Worked example 5.2

Solve the wave equation

PDE $\boxed{u_{tt} = c^2 u_{xx},}$ $\boxed{0 \leqslant x \leqslant L, \quad t \geqslant 0,}$ DOMAIN

subject to homogeneous boundary conditions and an inhomogeneous initial condition:

BOUNDARY CONDITIONS for all $t \geqslant 0$ INITIAL CONDITIONS. at $t=0$

$$\boxed{\underset{x=0}{u_x(0,t) = 0}, \quad \underset{x=L}{u_x(L,t) = 0},} \quad \boxed{u(x,0) = 0, \quad u_t(x,0) = g(x)}$$

for all $0 \leqslant x \leqslant L$

for a non-zero initial velocity $g(x)$.

$\frac{\partial u}{\partial x} = 0$ open ends. displacement velocity

5.2. The heat equation

Find the solution $u(x,t)$ of the heat equation

$$u_t = \alpha^2 u_{xx}, \qquad 0 \leqslant x \leqslant L, \qquad t > 0, \tag{13}$$

subject to boundary conditions (ends fixed at zero temperature)

$$u(0,t) = u(L,t) = 0 \tag{14}$$

and an initial temperature distribution

$$u(x,0) = h(x) \tag{15}$$

Ex 5.2

1. Guess a separable solution

$$u(x,t) = X(x)\,T(t)$$

2. Sub. into the PDE & separate the variables

$$X(x)\,T''(t) = c^2 X''(x)\,T(t)$$

$\div c^2 XT$

$$\underbrace{\frac{1}{c^2}\frac{T''(t)}{T(t)}}_{\text{function of } t \text{ only}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x \text{ only.}} = \underbrace{\text{constant}}_{x \,\&\, t \text{ are independent}} = -k^2 \qquad (k>0)$$

separate

$\Rightarrow$

$$X''(x) = -k^2 X(x)$$

$$T''(t) = -k^2c^2\,T(t)$$

solve.

$\Rightarrow$

$$\boxed{\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) \\ T(t) &= C\cancel{\cos}(kct) + D\sin(kct) \end{aligned}}$$

3. Separate the homogeneous bdy & init conds.

$\boxed{u_x = 0 \text{ at } x=0, L, \text{ for all } t}$ $\boxed{u = 0 \text{ at } t=0, \text{ for all } x.}$

$u_x = X'(x)T(t) = 0$ at $x=0$ $\Rightarrow$ $X'(0)T(t) = 0$ for all t

$\Rightarrow$ $\boxed{X'(0) = 0.}$

$x=L$ $\Rightarrow$ $\boxed{X'(L) = 0}$

$u = X(x)T(t)$ at $t=0$ $\Rightarrow$ $\boxed{T(0) = 0}$

$\cancel{X(x)}\,T(0) = 0.$ for all x. (as $X(x)$ can't be 0 for all x)

4. Apply them to find A, B, C, D, k.

$$X'(x) = -kA\sin(kx) + Bk\cos(kx)$$

$x=0$ $\quad X'(0) = +kB = 0 \quad \Rightarrow \quad B = 0.$

$x=L$ $\quad X'(L) = -kA\sin(kL) = 0 \quad \Rightarrow \quad \sin(kL) = 0$

$\Rightarrow \quad kL = n\pi$

$\Rightarrow \quad \boxed{k = \frac{n\pi}{L} \text{ for } n \text{ integer}}$

$$\Rightarrow \boxed{X(x) = A\cos\left(\frac{n\pi x}{L}\right)}$$

$t=0 \quad T(0) = C = 0 \quad \Rightarrow \quad C=0$

$$\Rightarrow \boxed{T(t) = D \sin\left(\frac{n\pi c t}{L}\right)}$$

5. Put the solution together & sum

the normal modes.

$$u(x,t) = X(x)T(t) = \underbrace{A.D.}_{\alpha} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right)$$

is a solution for any n integer

$\Rightarrow$ general soln is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right)}$$

6. Apply / solve the non-homogeneous cond.

$\frac{\partial u}{\partial t} = g(x)$ at $t=0$, for all x [g is known]

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \alpha_n \cdot \frac{n\pi c}{L} \cdot \cos\left(\frac{n\pi x}{L}\right) \left.\underbrace{\cos\left(\frac{n\pi c t}{L}\right)}_{=1}\right|_{t=0}$$

$$g(x) = \sum_{n=1}^{\infty} \underbrace{\alpha_n \cdot \frac{n\pi c}{L}}_{= a_n} \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \le x \le L$$

$$a_n = \alpha_n \cdot \frac{n\pi c}{L}$$

$$g(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \Rightarrow a_n = \frac{2}{L}\int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

A fourier ½-range cos series.

$$\Rightarrow \alpha_n = \frac{L}{n\pi c} \cdot a_n = \frac{L}{n\pi c} \cdot \frac{2}{L}\int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Plug these α_n's into the general soln to find the particular soln.

Step 1: separate the variables

The basic idea is once again to *try* to find a solution that is a function of x times a function of t. That is, we write

$$u(x,t) = X(x)T(t),$$

Substituting this form into the PDE we get

$$X(x)T'(t) = \alpha^2 X''(x)T(t),$$

which simplifies to

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}, \tag{16}$$

Step 1: separate the variables

Now, the left-hand side of (16) is a function of time t, while the right-hand side is a function of space x. The only way that this can be true for all x and t is if both functions are actually equal to a constant. Hence

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const.} \tag{17}$$

This constant is called the **separation constant**. The question remains what sign this constant should have. We proceed by trial and error to see what fits the boundary and initial conditions, and what makes sense physically.

Step 2: sign of the separation constant

Try first a *positive* constant. Hence we write (17) as:

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = k^2 > 0$$

Then we get two separate linear ODEs to solve:

$$T'(t) = (\alpha k)^2 T(t)$$
$$X''(x) = k^2 X(x)$$

Both are easy to solve

$$T(t) = A\,\mathrm{e}^{(\alpha k)^2 t}$$
$$X(x) = B\,\mathrm{e}^{-kx} + C\,\mathrm{e}^{kx}$$

but the solution for T tends to $+\infty$ as $t \to \infty$. This is not a diffusion-like process (heat decays, not blows up!)

