Compressible Flow Tips and Tricks: Isentropic Flow, Nozzles, Normal Shocks, Oblique Shocks and Expansion Fans

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Contents

1 Introduction

1.1 A Word of Caution

This document is intended to give you a clear idea about how to carry out compressible flow calculations in this course. You should know everything in this document *and* the lecture slides. Generally in the lectures we will discuss why something is true and how it is derived, while in these notes you will see where it fits in to a calculation. You will not ordinarily be expected to reproduce derivations (such as for the normal shock relations) but you *must* understand all of the assumptions that they are built on. Although the calculations presented here may seem mechanical, you will only be able to navigate through them if you understand the theoretical basis soundly.

Understanding of a subject always benefits from absorbing a range of viewpoints; if you are looking for suitable background reading I suggest either *Fundamentals of Aerodynamics*[1] (more introductory) or *Modern Compressible Flow*[2] (fairly detailed).

1.2 Forces in Compressible Flow

The key point in the compressible calculations for this course is often to find the pressure coefficient

$$C_{p_1} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_1}{p_\infty} - 1\right) \quad (1)$$

The pressure coefficient is nature's own measure of pressure. It is dimensionless. It measures pressure above or below the reference pressure p_∞ in multiples of the dynamic pressure.

By now you will have realised that all engineers love dimensionless numbers. This is because:

1. They are independent of the system of units we are using (principally, this makes is easier to compare results across the Atlantic...). It doesn't matter if your colleagues measure the pressure on their wind tunnel model in (the mass of a cupcake)(furlongs per fortnight per fortnight)/furlongs^2 providing that they understand the definition of C_p

2. Dimensionless analysis allows us to discover the smallest number of independent variables suitable for description of a particular phenomenon (the Buckingham π Theorem). This means we don't waste time worrying about an excessive number of degrees of freedom

3. Eliminating dimensions either leads to a framework where scale doesn't matter anymore, or, better still, one where scale can described by another non-dimensional number (the classic examples is the Reynolds number, but other similar numbers exist for other settings, for example the Froude number or the Nusselt number)

For a particular aircraft in steady flight, C_p at a point on the aeroplane is only a function of R_e, M, α (you could also include the roll and yaw angles too). R_e defines the influence of the physical size of the aircraft, and complicates things significantly. In these compressible flow notes we will assume C_p just depends on M, α. This is not actually true, but without this assumption we cannot easily derive analytical flow solutions. Providing the surface curvature and angle of attack both remain small then separation is unlikely and the results are reasonably accurate, but compared to experimental results we will see errors towards the trailing edge of aerofoils, where the boundary layer has had the time to thicken.

So, unless it is explicitly asked, do not calculate pressures themselves (i.e. in Pascals). If you find yourself calculating a pressure, stop, think, and revise your plan. To find a C_p you *always* find $\frac{p}{p_\infty}$ first. Fortunately for you, this is how the flow tables have been designed already.

The next few sections describe how you should go about carrying out a compressible flow calculation. I strongly suggest you memorise the ‘procedures’ and then reinforce this with practice on the tutorial sheets when we reach the appropriate part of the course.

A word of warning for those amongst you who (like me) prefer it if their ducks are in a row. The individual calculations described below are not difficult to understand or repeat in isolation; however, the expectation on completion of this course is not that you will be capable of working out what happens in an oblique shock or an expansion fan in isolation. The expectation is that **you will be able to promptly calculate C_p, M and any other relevant variables for an arbitrary configuration of normal/oblique shocks and expansion fans, applying and showing all necessary precautions you take during the calculation to prevent careless errors**. Questions on this part of the course will evidently involve all the operations below, that much is obvious, but stitching them together correctly takes experience and good judgement. You can only pick this up through plenty of practice.

Finally, you need to complete the calculation *promptly*. Getting the right answer is not enough, you need the right answer as fast as possible. This is a good attitude for exams. On any exam there will be easy and difficult marks (if there are no easy marks, I’m afraid you’re just out of luck), and you should aim to get the easy marks quickly to leave time to worry about things that may be genuinely tricky. On this note, make certain you are familiar with performing linear interpolations $\mathcal{L}$, and if not refer to section 13.

2 Flow Along Streamtubes

2.1 Isentropically Connected Regions

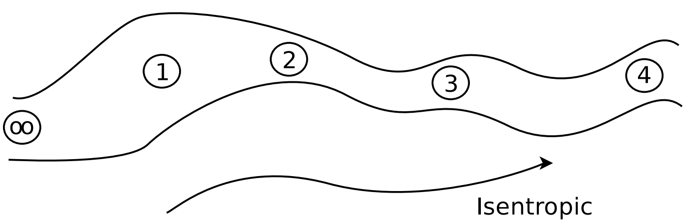

Figure 1: An honest streamtube

The isentropic flow tables provide, amongst other useful ratios, $\frac{p_0}{p}$ for a particular M. The C_p definition does not use total pressures, only static pressures, so these ratios need to be manipulated to be used. Consider the case where you’d like to find $\frac{p_4}{p_\infty}$, where region 4 is isentropically connected to ∞ via regions 1,2, and 3. We can write this as

$$\frac{p_4}{p_\infty} = \frac{p_4}{p_{04}} \frac{p_{03}}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_{0\infty}} \frac{p_{0\infty}}{p_\infty} \tag{2}$$

We've just said that ∞,1,2,3,4 are isentropically connected, so $p_{04} = p_{03} = p_{02} = p_{01} = p_{0\infty}$, therefore this simplifies to

$$\frac{p_4}{p_\infty} = \frac{p_4}{p_{04}}\frac{p_{0\infty}}{p_\infty} \tag{3}$$

So for any region n this could be found as

$$\frac{p_n}{p_\infty} = \frac{p_n}{p_{0n}}\frac{p_{0\infty}}{p_\infty} \tag{4}$$

providing the flow remains isentropic along a streamline $\infty \to n$*.*

A nice aspect to this is that since the total pressure remains unchanged along an isentropic path, you only need to work out the total pressure ratios at the start and finish; there is no need to work out any intermediate values unless you are specifically required to do so. Minimising the amount of linear interpolation you have to do will save you valuable time.

2.2 Non-isentropically Connected Regions

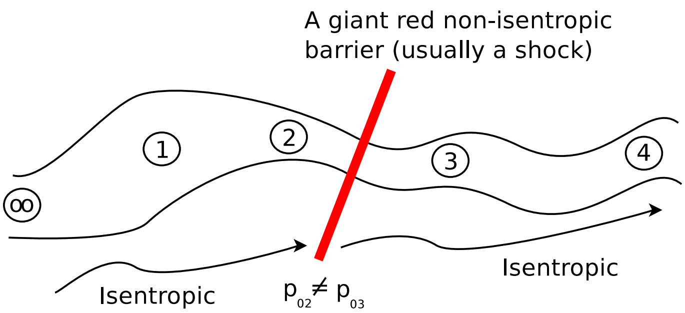

Figure 2: A dishonest streamtube

In the event that region $2 \to 3$ is not isentropic but $\infty \to 1 \to 2$ and $3 \to 4$ still are, you must retain the non-isentropic total pressure ratio

$$\frac{p_4}{p_\infty} = \frac{p_4}{p_{04}}\frac{p_{03}}{p_{02}}\frac{p_{0\infty}}{p_\infty} \tag{5}$$

where $p_{02} \neq p_{03}$. You will need to find $\frac{p_{02}}{p_{03}}$ through the shock tables (either normal or oblique as appropriate).

The easiest way to find the total pressure ratio from tables is as

$$\frac{p_{03}}{p_{02}} = \frac{p_{03}}{p_3}\frac{p_3}{p_2}\frac{p_2}{p_{02}} \tag{6}$$

and this is how you should calculate going through an oblique shock. You can see that three pressure ratios are needed, requiring three linear interpolations. However, for a normal shock the ratio $\frac{p_{03}}{p_2}$ can be retrieved directly from tables, so you could get the total pressure ratio as

$$\frac{p_{03}}{p_{02}} = \frac{p_{03}}{p_2}\frac{p_2}{p_{02}} \tag{7}$$

thereby saving $\frac{1}{3}$ of the effort.

