EMAT10100 Engineering Maths I
Lecture 9: Application of Vectors to Geometry

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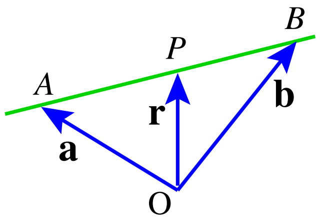
Looking back, looking forward

- Last time:
  - 3D vectors in component form
  - scalar (dot) product
  - vector (cross) product
  - applications in mechanics (moments, rotation)
- This time: geometry
  - vector equations of line, plane and sphere
  - scalar triple product
  - vector triple product

Vector equation of a line

- Straight line through points A and B:
- $\overrightarrow{\mathrm{AP}}$ is parallel to $\overrightarrow{\mathrm{AB}}$,
  so $\overrightarrow{\mathrm{AP}} = t\,\overrightarrow{\mathrm{AB}}$ (scalar t)
  so $\mathbf{r} - \mathbf{a} = t(\mathbf{b} - \mathbf{a})$.



- Vector equation of line:

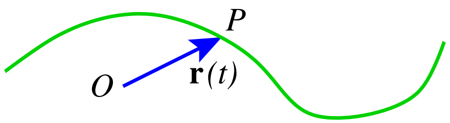
$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

- As t varies, P moves along line
  - E.g., when $t = 0$, P=A
  - Or when $t = 1$, P=B
  - t varies from $-\infty$ to $+\infty$

Exercise

- $\mathbf{a} = (1, 4, 6)$ and $\mathbf{b} = (3, 5, 7)$ lie on the same straight line.
- Find:
  - the equation of the line in vector form
  - the equation of the line in Cartesian form
  - where this line intersects the (x, z) plane
- Note this is *James* Worked example 4.38
  - read this and similar examples there
  - do similar exercises

Differentiation of vectors



- Curvy line $\mathbf{r}(t)$:
  natural interpretation as position of particle at time t
- Q: How to find velocity $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$?
- A: Just differentiate component by component.
- Example: Take $\mathbf{r} = (\log_e t, \sin t, t^3 + 3t^2)$. Find $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$.
- More complicated vector differentiation next year

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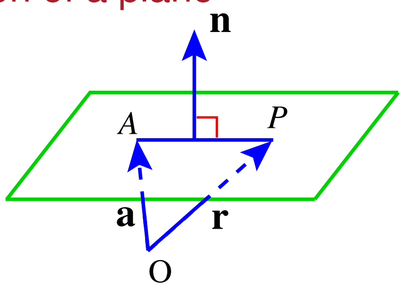
Vector equation of a plane



- AP lies in plane and $\vec{\mathrm{AP}} = \mathbf{r} - \mathbf{a}$
- So: $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$
- Vector equation of plane:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

or (alternative) $\quad \mathbf{r} \cdot \mathbf{n} = \rho \quad$ (ρ scalar constant)

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Exercise

- Find the point where the plane

$$\mathbf{r} \cdot (1, 2, 2) = 3$$

meets the line

$$\mathbf{r} = (2, 1, 1) + \lambda(0, 1, 2)$$

- Note this is *James* Worked example 4.46
  - read this and similar examples there
  - do similar exercises

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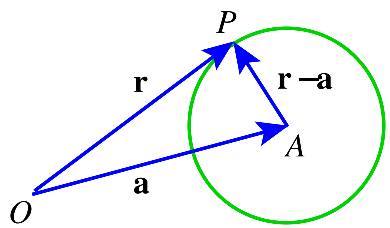
Vector equation of a sphere

