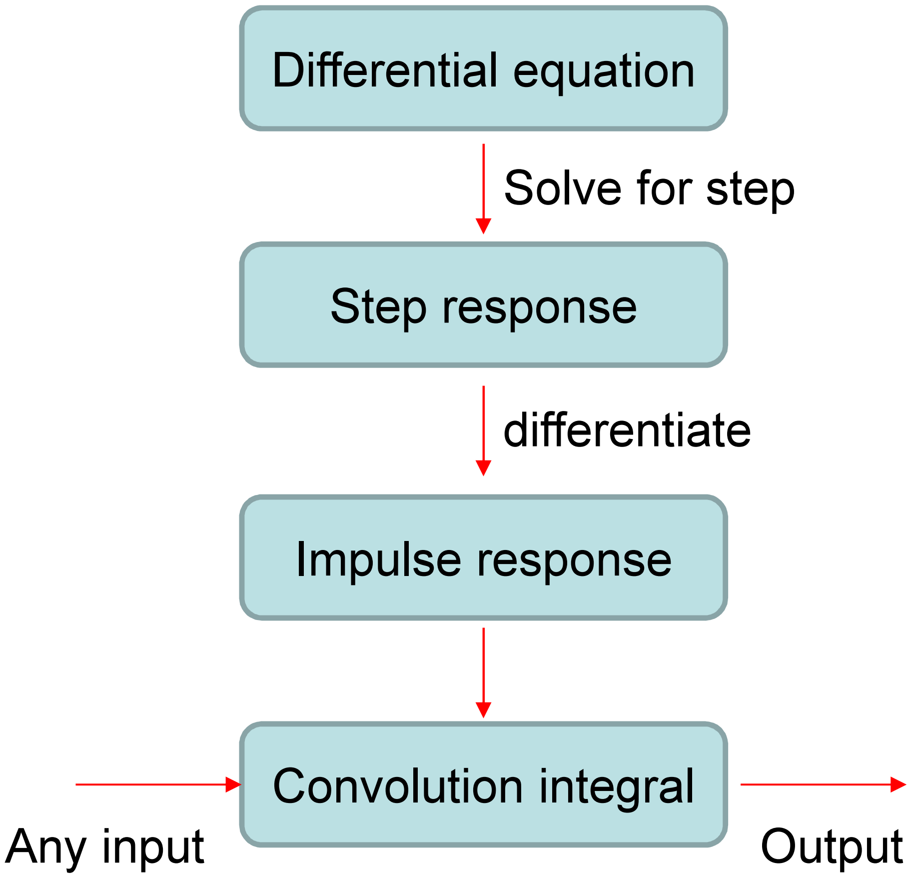
SYSTEMS Pt. 2
Modelling systems in the frequency domain

