EMAT10100 Engineering Maths I
Lecture 23: Improper integrals

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Looking back, looking forward

- Last time:
  - First principles definition of integral (Riemann integral)
  - Definite and indefinite integrals and the fundamental theorem of calculus
  - Formulae for simple functions
  - The bad news, there are many functions you can't integrate. (No product, quotient or chain rule for integration)
  - The substitution method:

$$\int_a^b g(u(x))dx = \int_{x=a}^{x=b} g(u)\frac{\mathrm{d}\,x}{\mathrm{d}\,u}du = \int_{x=a}^{x=b} \frac{g(u)}{u'(x)}du$$

- This time
  - A little more on the substitution method
  - The good news - integration across discontinuities and on infinite domains

Special cases of substitution method

- Rule 1 : stretched co-ordinate

$$\int_{x=a}^{x=b} f(kx)dx = \frac{1}{k}\int_{y=ak}^{y=bk} f(y)dy$$

- Rule 2: logarithm rule

$$\int \frac{f'(x)}{f(x)}dx = \ln[f(x)] + c$$

(to see this use the substitution, $u = f(x)$).

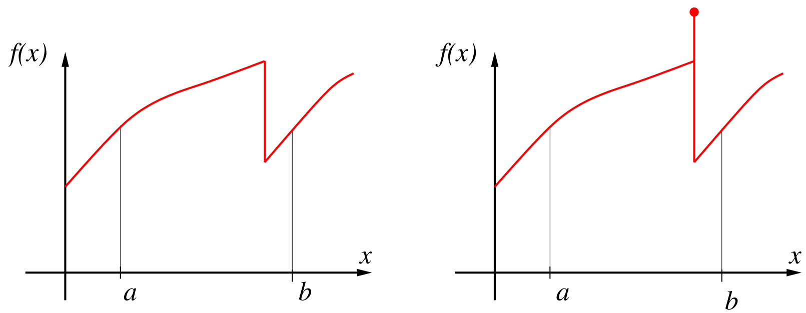
- Exercise use the logarithm rule to calculate

$$1.\quad \int_1^2 \frac{x^2 - 2x}{x^3 - 3x^2}dx, \qquad 2.\quad \int \tan(x)dx$$

(Hint, use $\tan(x) = \sin(x)/\cos(x)$)

Integration - the good news

- The integral is well defined for almost all “reasonable” functions for which the area under the curve is well defined and finite
- e.g discontinuous functions



- To calculate integral in practice, we can split it into separate pieces . . .

Example

- Find $\int_{-1}^{2} f(x)dx$ for the piecewise continuous function

$$f(x) = \begin{cases} 1+x & x<0 \\ 1-2x & 0 \le x < 1 \\ -2+x & x \ge 1 \end{cases}$$

- Start by drawing graph, check this makes sense ...
- ... then calculated piece by piece

$$\begin{aligned} \int_{-1}^{2} f(x)dx &= \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx \\ &= \int_{-1}^{0} (1+x)dx + \int_{0}^{1} (1-2x)dx + \int_{1}^{2} (-2+x)dx \\ &= \left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - x^2\right]_0^1 + \left[-2x + \frac{x^2}{2}\right]_1^2 \\ &= [0-(-1/2)] + [0-0] + [-2-(-3/2)] = 0 \end{aligned}$$

An application

Application to continuous probabilities (next term)

- In probably theory, the probability density function (p.d.f.) is defined as a continuous function $f(x)$, where the domain $x \in [a,b]$ is the set of all possible values of the continuous quantity.
- For this to be a well-defined p.d.f. we require that

$$\int_a^b f(x)dx = 1 \qquad \text{(sum of all probabilities must equal 1)}$$

- Exercise: Find the value of the parameter β so that the following is a well-defined probability density function for $x \in [-1,1]$

$$f(x) = \begin{cases} \beta(x+1) & -1 \le x < 0 \\ \beta & 0 \le x \le 1 \end{cases}$$

Engineering HOTSPOT

How to take experimental measurements

- Note that integration is smoothing. E.g. A function with a jump like

$$\int_{-1}^{x} f(\xi)d\xi, \qquad \text{where} \quad f(\xi) = \begin{cases} 1 & \xi < 0 \\ 2 & \xi \ge 0 \end{cases}$$

  integrates to a function that is continuous
- $\Rightarrow$ integration “smooths out” small errors
- $\Rightarrow$ differentiation amplifies small errors
- Q. In an experiment, suppose can only take measurements of displacement and acceleration at discrete instances of time. If I need to estimate velocity. Should I do this by:
  1. measuring position and differentiating in time?
  2. measuring acceleration and integrating in time?

