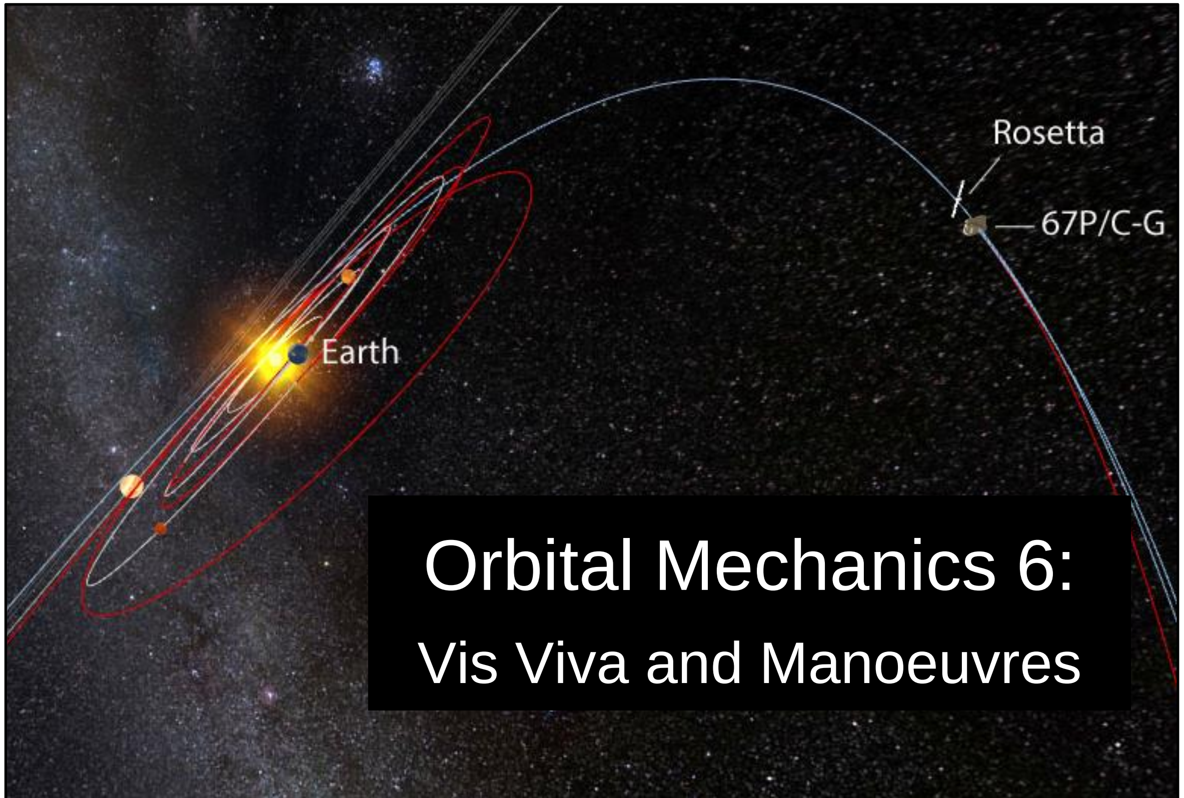

Orbital Mechanics 6:

Vis Viva and Manoeuvres

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Learning Objectives

1. Vis-viva equation to calculate velocities for all orbits
2. Kepler equation
3. Free Delta V due to the Earth's rotation
4. Launch losses

1. Derive and use Vis-viva equation to calculate velocities for all orbits
2. Know and be able to use the Kepler equation
3. Quantify the delta V due to the Earth's rotation
4. Be able to explain launch losses

The Vis-Viva Equation 1

- Velocity is important, because to change it requires fuel.
- Using some of the properties of an ellipse, and the conservation of energy, we can find an expression for the velocity at any point in an orbit.

The vis-viva equation, along with the rocket equation, is the most useful equation in spacecraft systems design.

The Vis-Viva Equation 2

- We have already derived an expression for the specific energy of an orbit:

$$\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = (e^2 - 1)\frac{\mu^2}{2h^2} \qquad (5A\text{-}15)$$

We know that the total energy at any point on the orbit is the sum of the kinetic and potential energy. We can write this for the Specific energy by dividing through by m.

