UNIVERSITY OF BRISTOL
FACULTY OF ENGINEERING

First Year Examination for the Degree of Master of Engineering

MAY/JUNE 2014 3 Hours

AENG11100

FLUIDS I

Solutions

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Q 1

$$p_G = \rho g h = 1000 \times 9.81 \times 100 = 0.981 \times 10^5\ N/m^2$$
$$p = p_a + \rho g h = 1.023 \times 10^5 + 0.981 \times 10^5 = 2.004 \times 10^5\ N/m^2$$

(4 *marks*)

Q 2

$$B = \tfrac{4}{3}\pi r^3 \rho_{air} \times g = \left(10 + \tfrac{4}{3}\pi r^3 \rho_{gas} + M\right) \times g$$
$$M = \tfrac{4}{3}\pi r^3 \left(\rho_{air} - \rho_{gas}\right) - 10 = \tfrac{4}{3}\pi 2^3 (0.8) - 10 = 16.81\, Kg$$

(4 *marks*)

Q 3

Steady, incompressible, inviscid,1D flow

(4 marks)

Q 4

Consider that the static pressure changes across the streamlines by $\rho g h$ with distance from the surface, the pitot measures static plus dynamic pressure so that:

$$P_{pitot\,@1} = p_a + \rho g h + \tfrac{1}{2}\rho V_1^2 \quad \rightarrow \quad V_1^2 = 2\frac{(P_0 - p_a)}{\rho} - 2gh = \frac{2 \times 65000}{1000} - 2 \times 9.81 \times 6 = 12.28$$

$$V_1 = 3.504 ms^{-1}$$

Use continuity so

$$A_1 V_1 = A_2 V_2 \quad \rightarrow \quad V_2 = \frac{A_1 V_1}{A_2} = \frac{h_1 \times w \times V_1}{h_2 \times w} = \frac{h_1 \times V_1}{h_2} = 3 \times 3.504 = 10.512$$

Again the pitot measures static plus dynamic pressure so that

$$P_{pitot\,@2} = p_a + \rho g h_2 + \tfrac{1}{2}\rho V_2^2 \quad \rightarrow \quad P_{pitot\,@2} - p_a = 1000 \times \left(9.81 \times 2 + \tfrac{1}{2} 10.512^2\right) = 74955.2$$

(*4 marks*)

Q 5

$$M = \frac{V}{a} = \frac{V}{\sqrt{\gamma R T}} = \frac{70}{\sqrt{1.403 \times 287 \times 293}} = 0.203$$

Compressibility effects will be small and may be neglected in some calculations.

(*4 marks*)

Q 6

$$\mathrm{Re}_x = \frac{Vx}{\nu} \quad \rightarrow \quad x = \frac{\mathrm{Re}_x\, \nu}{V} = \frac{5 \times 10^5 \times 1.14 \times 10^{-6}}{10} = 0.057\ \mathrm{m}$$

A more adverse pressure gradient (pressure gradient increasing) would lower the Reynolds number of transition.

Surface roughness produces early transition (or increased freestream turbulence but the water is defined as still in the question).

(4marks)

Q 7

From continuity we know that the velocity upstream of the sudden enlargement must be

$$A_1 V_1 = A_2 V_2 \quad \rightarrow \quad \frac{V_2}{V_1} = \frac{A_1}{A_2} = 2 \quad \rightarrow \quad V_1 = 2V_2$$

Applying Bernoulli

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g h_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g h_2 + C_{loss}\tfrac{1}{2}\rho V_1^2$$
$$p_1 = p_2 + \tfrac{1}{2}\rho V_2^2\left(1 - 2^2\right) + \rho g (h_2 - h_1) + C_{loss}\tfrac{1}{2}\rho V_1^2$$
$$p_1 = 3 \times 10^5 + \tfrac{1}{2} \times 900 \times 2^2 (-3) - 900 \times 9.81 \times 2 + 0.25 \times \tfrac{1}{2} \times 900 \times 4^2 = 278742\ \mathrm{N/m^2}$$

