Numerical methods

Lectures 7 and 8: Series

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Lecture 7: Series

A series is an infinite sum i.e.

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

For example we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots + \frac{1}{2^n} + \cdots = 1$$

Note that we sometimes begin a sum from $n = 1$ and sometimes from $n = 0$. Either way it's still a series but the value of the series will be different - the above series would be equal to 2 if we started from $n = 0$.

Sequences and series

In this part of the unit we look at two different but related things. Don't get them mixed up!

A *sequence* is a an *ordered set* of numbers that goes on forever:

$$\{a_n\}_{n=1}^{\infty} = a_1,\ a_2,\ a_3,\ \ldots, a_n, \ldots$$

A *series* is the *sum* of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

We already looked at *sequences* in the last lecture. Today we are talking about *series* (sums)...

Convergence of a series

Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

We are adding up *infinitely* many things, so shouldn't the answer be infinite?

It is possible to add infinitely many things and get a finite answer. In this case we say that a series *converges*.

Key question we often want to answer is "does it converge?".

Partial sums

Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

we can define the partial sums of the series as

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$

The kth partial sum is given by

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

Definition of convergence of a series

Given a series we can make a sequence of the partial sums of the series

$$\{s_k\} = s_1,\ s_2,\ s_3,\ \cdots,\ s_k,\ \cdots$$

If this *sequence* of partial sums converges to S we say that the *series* converges and equals S:

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} s_k = S$$

The geometric series

- One common series is the geometric series (also known as geometric progression)

$$\sum_{n=0}^{\infty} r^n$$

- In this case, we can find the partial sums explicitly:

$$\begin{aligned} s_k &= 1 + r + r^2 + r^3 + \cdots + r^k \\ r s_k &= r + r^2 + r^3 + \cdots + r^k + r^{k+1} \end{aligned}$$

$$(1-r)s_k = 1 - r^{k+1} \quad \Rightarrow \quad s_k = \frac{1-r^{k+1}}{(1-r)}$$

- We have an explicit formula for the partial sum of a geometric series.

The geometric series - convergence from the definition

Given a geometric series $\sum_{n=0}^{\infty} r^n$ the partial sums are given by $s_k = \sum_{n=0}^{k} r^n = \frac{1-r^{k+1}}{1-r}$. We have then that

$$\sum_{n=0}^{\infty} r^n = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r}$$

For a geometric *sequence* we know that $\lim_{k\to\infty} r^k = 0$ if $|r| < 1$ so

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

provided $|r| < 1$.

General geometric series

More general form of geometric series:

Sum of a geometric series

Let

$$s_k = \sum_{n=0}^{k} a r^n$$

The general expression for the sum of the first $k+1$ terms is

$$s_k = \frac{a(1-r^{k+1})}{(1-r)}$$

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$$

General geometric series - sum from 1 or 0

The formula for the partial or total sum of a geometric series varies dependinding on whether you sum from $n = 1$ or $n = 0$.

Compare the following formulas:

$$\sum_{n=0}^{k} a r^k = \frac{a(1-r^{k+1})}{(1-r)} \qquad \text{vs} \qquad \sum_{n=1}^{k} a r^k = \frac{a(r-r^{k+1})}{(1-r)}$$

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r} \qquad \text{vs} \qquad \sum_{n=1}^{\infty} a r^n = \frac{ar}{1-r}$$

Exercises

1. First example in Lecture 6:

$$s_k = \sum_{n=0}^{k} \frac{1}{2^n} \quad = 1 + (1/2) + (1/4) + (1/8) + \ldots$$

   find
   - the sum as $k \to \infty$
   - the sum of the first 80 terms
   - the lowest number k such that $s_k > 1.999$

2. Find $\sum_{n=1}^{\infty} a_n$ where $a_n = 5 \times (0.2)^n$
   What about $a_n = \frac{1}{2^n} + \left(\frac{5}{6}\right)^n$

Engineering Hotspot: mortgages

Suppose I buy a house costing $£250000$ and have a $£50000$ deposit. I will need to borrow $£200000$ from the bank and expect to pay this back over 25 years. If the interest rate is 3% what will be the monthly repayment?

The initial amount borrowed is called P (the *principal*). Each month the bank charges interest at a rate I and I pay a fixed amount F back to the bank. So if x_n is the amount owed we have the recurrence $x_{n+1} = x_n(1+I) - F$. Let's say that $r = 1 + I$ then

$$\begin{aligned} x_0 &= P \\ x_1 &= x_0 r - F = Pr - F \\ x_2 &= x_1 r - F = Pr^2 - Fr - F \\ x_3 &= x_2 r - F = Pr^3 - Fr^2 - Fr - F \end{aligned}$$

Can you recognise the pattern forming?

