4. Integration along curves

How do we integrate scalar & vector fields over straight lines.

How do we integrate scalar and vector fields over curves rather than straight lines? Before we do that, we better find out how to parametrise curves $C(t)$ in 2 and 3 dimensions. Also, what is the arclength $d\boldsymbol{r}$ along a curve in 3 dimensions? How do we use vector calculus to define the work done in moving along a path C. Why does calculating the work done in certain force fields lead to independence of path and in others does not? What is a conservative vector field?

4.1 Path integrals of scalar and vector fields

MOTIVATION: Consider the work done by a force in moving a load from $x = a$ to $x = b$

force component $F_x = |\underline{F}| \cos\theta$

"Work = Force $\times$ distance"
component

$W = |\underline{F}| \cos\theta \times |\underline{d}|$
$\underline{d} = (b-a)\underline{i}$

But Force is a vector. So

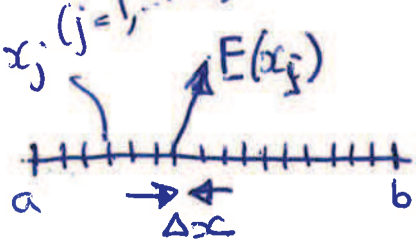
work done = force • displacement

$$\boxed{W = \boldsymbol{F} \cdot \mathbf{d}} = \boldsymbol{F} \cdot ((b-a)\boldsymbol{i})$$

Now, what if $\boldsymbol{F}$ were not constant, $\boldsymbol{F} = \boldsymbol{F}(x)$? Then

$\underline{F} \cdot \underline{d}$
$\underline{F}(x_j)$ $\quad \underline{i}\,\Delta x$

$x_j \; (j=1,\dots N)$



$$W = \lim_{N\to\infty} \sum_{j=1}^{N} (\boldsymbol{F}(x_j)\cdot \boldsymbol{i})(\Delta x)$$

$$= \int_{x=a}^{x=b} (\boldsymbol{F}(x) \cdot \boldsymbol{i})dx$$

$$W = \int_{t=0}^{t=(b-a)} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

$\Delta x = \dfrac{b-a}{N}$

approximation finite Δx

exact as $\Delta x \to 0$ (equivalently $N \to \infty$)

If ~~Can~~ parameterise path by

$$\boldsymbol{r}(t) = (a, 0, 0) + (t, 0, 0)$$

$t=0 \;\; \underline{r}(t) = (a,0,0) \qquad t=(b-a) \;\; \underline{r}(t) = (b,0,0)$

is the position vector of the load, so that in this case $d\boldsymbol{r} = dt\,\boldsymbol{i}$

More generally, if $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$ is a vector field which varies with spatial position $\boldsymbol{r} = (x, y, z)$ and distance moved is in direction $\boldsymbol{c}$, then motion occurs along the line

$$\boldsymbol{r}(t) = \underset{\text{start}}{\boldsymbol{a}} + \underset{\text{direction}}{\boldsymbol{c}}t, \qquad \text{for which} \quad d\boldsymbol{r}(t) = \boldsymbol{c}dt, \qquad \left(\frac{d\underline{r}}{dt} = \underline{c}\right)$$

so that

$$W = \int_{t=0}^{(b-a)} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \underbrace{\boldsymbol{c}\,dt}_{d\underline{r}}$$

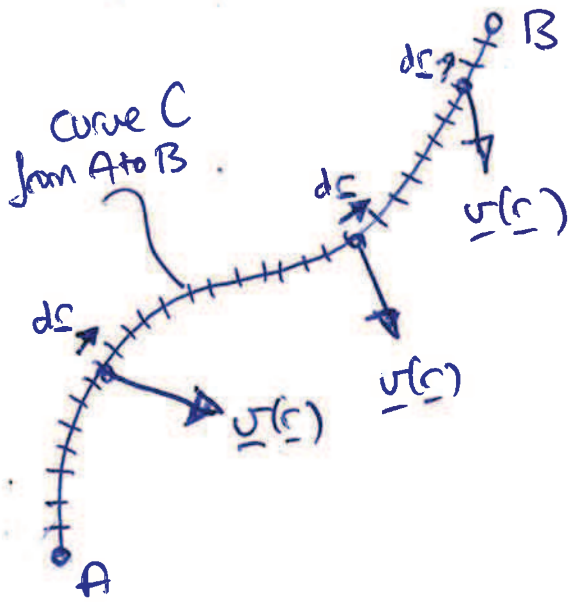
Worked example 4.1 *Calculate the work done by moving along the straight line from A at (2,1,1) to B at (3,2,2) in a force field*

$$\boldsymbol{F} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|^2} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{x^2 + y^2 + z^2}$$

But what if the path we move on is a general curve C rather than a straight line? We need to calculate at path integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ of the force along that curve.

Definition *The* **work integral** *is an example of a path integral of a vector field*

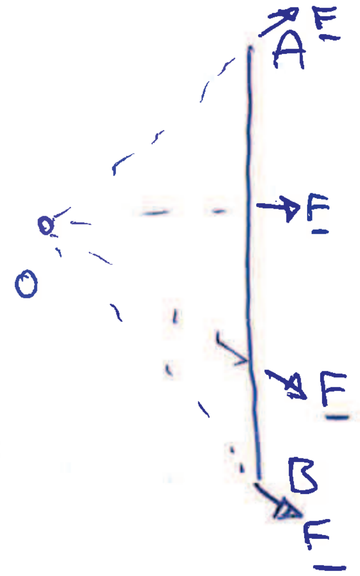
$$\boxed{\int_C \boldsymbol{v}(\boldsymbol{r}) \cdot d\boldsymbol{r}.} \tag{4.1}$$



Breaking curve into lots of little straight segments $d\underline{r}$

$$W = \underset{\text{sum}}{\int_A^B} \underset{\text{vector field of that segment}}{\underline{v}(\underline{r})} \cdot \underset{\text{little segments}}{d\underline{r}}$$

EXAMPLE 4.1 $\quad \underline{E} = \dfrac{\underline{r}}{|\underline{r}|^2}$



first parametrise line AB

$$\underline{r}(t) = \underset{\text{initial point}}{\underline{a}} + t\underset{\text{direction}}{\underline{b}} .$$

