StM3 – Composite Laminate Analysis

Lecture 5 : Classical Laminate Theory / ABD-matrix

2019/2020

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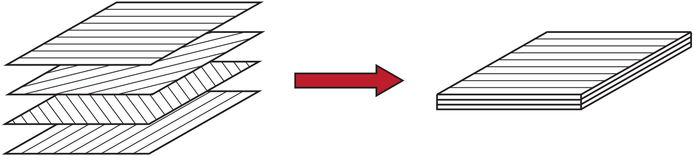
Lecture Outline

- continue Classical Laminate Theory
- derive ABD-matrix for composite laminate plate
- interpret ABD coupling terms
- explore standard laminate lay-ups
  (symmetric, balanced, quasi-isotropic, anti-symmetric)

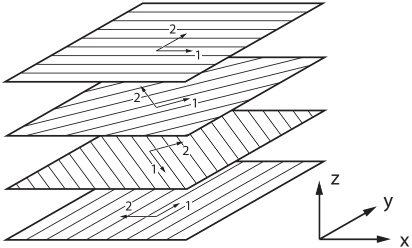
Composite Laminate – I

composite laminate:
combine multiple composite layers into single structural element



composite lay-up: **tailored structural properties**

Composite Laminate – II

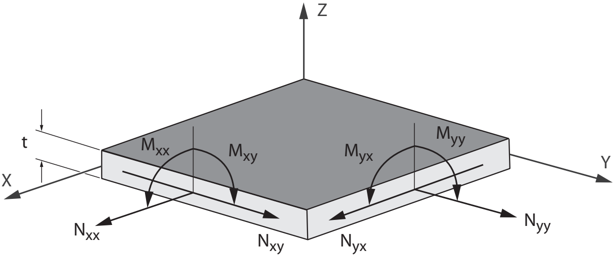


lay-up notation:

- ply numbering bottom-up in positive z-direction
- angle θ defines orientation of ply material axes (CCW)
- shorthand notation (θ_n, $\pm/\mp$, $[\theta]_S$)

Plate Model

plate model: in-plane and out-of-plane loads and deformations



formulate structural model for composite laminate plate

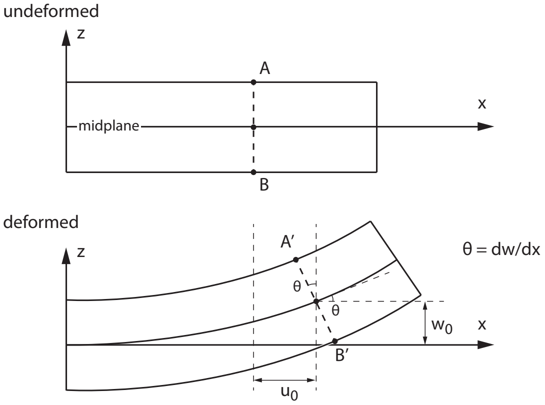
Laminate Structural Properties: ABD-matrix

analysis procedure:

1. calculate strains across laminate cross-section from $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\kappa}$
2. calculate ply stresses using $\bar{\boldsymbol{Q}}_k$ for each ply k
3. calculate stress resultants: N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}
4. formulate **ABD-matrix**

Strain and Deformations – I

Kirchoff-Love plate model: cross-sections remain plane



Strain and Deformations – II

strains along cross-section AB:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

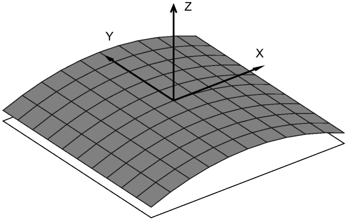
midplane strains and curvatures, and distance z from midplane

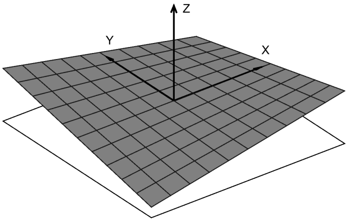
Strain and Deformations – III

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = - \begin{bmatrix} \partial^2 w_0 / \partial x^2 \\ \partial^2 w_0 / \partial y^2 \\ 2\partial^2 w_0 / \partial x \partial y \end{bmatrix}$$

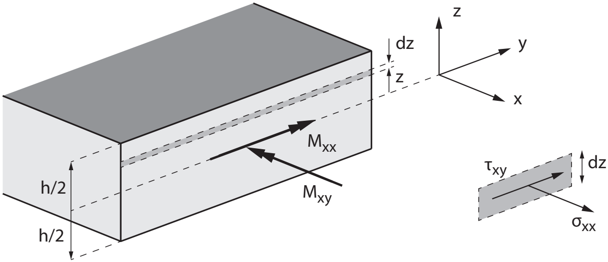
curvature κ_{xx} : rate of change of slope $\partial w / \partial x$ w.r.t. x

twist κ_{xy} : rate of change of slope $\partial w / \partial x$ w.r.t. y





Stress & Stress Resultants – I

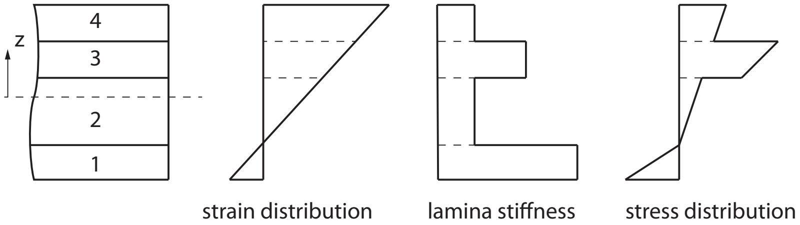


$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz \qquad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz \qquad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy}\, dz$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z\, dz \qquad M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z\, dz \qquad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz$$