The value of $\frac{p_{03}}{p_{02}}$ is going to turn out to be < 1, because a non-isentropic process involves a loss in total pressure.

Error check: making the error of using a total pressure ratio through a shock $= 1$ is a common and severe mistake.

You may also need to connect multiple non-istentropic regions. This may be done along similar lines

$$\frac{p_n}{p_\infty} = \frac{p_n}{p_{0n}}\frac{p_{0n}}{p_{0n'}}\frac{p_{0n'}}{p_{0n''}}\frac{p_{0n''}}{p_{0n'''}}\ldots\frac{p_{0n'''}}{p_{0\infty}}\frac{p_{0\infty}}{p_\infty} \tag{8}$$

where $p_{0n} \neq p_{0n'} \neq p_{0n''} \neq p_{0n'''}$.

Unfortunately, and in constrast to the previous isentropic case, you will need to find the non-unity intermediate total pressure ratios through the shock tables (either normal or oblique as appropriate in each case). You cannot leave out any of these intermediate ratios out because each one alters the total pressure, and any upstream change in total pressure propagates all the way downstream. However, if you are moving along an isentropic section of the path, once again you only need two total pressure ratios to find $\frac{p}{p_\infty}$.

3 Nozzles

Nozzles come in two flavours: unchoked and choked. A choked nozzle reaches $M = 1$ at the throat (narrowest point), while an unchoked nozzle is wholly subsonic. The sonic point will *always* occur at the narrowest point, as indicated by the relationship quoted in the lecture slides.

The most common calculation is to find M given the cross sectional area A. The key point to solving this is to understand A^*, which is the choking area. For any value of M occurring at a point in the nozzle with cross sectional area A there is also an area A^* that would accelerate/decelerate the flow to give $M = 1$.

The value A^* always exists, even if the flow throughout the nozzle is wholly supersonic or subsonic. So, it doesn't matter at all if choking doesn't take place; A^* is just an extremely useful quantity for calculating what happens.

Let's say you know M_1 at a point where the area is A_1. Your challenge is to find M_2 at a point where the area is A_2, but everything must use $\frac{A}{A^*}$ values because this is what is in your tables.

Begin in a similar vein to expanding isentropic pressure ratios, so

$$\frac{A_2}{A^*} = \frac{A_2}{A_1}\frac{A_1}{A^*} \tag{9}$$

The value $\frac{A_2}{A_1}$ we obviously know, but $\frac{A_1}{A^*}$ is something that must be found from tables as $\frac{A}{A^*}_{\mathcal{L}} = \mathcal{L}\left(\frac{A}{A^*}_a, \frac{A}{A^*}_b, M_a, M_b, M_{\mathcal{L}}\right)$. Then get $\frac{A_2}{A^*}$ from equation (9).

Once you have $\frac{A_2}{A^*}$ compute $M_{\mathcal{L}} = \mathcal{L}\left(M_a, M_b, \frac{A}{A^*}_a, \frac{A}{A^*}_b, \frac{A}{A^*}_{\mathcal{L}}\right)$.

Beware! This is not the end of the story. You will find that for the value of $\frac{A_2}{A^*}$ that you have, there are actually two possible values of M, one subsonic and one supersonic. To determine which is correct you must look at the problem and decide if the nozzle is choked. Then, if you are trying to find a value of M beyond the choking point, it is the supersonic one you need. If before the choking point, then you need the subsonic value.

This brings us to a useful sanity check.

Error check: subsonic flow in a converging (contracting) duct accelerates. Supersonic flow in a converging (contracting) duct decelerates. Subsonic flow in a diverging (expanding) duct decelerates. Supersonic flow in a diverging (expanding) duct accelerates.

This is simple to remember, so make certain you do.

3.1 Supersonic Expanding Nozzle

At station 1 we have M_1=1.5. Determine M_2 at a location where the nozzle has expanded so that $\frac{A_2}{A_1} = 2$.

Think: the nozzle is expanding and the flow is supersonic. The Mach number must be going to go up.

No need to interpolate here, as the result is straight from tables

$$\frac{A_1}{A^*} = 1.176 \tag{10}$$

which gives

$$\frac{A_2}{A^*} = 2.352 \tag{11}$$

leading us to

$$M_2 = \mathcal{L}\,(2.36, 2.38, 2.311, 2.353, 2.352) = 2.3795240 = 2.380 \tag{12}$$

Check: the Mach number has indeed risen.

Check: the result is between 2.36 and 2.38 (although it has rounded to the upper bound here).

3.2 Subsonic Expanding Nozzle

At station 1 we have M_1=0.5. Determine M_2 at a location where the nozzle has expanded so that $\frac{A_2}{A_1} = 2$.

Think: the nozzle is expanding and the flow is subsonic. The Mach number must be going to go down

No need to interpolate here, as the result is straight from tables

$$\frac{A_1}{A^*} = 1.340 \tag{13}$$

which gives

$$\frac{A_2}{A^*} = 2.680 \tag{14}$$

leading us to

$$M_2 = \mathcal{L}\left(0.22, 0.24, 2.707, 2.495, 2.680\right) = 0.22254716 = 0.222 \tag{15}$$

Check: the Mach number has indeed decreased.

Check: the result is between 0.22 and 0.24.

4 Normal Shocks

Before starting let's clear up one thing. Normal shocks are *perpendicular* to the flow direction (i.e. at right angles). In this context normal *means* perpendicular. The opposite of a normal shock is an oblique shock, which is inclined to the flow vector. There is no such thing as a 'strange' shock!

There is not much more to normal shocks than using $\mathcal{L}$ to get the value you need. It is useful to notice that a downstream total pressure ratio to upstream static $\frac{p_{02}}{p_1}$ is often included in tables, as this may save you some calculating effort in particular cases (see section 2.2).

Error check: a normal shock *always* has supersonic flow infront of it and subsonic flow behind.

Error check: any shock *always* produces an increase in pressure, so $\frac{p_2}{p_1} > 1$, and a drop in total pressure, so $\frac{p_{02}}{p_{01}} < 1$.

5 Oblique Shocks

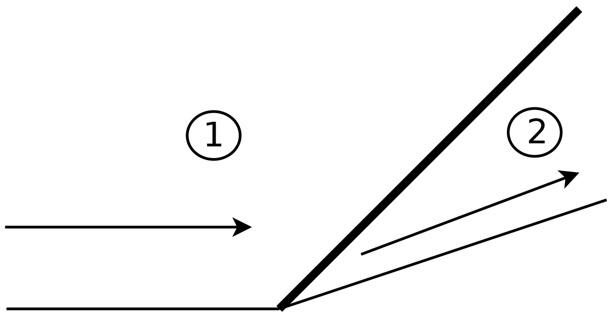

Figure 3: A hearty oblique shockwave

The oblique shock tables have two solutions for a particular deflection angle θ. The one that has the lowest β is the *weak* shock solution, and the largest β corresponds to the *strong* shock solution. For all calculations in this course you will be using the weak shock solution. Although strong shocks are physically possible, they only tend to occur in situations where there is a significant back pressure on the flow (i.e. a very severe adverse pressure gradient). This does not normally occur in the external flows we deal with, but may happen for complicated internal flows (where the gas is completely/mostly enclosed by solid walls).

1. Write down M_1 (Mach number infront of the object) and θ (the half-wedge angle). Remember, the half-wedge angle is the angle from the freestream to the surface, and that for a shock to

occur we must have $\mathbf{V_1} \cdot \mathbf{n} < 0$ where $\mathbf{n}$ is the outward normal vector for the surface. You can also think of this as though there is a component of the freestream vector that is pointing towards the *inside* of the shape. Of course, if this dot product is positive, we have an expansion, not a shock, so the Prandtl-Meyer expansion method must be used (this will be covered in the next section)

2. Confirm that there is indeed an oblique shock solution. If $\theta_{max} < \theta$ then you are out of luck; the only option is instead a normal shock. Reappraise your plan and turn the page to the normal shock tables

3. Check to see if the combination of M_1 and θ corresponds to an entry in the oblique shock table. If it does, write down $\frac{p_2}{p_1}$ immediately and go to step 5. Make certain the value of $\frac{p_2}{p_1}$ you are writing down is > 1. An oblique shock is a compression, so this ratio has to be more than unity!