A @ t=0 : $\underline{r}(0) = \underline{a} = (2,1,1)$

B @ t=1 : $\underline{r}(1) = \underline{a}+\underline{b} = (3,2,2) \Rightarrow \underline{b} = (1,1,1)$

$$\therefore \quad \underline{r}(t) = (2,1,1) + (1,1,1)t = (2+t, 1+t, 1+t)$$

now calculate integral

$$W = \int_A^B \underline{E}(\underline{r}).d\underline{r} = \int_{t=0}^{1} \underline{E}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\,dt$$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(2+t, 1+t, 1+t) = (1,1,1)$$

$$\underline{F}(\underline{r}(t)) = \frac{(2+t, 1+t, 1+t)}{(2+t)^2+(1+t)^2+(1+t)^2} = \frac{(2+t, 1+t, 1+t)}{3t^2+8t+6}$$

$$\therefore \quad W = \int_{t=0}^{1} \frac{(2+t, 1+t, 1+t)}{3t^2+8t+6} \cdot (1,1,1)\,dt$$

$$= \int_{t=0}^{1} \frac{4+3t}{3t^2+8t+6}\,dt$$

regular integral! Similar to $\int \frac{f'}{f} = [\ln f]$

$$= \frac{1}{2}\int_{t=0}^{1} \frac{6t+8}{3t^2+8t+6}\,dt$$

$$= \frac{1}{2}\left[\ln(3t^2+8t+6)\right]_0^1 = \tfrac{1}{2}\ln 17 - \tfrac{1}{2}\ln 6$$

$$= \ln\sqrt{17/6}$$

$$d\underline{r} = \frac{d\underline{r}(t)}{dt}dt$$

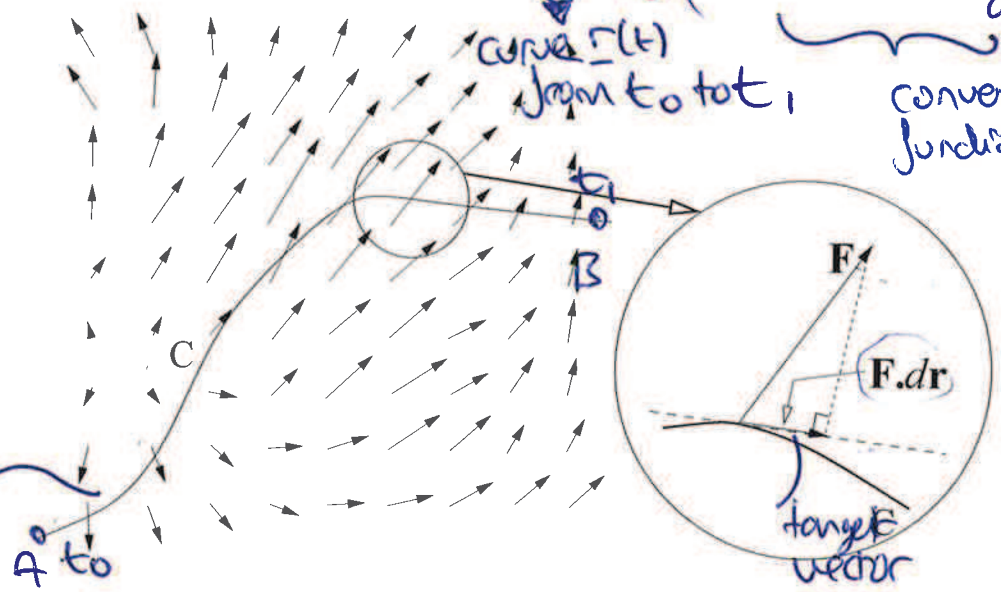
We can interpret this in 2D as summing the components of the vectors in the direction of the tangent to the curve C at each point along C whose position vector is $\boldsymbol{r}$.

$$W = \int_A^B \underline{F}\cdot d\underline{r} = \int_{t_0}^{t_1} \underbrace{\left(\underline{F}(\underline{r}(t))\cdot\frac{d\underline{r}}{dt}\right)}_{\text{conventional function of } t} dt$$

curve $\underline{r}(t)$ from t_0 to t_1

parametrise curve

$\underline{r}(t)$



The answer is a scalar

$$W = \int_C \boldsymbol{v}\cdot d\boldsymbol{r} = \int_C (v_1dx + v_2dy + v_3dz).$$

In order to evaluate such path integrals we need to have a parametrisation of the curve $C(t)$, just like we did for the straight line earlier ~~in worked example 4.1~~. That is we write

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} = (x(t), y(t), z(t))$$

Then

$$d\boldsymbol{r} = \frac{d\boldsymbol{r}(t)}{dt}dt = \left(\frac{dx(t)}{dt}\boldsymbol{i} + \frac{dy(t)}{dt}\boldsymbol{j} + \frac{dz(t)}{dt}\boldsymbol{k}\right)dt$$

ASIDE:

We shall see that for many physically important vector fields $\boldsymbol{v}$ (to be made precise shortly) the path integral is independent of the path taken. That is, if two curves C_1 and C_2 connect the same endpoints with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the integral of $\boldsymbol{v}$ over both paths is the same.

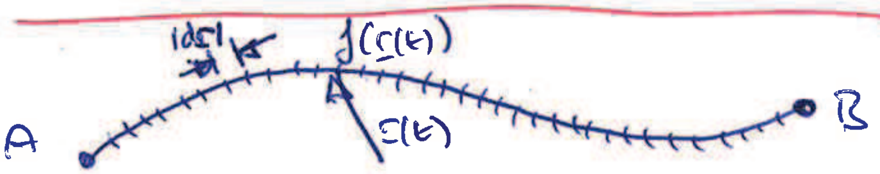
$$ds \equiv |d\underline{r}| = \left|\frac{d\underline{r}}{dt}\right| dt$$

Before considering how to parametrise various curves lets mention another type of path integral:

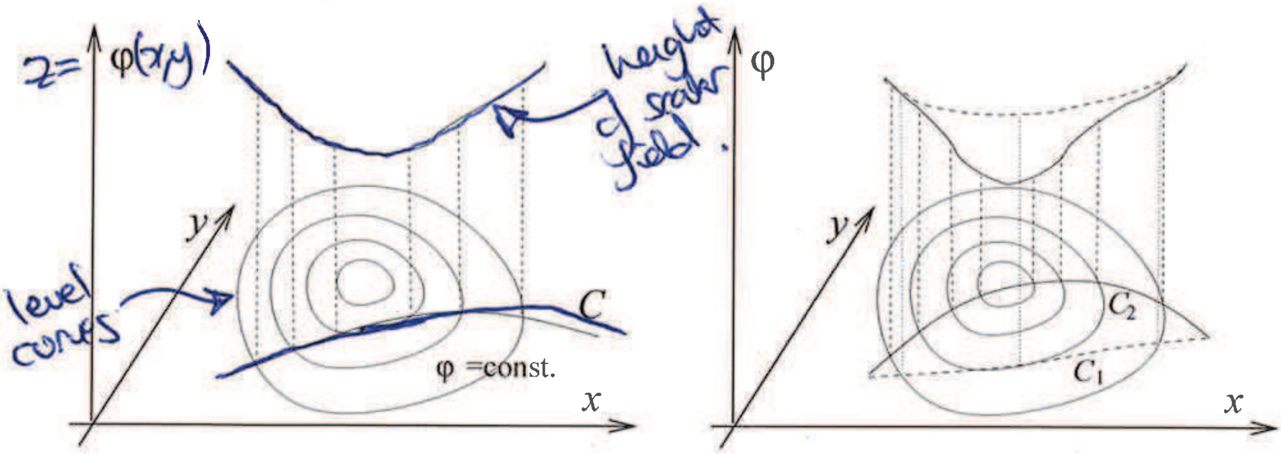
Definition: **path integral of a scalar field** $f(\boldsymbol{r})$

$$\int_C f(\boldsymbol{r})ds = \int_C f(\boldsymbol{r})|d\boldsymbol{r}| \tag{4.2}$$

where ds is the infinitesimal **arclength** along C (see later).



We can interpret this in 2D as taking a path C across the contours f =const. summing up the height.



