Stress, Strain and Deformation
Buckling

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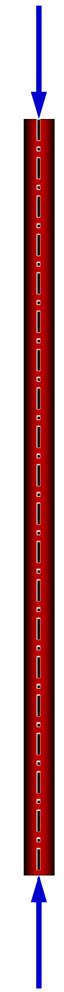
- Euler buckling theory
- End conditions
- Imperfections in geometry & end conditions
- Failure in bending

for struts

Failure of slender members

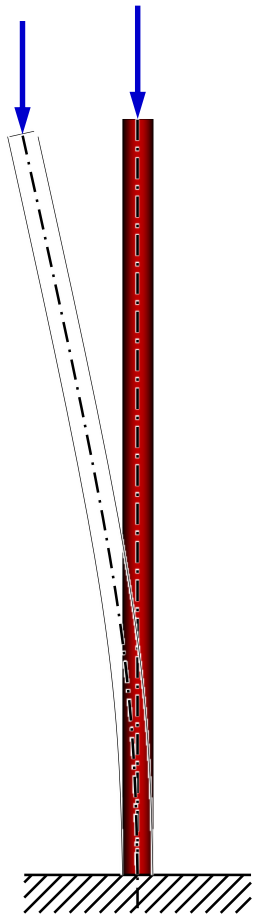
- Tension force
  - Failure by plastic yielding or fracture
- Compression force
  - Failure by buckling (before yielding or fracture)



Buckling

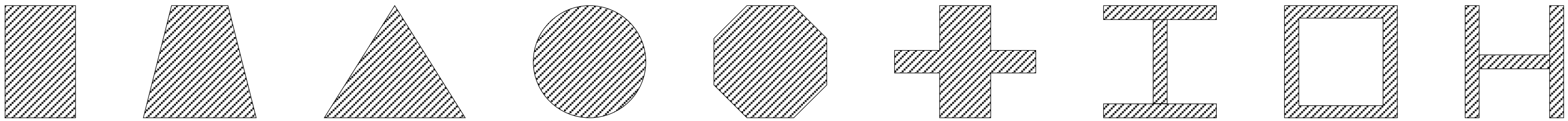
"Instability leading to increasing bending deflection"

- Depends on bending stiffness (EI) not strength
- Buckling can also lead to torsional deflection
  - Therefore may depend on torsional stiffness
  - But this will be considered later (StM2)



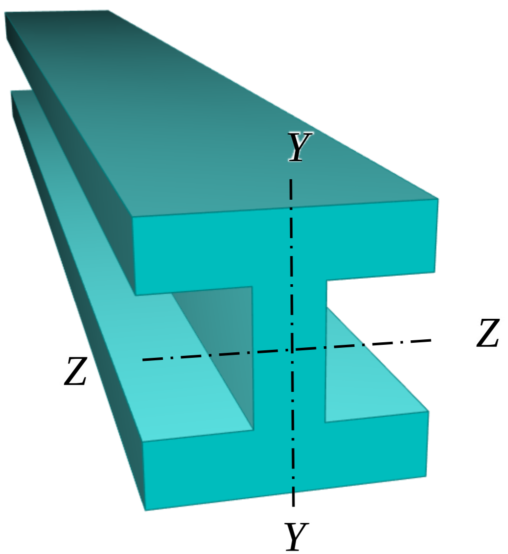
Cross-section dependency

- Buckling strength will depend on:
  - The shape of the cross-section (I)
  - The material modulus (E)



- Buckling will occur about the axis which has the least 2nd moment of area I

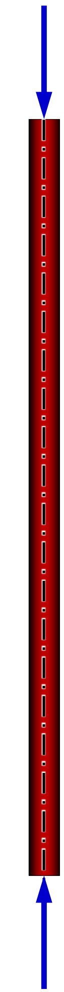
E.g. this strut would buckle about axis Y-Y

