Advanced Bending and Torsion

Assignment A - Solution

Dr Luiz Kawashita

Luiz.Kawashita@bristol.ac.uk

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Workflow

1. Split cross-section into distinct segments $i = 1 \dots 5$ and define individual functions ${}^{i}X_{(s)}$, ${}^{i}Y_{(s)}$ and ${}^{i}t_{(s)}$ for a 'local' $s = (0 \dots b_i)$

2. Calculate A, Q_{XX}, Q_{YY} using **definite integrals**, and find $\bar{X}$,$\bar{Y}$

3. Re-write equations w.r.t. the centroid:
   ${}^{i}x_{(s)} = {}^{i}X_{(s)} - \bar{X}$, ${}^{i}y_{(s)} = {}^{i}Y_{(s)} - \bar{Y}$

4. Calculate I_{xx}, I_{yy}, I_{xy} and J using **definite integrals**

5. Write expressions for ${}^{i}q_{(s)}$ using **indefinite integrals**

6. Find the constants $q_{(i-1)}$ using **definite integrals**

7. Consider the 'unit load' case $\{S_x, S_y\} = \{1,0\}$, calculate moments M_i using **definite integrals**, sum those to find e_Y

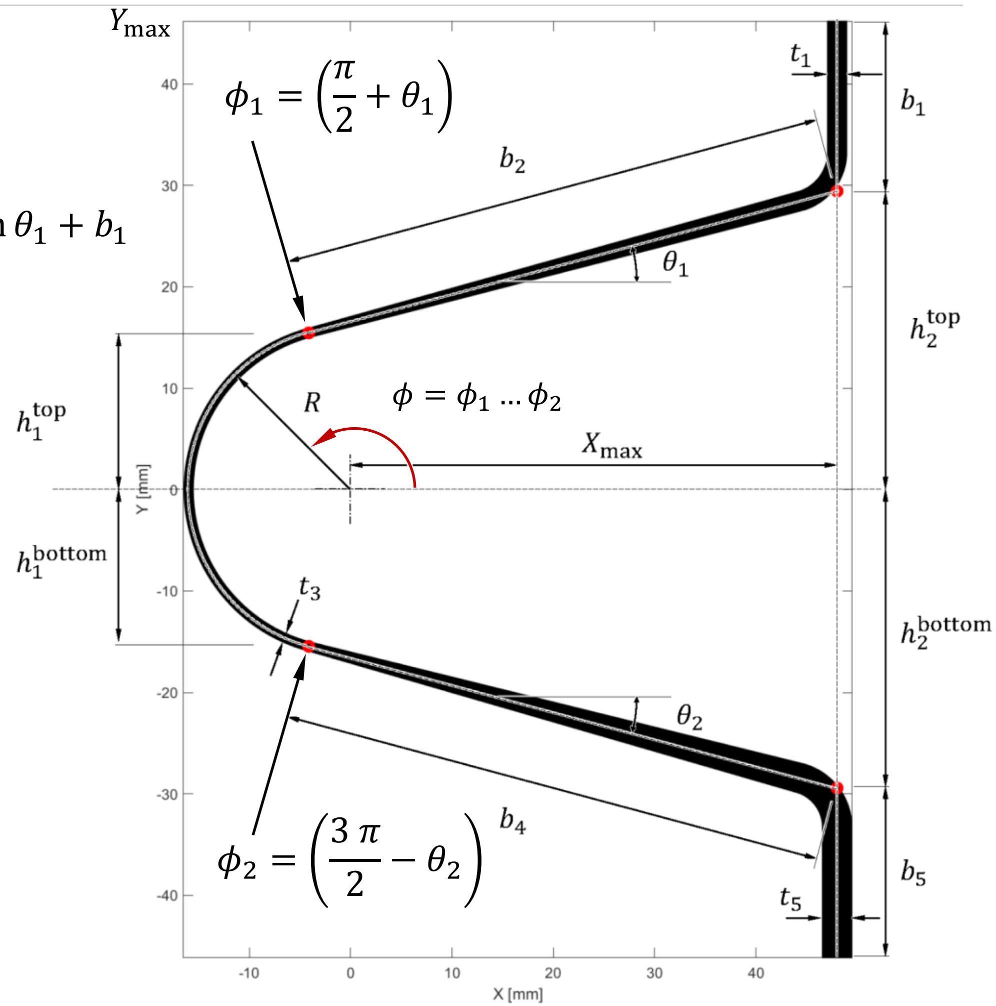
8. Consider the 'unit load' case $\{S_x, S_y\} = \{0,1\}$, calculate moments M_i using **definite integrals**, sum those to find e_X