Clearly in general, if two different curves C_1 and C_2 connect the same end points $\boldsymbol{a}$ and $\boldsymbol{b}$ then the integrals along C_1 and C_2 will be different.

ASIDE

One can also define other path integrals of vector quantities, e.g.

$$\int_C \boldsymbol{v}(\boldsymbol{r})ds = \int_C v_1(\boldsymbol{r})|d\boldsymbol{r}|\boldsymbol{i} + \int_C v_2(\boldsymbol{r})|d\boldsymbol{r}|\boldsymbol{j} + \int_C v_3(\boldsymbol{r})|d\boldsymbol{r}|\boldsymbol{k}$$

gives a vector

which is just 3 integrals of the form (4.2). So for the rest of this chapter we shall just consider integrals of the form (4.1) and (4.2).

treat components as scalar path integrals

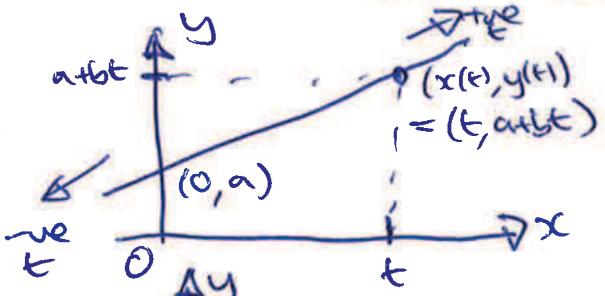
4.2 Parameterisation of Curves

The key to evaluating such integrals is to define a single (intrinsic) co-ordinate t that parametrises the curve C. Then use

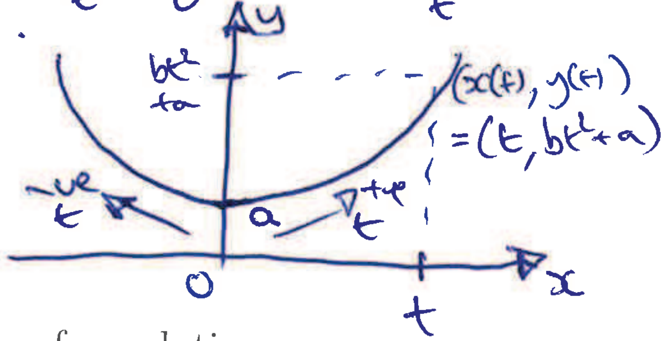
$$d\boldsymbol{r} = \frac{d\boldsymbol{r}}{dt}dt$$

Consider first curves in 2D. For some curves it is obvious how to do this, e.g. use the x-coordinate as the parameter: [curve s.t. only one y-value for every x-value]

- straight line $y = bx + a$, or $x = t$, $y = a + bt \Rightarrow$
  $\boldsymbol{r}(t) = (t, a + bt) = (0, a) + (1, b)t$ start direction
  $\Rightarrow \quad d\boldsymbol{r} = (1, b)\, dt$
  $\frac{dr}{dt} = \frac{d}{dt}(t, a+bt) = (1, b)$

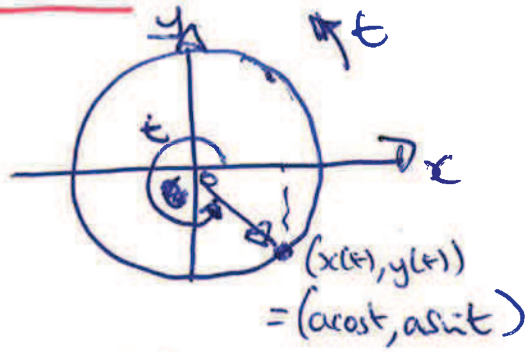


- parabola $y = bx^2 + a \Rightarrow$ $x = t$, $y = bt^2 + a$.
  $\boldsymbol{r}(t) = (t, bt^2 + a)$
  $\Rightarrow \quad d\boldsymbol{r} = (1, 2bt)\, dt$
  $\frac{dr}{dt} = \frac{d}{dt}(t, bt^2+a) = (1, 2bt)$

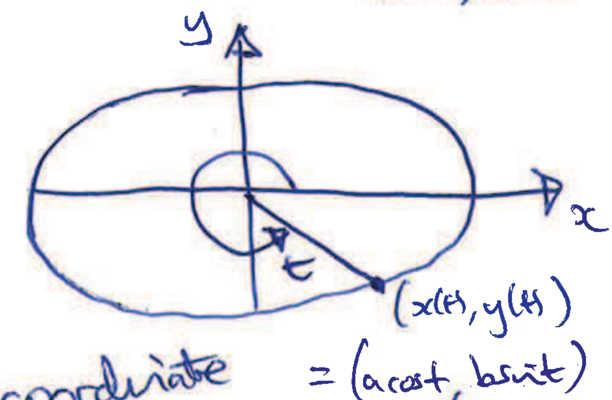


For other curves one can use an angular formulation

- circle $x^2 + y^2 = a^2 \Rightarrow$
  $\boldsymbol{r}(t) = (a\cos t, a\sin t)$, $t \in [0, 2\pi]$
  $\Rightarrow d\boldsymbol{r} = (-a\sin t, a\cos t)\, dt$



- ellipse $x^2/a^2 + y^2/b^2 = 1 \Rightarrow$
  $\boldsymbol{r}(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$
  $\Rightarrow d\boldsymbol{r} = (-a\sin t, b\cos t)\, dt$



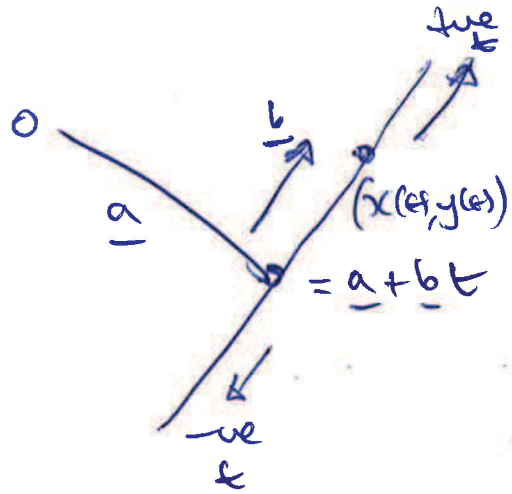
NOTE: CIRCLE / ELLIPSE: If use instead $x(t) = t$ $\Rightarrow$ half circle $\Rightarrow$ use an angular coordinate instead.

In 3D the concepts are similar; e.g.