Step 2: sign of the separation constant

Hence we should take the original separation constant to be *negative*. That is we write (4) in the form

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -k^2 < 0$$

Thus we get the two separate linear ODEs:

$$T'(t) = -(\alpha k)^2 T(t)$$
$$X''(x) = -k^2 X(x)$$

Both are easy to solve:

$$T(t) = A\,\mathrm{e}^{-(\alpha k)^2 t}$$
$$X(x) = B\cos(kx) + C\sin(kx)$$

Steps 3 & 4: separate & apply homogenous boundary conditions

The homogenous boundary conditions (14) become

$$\begin{aligned} 0 = u(0,t) = X(0)T(t) \quad & \text{for all } t > 0 \quad \Rightarrow \quad X(0) = 0 \\ 0 = u(L,t) = X(L)T(t) \quad & \text{for all } t > 0 \quad \Rightarrow \quad X(L) = 0 \end{aligned}$$

Applying them we get

$$0 = X(0) = B\cos(0) + C\sin(0) = B, \tag{18}$$

$$0 = X(L) = B\cos(kL) + C\sin(kL). \tag{19}$$

From (18) we get $B = 0$, hence from (19) we have

$$C\sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{L}, \quad n \in \mathbb{Z}.$$

Step 5: put it all together

Substitute the value for k into the solutions for $X(t)$ and $T(t)$:

$$X(x) = C\sin\left(\frac{n\pi x}{L}\right), \qquad T(t) = A\,\mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

Since $u(x,t) = X(x)T(t)$

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

solves the PDE (13) and homogenous boundary conditions (14) for any integer n. Since both are linear, any sum of the u_ns will also be a solution, so the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

Step 6: initial conditions

At this stage we should check that we satisfy the PDE and the boundary + initial conditions.

It sill remains to satisfy the initial condition $u(x, 0) = h(x)$. Setting $t = 0$ in the general solution we gat

$$h(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

This is just the Fourier half-range sine series expansion of the function $h(x)$. Hence we know that

$$b_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{20}$$

Step 7: the particular solution

So, the particular solution of the heat equation PDE (13) satisfying the boundary and initial conditions, (14) and (15), is:

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\alpha\pi}{L}\right)^2 t},$$

where the b_ns are the Fourier half-range sine series coefficients, given by

$$b_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Different boundary conditions

Subtle changes in the boundary conditions lead to different forms of solution. For example, we can replace these boundary conditions with conditions that the bar is *insulated* at each end. That is, there is no heat flux:

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

Such boundary conditions can be shown to lead to a similar general solution to the heat equation, but with *cosine* rather than sine terms:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\alpha\pi}{L}\right)^2 t},$$

where the a_ns are the Fourier half-range cosine coefficients of $h(x)$.

5.3. Laplace's equation

Since Laplace's equation involves only spatial co-ordinates, (x, y) (or (x, y, z) in three dimensions), it is quite natural to pose Laplace's equation on domains of any shape. E.g. circular domains (find the shape of a drumskin) or complex curvy shapes (find the incompressible, irrotational flow through a curved river bed).

However, in these lectures we shall concentrate only on the simplest case of a rectangular domain in 2D.

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \quad 0 < y < b. \qquad (21)$$

Homogenous boundary conditions

On each boundary ($x = 0$ or a, $y = 0$ or a) there are two types of homogeneous boundary condition we can pose, that either the solution is zero on the boundary (**Dirichlet** BCs):

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = 0; \qquad (22)$$

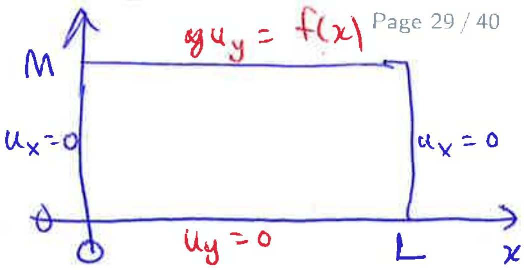
or its normal derivative is zero on the boundary (**Neumann** BCs):

$$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad u_y(x, 0) = 0, \quad u_y(x, b) = 0. \quad (23)$$

A mixture of Dirichlet on some parts of the boundary and Neumann on others is also possible.

Note that we usually need some kind of inhomogeneous boundary condition in order to get a non-trivial solution.

Worked example 5.3



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solve the Laplace equation (21) on a rectangular domain subject to the inhomogeneous Neumann boundary conditions

$x=0$ $\quad$ $x=L$ $\quad$ $y=0$ $\quad$ $y=M$

$$\boxed{u_x(0, y) = u_x(L, y) = 0, \quad u_y(x, 0) = 0, \quad u_y(x, M) = f(x)}$$

for all y $\quad$ for all x

for some given function $f(x)$. only boundary conds.

Such a problem could describe, for example, the electrostatic potential $u(x, y)$ in a rectangular device whose boundaries on three sides are insulated (electromagnetically shielded) but with an imposed field $f(x)$ on the boundary $y = M$.

To solve this problem we use the method of separation of variables.

Ex 5.3 // Use separation of variables.

1. Guess a sep. soln

$$u(x,y) = X(x)\,Y(y)$$

2. Sub into the PDE & separate the vars.

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

$\div XY$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{fn. of } x \text{ only}} = \underbrace{-\frac{Y''(y)}{Y(y)}}_{\text{fn of } y \text{ only}} = \underbrace{\text{constant} = \pm k^2}_{x \,\&\, y \text{ are independent.}}$$

either $X''(x) = +k^2 X(x)$ or $X''(x) = -k^2 X(x)$

$Y''(y) = -k^2 Y(y)$ $\quad$ $Y''(y) = +k^2 Y(y)$

$\Rightarrow$ either $X(x) = A e^{kx} + B e^{-kx}$ or $X(x) = A\cos(kx) + B\cancel{\sin(kx)}$

$Y(y) = C\cos(ky) + D\sin(ky)$ $\quad$ $Y(y) = C e^{ky} + D e^{-ky}$

3. Use the homogeneous bdy cnds to choose the sign of the sep. const.

$u_x = 0$ at $x = 0, L$ $\qquad$ $u_y = 0$ at $y = 0$

$$X'(0) = X'(L) = 0 \qquad\qquad Y'(0) = 0.$$

The fn X' has to be zero in two diff places

$\Rightarrow$ X must be cos & sin, not exp.

4. Apply the separated hom. bdy conds.

$$X'(x) = -kA\sin(kx) + \cancel{Bk\cos(kx)}$$

$x=0 \quad X'(0) = Bk = 0 \quad \Rightarrow \quad B=0.$

$x=L \quad X'(L) = -kA\sin(kL) = 0 \quad \Rightarrow \quad \sin(kL) = 0$

$kL = n\pi$

$$\boxed{k = \frac{n\pi}{L} \text{ for } n \text{ integer}}$$

$$\Rightarrow \quad \boxed{X(x) = A\cos\left(\frac{n\pi x}{L}\right)}$$

next $\quad Y(y) = \hat{C}e^{ky} + \hat{D}e^{-ky} = C\cosh(ky) + D\,\cancel{\sinh}(ky)$

$Y'(y) = \qquad = Ck\sinh(ky) + Dk\,\cancel{\cosh}(ky)$

$y=0 \quad Y'(0) = Dk = 0 \quad \Rightarrow \quad D=0.$

$$\Rightarrow \quad \boxed{Y(y) = C\cosh\left(\frac{n\pi y}{L}\right)} \quad \text{for any integer } n$$

5. Put the pieces together and sum: $u(x,y) = X(x)Y(y)$

$$u(x,y) = \sum_n \alpha_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)$$

is the gen. soln of PDE + hom. bdy. conds.