   Error check: pressure rises through a shock.

   You should always check this as it is easy to rashly write down a wrong number from tables in the heat of the moment, and you are expected to perform all obvious checks as your calculation proceeds

4. The combination of M_1 and β does not correspond to an entry in the tables. Sadly, it is simply not going to be worth your while attempting to solve the θ, β, M relation. It is possible, but it requires solution of a cubic, and the fact of the matter is it is much simpler to interpolate the value you need from the tables.

   Woe is me...a linear intepolation is required! You now need to determine which of the two possibilities below you have encountered:

   - The value of either M or θ is in the tables, but not both. Unfortunate, but no great worry. It could have been worse. You could have ended up at the bullet point below.

     Compute $\mathcal{L}$ and you will be done, so find $\frac{p_2}{p_1}_{\mathcal{L}} = \mathcal{L}\left(\frac{p_2}{p_1}_a, \frac{p_2}{p_1}_b, \theta_a, \theta_b, \theta_{\mathcal{L}}\right)$ or $\frac{p_2}{p_1}_{\mathcal{L}} = \mathcal{L}\left(\frac{p_2}{p_1}_a, \frac{p_2}{p_1}_b, M_a, M_b, M_{\mathcal{L}}\right)$ and got to step 5
   - Neither the value of M nor θ is in the tables. You are now faced with a double linear interpolation. Find the two bracketing Mach numbers in the tables for the M you are interested in, and for each of these compute $\frac{p_2}{p_1}_{\mathcal{L}} = \mathcal{L}\left(\frac{p_2}{p_1}_a, \frac{p_2}{p_1}_b, \theta_1, \theta_2, \theta_{\mathcal{L}}\right)$. You next just need to find $\frac{p_2}{p_1}_{\mathcal{L}} = \mathcal{L}\left(\frac{p_2}{p_1}_a, \frac{p_2}{p_1}_b, M_a, M_b, M_{\mathcal{L}}\right)$, but the values of $\frac{p_2}{p_1}_a$ and $\frac{p_2}{p_1}_b$ are now the two values you just found. Hurray! (You could also interpolate in M before θ, and the answer will be very slightly different)

5. Insert the value of $\frac{p_2}{p_1}$ into equation (1). If 1 is the same as ∞ then you have finished, otherwise you need to use the techniques in sections 2.1 and 2.2 to get $\frac{p_2}{p_\infty}$

6 Expansion Fans

The equation you cannot afford to forget is

$$\theta = \nu(M_2) - \nu(M_1) \tag{16}$$

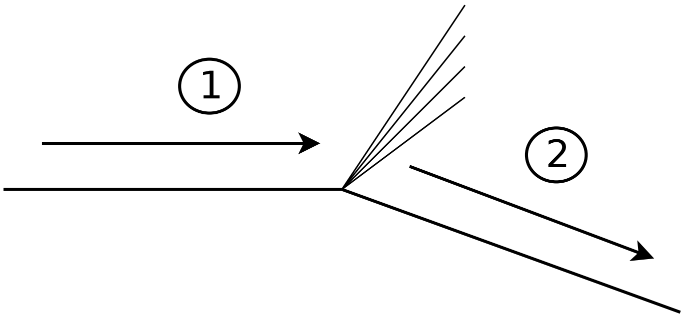

Figure 4: An expansion fan in its native environment

and it will not be quoted on any rubric so you have to memorise it. Why? Simply because if you do not know this off the top of your head, there is scant chance of you making any progress with this calculation.

Notice that θ here is positive away from the flow direction, so if it is negative, we actually have a compression. Now, I hear you cry, “You told me shocks were compressions. This confusing and apparently contradictory development is simply terrible news. I just cannot go on”. The reason why compression is also possible here is that isentropic compression (as opposed to the non-isentropic compression of shockwaves) is physically possible. It is however rather unlikely. A good example would be a surface curving *smoothly* into the flow (whereas a wedge is a *sharp* disconintuity). But beware, if the curvature is too steep, the isentropic compression waves coalesce to form a shock at some (small-ish) distance off the surface. If this doesn’t happen you can deal with the compression in the same way as an expansion but with a negative θ. You will always be warned if a compression is isentropic, though perhaps in loosely veiled language such as ‘the surface curves smoothly at small amplitude with no discontinuities...’.

On a minor point, $\nu\left(M\right)$ means ν operating on M, **not** $\nu \times M$, i.e. ν is a function of M. You may think this trivial, but it has cropped up a number of times.

Another interesting feature of $\nu\left(M\right)$ is that it is tabulated in *degrees*. Why this convention exists is beyond me; perhaps it is done simply to confuse, though the most likely answer is because it gives pleasant numbers. Make a mental note of this, and at the same time also remember that Ackeret’s linearised C_p *does use radians*. You should also remember that Ackeret’s C_p measures the deflection angle relative to the freestream flow, whereas deflection angles for expansion fans (or oblique shocks) are measured relative to the last flow direction (i.e. the velocity vector before the compression or expansion was encountered).

1. In equation (16) we know θ, but we don’t know $\nu\left(M_2\right)$ or $\nu\left(M_1\right)$. Refer to your tables and if necessary compute $\nu\left(M_{\mathcal{L}}\right) = \mathcal{L}\left(\nu\left(M\right)_a, \nu\left(M\right)_b, M_a, M_b, M_{\mathcal{L}}\right)$ to get $\nu\left(M_1\right)$

2. Add the expansion angle θ to $\nu\left(M_1\right)$ to get $\nu\left(M_2\right) = \theta + \nu\left(M_1\right)$

3. Calculate $M_2 = \nu^{-1}\left(\nu\left(M_2\right)\right)$. The function ν^{-1} does not, to my knowledge, exist in a simple, easy to use form. There are some explicit approximate inverses that are quite often used, but if faced with needing to find the exact inverse I use some direct iteration, or maybe even some Newton-Raphson iterations. It is not worthwhile doing this in hand calculations so instead you are going to do another linear interpolation using the tables. Just compute $\nu^{-1}\left(M_{\mathcal{L}}\right) = \mathcal{L}\left(\nu^{-1}\left(M\right)_a, \nu^{-1}\left(M\right)_b, M_a, M_b, M_{\mathcal{L}}\right)$ to get $\nu^{-1}\left(M_2\right)$

Error check: if you have an expansion, $M_2 > M_1$ and $\frac{p_2}{p_1} < 1$, otherwise for an isentropic compression you have $M_2 < M_1$ and $\frac{p_2}{p_1} > 1$.

6.1 Expansion Example

A flow at $M = 1.5$ passes of an expansion of $\theta = 10^o$. Find the downstream Mach number and pressure coefficient.

$$\nu\left(1.5\right) = 11.880 \tag{17}$$

$$\nu\left(M_2\right) = 11.880 + 10 = 21.880 \tag{18}$$

Now we need to compute the inverse

$$\nu^{-1}\left(21.880\right) = \mathcal{L}\left(1.84, 1.86, 21.830, 22.400, 21.880\right) = 1.8417544 = 1.842 \tag{19}$$

Check: the Mach number has gone up, which is consistent with air passing over an expansion.

Check: the result is between 1.84 and 1.86.