- straight line

  $\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b}t = (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t)$
  $\Rightarrow d\boldsymbol{r} = (b_1, b_2, b_3) = \boldsymbol{b}\, dt$

  $$\frac{d\underline{r}}{dt} = \frac{d}{dt}(\underline{a} + \underline{b}t) = \underline{b}$$

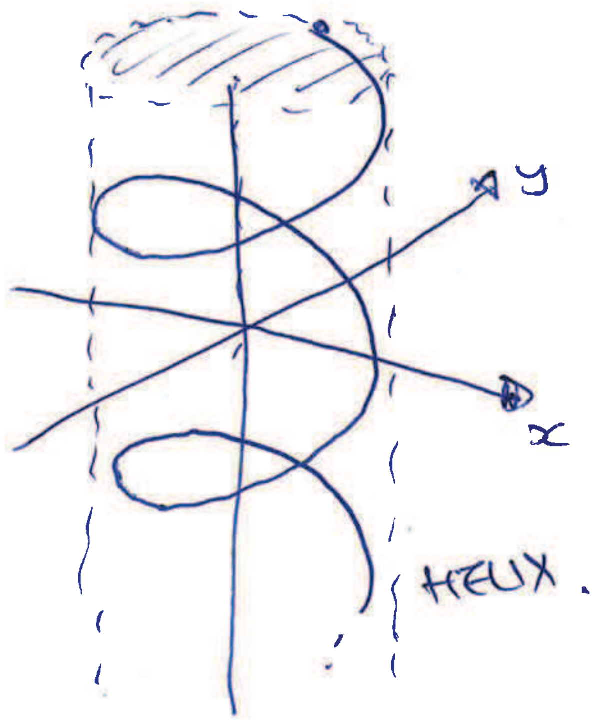


- helix (shape of classical light spring)

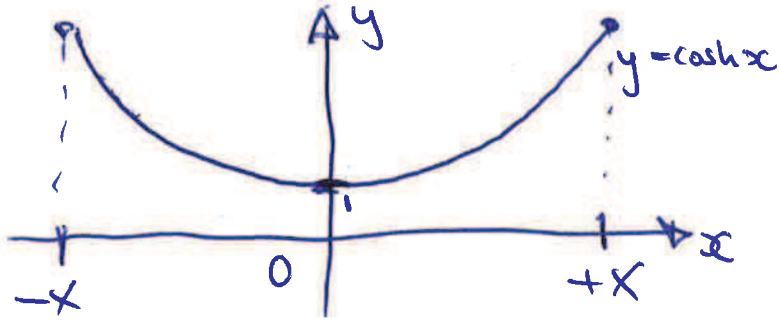
  x, y describe a circle, so that the helix lies on the surface of a cylinder, with linearly varying value of z. Here a is the radius of the helix and $b/a = \tan\theta$ where θ is the helix angle.

  $\boldsymbol{r} = (x, y, z) = (a\cos t, a\sin t, bt)$
  $\Rightarrow d\boldsymbol{r} = (-a\sin t, a\cos t, \boldsymbol{b})\, dt$

  $$\frac{d\underline{r}}{dt} = \frac{d}{dt}(a\cos t, a\sin t, bt)$$
  $$= (-a\sin t, a\cos t, b)$$



Example 4.2 (i)



write $y = \cosh x$ parametrically:

set $x(t) = t$

$\Rightarrow \quad y(t) = \cosh x = \cosh t$

$\therefore \quad \underline{r}(t) = (x(t), y(t)) = (t, \cosh t)$

endpoints: $x(t) = X = t$ ie $t = \pm X$.

$x(t) = -X = t$

arclength $\quad S = \int_C ds = \int_C |d\underline{r}| = \int_{t=-X}^{+X} \left| \frac{d\underline{r}}{dt}(t) \right| dt.$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(t, \cosh t) = (1, \sinh t)$$

$$\left| \frac{d\underline{r}}{dt} \right| = \sqrt{1^2 + \sinh^2 t} = \cosh t$$

$$S = \int_{t=-X}^{+X} \cosh t \, dt$$

$$= \left[\sinh t \right]_{-X}^{+X} = \sinh X - \sinh(-X)$$

$$= \underline{\underline{2 \sinh X}}$$

EXAMPLE 4.2(ii)

parametric curve

$$\underline{r}(t) = (a\cos t, a\sin t, bt)$$

one revolution:

start $t = 0$

finish $t = 2\pi$

arclength

$$S = \int |d\underline{r}| = \int_{t=0}^{2\pi} \left|\frac{d\underline{r}(t)}{dt}\right| dt$$

$$\frac{d\underline{r}}{dt} = (-a\sin t, a\cos t, b)$$

$$\left|\frac{d\underline{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2+b^2}$$

$$S = \int_{t=0}^{2\pi} \sqrt{a^2+b^2}\, dt = 2\pi\sqrt{a^2+b^2}$$

exercise in terms of $\tan\theta = b/a$

$$ds \equiv |d\underline{r}| = \left|\frac{d\underline{r}}{dt}dt\right| = \left|\frac{d\underline{r}}{dt}\right| dt$$

The length, or total **arclength** S of a curve C parametrised by $\boldsymbol{r} = (x(t), y(t), z(t))$ from $t = a$ to $t = b$ is the sum of little pieces of curve (infinitesimal arcs)

$\frac{d}{dt}(x(t), y(t), z(t))$

increments of arclength

$$ds \equiv |d\boldsymbol{r}| = \left|\frac{d\boldsymbol{r}(t)}{dt}dt\right| = \left|\frac{d\boldsymbol{r}}{dt}\right| dt$$

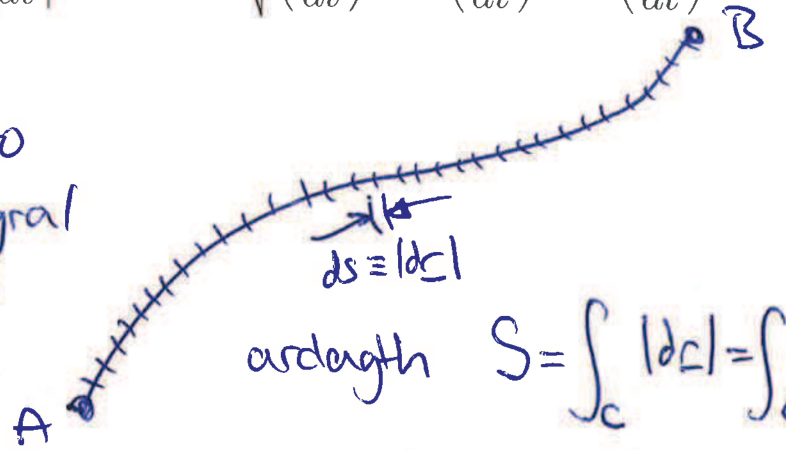
$$= \left|\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}\right| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt;$$

so that

$$S = \int_C ds = \int_C |d\boldsymbol{r}| = \int \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt;$$

total arclength

corresponds to scalar path integral with scalar field $f(\underline{r}) = 1$



arclength $S = \int_C |d\underline{r}| = \int_C ds$.

Worked example 4.2 *Calculate the total arclength of the following curves*

1. *The catenary (shape of a perfectly flexible heavy chain suspended between two points $x = -X$ and $x = X$) whose height is given by the dimensionless formula $y = \cosh x$.*

2. *One complete revolution of a helix of radius a and helix angle $\tan\theta = b/a$.*

$\underline{r}(t) = (a\cos t, a\sin t, bt)$

$|d\underline{r}| \equiv ds$; $S = \int_C ds = \int_C |d\underline{r}|$

total arclength

Application to arclength

Arclength gives a special choice of the parametrisation such that the curve is parametrised by its length. Given any parametrisation $\boldsymbol{r}(t)$ of a curve C, we can obtain the arclength parametrisation $\boldsymbol{r}(s)$ of C, where the parameter s represents the arclength, by defining

arclength parameter to point $\underline{r}(t)$ from $\underline{r}(0)$

$$s(t) = \int_0^t \left|\frac{d\boldsymbol{r}}{dt}\right| dt,$$

ie $\int_{\text{start}}^{\text{s point}} |d\underline{r}|$

So that we can reparametrise C as $\boldsymbol{r}(s)$ instead of $\boldsymbol{r}(t)$.