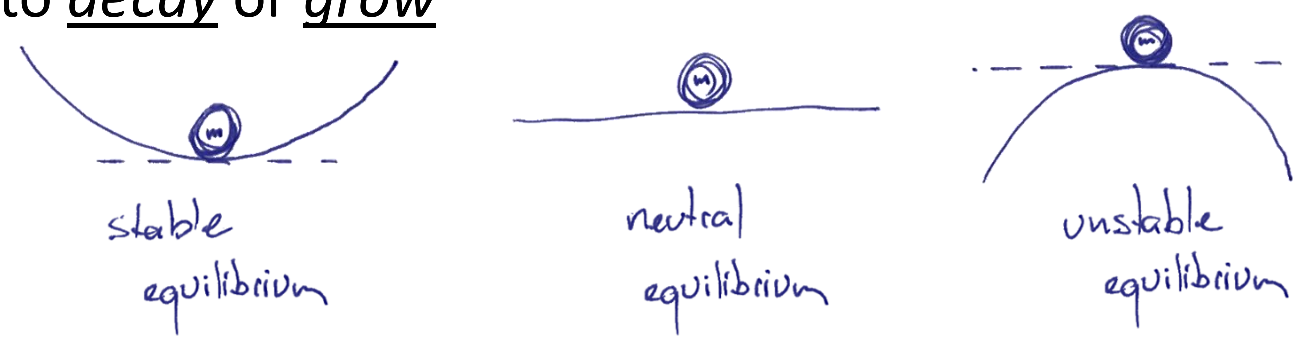


Classic (Euler) buckling theory

Assumptions:

- Linear elastic material response
- Initially perfectly straight strut
- Uniform cross-section
- Load applied along the centroidal axis
- Small deformations
- Pin-jointed ends
- Classic stability, *e.g.* consider whether a small disturbance tends to ***decay*** or ***grow***