Finding C_p requires $\frac{p_0}{p}$ for $M = 1.5$ and $M = 1.842$.

$$\left.\frac{p_0}{p}\right|_{M=1.5} = 3.675 \tag{20}$$

$$\left.\frac{p_0}{p}\right|_{M=1.842} = \mathcal{L}\left(6.115, 6.305, 1.84, 1.86, 1.842\right) = 6.1339993 = 6.134 \tag{21}$$

$$\frac{p_1}{p_\infty} = \frac{p_1}{p_{01}}\frac{p_{0\infty}}{p_\infty} = \frac{1}{6.134} \times 3.675 = 0.599 \tag{22}$$

$$C_p = \frac{2}{1.4 \times 1.5^2}\left(0.599 - 1\right) = -0.254 \tag{23}$$

Check: C_p is negative, consistent with expansion.

6.2 Compression Example

A flow at $M = 1.5$ passes through a compression of $\theta = 10^o$. The surface curves smoothly so no oblique shock forms. Find the downstream Mach number and pressure coefficient.

$$\nu\left(1.5\right) = 11.880 \tag{24}$$

$$\nu\left(M_2\right) = 11.880 - 10 = 1.880 \tag{25}$$

Notice that such a small value of ν means the flow is only just supersonic. Now we need to compute the inverse

$$\nu^{-1}\left(21.880\right) = \mathcal{L}\left(1.12, 1.14, 1.730, 2.160, 1.880\right) = 1.1269767 = 1.127 \tag{26}$$

Check: the Mach number has gone down, which is consistent with air passing through a compression.

Check: the result is between 1.12 and 1.14.

Check: as suspected the flow is only just superonic.

Finding C_p requires $\frac{p_0}{p}$ for $M = 1.5$ and $M = 1.127$.

$$\frac{p_0}{p}_{M=1.5} = 3.675 \tag{27}$$

$$\frac{p_0}{p}_{M=1.127} = \mathcal{L}\left(2.191, 2.247, 1.12, 1.14, 1.127\right) = 2.2105348 = 2.211 \tag{28}$$

$$\frac{p_1}{p_\infty} = \frac{p_1}{p_{01}}\frac{p_{0\infty}}{p_\infty} = \frac{1}{2.211} \times 3.675 = 1.662 \tag{29}$$

$$C_p = \frac{2}{1.4 \times 1.5^2}\left(1.662 - 1\right) = 0.420 \tag{30}$$

Check: C_p is positive, consistent with compression.

7 Complete Calculations

This section should give you a good idea about how to put shocks and expansions together in a single calculation. The basic idea is to daisy-chain ratios in the manner you have seen already, taking care not to accidentally cross a shock without letting the total pressure change. If there is a ratio you need but don't have, just write it as a product of two other ratios that you can find.

Sometimes there are a couple of alternative routes to get a particular answer, as for the total pressure ratios below. Clearly you should choose the one that minimises effort, as this saves you time and also minimises roundoff errors.

7.1 Shock-expansion-expansion

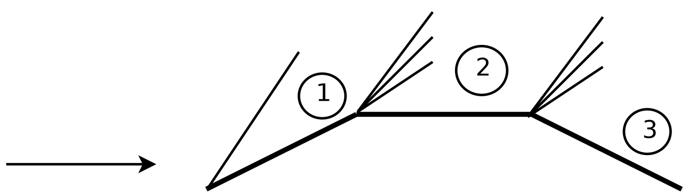

Figure 5: Shock-expansion-expansion

Mach 2 flow passes over three surfaces that produce through an oblique shock of wedge angle 10^o, an expansion of 10^o and then a final expansion of 10^o. Calculate C_p, M and $\frac{p_0}{p_{0\infty}}$ for each of the three surfaces.

Deal with the shock first.

$$\frac{p_1}{p_\infty} = 1.708 \tag{31}$$

Check: this is $>$1.

$$M_1 = 1.640 \tag{32}$$

Check: this is $<$2.

$$\frac{p_{0\infty}}{p_\infty} = 7.830 \tag{33}$$

Check: this is quite large due to the high Mach number.

$$\frac{p_{01}}{p_1} = 4.516 \tag{34}$$

Check: this lower because the Mach number is lower.

$$\frac{p_{01}}{p_{0\infty}} = \frac{p_{01}}{p_1}\frac{p_1}{p_\infty}\frac{p_\infty}{p_{0\infty}} = 4.516 \times 1.708 \times \frac{1}{7.830} = 0.985 \tag{35}$$

Check: this is $<$1 because a shock lowers the total pressure.

$$C_p = \frac{2}{1.4 \times 2^2}\left(1.708 - 1\right) = 0.253 \tag{36}$$

Check: this is $>$1 and of reasonable magnitude.

Now move to the 1st expansion.

Since expansions are isentropic, we could calculate the 2nd expansion first if needed; the flow variables only depend on the expansion angle and the initial input conditions. This is the opposite to when we have a series of shocks and all the intermediate shock pressure ratios need to be evaluated.

$$\nu\left(1.640\right) = 16.010 \tag{37}$$

$$\nu\left(M_2\right) = 16.010 + 10 = 26.010 \tag{38}$$

$$\nu^{-1}\left(21.010\right) = \mathcal{L}\left(1.98, 2.0, 25.77, 26.32, 26.010\right) = 1.989 \tag{39}$$

Check: this is $>$1.64. If we were using linear theory (Ackeret) then it would be exactly 2 because the flow has been turned back to be parallel to the freestream, but the total pressure loss through the oblique shock alters things, as total pressure loss is not included in linear models. As calculations move downstream deviations from linear results will increase as total pressure losses compound through any subsequent shocks.

$$\frac{p_{02}}{p_2} = \mathcal{L}\left(7.59, 7.83, 25.77, 26.32, 26.010\right) = 7.695 \tag{40}$$

Check: quite large again; the Mach number has gone up.

$$\frac{p_2}{p_\infty} = \frac{p_2}{p_{02}}\frac{p_{01}}{p_1}\frac{p_1}{p_\infty} = \frac{1}{7.695} \times 4.516 \times 1.708 = 1.002 \tag{41}$$

Check: close to 1 because the flow has turned parallel to the freestream again.

$$C_{p_2} = \frac{2}{1.4 \times 2^2}\left(1.002 - 1\right) = 0.001 \tag{42}$$

$$\frac{p_{02}}{p_{0\infty}} = 0.985 \tag{43}$$

Check: the expansion is isentropic, so no change in total pressure.

Now move to the 2^{nd} expansion.

$$\nu\left(M_3\right) = 16.01 + 20 = 36.01 \tag{44}$$

$$\nu^{-1}\left(36.01\right) = \mathcal{L}\left(2.36, 2.38, 35.67, 36.16, 36.01\right) = 2.374 \tag{45}$$

Check: the Mach number has gone up again through the expansion.

$$\frac{p_{03}}{p_3} = \mathcal{L}\left(13.732, 14.167, 35.67, 36.16, 36.01\right) = 14.034 \tag{46}$$

Check: a rather large number because the Mach number is high.

$$\frac{p_3}{p_\infty} = \frac{p_3}{p_{03}}\frac{p_{01}}{p_1}\frac{p_1}{p_\infty} = \frac{1}{14.034} \times 4.516 \times 1.708 = 0.550 \tag{47}$$

$$C_{p_3} = \frac{2}{1.4 \times 2^2}\left(0.550 - 1\right) = -0.161 \tag{48}$$

Check: the pressure has dropped through the expansion.

$$\frac{p_{03}}{p_{0\infty}} = 0.985 \tag{49}$$

Check: total pressure remains unchanged through an expansion.