(i) start with curve parametrised in terms of t (some arbitrary choice)

(ii) convert variables to parameterising with arclength $s(t)$

Note that

need to write dt in terms of ds

$$\frac{ds}{dt} = \frac{d}{dt}\int_0^t \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \left|\frac{d\boldsymbol{r}}{dt}\right|,$$

$ds = \frac{ds}{dt} dt$

so that with respect to the parameterisation s:

RESULT :

$$\left|\frac{d\boldsymbol{r}(s)}{ds}\right| = \left|\frac{d\boldsymbol{r}(t)}{\cancel{dt}} \Big/ \frac{ds(t)}{\cancel{dt}}\right| = \left|\frac{d\boldsymbol{r}}{dt} \Big/ \left|\frac{d\boldsymbol{r}}{dt}\right|\right| = 1.$$

$s(t) = \int_0^{s(t)} \left|\frac{d\underline{r}}{ds}\right| ds$

Implication:

~~Another way of saying it~~:

Scalar path integral $\int_{t=0}^{t} \varphi(\underline{r}(t))|d\underline{r}| = \int_{t=0}^{t} \varphi(\underline{r}(t))\left|\frac{d\underline{r}}{dt}\right|dt = \int_{s=0}^{s(t)} \varphi(\underline{r}(s))\cancel{\left|\frac{d\underline{r}}{ds}\right|}^{1} ds$

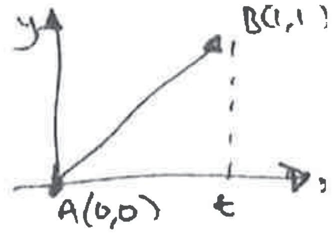
This enables us to define simple geometric properties of the curve (YOU'RE NOT EXPECTED TO MEMORISE THESE DEFINITIONS)

1. unit *tangent vector* $\hat{\boldsymbol{u}}(s) = \boldsymbol{r}'(s) = d\boldsymbol{r}(s)/ds$
2. *curvature* $\kappa(s) = |\hat{\boldsymbol{u}}'(s)| = |\boldsymbol{r}''(s)|$
3. unit (principle) *normal vector* $\boldsymbol{p(s)} = \frac{\boldsymbol{u}'}{|\boldsymbol{u}'|} = \frac{\boldsymbol{r}''}{\kappa}$
4. *torsion* (tortuosity) $\tau = -\boldsymbol{p(s)} \cdot \boldsymbol{b}'(s)$,
   where $\boldsymbol{b}(s) = (\boldsymbol{u}(s) \times \boldsymbol{p}(s))$ is so-called unit *bi-normal.*

$' \equiv \frac{d}{ds}$.

These definitions only work if we use the arclength parametrisation.

Example 4.3 (i)



STEP 1 : parameterize $y = x$

write $x(t) = t \Rightarrow y(t) = x(t) = t$

$\therefore \quad \underline{r}(t) = (x(t), y(t)) = (t, t)$

STEP 2 : find endpoints

$\underline{A} = (0,0) \Rightarrow t = 0$

$\underline{B} = (1,1) \Rightarrow t = 1$

STEP 3 : differential $|ds| = |d\underline{r}| = \left|\frac{d\underline{r}}{dt}\right| dt$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(t,t) = (1,1)$$

$$\therefore \quad \left|\frac{d\underline{r}}{dt}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

STEP 4

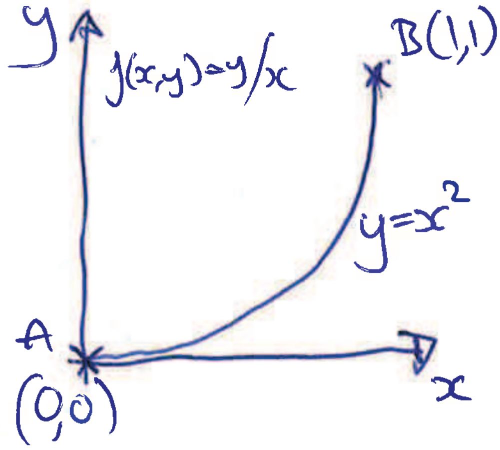
$$\int_C f(\underline{r}(t))\,ds = \int f(\underline{r}(t)) \left|\frac{d\underline{r}}{dt}\right| dt$$

$$f(\underline{r}(t)) = \frac{y(t)}{x(t)} = \frac{t}{t} = 1$$

$$\therefore \int_C f(\underline{r}(t))\,ds = \int_0^1 1 \times \sqrt{2}\, dt$$

$$= \sqrt{2}$$

EXAMPLE 4.3 (ii)



STEP 1 : parametrise $y = x^2$

write $x = t \Rightarrow y(t) = t^2$

$$\underline{r}(t) = (x(t), y(t)) = (t, t^2)$$

STEP 2 : find limits ie endpoints

$$A = (0,0) \Rightarrow t = 0$$
$$B = (1,1) \Rightarrow t = 1$$

STEP 3 : $|d\underline{r}| = \left|\frac{d\underline{r}}{dt}\right| dt$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(t, t^2) = (1, 2t)$$

$$\left|\frac{d\underline{r}}{dt}\right| = \sqrt{1+4t^2}$$

STEP 4 : $\int_{AB} f(\underline{r}(t)) \left|\frac{d\underline{r}}{dt}\right| dt$

$$f(\underline{r}(t)) = y(t)/x(t) = t^2/t = t.$$

$$\therefore \int_{AB} f(\underline{r}(t)) \left|\frac{d\underline{r}}{dt}\right| dt = \int_{t=0}^{1} t\sqrt{1+4t^2}\, dt.$$

$$= \left[\frac{1}{\frac{3}{2}\times 8}(1+4t^2)^{3/2}\right]_0^1$$

$$= \underline{(5\sqrt{5}-1)/12}$$

4.3 Evaluation of scalar path integrals

To evaluate the scalar path integrals of the form (4.2)

$$\int_C f(\boldsymbol{r})ds = \int_C f(\boldsymbol{r})|d\boldsymbol{r}|$$

from points A to B along the curve C:

0. DRAW A PICTURE

1. parametrise the curve C as $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ (NOTE: any parametrisation t will do provided we strictly increase from A to B along the curve)

2. work out the limits a and b on t

3. calculate

CALCULATE THIS

$$ds = |d\boldsymbol{r}| = \left|\frac{d\boldsymbol{r}(t)}{dt}\right|dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

4. evaluate the scalar field $f(\underline{r})$ along $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ and integrate with respect to t.

$$\int_C f(\boldsymbol{r})ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$f(\underline{r}(t))$ $\quad\quad$ $\left|\frac{d\underline{r}}{dt}(t)\right|$

We have already seen examples of this already: total arclength S is just the scalar path integral $S = \int_C 1\, ds$ of the field $f(\boldsymbol{r}) = 1$.