(4 marks)

Q 8 From continuity

$\dot{m} = \rho AV = 1000 \times \pi \times 0.05^2 \times 10 = 78.540\, kg/s$

Steady flow x-momentum equation

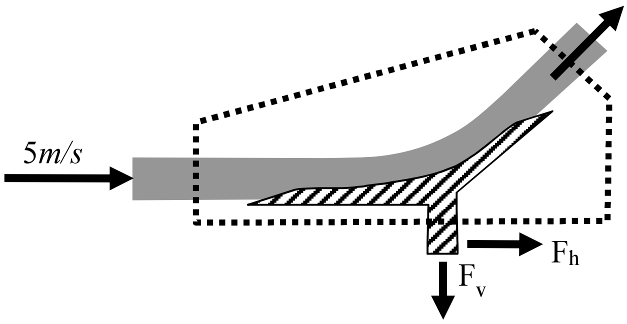
$F_x = \dot{m}(V\cos 45 - V) = \dot{m} \times 10 \times (\cos 45 - 1) = -230.038\, N$

$F_h = 230.038\, N$

Steady flow y-momentum equation

$F_y = \dot{m}(V\sin 45 - 0) = \dot{m} \times 10 \times \sin 45 = 555.36\, N$

$F_v = 555.36\, N$



(4 marks)

Q 9

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

p : local static pressure

p_∞ : freestream static pressure

ρ : density

U_∞ : freestream velocity

From Bernoulli's equation applied between the freestream and the local point

$$p_1 - p_\infty = \frac{1}{2}\rho\left(U_\infty^2 - V_1^2\right)$$

$$c_p = \frac{\frac{1}{2}\rho\left(U_\infty^2 - V_1^2\right)}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2}$$

(4 marks)

Q 10 The lifting flow over a cylinder can be modelled as a combination of: plane onset flow, a doublet and a vortex?

The non-lifting flow over an oval can be modelled as a combination of: plane onset flow, source and sink of equal strength.

(4 marks)

Q11 a) Dam is a triangular prism. Length of base of dam prism

$$l = \frac{2H}{\tan(\alpha)}$$

Vertical force, F_v, equals weight of water above dam surface

$$F_v = \rho_w g \times \tfrac{1}{2} \times \frac{h}{\tan(\alpha)} \times h \times L = \rho_w g \frac{h^2 L}{2\tan(\alpha)}$$

Clockwise moment from vertical thrust given by

$$M_v = -F_v\left(2\frac{H}{\tan(\alpha)} - \frac{h}{3\tan(\alpha)}\right) = -\rho_w g \frac{h^2 L}{\tan^2(\alpha)}\left(H - \frac{h}{6}\right)$$

Weight of dam prism

$$F_D = \rho_D g \times \tfrac{1}{2} \times \frac{2H}{\tan(\alpha)} \times H \times L = \rho_D g \frac{H^2 L}{\tan(\alpha)}$$

Clockwise moment from dam prism weight given by

$$M_D = -F_D\left(\frac{H}{\tan(\alpha)}\right) = -\rho_D g \frac{H^3 L}{\tan^2(\alpha)}$$

Horizontal component of thrust from the left is given by

$$F_H = \rho_w g \times \frac{h}{2} \times h \times L - = \rho_w g \frac{h^2 L}{2}$$

The horizontal forces act at depths

$$h_H = \frac{h}{2} + \frac{I_{xx}}{\frac{h}{2} \times h \times L} = \frac{h}{2} + \frac{Lh^3}{12 \times \frac{h}{2} \times h \times L} = \frac{h}{2} + \frac{h}{6} = \frac{2h}{3}$$

Clockwise moment from the horizontal thrust is given by

$$M_H = F_H \times \left(h - \tfrac{2}{3}h\right) = \rho_w g L \frac{h^2}{2}\left(\frac{h}{3}\right) = \rho_w g L\left(\frac{h^3}{6}\right)$$