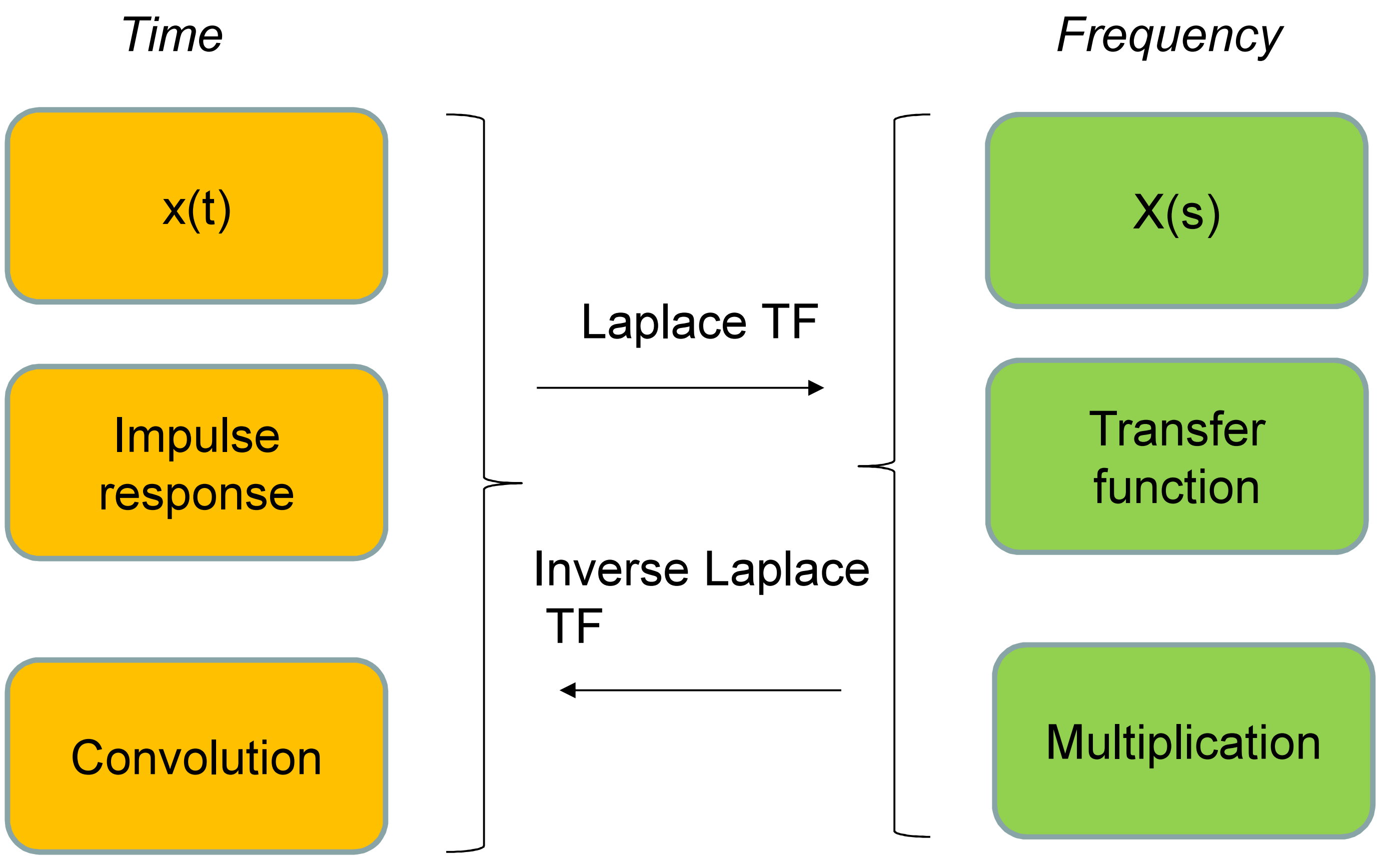
Solving in the time domain

In the last lecture we looked at the time domain approach to modelling LTI systems



- Solving in the time domain involves several steps.
- Typically the differential equation of the system is solved for a step input and this is then differentiated to determine the impulse response.
- Then the impulse response is used in the convolution integral, along with the input signal, to determine the output signal

Solving in the frequency domain



What do our systems look like?

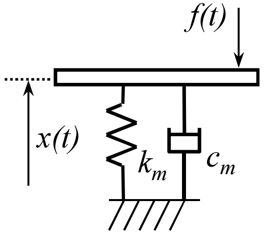
- Our systems need to conform to the LTI approximation.
  - Many mechanical systems
  - Many electrical systems
  - *Not so many acoustic/aerodynamic systems*

- *Consider:*
  - $y=x^2$
  - $y=mx + c$

Do these describe LTI systems? What could we do?

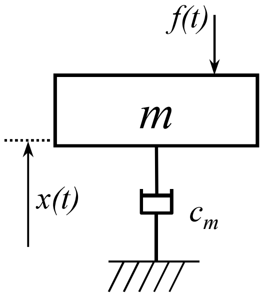
What do our systems look like?

