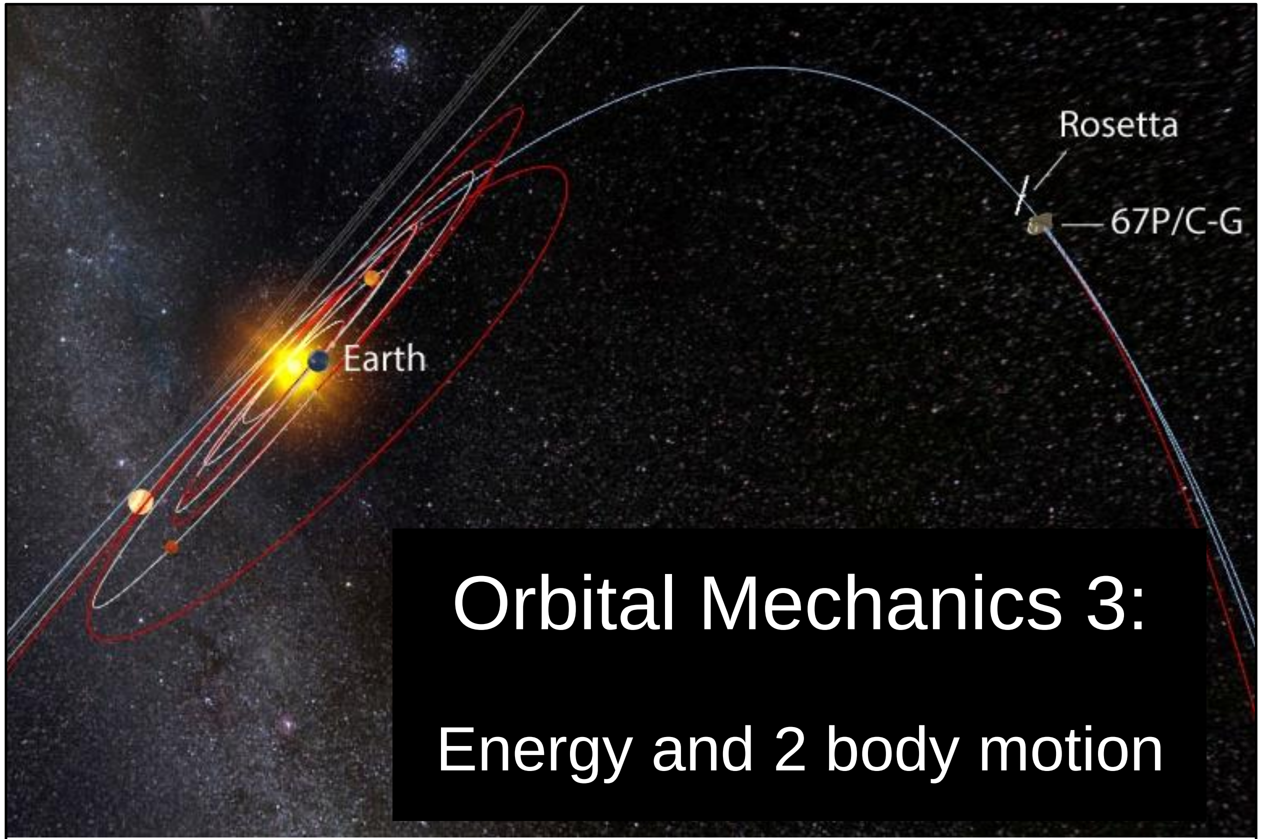

Orbital Mechanics 3:

Energy and 2 body motion

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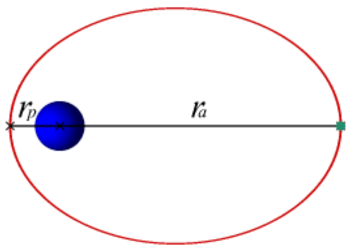
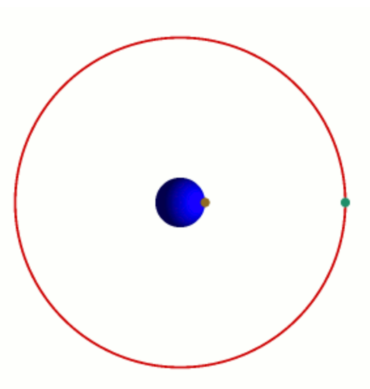
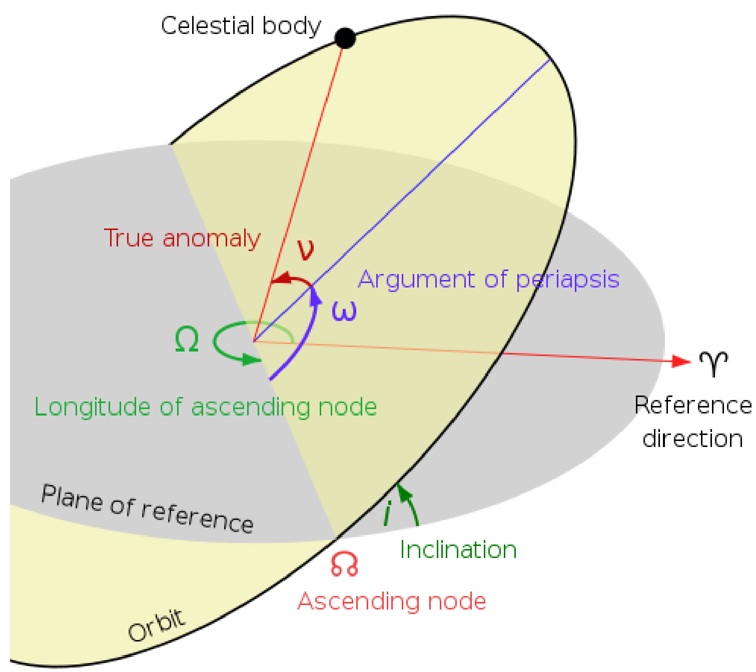
Revision