- Consider 2D case (circle) for simplicity

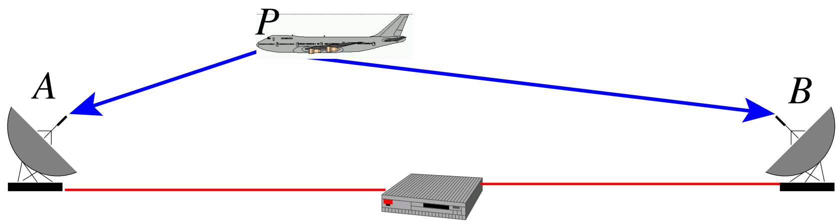


- P always same distance (radius ρ_s) from A
- NB: $\vec{\mathrm{AP}} = \mathbf{r} - \mathbf{a}$, so $|\vec{\mathrm{AP}}| = |\mathbf{r} - \mathbf{a}|$
- Vector equation of circle (or sphere in 3D):

$$|\mathbf{r} - \mathbf{a}| = \rho_s$$

Engineering HOT SPOT

- Positioning with ultrasound, radar pings etc.
- Detect landmines, breast cancer (Prof Ian Craddock)



- Elapsed time between signals: $t = t_A - t_B$.

$$|\mathbf{a}-\mathbf{r}| - |\mathbf{b}-\mathbf{r}| = ct \qquad \text{(wave speed } c\text{)}$$

- Vector equation of a hyperbola!

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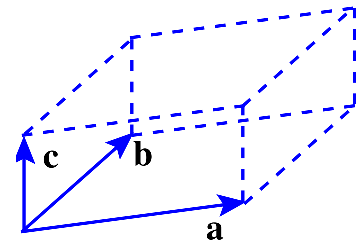
Scalar triple product

One of two ways to multiply three vectors together:
Scalar triple product $= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



(we will cover matrix determinants later)

- Note: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$
  So sometimes written simply $[\mathbf{a},\mathbf{b},\mathbf{c}]$
- $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \neq 0 \Rightarrow$ vectors are linearly independent
- $=$ volume of parallelepiped (give or take $\pm$)

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Vector triple product

- The other way to multiply three vectors together.
- vector$\times$(vector$\times$vector)$=$vector
- Two alternatives, short-cut formulae

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c},$$
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$$

(no need to learn this)

- NB: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$
  Not associative: not even parallel in general Why not?
- Applications beyond the scope of this course

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Homework

- Read *James*
  - Sections 4.3 & 4.2.12 (4.2.11 in 4th Edn)
- Do exercises (4th Edition)
  - 4.3.3 Q.52–55, 59-61, 63
  - 4.2.12 nos. 43–45, 49
- Do exercises (5th Edition)
  - 4.3.2 Q.66–69
  - 4.3.4 Q. 73, 77–79
  - 4.2.13 nos. 57-59, 63

Warning

We have gone through topic v. quickly

- If much of 3D vectors new to you, then
  1. Read the relevent sections of James
  2. Do the exercises set as Homework
  3. Get one-to-one tuition at the drop-in-classes
- There will be a in-class test on Monday 23rd Oct
  - topics covered = complex numbers & vectors
  - peer marked in class, marks do not count
  - but test paper collected and marks recorded
- Next week, new topic Matrices

Summary - of vectors

- Component form $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$
- $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, unit vector $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$
- addition, subtraction differentiation etc. term-by term
- Dot product $\mathbf{a} \cdot \mathbf{b}$ is a scalar

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Cross product $\mathbf{a} \times \mathbf{b}$ produces a vector in direction $\hat{\mathbf{n}}$ perpendicular to $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= |\mathbf{a}||\mathbf{b}|\sin(\theta)\,\hat{\mathbf{n}} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

- Note $\mathbf{a} \times \mathbf{a} = 0, \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

Summary - continued

- Vectors help us do things in 3D that we know how to do in one and two dimensions
- e.g. calculate work done, components of forces, moments and angular velocites
- general equations for straight line in 3D

$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

- equations for a plane in 3D $\quad \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
- equations geometric objects (spheres, curves etc.)
- Scalar triple product: $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ gives volume of parallelpiped: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$= $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$=$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$
- Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ strange beast!