Stress, Strain and Deformation
Compound Cross-Sections

Dr Luiz Kawashita

Luiz.Kawashita@bristol.ac.uk

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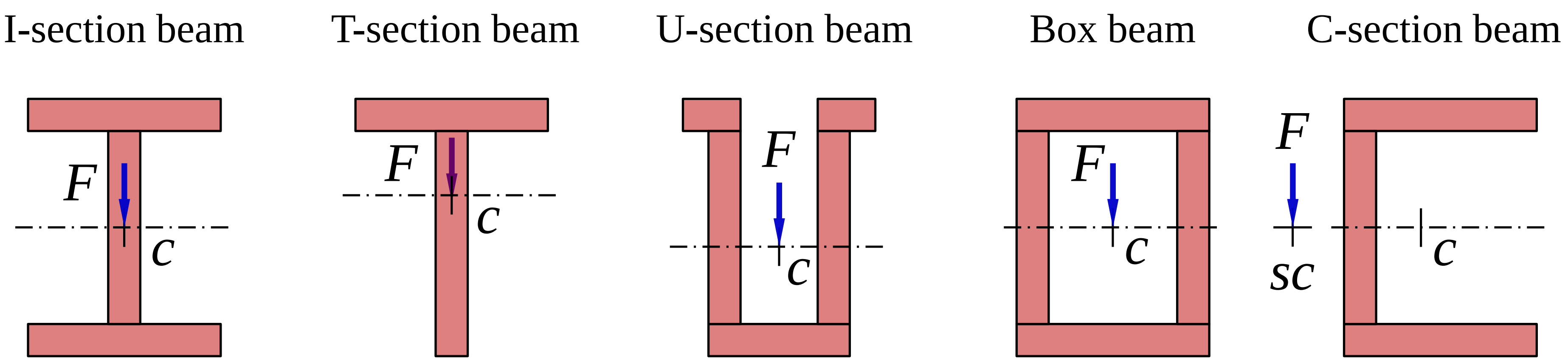
University of BRISTOL

Compound Cross-Sections

- Cross-sections of more complex shape, seen as an assembly of simpler sections – most often rectangular sections
- All ‘parts’ are perfectly bonded together – *i.e.* via welding or adhesive bonding

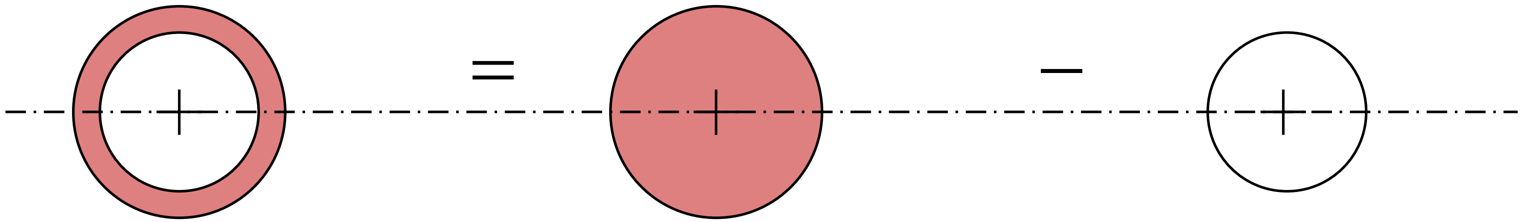
Classic compound cross-sections:



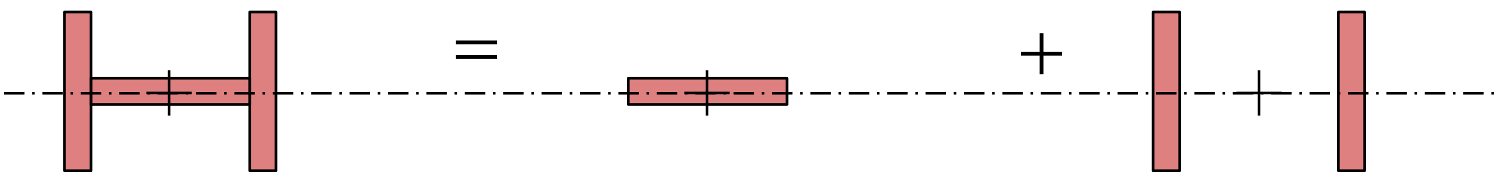
- **Webs** are elements **parallel** with main transverse loading
- **Flanges** are elements **normal** to main transverse loading