The total moment is therefore

$$M = M_D + M_V + M_H =$$

$$M = -\rho_D g \frac{H^3 L}{\tan^2(\alpha)} - \rho_w g \frac{h^2 L}{\tan^2(\alpha)}\left(H - \frac{h}{6}\right) + \rho_w g \frac{Lh^3}{6}$$

$$M = \frac{gLH^3}{6}\left(-\rho_D \frac{6}{\tan^2(\alpha)} - \rho_w \frac{h^2}{H^2\tan^2(\alpha)}\left(6 - \frac{h}{H}\right) + \rho_w \frac{h^3}{H^3}\right)$$

$$M = 0 \quad \rightarrow \quad -\frac{6\rho_D}{\tan^2(\alpha)} - \frac{\rho_w h^2}{H^2\tan^2(\alpha)}\left(6 - \frac{h}{H}\right) + \frac{\rho_w h^3}{H^3} = 0 \quad \rightarrow \quad \frac{6\rho_D}{\tan^2(\alpha)} + \frac{\rho_w h^2}{H^2\tan^2(\alpha)}\left(6 - \frac{h}{H}\right) = \frac{\rho_w h^3}{H^3}$$

$$\tan^2(\alpha) = \left(\frac{H^3}{\rho_w h^3}\right)\left(6\rho_D + \frac{\rho_w h^2}{H^2}\left(6 - \frac{h}{H}\right)\right) = \left(6\frac{\rho_D H^3}{\rho_w h^3} + 6\frac{H}{h} - 1\right)$$

$$\tan(\alpha) = \sqrt{\left(6SG\frac{H^3}{h^3} + 6\frac{H}{h} - 1\right)}$$

(14 marks)

(b)
The minimum volume of concrete has h=H and the smallest value of α that doesn't tip so from (a)
$\tan(\alpha) = \sqrt{(6 \times 2.5 + 6 - 1)} = \sqrt{20}$

Also from part (a)
$$volume = \frac{H^2 L}{\tan(\alpha)} = \frac{50^2 \times 200}{\sqrt{20}} = 1.118 \times 10^5 m^3$$

(6 marks)

Q12

(a) Assume steady inviscid flow. Also assume, as flow is water, the flow is incompressible. Finally assume 1D flow at locations 1 & 2, i.e. parallel uniform flow across the sections.

Apply continuity between 1 & 2

$$A_1V_1 = A_2V_2 \qquad \rightarrow \quad V_2 = \frac{A_1}{A_2}V_1$$

Apply Bernoulli's equation between 1 & 2

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho gL = p_2 + \tfrac{1}{2}\rho V_2^2 \qquad \rightarrow \quad \tfrac{1}{2}\rho V_1^2\left(\frac{A_1^2}{A_2^2} - 1\right) = p_1 - p_2 + \rho gL$$

Assuming static flow in arms of manometer we can use the hydrostatic equation between the fluid interfaces in each arm.

$$p_1 + \rho gh = p_2 + \rho g(h - L - \Delta h_m) + \rho_m g\Delta h_m \qquad \rightarrow \quad p_1 - p_2 = g\Delta h_m(\rho_m - \rho) - \rho gL$$

$$\tfrac{1}{2}\rho V_1^2\left(\frac{A_1^2}{A_2^2} - 1\right) = g\Delta h_m(\rho_m - \rho) - \rho gL + \rho gL$$

Rearranging $V_1 = \sqrt{2g\Delta h_m \dfrac{A_2^2(\rho_m - \rho)}{(A_1^2 - A_2^2)\rho}}$

Mass flow rate given by

$$\dot{m} = \rho A_1V_1 = \rho A_1\sqrt{2g\Delta h_m \frac{A_2^2(\rho_m - \rho)}{(A_1^2 - A_2^2)\rho}} = A_1A_2\sqrt{2g\Delta h_m \frac{\rho(\rho_m - \rho)}{(A_1^2 - A_2^2)}}$$

