EMAT10100 Engineering Maths I
Lecture 29: Inverse and periodic functions

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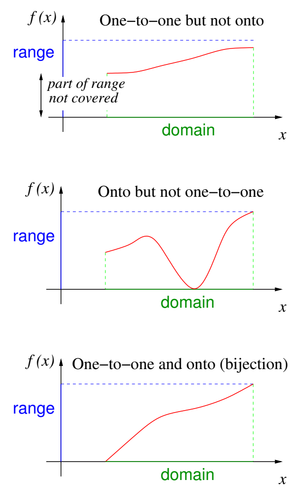
Properties of functions

A function $f : X \to Y$ is said to be:

- one-to-one (or injective) if:
  $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$
  i.e every value in Y is only mapped to by one point in X
- onto (or surjective) if:
  for all $y \in Y$ there exists an $x \in X$ such that $y = f(x)$
  i.e. every point in Y is mapped to by at least one point in X
- bijective if it is both one-to-one and onto.



Exercises

For each of the following functions, decide if it is one-to-one, onto, neither or both (bijective):

(hint: sketch the graph):

1. $$f : [0, \infty) \to (-\infty, \infty), \quad f(x) = \sqrt{x}$$
2. $$f : (-\infty, \infty) \to (-\infty, \infty), \quad f(x) = x^2$$
3. $$f : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad f(x) = \arctan(x)$$

Inverse functions

- Let $f : X \to Y$ be a function
- If f is bijective, then we say it is invertible and the inverse function $f^{-1}(x)$ exists

  the inverse function $f^{-1} : Y \to X$ is such that if $f(x) = y$, then $x = f^{-1}(y)$
- f^{-1} is also bijective
- Some examples:
  - $f : [0, \infty) \to [0, \infty)$ with $f(x) = x^2$ is invertible
  - $f : \mathbb{R} \to [0, \infty)$ with $f(x) = x^2$ is not invertible
  - $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(\mathbf{v}) = M\mathbf{v}$ is invertible if and only if $\det(M) \neq 0$

Finding the inverse

- If inverse exists we find it by solving $y = f(x)$ for x
- Example: Find the inverse of the function $f : \mathbb{R}\setminus\{-1\} \to \mathbb{R}\setminus\{1\}$ (i.e. ignoring $x = -1$ & $y = 1$) $\quad f(x) = 1 + \frac{1}{1+x}$
- We first simplify the fractional part:

$$y = f(x) = 1 + \frac{1}{1+x} = \frac{2+x}{1+x}$$

- Then we make y the subject:

$$\Rightarrow \quad (1+x)y = 2 + x \Rightarrow \quad (y-1)x = 2 - y$$

- Then when we have $x = f^{-1}(y)$ we swap the x and the y:

$$x = \frac{2-y}{y-1}, \quad \text{hence} \quad f^{-1}(x) = \frac{2-x}{x-1}$$

- Note. f^{-1} is the reflection of the graph of f in the line $y = x$ (next slide)

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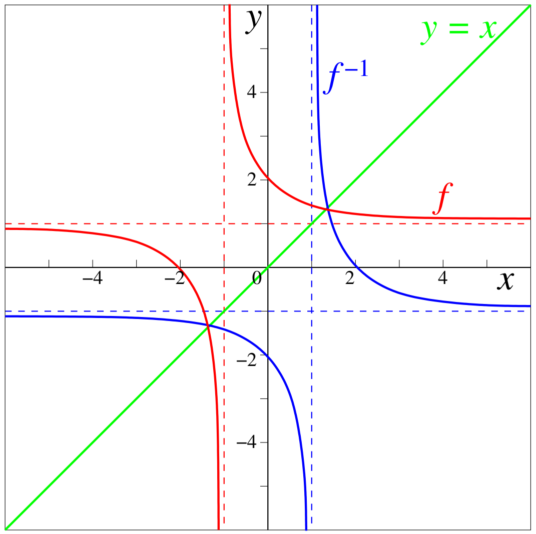

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Differentiation of inverse functions

- Let $y = f^{-1}(x)$ then $x = f(y)$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}\,x} f^{-1}(x) = \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \left(\frac{\mathrm{d}\,x}{\mathrm{d}\,y}\right)^{-1} = \frac{1}{f'(y)}$$

[Note $(dy/dx) = (dx/dy)^{-1}$ would not be true for partial derivatives: $(\partial y/\partial x) \neq (\partial x/\partial y)^{-1}$]

- Example

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\arccos(x) = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}}$$

- Exercise: use this method to show that:
  1. $\frac{\mathrm{d}}{\mathrm{d}\,x}\ln(x) = \frac{1}{x}$ using only that $\frac{\mathrm{d}}{\mathrm{d}\,x}e^x = e^x$
  2. $\frac{\mathrm{d}}{\mathrm{d}\,x}x^{1/3} = \frac{1}{3x^{2/3}}$ using only that $\frac{\mathrm{d}}{\mathrm{d}\,x}x^3 = 3x^2$

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Integration of inverse functions

- Let $y = f^{-1}(x)$, so that $x = f(y)$. What is $\int f^{-1}(x)\,\mathrm{d}\,x$?
- The answer is

$$\int f^{-1}(x)\,\mathrm{d}\,x = [xy] - \int f(y)dy, \qquad \text{but why?}$$

- integration by parts (with ‘u’ $= f^{-1}(x)$ and ‘$\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$’ = 1)

$$\begin{aligned}\int f^{-1}(x)\,\mathrm{d}\,x &= [yx] - \int x\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\,\mathrm{d}\,x \\ &= [yx] - \int f(y)\frac{\mathrm{d}\,y}{\mathrm{d}\,x}\,\mathrm{d}\,x \\ &= [yx] - \int f(y)\,\mathrm{d}\,x\end{aligned}$$