7.2 Expansion-shock-shock

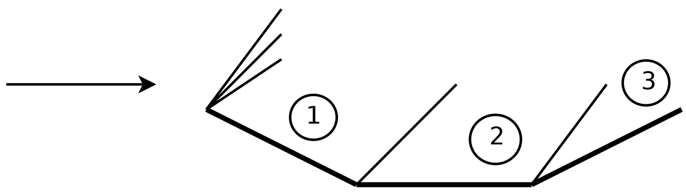

Figure 6: Expansion-shock-shock

Mach 2 flow passes over three surfaces that produce through an expansion over 10^o, a shock of wedge angle 10^o and then a final shock of wedge angle 10^o. Calculate C_p, M and $\frac{p_0}{p_{0\infty}}$ for each of the three surfaces.

$$\nu\left(M_1\right) = 26.320 + 10 = 36.32 \tag{50}$$

$$\nu^{-1}\left(36.320\right) = \mathcal{L}\left(2.38, 2.40, 36.16, 36.65, 36.32\right) = 2.387 \tag{51}$$

Check: this is $>$2 and bounded. This is an expansion so the pressure drops and the Mach number increases.

$$\frac{p_{01}}{p_1} = \mathcal{L}\left(14.167, 14.615, 36.16, 36.65, 36.32\right) = 14.313 \tag{52}$$

Check: this is quite large due to the high Mach number, and bounded.

$$\frac{p_1}{p_\infty} = \frac{p_1}{p_{01}}\frac{p_{0\infty}}{p_\infty} = \frac{1}{14.313} \times 7.830 = 0.547 \tag{53}$$

Check: this is $<$1. The pressure is dropping as this is an expansion.

$$C_{p_1} = \frac{2}{1.4 \times 2^2}(0.547 - 1) = -0.162 \tag{54}$$

Check: this is $<$0 and of reasonable magnitude. The pressure has indeed dropped as suspected.

$$\frac{p_{01}}{p_{0\infty}} = 1 \tag{55}$$

1^{st} shock now.

$$\frac{p_2}{p_1} = \mathcal{L}(1.798, 1.831, 2.3, 2.4, 2.387) = 1.827 \tag{56}$$

Check: this is $>$1 since this is a shock, and bounded.

$$M_2 = \mathcal{L}(1.91, 1.998, 2.3, 2.4, 2.387) = 1.987 \tag{57}$$

Check: this is $<$2.387 and bounded.

$$\frac{p_2}{p_\infty} = \frac{p_2}{p_1}\frac{p_1}{p_\infty} = 1.827 \times 0.547 = 0.999 \tag{58}$$

Check: this is close to 1. Linear theory would imply it should be exactly 1, but total pressure changes make the actual result slightly off this value.

$$C_{p_2} = \frac{2}{1.4 \times 2^2}(0.999 - 1) = 0.000 \tag{59}$$

Check: after rounding this is actually 0, which is sensible as the flow is parallel to freestream again.

$$\frac{p_{02}}{p_2} = \mathcal{L}(7.59, 7.83, 1.98, 2.0, 1.987) = 7.674 \tag{60}$$

$$\frac{p_{02}}{p_{0\infty}} = \frac{p_{02}}{p_{01}}\frac{p_{01}}{p_{0\infty}} = \frac{p_{02}}{p_2}\frac{p_2}{p_\infty}\frac{p_\infty}{p_{0\infty}} = 7.674 \times 0.999 \times \frac{1}{7.83} \times 1 = 0.979 \tag{61}$$

Check: less than 1 as the total pressure has dropped.

2^{nd} shock now.

$$\frac{p_3}{p_2} = \mathcal{L}(1.684, 1.708, 1.9, 2.0, 1.987) = 1.705 \tag{62}$$

Check: this is $>$1.

$$M_3 = \mathcal{L}(1.545, 1.64, 1.9, 2.0, 1.987) = 1.628 \tag{63}$$

Check: this is $<$1.987 and bounded.

$$\frac{p_3}{p_\infty} = \frac{p_3}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_\infty} = 1.705 \times 1.827 \times 0.547 = 1.704 \tag{64}$$

Check: this is $>$1.

$$C_{p_3} = \frac{2}{1.4 \times 2^2}(1.704 - 1) = 0.251 \tag{65}$$

Check: this is of reasonable magnitude.

$$\frac{p_{03}}{p_3} = \mathcal{L}(4.383, 4.516, 1.62, 1.64, 1.628) = 4.436 \tag{66}$$

$$\frac{p_{03}}{p_{0\infty}} = \frac{p_{03}}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_{0\infty}} = \frac{p_{03}}{p_3}\frac{p_3}{p_\infty}\frac{p_\infty}{p_{0\infty}} = 4.436 \times 1.704 \times \frac{1}{7.83} = 0.965 \tag{67}$$

Check: less than 1, and also less than 0.979, as the total pressure has dropped through the second shock.

8 Finding the Critical Mach Number

The critical Mach number is the freestream Mach number for which sonic flow first appears *anywhere* on an object. It is important because it is the point after which transonic effects *might* be seen, such as a large change in the pitch characteristics of a wing, aerofoil or aircraft. It is a conservative estimate for the highest safe Mach number of an aircraft designed to operate only subsonically and not transonically.

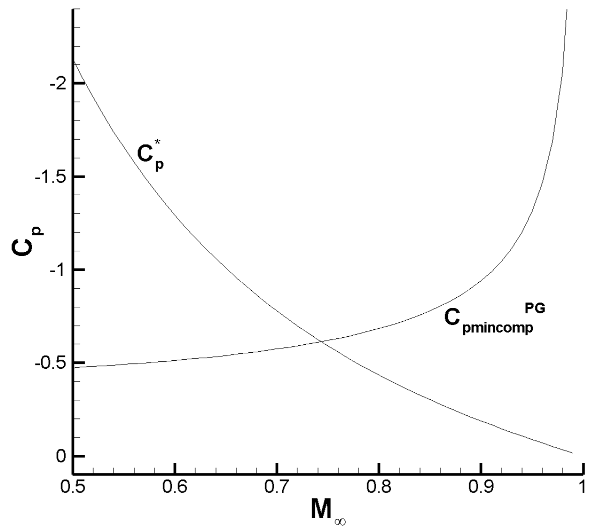

Figure 7: Variation of C_p^* and $C_{p_{min_{comp}}}^{PG}$ for $C_{p_{min_{incomp}}} = -0.41$

Nowadays transonic flight is the norm for commercial aircraft, but originally this regime caused many problems. The biggest changes that came about to allow this were (in addition to the jet engine!)

- Building plenty of pitch authority into aircraft to cope with the quite large changes in pitching moment. You can see this by observing that commercial aircraft have large, fully moveable tailplanes, and supersonic aircraft have similar (but thinner) all-moving tailplanes. Also, a good understanding of the movement of the aerodynamic centre/neutral point allowed correct estimates of longitudinal stability

- Understanding and avoiding shock-buffet and separation through good wing design, and allowing for sufficient wing stiffness to avoid control reversal (especially of the ailerons)

Historically the information available for finding M_{crit} would be a series of incompressible C_p measurements, which originally would have been experimentally derived. It is now more likely these would come from a panel code computation or a single incompressible CFD calculation, but the method remains the same. You would not usually perform a compressible CFD simulation to find M_{crit} because this cannot directly give such a result without wasteful adjustment of M_∞ and recomputation.

Fundamentally $M_{crit} = M_{crit}(C_{p_{min_{incomp}}}(\alpha))$ for a particular shape, but there is no simple explicit form for the function and it is most easily computed through some iterative strategy. The important point is that M_{crit} is the value of M_∞ which satisfies $C_p^* = C_{p_{min_{comp}}}$

If $C_{p_{min_{comp}}}$ is found from the Prandtl-Glauert scaling (others could be used, for example the Karman-Tsien scaling), then

$$C_p^*\left(M_{crit}\right) = C_{p_{min_{comp}}}^{PG}\left(M_{crit}\right) \tag{68}$$

So, we just need to describe how to find C_p^* and then solve this equation.

C_p^* is the pressure coefficient which occurs for sonic flow, and is given by

$$C_p^* = \frac{2}{\gamma M_\infty^2}\left(\left[\frac{p}{p_0}\right]_{M=1}\left[\frac{p_0}{p}\right]_{M=M_\infty} - 1\right) \tag{69}$$

From tables $\left[\frac{p}{p_0}\right]_{M=1} = \frac{1}{1.895}$. ***Remember this value***; it is always the same for each M_{crit} calculation you will do. $\left[\frac{p_0}{p}\right]_{M=M_\infty}$ may be found straight from tables given M_∞, maybe requiring a $\mathcal{L}$ interpolation. This value will change at every iteration.