Stress & Stress Resultants – II

laminate **stress distribution is discontinuous** across cross-section



note: difference between midplane and neutral axis

Stress & Stress Resultants – III

laminate stress resultant:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} dz \;\; = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k dz \right)$$

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} z\, dz = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k z\, dz \right)$$

integrate over each ply k, and sum over n laminate plies

Stress & Stress Resultants – IV

laminate with n plies and total thickness h

ply numbering $k = 1 \ldots n$ in positive z-direction

location h_k of top of each ply measured from *geometric* mid-plane

ply thickness: $t_k = h_k - h_{k-1}$

Stress & Stress Resultants – V

each ply k is assumed to be in plane stress: $\bar{\boldsymbol{Q}}_k$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

where:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} \qquad z \in [h_{k-1}, h_k]$$

Stress & Stress Resultants – VI

substitute strains to find the in-plane stress resultants:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} dz \right)$$

$$\ldots + \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} z\, dz \right)$$

Stress & Stress Resultants – VII

$$\begin{aligned} \text{midplane strains and curvatures} &: \quad \text{constant across all plies } n \\ \text{stiffness matrix } \bar{\boldsymbol{Q}}_k &: \quad \text{constant across ply } k \end{aligned}$$

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} dz \right) \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix}$$

$$\ldots + \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z \, dz \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

Stress & Stress Resultants – VIII

in-plane stress resultants are expressed as:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

extensional stiffness matrix $\boldsymbol{A}$ and coupling stiffness matrix $\boldsymbol{B}$

Stress & Stress Resultants – IX

similarly, for out-of-plane stress resultants

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z dz \right) \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix}$$

$$\ldots + \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z^2 \, dz \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

Stress & Stress Resultants – X

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

coupling stiffness matrix $\boldsymbol{B}$ and bending stiffness matrix $\boldsymbol{D}$

ABD-matrix – I

mechanics of composite laminate is described by ABD-matrix

$$
\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]
$$

$$
\left[\begin{array}{c} \boldsymbol{N} \\ \boldsymbol{M} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{array}\right] \tag{3.2}
$$

ABD-matrix – II

A is the **extensional stiffness matrix**

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \tag{3.3}$$

B is the **coupling stiffness matrix**

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \tag{3.4}$$

D is the **bending stiffness matrix**

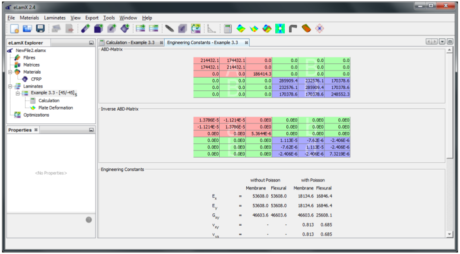
$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \tag{3.5}$$

Composite Laminate Software

repetitive calculations: software!

- eLamX^2 software program

  `https://tu-dresden.de/ing/maschinenwesen/ilr/lft/elamx2/elamx`



- write your own in MATLAB!

ABD Matrix - Coupling terms – I

	ε_{xx}^0	ε_{yy}^0	γ_{xy}^0	κ_{xx}	κ_{yy}	κ_{xy}
N_{xx}	A_{11}	A_{12}	A_{16}	B_{11}	B_{12}	B_{16}
N_{yy}	A_{12}	A_{22}	A_{26}	B_{12}	B_{22}	B_{26}
N_{xy}	A_{16}	A_{26}	A_{66}	B_{16}	B_{26}	B_{66}
M_{xx}	B_{11}	B_{12}	B_{16}	D_{11}	D_{12}	D_{16}
M_{yy}	B_{12}	B_{22}	B_{26}	D_{12}	D_{22}	D_{26}
M_{xy}	B_{16}	B_{26}	B_{66}	D_{16}	D_{26}	D_{66}

ABD Matrix - Coupling terms – II

extension-shear coupling

transverse strain
i.e. Poisson's ratio

extension-bend coupling

$$\left[\begin{array}{ccc|ccc}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
\hline
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{array}\right]$$

extension-twist coupling
shear-bend coupling

shear-twist coupling

transverse curvature
i.e. Poisson's ratio

in-plane/out-of-plane coupling

bend-twist coupling

ABD Matrix - Alternative Formulation – I

alternative formulation (Nettles, 1994) provides further insight

extensional stiffness matrix

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1})$$

$$= \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k t_k$$

where t_k is thickness of k-th ply

ABD Matrix - Alternative Formulation – II

coupling stiffness matrix

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^2 - h_{k-1}^2\right)$$

$$= \sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k - h_{k-1}\right)\frac{\left(h_k + h_{k-1}\right)}{2}$$

$$= \sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k t_k \bar{z}_k$$

where $\bar{z}_k$ is distance to middle of k-th ply

ABD Matrix - Alternative Formulation – III

bending stiffness matrix

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^3 - h_{k-1}^3\right)$$

$$= \ldots$$

$$= \sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(\frac{t_k^3}{12} + t_k\bar{z}_k^2\right)$$

second moment of area of ply : $t_k^3/12$

parallel axis effect from midplane : $t_k\bar{z}_k^2$

Example 3.2: Single Layer Isotropic – I

single layer of **isotropic** material (E, ν) of thickness t

reduced stiffness matrix:

$$\bar{Q}_{ij} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

ply positions: $h_1 = t/2$ and $h_0 = -t/2$

Example 3.2: Single Layer Isotropic – II

ABD matrices:

$$\boldsymbol{A} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) = \bar{Q}_{ij} (h_1 - h_0) = \bar{Q}_{ij} t$$

$$= \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Example 3.2: Single Layer Isotropic – III

$$\boldsymbol{B} = \frac{1}{2}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^2 - h_{k-1}^2\right) = \frac{1}{2}\bar{Q}_{ij}\left(h_1^2 - h_0^2\right)$$

$$= 0$$

$$\boldsymbol{D} = \frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^3 - h_{k-1}^3\right) = \frac{1}{3}\bar{Q}_{ij}\left(h_1^3 - h_0^3\right) = \frac{1}{3}\bar{Q}_{ij}\frac{t^3}{4}$$