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Learning objectives

1. Escape velocity
2. 2 body motion
3. Relative acceleration
4. Numerical modelling of 2 body motion

Be able to manipulate conservation of energy equation
Be able to derive escape velocity and calculate it for a body.
Be able to explain why we need equations for 2 body motion
Be able to derive relative acceleration between 2 bodies
Be able to describe how to model 2 body motion numerically and state its limitations

Reminder: Orbital Energy

From centripetal force and N2, we have:

$$F = ma = \frac{GMm}{r^2} = \frac{mv^2}{r} \qquad \text{(1-14)}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} \qquad \text{(1-15)}$$

So we have:

$$E = K + U$$

$$\boxed{E = \frac{1}{2}mv^2 - \frac{GMm}{r}} = \frac{GMm}{2r} - \frac{GMm}{r} \qquad \text{(1-16)}$$

$$E = -\frac{GMm}{2r} \qquad \text{(1-17)}$$

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If you remember your circular motion equation for centripetal force it looks like (1-14) above. We can rearrange this so that we have an expression for the Kinetic energy of a circular orbit. At the end we can see that the total energy E is larger for larger radii (as it is negative).

Escape velocity

From conservation of energy:

$$(K+U)_{initial} = (K+U)_{final} = 0 \qquad \text{(1-13)}$$

$$\boxed{E = \frac{1}{2}mv^2 - \frac{GMm}{r}} \qquad \text{(1-18)}$$

$$\frac{GMm}{r} = \frac{mv^2}{2} \qquad \text{(3-1)}$$

$$v_{esc} = \sqrt{\frac{2GM}{r}} \qquad \text{(3-2)}$$

Minimum velocity needed to "escape" the gravitational force.

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If the kinetic energy of an object launched from the Earth were equal in magnitude to the potential energy, then in the absence of friction resistance it could escape from the Earth.
Escape Velocity = Minimum velocity needed to "escape" the gravitational force. Actually, escape velocity is a speed not a velocity, as it does not specify a direction. We define it at a distance r from the centre of a body.

Numerical example

What is the escape velocity for the Earth?

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

$$v_{esc} = \sqrt{\frac{2x6.67x10^{-11}.5.97x10^{24}}{6378x10^{3}}} = 11.2x10^{3}\,ms^{-1}$$

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We can calculate escape velocity for Earth at its surface which is 11.2e3 m/s.

The Two Body Problem

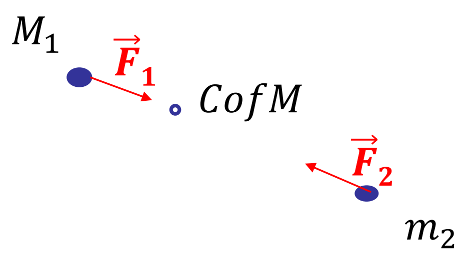
Equation of motion for two bodies interacting gravitationally with each other…

- eg: the Earth and a satellite, the Sun and the Earth, the Earth and the Moon
- WHY? Because then we can predict where they are at any point in their orbit or in time.
- Why do we need to do this?

To point the antenna of a ground station
To initiate an engine burn
To perform measurements
To time tag measurements
To rendez-vous

The Two Body Problem

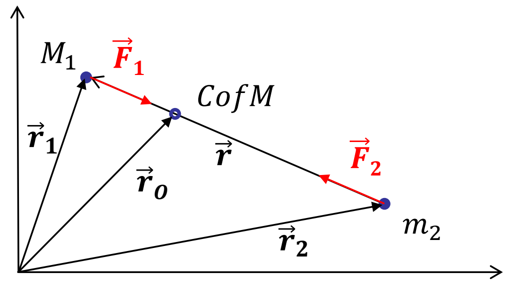
- Consider the gravitational force between two bodies, M_1 and $m_{2:}$



There are attractive forces between the two bodies $\mathbf{F}_1$ and $\mathbf{F}_2$. If M_1 has a larger mass than M_2, then the CoM will be nearer to M_1 than to m_2.