Worked example 4.3 *Integrate $f(\boldsymbol{r}) = y/x$ from $A = (0,0)$ to $B = (1,1)$ along the two curves $y = x$ and $y = x^2$. Are the two answers the same?*

VECTOR PATH INTEGRALS

4.4 Evaluation of work integrals

To evaluate the work integral of the form

$$W = \int_C \boldsymbol{v}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C \boldsymbol{v}(\boldsymbol{r}) \cdot \frac{d\boldsymbol{r}}{dt} dt$$

from points A to B along the curve C:

0. DRAW A PICTURE (IF YOU CAN!)

1. parametrise the curve C as $r(t) = (x(t), y(t), z(t))$ (AGAIN any reasonable parametrisation will do)

2. work out the limits a and b on t

3. evaluate the vector field $\boldsymbol{v}$ along $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, form the dot product and integrate w.r.t. t:

$$\int_C \boldsymbol{v}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{t=a}^{t=b} \boldsymbol{v}(x(t), y(t), z(t)) \cdot \left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k} \right) dt$$

$\underline{v}(\underline{r}(t))$ ← dot together → $\frac{d\underline{r}}{dt}$

Worked example 4.4

Find the work done $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *in moving a particle from* $(0,0,0)$ *to* $(1,1,1)$ *in the force field*

$$\boldsymbol{F} = (2x + y^2)\boldsymbol{i} - 3xy\boldsymbol{j} + \boldsymbol{k}$$

along the following paths

1. *the straight path* C_1*, the straight line joining* $(0,0,0)$ *to* $(1,1,1)$
2. *the path* C_2 *given parametrically by* $x = t$, $y = t^2$, $z = t^3$

Are the answers the same?

EXAMPLE 4.4(i)

STEP 1 parameterise the curve $x = y = z \; (= t)$

$$\underline{r}(t) = (x(t), y(t), z(t)) = (t, t, t)$$

STEP 2 endpoints

$$\underline{A} = (0,0,0) \Rightarrow t = 0$$

$$\underline{B} = (1,1,1) \Rightarrow t = 1$$

STEP 3 integrate

$$\int_A^B \underline{F}(\underline{r}) \cdot d\underline{r} = \int_{t=0}^1 \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt} \, dt$$

$$\underline{F}(\underline{r}(t)) = (2x(t) + y(t)^2, -3x(t)y(t), 1)$$

$$= (2t + t^2, -3t^2, 1)$$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(t,t,t) = (1,1,1)$$

$$\therefore \int_A^B \underline{F}(\underline{r}) \cdot d\underline{r} = \int_{t=0}^1 (2t + t^2, -3t^2, 1) \cdot (1,1,1) \, dt$$

$$= \int_{t=0}^1 (-2t^2 + 2t + 1) \, dt$$

$$= \left[\frac{-2t^3}{3} + \frac{2t^2}{2} + t \right]_0^1$$

$$= -\frac{2}{3} + 1 + 1 = 4/3$$

EXAMPLE 4.4(ii)

STEP 1   parametrise curve

$$\underline{r}(t) = (x(t), y(t), z(t)) = (t, t^2, t^3)$$

STEP 2 : endpoints.

$$\underline{A} = (0,0,0) \Rightarrow t = 0$$

$$\underline{B} = (1,1,1) \Rightarrow t = 1.$$

STEP 3 : $\int \underline{F}(\underline{r}(t)) \circ \frac{d\underline{r}}{dt}\,dt$

$$\underline{F}(\underline{r}(t)) = (2x+y^2, -3xy, 1)$$
$$= (2t+t^4, -3t^3, 1).$$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(t, t^2, t^3) = (1, 2t, 3t^2)$$

$$\therefore \int_{AB} \underline{F}(\underline{r}) \cdot d\underline{r} = \int_{t=0}^{1} (2t+t^4, -3t^3, 1) \circ (1, 2t, 3t^2)\,dt$$

$$= \int_{t=0}^{1} (2t \cancel{+t^4} - \overset{5}{\cancel{6}}t^4 + 3t^2)\,dt.$$

$$= \left[t^2 - t^5 + t^3\right]_{t=0}^{1}$$

$$= 1$$

vector & scalar path integrals

Properties of work and line integrals

- Linear:

α, β constants.

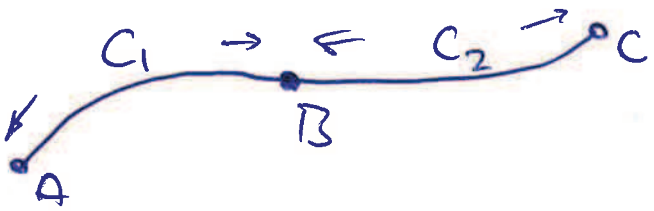
$$\int_C(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2)\cdot d\boldsymbol{r} = \alpha\int_C \boldsymbol{v}_1\cdot d\boldsymbol{r} + \beta\int_C \boldsymbol{v}_2\cdot d\boldsymbol{r}$$

$$\int_C(\alpha f_1 + \beta f_2)|d\boldsymbol{r}| = \alpha\int_C f_1|d\boldsymbol{r}| + \beta\int_C f_2|d\boldsymbol{r}|$$

- decomposition into pieces: if $C = C_1 + C_2$

$$\overset{AC}{\int_C} \boldsymbol{v}\cdot d\boldsymbol{r} = \overset{AB}{\int_{C_1}} \boldsymbol{v}\cdot d\boldsymbol{r} + \overset{BC}{\int_{C_2}} \boldsymbol{v}\cdot d\boldsymbol{r}$$

$$\int_C f|d\boldsymbol{r}| = \int_{C_1} f|d\boldsymbol{r}| + \int_{C_2} f|d\boldsymbol{r}|$$

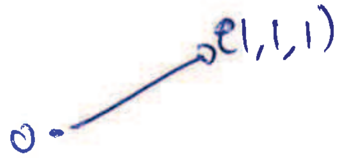


Worked example 4.5

Find the work done in moving from $(0,0,0)$ *to* $(1,1,1)$ *in the force field* $\boldsymbol{F}$ *given by*

$$\boldsymbol{F} = (2xy + z^3)\boldsymbol{i} + x^2\boldsymbol{j} + 3xz^2\boldsymbol{k}$$

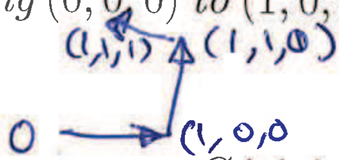
along the paths



1. *C_1 the straight line joining* $(0,0,0)$ *to* $(1,1,1)$

EXERCISE (HOME)

2. *C_2 composed of the three straight lines joining* $(0,0,0)$ *to* $(1,0,0)$ *to* $(1,1,0)$ *to* $(1,1,1)$.



Both answers are the same (as they would be for any curve C joining $(0,0,0)$ and $(1,1,1)$). This is because the force field $\boldsymbol{F}$ is conservative.

Worked example 4.5 continued...