1. Individual Segments (1/4)

Segment 1:

- $^{1}X_{(s)} = X_{\max}$
- $^{1}Y_{(s)} = Y_{\max} - s$
- $^{1}t_{(s)} = t_1$

$Y_{\max} = h_1^{\text{top}} + b_2 \sin\theta_1 + b_1$



Segment 2:

- $^{2}X_{(s)} = X_{\max} - s \cdot \cos\theta_1$
- $^{2}Y_{(s)} = h_2^{\text{top}} - s \cdot \sin\theta_1$
- $^{2}t_{(s)} = t_1 + \frac{(t_3 - t_1)}{b_2} s$

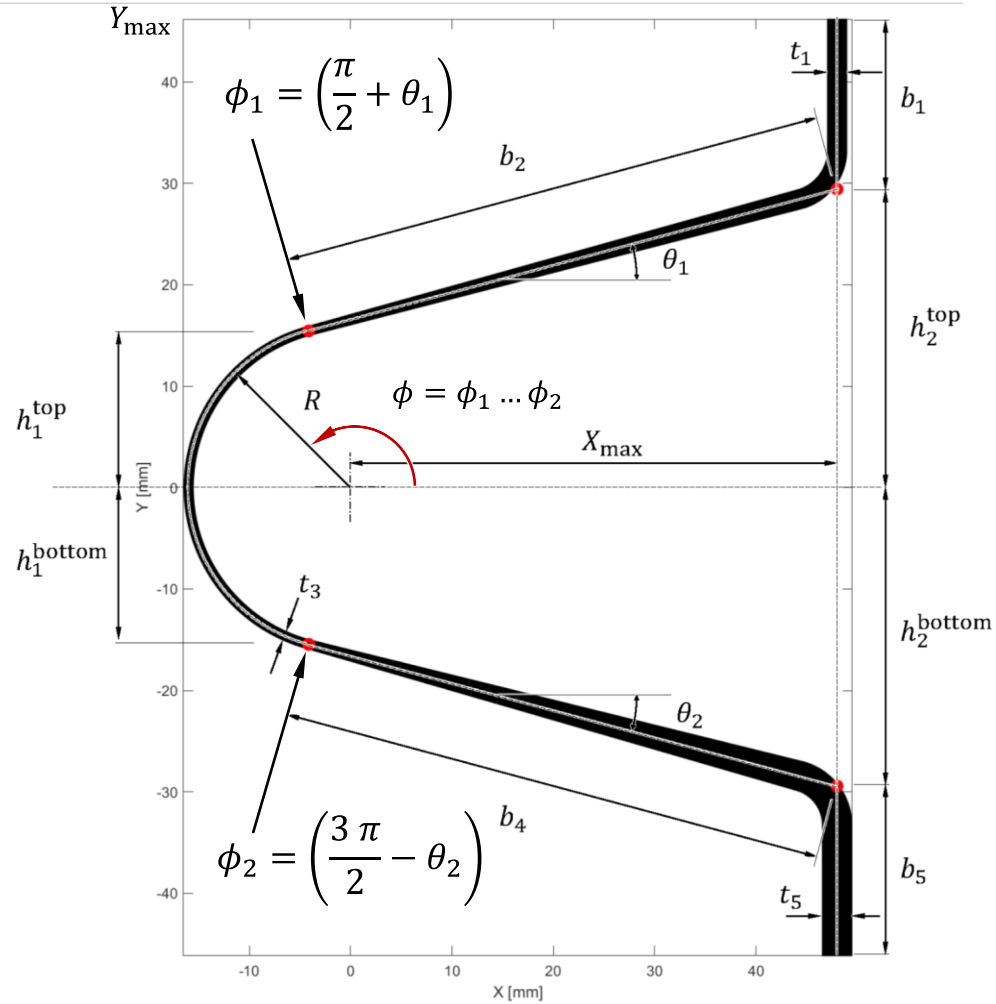
1. Individual Segments (2/4)

