Vibrations 2, Lecture 11
Introduction to 2 DOF systems

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Lecture 10 review

Steady-state response under multi-harmonic excitation:

$$m\ddot{x} + c\dot{x} + kx = \sum_{(j)} F_{C,j} \exp(i\omega_j t)$$

$$x(t) = \sum_{(j)} x_j(t)$$

$$x_j(t) = H(\omega_j) F_{C,j} \exp(i\omega_j t)$$

Response to impulse excitation ($T_P << 2\pi/\omega_D$):

$$x(t) = \frac{I}{m\omega_D} e^{-\zeta\omega_0 t} \sin(\omega_D t)$$

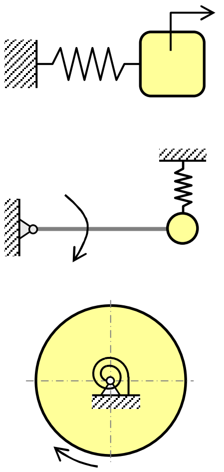
$$I = \int_0^{T_P} F(t)\, dt$$

Lecture 11

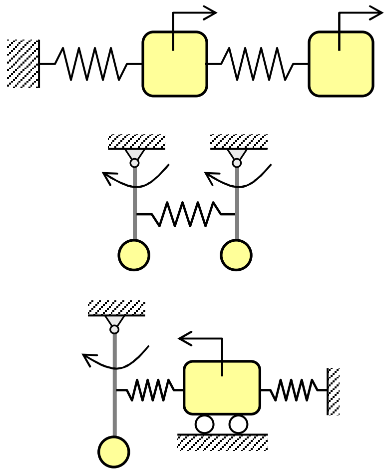
- 2 DOF examples and overview
- Newton's method
- Matrix form of EOMs

Degree of freedom

1 DOF examples



2 DOF examples

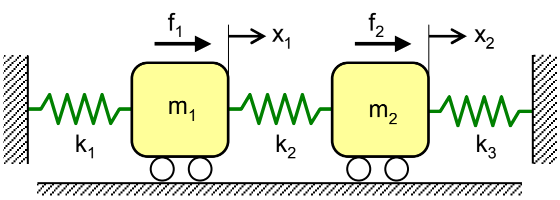


Overview

- EOMs and Newton's method
- Free response of 2 DOF systems
- Natural frequencies and mode shapes
- Initial conditions
- 2 DOF system with harmonic excitation
- Tuned Vibration Absorber
- Lagrange's equation and Energy methods
- Harmonic vibration of damped 2 DOF system
- Computational and approximate methods

2 DOF system

Consider the following 2 DOF system:

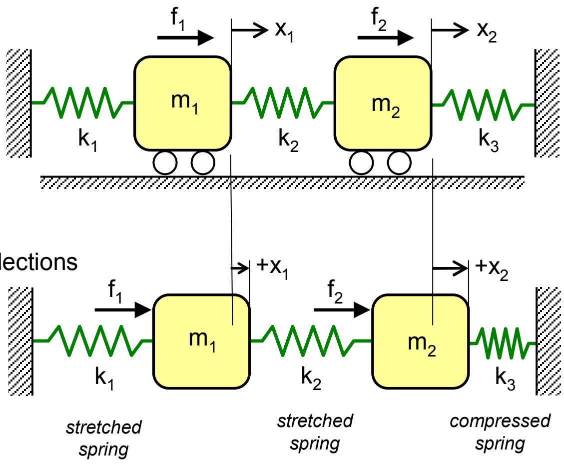


The main characteristics of this system are:

- two masses characterized by the mass m_1 and m_2
- two independent displacements described by the coordinates x_1 and x_2
- two DOFs and, therefore, two equations of motion (EOMs)
- two applied forces f_1 and f_2 acting on the lumped masses
- three springs with the stiffness parameters k_1, k_2 and k_3
- no damping (undamped system) (also no friction and no "rolling resistance")

2 DOF system

Preparation for the **free body diagram** (+/ve displacements):

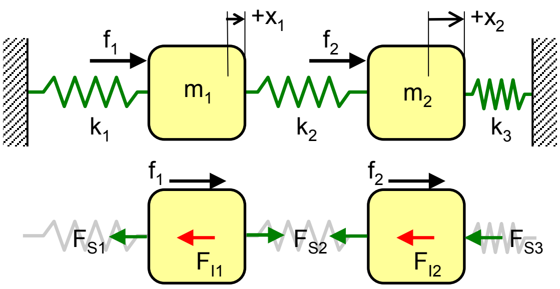


We assume:

- positive deflections
- $x_1 < x_2$

Newton's method

Free body diagram (+/ve displacements):



Equations of dynamic equilibrium (horizontal direction only!):

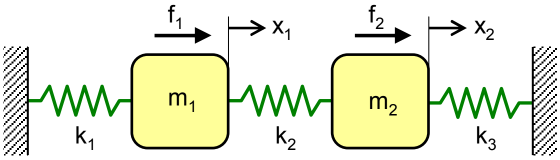
$$-F_{S1} - F_{I1} + F_{S2} + f_1 = 0 \qquad -F_{S2} - F_{I2} - F_{S3} + f_2 = 0$$

$$-k_1 x_1 - m_1 \ddot{x}_1 + k_2(x_2 - x_1) + f_1 = 0 \qquad -k_2(x_2 - x_1) - m_2 \ddot{x}_2 - k_3 x_3 + f_2 = 0$$

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1 \qquad m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = f_2$$

Newton's method

2 DOF system:



Equations of Motion:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1$$

$$m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = f_2$$

EOMs in *matrix form*:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

EOM in matrix form

EOMs:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

We will use the following notation:

$$\boxed{\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t)}$$

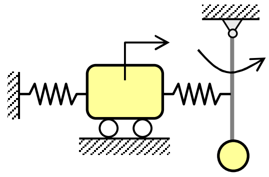
$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ is the mass matrix, $\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$ is the stiffness matrix

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the vector of displacements, $\ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$ is the vector of accelerations,

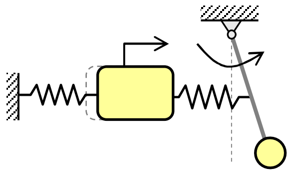
$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$ is the vector of applied (or external) loads (or forces)

Example

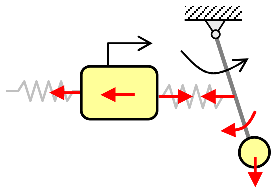
2 DOF system



+/ve deflection
deflection1 < deflection2



FBD



EOM

equation 1

equation 2

Summary

- Number of unique coordinates required to describe the configuration (deflection) of the vibrating system (DOFs)
- Newton's method for 2 DOF systems (assume positive deflections, relationship between assumed deflections)
- Matrix form of EOMs: mass and stiffness matrices; vectors of displacements, accelerations and applied loads