From inspection we see that we need to sum forces:



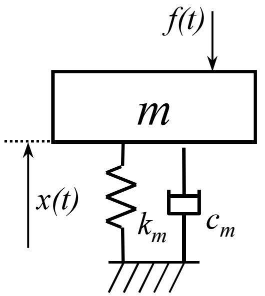
Spring with damping

$$f(t) = k_m x + c_m \frac{dx}{dt}$$



Mass with damping

$$f(t) = c_m \frac{dx}{dt} + m \frac{d^2x}{dt^2}$$

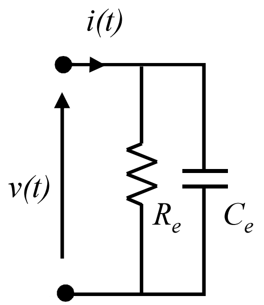


Mass and spring with damping

$$f(t) = m \frac{d^2x}{dt^2} + c_m \frac{dx}{dt} + k_m x$$

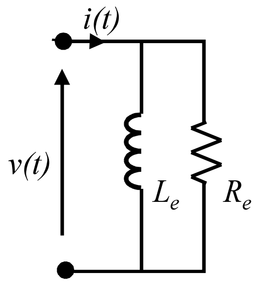
What do our systems look like?

In these cases we sum currents:



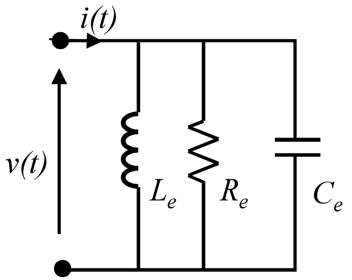
Capacitor with resistor

$$i(t) = \frac{1}{R_e}v + C_e\frac{dv}{dt}$$



Inductor with resistor

$$i(t) = \frac{1}{R_e}v + \frac{1}{L_e}\int vdt$$



Capacitor, inductor with damping

$$i(t) = C_e\frac{dv}{dt} + \frac{1}{R_e}v + \frac{1}{L_e}\int vdt$$

System order

- The majority of the systems we will look at can be described by ODE's
- The 'order' of a system describes the highest power to which the differential operator is raised, i.e.
  - $\frac{dx}{dt} + nx = y$ is a 1st order system
  - $\frac{d^2x}{dt^2} = y$ is a 2nd order system
  - $\frac{d^nx}{dt^n} + \ldots + \frac{d^2x}{dt^2} + \frac{dx}{dt} + nx = y$ is an nth order system
- In general, each order comes from an irreducible energy storage element

Describing in the frequency domain

- We have discussed how the impulse response of a system can be transformed into the frequency domain, via Laplace, to create the transfer function and how that reduces convolution to multiplication.
- However, if we still have to first derive the impulse response from the differential equations describing our system, then there is still significant effort.
- The answer is to convert the differential equation into the frequency domain and these can then be rearranged algebraically to find the transfer function.

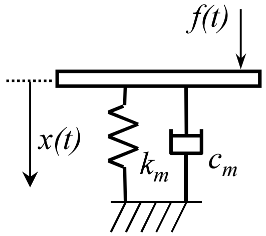
- Laplace transforms are normally performed by using tables of known transform pairs.
- In our case the two that are most useful are;

$$s = \frac{d}{dt} \quad \text{and for higher orders; } s^n = \frac{d^n}{dt^n}$$

$$\frac{1}{s} = \int x\,dt \quad \text{and for higher orders; } \frac{1}{s^n}$$

- This is the same as solving ODE’s using *differential operators*

Examples



Spring with damping

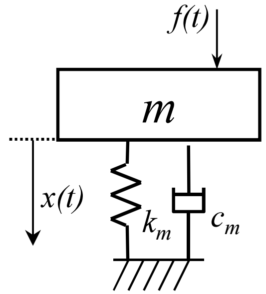
$$f(t) = c_m \frac{dx}{dt} + k_m x \quad \Rightarrow \quad F(s) = c_m s X(s) + k_m X(s)$$

$$\frac{F(s)}{X(s)} = c_m s + k_m$$

Now, by def.: Output = Transfer function x Input, So;

Transfer function $= \dfrac{output}{input} = \dfrac{X(s)}{F(s)} = \dfrac{1}{c_m s + k_m}$