The Vis-Viva Equation 3

- We have already derived an expression for the specific energy of an orbit:

$$\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = (e^2 - 1)\frac{\mu^2}{2h^2} \qquad \text{(5A-15)}$$

- For an ellipse, remember that $h^2 = \mu a(1 - e^2)$ so:

$$\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = (e^2 - 1)\frac{\mu^2}{2h^2} = (e^2 - 1)\frac{\mu^2}{2a\mu(1 - e^2)} \qquad \text{(6-1)}$$

$$\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = \frac{\mu^2}{2a\mu}\cancelto{=-1}{\frac{e^2 - 1}{1 - e^2}} = -\frac{\mu}{2a} \qquad \text{(6-2)}$$

In the annex, we derived expression 5A-15. This shows that specific energy is independent of eccentricity and depends only on semi major axis.
The resulting specific energy is valid only for ellipses.

The Vis-Viva Equation 4

- Given:

$$\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \qquad (6\text{-}3)$$

Rearranging:

$$v = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]} \qquad (6\text{-}4)$$

- This is a very useful equation that enables us to quickly calculate "delta velocity budgets"

These are both different forms of the vis-viva. I find the second one more useful practically. But if you were asked in an interview or exam you could write either. Note that the velocity changes as you go around the orbit, but it wouldn't cost any delta V to stay in the orbit in this ideal case. In reality for an Earth orbiting satellite there are luni-solar perturbations, solar radiation pressure etc. which mean that staying in an orbit (station-keeping) requires a small amount of delta V to maintain its orbit.

Circular Orbits

- Circular orbits are a particular case of an elliptical orbit, so we can use the vis-viva equation with a constant radius, i.e. $r \equiv a$, so:

$$v_c^2 = \mu\left[\frac{2}{r} - \frac{1}{r}\right] = \frac{\mu}{r} \qquad (6\text{-}5)$$

- Therefore, the velocity is inversely proportional to the square root of the altitude.

Numerical example

What is the velocity of a spacecraft in a Earth circular orbit with a period of 23.934hrs? μ=3.986e14Nm^{2}kg^{-1}

- Previously we used $T = \frac{2\pi}{\sqrt{\mu}} a^{3/2}$ to work out a=42.164e6 m
- Then use r=a for circular orbit:

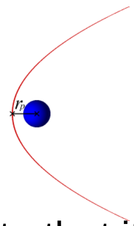
$$v_c^2 = \frac{\mu}{r} = \frac{3.986e14}{42.164e6}$$

- So v_c =3074m/s

The strategy is always to go after the primary orbital parameters: eccentricity, semi-major axis and angular momentum, first. Assuming that the orbit is equatorial, this is the velocity of a spacecraft in geostationary orbit.

Parabolic Orbits

- Remembering v_{esc} escape velocity:



$$v_{esc} = \sqrt{\frac{2\mu}{r}} \qquad \text{(3-7)}$$

- Note that if you compare v_{esc} with v_c, there is only a factor of $\sqrt{2}$ difference:

$$v_{esc} = \sqrt{\frac{2\mu}{r}} \quad \text{c.f.} \quad v_c = \sqrt{\frac{\mu}{r}} \qquad \text{cf (6-5)}$$

- Thus the necessary ΔV to escape on a parabolic orbit from a circular orbit is $\left(\sqrt{2}-1\right)v_c$

3-7 was in the third orbits lecture. A parabola is the shape of the orbit we need as a minimum to escape a planet's gravity.
Remember that the parabola is the demarcation between negative and positive energy orbits.

Taking the orbit equation to the Kepler equation

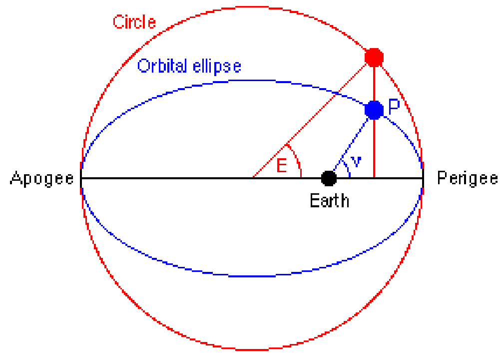
$$r(\theta) = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)} \quad \text{Orbit equation}$$

- It would be nice to find an expression for $r(t)$, this can be done by integrating an expression for $\frac{dr}{dt}$, this gets a bit tricky(!)
- It is easier to solve geometrically by introducing concept of Eccentric anomaly 'E' and mean motion 'n'.
- For the derivation have a look at reference 1 at the end of this lecture
- You do not need to know it for the exam.