The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and $m_{2.}$



  - Define a coordinate system such that:
    - $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2 = \vec{\boldsymbol{r}}_{2,1}$ is the *relative position vector* pointing from m_2 to M_1.
    - $\hat{\boldsymbol{r}}_{2,1}$ is the unit vector pointing from m_2 to M_1.
    - $r = |\vec{\boldsymbol{r}}|$, i.e. the magnitude of $\vec{\boldsymbol{r}}$.

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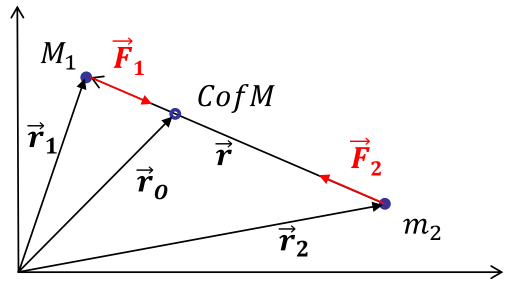
Let's put the bodies in a system of reference: $\mathbf{r}_1$ is the position vector for M1, $\mathbf{r}_2$ is the position vector for m2, $\mathbf{r}_0$ is the position vector for the centre of mass and $\mathbf{r}$ is the relative position vector pointing from m2 to M1.

The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and m_2.



- Questions:
  1. Define the forces on M_1 and m_2?
  2. What is the position of the centre of mass?

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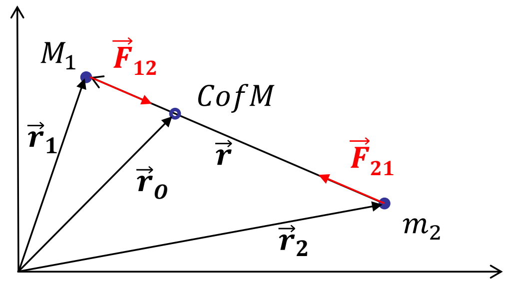
F1 is the force that M1 exerts on m2 and F2 is the force that m2 exerts on M1.

The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and m_2.



$$\boxed{\ddot{\mathbf{r}} = \frac{d^2\vec{\mathbf{r}}}{dt^2}}$$

$$\boldsymbol{F}_{1,2} = M_1\ddot{\boldsymbol{r}}_1 = -\frac{GM_1m_2}{r^2}\hat{\boldsymbol{r}}_{1,2} \qquad (3\text{-}3)$$

$$\boldsymbol{F}_{2,1} = m_2\ddot{\boldsymbol{r}}_2 = -\frac{GM_1m_2}{r^2}\hat{\boldsymbol{r}}_{2,1} = \frac{GM_1m_2}{r^2}\hat{\boldsymbol{r}}_{1,2} \qquad (3\text{-}4)$$

As $\hat{\boldsymbol{r}}_{2,1} = -\hat{\boldsymbol{r}}_{1,2}$ then…. $\boldsymbol{F}_{1,2} = -\boldsymbol{F}_{2,1}$ (3-5)

If we look at the forces on each object, let's say planet, we can use N2 and the Universal Law of Gravitation to state 3-3 and 3-4. The negative just means attractive force and is to do with which way we define as positive. Ideally, the dot notation wouldn't lose the "over-arrow" but that looks cluttered in Office so here we will retain just the bold notation to denote vectors.

The Two Body Problem

The governing equations for $\ddot{\boldsymbol{r}}_1$ and $\ddot{\boldsymbol{r}}_2$ are (3-3) and (3-4). Subtracting these gives an expression for $\ddot{\boldsymbol{r}}$:

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = -\frac{G\cancel{M_1}m_2}{\cancel{M_1}r^2}\hat{\boldsymbol{r}}_{1,2} - \frac{GM_1\cancel{m_2}}{\cancel{m_2}r^2}\hat{\boldsymbol{r}}_{1,2} \quad (3\text{-}6)$$

$$\ddot{\boldsymbol{r}} = -\frac{Gm_2}{r^2}\hat{\boldsymbol{r}}_{1,2} - \frac{GM_1}{r^2}\hat{\boldsymbol{r}}_{1,2} = -\frac{G(M_1+m_2)}{r^2}\hat{\boldsymbol{r}}_{1,2} \quad (3\text{-}7)$$

$\mu = G(M_1 + m_2)$ is '*gravitational parameter*'
(note that if $M_1 \gg m_2$ then $\mu \approx GM_1$).

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We know that $r=r_1-r_2$, so also $d^2r/dt^2 = d^2r_1/dt^2 - d^2r_2/dt^2$.
This is the second order differential equation that governs the motion of m2 relative to m1. We will see that it has two vector constants of integration, each having three scalar components. Therefore, the equation has six constants of integration.