6. Use the non-homogeneous bdy. cond to find α_n

$$\frac{\partial u}{\partial y} = f(x) \text{ at } y = M$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=M} = \sum_n \alpha_n \cos\left(\frac{n\pi x}{L}\right)\cdot\frac{n\pi}{L}\sinh\left(\frac{n\pi y}{L}\right)\Bigg|_{y=M}.$$

$$f(x) = \sum_n \underbrace{\alpha_n\cdot\frac{n\pi}{L}\cdot\sinh\left(\frac{n\pi M}{L}\right)}_{a_n}\cos\left(\frac{n\pi x}{L}\right) = \sum_n a_n\cos\left(\frac{n\pi x}{L}\right)$$

a ½ range Fourier sin series.

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$a_n \longrightarrow \alpha_n \longrightarrow$ general solution $\longrightarrow$ particular soln.

Step 1: separate the variables

Look for a solution of the form

$$u(x,y) = X(x)Y(y)$$

Substituting this into the PDE (21) gives us

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

which simplifies to

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const}$$

Step 2: decide on sign of separation constant

Note that if we choose const $= k^2 > 0$ then we get exponential solutions for $X(x)$ and sinusoidal solutions for $Y(y)$. Alternatively, const $= -k^2 < 0$ then we get exponential solutions for $Y(y)$ and sinusoidal solutions for $X(x)$.

You could just try both and see which works (this is a perfectly valid approach in a 'trial and error' method). Alternatively, you could appeal to the guiding principle that when we come to pose the inhomogeneous boundary condition $u_y(x,b) = f(x)$ we are going to be looking for a function of x, and we want to end up by expressing this function of x as a sum of sines or cosines.

Hence we choose const $= -k^2 < 0$.

Step 3: solve the separated ODEs

The separated ODEs, with const $= -k^2 < 0$, are

$$X''(x) = -k^2 X, \qquad Y''(y) = k^2 Y$$

Solving the equation for X gives

$$X(x) = A \sin kx + B \cos kx,$$

for arbitrary constants A and B. For the Y equation we get

$$Y(y) = \tilde{C}\, \mathrm{e}^{-ky} + \tilde{D}\, \mathrm{e}^{ky},$$

for arbitrary constants $\tilde{C}$ and $\tilde{D}$. However it is useful to express the solution $Y(y)$ in another way...

Alternative form for $Y(y)$

Using the fact that

$$\cosh(z) = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z}) \qquad \sinh(z) = \frac{1}{2}(\mathrm{e}^z - \mathrm{e}^{-z})$$

we get an alternative form for $Y(y)$:

$$Y(y) = C \cosh(ky) + D \sinh(ky)$$

(we do this for convenience because cosh is an even function and sinh is an odd function).

Hence, so far we have $u(x, t) = X(x)Y(y)$ given by

$$u(x, t) = (A \sin(kx) + B \cos(kx))\,(C \cosh(ky) + D \sinh(ky))$$

Step 4: solve the homogeneous boundary conditions

First, we separate the homogeneous boundary conditions

$$u_x(0,y) = X'(0)Y(y) = 0$$
$$u_x(a,y) = X'(L)Y(y) = 0$$
$$u_y(x,0) = X(x)Y'(0) = 0$$

Since the first two are true for all values y, and the last for all values of x, we must have that

$$X'(0) = X'(L) = 0, \qquad Y'(0) = 0$$

Step 4: solve the homogeneous boundary conditions

Let us first pose the boundary conditions at $x = 0$ and $x = a$:

$$X'(0) = Ak\cos 0 - Bk\sin 0 = Ak = 0$$
$$X'(a) = Ak\cos Lk - Bk\sin Lk = 0.$$

The first equation gives us $A = 0$, and the second that $Bk\sin Lk = 0$, hence $Lk = n\pi$ for some integer n, and so $k = \frac{n\pi}{L}$. Thus $X(x) = B\cos\frac{n\pi x}{L}$.

Now, consider the boundary condition at $y = 0$:

$$Y'(0) = Ck\sinh(0) + Dk\cosh(0) = Dk,$$

Hence $D = 0$, and so $Y(y) = C\cosh\frac{n\pi y}{L}$.

Step 5: put it all together

Since $u(x,y) = X(x)Y(y)$, we have that (letting $AC = A_n$)

$$u_n(x,y) = A_n \cos\frac{n\pi x}{a} \cosh\frac{n\pi y}{L}$$

is a solution of the heat equation (13) and the homogeneous boundary conditions for any $n \in \mathbb{Z}$.

Using linearity, any sum of the u_ns is also a solution, and so the general solution is

$$\begin{aligned} u(x,y) &= \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} \cosh\frac{n\pi y}{L} \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \cosh\frac{n\pi y}{L} \end{aligned}$$

Step 6: inhomogeneous boundary condition

The inhomogeneous boundary condition is $u_y(x,M) = f(x)$, i.e.

$$f(x) = u_y(x,M) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \frac{n\pi}{L} \sinh\left(\frac{n\pi M}{L}\right)$$

Hence if we let

$$A_n \frac{n\pi}{L} \sinh\frac{n\pi M}{L} = a_n,$$

then we'd have

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

which is a Fourier half-range cosine series* for $f(x)$, so

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

*with $a_0 = 0$

0.38

Worked example 5.4

Solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < \overset{L}{\cancel{a}}, \quad 0 < y < \overset{M}{\cancel{b}}.$$

on a rectangular domain subject to the inhomogeneous Neumann boundary conditions

$$u_x(0, y) = u_x(\overset{L}{\cancel{a}}, y) = 0, \quad u_y(x, 0) = 0, \quad u_y(x, \overset{M}{\cancel{b}}) = f(x)$$

for the particular case $L = 4$ and $M = 2$ and

$$\left.\frac{\partial u}{\partial y}\right|_{y=M} = f(x) = \cos\left(\frac{\pi x}{4}\right) - \frac{1}{9}\cos\left(\frac{3\pi x}{4}\right) \tag{24}$$

5.4. Summary

- The separation of variables is a trial and error method. We **try** $u(x, t) = X(x)T(t)$ (or $u(x, y) = X(x)Y(y)$).
- The choice of the *sign* of the separation constant is crucial. The key idea is that we want u on the inhomogeneous boundary to be a sum of sines or cosines, e.g:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{or} \quad \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad .$$

- We then use half-range Fourier series to compute the coefficients a_n and b_n.

