Lecture 25: Partial Fractions

- Last few lectures
  - the meaning of integration as area under the curve
  - indefinite integrals and definite integrals
  - integration by substitution: $\int_a^b u'(x)f(u(x))dx = \int_{x=a}^{x=b} f(u)du$
  - integrating piecewise functions by chopping into different bits
  - improper integrals: take limits, integrate and evaluate
  - integration by parts
- This time
  - how to integrate $p(x)/q(x)$ where p and q are polynomials
  - last lecture on integration!
- Next week
  - Partial differentiation

A trick for integrating rational functions

- Most integrals of simple functions don't have closed form expressions, e.g. can't integrate (directly)

$$\int_0^1 \frac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3}\,\mathrm{d}\,x$$

- But can always use linear property:

$$\int [f(x) + g(x)]\,\mathrm{d}\,x = \int f(x)\,\mathrm{d}\,x + \int g(x)\,\mathrm{d}\,x$$

- So how can we find f_1, f_2, f_3 etc. so that

$$f(x) = \frac{p(x)}{q(x)} = f_1(x) + f_2(x) + f_3(x) + \ldots?$$

- Answer: use partial fractions ...

Partial fractions: the idea

- given $f(x) = \frac{p(x)}{q(x)}$ let's try to simplify
- Q. what is degree (highest power) of $p(x)$ and $q(x)$?
- If deg $p(x) <$ deg $q(x)$ then factorise $q(x)$:

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x - x_1)(x - x_2)\ldots(x - x_n)}$$

and find constants A_1, A_2, $A_3, \ldots A_n$ such that

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x - x_1)(x - x_2)\ldots(x - x_n)} = \frac{A_1}{(x - x_1)} + \frac{A_2}{(x - x_2)} + \ldots \frac{A_n}{(x - x_n)}$$

- then $\int \frac{p(x)}{q(x)}\,\mathrm{d}\,x = A_1 \ln|x - x_1| + A_2 \ln|x - x_2| + \ldots A_n \ln|x - x_n|$
- but how to find the constants $A_1, \ldots A_n$?

A simple method: the cover-up rule

for finding the coefficients A_i (doesn't work in every case)

- Example Express $\frac{2x+1}{x^2-4x+3}$ in partial fractions
- Step 1: Factorise: $x^2 - 4x + 3 = (x - 1)(x - 3)$, and write

$$\frac{2x+1}{(x-1)(x-3)} = \frac{A_1}{x-1} + \frac{A_2}{x-3}$$

- Step 2a: for A_1 evaluate LHS when $x = 1$, covering up the factor $(x - 1)$:
  $\frac{2\cdot1+1}{(1-3)} = A_1 \quad \Rightarrow A_1 = -\frac{3}{2}$
- Step 2b: evaluate LHS when $x = 3$, covering up the factor $(x - 3)$:
  $\frac{2\cdot3+1}{(3-1)} = A_2 \quad \Rightarrow A_2 = \frac{7}{2}$
- Hence

$$\frac{2x + 1}{x^2 - 4x + 3} = \frac{(7/2)}{x - 3} - \frac{(3/2)}{x - 1}$$

- Final step Check!
  - Either: expand out:

$$\frac{(7/2)}{x-3}-\frac{(3/2)}{x-1}=\frac{1}{2}\frac{7(x-1)-3(x-3)}{x^2-4x+3}=\frac{2x+1}{x^2-4x+2}$$

  - Or, evaluate for some fixed x. E.g. $x=2$:

$$\frac{(7/2)}{2-3}-\frac{(3/2)}{2-1}=-(7/2)-(3/2)=-5=\frac{4+1}{1-4+2}$$

- Exercises: Express as partial fractions:

1. $\dfrac{x}{x^2+5x+4}$
2. $\dfrac{1}{(x+2)(x-5)(x-1)}$

Why does cover-up method work?