Segment 3:

- ${}^{3}X_{(s)} = R \cdot \cos\phi$
- ${}^{3}Y_{(s)} = R \cdot \sin\phi$
- ${}^{3}t_{(s)} = t_3$

Segment 4:

- ${}^{4}X_{(s)} = R \cdot \cos\phi_2 + s \cdot \cos\theta_2$
- ${}^{4}Y_{(s)} = R \cdot \sin\phi_2 - s \cdot \sin\theta_2$
- ${}^{4}t_{(s)} = t_3 + \frac{(t_5 - t_3)}{b_4} s$



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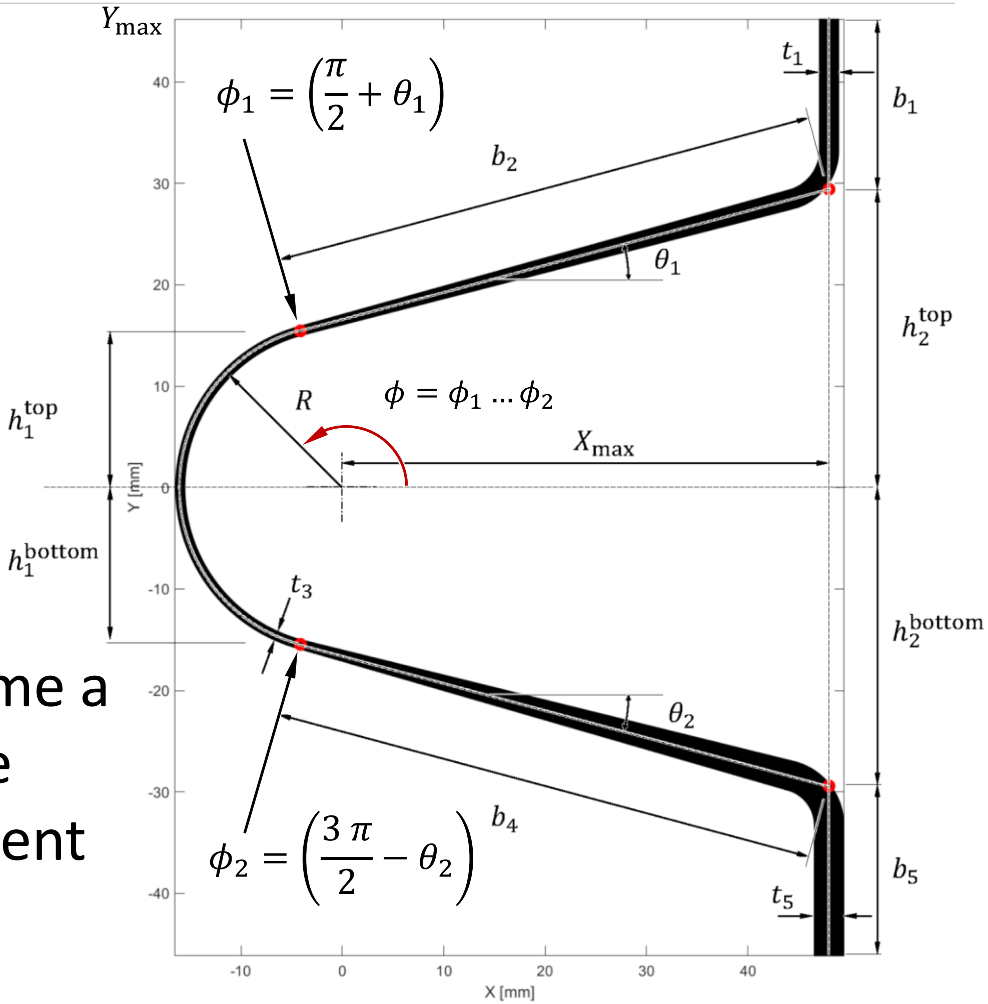
1. Individual Segments (3/4)

Segment 5:

- $^{5}X_{(s)} = X_{\max}$
- $^{5}Y_{(s)} = -h_2^{\text{bottom}} - s$
- $^{5}t_{(s)} = t_5$