The Two Body Problem

- We have found expressions for the acceleration of M_1 and m_2:

$$\ddot{\boldsymbol{r}}_1 = -\frac{Gm_2}{r^2}\hat{\boldsymbol{r}}_{1,2}$$

$$\ddot{\boldsymbol{r}}_2 = -\frac{GM_1}{r^2}\hat{\boldsymbol{r}}_{2,1} \qquad (3\text{-}8)$$

- As we know, these orbit around the barycentre.

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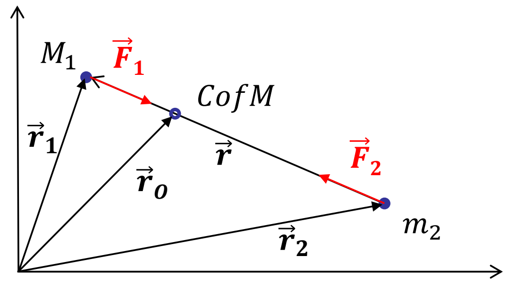
The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and m_2.



What would the resultant motion of the bodies be?

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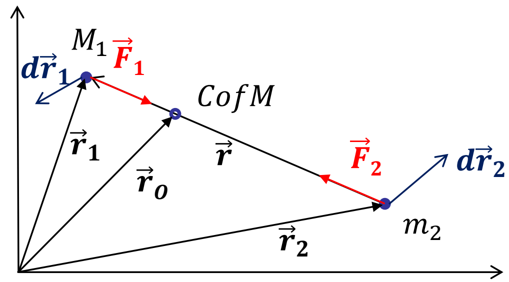
The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and m_2.

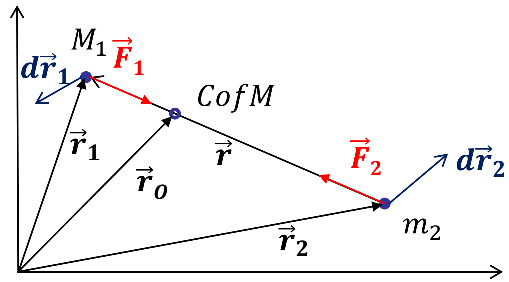


What would the resultant motion of the bodies be?

Without loss of generality, we can include a non-zero initial velocity.

The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and m_2.



What is the position of the centre of mass?

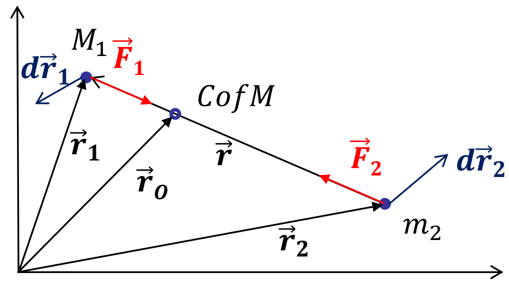
It is defined as:

$$\vec{r}_o = \frac{M_1\vec{r}_1 + m_2\vec{r}_2}{M_1 + m_2} \qquad (3\text{-}9)$$

Why are we interested in the position of the centre of mass? Because both the bodies move around this position (also known as 'barycentre'). This point acts as a fulcrum, like that on a seesaw. At any time the bodies on opposite sides of the barycentre moving in opposite directions. The barycentre is defined as the point at which the weighted sum of the two position vectors of the two masses relative to the barycentre is zero. So M1(r1-r0) + m2(r2-r0)=0. From this we can deduce 3-9.

The Two Body Problem

- Consider the gravitational force between two bodies, M_1 and m_2.



$$\vec{r}_o = \frac{M_1\vec{r}_1 + m_2\vec{r}_2}{M_1 + m_2} \qquad (3\text{-}10)$$

Note that:

- if $M_1 \gg m_2$, then $\vec{r}_0 \approx \vec{r}_1$.
- $\dot{r}_o$ = constant ie: no net force acts on it
- $\ddot{r}_o = 0$, therefore, the CoM can be found from initial conditions

In many cases, however, one particle is significantly heavier than the other, e.g., the Earth and the Sun. In such cases, the heavier particle is approximately the center of mass. Hence, the heavier mass may be treated roughly as a fixed centre of force, and the motion of the lighter mass may be solved for directly by one-body methods. This is what we will look at next lecture.