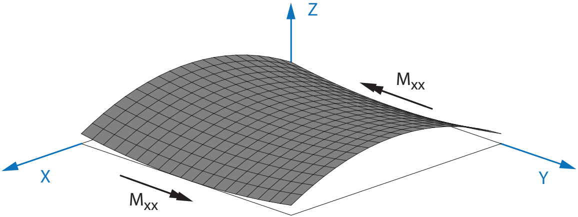
$$= \frac{Et^3}{12\left(1-\nu^2\right)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Example 3.2: Single Layer Isotropic – IV

no coupling between bending and extension: $B_{ij} = 0$

no bend-twist coupling: $D_{16} = D_{26} = 0$

coupling between bending in two directions: D_{12}



results in anticlastic curvature for applied bending moment M_{xx}

StM3 – Composite Laminate Analysis

Lecture 6 : Classical Laminate Theory / ABD-matrix

2019/2020

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Admin

next week:

example class, working through a representative exam problem

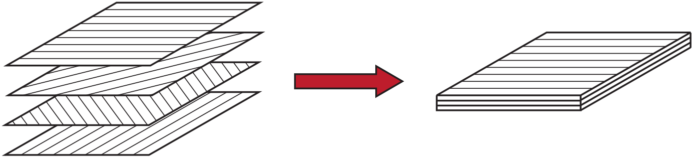
- **please revise ABD matrix calculations**!
- a set of revision slides will be available on Blackboard
- look through Example 3.4 and Example 3.5

Lecture Outline

- describe features of common laminates
  (balanced, symmetric, anti-symmetric, quasi-isotropic)
- invert ABD matrix (solve for applied loads)
- calculate laminate strength for first-ply failure
- appreciate ABD coordinate transformations
- laminate engineering constants
- thermal effects in composite analysis

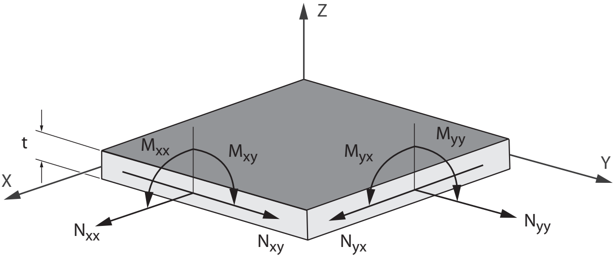
Composite Laminate – I

composite laminate:
combine multiple composite layers into structural element



Composite Laminate – II

plate model: in-plane and out-of-plane loads and deformations



structural model for composite laminate plate: **ABD-matrix**

ABD-matrix – I

mechanics of composite laminate described by ABD-matrix

$$\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]$$

$$\left[\begin{array}{c} \boldsymbol{N} \\ \boldsymbol{M} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{array}\right] \tag{3.2}$$

ABD-matrix – II

A is **extensional stiffness matrix**

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right) \tag{3.3}$$

B is **coupling stiffness matrix**

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \tag{3.4}$$

D is **bending stiffness matrix**

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \tag{3.5}$$

ABD-matrix – III

	ε_{xx}^0	ε_{yy}^0	γ_{xy}^0	κ_{xx}	κ_{yy}	κ_{xy}
N_{xx}	A_{11}	A_{12}	A_{16}	B_{11}	B_{12}	B_{16}
N_{yy}	A_{12}	A_{22}	A_{26}	B_{12}	B_{22}	B_{26}
N_{xy}	A_{16}	A_{26}	A_{66}	B_{16}	B_{26}	B_{66}
M_{xx}	B_{11}	B_{12}	B_{16}	D_{11}	D_{12}	D_{16}
M_{yy}	B_{12}	B_{22}	B_{26}	D_{12}	D_{22}	D_{26}
M_{xy}	B_{16}	B_{26}	B_{66}	D_{16}	D_{26}	D_{66}

ABD-matrix – IV

extension-shear coupling

transverse strain
i.e. Poisson's ratio

extension-bend coupling

$$\left|\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right|$$

extension-twist coupling
shear-bend coupling

shear-twist coupling

transverse curvature
i.e. Poisson's ratio

in-plane/out-of-plane coupling

bend-twist coupling

Appendix A: Bistable Deployable Boom – I

composites enable structural behaviour not possible with isotropic materials, such as bistability

open-section deployable boom can be coiled up and stowed compactly; stable in both deployed and coiled configuration

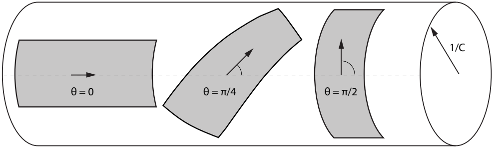
Appendix A: Bistable Deployable Boom – II

Guest and Pellegrino (2006) assume only inextensional deformation

change in curvature $\Delta\boldsymbol{\kappa}$ from initial configuration:

$$\Delta \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = \frac{C}{2} \begin{bmatrix} 1 - \cos 2\theta \\ \cos 2\theta + 1 \\ 2 \sin 2\theta \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{R} \\ 0 \end{bmatrix}$$

where $1/R$ is original transverse curvature of the boom, C is imposed curvature at angle θ to original axes



Appendix A: Bistable Deployable Boom – III

non-dimensionalised elastic strain energy:

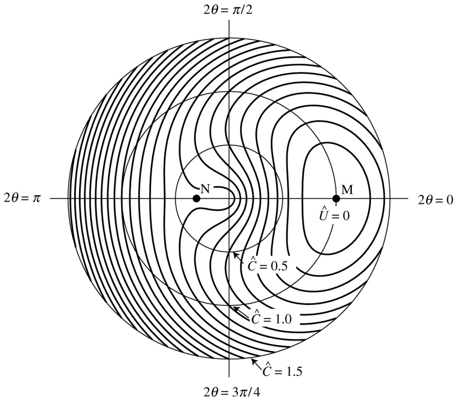
$$\hat{U} = \hat{\boldsymbol{\kappa}}^T \hat{\boldsymbol{D}} \hat{\boldsymbol{\kappa}}$$

equilibrium position: $\partial U / \partial \theta = \partial U / \partial C = 0$

by inspection determine if equilibrium position is local energy maximum (unstable) or energy minimum (stable)

Appendix A: Bistable Deployable Boom – IV

<u>isotropic</u>: stable (M) and unstable (N) equilibrium position

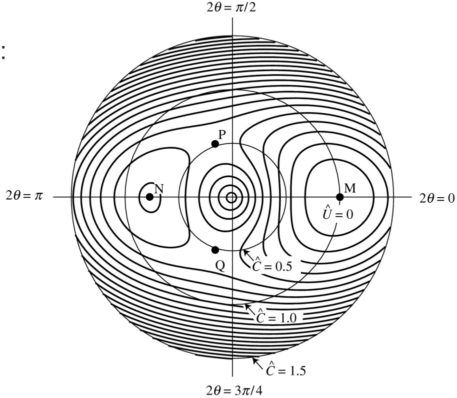


Appendix A: Bistable Deployable Boom – V

antisymmetric $[\pm 45/0/\pm 45]$:

- $B_{16} \neq 0,\ B_{26} \neq 0$
- $D_{16} = D_{26} = 0$

bistable: second stable equilibrium point N; unstable P and Q

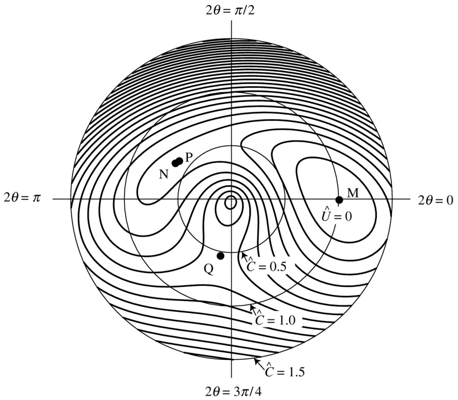


Appendix A: Bistable Deployable Boom – VI

symmetric $[\pm 45/0/\mp 45]$:

- $\boldsymbol{B} = 0$
- $D_{16} \neq 0,\ D_{26} \neq 0$

bistable, but symmetry
broken due to
bend-twist coupling



Appendix A: Bistable Deployable Boom – VII

minimise bend/twist coupling (D_{16}, D_{26}) to avoid helical coiling

solution: used woven fabric in $[\pm 45/0/\pm 45]$ lay-up

two 'plies' in ± 45 fabric lie at 'same' distance from midplane, eliminating bend-twist coupling; $\boldsymbol{B} = 0$ due to lay-up symmetry

---

unusual mechanical properties due to composite laminate lay-up!

Laminate Compliance Equations – I

compliance matrix:
calculate strains and curvatures for known applied loads

$$\begin{bmatrix} \varepsilon^0 \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}$$

general inversion $(ABD)^{-1}$ done by inverting sub-matrices

---

MATLAB

```
>> inv(ABD)
```

Laminate Compliance Equations – II

from ABD-matrix

$$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon}^0 + \boldsymbol{B}\boldsymbol{\kappa}$$

find midplane strains:

$$\boldsymbol{\varepsilon}^0 = \boldsymbol{A}^{-1}\boldsymbol{N} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{\kappa}$$

substitute $\boldsymbol{\varepsilon}^0$ back into ABD

$$\begin{aligned}
\boldsymbol{M} &= \boldsymbol{B}\boldsymbol{\varepsilon}^0 + \boldsymbol{D}\boldsymbol{\kappa} \\
&= \boldsymbol{B}\left(\boldsymbol{A}^{-1}\boldsymbol{N} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{\kappa}\right) + \boldsymbol{D}\boldsymbol{\kappa} \\
&= \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{N} + \left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)\boldsymbol{\kappa}
\end{aligned}$$

Laminate Compliance Equations – III

combine into **partially inverted** form:

$$\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^* & \boldsymbol{B}^* \\ \boldsymbol{C}^* & \boldsymbol{D}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{bmatrix}$$

$$\boldsymbol{A}^* = \boldsymbol{A}^{-1} \qquad \boldsymbol{B}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}$$

$$\boldsymbol{C}^* = \boldsymbol{B}\boldsymbol{A}^{-1} \qquad \boldsymbol{D}^* = \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}$$

$\boldsymbol{D}^*$ is known as reduced bending stiffness

Laminate Compliance Equations – IV

continue inversion; solve for curvatures:

$$\boldsymbol{\kappa} = \boldsymbol{D}^{*^{-1}}\boldsymbol{M} - \boldsymbol{D}^{*^{-1}}\boldsymbol{C}^{*}\boldsymbol{N}$$

and substitute into expression for mid-plane strain:

$$\begin{aligned}\boldsymbol{\varepsilon}^{0} &= \boldsymbol{A}^{*}\boldsymbol{N} + \boldsymbol{B}^{*}\boldsymbol{\kappa} \\ &= \left(\boldsymbol{A}^{*} - \boldsymbol{B}^{*}\boldsymbol{D}^{*^{-1}}\boldsymbol{C}^{*}\right)\boldsymbol{N} + \boldsymbol{B}^{*}\boldsymbol{D}^{*^{-1}}\boldsymbol{M}\end{aligned}$$