- Note that these equations assume a local length $s = (0 \ldots b_i)$, *i.e.* we ‘reset’ s every time a new segment starts
- This is OK because we will add the contributions of ‘previous’ segments in our shear flow calculation

1. Individual Segments (4/4)

In summary:

Segment 1:

- ${}^{1}X_{(s)} = X_{\max}$
- ${}^{1}Y_{(s)} = Y_{\max} - s$
- ${}^{1}t_{(s)} = t_1$

Segment 2:

- ${}^{2}X_{(s)} = X_{\max} - s \cdot \cos\theta_1$
- ${}^{2}Y_{(s)} = h_2^{\text{top}} - s \cdot \sin\theta_1$
- ${}^{2}t_{(s)} = t_1 + \frac{(t_3 - t_1)}{b_2} s$

Segment 3:

- ${}^{3}X_{(s)} = R \cdot \cos\phi$
- ${}^{3}Y_{(s)} = R \cdot \sin\phi$
- ${}^{3}t_{(s)} = t_3$

Segment 4:

- ${}^{4}X_{(s)} = R\ \cos\phi_2 + s\ \cos\theta_2$
- ${}^{4}Y_{(s)} = R\ \sin\phi_2 - s\ \sin\theta_2$
- ${}^{4}t_{(s)} = t_3 + \frac{(t_5 - t_3)}{b_4} s$

Segment 5:

- ${}^{5}X_{(s)} = X_{\max}$
- ${}^{5}Y_{(s)} = -h_2^{\text{bottom}} - s$
- ${}^{5}t_{(s)} = t_5$

2. Area Properties and Centroid

- Area:

$$A = \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} {}^{i}t_{(s)} \, \mathrm{d}s \right]$$

- First moments of area:

$$Q_{XX} = \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} {}^{i}Y_{(s)} \; {}^{i}t_{(s)} \, \mathrm{d}s \right]$$

$$Q_{YY} = \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} {}^{i}X_{(s)} \; {}^{i}t_{(s)} \, \mathrm{d}s \right]$$

For segment 3 use:

$$\int_{s=0}^{b_3} \mathrm{d}s = \int_{\phi_1}^{\phi_2} R \, \mathrm{d}\phi$$

- Centroid:

$$\bar{X} = \frac{Q_{YY}}{A}, \bar{Y} = \frac{Q_{XX}}{A}$$

2. Area Properties and Centroid


```
A = double( ...           % definite
    int(t1,        s,    0,b1       ) ...
  + int(ts2,       s,    0,b2       ) ...
  + int(t3*R,      phi,  phi1,phi2) ...
  + int(ts4,       s,    0,b4       ) ...
  + int(t5,        s,    0,b5       ) );

Q_XX = double( ...        % definite
    int(Y_1*t1,    s,    0,b1       ) ...
  + int(Y_2*ts2,   s,    0,b2       ) ...
  + int(Y_3*t3*R,  phi,  phi1,phi2) ...
  + int(Y_4*ts4,   s,    0,b4       ) ...
  + int(Y_5*t5,    s,    0,b5       ) );

Q_YY = double( ...        % definite
    int(X_1*t1,    s,    0,b1       ) ...
  + int(X_2*ts2,   s,    0,b2       ) ...
  + int(X_3*t3*R,  phi,  phi1,phi2) ...
  + int(X_4*ts4,   s,    0,b4       ) ...
  + int(X_5*t5,    s,    0,b5       ) );

X_bar = Q_YY/A;
Y_bar = Q_XX/A;
```

3. Parameters w.r.t. the Centroid

General:

- $x_{\max} = (X_{\max} - \bar{X})$
- $y_{\max} = (Y_{\max} - \bar{Y})$

Segment 1:

- ${}^{1}x_{(s)} = x_{\max}$
- ${}^{1}y_{(s)} = y_{\max} - s$

Segment 2:

- ${}^{2}x_{(s)} = x_{\max} - s \cdot \cos\theta_1$
- ${}^{2}y_{(s)} = \left(h_2^{\text{top}} - \bar{Y}\right) - s \cdot \sin\theta_1$

Segment 3:

- ${}^{3}x_{(s)} = R\ \cos\phi - \bar{X}$
- ${}^{3}y_{(s)} = R\ \sin\phi - \bar{Y}$

Segment 4:

- ${}^{4}x_{(s)} = R\ \cos\phi_2 + s\ \cos\theta_2 - \bar{X}$
- ${}^{4}y_{(s)} = R\ \sin\phi_2 - s\ \sin\theta_2 - \bar{Y}$

Segment 5:

- ${}^{5}x_{(s)} = x_{\max}$
- ${}^{5}y_{(s)} = \left(-h_2^{\text{bottom}} - \bar{Y}\right) - s$

$$I_{xx} = \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} \left[{}^{i}y_{(s)} \right]^2 \; {}^{i}t_{(s)} \, \mathrm{d}s \right]$$

$$I_{yy} = \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} \left[{}^{i}x_{(s)} \right]^2 \; {}^{i}t_{(s)} \, \mathrm{d}s \right]$$

$$I_{xy} = \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} \left[{}^{i}x_{(s)} \; {}^{i}y_{(s)} \right] \; {}^{i}t_{(s)} \, \mathrm{d}s \right]$$

$$J = \frac{1}{3} \sum_{i=1}^{5} \left[\int_{s=0}^{b_i} \left[{}^{i}t_{(s)} \right]^3 \mathrm{d}s \right]$$

For segment 3 use:

$$\int_{s=0}^{b_3} \mathrm{d}s = \int_{\phi_1}^{\phi_2} R \, \mathrm{d}\phi$$

4. Second Moments of Area


```
Ixx = double ( ...          % definite
    int(ys1^2*t1,   s,    0,b1       ) ...
  + int(ys2^2*ts2,  s,    0,b2       ) ...
  + int(ys3^2*t3*R, phi,  phi1,phi2) ...
  + int(ys4^2*ts4,  s,    0,b4       ) ...
  + int(ys5^2*t5,   s,    0,b5       ) );

Iyy = double ( ...          % definite
    int(xs1^2*t1,   s,    0,b1       ) ...
  + int(xs2^2*ts2,  s,    0,b2       ) ...
  + int(xs3^2*t3*R, phi,  phi1,phi2) ...
  + int(xs4^2*ts4,  s,    0,b4       ) ...
  + int(xs5^2*t5,   s,    0,b5       ) );

Ixy = double ( ...             % definite
    int(xs1*ys1*t1,    s,    0,b1       ) ...
  + int(xs2*ys2*ts2,   s,    0,b2       ) ...
  + int(xs3*ys3*t3*R,  phi,  phi1,phi2) ...
  + int(xs4*ys4*ts4,   s,    0,b4       ) ...
  + int(xs5*ys5*t5,    s,    0,b5       ) );

J = double ( ...          % definite
    int( t1^3,    s,    0,b1       ) ...
  + int(ts2^3,    s,    0,b2       ) ...
  + int( t3^3*R,  phi,  phi1,phi2) ...
  + int(ts4^3,    s,    0,b4       ) ...
  + int( t5^3,    s,    0,b5       ) ) / 3. ;
```

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- Defining the two constants as:

$$C_x = \frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}} \qquad\qquad C_y = \frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}$$

- We get:

$$-\ {}^{i}q_{(s)} = C_x \int_0^s {}^{i}x_{(s)}\ {}^{i}t_{(s)}\, \mathrm{d}s + C_y \int_0^s {}^{i}y_{(s)}\ {}^{i}t_{(s)}\, \mathrm{d}s \ \boxed{-\, q_{(i-1)}}$$

Add the definite integral obtained for the previous segment

- The shear flow is continuous along the cross-section, so we **must** add the definite integral $q_{(i-1)}$ obtained with the previous segment

- Segment 1 starts at a free edge where $q_0 = 0$

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```
% Subroutine_Update_ShearFlow:

qs1 = ...                   % indefinite
    Cx * int(xs1*t1,  s) ...
  + Cy * int(ys1*t1,  s) ...
  + 0;

q1 = double ( ...           % definite
    Cx * int(xs1*t1,  s, 0,b1) ...
  + Cy * int(ys1*t1,  s, 0,b1) ...
  + 0);