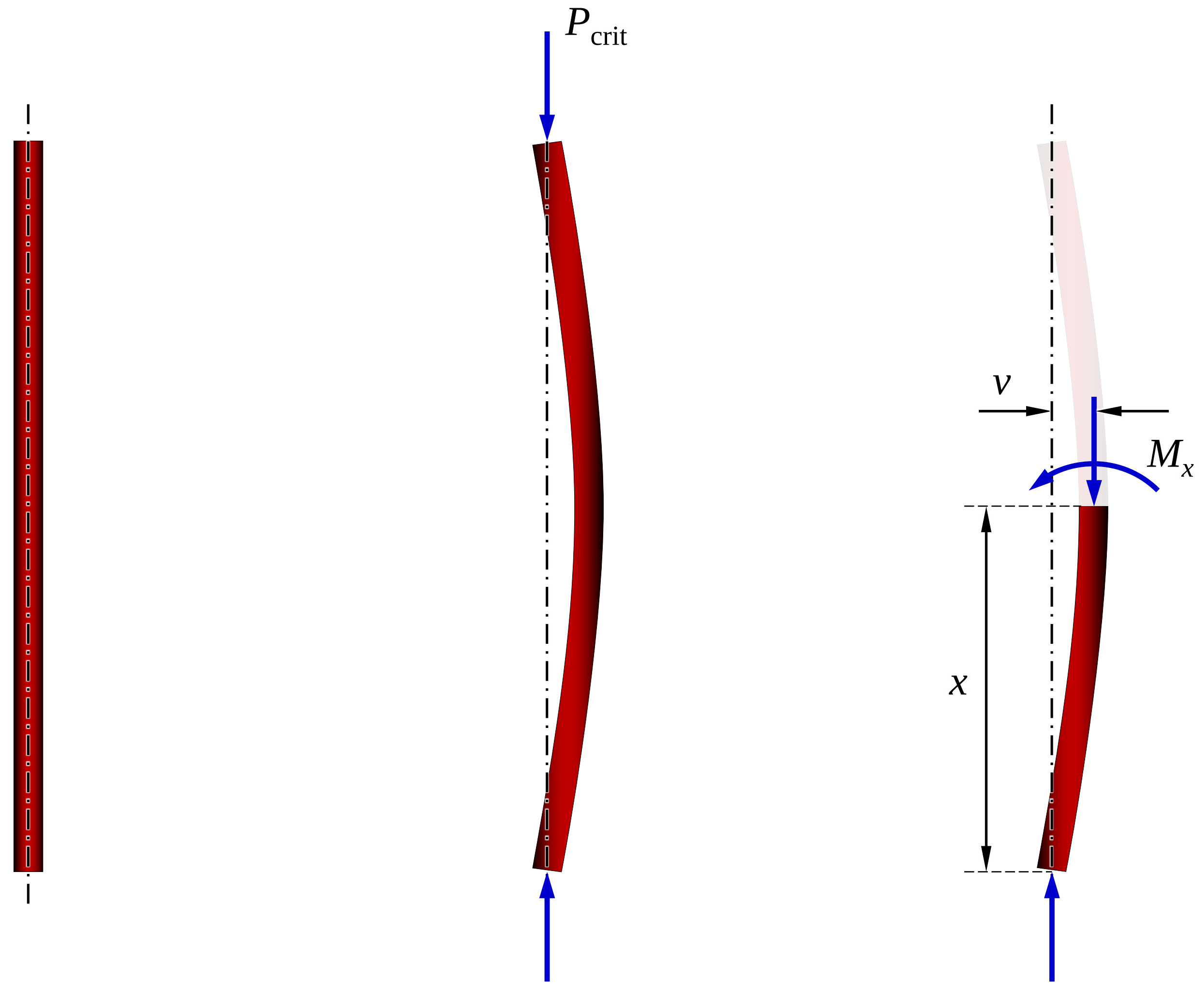




Critical buckling load, P_{crit}

"Load at which the strut is ***neutrally stable*** for small deflections"

- *I.e.* the strut is still in equilibrium in its deformed state



Deflected shape

- Remember: $v = f(x)$

- Using the 2nd order differential equation of bending for small deflections:

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = M$$

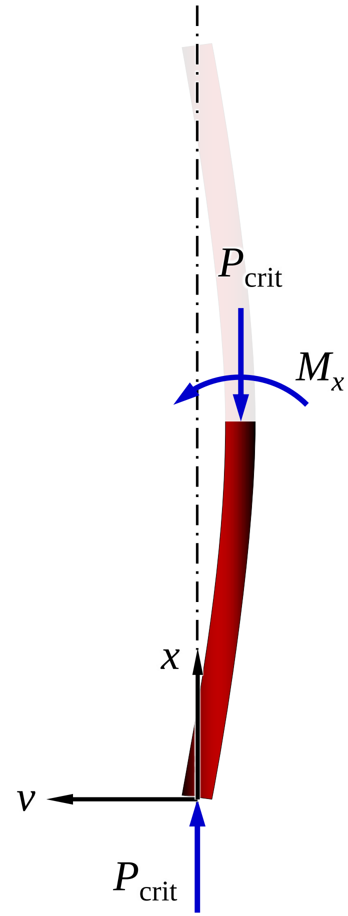
- Rotational equilibrium :

$$\sum M_{@x}^{CCW} = 0$$

$$M - (P_{\text{crit}})(-v) = 0$$

$$M = -P_{\text{crit}}\, v$$

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -P_{\text{crit}}\, v$$



Solution for P_{crit} (1/5)

- Rearranging it as a <u>linear differential equation</u>:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\,x^2} - \frac{P_{\text{crit}}}{EI} v = 0$$

- We can define a variable μ as:

$$\mu^2 = \frac{P_{\text{crit}}}{EI}$$

- So that:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\,x^2} - \mu^2\, v = 0$$

- Which has solution of the form:

$$v = A \sin \mu x + B \cos \mu x$$

- Where A and B are constants of integration

$$v = A\sin\mu x + B\cos\mu x \qquad (1)$$

$$\frac{dv}{dx} = A\mu\cos\mu x - B\mu\sin\mu x$$

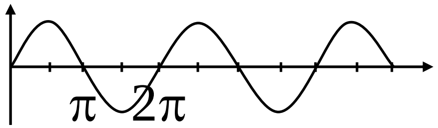
i.e. differentiating to get back to the differential equation of bending

$$\frac{d^2v}{dx^2} = -A\mu^2\sin\mu x - B\mu^2\cos\mu x \qquad (2)$$

So $\dfrac{d^2v}{dx^2} + \mu^2 v = 0$

$$\underbrace{(-A\mu^2\sin\mu x - B\mu^2\cos\mu x)}_{(2)} + \mu^2\underbrace{(A\sin\mu x + B\cos\mu x)}_{(1)} = 0$$