- Return to example:

$$\frac{2x+1}{(x-1)(x-3)}=\frac{A_1}{x-1}+\frac{A_2}{x-3}$$

multiply both sides by $(x-1)(x-3)$:

$$(2x+1)=A_1(x-3)+A_2(x-1)$$

- evaluating at $x=1$, A_2-term disappears and we get:
  $(2x+1)=A_1(x-3)$, which is $\frac{2x+1}{x-3}=A_1$,
  which we evaluate at $x=1$ to get: $A_1=-3/2$.
- similarly, evaluating at $x=3$ eliminates A_1-term & is like covering-up the $(x-3)$ term: $A_2=\frac{2x+1}{x-1}|_{x=3}=(7/2)$

Complication I: repeated roots

- What if we had for example $\frac{1}{(x-2)^2(x-1)}$?
- In general, I should write this as

$$\frac{1}{(x-2)^2(x-1)}=\frac{A_1}{(x-2)}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-1)}$$

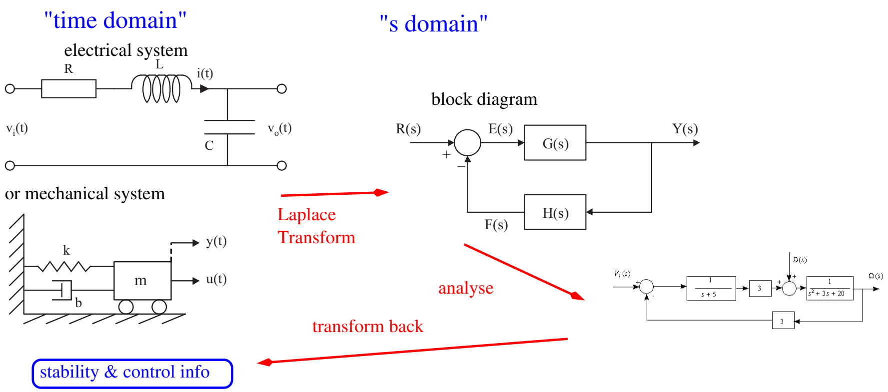
- But how do we evaluate the coefficients A_i in this case?
  $\Rightarrow$ general method (always works!):
  - step 1. multiply by denominator of LHS
  - step 2. use cover-up method where you can
  - step 3. compare coefficients of powers of x
  - step 4. solve any simultaneous equations that arise

Example

- How to express $\frac{1}{(x-2)^2(x-1)}$ as partial fractions ?
- Write $\frac{1}{(x-2)^2(x-1)}=\frac{A_1}{(x-2)}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-1)}$
- Step 1. Multiply by the denominator of LHS
  $1=A_1(x-2)(x-1)+A_2(x-1)+A_3(x-2)^2$
- Step 2. Use cover-up method for $x=1$ and $x=2$:
  $1=0+0+A_3(-1)^2, \qquad 1=0+A_2$
  Hence $A_3=1, \quad A_2=1$
- Step 3. compute coefficient of x^2:
  $0=A_1+0+A_3, \qquad \Rightarrow A_1=-1$
- Finally, multiply out to check:
  $1=-(x-2)(x-1)+(x-1)+(x-2)^2=$
  $-(x^2-3x+2)+(x-1)+(x^2-4x+4)=1$

Engineering HOT SPOT I:

Laplace Transforms (used lots in control engineering)



Engineering HOT SPOT II:

- Laplace Transform
$$F(s) = \int e^{-st} f(t)\, \mathrm{d}\, t$$
- Get a transfer function
$$\text{ouput}(s) = Y(s) \times \text{input}(s)$$
- $Y(s)$ usually a rational function e.g.
$$\frac{s+1}{s^3 + 2s^2 + s + 1}$$
- need to use partial fractions to transform back