The straightforward approach is clear but leads to a difficult integral. Can be treated numerically, but then one might just as well give up the idea of using Kepler orbits and switch to numerical representations altogether. So it is generally solved geometrically using the concept of Eccentric and Mean anomaly and mean motion.
Do not confuse E ("eccentric anomaly") with E ("energy")!

The Kepler equation

- Introduces concept of Eccentric anomaly ‘E’ and mean motion ‘n’.
- The position in orbit as a function of time t, is known as the ‘Kepler equation’.
- There are actually 3 equations we need: (6-10) to (6-12)



Kepler Equation

$$\boxed{M = n(t - t_0) = E - esinE} \qquad \text{(6-10)}$$

Where:

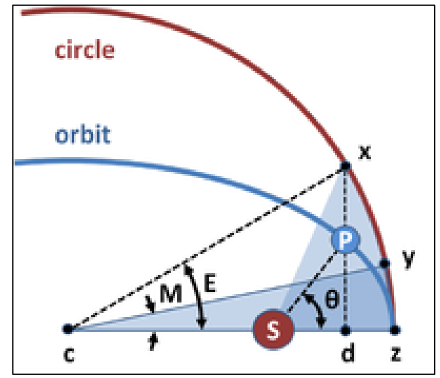
M=mean anomaly (rad)

n= mean motion (rad/s)

$t-t_0$= time since periapsis (s)

E= eccentric anomaly (rad)

e= eccentricity



You do not need to know this diag. for exam!

M=mean anomaly is the angle from periapsis if body in a circular orbit with the same period
n=The mean motion 'n' is the average angular velocity needed to complete one orbit.
E= The angle xcz where x is the projection of a point P onto a circular orbit with the same period, c is the centre of the circle and z is the point where the circle and orbit intersect. Defined in diagram above. Don't confuse E eccentric anomaly with E total energy!
Kepler's equation gives the relation between time (t, in [s]) and position (M and/or E, in [rad]) for an ellipse. Other formulations exist for hyperbola and parabola.

Kepler equation

- **Mean anomaly M**: angle from periapsis if body in a circular orbit with the same period

- If the spacecraft were travelling on the circle, its angular velocity would be equal to the **mean motion 'n'**:

$$\boxed{n = \sqrt{\frac{\mu}{a^3}}} \quad \text{rad/s} \qquad (6\text{-}11)$$

- And true anomaly θ is related to **Eccentric anomaly 'E'** through:

$$\boxed{tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}} \qquad (6\text{-}12)$$

We actually need these two relationships as well as Kepler's equation in order to calculate time from orbital elements.

Num. Example to find t from a,e,θ

Question:

a = 7000 km; e = 0.1; μ = 3.986e14 m^3/s^2 ; θ = 35°

What is the time taken?

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \qquad n = \sqrt{\frac{\mu}{a^3}}$$

$$M = n(t - t_0) = E - e\sin E$$

Answer: So we need $t-t_0$

1. We can find $n = \sqrt{\frac{\mu}{a^3}}$=1.078×$10^{-3}$rad/s
2. Converting θ = 35°=0.61087 rad
3. From $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$ we have E = 0.55565 rad
4. $M = E - e\sin E$= 0.50290 rad
5. $M = n(t - t_0)$ gives $t-t_0$ = 466.5 s

This is an example where we find the time t from the position ie: orbital elements a, e and theta.

Num. Example to find θ from e, a, t

Question

a = 7e6 m, e = 0.1 and t-tp = 900s, what is true anomaly θ?

Answer

1. We can find $n = \sqrt{\frac{\mu}{a^3}}$ =1.078×10^{-3}rad/s
2. $\mathrm{M} = \mathrm{n}(\mathrm{t} - \mathrm{t}_0)$ = 1.078×10^{-3}.900=0.9702 rad
3. M=$\mathrm{E} - \mathrm{esinE}$= 0.9702. But how do we extract E???