Numerical example

Find the centre of mass (CoM) of a system with 3 bodies: Sun (S), Saturn (sa) and its moon phoebe (p). Assume alignment:

$\boldsymbol{r}_{S,0}$

CoM $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \hat{\boldsymbol{r}}$

Sun-----x--------------------------------------saturn-------phoebe

$\boldsymbol{r}_{S,sa} \qquad\qquad \boldsymbol{r}_{sa,p}$

$\boldsymbol{r}_{S,p}$

$GM_{phoebe} = 0.3\ km^3 s^{-2}$

$GM_{Saturn} = 4x10^7 km^3 s^{-2}$

$GM_{Sun} = 1.3x10^{11} km^3 s^{-2}$

$\boldsymbol{r}_{S,p} = 1.428356\ x\ 10^9 . \hat{\boldsymbol{r}}\ km$

$\boldsymbol{r}_{S,sa} = \sqrt{2}\ x\ 10^9 . \hat{\boldsymbol{r}}\ km$

$\boldsymbol{r}_{sa,p} = \sqrt{2}\ x\ 10^7 . \hat{\boldsymbol{r}}\ km$

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We are assuming that all bodies are collinear in this drawing which is only very rarely true but which serves to simplify the calculation. To find the CoM we need the masses (given here in the form of GM=Gravitational constant x mass of body). We also need the distances between the bodies. We can take the Sun as the origin for these vectors.

Numerical example continued…

$$GM_{tot} = GM_S + GM_{sa} + GM_p = 1.3004x10^{11}km^3s^{-2}$$

From (3-13): $\vec{\boldsymbol{r}}_{\boldsymbol{o}} = \dfrac{M_1\vec{\boldsymbol{r}}_{\boldsymbol{1}} + m_2\vec{\boldsymbol{r}}_{\boldsymbol{2}}}{M_1 + m_2}$

$$\boldsymbol{r}_0 = \frac{\overset{=0}{\cancel{GM_S\boldsymbol{r}_S}} + GM_{sa}\boldsymbol{r}_{sa} + GM_p\boldsymbol{r}_p}{GM_{tot}}$$

$$\boldsymbol{r}_0 = \frac{0 + 4x10^7x\sqrt{2}\ x\ 10^9\hat{\boldsymbol{r}} + 0.3x1.428356x10^9.\hat{\boldsymbol{r}}}{1.3004x10^{11}}$$

$\boldsymbol{r}_0 = 4.35009x10^5\hat{\boldsymbol{r}}$ km

We can sum the total GM for all the bodies in the system and then calculate the CoM (wrt the Sun - as we have assumed this as the origin).

Numerical example continued…

What is the differential equation of motion for Phoebe wrt CoM? (gives accelerations acting on Phoebe)

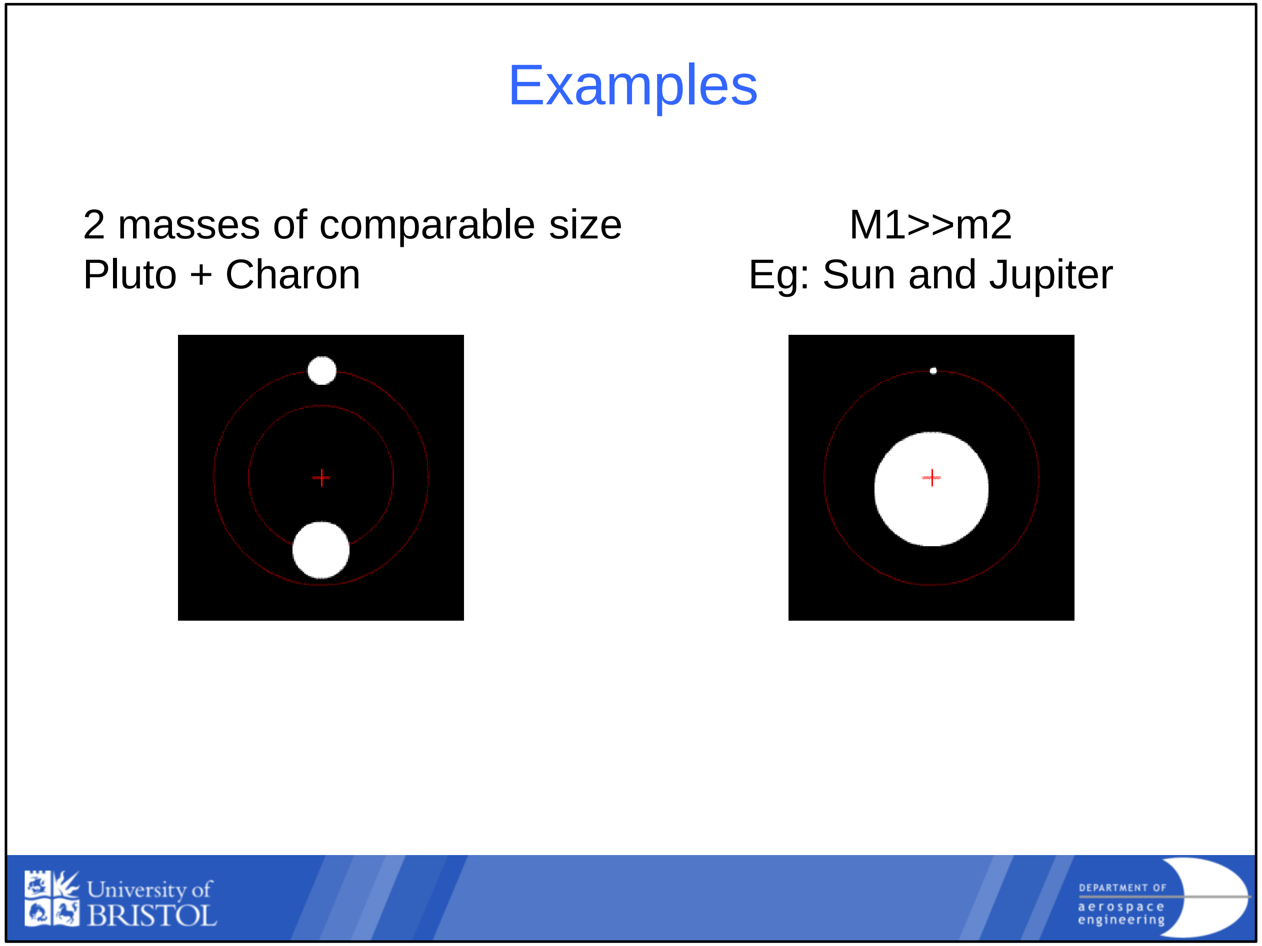
$$\ddot{\mathbf{r}}_{0,p} = -\frac{GM_S}{r_{S,p}^3}\mathbf{r}_{S,p} - \frac{GM_{sa}}{r_{sa,p}^3}\mathbf{r}_{sa,p}$$