Ex 5.4

general soln.

$$\boxed{u(x,y) = \sum_n \alpha_n \cos\left(\frac{n\pi x}{4}\right)\cosh\left(\frac{n\pi y}{4}\right)}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=2} = \sum_n \alpha_n \cos\left(\frac{n\pi x}{4}\right)\cdot\frac{n\pi}{4}\sinh\left(\frac{n\pi y}{4}\right)\Bigg|_{y=2}$$

$$\underbrace{\cos\left(\frac{\pi x}{4}\right)}_{n=1}\underbrace{-\frac{1}{9}\cos\left(\frac{3\pi x}{4}\right)}_{n=3} = \sum_n \alpha_n \cdot \frac{n\pi}{4}\cdot\sinh\left(\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi x}{4}\right)$$

$$\Rightarrow \quad \alpha_1 \cdot \frac{1\pi}{4}\cdot\sinh\left(\frac{1.\pi}{2}\right) = 1$$

$$\alpha_3 \cdot \frac{3\pi}{4}\cdot\sinh\left(\frac{3\pi}{2}\right) = -\frac{1}{9}$$

$$\alpha_j = 0 \quad \text{for all} \quad j \neq 1, 3$$

$$\alpha_1 = \frac{4}{\pi\sinh(\pi/2)}$$

$$\alpha_3 = -\frac{4}{27\pi\sinh(3\pi/2)}$$

particular soln

$$\boxed{u(x,y) = \alpha_1 \cos\left(\frac{\pi x}{4}\right)\cosh\left(\frac{\pi y}{4}\right) + \alpha_3\cos\left(\frac{3\pi x}{4}\right)\cosh\left(\frac{3\pi y}{4}\right)}$$

→ variants for other hyperbolic PDES.

6. d'Alembert's method

↳ method to find travelling waves.

↳ set up for very large (infinite) domains

↳ works only for the wave equation

The separation of variables method is one way of finding solutions of the wave equation.

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.}$$

It is well suited to the case where we have boundary conditions. Then the 'wavelength' is determined by the boundary conditions. For example, what is the note played by the guitar string, the organ pipe or a percussive instrument? The solution is said to be a *standing wave*.

6.1. General solution of the wave equation

This is not a viable method if the domain over which we solve the PDE is infinite. Of course, nothing is truly infinite. Really; what we mean by an infinite domain is a long domain in which the boundaries are far away and cannot influence the wave length.

For example, what is the shape of ripples if we drop a stone into a pond? How do waves propagate along a long cable — in a cable stayed bridge, bacterial flagellum or a whip?

The kind of solution we are looking for is a *travelling wave*.

[James Advanced MEM (4th Edn) §9.3.1]

Travelling wave solution of the wave equation

$u_{tt} = c^2 u_{xx}$

In order to find travelling wave solutions we note the following.

Theorem *A general solution of the wave equation can be expressed in the form*

$$\boxed{u(x,t) = \boxed{f(x-ct)} + \boxed{g(x+ct)}}$$

for arbitrary functions f and g.

Proof just differentiate twice with respect to t and x.

the graph of $f(x-ct)$ is
the same as the graph of $f(x)$,
shifted right by a distance ct
travelling to the right, speed c
it is a travelling wave!

a travelling wave,
same shape as $g(x)$
travelling to the left,
at speed c.

Travelling wave solution: proof

plug it into the wave eqn. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

chain rule

Let $u(x,t) = f(x-ct) + g(x+ct)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[f(x-ct) + g(x+ct)\right] = f'(x-ct)\cdot\underbrace{\frac{\partial}{\partial x}(x-ct)}_{1} + g'(x+ct)\cdot\underbrace{\frac{\partial}{\partial x}(x+ct)}_{1}$$

$$u_x = f'(x-ct) + g'(x+ct)$$

$$\partial/\partial x \qquad \therefore \quad u_{xx} = f''(x-ct) + g''(x+ct)$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[f(x-ct) + g(x+ct)\right] = f'(x-ct)\cdot\overbrace{\frac{\partial}{\partial t}(x-ct)}^{-c} + g'(x+ct)\cdot\underbrace{\frac{\partial}{\partial t}(x+ct)}_{+c}$$

$$u_t = -cf'(x-ct) + cg'(x+ct)$$

$$\therefore \quad u_{tt} = c^2 f''(x-ct) + c^2 g''(x+ct) = c^2 u_{xx}$$

Hence $u_{tt} = c^2 u_{xx}$ $(= c^2 f''(x-ct) + c^2 g''(x+ct))$.

Remarks

in an exam, just write it down.

1. This is known as **d'Alembert's solution** to the wave equation.
2. Note what functions $f(x - ct)$ and $g(x + ct)$ look like. Functions of the form $f(x - ct)$ represent waves travelling *to the right* and $g(x + ct)$ waves travelling *to the left*.
3. The form that f and g take is determined by the boundary + initial conditions. We make a distinction between waves on infinite domains (dropping a stone in a pond) and on semi-infinite domains (wave propagation along a whip).

case 1 (easy)

case 2 (bit harder)