Solution for P_{crit} (1/5)



$$(-A\mu^2 \sin\mu x - B\mu^2 \cos\mu x) \quad + \mu^2(A \sin\mu x + B \cos\mu x) = 0$$

Apply boundary conditions to solve for constants A and B

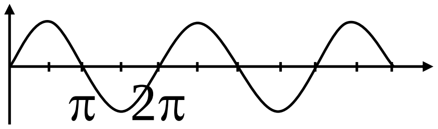
$@\ x = 0, v = 0 \Rightarrow B = 0 \Rightarrow v = A \sin\mu x$

$@\ x = L, v = 0 \Rightarrow A \sin\mu L = 0 \Rightarrow A = 0$ or $\sin\mu L = 0$

Either $A = 0$ gives zero deflection for all x – not relevant

Or $\sin\mu L = 0$ is possible for $\mu L = 0, \pi, 2\pi, 3\pi, \ldots$

Solution for P_{crit} (1/5)



Consider values of μL in turn:

For $\mu L = 0$, either $L = 0$ or $P_c = 0$, which is trivial

For $\mu L = \pi$, P_c = lowest non-zero value*
i.e. most critical* buckling strength

Substituting $\mu = \pi/L$ into : $\mu^2 = \dfrac{P_{\text{crit}}}{EI}$

Gives the "Euler buckling load" $\boxed{P_{\text{crit}} = \dfrac{\pi^2 EI}{L^2}}$

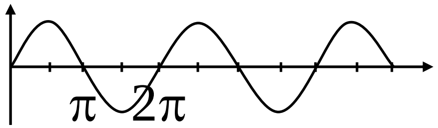
$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

- Note, P_{crit} depends on EI, *i.e.* 'bending rigidity'
- Therefore, buckling will take place about the axis with least 2nd moment of area, I_{min}

Also, note relationship to L^2

↳ Long struts have low P_{crit}
(many times lower than the load required to yield the strut)

Deflected shape



⑤ $v = A \sin \mu x$ where $\mu = \pi / L$
And $A = v_{max}$
$\Rightarrow$ $v = v_{max} \sin \dfrac{\pi x}{L}$

@ x =0 or L, v=0

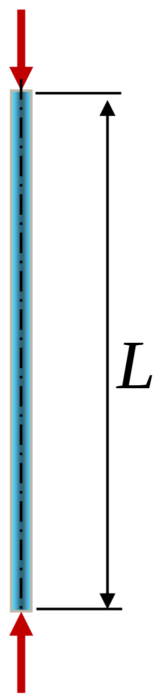
Note:
Different to circular arc

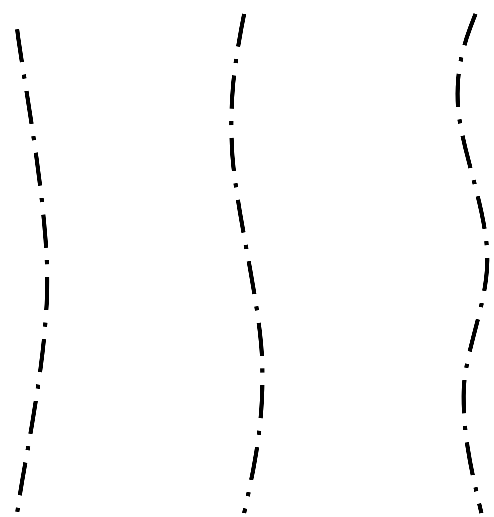
Note: we can determine the deflected shape,
but not the amplitude at the critical buckling condition
because neutral equilibrium is not associated with a specific displacement

$v = A \sin \mu x$

Mode

1st	2nd	3rd	
$\mu L = \pi$	$\mu L = 2\pi$	$\mu L = 3\pi$	…





1^{st} mode is most critical and will occur at lowest buckling loads.

Other modes will only appear if earlier modes are restrained