Examples



Mass and spring with damping

$$f(t) = m\frac{d^2x}{dt^2} + c_m\frac{dx}{dt} + k_m x \quad \Rightarrow \quad F(s) = ms^2X(s) + c_m sX(s) + k_m X(s)$$

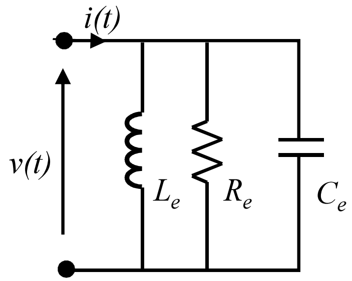
$$\frac{F(s)}{X(s)} = ms^2 + c_m s + k_m$$

Transfer function $= \dfrac{output}{input} = \dfrac{X(s)}{F(s)} = \dfrac{1}{ms^2 + c_m s + k_m}$

What about if the input/outputs were swapped?

TF $= \dfrac{output}{input} = \dfrac{F(s)}{X(s)} = ms^2 + c_m s + k_m$

Examples



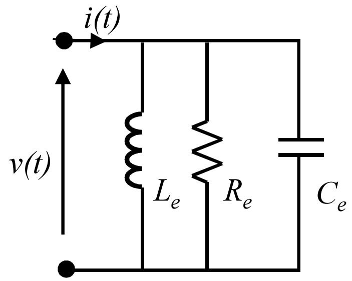
Capacitor, inductor with damping

$$i(t) = C_e \frac{dv}{dt} + \frac{1}{R_e} v + \frac{1}{L_e} \int v dt \quad \Rightarrow \quad I(s) = C_e s V(s) + \frac{1}{R_e} V(s) + \frac{1}{L_e s} V(s)$$

$$\frac{I(s)}{V(s)} = C_e s + \frac{1}{R_e} + \frac{1}{L_e s}$$

With electrical circuits we don't normally follow this approach……..