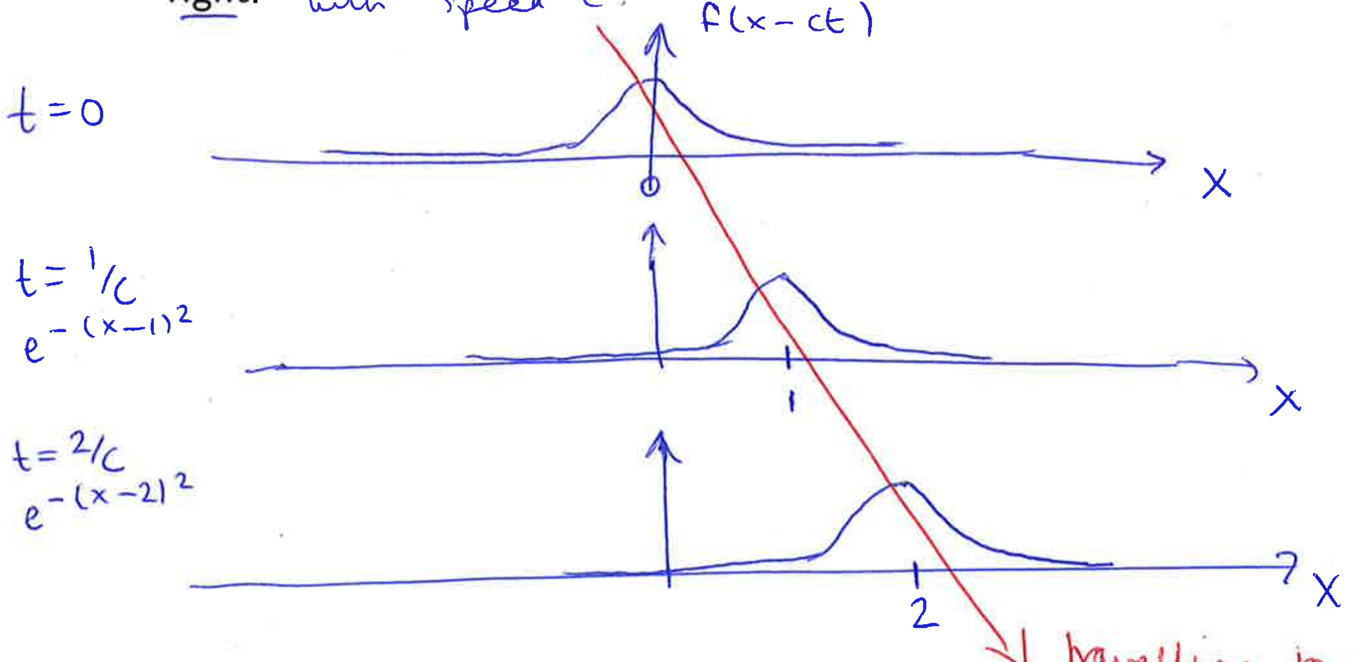
Worked example 6.1

if $f(x) = e^{-x^2}$

$= f(x - ct)$

$ct = 1$ $ct = 2$

Sketch the function $e^{-(x-ct)^2}$ for $t = 0$, $t = 1/c$, $t = 2/c$, and $t = 3/c$. Show that this represents a wave that travels to the right. with speed c.



travelling to the right, speed $\frac{1}{1/c} = c$

6.2. Method for an infinite domain

→ d'Alembert method

Example: *Consider the wave equation on an infinite domain*

PDE $$\boxed{u_{tt} = c^2 u_{xx},} \quad \boxed{-\infty < x < \infty, \quad t > 0,} \qquad (1)$$

DOMAIN

subject to the plucked *initial conditions*

TWO INITIAL COND

velocity, displacement

$$u_t(x,0) = 0, \qquad u(x,0) = F(x) = \begin{cases} 1+x & x \in [-1,0) \\ 1-x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

F(x)

at $t=0$ for all x

TWO BDY CONDS

Note there are no boundary conditions, just initial conditions. The boundary conditions at $x = \pm\infty$ are implicit (the solution should be finite as $x \to \pm\infty$). $u \to 0$ as $x \to \pm\infty$ (no energy at $x = \pm\infty$)

Step 1: state the d'Alembert solution

The d'Alembert (travelling wave) solution of the wave equation (1)

$$u(x,t) = f(x-ct) + g(x+ct)$$

Q: what are f & g? use the initial conds.

- No need to prove this every time you use it
- It only works for infinite (or semi-infinite) domains
- For finite domains (e.g. $0 < x < a$) with two boundary conditions you have to use separation of variables instead

Step 2: use the initial conditions

$\frac{\partial u}{\partial t} = 0$ at $t = 0$ for all x

First we take the zero-derivative condition

$$0 = u_t(x,0) = [-cf'(x - \underset{0}{\underset{\shortparallel}{ct}}) + cg'(x + \underset{0}{\underset{\shortparallel}{ct}})]_{t=0},$$

which implies that

$$-f'(x) + g'(x) = 0. \quad \text{for all } x$$

We can integrate this expression with respect to x and we get

Integrate w.r.t. x

1 eqn for f & g.

$$\boxed{g(x) - f(x) = K}$$

for some constant K. constant of integration

Step 2: use the initial conditions

at $t=0$ $u = F(x) = [f(x - \underset{0}{\underset{\shortparallel}{ct}}) + g(x + \underset{0}{\underset{\shortparallel}{ct}})]_{t=0}$

Now, solving the initial condition on displacement we get

$$\boxed{F(x) = u(x,0) = f(x) + g(x),}$$

2nd eqn for f & g

known

where $F(x)$ ~~was the known function given by the initial~~ condition.

Hence we have two simultaneous equations for the two unknown functions f and g:

$$\boxed{\begin{aligned} g(x) - f(x) &= K, & (2)\\ f(x) + g(x) &= F(x). & (3)\end{aligned}}$$

Solve for f & g.

Step 3: solve for f and g

To solve the two simultaneous equations (2) and (3) for the unknown functions f and g, first substitute $g(x) = K + f(x)$ from (2) into (3). Then

$$2f(x) + K = F(x)$$

Therefore

$$\boxed{f(x) = \frac{1}{2}F(x) - \frac{K}{2} \quad \text{and} \quad g(x) = \frac{1}{2}F(x) + \frac{K}{2}.}$$

So we know u!

Step 4: recombine to get general solution

We have the d'Alembert solution $u(x,t) = f(x-ct) + g(x+ct)$, and expressions for f and g. So, substituting, we get

$$u(x,t) = \underbrace{f(x-ct)}_{} + \underbrace{g(x+ct)}_{}$$

f(x − ct)   g(x + ct)

$$= \boxed{\frac{1}{2}F(x-ct) - \frac{K}{2}} + \boxed{\frac{1}{2}F(x+ct) + \frac{K}{2}}$$

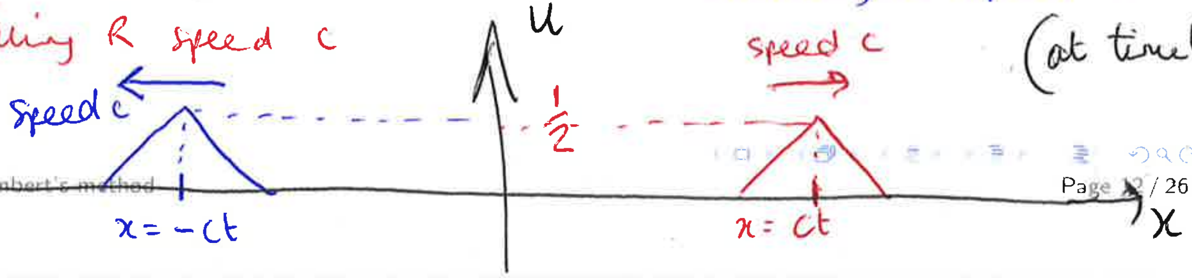
$$\boxed{u(x,t) = \boxed{\frac{1}{2}F(x-ct)} + \boxed{\frac{1}{2}F(x+ct)}}$$

two copies of $\frac{1}{2}F$ (the init. disp).

where $F(x) = \begin{cases} 1+x & -1 \leqslant x < 0 \\ 1-x & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