%-------------------------------

qs2 = ...                   % indefinite
    Cx * int(xs2*ts2, s) ...
  + Cy * int(ys2*ts2, s) ...
  + q1;

q2 = double ( ...           % definite
    Cx * int(xs2*ts2, s, 0,b2) ...
  + Cy * int(ys2*ts2, s, 0,b2) ...
  + q1);

%-------------------------------
```

Computing q as <u>positive</u> numbers – will be multiplied by -1 later on

```
qs3 = ...                   % indefinite
    Cx * int(xs3*t3*R, phi) ...
  + Cy * int(ys3*t3*R, phi) ...
  + q2;

q3 = double ( ...           % definite
    Cx * int(xs3*t3*R, phi, phi1,phi2) ...
  + Cy * int(ys3*t3*R, phi, phi1,phi2) ...
  + q2);

%-------------------------------

qs4 = ...                   % indefinite
    Cx * int(xs4*ts4, s) ...
  + Cy * int(ys4*ts4, s) ...
  + q3;

q4 = double ( ...           % definite
    Cx * int(xs4*ts4, s, 0,b4) ...
  + Cy * int(ys4*ts4, s, 0,b4) ...
  + q3);

%-------------------------------
qs5 = ...                   % indefinite
    Cx * int(xs5*t5,  s) ...
  + Cy * int(ys5*t5,  s) ...
  + q4;