We seed a solution thus:

E_{i+1} = M + e sinE_i

E_0 = 0.9702 + 0.1 sin0 =0.9702

E_1 = 0.9702 + 0.1 sin0.9702 =1.0527

We are effectively finding the root of the function f(t)= E-esinE-M.
This was the Everest of maths for hundreds of years but no one solved it and we have to do it numerically.

Num. Example to find θ from e, a, t cont…

4. E =1.0573 rad
5. Then use:

$$tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \sqrt{\frac{1+0.1}{1-0.1}}\tan\frac{1.0573}{2}$$

to give θ = 1.1468 rad

iterate $E_{i+1} = M + e \sin E_i$

E0	0.9702
E1	1.0527
E2	1.0570
E3	1.0572
E4	1.0530
E5	1.0573
E6	1.0573

Or use Newton Raphson:

$$E_{i+1} = E - \frac{E - esinE - M}{1 - ecosE}$$

This kind of iteration is the simplest and is called fixed point iteration, but you could equally well use Newton-Raphson method (otherwise known as Newton's method).

Manoeuvres

What is wrong with this?

- https://www.youtube.com/watch?v=wjXhHEXqMKE

We like: Throwing the extinguisher moves her along and the last part of rendezvous is very slow as it ought to be.
We don't like: This is not how you perform rendezvous! It's boring to do rendezvous properly with Hohmann transfers. No backpack? Where is her air supply?
CSS at 42deg inclination, ISS at 52deg, you work out the delta V…
Have you ever seen a fire extinguisher last as long as that?
Atmospheric drag does not start at 330km (altitude of space station)

Launch

- Do we wish to go vertically up? Or horizontally?

  http://www.youtube.com/watch?v=vPQvTgD2quQ

- Time lapse image of shuttle showing path after lift off.

The main objective of the rocket engine is not only to get the cargo above the atmosphere, but more importantly to accelerate it in horizontal direction to the orbital speed (7.5 km/s for the orbital altitudes of the Shuttle and International Space Station).
A ‘gravity turn’ is when the rocket is turned from vertical to horizontal using only gravitational force, no thrust is applied to bend the trajectory. The angle (not the heading which is up to the desired trajectory) is dependent on the vehicle.

Launch losses

- Large Impulse takes finite time, so extra propellant is needed to counteract **gravity losses**
- Atmosphere means we need to go up before we go horizontally to minimise **drag losses**
- **Steering losses** due to axis of rocket not aligned with **V.**

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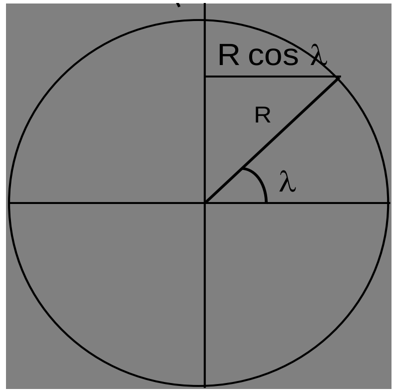
Gravity losses account for 15% of fuel, drag losses only 0.5%, steering even less. Drag increases as square root of speed
If you play Kerbal you will have noticed that low velocity wastes delta V on gravity (gravity losses), high velocity wastes fuel on air resistance ie: drag losses.

