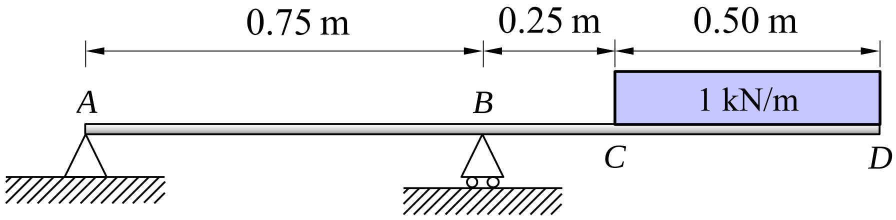
Example 2.3.3 – Plot the bending moment, slope and deflection diagrams for the beam below, showing the value and location of any maxima and/or minima. The beam has a constant flexural modulus $EI = 1\,\mathrm{kN\,m^2}$.



We start by finding support reactions:

$$\sum M_{@A}^{CW} = 0,$$

$$M_A - (R_B)\left(\frac{3}{4}\mathrm{m}\right) + \left[\left(1\frac{\mathrm{kN}}{\mathrm{m}}\right)\left(\frac{1}{2}\mathrm{m}\right)\right]\left(\frac{5}{4}\mathrm{m}\right) = 0.$$

The extremity A is pinned, therefore $M_A = 0$ and:

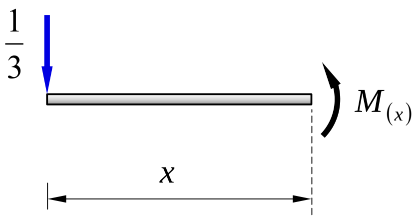
$$(R_B)\left(\frac{3}{4}\mathrm{m}\right) = \left[\left(1\frac{\mathrm{kN}}{\mathrm{m}}\right)\left(\frac{1}{2}\mathrm{m}\right)\right]\left(\frac{5}{4}\mathrm{m}\right) \qquad \therefore \qquad R_B = \frac{5}{6}\,\mathrm{kN}.$$

Vertical equilibrium gives:

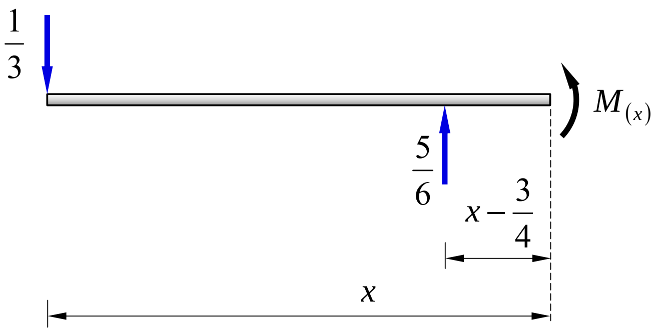
$$\sum F = 0,$$

$$R_A + R_B - \left(1\frac{\mathrm{kN}}{\mathrm{m}}\right)\left(\frac{1}{2}\mathrm{m}\right) = 0 \qquad \therefore \qquad R_A = -\frac{1}{3}\,\mathrm{kN}.$$

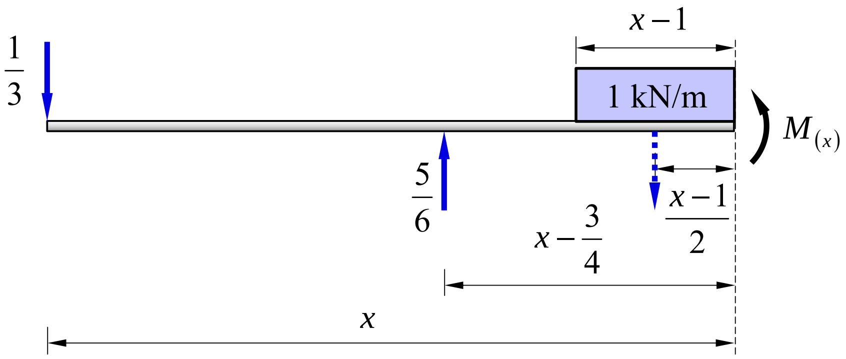
There are three equations of moment, depending on where we section the beam:



$$M_{(x)} + \frac{1}{3}x = 0 \qquad \therefore \qquad M_{(x)} = -\frac{1}{3}x$$



$$M_{(x)} + \frac{1}{3}x - \frac{5}{6}\left(x - \frac{3}{4}\right) = 0 \qquad \therefore \qquad M_{(x)} = -\frac{1}{3}x + \frac{5}{6}\left(x - \frac{3}{4}\right)$$



$$M_{(x)}+\frac{1}{3}x-\frac{5}{6}\left(x-\frac{3}{4}\right)+[(1)(x-1)]\left(\frac{x-1}{2}\right)=0 \quad \therefore \quad M_{(x)}=-\frac{1}{3}x+\frac{5}{6}\left(x-\frac{3}{4}\right)-\frac{(x-1)^2}{2}$$

In order to **combine the three equations into one**, we use the Heaviside step function:

$$M_{(x)}=-\frac{1}{3}x+\left[\frac{5}{6}\left(x-\frac{3}{4}\right)H\left(x-\frac{3}{4}\right)\right]-\left[\frac{(x-1)^2}{2}H(x-1)\right].$$

This is our curvature equation:

$$M_{(x)}=EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}=-\frac{1}{3}x+\left[\frac{5}{6}\left(x-\frac{3}{4}\right)H\left(x-\frac{3}{4}\right)\right]-\left[\frac{(x-1)^2}{2}H(x-1)\right]. \qquad (1)$$

Integrating once gives the slope:

$$EI\,\phi_{(x)}=EI\frac{\mathrm{d}v}{\mathrm{d}x}=-\frac{1}{6}x^2+\left[\frac{5}{12}\left(x-\frac{3}{4}\right)^2 H\left(x-\frac{3}{4}\right)\right]-\left[\frac{(x-1)^3}{6}H(x-1)\right]+A. \qquad (2)$$

Integrating again gives the deflection:

$$EI\,v_{(x)}=-\frac{1}{18}x^3+\left[\frac{5}{36}\left(x-\frac{3}{4}\right)^3 H\left(x-\frac{3}{4}\right)\right]-\left[\frac{(x-1)^4}{24}H(x-1)\right]+Ax+B. \qquad (3)$$

The first boundary condition is:

$$x=0,\ v=0 \qquad \therefore \qquad B=0.$$

And the second boundary condition is:

$$x=\frac{3}{4},\ v=0 \qquad \therefore \qquad -\frac{1}{18}\left(\frac{3}{4}\right)^3+A\left(\frac{3}{4}\right)=0 \qquad \therefore \qquad A=\frac{1}{32}\ \mathrm{kN\,m^2}$$

In order to draw the three graphs we need to find some points of interest.

- Point of maximum upward deflection. The span *AB* will 'hog' under the applied load (*i.e.* it will have a 'sad face' type deformation) and the remainder of the beam with deflect downwards. So the maximum upward deflection must occur within span *AB*. Deflection maxima and minima are characterised by a zero slope, therefore we write:

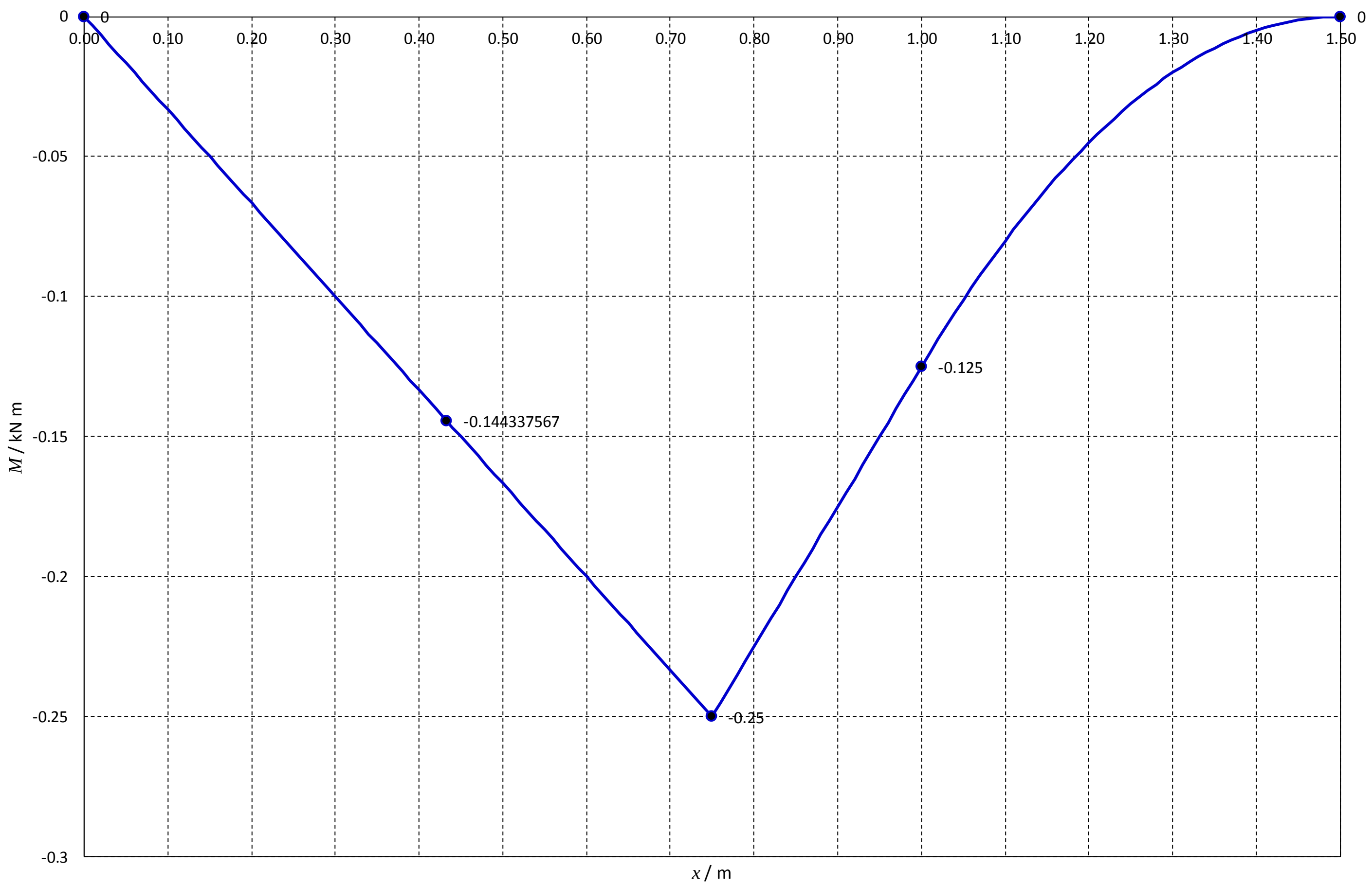
$$EI\,\phi_{x_{(v_{\max})}}=0=-\frac{1}{6}x_{(v_{\max})}{}^2+\frac{1}{32} \qquad \therefore \qquad x_{(v_{\max})}=\frac{\sqrt{3}}{4}\ \mathrm{m}\cong 0.433\ \mathrm{m}.$$