q5 = double ( ...           % definite
    Cx * int(xs5*t5,  s, 0,b5) ...
  + Cy * int(ys5*t5,  s, 0,b5) ...
  + q4);
```

- Segment 1:

$$-\,{}^{1}q_{(s)} = C_x \int_0^s {}^{1}x_{(s)}\,{}^{1}t_{(s)}\,\mathrm{ds} + C_y \int_0^s {}^{1}y_{(s)}\,{}^{1}t_{(s)}\,\mathrm{ds} \;-q_0$$

$$-\,{}^{1}q_{(s)} = C_x \int_0^s x_{\max}\,t_1\,\mathrm{ds} + C_y \int_0^s (y_{\max} - s)\,t_1\,\mathrm{ds} \;-0$$

$$-\,{}^{1}q_{(s)} = t_1\left(C_x\,x_{\max}\,s + C_y\,y_{\max}\,s - C_y \int_0^s s\,\mathrm{ds}\right)$$

$$-\,{}^{1}q_{(s)} = t_1\left[\left(C_x\,x_{\max} + C_y\,y_{\max}\right)s - \frac{C_y}{2}s^2\right]$$

Indefinite integral = function ${}^{1}q_{(s)}$

$$-q_1 = t_1\left[\left(C_x\,x_{\max} + C_y\,y_{\max}\right)s - \frac{C_y}{2}s^2\right]_0^{b_1}$$

$$-q_1 = t_1\left[\left(C_x\,x_{\max} + C_y\,y_{\max}\right)b_1 - \frac{C_y}{2}{b_1}^2\right]$$

Definite integral = constant q_1

- Segment 2:

$$-\,{}^{2}q_{(s)} = C_x \int_0^s {}^{2}x_{(s)}\,{}^{2}t_{(s)}\,\mathrm{d}s + C_y \int_0^s {}^{2}y_{(s)}\,{}^{2}t_{(s)}\,\mathrm{d}s \;-\, q_1$$

$$-\,{}^{2}q_{(s)} + q_1 = \begin{cases} \quad C_x \int (x_1 - s\cdot\cos\theta_1)\left(\dfrac{b_2 - s}{b_2}t_1 + \dfrac{s}{b_2}t_3\right)\mathrm{d}s \\ + \quad C_y \int (y_1 - s\cdot\sin\theta_1)\left(\dfrac{b_2 - s}{b_2}t_1 + \dfrac{s}{b_2}t_3\right)\mathrm{d}s \end{cases}$$

$$-\,{}^{2}q_{(s)} + q_1 = \begin{cases} \quad C_x \int \left(\dfrac{b_2 - s}{b_2}t_1\,x_1 + \dfrac{s}{b_2}t_3\,x_1 - \dfrac{b_2 - s}{b_2}t_1\,s\cdot\cos\theta_1 - \dfrac{s^2}{b_2}t_3\cdot\cos\theta_1\right)\mathrm{d}s \\ + \quad C_y \int \left(\dfrac{b_2 - s}{b_2}t_1\,y_1 + \dfrac{s}{b_2}t_3\,y_1 - \dfrac{b_2 - s}{b_2}t_1\,s\cdot\sin\theta_1 - \dfrac{s^2}{b_2}t_3\cdot\sin\theta_1\right)\mathrm{d}s \end{cases}$$

- Segment 2:

$$-\,{}^{2}q_{(s)} + q_1 = \begin{cases} & C_x \int \left(t_1\,x_1 - s\,\dfrac{t_1\,x_1}{b_2} + s\,\dfrac{t_3\,x_1}{b_2} - s\,t_1 \cos\theta_1 + s^2\,\dfrac{t_1 \cos\theta_1}{b_2} - s^2\,\dfrac{t_3 \cos\theta_1}{b_2} \right) \mathrm{d}s \\ + & C_y \int \left(t_1\,y_1 - s\,\dfrac{t_1\,y_1}{b_2} + s\,\dfrac{t_3\,y_1}{b_2} - s\,t_1 \sin\theta_1 + s^2\,\dfrac{t_1 \sin\theta_1}{b_2} - s^2\,\dfrac{t_3 \sin\theta_1}{b_2} \right) \mathrm{d}s \end{cases}$$

$$-\,{}^{2}q_{(s)} + q_1 = \begin{cases} & C_x \left[s\,t_1\,x_1 - s^2\,\dfrac{t_1\,x_1}{2\,b_2} + s^2\,\dfrac{t_3\,x_1}{2\,b_2} - s^2\,\dfrac{t_1 \cos\theta_1}{2} + s^3\,\dfrac{t_1 \cos\theta_1}{3\,b_2} - s^3\,\dfrac{t_3 \cos\theta_1}{3\,b_2} \right] \\ + & C_y \left[s\,t_1\,y_1 - s^2\,\dfrac{t_1\,y_1}{2\,b_2} + s^2\,\dfrac{t_3\,y_1}{2\,b_2} - s^2\,\dfrac{t_1 \sin\theta_1}{2} + s^3\,\dfrac{t_1 \sin\theta_1}{3\,b_2} - s^3\,\dfrac{t_3 \sin\theta_1}{3\,b_2} \right] \end{cases}$$

$$-\,{}^{2}q_{(s)} = \left\{ \begin{array}{ll} & C_x \left[s\,t_1\,x_1 + s^2 \left(\dfrac{t_3 - t_1}{2\,b_2}\,x_1 - \dfrac{t_1 \cos\theta_1}{2} \right) + s^3 \left(\dfrac{t_1 - t_3}{3\,b_2} \cos\theta_1 \right) \right] \\ + & C_y \left[s\,t_1\,y_1 + s^2 \left(\dfrac{t_3 - t_1}{2\,b_2}\,y_1 - \dfrac{t_1 \sin\theta_1}{2} \right) + s^3 \left(\dfrac{t_1 - t_3}{3\,b_2} \sin\theta_1 \right) \right] \end{array} \right\} - q_1$$

Indefinite integral (function)

$$-q_2 = \left\{ \begin{array}{ll} & C_x \left[s\,t_1\,x_1 + {b_2}^2 \left(\dfrac{t_3 - t_1}{2\,b_2}\,x_1 - \dfrac{t_1 \cos\theta_1}{2} \right) + {b_2}^3 \left(\dfrac{t_1 - t_3}{3\,b_2} \cos\theta_1 \right) \right] \\ + & C_y \left[s\,t_1\,y_1 + {b_2}^2 \left(\dfrac{t_3 - t_1}{2\,b_2}\,y_1 - \dfrac{t_1 \sin\theta_1}{2} \right) + {b_2}^3 \left(\dfrac{t_1 - t_3}{3\,b_2} \sin\theta_1 \right) \right] \end{array} \right\} - q_1$$

Definite integral (constant)

- Each segment contributes with a moment M_i:

$$M_i = \int_0^s {}^i r_{(s)} \, \boxed{{}^i q_{(s)}} \, \mathrm{d}s$$

Indefinite integral (function)

For segment 3 use:

$$\int_{s=0}^{b_3} \mathrm{d}s = \int_{\phi_1}^{\phi_2} R \, \mathrm{d}\phi$$

- Note:
  - ${}^i q_{(s)}$ are written in terms of centroid coordinates x, y
  - ${}^i r_{(s)}$ are 'local moment arms' w.r.t. the origin of X, Y

- In this case these 'moment arms' ${}^i r_{(s)}$ are **constants**:

$$\boxed{{}^1 r_{(s)} = {}^5 r_{(s)} = -X_{\max}}$$

$$\boxed{{}^2 r_{(s)} = {}^3 r_{(s)} = {}^4 r_{(s)} = R}$$

To find e_X :

- Make $\{S_x, S_y\} = \{0,1\}$, compute C_x and C_y
- Write the 5 functions ${}^{i}q_{(s)}$ (not forgetting to add the constants $q_{(i-1)}$)
- Evaluate the **definite integrals**:

$$M_1 = \int_0^{b_1} X_{\max}\ {}^{1}q_{(s)}\ \mathrm{d}s$$

$$M_2 = \int_0^{b_2} -R\ {}^{2}q_{(s)}\ \mathrm{d}s$$

$$M_3 = \int_{\phi_1}^{\phi_2} -(R^2)\ {}^{3}q_{(s)}\ \mathrm{d}\phi$$

$$M_4 = \int_0^{b_4} -R\ {}^{4}q_{(s)}\ \mathrm{d}s$$

$$M_5 = \int_0^{b_5} X_{\max}\ {}^{5}q_{(s)}\ \mathrm{d}s$$

$$e_X = M_1 + M_2 + M_3 + M_4 + M_5$$

To find e_Y :

- Make $\{S_x, S_y\} = \{1,0\}$, compute C_x and C_y
- Write the 5 functions ${}^{i}q_{(s)}$ (not forgetting to add the constants $q_{(i-1)}$)
- Evaluate the **definite integrals**:

$$M_1 = \int_0^{b_1} X_{\max}\ {}^{1}q_{(s)}\ \mathrm{d}s$$

$$M_2 = \int_0^{b_2} -R\ {}^{2}q_{(s)}\ \mathrm{d}s$$

$$M_3 = \int_{\phi_1}^{\phi_2} -(R^2)\ {}^{3}q_{(s)}\ \mathrm{d}\phi$$

$$M_4 = \int_0^{b_4} -R\ {}^{4}q_{(s)}\ \mathrm{d}s$$

$$M_5 = \int_0^{b_5} X_{\max}\ {}^{5}q_{(s)}\ \mathrm{d}s$$

$$e_Y = M_1 + M_2 + M_3 + M_4 + M_5$$

6. Shear Centre