Laminate Compliance Equations – V

combine into **compliance matrix**:

$$\begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}$$

$$\boldsymbol{a} = \boldsymbol{A}^* - \boldsymbol{B}^* \boldsymbol{D}^{*^{-1}} \boldsymbol{C}^* \qquad \boldsymbol{b} = \boldsymbol{B}^* \boldsymbol{D}^{*^{-1}}$$

$$\boldsymbol{c} = -\boldsymbol{D}^{*^{-1}} \boldsymbol{C}^* = \boldsymbol{b}^T \qquad \boldsymbol{d} = \boldsymbol{D}^{*^{-1}}$$

MATLAB

```
>> inv(ABD)
```

Laminate Compliance Equations – VI

calculate strains and curvatures for known applied loads

$$\begin{bmatrix} \varepsilon^0 \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}$$

once ε^0 and $\boldsymbol{\kappa}$ are known, stress $\boldsymbol{\sigma}_k$ in plies can be calculated

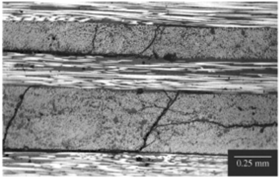
$$\boldsymbol{\sigma}_k = \bar{\boldsymbol{Q}}_k \, \varepsilon_k$$

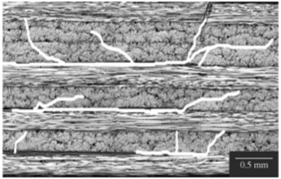
where

$$\varepsilon_k = \boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa} \qquad z \in [h_{k-1}, h_k]$$

Laminate Strength

laminate strength is complex topic: ply failure, inter-laminar stresses (ply delamination), crack propagation, *etc.*



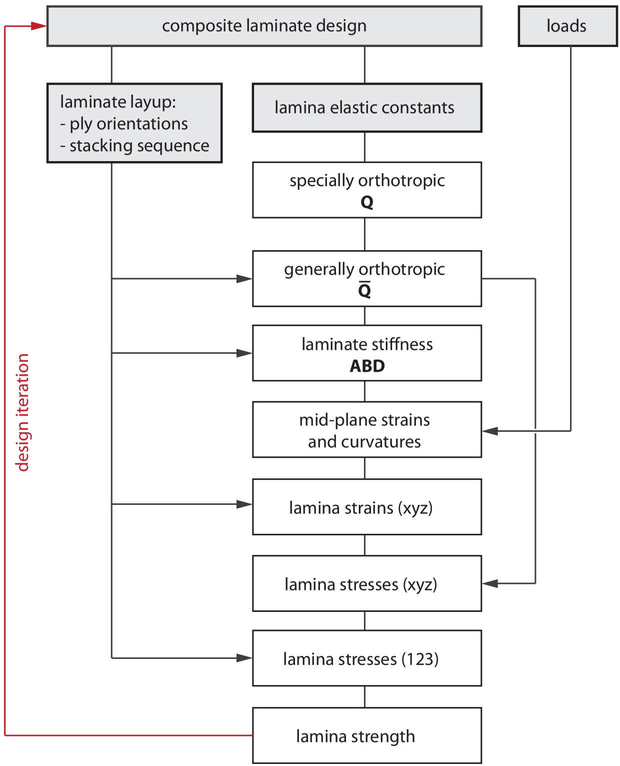


fourth year option: Advanced Composite Analysis (ACA)

---

simplified laminate failure criterion: **first ply failure**

ply failure criteria: maximum stress, Tsai-Hill, *etc.*


composite laminate design
loads
laminate layup:
- ply orientations
- stacking sequence
lamina elastic constants
specially orthotropic
Q
generally orthotropic
$\bar{\mathbf{Q}}$
laminate stiffness
ABD
mid-plane strains
and curvatures
lamina strains (xyz)
lamina stresses (xyz)
lamina stresses (123)
lamina strength
design iteration

Laminate Design – I

laminate design: tailoring of ABD-matrix enables novel structural functionality

morphing and multistable structures, aeroelastic tailoring, *etc.*

---

not complete freedom in tailoring coupling terms, as will be highlighted with *lamination parameters* (non-examinable; H4)

Laminate Design – II

explore and appreciate link between lay-up and ABD matrix

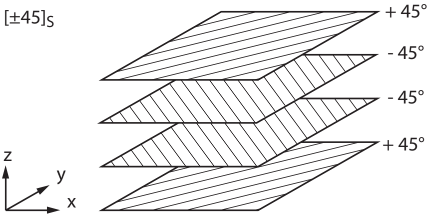
---

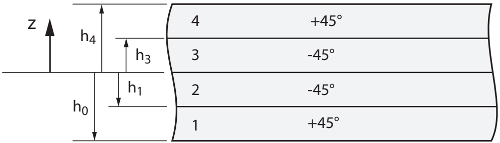
conventional laminate lay-ups:

- balanced & symmetric laminate
- anti-symmetric angle-ply laminate
- quasi-isotropic laminate

Example 3.3: Symmetric & Balanced – I

symmetric and *balanced* laminate, *e.g.* $[\pm 45]_S$





Example 3.3: Symmetric & Balanced – II

symmetric laminate:
symmetric about mid-surface, in geometry and material properties

---

all bending-extension coupling cancels:

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij} \right)_k \left(h_k^2 - h_{k-1}^2 \right) = 0$$

commonly used for simplicity of analysis, but also manufacturability (bending/warping after cure due to ΔT; Section 3.7)

Example 3.3: Symmetric & Balanced – III

balanced laminate:
each $+\theta$ ply is balanced by a $-\theta$ ply of equal stiffness and thickness

---

shear-extension components cancel out: $A_{16} = A_{26} = 0$

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right) = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$$

Example 3.3: Symmetric & Balanced – IV

terms $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are *odd* functions of θ

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta$$

therefore:

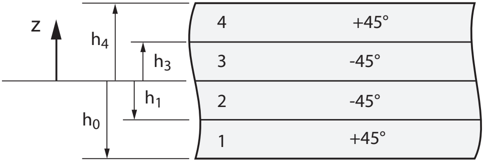
$$\bar{Q}_{16}(-\theta) = -\bar{Q}_{16}(\theta) \qquad \bar{Q}_{26}(-\theta) = -\bar{Q}_{26}(\theta)$$

thus, $A_{16} = A_{26} = 0$ for a balanced laminate

Example 3.3: Symmetric & Balanced – V

numerical example: symmetric & balanced $[\pm 45]_S$, plies 1 mm,
$E_{11} = 180$ GPa, $E_{22} = 10$ GPa, $G_{12} = 5$ GPa, and $\nu_{12} = 0.2$

(i) sketch laminate cross-section



Example 3.3: Symmetric & Balanced – VI

(ii) calculate laminate reduced stiffness components

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \qquad Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} \qquad Q_{66} = G_{12}$$

$$\boldsymbol{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \begin{bmatrix} 180.4 & 2 & 0 \\ 2 & 10.02 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ GPa}$$

Example 3.3: Symmetric & Balanced – VII

(iii) calculate generally orthotropic ply stiffness matrix $\bar{\boldsymbol{Q}}$

$$\bar{\boldsymbol{Q}}_{(45^\circ)} = \begin{bmatrix} 53.6 & 43.6 & 42.6 \\ 43.6 & 53.6 & 42.6 \\ 42.6 & 42.6 & 46.6 \end{bmatrix}$$

$$\bar{\boldsymbol{Q}}_{(-45^\circ)} = \begin{bmatrix} 53.6 & 43.6 & -42.6 \\ 43.6 & 53.6 & -42.6 \\ -42.6 & -42.6 & 46.6 \end{bmatrix}$$

Example 3.3: Symmetric & Balanced – VIII

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta$$

$$\bar{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta$$

Example 3.3: Symmetric & Balanced – IX

(iv) from lay-up and $\bar{\boldsymbol{Q}}$ matrices, construct ABD matrix:

$$\boldsymbol{A} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1})$$

$$= \bar{\boldsymbol{Q}}_{45} (-1 - (-2)) + \bar{\boldsymbol{Q}}_{-45} (0 - (-1)) \dots$$

$$+ \bar{\boldsymbol{Q}}_{-45} (1 - 0) + \bar{\boldsymbol{Q}}_{45} (2 - 1)$$

$$= \begin{bmatrix} 214 & 174 & 0 \\ 174 & 214 & 0 \\ 0 & 0 & 186 \end{bmatrix} \text{ GPa mm}$$

Example 3.3: Symmetric & Balanced – X

$$
\begin{aligned}
\boldsymbol{B} &= \frac{1}{2}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^2 - h_{k-1}^2\right) \\
&= \frac{1}{2}\bar{\boldsymbol{Q}}_{45}\left((-1)^2 - (-2)^2\right) + \frac{1}{2}\bar{\boldsymbol{Q}}_{-45}\left(0 - (-1)^2\right)\dots \\
&\qquad + \frac{1}{2}\bar{\boldsymbol{Q}}_{-45}\left(1^2 - 0\right) + \frac{1}{2}\bar{\boldsymbol{Q}}_{45}\left(2^2 - 1^2\right) \\
&= 0
\end{aligned}
$$

Example 3.3: Symmetric & Balanced – XI

$$
\begin{aligned}
\boldsymbol{D} &= \frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^3 - h_{k-1}^3\right) \\
&= \frac{1}{3}\bar{\boldsymbol{Q}}_{45}\left((-1)^3-(-2)^3\right)+\frac{1}{3}\bar{\boldsymbol{Q}}_{-45}\left(0-(-1)^3\right)\ldots \\
&\qquad\qquad +\frac{1}{3}\bar{\boldsymbol{Q}}_{-45}\left(1^3-0\right)+\frac{1}{3}\bar{\boldsymbol{Q}}_{45}\left(2^3-1^3\right) \\
&= \begin{bmatrix} 286 & 233 & 170 \\ 233 & 286 & 170 \\ 170 & 170 & 249 \end{bmatrix} \text{ GPa mm}^3
\end{aligned}
$$

---

example symmetric and balanced laminate has full $\boldsymbol{D}$ matrix

ABD Matrix Coordinate Transformation – I

ABD matrix coupling terms vary with coordinate system!

coordinate transformation (strain and curvature are tensors):

$$\boldsymbol{A}' = \boldsymbol{T}\boldsymbol{A}\boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}$$

$$\boldsymbol{B}' = \boldsymbol{T}\boldsymbol{B}\boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}$$

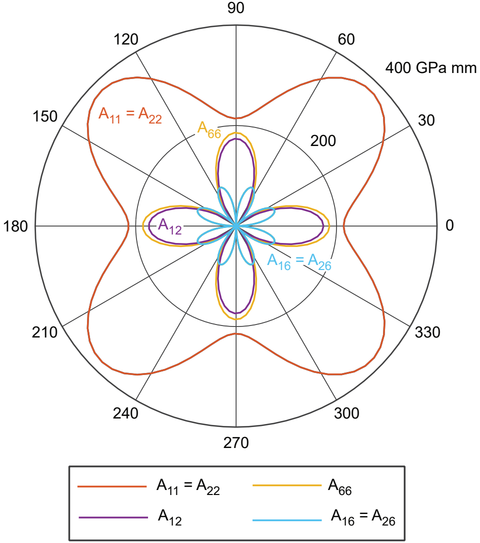
$$\boldsymbol{D}' = \boldsymbol{T}\boldsymbol{D}\boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}$$