Effect of Earth's rotation

λ=latitude



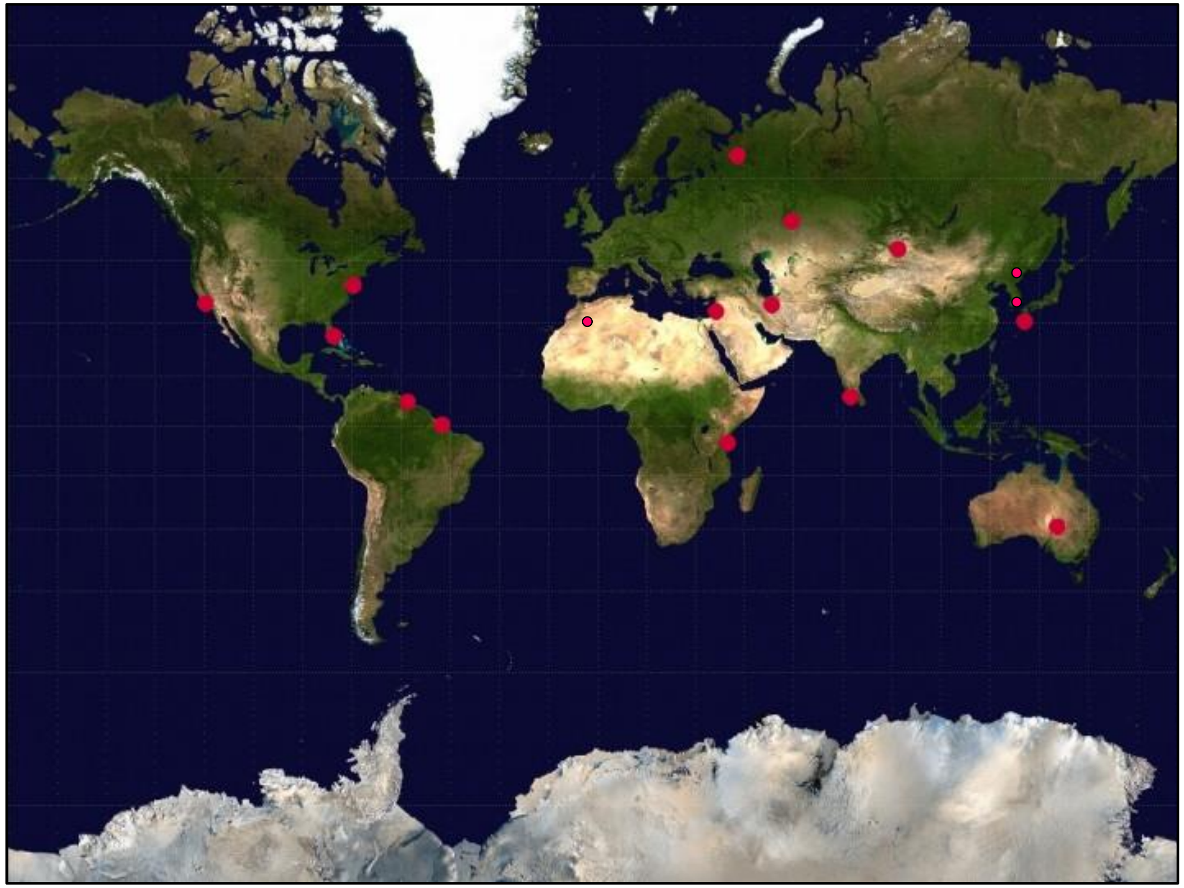
$$\Delta V = \frac{distance}{time} = v\cos\lambda \quad \text{(6-13)}$$

$$\Delta V = \frac{2\pi.6378e3}{86164}\cos\lambda$$

$$= 465\cos\lambda \ (m/s)$$

Site	Latitude λ°	Cos(λ)	ΔV ms^{-1}
UK	52	0.616	286
ETR	28.5	0.883	411

The rotational velocity of the Earth at the launch site will give the launch vehicle a valuable extra increment (or undesirable decrement if required orbit is retrograde) of velocity. Since this effect is greatest at low geographic latitudes, it is desirable to launch as near the Equator as possible.The speed of the equatorial points can be calculated as above at 465m/s. This is the maximum gain you can have from an eastward launch from the equator. For polar launches the gain is insignificant. ETR-Cape Canaveral (Kennedy Space Centre). 86164s is length of a sidereal day.



Location of the world's launch sites . Note that many are as near to the equator as possible to take advantage of the additional delta V from the Earth's rotation. Launch site latitude limits the inclination of the satellite's orbit. The minimum inclination of the orbit is equal to the latitude of the launch site. To get to a lower inclination, satellites need to go through an orbit plane change.