```
Sx = 0.; Sy = 1.;

Cx = (Sx*Ixx + Sy*Ixy)/(Ixy^2 - Ixx*Iyy);
Cy = (Sy*Iyy + Sx*Ixy)/(Ixx*Iyy - Ixy^2);

Subroutine_Update_ShearFlow;

e_X = double (...        % definite
        int(-1*r1*qs1,    s,    0,b1      ) ...
      + int(-1*r2*qs2,    s,    0,b2      ) ...
      + int(   r3*qs3*R, phi, phi1,phi2) ...
      + int(-1*r4*qs4,    s,    0,b4      ) ...
      + int(-1*r5*qs5,    s,    0,b5      ) );

%----------------------------------------------------------------------------
Sx = 1.; Sy = 0.;

Cx = (Sx*Ixx + Sy*Ixy)/(Ixy^2 - Ixx*Iyy);
Cy = (Sy*Iyy + Sx*Ixy)/(Ixx*Iyy - Ixy^2);

Subroutine_Update_ShearFlow;

e_Y = double (...        % definite
        int(-1*r1*qs1,    s,    0,b1      ) ...
      + int(-1*r2*qs2,    s,    0,b2      ) ...
      + int(   r3*qs3*R, phi, phi1,phi2) ...
      + int(-1*r4*qs4,    s,    0,b4      ) ...
      + int(-1*r5*qs5,    s,    0,b5      ) );
```