- Second moments of area can be added or subtracted **only** when the centroid of the 'part' (or sub-section) **lies on the neutral axis**, *e.g.*:

$$I_{NA} = I_{NA,\text{ outer}} - I_{NA,\text{ inner}}$$

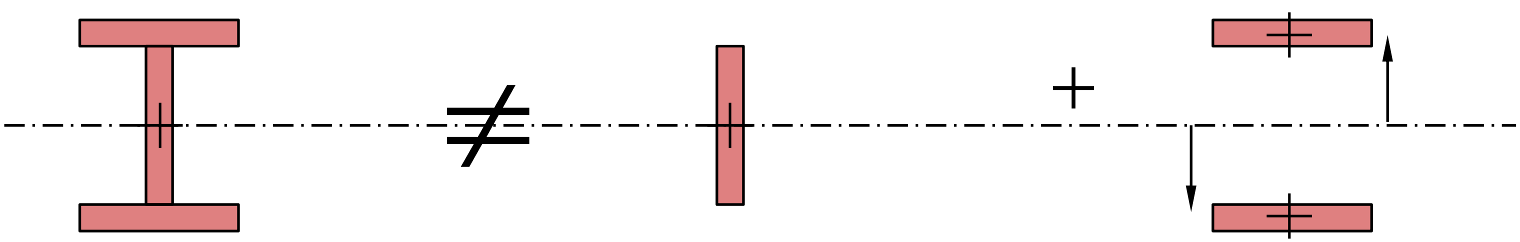


$$I_{NA} = I_{NA,\text{ flange}} + 2\times I_{NA,\text{ web}}$$



- However, if the centroid of a sub-section is **offset** from the neutral axis, the **parallel axis theorem** **must** be used!!

$$I_{NA} \neq I_{NA,\text{ web}} + 2\times I_{NA,\text{ flange}}$$

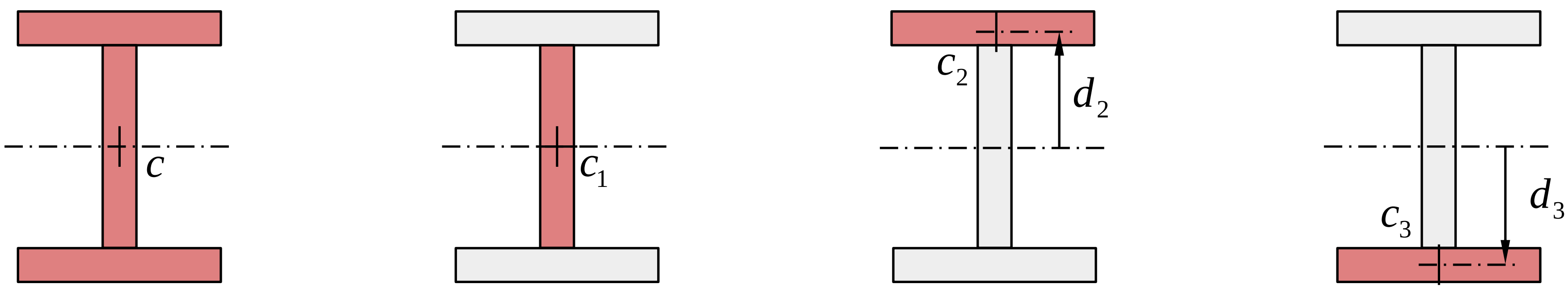


Parallel Axis Theorem (2/3)

KEY POINTS

- First we need to analyse each 'part' separately. We number them with an index i:

Composite section	Part 1	Part 2	Part 3



	Area A_1	Area A_2	Area A_3
	Centroid c_1	Centroid c_2	Centroid c_3
	SMA I_1	SMA I_2	SMA I_3
	Distance $d_1 (= 0)$	Distance d_2	Distance d_3