(10 marks)

(b) Applying Bernoulli's equation between the reservoir surface (subscript 0) and 2 in the Venturi throat, just at the point of cavitation:

$$p_0 + 0 + \rho g \times 20 = p_2 + \tfrac{1}{2}\rho V_2^2 + 0$$

$$V_2 = \sqrt{2\times\frac{(1.02\times10^5 - 2.4\times10^3 + 1000\times9.81\times20)}{1000}} = \sqrt{2\times(1.02\times10^2 - 2.4 + 9.81\times20)}$$

$$V_2 = 24.323 m/s$$

The mass flow rate is given by

$$\dot{m} = \rho A_2V_2 = 1000\times\pi\times0.25^2\times24.323 = 4775.78\ kg/s$$

Using result from a)

$$\Delta h_m = \frac{(A_1^2 - A_2^2)}{2g\rho(\rho_m - \rho)}\left(\frac{\dot{m}}{A_1A_2}\right)^2 = \frac{\pi^2(2^4 - 0.25^4)}{2\times9.81\times1000\times(9000 - 1000)}\left(\frac{4775.78}{\pi^2\times2^2\times0.25^2}\right)^2$$

$$\Delta h_m = \frac{\pi^2(2^4 - 0.25^4)}{2\times9.81\times1000\times8000}\left(\frac{4775.78}{\pi^2\times2^2\times0.25^2}\right)^2 = 3.768m$$

(8 marks)

(c) Looking at the solution for V at 2 in (b) above. If we have height of reservoir surface H then $V_2 = \sqrt{2\times\dfrac{(1.02\times10^5 - 2.4\times10^3 + 1000\times9.81\times H)}{1000}}$

Hence maximum mass flow rate would increase.

(2 marks)

Q13

(a) Δh is due to the difference in static pressure between the wind tunnel pressure tapping and ambient pressure. Applying Bernoulli's equation between ambient and working section locations and neglecting hydrostatic terms as air density is small

$$p_a = p_w + \tfrac{1}{2}\rho V_w^2$$

Applying the hydrostatic equation across the manometer fluid surfaces

$$p_a = p_w + \rho_m g \Delta h$$

Rearranging

$$\rho_m g \Delta h = \tfrac{1}{2}\rho V_w^2 \qquad \rightarrow \Delta h = \frac{\rho}{2\rho_m g} V_w^2$$

(7 marks)

(b) From continuity and assuming the flow is incompressible, we know that the volume flow rate at the exit must be the same as at the working section.

$$A_e V_e = A_w V_w \qquad \rightarrow \quad V_e = \frac{A_w}{A_e} V_w$$

Similarly the average velocity downstream of the fan must be the same as in the working section (the cross section is constant).

Taking Bernoulli's equation from just downstream of the fan (subscript fd) to the exit, where the pressure is equal to the ambient pressure

$$p_{fd} + \tfrac{1}{2}\rho V_{fd}^2 = p_a + \tfrac{1}{2}\rho V_e^2$$

Applying continuity

$$p_{fd} + \tfrac{1}{2}\rho V_w^2 = p_a + \tfrac{1}{2}\rho \left(\frac{A_w}{A_e}\right)^2 V_w^2 \qquad \rightarrow \qquad p_{fd} = p_a + \tfrac{1}{2}\rho \left(\left[\frac{A_w}{A_e}\right]^2 - 1 \right) V_w^2$$

Therefore

$$p_{fd} - p_w = p_a + \tfrac{1}{2}\rho \left(\left[\frac{A_w}{A_e}\right]^2 - 1 \right) V_w^2 - p_a + \tfrac{1}{2}\rho V_w^2$$

