Vibrations 2, Lecture 6
Introduction to forced vibration

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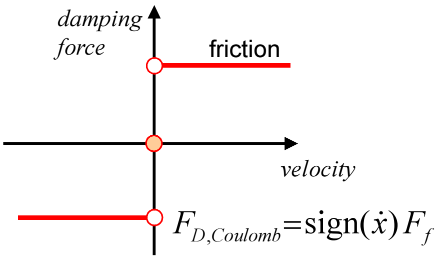
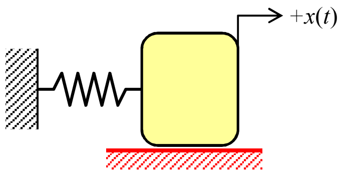
Lecture 5

Coulomb model of dry friction



Equation of motion



$$m\ddot{x} + kx = -\mathrm{sign}(\dot{x}) F_f$$

Free vibration with dry friction:

- free vibration at ω_0
- linear vibration decay, $x_{(j)} - x_{(j+2)} = 4F_f/k$
- vibration stops when $F_f > kx$

Lecture 6

- Forced vibration response of 1 DOF systems
- Forced vibration with constant excitation
- Solved example

Forced response of 1 DOF system

Forced response of a vibrating 1 DOF system consists of:

- a component due to free response of the system at ω_D,
- a component due to applied force F(t).

Complementary solution (CS), solution when RHS=0:

$$x_H = X e^{-\zeta\omega_0 t}\sin(\omega_D t + \varphi) = e^{-\zeta\omega_0 t}(X_1 \sin(\omega_D t) + X_2 \cos(\omega_D t))$$

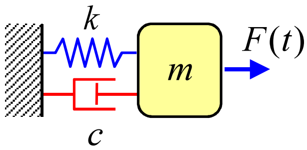
Particular solution (PS), solution when RHS=F(t).
We will consider these types of excitation:

- constant force: F(t)=F_0=const. (example: helicopter landing)
- harmonic force: F(t)=F_0sin(ωt) (example: aero engine bladed disc)

Total solution = CS + PS

$$\boxed{x = x_H + x_P = e^{-\zeta\omega_0 t}(X_1 \sin(\omega_D t) + X_2 \cos(\omega_D t)) + x_{forced}}$$

Total vibration response



$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$x(t) = x_H(t) + x_P(t)$$

a.k.a. principle of superposition

Free response:

$$m\ddot{x}_H + c\dot{x}_H + kx_H = 0$$

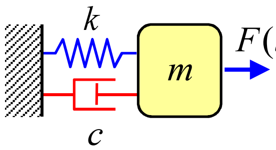
Associated with transient and decaying vibrations

Forced response:

$$m\ddot{x}_P + c\dot{x}_P + kx_P = F(t)$$

Associated with forced vibrations

Constant force: $F(t)=F_0$



$$F(t) = F_0 = const.$$

$$m\ddot{x} + c\dot{x} + kx = F_0$$

- find the particular solution x_P by "guessing" the trial solution
- the trial solution used here is x_P=C, where C is the unknown constant
- use $x=x_P$=C is the EOM:

$$m0 + c0 + kC = F_0 \Rightarrow x_P = C = F_0/k$$

The total solution:

$$\boxed{x = x_H + x_P = e^{-\zeta\omega_0 t}(X_1\sin(\omega_D t) + X_2\cos(\omega_D t)) + F_0/k}$$

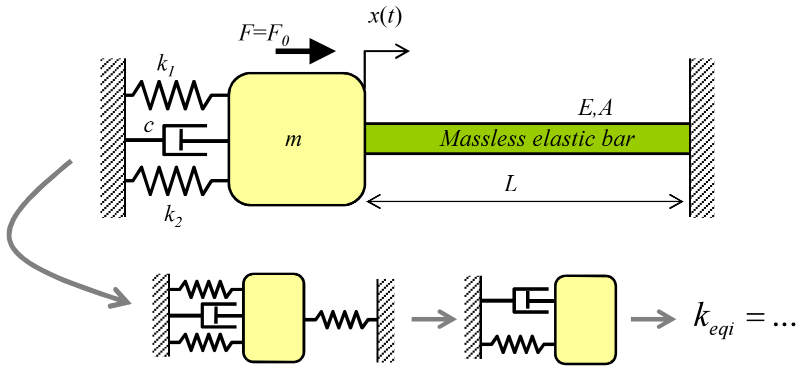
X_1 and X_2 from the initial conditions.