(see also *James* for geometric interpretation)

- Exercise: use this method to integrate of $\arccos(x)$

More types of functions

- Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is
  - periodic, with period T, if and only if

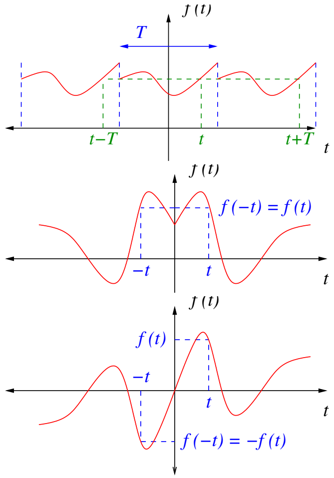
$$f(t+T) = f(t) \qquad \forall\, t \in \mathbb{R}$$

  - even if and only if

$$f(-t) = f(t) \qquad \forall\, t \in \mathbb{R}$$

  - odd if and only if

$$f(-t) = -f(t) \qquad \forall\, t \in \mathbb{R}$$



periodic, odd and even functions are important in signal processing

Exercises

Decide whether each of the following functions defined for $-\infty < t < \infty$ is odd, even and/or periodic. If it is periodic, state the (minimal) period. (hint: sketch the graph):

1. $$f_1(t) = t^2, \qquad f_2(t) = t^3$$
2. $$f_1(t) = \cos(3t), \qquad f_2(t) = \sin(3t)$$
3. $$f(t) = \tan(t)$$
4. $$f(t) = (\sin(t))^2$$

Calculus and periodic functions

- Q. Which of the following statements is true?
  1. The derivative of a periodic function$^{(1)}$ is itself periodic
  2. The integral of a periodic function is periodic
  3. The derivative of an odd function$^{(1)}$ is even (and vice versa)
  4. The integral of an even function is odd (and vice versa)

  (1)=provided it is sufficiently smooth - always read the small print
- 1. TRUE,
  2. FALSE
  3. TRUE
  4. TRUE
- to see why, try sketching a graph

Engineering HOT SPOT

Fourier Series (a way of approximating periodic functions)

- Let $f(x)$ be a periodic function $f(x)$ with period $2L$
- The Fourier series of $f(x)$ is then given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

- where the Fourier coefficients are defined by:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}\,x, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}\,x \text{ for } n = 1, 2\ldots$$

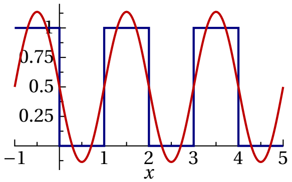
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}\,x \text{ for } n = 1, 2\ldots$$

- You do not need to learn these formulae, we will do this properly next year.

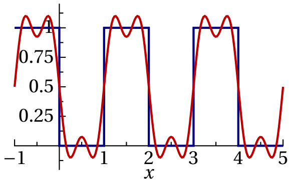
Calculating a Fourier series

Fourier series example:

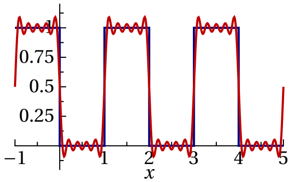
$$f(x) = \begin{cases} 1 & \text{if } -1 \leqslant x < 0 \\ 0 & \text{if } 0 \leqslant x < 1 \end{cases} \quad \text{with periodic extension}$$



1 term



3 terms



9 terms

- Notice that the series converges to the original function
- Fourier series are a special example of Fourier transforms (fft) . . .
- which converts time domain signals into frequency domain

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Exercise

- Exercise: Use the formulae

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx \quad \text{for } n = 0, 1, 2, \ldots$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx \quad \text{for } n = 1, 2 \ldots$$

to calculate the coefficients a_n and b_n for the function used in the above hotspot:

$$f(x) = \begin{cases} 1 & \text{if } -1 \leqslant x < 0 \\ 0 & \text{if } 0 \leqslant x < 1 \end{cases} \quad \text{with periodic extension}$$

- What do you notice? What does this say about whether the function $(f(x) - 1/2)$ is even or odd?
- Moral: odd functions only have $\sin$ terms in the Fourier series; even functions only have $\cos$ terms.

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Homework

- Properties of functions 4th and 5th Eds.
  - read *James* sections 2.2.3 and 2.2.6
  - do exercises 2.2.5 Q. 10. 2.2.7 Qns. 14, 15
- Calculus of inverse functions etc.
  - *James* 4th edition:
    - read sec. 8.3.7 and 8.8.1 (esp. example 8.41)
    - do exercises 8.3.11 Q. 33, 8.8.2 Q. 103, 8.8.6 Q. 109
  - *James* 5th edition:
    - read sec. 8.3.7 and 8.8.1 (esp. example 8.44)
    - do exercises 8.3.11 Q. 35, 8.8.3 Q. 108, 8.8.7 Qns. 113,116

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The midessional exam

- Monday 15th Jan 9.30 a.m. 1.5 hours
- 12 $\times$ 5-mark questions
- on all we have done this term:
  - not Maple
  - nor Engineering HOTSPOTs
- it's worth 20% unit, Summer exam is worth 80% and covers all the material (including this term's).
- To revise:
  - $\rightarrow$ read the notes
  - $\rightarrow$ do James exercises
  - $\rightarrow$ test learning using QuestionMark
  - $\rightarrow$ Practice past papers on blackboard
  - If stuck: use Drop-ins this week . . . and Support Forum on Blackboard in the vacation.
- Good News: in last 3 years, no one has failed the unit by failing the Jan exam, provided they pass the summer exam