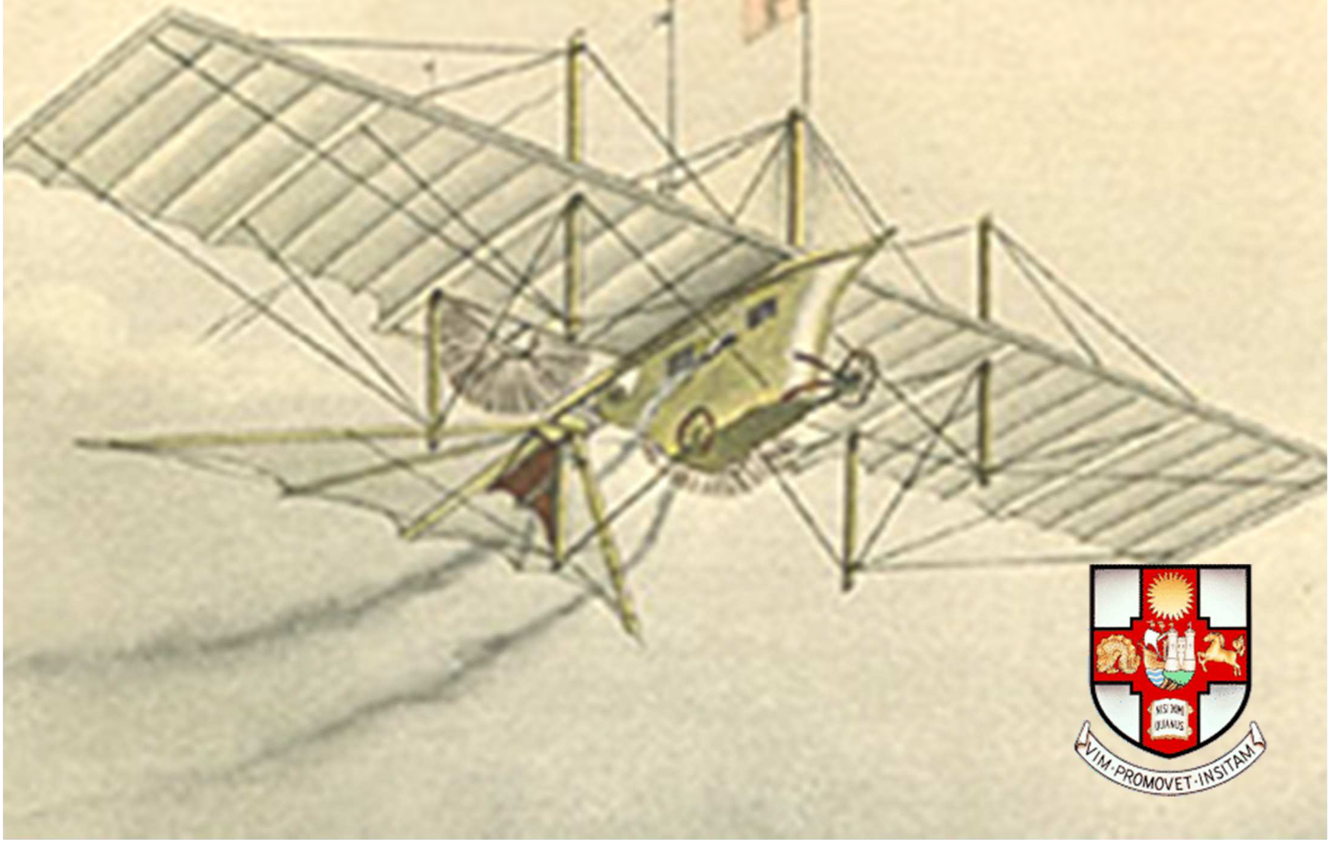
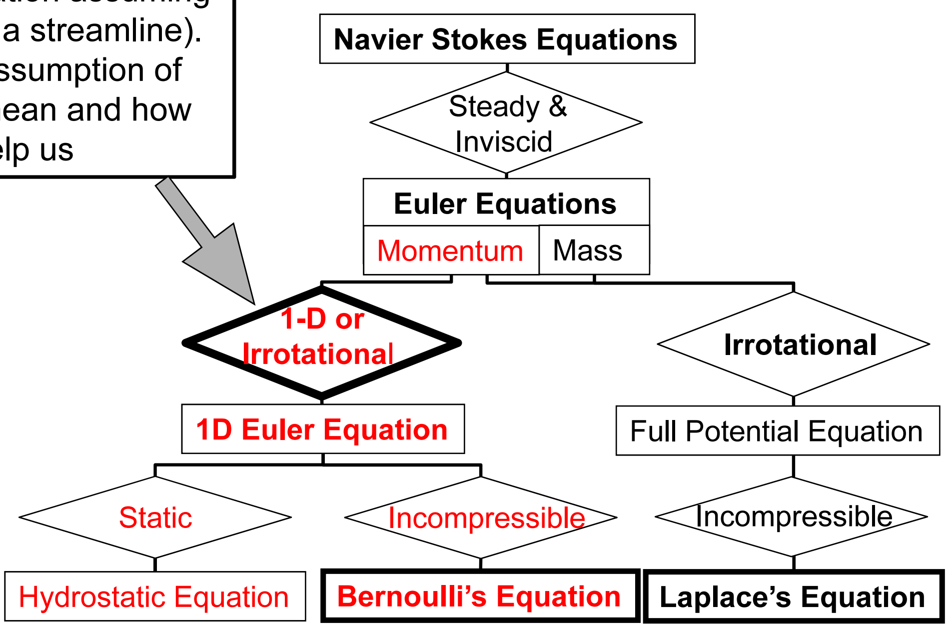
Potential Flow



