EMAT10100 Engineering Maths I Lectures 1&2, Introduction to Probability: Introduction

Rosalyn Moran & Nikolai Bode

Randomness . . .

- . . . is a lack of pattern or certainty in events.
- Some examples in Engineering:
  - Component reliability - I cannot predict with certainty when a single component (a light bulb, a spark plug, a rotor blade) will fail.
  - Will a packet of information sent over the Internet reach its destination?
  - What is the likelihood of a fault occurring on a power line? Or in a production line? Or in a plane flying through a storm of a certain intensity?
- Some examples from everyday life:
  - Coin toss - Will an individual toss be heads or tails? What if I toss the coin several time in the row?
  - Roll a die - What number will come up? What if I roll two dice or more?
  - Long range weather forecasting - Will tomorrow rain? What will be the weather like in 3 months from now?

Lots and lots of coin tossing

- Although there is no predictability for a single event — we might say something about:
  - large numbers of coin tosses or die rolls
  - what the weather is typically like in June;
  - or what is the typical life of a component.

Allow me to “toss some coins”:

- HHTHTTTHHH (6 heads, 4 tails)
- TTTHTHHTTTHTHTTHTTTH (7 heads, 13 tails)
- THTTHHTHHTTHHTHHHHTHHTHHTTTHHTHTTTTTTHHTT HTTTTTTTTTTHHTTHTHHHTHTTTTHTTHHTTTHTHTTHT HHHTHTTTTHHHTHHHTH (45 heads, 55 tails)

If we toss for long enough — the proportion of heads will eventually settle down to $1/2$ (* unless we have criminal tendencies).

Probability and Statistics

Very loosely:

- The *probability* of something is how often it will occur as a long-run average. The fact that long-run averages typically settle down is sometimes called the *Law of Large Numbers*.
- *Probability* is a theory helping engineers making predictions of how often things will occur, assuming probabilities of some basic events are known.
- *Statistics* (the subject) is about inferring the probability of something (i.e., its long-run frequency) — from a relatively small set of observations. **And we won't do any Statistics this year!**

Random Trials; Elementary Outcomes; Sample Space

Some terminology:

- *Random Trial*, or *Random Experiment*:
  - Toss a coin
  - Throw a six-sided die
- *(Elementary) Outcome*: the result of a single trial
  - e.g.: heads (H)
  - e.g.: 5
- *Sample Space*: the set of all possible outcomes
  - $\{H, T\}$
  - $\{1, 2, 3, 4, 5, 6\}$

Notation: Sample Space is called S or sometimes Ω.
Outcomes are referred to as $\omega \in S$.

More Examples of Sample Spaces

- Trial: we toss a coin three times in sequence

  $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$

- Trial: we toss three indistinguishable coins at the same time

  $S =$ Exercise

- Trial: we throw two indistinguishable six-sided dice

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$$

Probabilities of Elementary Outcomes

Loosely: the *probability* of an elementary outcome is the proportion of times it would occur in a very, very large number of trials.

- Trial: we toss a single (fair, unbiased) coin
  $S = \{H, T\}$
  $P(H) = 1/2$, $P(T) = 1/2$.
- Trial: we roll a single (fair, unbiased) six-sided die
  $S = \{1, 2, 3, 4, 5, 6\}$
  $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6.$

Notation: here I will use $P(\cdot)$ to denote the probability of something. Other people use $Pr(\cdot)$. It means exactly the same thing.

Warning: Probabilities should really be defined on *events*, i.e. on subsets of the sample space and not on individual *outcomes*! More later on this.

Events and the Sample Space

Terminology: (for a given random trial) an *event* is a subset of the sample space.
Examples:

- Random trial: toss a single coin
  - Sample space $S = \{H, T\}$
  - Possible events:
    $\{H\}$, or $\{T\}$
    $\{H, T\}$ (the whole sample space)
    $\{\} =: \emptyset$
- Random trial: throw a single die
  - Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
  - Possible events, e.g.:
    $\{1\}$, $\{2\}$, etc.
    $\{1, 2, 3, 4, 5, 6\}$, $\emptyset$, (the whole sample space, and the null set)
    the outcome is even: $\{2, 4, 6\}$, the outcome is odd: $\{1, 3, 5\}$
    Exercise: outcome is greater than 3:
    Exercise: outcome is even and greater than 3:
    Exercise: outcome is even or greater than 3:

Events and Sets

So events might consist of either single elementary outcomes or combinations of elementary outcomes. The empty set and the whole sample space are also considered to be events.

Notice how the language of sets is taking over here.

* Advanced topic - if the sample space is finite - then the set of possible events is the set of all possible subsets of the sample space - this is called the *power set*. Question: how big is it?

Simple Man's (or Woman's) Approach to Probability

How to compute the *probability* of a given *event*?

- Count up the outcomes that are included in (i.e, are set elements of) the event.
- Add up the probabilities of those outcomes.
- Erm, that's it!

Why does this work?

Loosely. The probability of an event is the proportion of times it occurs in a very, very large number of trials. An event can occur via several different outcomes — so you just need to add up the proportion of times (i.e., the probabilities) that each of those outcomes occurs.

Important: outcomes (unlike events) are disjoint and indivisible. One trial only ever has one outcome. That is why we are allowed to add their probabilities.

Simple Example using the Simple Man's Approach

James: Example 13.3 page 983 (5th ed., pp. 1009).