Finally the PG scaling is just

$$C_{p_{min_{comp}}}^{PG} = \frac{C_{p_{min_{incomp}}}}{\sqrt{1 - M_\infty^2}} \tag{70}$$

That's a whole load of symbols to digest so let's recap.

We know a minium pressure coefficient for the incompressible case. This can be PG scaled to give the minimum pressure coefficient for the compressible case at any Mach number. At the same time, we know the pressure coefficient that sonic flow produces. Thus, we just need to find a value of M_∞ that makes these two values are the same, and that is the critical Mach number.

Before continuing, look at the graphs of these two functions in figure 7, plotted with negative upwards. C_p^* starts off very negative and becomes more positive as M_∞ goes up, while $C_{p_{min_{comp}}}^{PG}$ starts off more positive and gets more negative as M_∞ goes up. Somewhere in between the graphs will cross, at M_{crit}. There are a range of iterative schemes you could use, all of which you will be familiar with, principally: bisection, direct iteration and Newton-Raphson iteration. There is also the possibility of fitting a polynomial to each curve and solving for their intersection. Of all these methods, I'd recommend bisection as it is the simplest and allows easy spotting and correction of errors at any point in the calculation. An alternative uses linear interpolation over a small increment; choose the method you prefer, but remember you must satisfy yourself at the end of the procedure that the accuracy is adequate.

8.1 Bisection

Bisection is a straightforward process, shown in table 1 for $C_{p_{min_{incomp}}}$=-0.41. There are two values, M_{lower} and M_{upper}, which are initially chosen. These must bracket M_{crit}, but there are no other requirements. In each row of table 1 M_{middle} is found, which is the average of these two values, and C_p^* and $C_{p_{min_{comp}}}^{PG}$ are found for this average point.

M_{lower}	M_{upper}	M_{middle}	$C_{p_{min_{comp}}}^{PG}$	C_p^*
0.7000	0.8000	0.7500	-0.6199	-0.5905
0.7000	0.7500	0.7250	-0.5953	-0.6799
0.7250	0.7500	0.7375	-0.6071	-0.6342
0.7375	0.7500	0.7438	-0.6134	-0.6121
0.7375	0.7438	0.7406	-0.6102	-0.6231
0.7406	0.7438	0.7422	-0.6118	-0.6176
0.7422	0.7438	0.7430	-0.6126	-0.6148
0.7430	0.7438	0.7434	-0.6130	-0.6135
0.7434	0.7438	0.7436	-0.6132	-0.6128
0.7434	0.7436	0.7435	-0.6131	-0.6131
0.7435	0.7436	0.7435	-0.6131	-0.6129

Table 1: Bisection for $C_{p_{min_{incomp}}}$=-0.41

M	C_p^*	$C_{p_{min_{comp}}}^{PG}$
0.7	-0.7782	-0.5741
0.8	-0.4341	-0.6833

Table 2: Data needed for linear interpolation method

The next step is to either set $M_{lower} \leftarrow M_{middle}$ if $C_p^* < C_{p_{min_{comp}}}^{PG}$ or $M_{upper} \leftarrow M_{middle}$ if $C_p^* > C_{p_{min_{comp}}}^{PG}$. M_{middle} is then recomputed and the process repeats until convergence. Normally an accuracy of 2dp is achieved after about four iterations. Bisection is easy to use; you can start from quite a wide guess for the value of M_{crit} and the method will find the right answer. As you complete a row, check it looks correct before moving on. Any errors only propagate down the table so this will limit how much damage a mistake can do.

8.2 Linear Interpolation

First you need to fill out the data shown in table 2. Use the data in table 2 solve for the intersection of two straight lines, so that

$$\begin{aligned} & \frac{-0.4341 - -0.7782}{0.8 - 0.7}(M_{crit} - 0.7) + -0.7782 \\ = & \frac{-0.6833 - -0.5741}{0.8 - 0.7}(M_{crit} - 0.7) + -0.5741 \end{aligned} \tag{71}$$

giving M_{crit}=0.7450 and comparing reasonably to the bisection result.

There is slightly less work involved in this method compared to bisection, but the chances of making an error are higher, and if you do, it will take longer to correct it than for bisection as you will have to completely recalculate the intersection point. Also, you have be be sure the two values of M (here 0.7 and 0.8) are fairly close to M_{crit}, otherwise the result may be inaccurate.

9 Force Coefficients

The force per unit depth on an aerofoil from pressures normal to the surface (i.e. ignoring viscous shear stresses) is given by

$$\mathbf{F} = \int p\mathbf{n}ds \tag{72}$$

over the surface of the aerofoil, where $\mathbf{n}$ is the inward pointing unit normal (points towards the inside of the aerofoil, magnitude unity). p is not a variable we like to work with; C_p is preferable. So

$$p = C_p q_\infty + p_\infty \tag{73}$$

where q_∞ is dynamic pressure. So

$$\mathbf{F} = \int \left(C_p q_\infty + p_\infty\right)\mathbf{n}ds \tag{74}$$

A constant pressure integrated around the boundary generates no lift, so this is the same as

$$\mathbf{F} = \int C_p q_\infty \mathbf{n}ds \tag{75}$$

Finally this would be better if it were dimensionless, so

$$\mathbf{C}_F = \frac{\mathbf{F}}{q_\infty c} = \int C_p \mathbf{n} d\left(\frac{s}{c}\right) \tag{76}$$

where c is chord and $\mathbf{C}_F = (C_X, C_Y)$, where C_X, C_Y are the non-dimensional force coefficients in X, Y.

$$C_X = \int C_p \mathbf{n}\cdot\mathbf{i} d\left(\frac{s}{c}\right) \tag{77}$$

$$C_Y = \int C_p \mathbf{n}\cdot\mathbf{j} d\left(\frac{s}{c}\right) \tag{78}$$

If we have a straight section of surface defined by $(\Delta x, \Delta y)$ then the normal vector is $(\Delta y, -\Delta x)$. If you are unsure of this, consider that the dot product of the two vectors must be zero as they are perpendicular, so swapping the components and inserting a minus works, and the only decision remaining is whether to point the normal towards or away from the segment.

Now swap to using $\mathbf{n}'$, which is the same as $\mathbf{n}$, but has a magnitude equal to the chord-normalised distance between the start point and the end point of the straight segment. This means

$$C_X = \sum_{i=1}^{i=N} C_{p_i}\mathbf{n}'_{x_i} = \sum_{i=1}^{i=N} C_{p_i}\Delta y_i \tag{79}$$

$$C_Y = \sum_{i=1}^{i=N} C_{p_i}\mathbf{n}'_{y_i} = -\sum_{i=1}^{i=N} C_{p_i}\Delta x_i \tag{80}$$

where the summation is over all the straight surface segments of the aerofoil. These are simply summations of C_p multiplied by the projected areas of the straight segments in each axis direction. Don't worry too much about the minus sign; it is advisable to insert this on a 'common sense' basis in the your calculations when needed, rather than sticking strictly to a given formula. For example, a positive C_p on the upper surface always gives a negative contribution to C_L, and a positive C_p on a forward face must increase C_D.

Interestingly, because the shock-expansion theory used in this part of the course gives a constant pressure on any straight line segment, these summations are the exact integrals, with no error. This is not true for subsonic flow where the smooth variations in pressure will introduce a slight error in the integrals (it is then common to use a trapezoidal method instead).

However, this is not quite what we want still, as C_L and C_D are defined relative to the flow vector at ∞, so we need to rotate through α

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{81}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{82}$$

As an exercise you may want to show that

$$C_M = \sum_{i=1}^{i=N} C_{p_i} \Delta y_i \left(y_{mid_i} - y_{ref}\right) + \sum_{i=1}^{i=N} C_{p_i} \Delta x_i \left(x_{mid_i} - x_{ref}\right) \tag{83}$$

about (x_{ref}, y_{ref}) where nose-up is positive and all lengths are in fractions of the chord, with (x_{mid}, y_{mid}) the mid point of a straight segment. Commonly $(x_{ref}, y_{ref}) = (0.25, 0.0)$, but $(0,0)$ and $(0.5, 0)$ are also used. $(0.25, 0.0)$ is the incompressible aerodynamic centre, $(0.5, 0.0)$ is the compressible aerodynamic centre, and $(0,0)$ is just convenient as it normally refers to the leading edge.