From Navier-Stokes to Laplace & Bernoulli's equations

We have looked at the derivation for Bernoulli's equation assuming 1-D flow (or along a streamline). What does the assumption of irrotational flow mean and how does it help us



From Navier-Stokes to Bernoulli's equation: The assumptions

1. **Viscous effects are negligible** - Valid for many real flows where viscous effects are confined to a narrow band near the body surface.
2. **Steady Flow**
3. **No Body forces**. - Such as gravity
4. **a) Flow along a streamline**

$$\underline{\mathbf{ds}} \times \underline{\mathbf{V}} = \underline{\mathbf{0}}$$

$$(dx, dy, dz) \times (u, v, w) = (0,0,0)$$

4. **b) Irrotational Flow -** The vorticity (or rotational velocity) is defined as zero everywhere in the flow field for irrotational flow.

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u, v, w) = (0,0,0)$$

5. **Incompressible Flow**
6. **Adiabatic Flow**

This gives Bernoulli's equation without the hydrostatic term (as we have ignored body forces) valid for

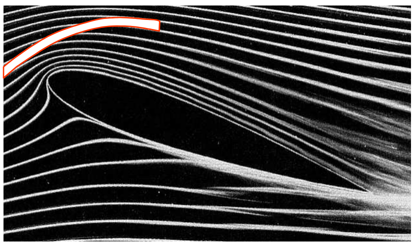
i) any two points along a streamline **OR**

ii) two points anywhere in an irrotational flow

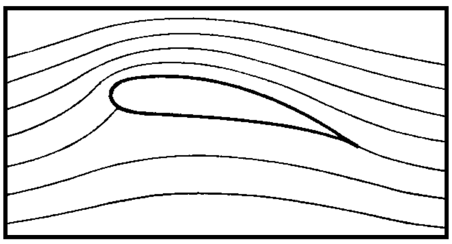
$$\boxed{p + \frac{1}{2}\rho V^2 = \text{ constant}}$$

Why isn't Bernoulli's Equation Enough

- Consider applying Bernoullis equation along each streamline for steady inviscid incompressible flow
- Even if we know the initial pressure and velocity along each streamline, Bernoulli's equation does not give us the streamline position so that we can integrate along it.



- If we make the further assumption of irrotational flow then we still need the pressure throughout the domain to give us the velocity field or the velocity field to give us the pressure distribution.