$$\ddot{\mathbf{r}}_{0,p} = -\frac{1.3\text{x}10^{11}}{(1.428356\ \text{x}10^{9})^3}1.428356\text{x}10^{9}\hat{r} - \frac{4\text{x}10^{7}}{(\sqrt{2}\text{x}10^{7})^3}\sqrt{2}\text{x}10^{7}\hat{r}$$

$$\ddot{\mathbf{r}}_{0,p} = -6.371921627x10^{-8}\hat{r} - 2x10^{-7}\hat{r}$$

$$\ddot{\mathbf{r}}_{0,p} = -2.637192163x10^{-7}\hat{r}$$

The differential equation of motion of Phoebe wrt the CoM is as written in the first line. Note that this is not relative acceleration of eg: one body relative to another, it is the acceleration relative to a fixed inertial base point. Evaluating each term enables us to see which body influences Phoebe's orbit the most. We can see that Saturn contributes most to the total acceleration acting on Phoebe. This might make us wonder if other nearby bodies should be included in the calculation. Acceleration is along the negative rhat axis as the bodies are orbiting the CoM.



The left hand animation shows the motion of 2 masses of comparable size around the barycentre. The right hand animation shows that m2 does still affect the motion of M1 even if it is more of a slight wobble than a mutual orbit (like the left hand animation). This motion can be detected and is one of the methods we use to detect exoplanets.

Examples

- We have found expressions for the acceleration of M_1 and m_2:

$$\ddot{\boldsymbol{r}}_1 = -\frac{Gm_2}{r^2}\hat{\boldsymbol{r}} \qquad \text{(3-8)}$$

$$\ddot{\boldsymbol{r}}_2 = -\frac{GM_1}{r^2}\hat{\boldsymbol{r}}$$

- Thus, if we know the position, $\boldsymbol{r}$, and velocity, $\dot{\boldsymbol{r}}$, of two objects at a given time, we can perform a numerical integration to determine their position after one 'time-step':

$$\dot{\boldsymbol{r}}(t+1) = \dot{\boldsymbol{r}}(t) + \ddot{\boldsymbol{r}}(\mathrm{t})\Delta t \qquad \text{(3-11)}$$

Integrate…

$$\boldsymbol{r}(t+1) = \dot{\boldsymbol{r}}(t)\Delta t + \frac{1}{2}\ddot{\boldsymbol{r}}(\mathrm{t})\Delta t^2 \qquad \text{(3-12)}$$

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(3-14) and (3-15) are the same as v=u+at and s=ut+1/2at^2

Examples

- Performing this operation multiple times we can model how the objects will move.
- Let us consider some common examples from our solar system.
- We will only consider planar motion, i.e. 2-dimensions (we will later see this is a valid assumption), but the method applies equally well to 3-D

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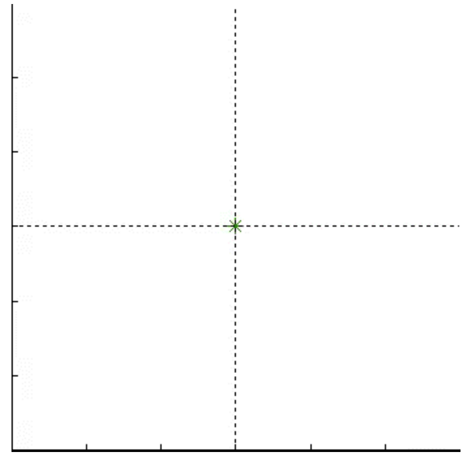
Example 1 – Sun & Earth