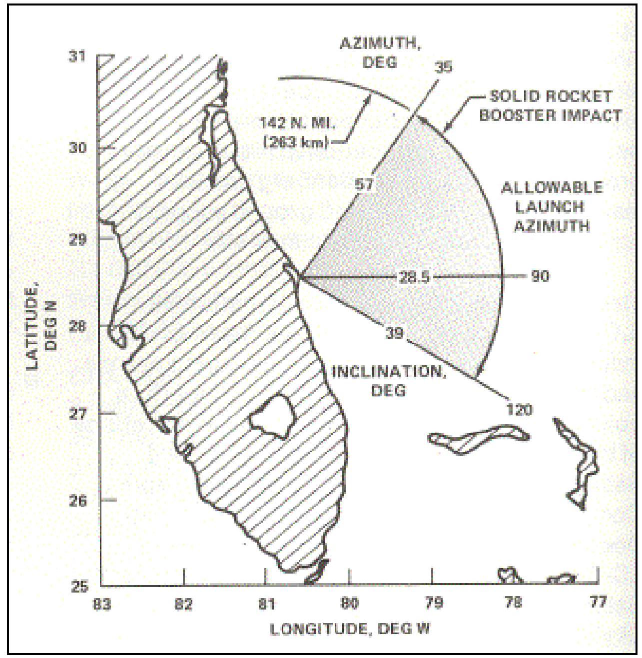
62.8, 67.1, 73-74 and 82-83

Launch direction

Site	Latitude°	Longitude	Launch Azimuth Restrictions
Tyuratam (now Baikonur) 1957	45.9	63.3E	Overland
ETR Cape Canaveral 1957	28.5	80.6W	35-120
WTR Vandenberg AFB 1959	34.6	120.6W	170 – 300, polar orbit possible
Wallops Island 1964	37.9	75.8W	E
Hammaguir 1965	30.9	3.0W	Overland
Plesetsk 1966	62.8	40.4E	62-83
Broglio (San Marco) 1967	-3.0	40.2E	E
Uchinoura (Kagoshima) 1970	31.5	131.1E	E
Jiuquian 1970	41.0	100.3E	Overland
Woomera 1971	-31.0	136.8E	Overland
Kourou 1979	5.5	52.7W	-45 – 90, polar orbit possible
Sriharikota 1980	13.8	80.3E	E
Palmachim 1988	31.5	34.5E	Overland
Semnan 2009	35.3	53.7E	E
Sohae (Tongch'ang-ri) 2012	39.7	124.7E	Over inhabited lands
Naro 2013	34.5	127.5E	Over inhabited island
Alcantara (1990)	-2.3	44.3W	-45 – 90, polar orbit possible

More northerly launch sites are usually used for polar launches. But Kourou in South America, for instance, has 80% launches to GTO, can you see why?

Kennedy Space Center launch restrictions



The useable range of launch azimuth values for Kennedy Space Center (KSC) is dictated by the range safety constraints. These constraints dictate launch azimuths to avoid launching over populated areas and place launch trajectories over the Atlantic Ocean. The allowable range of launch azimuths from KSC is from 35° to 120°. These result in accessible direct-launch orbit inclinations from 57° to 39° (Wertz, 734)

Summary

1. We have derived the Vis-viva equation to calculate velocities for all orbits: $\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$

2. Kepler equations:

$$tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \qquad n = \sqrt{\frac{\mu}{a^3}}$$

$$M = n(t - t_0) = E - esinE$$

3. Bonus delta V due to Earth's rotation is worth up to 0.5km/s
4. Launch losses can be due to gravity, drag and steering losses

Test Yourself! (Feedback)

1. Prove using $r_a v_a = r_p v_p$ and $\frac{mv_p^2}{2} - \frac{GMm}{r_p} = \frac{mv_a^2}{2} - \frac{GMm}{r_a}$

that: $v_p^2 = 2\mu(\frac{r_a}{r_p(r_a+r_p)})$

2. An artificial Earth satellite is in an elliptical orbit which has an altitude of 250 km at perigee and an altitude of 500 km at apogee. Use the above result to find the perigee velocity.
3. A satellite is in an orbit with a=7,500 km, e=0.1 Calculate the time it takes to move from a position 30º past perigee to 90º past perigee.
4. The satellite in problem 3 has a true anomaly of 90º. What will be the satellite's position, i.e. the true anomaly, 20 minutes later?

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