Laplace's Equation

- Kelvin's Theorem: "in the absence of viscous forces and discontinuities the flow will remain irrotational"
- Irrotational flow defined by

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(u,v,w)=\left(\left[\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right],\left[\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right],\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]\right)=(0,0,0)$$

- Discontinuities come from free surfaces (multi-phase flows) or shock waves (only present in compressible flows). Inviscid incompressible flows are irrotational.
- Irrotational flow is a good approximation other than at the surface

- For incompressible flow the mass conservation equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

- This is combined with the definition of irrotational flow to give us our flow equation

Laplace's Equation 2 :(Velocity Potential)

- **irrotationality** guarantees the existence of a scalar 'velocity potential' function ϕ.
- ie $u=\frac{\partial \phi}{\partial x},\ v=\frac{\partial \phi}{\partial y},\ w=\frac{\partial \phi}{\partial z}$ if the flow is irrotational.
- We have changed the problem, now need to find ϕ instead of u, v & w
- name 'potential' is significant
  - direct analogue to 'potential' in electrodynamics
  - flow only occurs where there is a difference (gradient) in potential
- only defined for irrotational flows
  - therefore such flows usually referred to as 'potential flows'
  - exists in 3D
  - exists in unsteady and compressible flows

Laplace's Equation (3)

- for **incompressible** and irrotational flow consider continuity applied to a velocity field defined by a velocity potential

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x}+\frac{\partial}{\partial y}\frac{\partial \phi}{\partial y}+\frac{\partial}{\partial z}\frac{\partial \phi}{\partial z}=0$$

or
$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

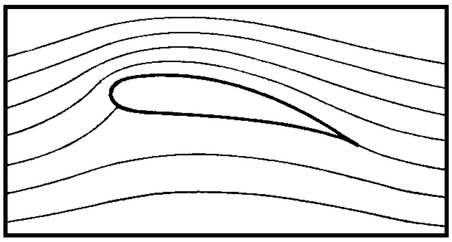
- **linear** 2nd order partial differential equation
  - well-understood equation – over 2 centuries of study!
  - 1 differential equation to solve for 3 velocity components
- solutions can be superimposed
  - flow models built up from 'elementary' flow solutions

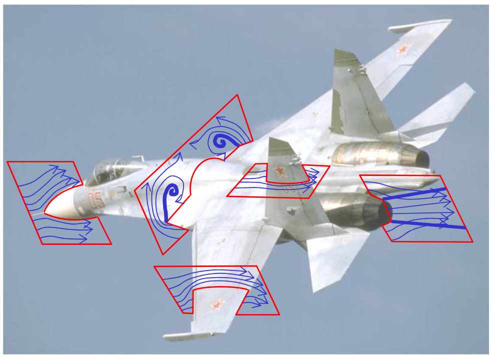
If $\frac{\partial^2 \phi_1}{\partial x^2}+\frac{\partial^2 \phi_1}{\partial y^2}+\frac{\partial^2 \phi_1}{\partial z^2}=0$ and $\frac{\partial^2 \phi_2}{\partial x^2}+\frac{\partial^2 \phi_2}{\partial y^2}+\frac{\partial^2 \phi_2}{\partial z^2}=0$ then $\frac{\partial^2 (\phi_1+\phi_2)}{\partial x^2}+\frac{\partial^2 (\phi_1+\phi_2)}{\partial y^2}+\frac{\partial^2 (\phi_1+\phi_2)}{\partial z^2}=0$

- boundary conditions required

How to Solve Potential Flows

- Solve Laplace's equation to find the potential in the domain
- Differentiate to find the velocity at all points
- Use Bernoulli to find the pressure at all points given the pressure and velocity at a single point (far field conditions)
- Same process in 2D & 3D, just 2D this year





Why Start in 2D

- Clearer demonstration of fundamentals
- Applicable to many practical 3D flows
- Mechanism for lift generation
- Stream function available in 2D only

Introduction of Stream Function

- In a 2D incompressible flow a function $\psi(x,y)$ exists such that on streamlines:

$$\psi(x,y) = c$$

where c is an arbitrary constant. This function is called the stream function (Note this equation is an alternative integrated form of the 2D equation for a streamline).