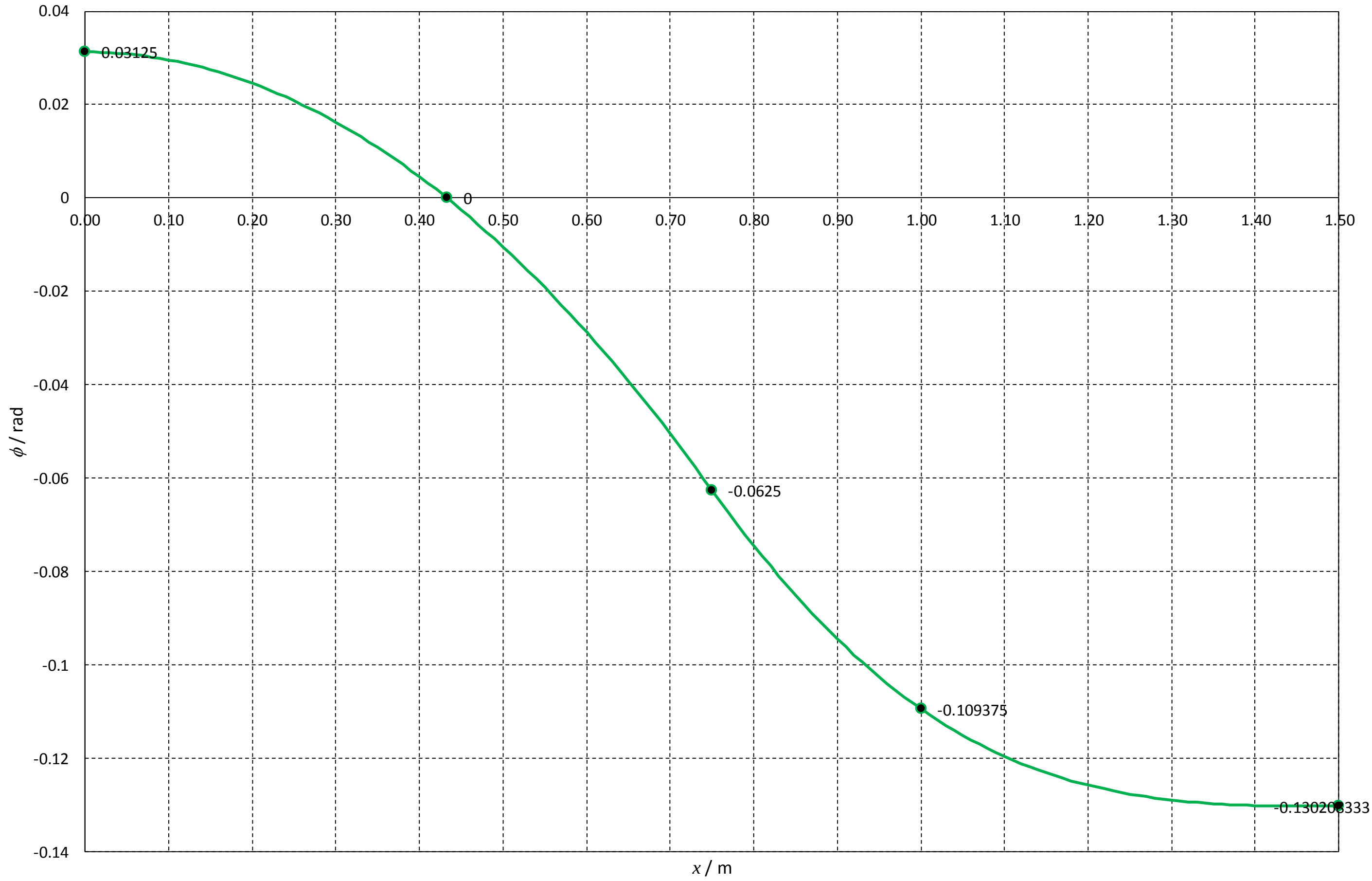
- Maximum and minimum deflections. These are found by substituting the now known values of *x* into equation (3):