Examples



Capacitor, inductor with damping

$$\frac{I}{V} = \frac{1}{Z_p}$$

From Impedance form of Ohm's law

$$Z_p = \frac{1}{(\frac{1}{Z_L} + \frac{1}{Z_R} + \frac{1}{Z_C})}$$

Rule to sum parallel impedances

$$Z_L = j\omega L \qquad Z_R = R \qquad Z_c = \frac{1}{j\omega C}$$

Impedances of individual components

$$Z_p = \frac{1}{(\frac{1}{j\omega L_e} + \frac{1}{R_e} + j\omega C_e)}$$

$$let\ s = j\omega$$

$$\frac{I(s)}{V(s)} = C_e s + \frac{1}{R_e} + \frac{1}{L_e s}$$

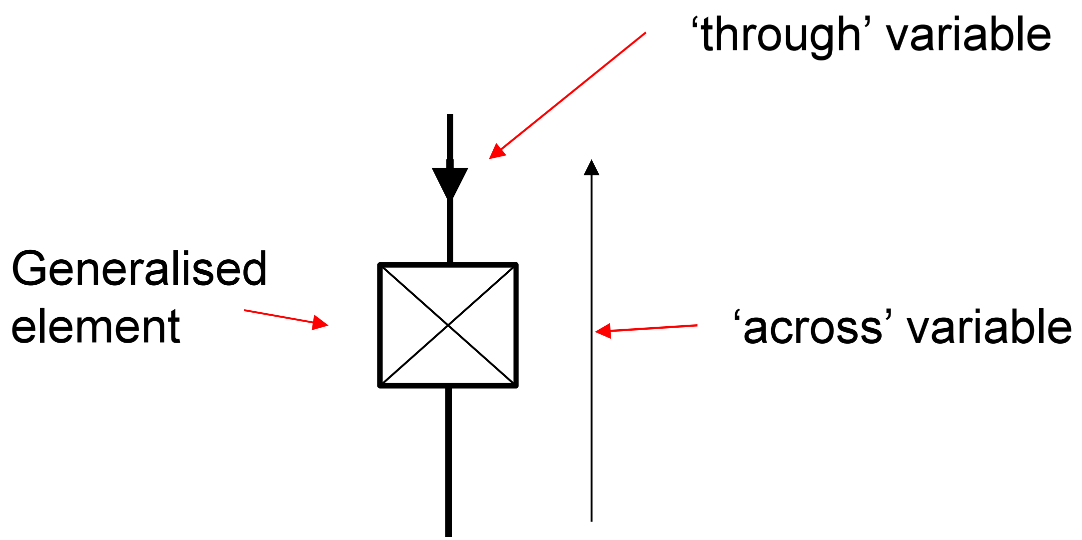
Impedance modelling

- Once you are familiar with working with impedances – just remembering that series impedances add and for parallel impedances it is the reciprocal of the reciprocals added is i.e.:

$$Z_{Total\_Series} = Z_1 + Z_2 + \cdots Z_3 \qquad Z_{Total\_Parrallel} = \frac{1}{(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots \frac{1}{Z_n})}$$

- - then it is much quicker to solve more complex systems. We have applied our Laplace transform at a component level and then easily solve the system using algebraic manipulation.
- A TF is the ratio of two complex qualities, defined as output/input. But impedance is a little more constrained as it the ratio of voltage/current. However even if our TF is not a voltage/current ratio we can still use impedances to derive the TF we want.
- We can apply the same thinking to mechanical systems: The mechanical equivalent is called mechanical impedance.

Impedance modelling



If we describe a components as having a ‘through’ and ‘across’ variables, then impedance describes the ratio of *across/through*; admittance describes the ratio of *through/across,* or the inverse of impedance

When described in this way the system diagram itself becomes a tool in solving the system – we can group elements and give them a combined impedance

Impedance modelling

- The complex ratio of ***Volts to Amps*** is called ***impedance*** and measured in ***ohms***

- The complex ratio of ***Amps to Volts*** is called ***admittance*** and measured in ***siemens***

- The mechanical equivalents are called ***'mechanical impedance***' and '***mechanical admittance'*** i.e.

- The complex ratio of ***Force to Velocity*** is called ***mech. impedance*** and measured in ***ohms***

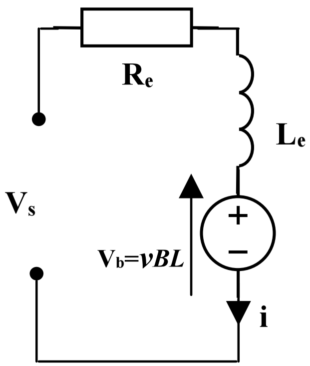
- The complex ratio of ***Velocity to Force*** is called ***mech. admittance*** (or '***mobility'***) and measured in ***siemens***

Electro-mechanical analogies

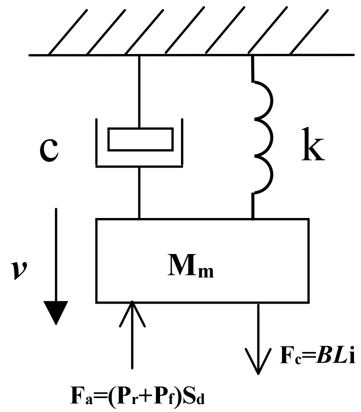
- Describing mechanical systems with impedances can help us solve mechanical systems
  - We tend to view this as electrical techniques applied to mechanical systems, because although the maths is the same, the tools for circuit reduction and manipulation have already been developed for circuits.
  - Circuit theory is just a ‘tool box’ for frequency domain problems; the frequency domain here is an approach to solve differential equations.
- But where electro-mechanical analogies are really useful is for dealing with combined systems i.e. a system that we want to model which has both mechanical and electrical components.