Engineering Hotspot: mortgages

The rhs has a geometric series so the amount owed after n months is

$$x_n = Pr^n - F\sum_{i=0}^{n-1} r^i = Pr^n - F\frac{1-r^n}{1-r}$$

The payment F is chosen so that after N months the amount owed $x_N = 0$ giving

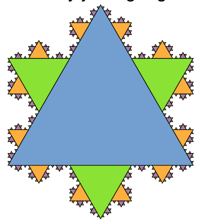
$$F = P(r-1)\frac{r^N}{r^N-1}$$

So if the (nominal) interest rate is 3% per year the monthly rate is $I = 0.25\%$ and $r = 1.0025$. After 25 years $N = 12 \times 25 = 300$ so

$$F = £200,000 \times 0.0025 \times \frac{1.0025^{300}}{1-1.0025^{300}} \approx £948.42$$

Exercise: Koch snowflake

Koch snowflakes are formed by joining together a series of triangles;



1. If the area of the central triangle (blue) is 1, what is the area of the snowflake?
2. What is the length of the perimeter of the snowflake?

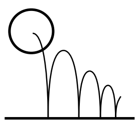
This is an example of a Fractal

Solution 1

Solution 2

Height of a bouncing ball

Remember the bouncing ball



The heights and velocities of the bounces form sequences $\{h_n\}$ and $\{v_n\}$ with

$$\{v_n\} = v_0,\ ev_0,\ e^2v_0,\ e^3v_0,\ \cdots$$

and $v_0 = \sqrt{2gh_0}$. From $v = u + at$ we have that the time for the initial fall is given as $t_0 = \frac{v_0}{g} = \sqrt{\frac{2h_0}{g}}$.

Height of a bouncing ball

Then the ball bounces up and down taking time t_1 and bounces again taking t_2. The time to finish bouncing is

$$T = t_0 + t_1 + t_2 + \cdots$$

Using $v = u + at$ again we have

$$t_n = \frac{2v_n}{g} = \frac{2v_0}{g}e^n = 2t_0e^n$$

We have then that

$$T = t_0 + \sum_{n=1}^{\infty} t_n = t_0 + 2t_0\sum_{n=1}^{\infty} e^n$$

So how long does it take if $h_0 = 1\,\mathrm{m}$ and $g = 10\,\mathrm{ms}^{-2}$ and $e = 0.9$?

Other elementary series . . .

- . . . for which we can express partial sum — James Section 7.3
  - Arithmetic progression

$$s_k = \sum_{n=0}^{k}(a+nd) = \frac{k+1}{2}(2a+kd) = \frac{(k+1)}{2}(a_0+a_n)$$

special case $\sum_{n=1}^{k} n = (1/2)k(k+1)$

  - Sum of squares

$$s_k = \sum_{n=0}^{k} n^2 = \frac{1}{6}k(k+1)(2k+1)$$

  - Arithmetic-geometric series

$$s_k = \sum_{n=1}^{k} nr^{n-1} = \frac{1-(k+1)r^k+nr^{k+1}}{(1-r)^2}$$

- Generally these are derived using *ad hoc* methods
- For most series there is no explicit expression for s_k

Lecture 8: Convergence of series

Series, the story so far

- $s_k = \sum_{n=0}^{k} a_n, (k = 0, 1, 2...)$ is called the *sequence of partial sums*
- We say the series *converges* if $\{s_k\}$ converges as $k \to \infty$
- Two important examples:
  - the geometric series

$$\sum_{n=0}^{k} ar^n = \frac{a(1-r^{k+1})}{1-r}$$

converges if $|r| < 1$ to limit $L = \frac{a}{1-r}$

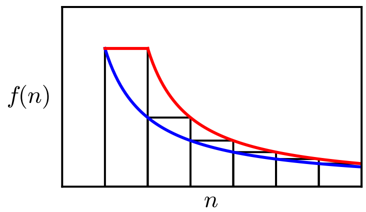
  - Now we'll see that the harmonic seres

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

diverges to infinity. . .