(7 marks)

$$\Delta p_{fan} = \tfrac{1}{2}\rho \left(\frac{A_w}{A_e}\right)^2 V_w^2$$

(c) Take a control volume around the model and upstream of the fan. Applying the conservation of linear momentum and neglecting viscous forces (see figure below)

$$p_m A_w - p_{fu} A_w - D = \dot{m} \times \left(V_m - V_m\right) = 0$$

$$p_{fu} = p_m - \frac{D}{A_w} = p_m - C_d \times \tfrac{1}{2}\rho V_m^2 \times \frac{A_m}{A_w}$$

Applying Bernoulli's equation from ambient conditions to just upstream of the model

$$p_a = p_m + \tfrac{1}{2}\rho V_w^2 \quad p_{fu} = p_a - \tfrac{1}{2}\rho V_w^2 - C_d \times \tfrac{1}{2}\rho V_w^2 \times \frac{A_m}{A_w}$$

Applying continuity and Bernoulli's equation downstream of the fan, as velocity unchanged then downstream pressure is unchanged

$$p_{fd} = p_a + \tfrac{1}{2}\rho_a \left[\left(\frac{A_w}{A_e}\right)^2 - 1 \right] V_w^2$$

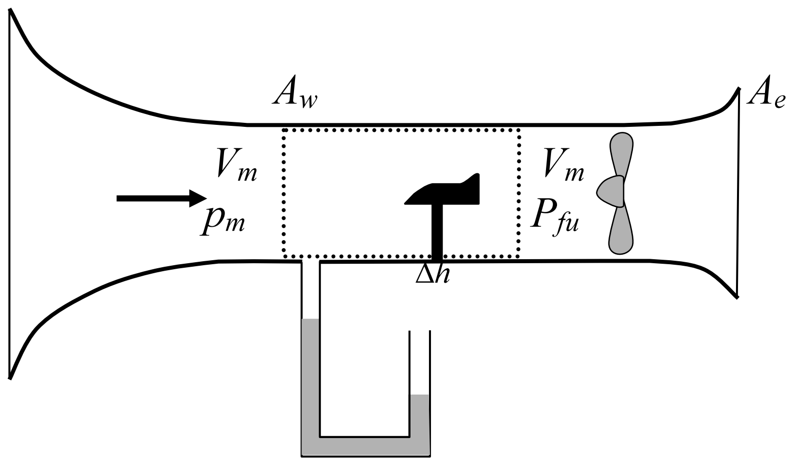
So the pressure change across the fan is now

$$p_{fd} - p_{fu} = \tfrac{1}{2}\rho\left[\left(\frac{A_w}{A_e}\right)^2 - 1\right]V_w^2 + \tfrac{1}{2}\rho V_w^2 + C_d \times \tfrac{1}{2}\rho V_w^2 \times \frac{A_m}{A_w} = \tfrac{1}{2}\rho\left(\frac{A_w}{A_e}\right)^2 V_w^2 + C_d \times \tfrac{1}{2}\rho V_w^2 \times \frac{A_m}{A_w}$$

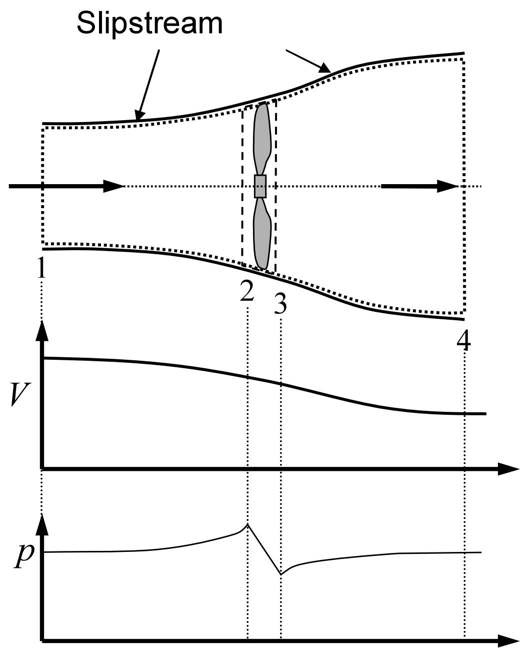
Considering a control volume around the fan, the force on the fan is given by

$$F_f = \tfrac{1}{2}\rho V_w^2\left(A_w\left(\frac{A_w}{A_e}\right)^2 + C_d A_m\right)$$