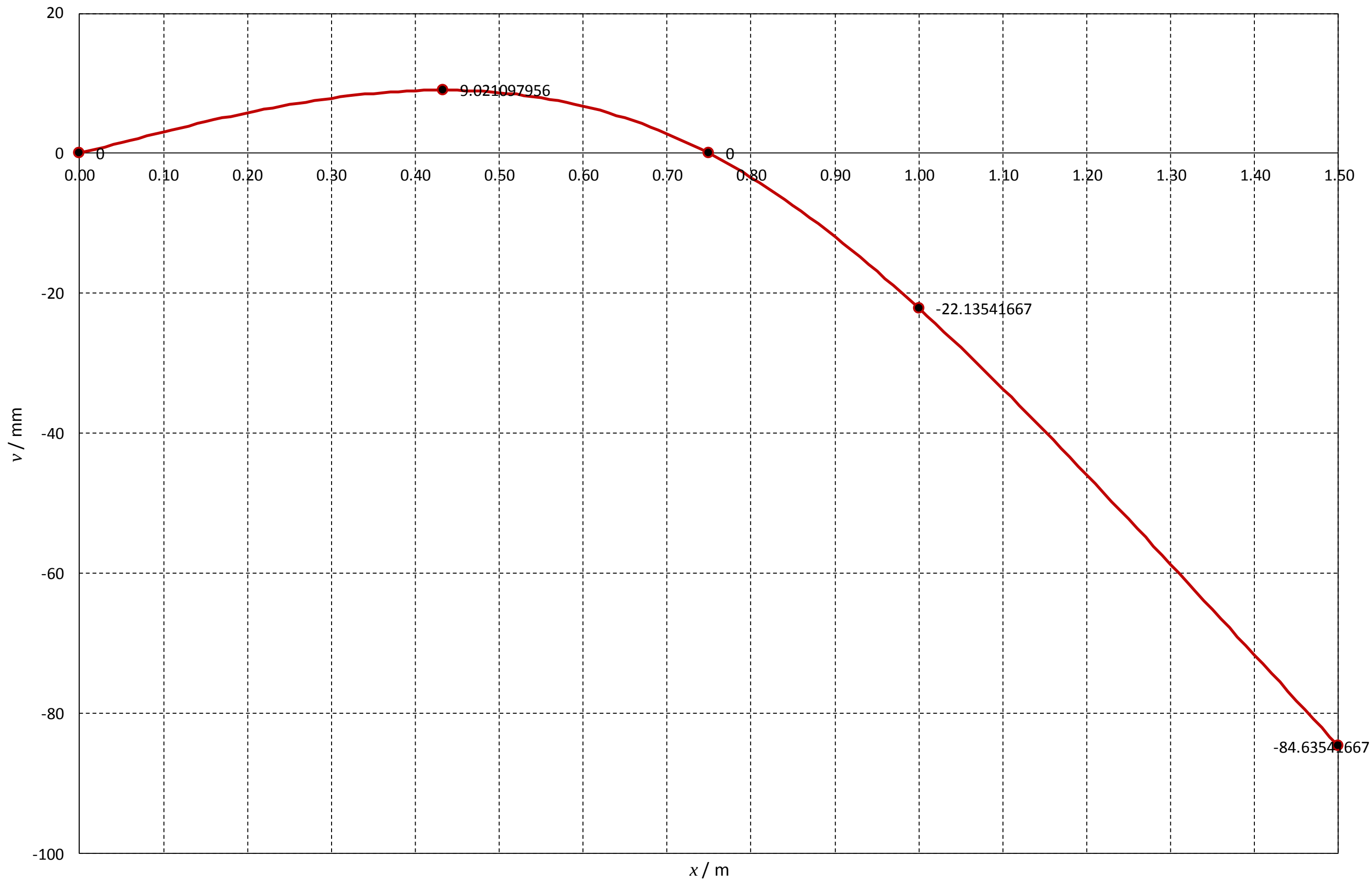
$$EI\,v_{\max}=-\frac{1}{18}\left(\frac{\sqrt{3}}{4}\right)^3+\left(\frac{1}{32}\right)\left(\frac{\sqrt{3}}{4}\right) \qquad \therefore \qquad v_{\max}=9.021\ \mathrm{mm}$$

$$EI\ v_{\min} = -\frac{1}{18}\left(\frac{3}{2}\right)^3 + \frac{5}{36}\left(\frac{3}{2}-\frac{3}{4}\right)^3 - \frac{\left(\frac{3}{2}-1\right)^4}{24} + \left(\frac{1}{32}\right)\left(\frac{3}{2}\right) \quad \therefore \quad \boxed{v_{\min} = -84.635 \text{ mm}}$$

Further relevant points may be found by substituting *x* values into equations (1), (2) and (3). The final graphs are shown below.




0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
1.10
1.20
1.30
1.40
1.50
0.03125
0
-0.0625
-0.109375
-0.130208333
ϕ / rad
x / m


20
0
-20
-40
-60
-80
-100
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
1.10
1.20
1.30
1.40
1.50
9.021097956
0
-22.13541667
-84.63541667
v / mm
x / m