Improper integrals

We can sometimes integrate things that aren’t even formally functions, or for which area isn’t well defined:

1. functions defined on infinite intervals, e.g.

$$\int_0^\infty e^{-10x}\, dx$$

2. functions that are infinite at an end point, e.g.

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

3. functions that are infinite within their domain, e.g.

$$\int_{-1}^1 \frac{1}{|x|}\, dx$$

Improper integrals - infinite domains

- Example: Compute $\int_0^\infty e^{-10x}dx$
- Method: calculate with finite limit X and take limit $X \to \infty$

$$\begin{aligned}
\int_0^\infty f(x)dx &= \lim_{X\to\infty}\int_0^X e^{-10x}dx \\
&= \lim_{X\to\infty}\left[-\frac{1}{10}e^{-10x}\right]_0^X \\
&= \lim_{X\to\infty}\left(\left[-\frac{1}{10}e^{-10X}\right]-\left[-\frac{1}{10}\right]\right) \\
&= [0]-[-1/10]=(1/10)
\end{aligned}$$

- Exercise: Find $\int_1^\infty \frac{1}{x^2}dx$

Improper integrals - infinite endpoints

- Example: Compute $\int_0^1 \frac{1}{\sqrt{x}}dx$
  (note $1/\sqrt{x} \to \infty$ as $x \to 0$)
- Method: we again take a limit

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{X\to 0}\int_X^1 \frac{1}{\sqrt{x}}dx \\
&= \lim_{X\to 0}\left[2\sqrt{x}\right]_X^1 \\
&= \lim_{X\to 0}\left[2(1-\sqrt{X})\right] \\
&= 2
\end{aligned}$$

- Exercise Evaluate the following integrals, if they exist:

$$1.\ \int_0^1 \frac{1}{x}dx \qquad 2.\ \int_0^1 \frac{2}{\sqrt{1-x}}dx$$

Improper integrals - integration through “bad” points

- Consider $\int_a^b f(x)dx$. Suppose integrand $f(x)$ is infinite at $x=c$, $a<c<b\ldots$
- . . . then we use the piecewise approach

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^c f(x)dx+\int_c^b f(x)dx \\
&= \lim_{X\to c^-}\int_a^X f(x)dx+\lim_{X\to c^+}\int_X^b f(x)dx
\end{aligned}$$

- Example:

$$\begin{aligned}
\int_{-1}^1 \frac{1}{\sqrt{|x|}}dx &= \lim_{X\to 0^-}\int_{-1}^X \frac{1}{\sqrt{-x}}dx+\lim_{X\to 0^+}\int_X^1 \frac{1}{\sqrt{x}}dx \\
&= \lim_{X\to 0^-}\left[-2\sqrt{-X}\right]-[-2] \\
&\quad +[2]-\lim_{X\to 0^+}[2\sqrt{X}] \\
&= 2+2 \\
&= 4.
\end{aligned}$$

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Lecture 24: Integration by parts

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Wouldn't it be nice . . .

- . . . if there was a product rule for integration $\int u(x)v(x)\,\mathrm{d}\,x = ?$
- Actually, there is (well, sort of!)
- consider product rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}\,x}(uv) = \frac{\mathrm{d}\,u}{\mathrm{d}\,x}v + u\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$$

- which rearranges to $u\frac{\mathrm{d}\,v}{\mathrm{d}\,x} = \frac{\mathrm{d}}{\mathrm{d}\,x}(uv) - v\frac{\mathrm{d}\,u}{\mathrm{d}\,x}$
- then integrate both sides

$$\int u\frac{\mathrm{d}\,v}{\mathrm{d}\,x}\,\mathrm{d}\,x = [uv] - \int v\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x$$