Verify the force field is conservative. $\nabla\times\boldsymbol{F} = 0$

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = (0, -3z^2+3z^2, 2x-2x)$$

$$= \underline{0}$$

(4.3) vector analogue of fund theorem of calculus $\int_a^b dt \frac{df}{dt} = f(b) - f(a)$

4.5 Conservative vector fields & path independence

Recall from Chapter 2, a vector field $\boldsymbol{v}$ is conservative if and only if curl $\boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{\nabla}\phi$, for some scalar function ϕ.

We can now add the following result:

If the vector field $\boldsymbol{F}$ is **conservative** then the work integral $\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r}$ between two points A and B is **independent of the path** C chosen between A and B (provided $\boldsymbol{F}$ remains finite within a simple domain containing A and B).

Moreover, writing $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ we have

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_A^B \boldsymbol{F}\cdot d\boldsymbol{r} = \int_A^B \boldsymbol{\nabla}\phi\cdot d\boldsymbol{r} = \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}) \qquad (4.3)$$

ie independent of path.

Proof. We showed in Chapter 2 that if $\boldsymbol{F}$ is conservative then there exists a scalar field ϕ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$. So

parametric curve $\underline{c}(t)$

$$\begin{aligned}
\int_A^B \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_A^B \boldsymbol{\nabla}\phi\cdot d\boldsymbol{r} = \int_{t_A}^{t_B} \boldsymbol{\nabla}\phi\cdot\frac{d\boldsymbol{r}}{dt}\,dt \\
&= \int_{t_A}^{t_B}\left(\frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}\right)\cdot\left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}\right)dt \\
&= \int_{t_A}^{t_B}\left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right)dt \\
&= \int_{t_A}^{t_B}\frac{d\phi}{dt}(\boldsymbol{r}(t))\,dt = \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}).
\end{aligned}$$

analogue of chain rule $\frac{d}{dx}f(g(x)) = \frac{df}{dg}\frac{dg}{dx}$

Hence the integral only depends on the value of ϕ at A and B and is independent of the the path taken between them. $\square$

EXERCISE.

Worked example 4.6 *Calculate curl* $\boldsymbol{F}$ *for the force field* $\boldsymbol{F}$ *taken in worked example 4.5, hence show the field is conservative. Find the scalar potential field* ϕ *and hence calculate the work done in moving from* $(0,0,0)$ *to* $(1,1,1)$.

4.5(i)

STEP 1 : $\underline{r}(t) = (t, t, t)$

STEP 2 : endpoints $(0,0,0) \Rightarrow t=0$

$(1,1,1) \Rightarrow t=1$

STEP 3 : $\int \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt}\, dt$

$$\underline{F}(\underline{r}(t)) = (2x(t)y(t) + z(t)^3,\ x(t)^2,\ 3x(t)z(t)^2)$$

$$= (2t^2 + t^3,\ t^2,\ 3t^3)$$

$$\frac{d\underline{r}}{dt} = (1,1,1)$$

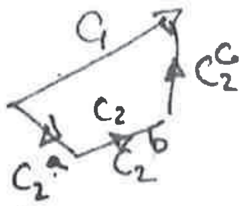
$$\int_{t=0}^{1} (2t^2 + t^3,\ t^2,\ 3t^3) \cdot (1,1,1)\, dt$$

$$= \int_{t=0}^{1} (3t^2 + 4t^3)\, dt$$

$$= \left[t^3 + t^4 \right]_0^1$$

$$= 2$$

Example 4.5 (ii)



Step 1: parameterize curves

C_2^a: $\underline{r}(t) = \underline{0} + t(1,0,0)$, $\frac{d\underline{r}}{dt} = (1,0,0)$

C_2^b: $\underline{r}(t) = (1,0,0) + t(0,1,0)$, $\frac{d\underline{r}}{dt} = (0,1,0)$

C_2^c: $\underline{r}(t) = (1,1,0) + t(0,0,1)$, $\frac{d\underline{r}}{dt} = (0,0,1)$

Step 2: endpoints

C_2^a: $t \in [0,1]$ ie $\underline{0}$ to $(1,0,0)$

C_2^b: $t \in [0,1]$ ie $(1,0,0)$ to $(1,1,0)$

C_2^c: $t \in [0,1]$ ie $(1,1,0)$ to $(1,1,1)$.

Step 3: integrate $\int_{C_2} \underline{F} \cdot d\underline{r} = \int_{C_2^a} \underline{F} \cdot d\underline{r} + \int_{C_2^b} \underline{F} \cdot d\underline{r} + \int_{C_2^c} \underline{F} \cdot d\underline{r}$

C_2^a: $\underline{F}(\underline{r}(t)) = (2xy + z^3, x^2, 3xz^2)$ $\quad x(t) = t$, $y(t) = z(t) = 0$

$= (0, t^2, 0)$

C_2^b: $\underline{F}(\underline{r}(t)) = (2t, 1, 0)$ $\quad x(t) = 1$, $y(t) = t$, $z(t) = 0$

C_2^c: $\underline{F}(\underline{r}(t)) = (2 + t^3, 1, 3t^2)$ $\quad x(t) = y(t) = 1$, $z(t) = t$

$$\int_{C_2} \underline{F} \cdot d\underline{r} = \int_{t=0}^{1} (0,t^2,0) \cdot (1,0,0)\, dt + \int_0^1 (2t,1,0) \cdot (0,1,0)\, dt$$

$$+ \int_0^1 (2+t^3, 1, 3t^2) \cdot (0,0,1)\, dt$$

$$= \cancelto{0}{\int_0^1 0\, dt} + \int_0^1 1\, dt + \int_0^1 3t^2\, dt$$

$$= [t]_0^1 + [t^3]_0^1$$

$$= 2$$

EXAMPLE 4.6 $\quad \underline{F} = (2xy + z^3, x^2, 3xz^2)$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = 0\,\underline{i} - (3z^2 - 3z^2)\underline{j} + (2x-2x)\underline{k}$$

$$= \underline{0} \quad \text{conservative}.$$

write $\quad \underline{F} = \underline{\nabla}\varphi = \begin{pmatrix} \partial\varphi/\partial x \\ \partial\varphi/\partial y \\ \partial\varphi/\partial z \end{pmatrix} = \begin{pmatrix} 2xy + z^3 \\ x^2 \\ 3xz^2 \end{pmatrix}$

components: $\quad \dfrac{\partial\varphi}{\partial x} = 2xy + z^3 \Rightarrow \varphi = \int dx\,(2xy + z^3)$

$$\therefore \varphi = x^2 y + z^3 x + \underset{\text{const } \int}{c(y,z)}$$

subst in: $\quad \dfrac{\partial\varphi}{\partial y} = x^2 \Rightarrow x^2 + \dfrac{\partial c}{\partial y} = x^2$

$$\therefore \frac{\partial c}{\partial y} = 0 \Rightarrow c(y,z) = \underset{\text{const } \int}{d(z)}$$

subst in: $\quad \dfrac{\partial\varphi}{\partial z} = 3xz^2 \Rightarrow \cancel{3z^2x} + \dfrac{\partial d}{\partial z} = \cancel{3xz^2}$

$$\therefore d = \text{const}.$$

$$\therefore \varphi(x,y,z) = x^2 y + z^3 x + d$$

Work done $= \varphi(3,1,4) - \varphi(1,-2,1)$

$$= (9 + 64 \times 3 + \cancel{d}) - (-2 + 1 + \cancel{d}) = 202$$

EXAMPLE 4.5 : Work done $= \varphi(1,1,1) - \varphi(0,0,0)$

$$= (1 + 1 + \cancel{d}) - (0 + 0 + \cancel{d}) = 2$$

How: $\underline{F}$ curl free, $\nabla\times\underline{F}=\underline{0}$, conservative $\Longleftrightarrow$ work integral $W=\int_C \underline{F}\cdot d\underline{r}$ independent of path C taken $=\varphi(\underline{b})-\varphi(\underline{a})$