(6 marks)



Q14 **(a)** Use the actuator disc theory for an ideal windmill, see figure below



Assumptions: Frictionless & incompressible, Steady 1D flow (neglect rotation and variation across the disc radius). Actuator disc is thin so $A_2=A_3=A_d$ & $V_2=V_3=V_d$. $p=p_a$ at all points on slipstream boundary & 1 & 4

Continuity: $Q=V_dA_d$

Bernoulli's equation for CV 1-2 & CV 3-4

$$\begin{aligned} p_1+\tfrac{1}{2}\rho V_1^2 &= p_2+\tfrac{1}{2}\rho V_d^2 \\ p_3+\tfrac{1}{2}\rho V_d^2 &= p_4+\tfrac{1}{2}\rho V_4^2 \end{aligned} \quad \rightarrow \quad p_3-p_2=\tfrac{1}{2}\rho\left(a^2V^2-V^2\right)$$

Steady Flow momentum for CV 2-3:

$$(p_2-p_3)A_d+F_{CV}=\rho Q(V_d-V_d)=0 \qquad \rightarrow \quad F_{CV}=(p_3-p_2)A_d$$

Where F is the force on the control volume

Applying results from Bernoulli's equation above

$$F_{CV}=\tfrac{1}{2}\rho A_dV^2\left(a^2-1\right)=\tfrac{\pi}{8}\rho d^2V^2\left(a^2-1\right)$$

Force on the windmill is equal and opposite to the force on the CV so

$$F=\rho\tfrac{\pi}{8}d^2V^2\left(1-a^2\right)$$

(8 marks)

(b) Steady Flow momentum for CV 1-4:

$$0+F_{CV}=\rho Q(V_4-V_1) \qquad \rightarrow \quad F_{CV}=\rho V_dA_dV(a-1)$$

From momentum & continuity

$$(p_3-p_2)A_d=\rho V_dA_d(V_4-V_1)$$

Eliminating $(p_3\text{-}p_2)$ using Bernoulli's equation above

$$\rho V_d(aV-V)=\tfrac{1}{2}\rho\left(a^2V^2-V^2\right)$$
$$V_dV(a-1)=\tfrac{1}{2}V^2(a+1)(a-1)$$
$$V_d=\tfrac{1}{2}V(a+1)$$

The power drawn from the air by the disc is

$$P_{\text{disc}}=-F_{CV}V_d=-\rho Q(V_4-V_1)V_d=\tfrac{1}{2}\rho A_dV_d(V_1-V_4)V_d=\tfrac{1}{4}\rho A_d(aV+V)\left(V^2-a^2V^2\right)$$

Power in the wind passing through the disc area, if the disc were not present.

$$P_{\text{wind}}=\tfrac{1}{2}\dot{m}V_1^2=\tfrac{1}{2}\rho A_dV_1\,V_1^2=\tfrac{1}{2}\rho A_dV^3$$

The efficiency of the turbine is therefore

$$\eta=\frac{P_{\text{disc}}}{P_{\text{wind}}}=\frac{\tfrac{1}{4}\rho A_d(aV+V)\left(V^2-a^2V^2\right)}{\tfrac{1}{2}\rho A_dV^3}=\frac{(aV+V)\left(V^2-a^2V^2\right)}{2V^3}=\frac{1}{2}(1+a)\left(1-a^2\right)=\frac{1+a-a^2-a^3}{2}$$

(6 marks)