Integral test

Suppose $a_n = f(n)$ where f is a decreasing function.



$$\int_1^{\infty} f(x)\mathrm{d}x \le \sum_{n=1}^{\infty} f(n) \le f(1) + \int_1^{\infty} f(x)\mathrm{d}x$$

Integral test

- If f is decreasing then the series $\sum_{n=1}^{\infty} f(n)$ is bounded from below *and* above by the integral $\int_1^{\infty} f(x)\mathrm{d}x$.
- The series converges *if and only if* the integral does.
- Return to the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

- Despite the fact that $\frac{1}{n} \to 0$ as $n \to \infty$, this **diverges!**

Proof:

$$\sum_{n=1}^{\infty}\frac{1}{n} \ge \int_1^{\infty}\frac{1}{x}\mathrm{d}x = \ln(\infty) - \ln(1) = \infty$$

Exercise: integral test

The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is of the form

$$\sum_{n=1}^{\infty} n^{\alpha} \qquad (1)$$

with $\alpha = -1$. We've seen that this diverges.

Show that these series always

- diverge if $\alpha > -1$.
- converge if $\alpha < -1$.

Hence for series (1) the harmonic series is the boundary between convergence and divergence.

Solution 1

Solution 2

Three simple criteria for convergence/divergence

Terms must go to zero

$\sum_{n=0}^{\infty} a_n$ diverges if $\lim_{n\to\infty} |a_n| \neq 0$

Alternating series

$\sum_{n=0}^{\infty} a_n$ converges if

- a_n alternate in sign, and
- $\lim_{n\to\infty} |a_n| = 0,$
- and $|a_{n+1}| < |a_n|$ for all n

The comparison test

(for positive series $a_n \geqslant 0$) $\sum_{n=0}^{\infty} a_n$

- converges if $b_n \geqslant a_n \geqslant 0$ and $\sum_{n=0}^{\infty} b_n$ converges
- diverges if $a_n \geqslant b_n \geqslant 0$ and $\sum_{n=0}^{\infty} b_n$ diverges

Examples

- Find whether the following series converge or diverge:
  1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$
  2. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^2},$
  3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Example 1

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

- the terms alternate in sign,
- $\lim_{n\to\infty} \frac{1}{n} = 0,$
- $|a_{n+1}| < |a_n|$ (since $\frac{1}{n+1} < \frac{1}{n}$)
- hence the series converges

Example 2

$$\sum_{n=1}^{\infty} \frac{n+2}{n^2}$$

- each term in the series is positive, and bigger than $\frac{1}{n}$ since

$$\frac{n+2}{n^2} = \frac{1}{n} + \frac{2}{n^2} > \frac{1}{n}$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)
- hence the series diverges by the comparison test

Example 3

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

- each term in the series is positive, and less than $\frac{1}{2^{n-1}}$ since

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \geq 2 \cdot 2 \cdot 2 \cdots 2 \cdot 1 = 2^{n-1}$$
$$\Leftrightarrow \quad \frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

- $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (geometric series)
- hence the series converges by the comparison test

D'Alembert's ratio test

- The most useful test for series convergence is the ratio test:

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent*

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum_{n=0}^{\infty} a_n$ is *divergent*

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test gives *no information*

Absolute convergence

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent

- Absolutely convergent series are themselves convergent and may be rearranged to make new absolutely convergent series

Ratio test examples

1. $\sum_{n=0}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)!}\frac{n!}{1}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

$\Rightarrow$ converges

2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)^3}\frac{n^3}{3^n}\right| = \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^3 = 3 > 1$$

$\Rightarrow$ diverges

3. for $\sum_{n=1}^{\infty} \frac{1}{n}$ (the harmonic series) we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1 \qquad \Rightarrow \text{ no information}$$

A word of caution

The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

converges (to $\ln 2$); a little rearrangement gives

$$\underbrace{1 - \frac{1}{2}}_{1/2} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{1/6} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}}_{1/10} - \frac{1}{12} + \underbrace{\frac{1}{7} - \frac{1}{14}}_{1/14} - \frac{1}{16} + \cdots$$

which gives

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right)$$

half its original value!

Infinite series can only be rearranged if they are absolutely convergent!

Exercises

Find whether or not the following series converge as $n \to \infty$

1. $$\sum_{n=1}^{\infty}\frac{e^n}{n!}$$

2. $$\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}$$

3. $$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Homework for series

- Read Section 7.2.2
- Exercises 7.2.3: 9 & 11
- Section 7.3
- Exercises 7.3.4: 20, 24 & 25
- Section 7.6
- Exercises 7.6.4: 41, 44–49.