of general square matrices
- Inverse of general square matrices
- **Warning: This lecture has a lot of theory in it**

Three by three determinants

- Main idea: any (big) determinant can be built up out of smaller ones:
- Here's how it works for $n = 3$:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

- General principle? How to do for bigger matrices?
- Are there short cuts?

Four by four determinants

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix}$$
$$- a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,3} & a_{4,4} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,4} \end{vmatrix}$$
$$- a_{1,4} \begin{vmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{vmatrix}$$

- General principle is in terms of minors and cofactors

Minors and Cofactors

- Minor $M_{i,j}$ of matrix $\mathbf{A}$ is determinant of matrix obtained by deleting ith row and jth column of $\mathbf{A}$
- Cofactor $A_{i,j}$ defined by

$$A_{i,j} = (-1)^{i+j} M_{i,j}$$

- Determinant expanded by ith row is given by

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{i,j} A_{i,j} \qquad \left(= \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} \right)$$

  - previous examples used first row, i.e. had $i = 1$

Exercise 1

- Use several different ways to find the determinant of

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 3 \\ -1 & 0 & -3 \\ 2 & 3 & -2 \end{pmatrix}$$

and identify the minors, cofactors etc. involved in the calculation

Example

- Show that (schematically)

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{a} \times \mathbf{b}$$

- Useful as *aide de memoire* for cross product.

Pattern of signs in cofactors

- How to remember the pattern of signs in cofactor calculations:

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}, \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}, \quad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix},$$

$$\begin{pmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{pmatrix} \qquad \text{and so on.}$$

Facts about the determinant

All these can be shown (hard) by using minors formula:

- $\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})$. $\quad \det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.
- If two rows of $\mathbf{A}$ are swapped
  - the sign of $\det \mathbf{A}$ flips
- If one row of $\mathbf{A}$ is multiplied by a scalar λ
  - $\det \mathbf{A}$ is mulplied by λ
- If $\mathbf{A}$ has two identical rows
  - $\det \mathbf{A} = 0$
- If two rows of $\mathbf{A}$ are added together:
  - $\det \mathbf{A}$ is unchanged

Example

- Q. Find the determinant of

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

- HINT: a determinant is zero if it has complete row or column of zeros
- A. determinant $= 3 \times (-2) \times 1 \times 1 = 6$
- UPSHOT: the determinant of any upper (lower) triangular matrix is product of diagonal elements

Exercise 2

- Where possible, use shortcuts to work out determinants of following:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 5 \\ 3 & 0 & 0 \end{pmatrix}$$

Matrix inverse (I)

- Identity matrix: $n \times n$ matrix which leaves $n \times 1$ and other $n \times n$ matrices unchanged under multiplication

$$\text{E.g.} \quad \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{etc.}$$

- Inverse matrix: given a square $n \times n$ matrix $\mathbf{A}$, its inverse $\mathbf{A}^{-1}$ (if it exists) is the $n \times n$ matrix for which

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

As a transformation, $\mathbf{A}^{-1}$ does the opposite of $\mathbf{A}$, so that combined they have no effect

Matrix inverse (II)

- How to compute the inverse matrix?
- Cramer's rule: (don't worry, it just works)
- Given

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \quad \text{then} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \\ A_{1,2} & A_{2,2} & \dots & A_{n,2} \\ \vdots & \vdots & \dots & \vdots \\ A_{1,n} & A_{2,n} & \dots & A_{n,n} \end{pmatrix}$$

where $A_{i,j}$ denotes cofactor

- NB: inverse $\mathbf{A}^{-1}$ only exists when $\det \mathbf{A} \neq 0$
  - hence why we say $\mathbf{A}$ is non-invertible if $\det \mathbf{A} = 0$

Two by two inverse

- LEARN THIS: If $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ then

$$\mathbf{A}^{-1} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}$$

- Exercise: find inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

and verify by multiplication that it is the inverse

Matrix inverse (III)

- Exercise: Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{pmatrix}$$

- Bit tedious isn't it? We will return to this later
- Take home message: (repeat out loud!)

"The Inverse of a square matrix is the transpose of the matrix of cofactors divided by the determinant"

- "transpose of matrix of co-factors" also known as the adjoint of A, written adj(A). So the above matra can be written mathematically as:

$$A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A)$$

Homework

- Read relevant sections of *James* on determinants
- Do *James* (4th edition) or (5th edition)
  - Exercises 5.3.1, Q. 35, 38
  - Exercises 5.4.1, Q. 51, 52
- Revise for the class test (Monday 23rd Oct) by:
  - Only questions on complex numbers and vectors
  - Read through notes and relevent bits of *James*
  - Use online QMP tests to check basic understanding
  - Revise using more basic examples from *James*
  - Go to Thurs and Fri drop-in sessions to get help
- But remember, the class test does not contribute to the unit mark, it's just to check progress.
  - It's important you also keep up with regular homeworks and QMP tests
  - Especially as this lecture and next few have a lot of theory in them.