(c) from the definition of efficiency in (b) we have

$$\eta=0.5=\frac{1+a-a^2-a^3}{2}\qquad\rightarrow\qquad a^2+a-1=0$$

$$a=\begin{cases}\frac{-1-\sqrt{5}}{2}\\ \frac{-1+\sqrt{5}}{2}\end{cases}$$

$a=\frac{\sqrt{5}-1}{2}$ is the only feasible solution

$$F=\rho\tfrac{\pi}{8}d^2V^2\left(1-a^2\right)$$
$$F=1.2\times\tfrac{\pi}{8}\times20^2\times10^2\left(1-\left(\tfrac{\sqrt{5}+1}{2}\right)^2\right)=116540N$$
$$V_d=\tfrac{1}{2}V(a+1)=\tfrac{1}{2}\times10\times\left(\frac{\sqrt{5}-1}{2}+1\right)=5\times\left(\frac{\sqrt{5}+1}{2}\right)=8.090m/s$$

From Bernoulli's equations in (a)

$$p_2-p_a=\tfrac{1}{2}\rho\left(V^2-V_d^2\right)=\tfrac{1}{2}\times1.2\times\left(10^2-8.090^2\right)=20.73N/m^2$$
$$p_3-p_a=\tfrac{1}{2}\rho\left(a^2V^2-V_d^2\right)=\tfrac{1}{2}\times1.2\times\left(\left(\sqrt{5}-1\right)^2 10^2-8.090^2\right)=-16.35N/m^2$$

(6 marks)

Q15

(a) The velocity components are given by

$$V_r = \left(1 - \frac{\kappa}{2\pi U_\infty r^2}\right) U_\infty \cos\theta, \quad V_\theta = -\left(1 + \frac{\kappa}{2\pi U_\infty r^2}\right) U_\infty \sin\theta - \frac{\Gamma}{2\pi r}$$

Now the cylinder is a stream line of the flow so there is no flow normal to the cylinder i.e.

$$V_r = 0$$

This means that

$$\left(1 - \frac{\kappa}{2\pi U_\infty r^2}\right) U_\infty \cos\theta = 0$$

for all θ so the doublet strength must be given by

$$\kappa = 2\pi U_\infty R^2$$

The velocity components are then

$$V_r = \left(1 - \frac{R^2}{r^2}\right) U_\infty \cos\theta, \quad V_\theta = -\left(1 + \frac{R^2}{r^2}\right) U_\infty \sin\theta - \frac{\Gamma}{2\pi r}$$

On the cylinder

$$V_r = 0, \quad V_\theta = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

The pressure coefficient on the cylinder is given by

$$C_p = 1 - \left(\frac{V}{U_\infty}\right)^2 = 1 - 4\sin^2\theta - \frac{2\Gamma \sin\theta}{\pi R U_\infty} - \left(\frac{\Gamma}{2\pi R U_\infty}\right)^2$$

The pressure distribution on the cylinder is then

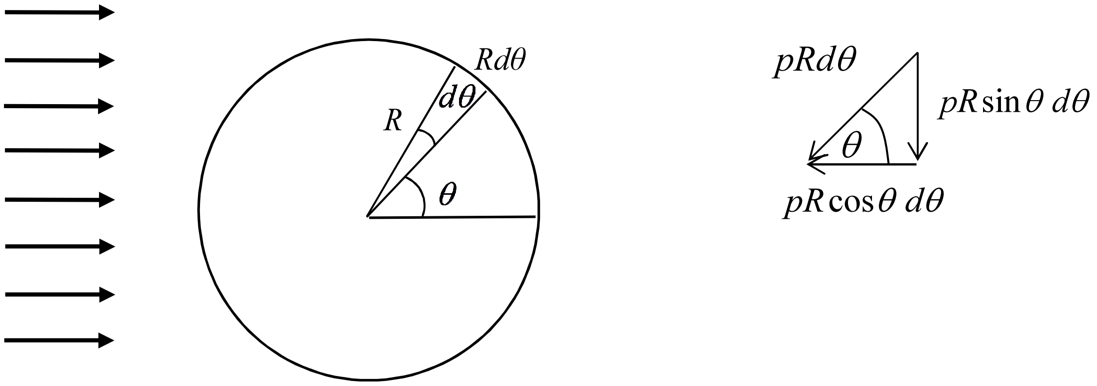
$$p(\theta) = p_\infty + \frac{1}{2}\rho U_\infty^2 \left(1 - 4\sin^2\theta\right) - \left(\frac{\rho U_\infty \Gamma \sin\theta}{\pi R}\right) - \frac{1}{8}\rho\left(\frac{\Gamma}{\pi R}\right)^2$$