Modelling **-** Loudspeaker example

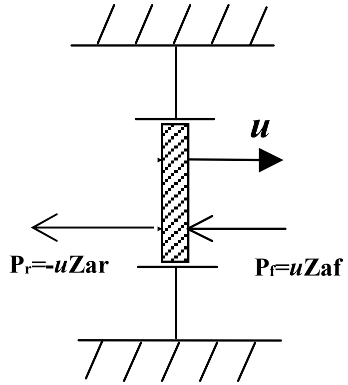
Electrical



Mechanical



Acoustical



R_e = Resistance.
L_e = Inductance.
e_b = Back emf.
i = Coil current.
e_s = Input signal.
BL = Tesla/metre product.

c = Mechanical loss.
k = Suspension stiffness
m = Moving mass.
v = Cone velocity.
F_c = Force from voice coil.
F_a = Force reflected back through cone.
S_d = Cone surface area.

u = Volume velocity.
P = Pressure created on speaker cone from volume velocity
Z_{ar} = Radiation impedance acting on rear of cone.
Z_{af} = Radiation impedance acting on front of cone.

Loudspeaker represented in 3 domains

Modelling - Impedance analogy

- Comparing the expression for a mechanical system:

$$f = M\frac{dv}{dt} + Cv + K\int vdt$$

Variables;
Force, f = voltage, e
Velocity, v = current, i

Parameters;
Mass, M = inductance, L
Damping, C = resistance, R
Compliance, 1/K = Capacitance, C

- With this expression of an electrical system:

$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt$$

- Results in the 'impedance analogy'

The trouble is we know from the physics of the voice coil transducer that a current in the electrical domain produces a force in the mechanical domain! We cant draw a circuit to represent this analogy (that is easily understandable with real electrical components).

Modelling - mobility analogy

- Comparing the expression for a mechanical system:

$$f = M\frac{dv}{dt} + Cv + K\int vdt$$

Variables;
Force, f = current, i
Velocity, v = voltage, e

Parameters;
Mass, M = Capacitance, C
Damping, C = conductance, 1/R
Compliance, 1/K = Inductance, L

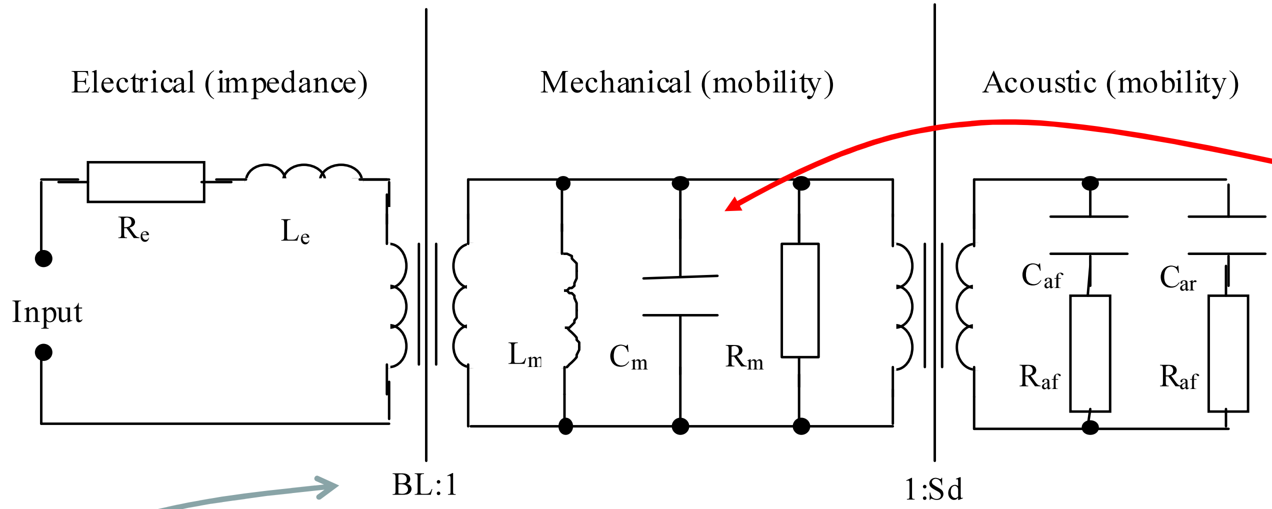
- With this expression of an electrical system:

$$i = C\frac{de}{dt} + \frac{1}{R}e + \frac{1}{L}\int edt$$

- Results in the 'mobility analogy'