10 Linearised Theory

In the lectures we shall derive Ackeret's linearised pressure coefficient

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} \tag{84}$$

The two most important points to appreciate, and which commonly lead to errors, are

- The angle θ is measured *relative to the freestream vector* and *in radians*. This is different to Prandtl-Meyer expansion calculations that use ν, which is measured *relative to the last flow vector* and is given in *degrees*
- The Mach number used is the freestream Mach number M_∞, *not* the local Mach number. This is consistent with the definition of C_p

10.1 Straight surfaces

Find C_p, C_L, C_M and C_D for a wedge aerofoil with total included angle at the leading edge of 10^o, with $\alpha = 0^o$

The C_p values are

$$C_p = \pm \frac{2 \times 5 \times \frac{\pi}{180}}{\sqrt{2^2 - 1}} = \pm 0.101 \tag{85}$$

Since the aerofoil is at $\alpha = 0^o$, all the C_p values have the same magnitude. On the front two faces C_p is positive, and on the rear two faces is it negative. This clearly means $C_L = 0$ and $C_M = 0$, however, $C_D \neq 0$. The working is a little shorter since $\alpha = 0$, so

$$C_X = C_D = \sum_{i=1}^{i=N} C_{p_i} \Delta y_i = 4 \times 0.101 \times 0.0437 = 0.018 \tag{86}$$

remembering that on the rear two faces, where C_p is negative, Δy is also negative. You can also reason this out by considering that a positive C_p on the front faces and a negative C_p on the rear faces must both act to make C_D positive.

10.2 Curved surfaces

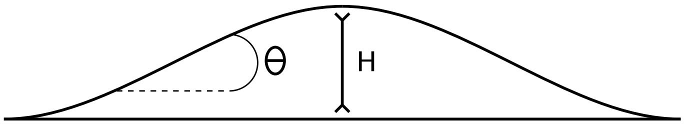

Figure 8: Wave-type aerofoil

Linear theory relates the inclination of the surface to C_p. This allows coefficients to be evaluated in some cases where the surface is defined by an analytical curve. For example, consider the aerofoil in figure 8 at zero incidence. This is defined in terms of the angle of inclination of the surface θ, which runs from 0 to a maximum θ_k and then back to zero, as a function of the fractional height $\frac{y}{H}$.

Since the aerofoil is symmetric in the chordwise direction, the integral can run to half chord then then be doubled. This gives

$$\theta = \theta_k \sin\left(\pi \frac{y}{H}\right) \tag{87}$$

$$\begin{aligned}
C_D &= 2 \int_0^{\frac{S_{final}}{2H}} \frac{2\theta_k \sin\left(\pi \frac{y}{H}\right)}{\sqrt{M_\infty^2 - 1}} \sin(\theta)\, d\left(\frac{s}{H}\right) \qquad (88)\\
&= \frac{2\theta_k}{\sqrt{M_\infty^2 - 1}} \int_0^{\frac{S_{final}}{2H}} 2 \sin\left(\pi \frac{y}{H}\right) \sin(\theta)\, d\left(\frac{s}{H}\right)\\
&= \frac{4\theta_k}{\sqrt{M_\infty^2 - 1}} \int_0^1 \sin\left(\pi \frac{y}{H}\right) d\left(\frac{y}{H}\right)\\
&= \frac{4\theta_k}{\sqrt{M_\infty^2 - 1}} \left[-\frac{1}{\pi} \cos\left(\pi \frac{y}{H}\right)\right]_0^1\\
&= \frac{8\theta_k}{\pi\sqrt{M_\infty^2 - 1}}
\end{aligned}$$

with a substitution of $d\left(\frac{s}{H}\right) \sin(\theta) = d\left(\frac{y}{H}\right)$, and with C_D based on H rather than c for convenience (a multiplication by $\frac{H}{c}$ would switch C_D to be based on c). So, for $\theta_k = 3^o$ we find C_D=0.077. Note that the $\sin(\theta)$ contribution has been taken as we are calculating drag.

A calculation like this still depends on the validity of the underlying linear model for C_p, so the flow must remain isentropic, and the results are more accurate if the angles of inclination are small. This also means that $\frac{dy}{dx} \approx \theta$. A final limitation is that the curve must give an integrable form for the force coefficients, although a numerical scheme could be used if needed.

You can do a similar calculation using cosine, as

$$\theta = \theta_k \cos\left(\pi \frac{y}{2H}\right) \tag{89}$$

$$\begin{aligned} C_D &= 2 \int_0^{\frac{S_{final}}{2H}} \frac{2\theta_k \cos\left(\pi \frac{y}{2H}\right)}{\sqrt{M_\infty^2 - 1}} \sin(\theta)\, d\left(\frac{s}{H}\right) \\ &= \frac{2\theta_k}{\sqrt{M_\infty^2 - 1}} \int_0^{\frac{S_{final}}{2H}} 2\cos\left(\pi \frac{y}{2H}\right) \sin(\theta)\, d\left(\frac{s}{H}\right) \\ &= \frac{4\theta_k}{\sqrt{M_\infty^2 - 1}} \int_0^1 \cos\left(\pi \frac{y}{2H}\right) d\left(\frac{y}{H}\right) \\ &= \frac{4\theta_k}{\sqrt{M_\infty^2 - 1}} \left[\frac{2}{\pi} \sin\left(\pi \frac{y}{2H}\right)\right]_0^1 \\ &= \frac{8\theta_k}{\pi\sqrt{M_\infty^2 - 1}} \end{aligned} \tag{90}$$

11 Complete nozzle example

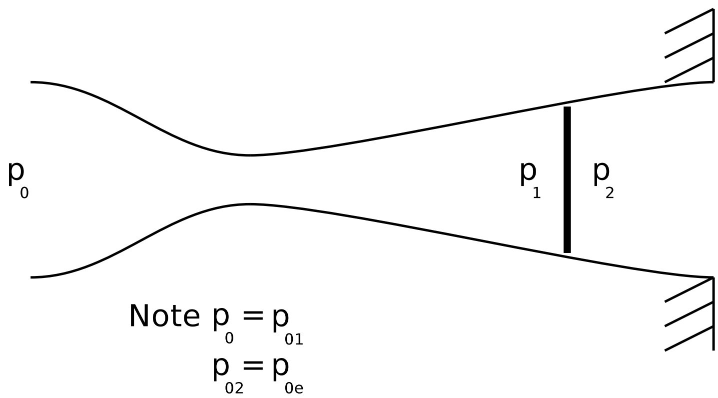

Figure 9: Nozzle geometry

Nozzles have a tendency towards the mysterious for newcomers to aerodynamics. The objective of this section is to set in to context everything you know about nozzles already, and how you would see it in practice in the real World. If you're confused, just remember it all really boils down to how big the stagnation pressure is going in to the nozzle (the 'push') and how quickly it is being sucked out of the exit (the 'pull'). The ratio push:pull determines what happens in the nozzle.

We shall consider a nozzle as figure 9 where

$$\frac{A}{A_t} = 2 - \sin^2(\pi x) \tag{91}$$

where x is from 0 to 1. There's nothing special about this function; I just chose it to make decent graphs. Actually, all that matters is that we know $\frac{A}{A_t}$. To be completely accurate, if you know p_0 going in to the nozzle and the exit static pressure, you can calculate everything that happens, including the shock position, as a function of $\frac{A}{A_t}$. However, this is a little too abstract for teaching purposes, although we will return to the idea later.