- The contribution of each 'part' i to the compound I_{NA} is:

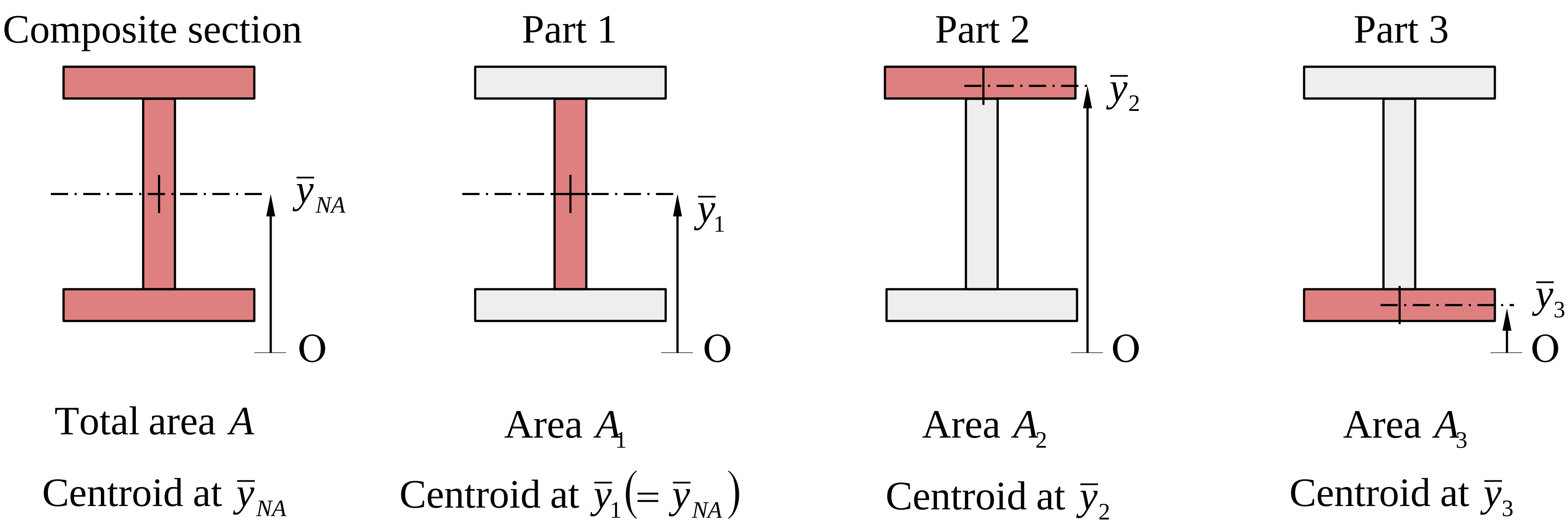
$$I_{i,ZZ} = I_i + A_i \left(d_i\right)^2$$

- And the final compound I_{NA} is:

$$\boxed{I_{NA} = \sum_i \left(I_{i,ZZ}\right) = \sum_i \left[I_i + A_i \left(d_i\right)^2\right]}$$

Parallel Axis Theorem (3/3)

- The **centroid of the compound section** is found by considering individual centroids w.r.t a 'vertical' coordinate $\bar{y}$ of arbitrary origin:



- It can be proven (homework) that the first moment of area gives:

$$A\, y_{NA} = \sum_i (A_i\, y_i) \quad \therefore \quad \boxed{\bar{y}_{NA} = \frac{\sum_i (A_i\, \bar{y}_i)}{\sum_i (A_i)}}$$