Complication II: complex roots

- What if there are complex roots in the denominator?
$$\text{e.g. } \frac{2x+1}{x^2 - 2x + 3}, \qquad \text{roots are } 2 \pm \sqrt{2}j$$
- Could write this as $\frac{A_1}{x-(2+\sqrt{2}j)} + \frac{A_2}{x-(2-\sqrt{2}j))}$

  not ideal as get complex logs if we integrate
- instead, best to leave this term unsimplified.
- But how to integrate? A. Complete the square & use $\arctan$

$$\begin{aligned}\int \frac{2x+1}{x^2-2x+3}\,\mathrm{d}\,x &= \int \frac{2x-2}{x^2-2x+3}\,\mathrm{d}\,x + \int \frac{3}{x^2-2x+3}\,\mathrm{d}\,x \\ &= \ln|x^2-2x+3| + 3\int \frac{\mathrm{d}\,x}{(x-1)^2+2} \\ &= \ln|x^2-2x+3| + \frac{3}{\sqrt{2}}\arctan\left(\frac{x-1}{\sqrt{2}}\right)\end{aligned}$$

Example with real and complex roots

Illustrating the general method

- To express $\frac{5x}{(x^2+x+1)(x-2)}$ as partial fractions
- Write $\qquad \frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$
- Step 1. multiply both sides by the denominator of LHS
$$5x = (Ax+B)(x-2) + C(x^2+x+1)$$
- Step 2. find C using the cover-up method with $x = 2$
$$5 \times 2 = C(4+2+1) \quad \Rightarrow C = \frac{10}{7}$$
- Step 3. equate coefficients of x^2 and 1:
$$0 = A + C, \quad 0 = -2B + C, \quad \Rightarrow A = -\frac{10}{7},\ B = \frac{5}{7}$$
- Hence $\qquad \frac{5x}{(x^2+x+1)(x-2)} = \frac{5-10x}{7(x^2+x+1)} + \frac{10}{7(x-2)}$

Exercise

- Express as partial fractions

$$\frac{1}{(x^2+9)(x+1)}$$

- Hence evaluate

$$\int_0^1 \frac{1}{(x^2+9)(x+1)}\,\mathrm{d}\,x$$

Complication III: improper fractions

- what if we have $\int \frac{p(x)}{q(x)}\,\mathrm{d}\,x$ where deg$(p(x)) >$ deg$(q(x))$?
- Example: $\frac{3x^4+2x^3-5x^2+6x-7}{x^2-2x+3}$
- Use polynomial division

$$\begin{aligned}(3x^4+2x^3-5x^2+6x-7) &= (x^2-2x+3)(3x^2+\dots\\ \text{match coef. of } x^3: &= (x^2-2x+3)(3x^2+8x+\dots\\ \text{match coef. of } x^2: &= (x^2-2x+3)(3x^2+8x+2)+\text{Remainder}\end{aligned}$$

- Remainder $=$ linear & const. terms of (LHS $-$ RHS):
  Remainder $= (6x-7)-20x+6 = -14x-13.$
- Hence (expand out, or evaluate at $x=1$ to check!)

$$\frac{3x^4+2x^3-5x^2+6x-7}{x^2-2x+3} = 3x^2+8x+2-\frac{14x+13}{x^2-2x+3}$$

- Note that x^2-2x+3 has complex roots (its irreducible)
  i.e. we can't simplify further
- Exercise: Use the answer to this example to evaluate

$$\int_0^1 \frac{3x^4+2x^3-5x^2+6x-7}{x^2-2x+3}\,\mathrm{d}\,x$$

- Hint: remember useful formula

$$\int \frac{1}{(x-b)^2+a^2}\,\mathrm{d}\,x = \frac{1}{a}\arctan\left(\frac{x-b}{a}\right)$$

Homework

- Read sect. 2.5.1 of James, including Summary of method table at end
  Also read sect. 8.8.1
- Do James exercises
- 4th edition:
  - Ex. 2.5.2 Q.40(a),(c),(e), Q.41(a),(c),(e) 8.8.2 Q.98(a),(c),(e),(g),(i),(k)
- 5th edition:
  - Ex. 2.5.2 Q.40(a),(c),(e), Q.41(a),(c),(e) 8.8.9 Q.117(a),(c),(e),(g),(i),(k)
- GOOD NEWS by popular demand all solutions to James now available
- Don't forget to keep up with qmp.bris.ac.uk