(6 marks)

(b)

Consider a small arc of the surface, as shown in the sketch, of size $Rd\theta$. The force acting on this element is given by

$$p(\theta) R d\theta$$



this acts normal to the surface and must be resolved to get the components (see above)

So the force in the vertical direction over the entire surface is given by

$$l = -\int_0^{2\pi} p(\theta) R \sin\theta d\theta$$

Now using the results from part (a)

$$p(\theta) = p_\infty + \frac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2\theta\right) - \left(\frac{\rho U_\infty \Gamma \sin\theta}{\pi R}\right) - \frac{1}{8}\rho\left(\frac{\Gamma}{\pi R}\right)^2$$

We find that

$$l = -\int_0^{2\pi}\left[p_\infty + \frac{1}{2}\rho U_\infty^2\left(1 - 4\sin^2\theta\right) - \left(\frac{\rho U_\infty \Gamma \sin\theta}{\pi R}\right) - \frac{1}{8}\rho\left(\frac{\Gamma}{\pi R}\right)^2\right] R\sin\theta d\theta$$

$$= R\int_0^{2\pi}\left[\left(-p_\infty - \frac{1}{2}\rho U_\infty^2 + \frac{1}{8}\rho\left(\frac{\Gamma}{\pi R}\right)^2\right)\sin\theta + 2\rho U_\infty^2 \sin^3\theta + \frac{\rho U_\infty \Gamma \sin^2\theta}{\pi R}\right] d\theta$$

$$= R\left(-p_\infty - \frac{1}{2}\rho U_\infty^2 + \frac{1}{8}\rho\left(\frac{\Gamma}{\pi R}\right)^2\right)\int_0^{2\pi}\sin\theta d\theta + 2\rho U_\infty^2 R\int_0^{2\pi}\sin^3\theta d\theta + \frac{\rho U_\infty \Gamma}{\pi}\int_0^{2\pi}\sin^2\theta d\theta$$

$$= \frac{\rho U_\infty \Gamma}{\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{\rho U_\infty \Gamma}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \rho U_\infty \Gamma$$

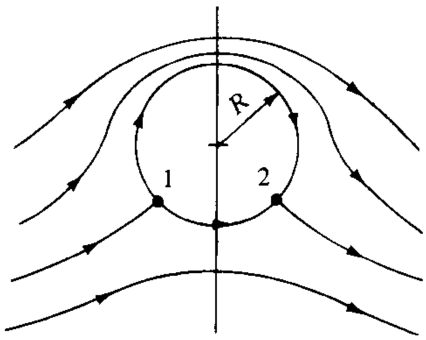
So finally the lift force acting is equal to

$$l = \rho U_\infty \Gamma \qquad \text{or} \qquad \Gamma = \frac{l}{\rho U_\infty}$$

(9 *marks)*

The stagnation point at $\theta = 225^o$ so $\sin\theta = -0.7071$, substitute into equation for radial velocity with $r{=}R$

From previous



$$V_r = 0, \quad V_\theta = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R} = 0$$

$$2 \times 0.7071 \times 10 = \frac{\Gamma}{2\pi \times 0.5} \qquad \rightarrow \qquad \Gamma = 44.429$$

Hence lift per unit length is

$l = 1.2 \times 10 \times 44.429 = 533.15N$

(5 *marks)*