A fair six-sided die is tossed. Find the probability of the event 'even number or number less than four'.

Outcomes which match these criteria are $1, 2, 3, 4$ and 6 and each has probability $1/6$.

The probability of the required event is thus
$1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 5/6$, DONE.

Unfortunately, this counting approach is very clumsy for more complicated problems and we really need an approach which exploits the full power of set theory.

Axioms of Probability (1)

Axiom $=$ building block of formal Mathematical reasoning.

At some level, the Axioms of Probability are statements which are blindingly obvious if you think about their interpretations in terms of the long-run proportions of given outcomes.

1. The certain event has probability one: $P(S) = 1$.
2. All probabilities are non-negative: $P(A) \geq 0$ for any event A.
3. Addition rule: if A and B are disjoint events (so $A \cap B = \emptyset$) then $P(A \cup B) = P(A) + P(B)$.

Here set union "$\cup$" is equivalent to (inclusive) "or" when problems are phrased in plain English; and similarly set intersection "$\cap$" is equivalent to the word "and".

Axioms of Probability (2)

axioms continued . . .

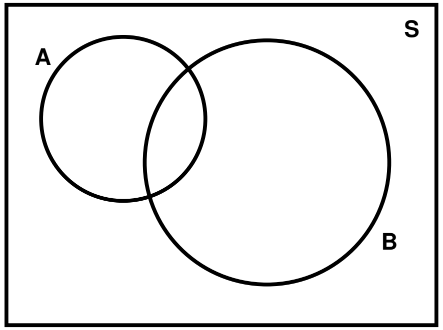
4. Complement rule: $P(S-A) = 1 - P(A)$ for any event A.
5. $P(\emptyset) = 0$.
6. If $A \subseteq B$, then $P(A) \leq P(B)$.
7. General addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for *any* given events A and B.

Here the plain English interpretation of $S - A$ is "not A" (i.e., A doesn't happen)

These axioms have very natural interpretations in geometry and areas — next slide.

Probabilities as Areas; and the General Addition Rule

Loosely speaking: think of the probability of an event as its area when illustrated in a Venn diagram.



Clearly to compute $P(A \cup B)$, we need to avoid double counting the intersection $P(A \cap B)$.
That is why $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Simple Example Revisited using Sets and Axioms

James: Example 13.3 page 983 (5th ed., pp. 1009).

A fair six-sided die is tossed. Find the probability of the event 'even number or number less than four'.

Let $A = \{2, 4, 6\}$ be the event 'even number'. $P(A) = 1/2$.
Let $B = \{1, 2, 3\}$ be the event 'number less than four'. $P(B) = 1/2$.

We should compute $P(A \cup B)$, where $\cup$ (union) means "or".
The axioms say $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
$A \cap B = \{2\}$, so $P(A \cap B) = 1/6$.
So $P(A \cup B) = 1/2 + 1/2 - 1/6, \ = 5/6,$ DONE.

This might seem like a silly long way round compared to what we did before. For more complicated problems, however, this is a better approach.

Exercise

James: Example 13.4 page 984 (5th ed., pp. 1010).

During the assessment of a class of students, 80% passed the examination in mathematics, 85% in laboratory work, and 75% passed both. For a student chosen at random from the class, find the probabilities that the student

(a) passed either in mathematics or laboratory work,

(b) passed in mathematics but failed in laboratory work,

(c) failed in both.

Technically speaking . . .

We should *really* define the probabilities on events (subsets of the sample space) - not on outcomes — although for the simplest examples it makes no difference. But consider the following problem.

For a random trial, pick any real decimal number between 0 and 1 (randomly). What is the sample space and the probability of each elementary outcome?

- Because the sample space is infinite, each elementary outcome has probability 0. Ooops!
- But: let A be the event that the number x that is chosen is between (say) a and b, so that $0 \leq a \leq x \leq b \leq 1$. ($A$ is called an interval and we would say $x \in [a, b]$.)
- Quite naturally — we can say $P(A) = b - a$.

So in this case — probabilities of events are ok, even though probabilities of individual outcomes are not. This is why the *axioms of probability* are defined on events rather than outcomes.

An Introduction to Conditional Probability (1)

See *James*: section 13.3.4.

Sometimes we are interested in the probability of an event A, given that we *know* some other event B has happened. This is called *conditional probability*.

We write $P(A|B)$, pronounced "the probability of A given B".

Examples.

- What is the probability that it will rain this PM — given that it is very cloudy?
- What is the probability that my house will be burgled today — given that I forgot to close the front door? (etc.)
- What is the probability that a die roll is a 6 — given that we know it is even?
- What is the probability of tossing a coin three times and getting two heads — given that the first toss is a head?

An Introduction to Conditional Probability (2)

In fact:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

(NB we require $P(B) > 0$.)

This equation has a natural interpretation in terms of areas in a Venn diagram — more on this next time.

Housekeeping arrangements

- See you Wednesday
- Homework (next couple of days)
  - Read these notes and *James* sections 13.3.1, 13.3.2, 13.3.3 to re-enforce this lecture.
  - Attempt some *James* exercises 13.3.6 page 991 (5th ed., pp. 1017). Question numbers 6–13 have been covered by this lecture.
  - Try to skim through *James* sections 13.3.4, 13.3.5 to prepare for Weds.
  - QMP questions should be available.