recall: transformation matrix **T** and Reuter's matrix **R**

ABD Matrix Coordinate Transformation – II

<u>Example 3.3:</u>
balanced, symmetric laminate, $[\pm 45]_S$

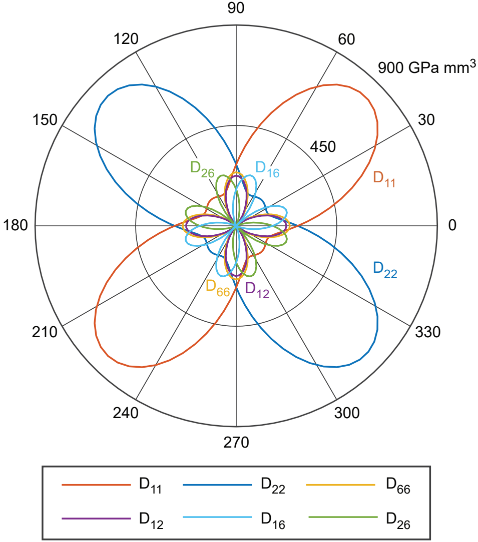
anisotropic in-plane properties



ABD Matrix Coordinate Transformation – III

anisotropic
out-of-plane properties

note: $B_{ij} = 0$ due to
lay-up symmetry



Laminate Engineering Constants – I

laminate **engineering constants** from compliance matrix

$$E_{xx} = \frac{1}{a_{11} h} \qquad E_{yy} = \frac{1}{a_{22} h}$$

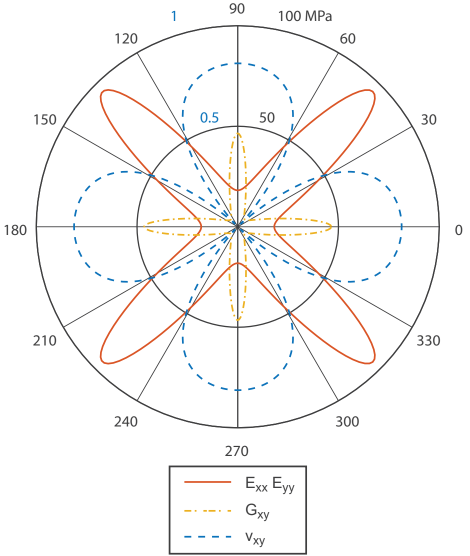
$$\nu_{xy} = -\frac{a_{12}}{a_{11}} \qquad G_{xy} = \frac{1}{a_{66} h}$$

averaged laminate stresses (σ_{xx}, σ_{yy}, τ_{xy}), divide in-plane stress resultants (N_{xx}, N_{yy}, N_{xy}) by laminate thickness h

Laminate Engineering Constants – II

Example 3.3:
balanced, symmetric laminate, $[\pm 45]_S$

anisotropic in-plane properties



Example 3.6: Quasi-Isotropic Laminate – I

quasi-isotropic laminate:
a combination of plies can achieve isotropic in-plane properties

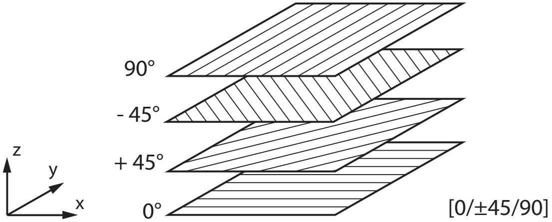
$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & (A_{11} - A_{12})/2 \end{bmatrix}$$

requires at least three axes of material symmetry (and thus plies);

plies of equal Q_{ij} and t oriented at equal angles π/n

Example 3.6: Quasi-Isotropic Laminate – II

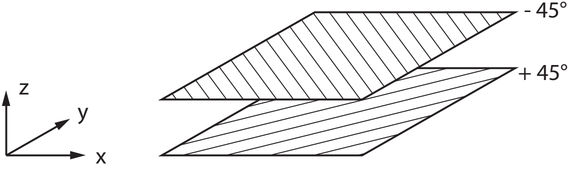
quasi-isotropic laminates: $[0/\pm 60]$ and $[0/\pm 45/90]$



in-plane isotropy only; bending-extension coupling matrix $B_{ij} \neq 0$, as well as non-zero bend-twist coupling (D_{16}, D_{26})

Example 3.4: Anti-Symmetric Laminate – I

anti-symmetric laminate: each $+\theta$ ply above midplane has $-\theta$ counterpart of equal thickness and stiffness below midplane



[±45]

anti-symmetric laminate is also balanced: $A_{16} = A_{26} = 0$

non-zero $\boldsymbol{B}$ matrix due to asymmetry

Example 3.4: Anti-Symmetric Laminate – II

consider bending stiffness matrix:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

- geometric components $\left(h_k^3 - h_{k-1}^3\right)$ same for layers of equal thickness and distance above/below the midplane
- recall that $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are *odd* functions of θ

thus, $D_{16} = D_{26} = 0$ for anti-symmetric laminates

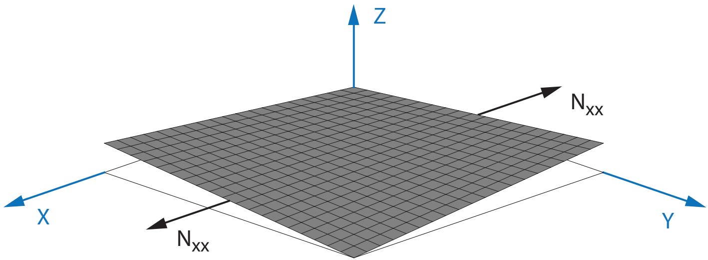
Example 3.4: Anti-Symmetric Laminate – III

<u>regular anti-symmetric angle-ply</u>:
all plies of equal thickness and stiffness, and single ply angle θ

$$
\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ \hline 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]
$$