Static deformation due to constant force F_0

Example: step force input

Determine the maximum amplitude of vibration of the system with F_0=130 N applied in *t*=0. Find the undamped natural frequency and the critical damping c_{cr}. Sketch the response for ζ=0 and ζ<1. Assume zero initial conditions.

The stiffness is k_1=k_2=15 kN/m, mass m=8 kg and damping ratio is ζ<1. The *massless* elastic bar has Young's modulus E=5 GPa, cross-sectional area A=10 mm^2 and length L=0.5 m.



Example

Equation of motion: $m\ddot{x} + c\,\dot{x} + k_{eqi}\,x = F$ $\quad k_{eqi} = k_1 + k_2 + EA/L$

The total solution: $x = x_H + x_P = e^{-\zeta\omega_0 t}(X_1\sin(\omega_D t) + X_2\cos(\omega_D t)) + F_0/k_{eqi}$

The unknown constants X_1 and X_2 and zero initial conditions:

$$t=0:\ x_0 = 0 = e^{-\zeta\omega_0 0}(X_1\sin(\omega_D 0) + X_2\cos(\omega_D 0)) + F_0/k_{eqi} \qquad X_2 = -F_0/k_{eqi}$$

$$t=0:\ \dot{x}_0 = 0 = (-\zeta\omega_0)e^{-\zeta\omega_0 0}(X_1\sin(\omega_D 0) + X_2\cos(\omega_D 0))$$

$$+e^{-\zeta\omega_0 0}(\omega_D X_1\cos(\omega_D 0) - \omega_D X_2\sin(\omega_D 0))$$

$$0 = (\zeta\omega_0)(F_0/k_{eqi}) + \omega_D X_1 \Rightarrow X_1 = -(\zeta\omega_0/\omega_D)(F_0/k_{eqi})$$

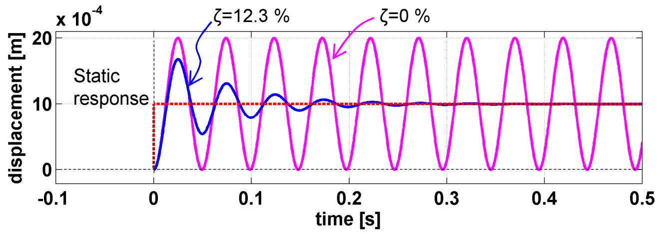
The total solution:

$$x = \frac{F_0}{k_{eqi}}\left(1 - e^{-\zeta\omega_0 t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_D t) + \cos(\omega_D t)\right)\right)$$

Example 5

If ζ=0 then exp(-ζω$_0$t)=1 and the total response and its maximum is:

$$x=\frac{F_0}{k_{eqi}}\left(1-\cos(\omega_D t)\right),\ \cos(\omega_D t)=-1\Rightarrow \max(x(t))=2\frac{F_0}{k_{eqi}}=2x_{static}$$



Stiffness: $k_{eqi}=k_1+k_2+EA/L=2\times15000+(5\times10^9)\times(10\times10^{-6})/0.5=130\ kN/m$

Undamped natural frequency: $f_0=(1/2\pi)\sqrt{k_{eqi}/m}=(1/2\pi)\sqrt{130000/8}=20.29\ Hz$

Critical damping: $c_{cr}=2\sqrt{mk_{eqi}}=2\sqrt{8\ kg\times130\ kN/m}=2039.6\,kg.s^{-1}$

Maximum undamped response: $\max(x(t))=2F_0/k_{eqi}=2\times(130/130000)=2\,mm$

Summary

- Free vibration response occurs at ω_D
- Forced vibration response = CS (related to free response) + PS (related to the applied load)
- CS component dies out in damped systems
- Constant force produces steady-state static deflection (a new equilibrium position)