Integration by parts

$$\int_a^b u\frac{\mathrm{d}\,v}{\mathrm{d}\,x}\,\mathrm{d}\,x = [uv]_a^b - \int_a^b v\frac{\mathrm{d}\,u}{\mathrm{d}\,x}\,\mathrm{d}\,x$$

- works when u is function we can differentiate easily
- Example: $\int_0^1 x\ln(x)\,\mathrm{d}\,x \Rightarrow \ln(x)$ easily differentiates
- So, let 'u' be $\ln(x)$ and '$\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$' be x. So:

$$\begin{aligned}
\int_0^1 x\,\ln(x)\,\mathrm{d}\,x &= \left[\frac{x^2}{2}\ln(x)\right]_0^1 - \int_0^1 \frac{x^2}{2}\times\frac{1}{x}\,\mathrm{d}\,x \\
&= \left[\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right]_0^1 \\
&= [0-(1/4)] - [0] = -(1/4)
\end{aligned}$$

Exercises

- Use integration by parts to evaluate

$$1.\quad \int_{-\infty}^0 xe^x\,\mathrm{d}\,x,$$

$$2.\quad \int_0^{\pi/2} x^2\sin(x)\,\mathrm{d}\,x$$

- Note in the second example we have to use parts twice
- But not all products can be integrated by parts

$$\text{e.g.}\quad \int x^2e^{-x^2}\,\mathrm{d}\,x$$

do you see why?

A more complicated case

Integration by parts can get you back to where you started:

- Consider $\int e^x \sin(x)\,\mathrm{d}\,x$
- let 'u' $= e^x$, '$\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$' $= \sin(x)$ (actually, other choice $u = \sin(x)$, $\frac{\mathrm{d}\,v}{\mathrm{d}\,x} = e^x$ also works in this case)
- get $\int e^x \sin(x)\,\mathrm{d}\,x = [-e^x\cos(x)] + \int e^x \cos(x)\,\mathrm{d}\,x$
- so, use parts again: $\int e^x \sin(x)\,\mathrm{d}\,x = [-e^x\cos(x)] + [e^x\sin(x)] - \int e^x \sin(x)\,\mathrm{d}\,x$
- Oh dear, we've got back to where we started . . . but wait!
- Let $I = \int e^x \sin(x)\,\mathrm{d}\,x$. Then we have $I = [e^x(\sin(x) - \cos(x))] - I$
- Hence $2I = [e^x(\sin(x) - \cos(x))]$, which gives $I = \frac{1}{2}e^x[\sin(x) - \cos(x)] + c$

Recursive integration by parts

- Example let $I_n = \int x^n e^x dx$ where n is positive integer
- integrating by parts with 'u' $= x^n$ and '$\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$' $= e^x$:

$$
\begin{aligned}
I_n &= [x^n e^x] - \int n x^{n-1} e^x \\
&= [x^n e^x] - nI_{n-1} \\
&= [x^n e^x] - n([x^{n-1}e^x] - (n-1)I_{n-2}) \\
&= [x^n e^x] - [nx^{n-1}e^x] + [n(n-1)x^{n-2}e^x] + \ldots
\end{aligned}
$$

- need to treat I_0 as a special case: $I_0 = \int e^x\,\mathrm{d}\,x = [e^x]$
- Hence

$$I_n = (x^n - nx^{n-1} + n(n-1)x^{n-2} + \ldots + (-1)^{n-1}n(n-1)\ldots 2x + (-1)^n n!)e^x$$

Exercise

- Use recurrence integration by parts to find a general expression for

$$I_n = x^n \cos(x)$$

  when
  - (a) n is even,
  - (b) n is odd
- Note that in this example we have to perform integration by parts twice in order to get a recurrence formula.

Homework

- *James* 4th edition:
  - Improper integrals: Read Sec. 9.2
  - Do exercises 9.2.3 1(a)-(c),(e),(f)
  - Integtation by parts: read Sec. 8.8.3
  - Do exercises 8.84 Qs. 105 & 107
- *James* 5th edition:
  - Improper integrals: Read Sec. 9.2
  - Do exercises 9.2.3 1(a)-(c),(e),(f)
  - Integtation by parts: read Sec. 8.8.4
  - Do exercises 8.85 Qs. 110 & 111