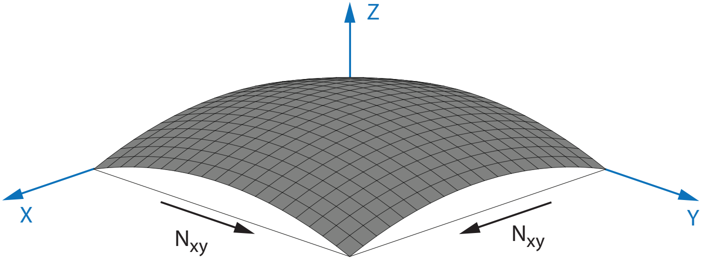
Example 3.4: Anti-Symmetric Laminate – IV

coupling of in-plane strain and out-of-plane twist (B_{16}, B_{26}):



Example 3.4: Anti-Symmetric Laminate – V

coupling of in-plane shear and out-of-plane bending (B_{16}, B_{26}):



Example 3.5: Bend-Twist Coupling – I

minimise/eliminate bend-twist coupling (D_{16}, D_{26}) of laminate

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij} \right)_k \left(h_k^3 - h_{k-1}^3 \right)$$

expressions for $\bar{Q}_{16}$ and $\bar{Q}_{26}$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$

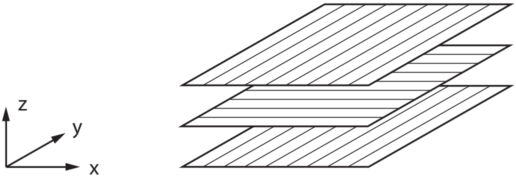
$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \sin\theta \cos^3\theta$$

Example 3.5: Bend-Twist Coupling – II

two options to achieve $D_{16} = D_{26} = 0$

- anti-symmetry: for every layer θ above the midplane, there is a layer with equal stiffness oriented at $-\theta$ below the midplane

- cross-ply laminate: all layers oriented at $\theta = 0^\circ$ and $\theta = 90^\circ$



Composite Thermal Analysis – I

composites cured at high temperatures (*e.g.* $T = 160^\circ$)

anisotropic coefficients of thermal expansion (CTE) α_{11}, α_{22}

$$\varepsilon_{11}^{T} = \alpha_{11}\Delta T$$

$$\varepsilon_{22}^{T} = \alpha_{22}\Delta T$$

CTEs in structural coordinate system:

$$\begin{pmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{pmatrix} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1} \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{pmatrix}$$

Composite Thermal Analysis – II

ply strains consists of mechanical and thermal components

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx}^M \\ \varepsilon_{yy}^M \\ \gamma_{xy}^M \end{pmatrix} + \begin{pmatrix} \varepsilon_{xx}^T \\ \varepsilon_{yy}^T \\ \gamma_{xy}^T \end{pmatrix}$$

$$= \begin{pmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} + \begin{pmatrix} \alpha_{xx}\Delta T \\ \alpha_{yy}\Delta T \\ \alpha_{xy}\Delta T \end{pmatrix}$$

Composite Thermal Analysis – III

inverted for ply stiffness equations

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \gamma_{xy} - \alpha_{xy}\Delta T \end{pmatrix}$$

combined into laminate:

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} - \begin{bmatrix} \boldsymbol{N}^T \\ \boldsymbol{M}^T \end{bmatrix}$$

Composite Thermal Analysis – IV

thermal loads and moments:

$$\left[\boldsymbol{N}^T\right] = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (\alpha)_k \left(h_k - h_{k-1}\right) \Delta T$$

$$\left[\boldsymbol{M}^T\right] = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (\alpha)_k \left(h_k^2 - h_{k-1}^2\right) \Delta T$$

thermal moments due to non-zero $\boldsymbol{B}$-matrix

calculate warping of flat plate due to cure - zero applied loads

Summary – I

- composite laminate: combine plies into structural element
- plate model: in-plane and out-of-plane loads and deformations
- mechanics of composite laminate plate: ABD-matrix
- intricate coupling between loads/deformations

Summary – II

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix}$$

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

Summary – III

- discussed features of common laminates
  (balanced, symmetric, quasi-isotropic, anti-symmetric)
- should be able to recognise these laminates!
- ABD coupling terms depend on coordinate system
- laminate engineering constants
- thermal effects in composites

Example Paper: Q4 – Q9 (+ Q10)

Summary – IV

summary of lecture series

- introduced Composite Laminate Analysis
- micromechanics:
  elastic properties of ply emerge from fibre/matrix constituents
- macromechanics of lamina:
  in-plane material model dependent on fibre angle θ
- macromechanics of laminate:
  combine laminate lay-up and orientations into ABD-matrix

Revision Objectives – I

Revision Objectives:

- describe assumptions underpinning Classical Laminate Theory;
- use the lamination notation for composite laminates (including shorthand);
- interpret the geometry of bending/twisting curvatures;
- recall the sign convention for applied loads/moments;

Revision Objectives – II

Revision Objectives:

- explain the derivation of the ABD-matrix;
- recall the equations for the ABD-matrix

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

- explain and interpret the ABD coupling terms;

Revision Objectives – III

Revision Objectives:

- recognise standard laminate types (balanced, symmetric, anti-symmetric, quasi-isotropic) and explain the link between the laminate lay-up and their corresponding ABD-matrices;
- calculate ABD-matrices from ply properties (E_{11}, E_{22}, G_{12}, ν_{12}) and laminate lay-up;
- calculate midplane strains and curvatures, and resulting lamina stresses and strains for applied loads (given the compliance matrix inversion);
- derive expressions for ABD-matrices in offset loading directions;
- calculate equivalent laminate engineering constants;
- check for laminate failure using first-ply failure criterion;
- calculate the thermal warping of a flat laminate plate;