Parallel Axis Theorem – Derivation

- The theorem is derived directly from the **2nd moment of area**:

$$I_i = \int_{y_1}^{y_2} y^2 \, b_{(y)} \, \mathrm{d}y$$

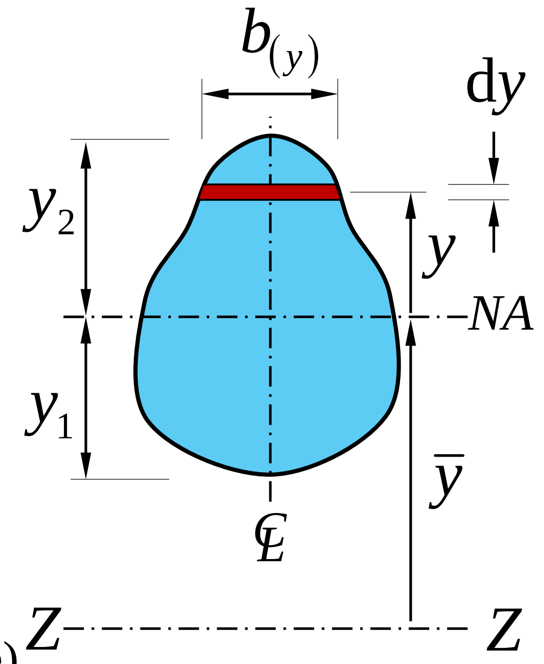
(2nd moment of area of a 'part' w.r.t. its own centroid)

$$I_{ZZ} = \int_{y_1}^{y_2} \left(y + \bar{y}\right)^2 b_{(y)} \, \mathrm{d}y$$

(2nd moment of area of a 'part' w.r.t. the compound centroid)

$$I_{ZZ} = \int_{y_1}^{y_2} \left(y^2 + 2\,y\,\bar{y} + \bar{y}^2\right) b_{(y)} \, \mathrm{d}y$$

(Imagine that this shape is now only one 'part' *i* of a compound section)



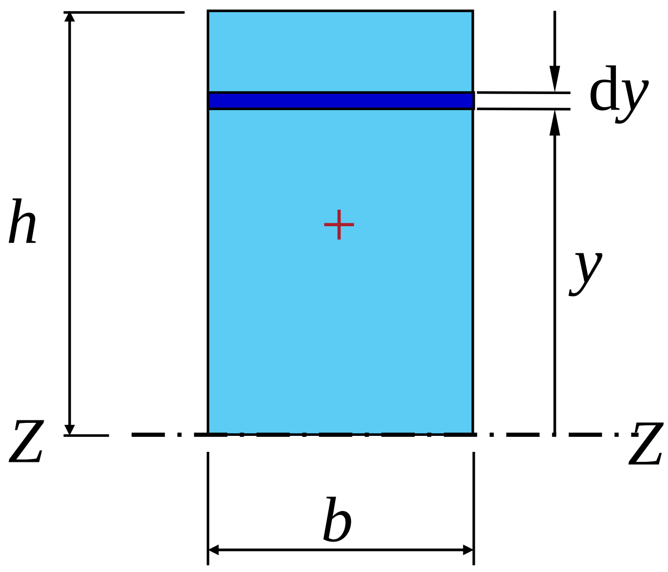
(Origin of $\bar{y}$, for convenience)

$$I_{ZZ} = \underbrace{\int_{y_1}^{y_2} y^2 \, b_{(y)} \, \mathrm{d}y}_{I_i} + 2\,\underbrace{\bar{y}^{\vphantom{2}}}_{} \underbrace{\cancelto{0}{\int_{y_1}^{y_2} y \, b_{(y)} \, \mathrm{d}y}}_{\text{1st moment of area}} + \underbrace{\bar{y}^2}_{d_i^{\,2}} \underbrace{\int_{y_1}^{y_2} b_{(y)} \, \mathrm{d}y}_{A_i}$$

$$\boxed{I_{i,ZZ} = I_i + A_i \left(d_i\right)^2}$$

PAT Example – Solid Rectangular Section (1/2)