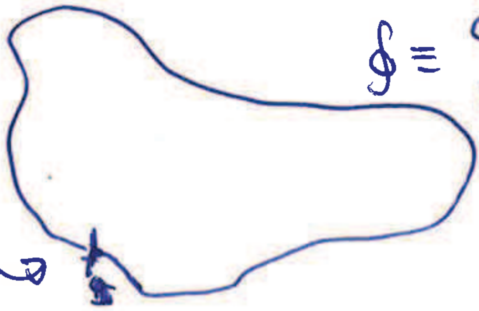
Remarks

- The converse is also true; if the work integral is independent of path taken between any two points then there must exist a scalar potential function ϕ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

- If C is a closed curve, then we can show that for a conservative vector field

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$$

(for any closed curve C inside a simple domain in which $\boldsymbol{F}$ is finite)



$\oint \equiv$ closed loop integral

can pick any point as beginning & end point

if $\underline{F}$ is conservative

$\oint_C \underline{F}\cdot d\underline{r} = \varphi(\underline{a})-\varphi(\underline{a}) = 0$

any point

NOTE the special notation $\oint$ for integrals around closed paths.

- So we have (provided $\boldsymbol{F}$ is finite everywhere of interest)

$$\boxed{\text{Conservative} \Leftrightarrow \text{curl}\,\boldsymbol{F} = \underline{0} \Leftrightarrow \boldsymbol{F} = \boldsymbol{\nabla}\phi \Leftrightarrow \oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0.}$$

- Physical interpretation: ϕ is the scalar potential function associated with the force $\boldsymbol{F} = \underline{\nabla}\phi$. Consider the earth's gravitational field. The gravitational force $\boldsymbol{F}$ associated with a body of mass m at position vector $\boldsymbol{r}$ from the center of the earth is

$$\boldsymbol{F} = -\boldsymbol{r}\frac{k}{|\boldsymbol{r}|^3} = -\frac{k}{|\boldsymbol{r}|^2}\hat{\boldsymbol{r}} \qquad (4.4)$$

$\underline{\nabla}\times\underline{F}=\underline{0} \Rightarrow \underline{F}=\underline{\nabla}\varphi$ when work out $\varphi=\frac{k}{|\underline{r}|}$

WD moving $\underline{r}_1$ to $\underline{r}_2$ $= \varphi(\underline{r}_2)-\varphi(\underline{r}_1) = \frac{k}{|\underline{r}_2|}-\frac{k}{|\underline{r}_1|}$

Since $\boldsymbol{F}$ is conservative ($\phi = k/|\boldsymbol{r}|$, see Worked example 2.3), the independence of path means that the energy required to overcome the gravitational potential in order to escape the atmosphere is independent of the path taken (e.g. by a space rocket).

- More generally, **central** force fields can be expressed as a force in the direction of the origin which is with magnitude a function only of distance $r := |\boldsymbol{r}|$ from the origin:

CENTRAL FORCE FIELD ?

$$\boldsymbol{F} = \frac{df(r)}{dr}\hat{\boldsymbol{r}}, \quad \text{for some scalar function } f.$$

Then $\boldsymbol{F}$ is conservative, because it can be shown that

$$\boldsymbol{F} = \boldsymbol{\nabla}\phi, \quad \text{where} \quad \phi(\boldsymbol{r}) = f(r) = f(|\boldsymbol{r}|). \tag{4.5}$$

Taking the special case $f = k/r$ we get the $1/r^2$ force law (4.4) for gravity (applies also to electrostatics).

EXERCISE

Worked example 4.7 *Verify the result* (4.5) *for: (i) the particular case* $f = r^3$*; and (ii) the general case* $f(\boldsymbol{r}) = f(|\boldsymbol{r}|)$.

- Force fields that are not conservative ($\boldsymbol{\nabla} \times \boldsymbol{F} \neq \underline{0}$) are called **non-conservative** or **dissipative**. An example is the force on a mass in a rotating frame due to an angular acceleration $\boldsymbol{a}$

$$\boldsymbol{F} = m\boldsymbol{a} = m\boldsymbol{\omega} \times \boldsymbol{r}$$

$\underline{\nabla} \times \underline{F} = 2\underline{\omega}$
non-conservative.

Then curl $F = 2m\boldsymbol{\omega}$ (see Worked example 3.2) and clearly from the physics the work done in moving in such a force field is not independent of path.

Example 4.7 $\quad \varphi = f(r) = r^3$

$$\therefore \underline{F} = \underline{\nabla}\varphi = \left(\frac{\partial}{\partial x}r^3, \frac{\partial}{\partial y}r^3, \frac{\partial}{\partial z}r^3\right)$$

$$= 3r^2\left(\partial r/\partial x, \partial r/\partial y, \partial r/\partial z\right)$$

$$r = (x^2+y^2+z^2)^{1/2} \Rightarrow \partial r/\partial x = \tfrac{1}{2}(x^2+y^2+z^2)^{-1/2} \times 2x = x/r$$

$$\partial r/\partial y = y/r, \quad \partial r/\partial z = z/r$$

$$\therefore \underline{F} = 3r^2(x/r, y/r, z/r) = 3r^2\,\underline{r}/r = 3r^2\,\hat{\underline{r}}$$

$\frac{df}{dr}$

GENERAL PROOF : $\quad \varphi = f(r) \qquad r = |\underline{r}|$

$$\underline{\nabla}\varphi = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{df}{dr}\frac{\partial r}{\partial x}, \frac{df}{dr}\frac{\partial r}{\partial y}, \frac{df}{dr}\frac{\partial r}{\partial z}\right)$$

$$= \frac{df}{dr}\underline{\nabla} r$$

$$r = (x^2+y^2+z^2)^{1/2} \Rightarrow \partial r/\partial x = x/r, \ \partial r/\partial y = y/r, \ \partial r/\partial z$$

(as above)

$$\therefore \underline{\nabla} r = \underline{r}/r = \hat{\underline{r}}$$

$$\therefore \underline{\nabla}\varphi = \frac{df}{dr}\hat{\underline{r}}$$

QED

Summary

- Two forms of integrals:

$$\text{scalar path integral} \quad \int_C f(\boldsymbol{r})ds$$

$$\text{work integral} \quad \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

- Evaluate by parametrising the curve; $C := \boldsymbol{r}(t)$. Note special parameterisation by arclength $\boldsymbol{r}(s)$: $ds = |d\boldsymbol{r}|$.
- Work integrals of conservative fields are independent of path

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \phi(B) - \phi(A) \quad \Leftrightarrow \quad \oint \boldsymbol{F} \cdot d\boldsymbol{r} = 0.$$