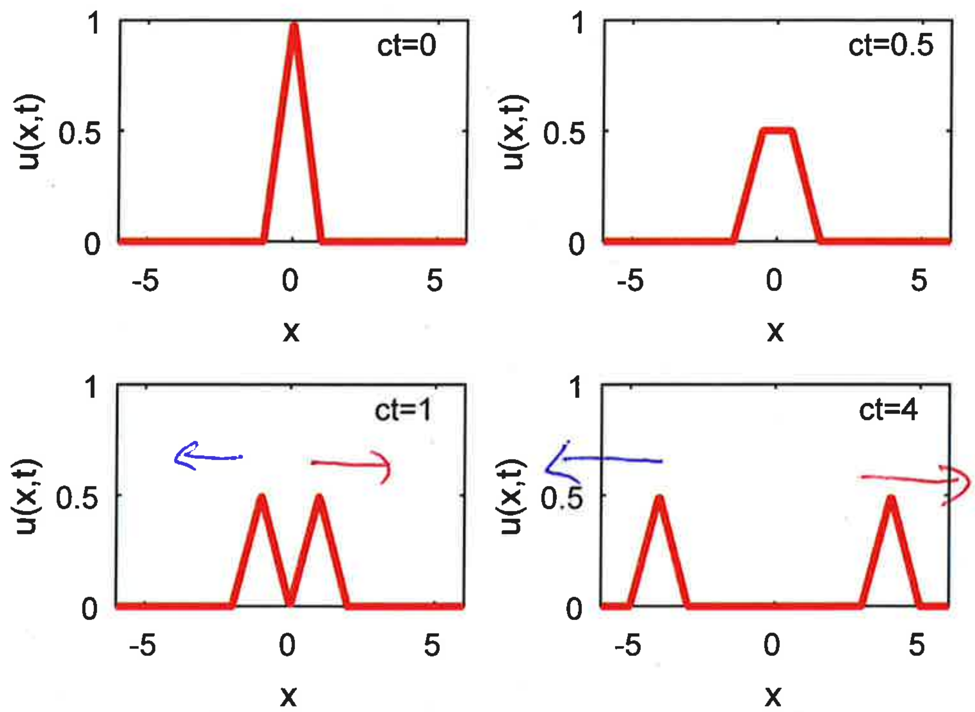
travelling R speed c   travelling L speed c.

(at time t)

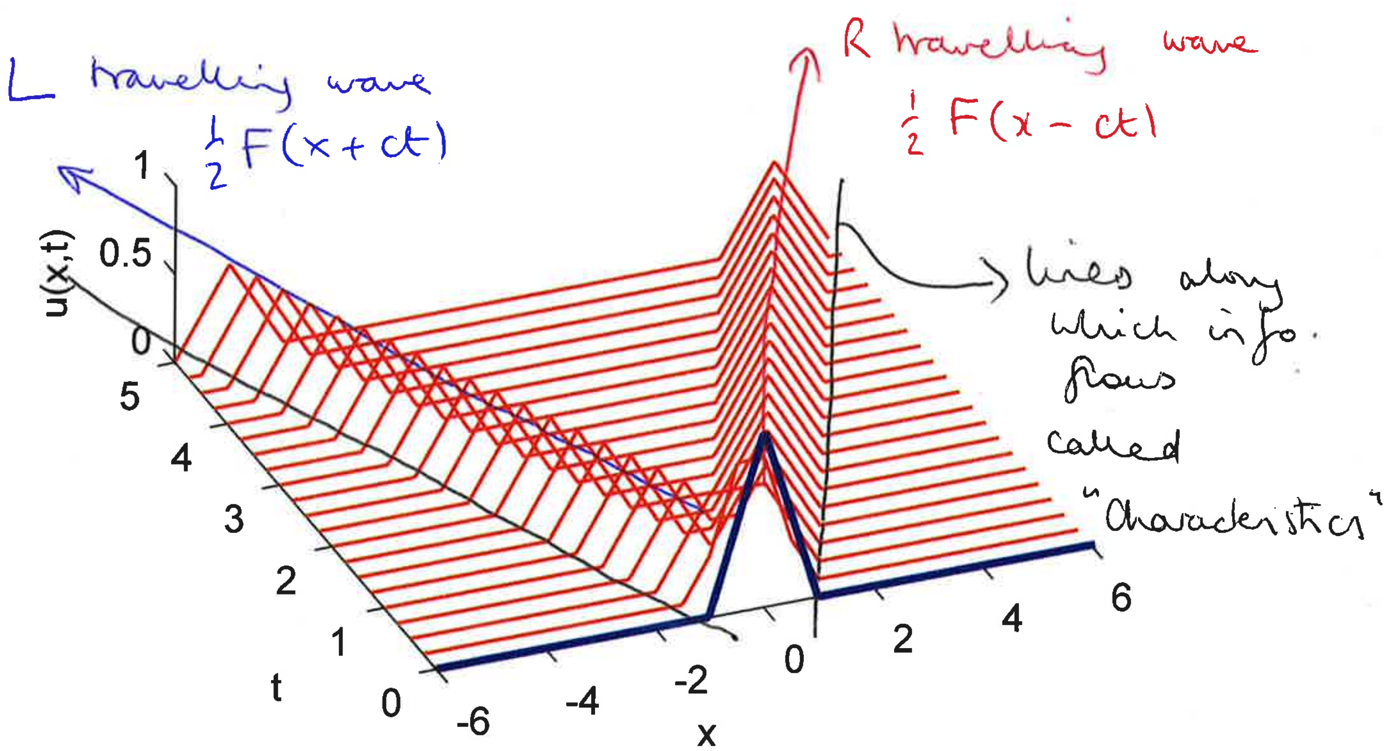


Step 5: plot the solution profile

Plots of $u(x,t)$ for $ct = 0, 0.5, 1, 4$:



Step 5: plot the solution profile



Worked example 6.2

d'Alembert soln.

Show that the general solution to the wave equation on an infinite domain

PDE $\boxed{u_{tt} = c^2 u_{xx},}$ $\boxed{-\infty \leqslant x \leqslant \infty, \quad t \geqslant 0,}$ DOMAIN