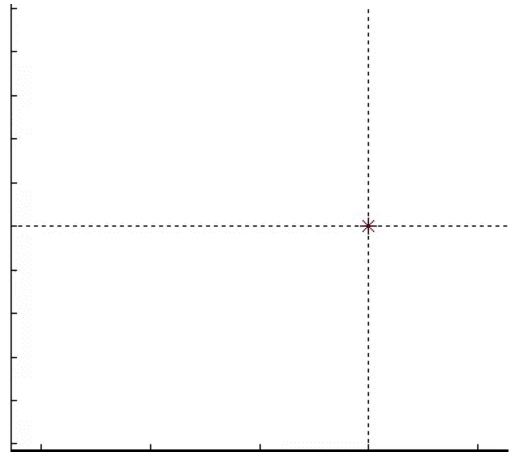
- Assume that:
  - M_1 = Sun = 1.989e30 kg; $\boldsymbol{r}_1(t=0) = [0,0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0,0]$
  - m_2 = Earth = 5.97219e24 kg; $\boldsymbol{r}_1(t=0) = [1.496\text{e}8\text{km}, 0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0, 29.788\ km/s]$



You do not need to learn these for the exam! They are just examples to aid understanding.
You can see here a 2D model of the Earth orbiting the Sun. The Sun is fixed at the origin with the Earth placed at its usual distance for t=0. The Earth's initial velocity is derived from vis-viva for circular orbits (we will do this later).

Example 2 –Earth & LEO Satellite

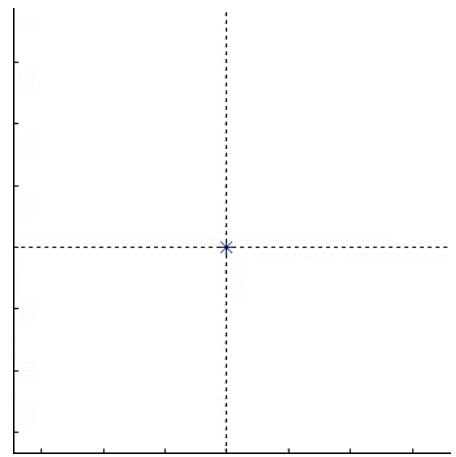
- Assume that:
  - M_1 = Earth = 5.97219e24 kg; $\boldsymbol{r}_1(t=0) = [0{,}0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0{,}0]$
  - m_2 = Satellite = 1000 kg; $\boldsymbol{r}_1(t=0) = [\mathit{300}\ \text{km}, 0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0{,}7.9044\ km/s]$



You do not need to learn these for the exam! They are just examples to aid understanding.
This example shows the Earth fixed at the origin with a satellite of 1000kg at 300km altitude. Why do we need the masses to do these calculations?

Example 3 –Earth & Moon

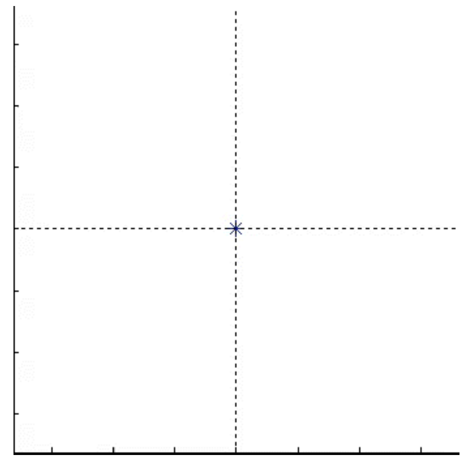
- Assume that:
  - M_1 = Earth = 5.97219e24 kg; $\boldsymbol{r}_1(t=0) = [0,0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0,0]$
  - m_2 = Moon = 7.3477e22 kg; $\boldsymbol{r}_1(t=0) = [362600\text{ km}, 0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0, 1.0484\ km/s]$



You do not need to learn these for the exam! They are just examples to aid understanding. This example shows the Earth at the origin and with the Moon orbiting. The movement of the Earth is caused by the Moon but is not circular as we don't give it any initial velocity.