- Let us first derive the second moment of area I_{ZZ} for the solid rectangular section using the 'original' method:



$$I = \int_0^h y^2 \, \mathrm{d}A$$

$$I = \int_0^h y^2 \, b \, \mathrm{d}y$$

$$I = \left[\frac{y^3}{3} b \right]_0^h$$

$$\boxed{I = \frac{b\,h^3}{3}}$$

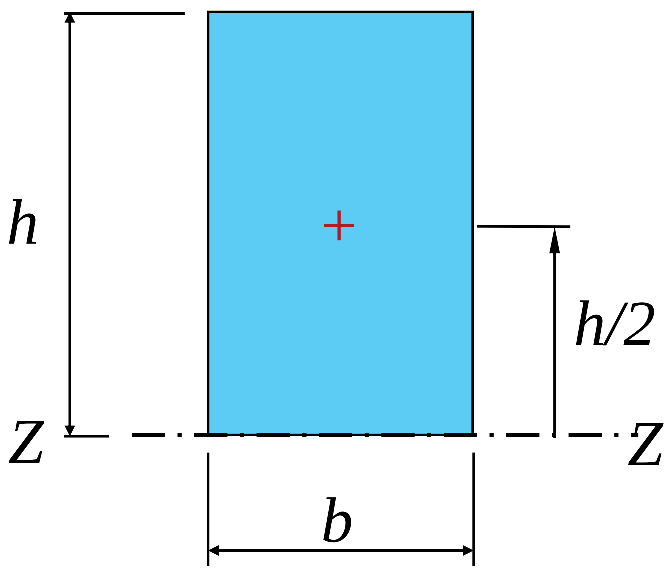
- Now, let us use the **parallel axis theorem** to verify it:

PAT Example – Solid Rectangular Section (2/2)

- Last week we derived the second moment of area about the neutral axis I_{NA} :

$$\boxed{I_{NA} = \frac{b\,h^3}{12}}$$

- Now, applying the theorem:



$$I_{ZZ} = I_{NA} + A\,d^2$$

$$I_{ZZ} = \frac{b\,h^3}{12} + (b\,h)\left(\frac{h}{2}\right)^2$$

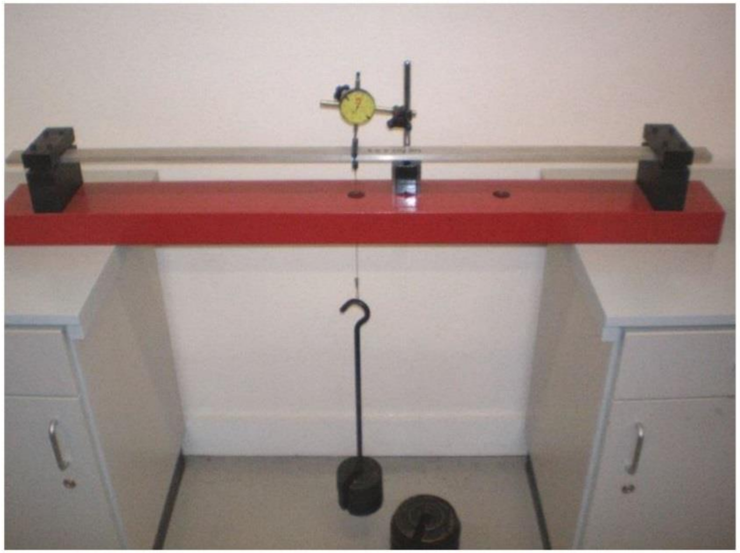
$$I_{ZZ} = \frac{b\,h^3}{12} + \frac{b\,h^3}{4}$$

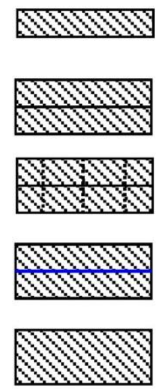
$$\boxed{I_{ZZ} = \frac{b\,h^3}{3}}$$

- As we wanted to prove!

PAT Example – Structures 1 Lab

- In the Structures lab beams of different cross-sections are tested:





(a) Single

(e) Double, loose

(f) Double, riveted

(g) Double, bonded

(h) Double, monolithic

- If the adhesive bonding were perfect, cases (g) and (h) should be identical
- So assuming perfect bonding:
  - Case (g): use the parallel axis theorem to find the combined I_{NA}
  - Case (e): simply make $I_{NA} = I_1 + I_2$
    - *i.e.* it behaves like a 'leaf spring'

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