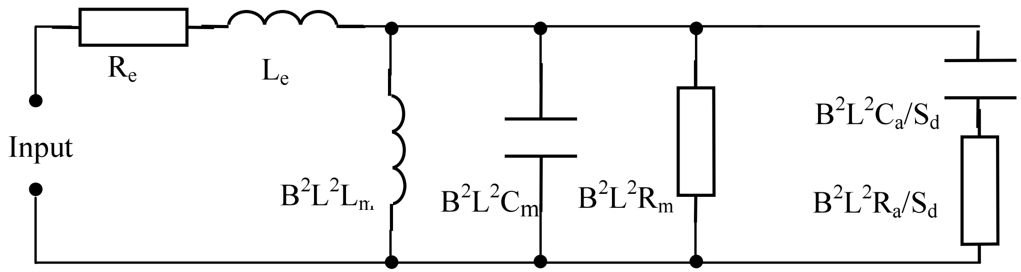
This works – but we have been forced to use impedance in one domain and admittance in the other! By convention we use admittance in the mechanical domain.

Modelling **-** Loudspeaker example



Force = through variable
Velocity = across variab

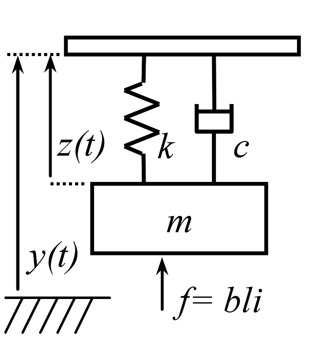
As an electrical circuit

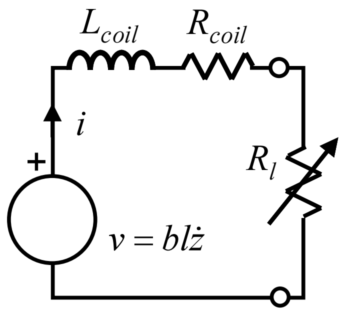


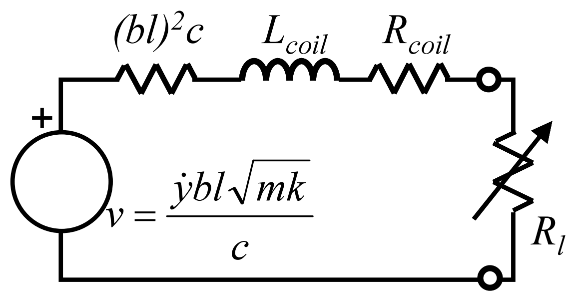
What's this?

As a simplified electrical circuit

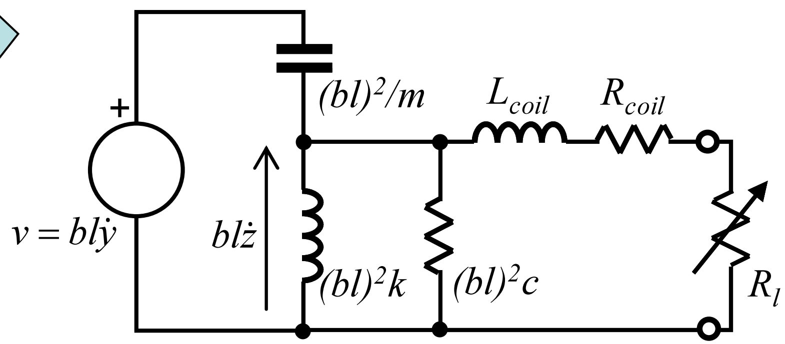
Example: electrically damped base-excited mass-spring system







1) Use analogies to describe an equivalent circuit



2) Use circuit manipulation to simplify problem and ease solution

Summary

- Once described in the frequency domain it is easier to manipulate expressions describing a system;
  - Either convert the differential expression derived from force or velocity balance approach or
  - Use impedances to describe directly
- System diagrams can be used to simplify the problem – often this is more intuitive than reducing the equations (although you should be aware that sometimes the opposite it true and the maths gives you an equivalence that is not obvious)
- We typically think of analogies as the representation of components in the electrical domain – this is where most techniques have been refined.
- The transducer will often determine the modelling approach for multi-domain systems