Example 3 –Earth & Moon v2

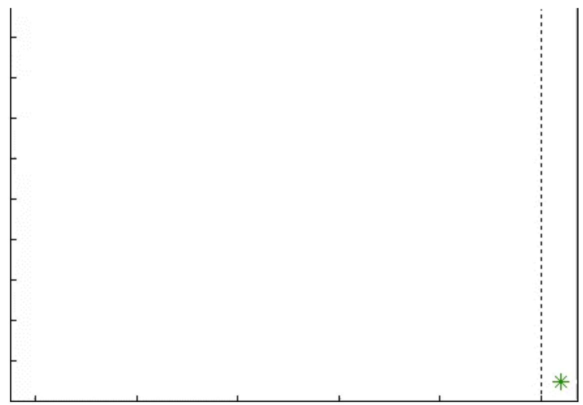
- Assume that:
  - M_1 = Earth = 5.97219e24 kg; $\boldsymbol{r}_1(t=0) = [0{,}0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0, -0.01274\ km/s]$
  - m_2 = Moon = 7.3477e22 kg; $\boldsymbol{r}_1(t=0) = [362600\text{ km}, 0]$; $\dot{\boldsymbol{r}}_1(t=0) = [0, 1.0484\ km/s]$



You do not need to learn these for the exam! They are just examples to aid understanding. In this example, the same data is used, but the Earth is given an initial velocity.

Example 4

- Assume that:
  - $M_1 = 4$; $\boldsymbol{r}_1(t=0) = [-2, 0]$; $\dot{\boldsymbol{r}}_1(t=0) = [-2, 0]$
  - $m_2 = 1$; $\boldsymbol{r}_1(t=0) = [\ 1\ , 0]$; $\dot{\boldsymbol{r}}_1(t=0) = [\ \ 2, 3]$



  - Looks ok…

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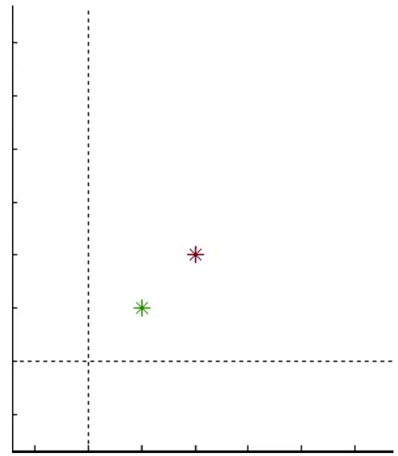
You do not need to learn these for the exam! They are just examples to aid understanding.
In this example we have a larger mass M1 (green) and a smaller mass m2 (red) which both have an initial velocity.

Example 4

- However, if we plot their motion relative to the barycentre (about which they should "orbit") we see a problem:



- This is due to our numerical approximation and shows a fundamental limitation of the method.

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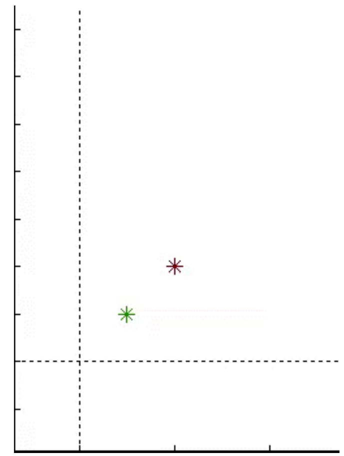
You do not need to learn these for the exam! They are just examples to aid understanding.
To improve the approximation we need to decrease the step size, but this will increase the calculation time.

Example 4

- We can alleviate the problem in this example by reducing the step size:



You do not need to learn these for the exam! They are just examples to aid understanding.
The “speed” of the bodies in the video is not indicative of their actual, absolute speed but is just a function of the number of frames in the video, i.e. step size. An order of magnitude change in step size, means order of magnitude more calculations

Example 4

- This requires a greater number of iterations to compute our trajectories, ok in this situation but limited by computational resources.
- In practice, modern software packages use some form of numerical method.

You do not need to learn these for the exam! They are just examples to aid understanding.

Summary

1. We can derive escape velocity from centripetal force and N2.
2. We can write the equation of motion for two bodies interacting gravitationally so that we can predict where they are at any point in their orbit or in time.
3. With some simplifications, the equation of motion can be written for a body and used to calculate accelerations on it wrt CoM.
4. Numerical approximations of the integration of the equation of motion improve with decreasing step size (but calculation time increases).

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Equations to remember

1. $\boxed{E = \frac{1}{2}mv^2 - \frac{GMm}{r}} = \frac{GMm}{2r} - \frac{GMm}{r}$

2. $v_{esc} = \sqrt{\frac{2GM}{r}}$

3. $\ddot{\boldsymbol{r}} = -G\frac{(M_1+m_2)}{r^2}\hat{\boldsymbol{r}} = -\frac{\mu}{r^2}\hat{\boldsymbol{r}}$

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Test Yourself!

1. If the Moon-Earth distance were to shrink, what would happen to the Moon's Period? Increase/decrease/stay the same?
2. Which planet has the lowest escape velocity?

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
diameter (Earth=1)	0.382	0.949	1	0.532	11.209	9.44	4.007	3.883
diameter (km)	4,878	12,104	12,756	6,787	142,800	120,000	51,118	49,528
mass (Earth=1)	0.055	0.815	1	0.107	318	95	15	17

3. In the Phoebe example, does it make sense that the resultant acceleration is along –r axis? Why?
4. Where will a spacecraft go if it has exactly the escape velocity of the planet?

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