TWO INITIAL CONDS *subject to the initial conditions*

displacement velocity

$$\boxed{u(x,0) = 0, \qquad u_t(x,0) = x\,\mathrm{e}^{-x^2},}$$

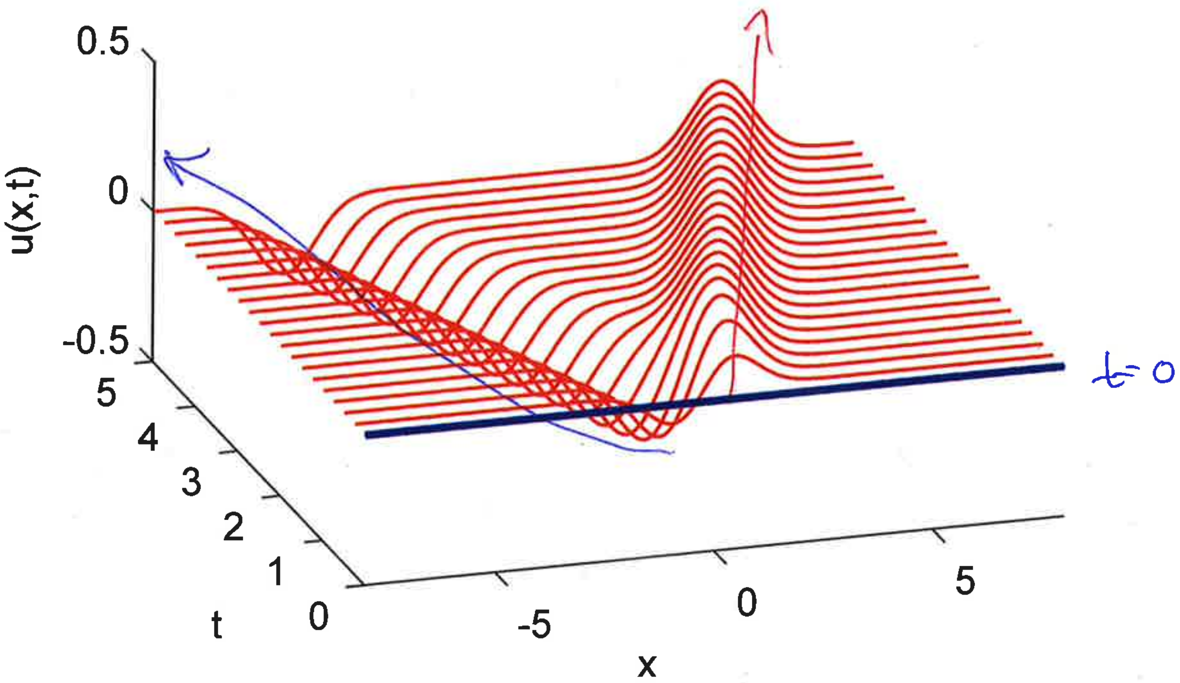
at $t=0$ for all x

is

$$u(x,t) = \frac{1}{4c}\,\mathrm{e}^{-(x-ct^2)} - \frac{1}{4c}\,\mathrm{e}^{-(x+ct^2)}$$

+ TWO BOUNDARY CONDS $u \to 0$ as $x \to \pm\infty$

Worked example 6.2: Solution profile



Ex 6.2

$$u(x,t) = f(x-ct) + g(x+ct)$$

d'Alembert sol'n of the wave eqn.

• use initial conds to find f, g

$t=0$

$$u = f(x) + g(x) = 0$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = -cf'(x) + cg'(x) = xe^{-x^2}$$

integrate wrt x

$$-cf(x) + cg(x) = \int xe^{-x^2}\,dx$$

$$= -\frac{1}{2}e^{-x^2} + K.$$

spot, or sub $v = x^2$

$$\boxed{\begin{aligned} f(x) + g(x) &= 0 \\ -\cancel{c}f(x) + \cancel{c}g(x) &= -\frac{1}{2c}e^{-x^2} + \frac{K}{c} \end{aligned}}$$

2 eqns for 2 unknowns f, g.

$+$

$$\boxed{\begin{aligned} \cancel{2}\, g(x) &= -\frac{1}{4c}e^{-x^2} + \frac{K}{2c} \\ f(x) &= -g(x) \\ &= +\frac{1}{4c}e^{-x^2} - \frac{K}{2c} \end{aligned}}$$

f and g!

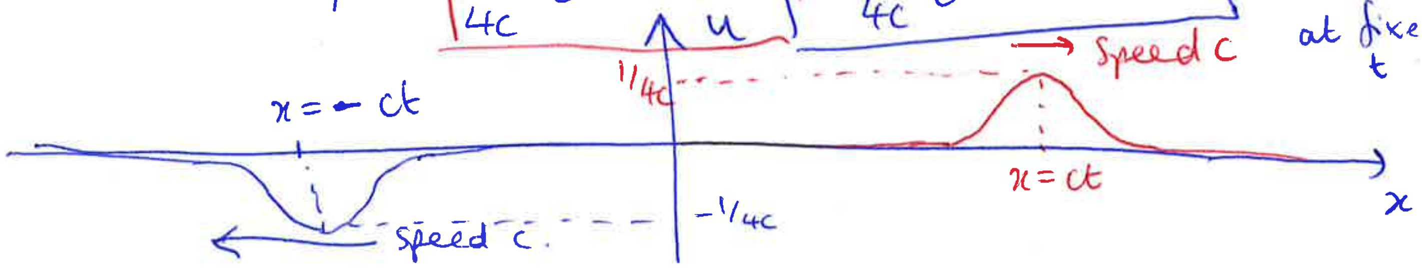
• sub into solution

$$u(x,t) = f(x-ct) + g(x+ct)$$

$$= \underbrace{\frac{1}{4c}e^{-(x-ct)^2} - \cancel{\frac{K}{2c}}}_{f(x-ct)}$$

$$\underbrace{-\frac{1}{4c}e^{-(x+ct)^2} + \cancel{\frac{K}{2c}}}_{g(x+ct)}$$

$$u(x,t) = \boxed{\frac{1}{4c}e^{-(x-ct)^2}} - \boxed{\frac{1}{4c}e^{-(x+ct)^2}}$$



at fixed t

6.3. Method for a semi-infinite domain

On a semi-infinite domain, the process is slightly more involved.

Example: *Solve the wave equation on a semi-infinite domain*

d'Alembert method.

PDE

$$\boxed{u_{tt} = c^2 u_{xx},} \quad \boxed{0 < x < \infty, \quad t > 0,}$$

subject to the initial conditions

2 INITIAL CONDS

displacement velocity

$$u(x, 0) = 0, \quad u_t(x, 0) = 0 \qquad \text{for } x > 0$$

at $t=0$ for all $x>0$

and boundary condition

2 BOUNDARY CONDS

L.H side

$$u(0, t) = \sin(\omega t), \qquad \text{for } t > 0$$

$x=0$ for all t

$u \to 0$ as $x \to +\infty$

which corresponds to a long string having one end subjected to a time-dependent (sinusoidal) excitation.

Step 1: state the d'Alembert solution

Note there is now only one boundary condition (in addition to the two initial conditions). Ordinarily for the wave equation we would expect 2 boundary conditions. The other boundary condition is an implicit one at $x = +\infty$ that the solution should be finite as $x \to \infty$.

The solution method initially proceeds as for the fully infinite domain, but with care to only allow x to be positive.

$$u(x, t) = f(x - ct) + g(x + ct)$$

B.3

$$u(x,t) = f(x-ct) + g(x+ct)$$

d'Alembert solution

- use initial conds to find f & g.

$t=0$ $\quad u = f(x) + g(x) = 0$

$t=0$ $\quad \frac{\partial u}{\partial t} = -\cancel{c}f'(x) + \cancel{c}g'(x) = 0$

for all $x>0$

int. wrt x

$$\boxed{\begin{aligned} -f(x) + g(x) &= K \\ f(x) + g(x) &= 0 \end{aligned}}$$

2 eqns for f & g.

solve.