- The stream function is related to flow velocities via

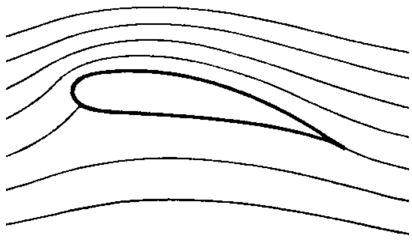
$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

- Physical interpretation of the stream function is that the increment in the stream function between two streamlines in the flow corresponds to the volume flow rate between the lines

$$\boxed{\Delta\psi = \text{Volume flow rate}}$$

or since density is constant

$$\boxed{\rho\,\Delta\psi = \text{Mass flow rate}}$$



Example, find stream function from velocity

If the fluid velocity components are given by $u = -\omega\, y, \quad v = \omega\, x$

then find an expression for the stream function and hence an equation for the stream lines.

$$u = \partial\psi/\partial y = -\omega y \qquad \text{Integrating w.r.t. } y \Rightarrow \psi = -\frac{1}{2}\omega y^2 + f(x)$$

$$\text{Differentiate w.r.t. } x \Rightarrow \partial\psi/\partial x = df(x)/dx = -\omega x$$

$$\text{Integrating w.r.t. } x \Rightarrow \quad f(x) = -\frac{1}{2}\omega x^2 + const \quad \text{(can ignore const)}$$

$$\psi = -\frac{1}{2}\omega(y^2 + x^2)$$

$$\text{Streamlines}: \quad \psi = const = -\frac{1}{2}\omega(y^2 + x^2) \ \Rightarrow \ y^2 + x^2 = const$$

$$\text{Alternatively Streamlines} \quad \frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y}$$

$$ydy + xdx = 0$$

$$\text{Integrating} \Rightarrow y^2 + x^2 = const$$

BUT this doesn't give us the stream function as requested.

Relationship of Stream function and Potential

$$\boxed{\begin{array}{ll} u = \dfrac{\partial \psi}{\partial y}, & v = -\dfrac{\partial \psi}{\partial x} \\ u = \dfrac{\partial \phi}{\partial x}, & v = \dfrac{\partial \phi}{\partial y} \end{array}}$$

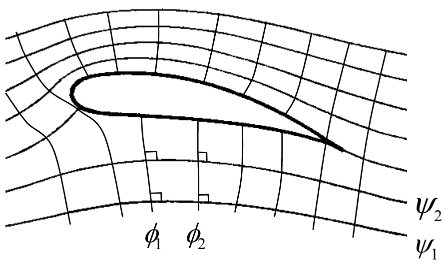
Potential and stream function are clearly related. In fact they are ORTHOGONAL functions i.e. are perpendicular to each other where they cross (except at stagnation points where $\mathbf{V} = \mathbf{0}$).

Both satisfy Laplace's equation so can have superposition of solutions

$$\nabla^2 \psi = \nabla^2 \phi = 0$$

Can be combined as a COMPLEX POTENTIAL $W(Z) = \phi + i\psi$

Lines cross perpendicularly. Note it is often easier to sketch the stream lines of a flow rather than equi-potentials because of the tangential relationship to the velocity vector.



Learning Outcomes:
"What you should have learnt so far"

- State the assumptions needed to reduce the 3 momentum equations of the Euler equations to a 1D equation (note two cases).
- Understand that a velocity potential may only be defined for an irrotational flow which satisfies Laplace's equation. This means only 1 equation needs to be solved for the three components of velocity.
- Be aware that because Laplace's equation is a linear equation so solutions can be found by superposition of other solutions.
- Give a physical interpretation of the stream function, know that it is constant on a stream line
- Find the stream function and stream lines from a velocity distribution.

Interested students should take a look at

http://www.grc.nasa.gov/WWW/K-12/airplane/foil3.html