$\oplus$

$$\boxed{\begin{aligned} g(x) &= K/2 \\ f(x) &= -K/2 \end{aligned}}$$

for all $x>0$

- Sub in to find u

$$u(x,t) = f(x-ct) + g(x+ct)$$

$$= -\frac{K}{2} + \frac{K}{2}$$

$$\boxed{u(x,t) = 0 \quad \text{if } x > ct}$$

not always true

if $x-ct>0$

if $x+ct>0$

always true

- What about $0 < x < ct$? We use the boundary condition: at $x=0$ $u = \sin(\omega t)$ for all $t>0$

$$u(x,t) = f(x-ct) + g(x+ct)$$

$x=0$ $\quad \sin \omega t = f(-ct) + g(ct)$

$$\boxed{\sin(\omega t) = f(-ct) + \frac{K}{2}}$$

as $ct > 0$.

The rule for f at $-$ve inputs.

Let ~~t~~ $z = -ct < 0$ everywhere

$$\sin\left(\frac{\omega . z}{c}\right) = f(z) + \frac{K}{2}$$

$$\boxed{f(z) = \sin\left(-\frac{\omega z}{c}\right) - \frac{K}{2}} \quad \text{for } z<0 .$$

So for $0 < x < ct \Rightarrow x - ct < 0$

$$u(x,t) = f(x-ct) + \underbrace{g(x+ct)}_{K/2} \qquad x + ct > 0$$

$$= \sin\left(-\frac{\omega(x-ct)}{c}\right) - \cancel{\frac{K}{2}} + \cancel{\frac{K}{2}}$$

$$u(x,t) = \begin{cases} \sin\left(-\dfrac{\omega(x-ct)}{c}\right) & \text{when } 0 < x < ct \\ 0 & \text{" } \quad x > ct \end{cases}$$

Step 2: use the initial conditions

First we take the zero-derivative condition

$$
\begin{aligned}
& 0 = u_t(x,0) = \left[-cf'(x-ct) + cg'(x+ct)\right]_{t=0}, \\
\therefore \quad & 0 = -f'(x) + g'(x) \\
\therefore \quad & K = g(x) - f(x), \qquad \text{for } x > 0 \qquad (4)
\end{aligned}
$$

for some constant K.

Setting the initial displacement to zero, we get

$$0 = u(x,0) = f(x) + g(x), \qquad \text{for } x > 0. \qquad (5)$$

Step 3: solve the simultaneous equations

Equations (4) and (5) are easily solved to give

$$f(x) = -\frac{K}{2}, \quad g(x) = \frac{K}{2} \qquad \textbf{but only for } x > 0$$

from which we get

$$
\begin{aligned}
f(x-ct) &= -\frac{K}{2} \qquad \text{for } x - ct > 0 \text{ (true for } x > ct) \\
g(x+ct) &= \frac{K}{2} \qquad \text{for } x + ct > 0 \text{ (true for all } x > 0,\ t > 0)
\end{aligned}
$$

So, for $x > ct$ we have

$$u(x,t) = f(x-ct) + g(x+ct) = 0$$

But we still have to find $f(x-ct)$ for $x < ct$.

Step 4: using the initial condition

We have $u(0,t) = \sin(\omega t)$. Hence

$$\begin{aligned}\sin(\omega t) = u(0,t) &= f(-ct) + g(ct)\\ &= f(-ct) + \frac{K}{2},\end{aligned}$$

from which we get that $f(-ct) = \sin(\omega t) - K/2$. Note that $-ct < 0$. Hence, if we let $z = -ct$, we have, for $z < 0$

$$f(z) = \sin\left(\frac{-\omega z}{c}\right) - \frac{K}{2} = -\sin\left(\frac{\omega z}{c}\right) - \frac{K}{2}$$

So, for $x - ct < 0$

$$f(x-ct) = -\sin\left(\frac{\omega(x-ct)}{c}\right) - \frac{K}{2}$$

Step 5: recombine to get general solution

For $x - ct < 0$ we have the solution

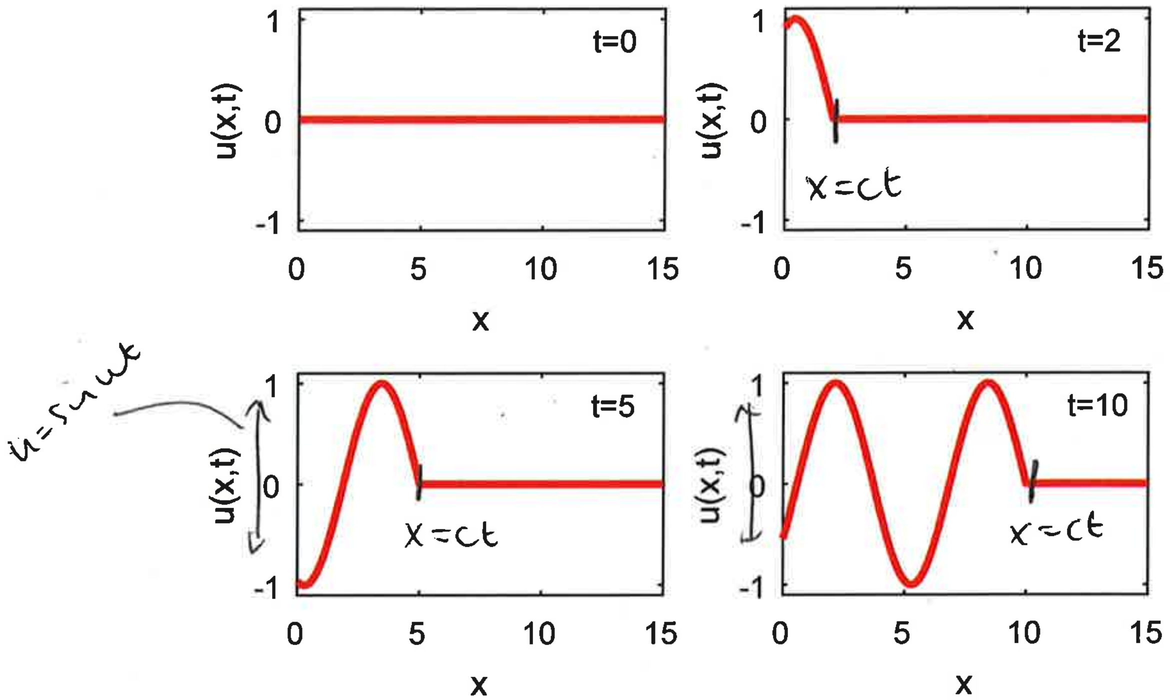
$$\begin{aligned}u(x,t) &= f(x-ct) + g(x+ct)\\ &= -\sin\left(\frac{\omega(x-ct)}{c}\right) - \frac{K}{2} + \frac{K}{2}\\ &= -\sin\left(\frac{\omega(x-ct)}{c}\right)\end{aligned}$$

So, the general solution is

$$u(x,t) = \begin{cases} -\sin\left(\frac{\omega(x-ct)}{c}\right) & x < ct\\ 0 & \text{otherwise}\end{cases}$$

Step 6: plot the solution profile

Plots for $\omega = 1$, $c = 1$ and $t = 0, 2, 5, 10$:



Step 6: plot the solution profile