We will consider a case where $p_0 = 2b$ upstream of the nozzle, and where the exit pressure p_e is varied to arrive at different flow configurations. If any normal shock is present (it may or may not be, of course) the conditions before the shock are labelled 1 and those after are labelled 2. We shall now consider three cases, $p_e = 1.9b$, $p_e = 1.5b$ and $p_e = 1b$. Obviously, these are carefully chose for your educational delectation.

11.1 Case $p_e = 1.9b$

The exit pressure is not much lower than the input total pressure, so we might expect this to be quite a slow nozzle.

If we want to know a Mach number, we need a location where we know the static pressure. We will adopt the hypothesis that the nozzle is unchoked and subsonic at exit, which means the static pressure at the nozzle exit can be assumed to be equal to p_e, thus $\frac{p_0}{p_e} = \frac{2}{1.9} = 1.0526$. Checking tables, this means the exit Mach number must be below 0.28, and that $\frac{A}{A^*}$ *cannot* be less than 2.165. Hence, choking would only occur if the throat ratio rose to at least $\frac{A}{A_t} = 2.165$, ie. the throat dropped to below $A_t = 0.924$. We can calculate the exact choking ratio as

$$\frac{A}{A_t} = 2.317 + \frac{2.165 - 2.317}{1.056 - 1.048}(1.0526 - 1.048) = 2.2230 \tag{92}$$

so the actual choking throat area would be 0.897, which is below the actual throat area of 1, so the nozzle is unchoked, even if it is perhaps slightly closer to choking than you might have expected.

11.2 Case $p_e = 1.5b$

This is the tricky case. Let's try the unchoked assumption again and see what happens. The area for choking would be for $\frac{p_0}{p_e} = \frac{2}{1.5} = 1.3333$

$$\frac{A}{A_t} = 1.145 + \frac{1.126 - 1.145}{1.34 - 1.318}(1.3333 - 1.318) = 1.1318 \tag{93}$$

ie. if A_t is below $\frac{2}{1.1318} = 1.77$ then the nozzle is choked. Since $A_t = 1$, the nozzle is indeed choked.

Knowing the choked condition, there are then two possibilities: (i) a normal shock exists in the nozzle or (ii) the nozzle is fully supersonic, with shocks or expansions at the exit, or possibly nothing at the exit if it is ideally expanded. Determining (i) or (ii) is easily accomplished by calculating a special case where there is a normal shock *exactly* at the nozzle exit, and then comparing the calculated exit static pressure to the real one. The first step is to the the total pressure ratio for supersonic flow at the exit (after which we will calculate the normal shock change!)

$$\frac{p_0}{p} = 10.366 + (2 - 1.966)\frac{10.695 - 10.366}{2.001 - 1.966} = 10.6856 \tag{94}$$

Incidentally, this means the ideally expanded case would require $p_e = \frac{2}{10.6856} = 0.1872b$.

Now consider if there was a shock exactly at the nozzle exit from this flow condition. Then we can find the downstream total pressure as a fraction of upstream static from the normal shock tables. First we need the Mach number going in to the shock at the exit, which is

$$M = 2.18 + 0.02\frac{2 - 1.966}{2.001 - 1.966} = 2.1994 \tag{95}$$

From this the normal shock tables give the downstream total to upstream static pressure ratio

$$\frac{p_{02}}{p_1} = 6.614 + (2.1994 - 2.18)\frac{6.726 - 6.614}{0.02} = 6.7226 \tag{96}$$

exit Mach number would then be

$$M = 0.55 + (2.199 - 2.18)\frac{0.548 - 0.55}{0.02} = 0.5481 \tag{97}$$

$$\frac{p_{0e}}{p_e} = 1.22 + (0.5481 - 0.54)\frac{1.238 - 1.22}{0.02} = 1.2273 \tag{98}$$

$$\frac{p_e}{p_{01}} = \frac{p_e}{p_{0e}}\frac{p_{0e}}{p_1}\frac{p_1}{p_{01}} = \frac{1}{1.2273}6.7204\frac{1}{10.6856} = 0.5124 \tag{99}$$

giving $p_e = 2 \times 0.5124 = 1.0248$. Evidently, 1.5b is above this, so the shock must be further up the nozzle, if it exists.

We can double check using the case where the nozzle chokes instantaneously at the throat, and then goes subsonic again, given by

$$\frac{p_0}{p} = 1.065 + (2 - 2.035)\frac{1.074 - 1.065}{1.921 - 2.035} = 1.0678 \tag{100}$$

so that $p = \frac{2}{1.0678} = 1.8731$. Clearly, 1.5b is below this, so there is definitely a shock *within* the diverging part of the nozzle. This last calculation is an alternative to checking the areas, which we did at the start.

But where is this shock? If you know it is there, how do you find it?

Sorry, but you can't work that out very easily. The correct physical argument is that the shock is located such that it produces a pressure increase that is just sufficient to ensure that when the air leaves the nozzle at the exit plane, its pressure is equal to p_e, ie.

$$p_e = p_{01}\frac{p_e}{p_{0e}}\frac{p_{0e}}{p_1}\frac{p_1}{p_{01}} \tag{101}$$

All three of the fractions on the right hand side are functions of the shock location: $\frac{p_e}{p_{0e}}$ because expansion after the shock depends on the Mach number after the shock, $\frac{p_{0e}}{p_1}$ because the pressure ratio through the shock depends on the upstream Mach number (which depends on shock location) and $\frac{p_1}{p_{01}}$ because the Mach number going in to the shock depends on where it is located (of course, up to the shock, normal nozzle isentropic flow applies).

Getting the right value in each of these three ratios to satisfy a given p_e requires iterating on the shock location. Equivalently, you could also iterate on the shock location value of $\frac{A}{A^*}$, which brings us back to the point made at the start of the section: fundamentally, everything about the nozzle is determined by the upstream total pressure and downstream static pressure.

The Matlab tool uses bisection to solve equation (101) correctly; to do this algebraically would require solving a substantial (and most likely analytically insoluble) polynomial.

11.3 Case $p_e = 1b$

Well, this is going to be straightforward. $1b$ is below $1.0248b$ showing that the nozzle must be fully supersonic (just!). It is also clearly overexpanded, because $1b$ is greater than $0.1872b$, which means there will be a shock system (perhaps complicated) at the nozzle exit.

If the pressure dropped to below $0.1872b$ it would become underexpanded (common for rocket exhausts), and there would be an expansion system at the exit plane.

Now have a play with the slider on the Matlab tool!

12 Complete aerofoil example

You may be wondering what the best way to solve a compressible aerofoils is.

I am going to give you a recipe. I suggest you stick to it but you are free to do as you please.

The acronym is P-S-C-I, which may be conveniently remembered as 'pedantic students cash in'. I hope you are and that you will.

- P is for PICTURE. Begin by drawing a picture of your aerofoil. Label all the regions. Draw shocks or expansions from every corner where relevant and label the turning angle of the flow at each of these points. This is so that you don't get confused later on, when you are busily interpolating. Someone once drew a beautiful unicorn on their exam script, but I recommend sticking to aerofoils only.
- S is for STRATEGISE. Write down your plan to calculate each of the pressure ratios as a product of other pressure ratios. Once done, this is a logical juncture to choose which ones to do first. You will always find the ones at the front of the aerofoil are quicker as there are fewer changes from p_{inf} to take account of. I suggest you do the easier pressure ratios first because this improves your chances of getting more marks in a shorter period of time. Once your strategy is established, find the inevitable mistake regarding total pressures, correct it, crack your knuckles, mutter 'I can definitely get this 100% correct to three decimal places because all mistakes are avoidable through procedure and failing that detectable using common sense' etc. etc. and continue.
- C is for COMPUTE. Working according to your strategy, find all necessary pressure ratios. From pressure ratios, find pressure coefficients. Sanity check your pressure coefficients and correct the obligatory mistake. Make certain your linear interpolations are within their bounds.
- I is for INTEGRATE. Summate pressure coefficients times projected edge lengths to find force coefficients in the axis directions, then rotate if necessary to find lift and drag.

Right, here we go. The (somewhat terse) exam question reads: an infinitely thin section, shown in figure 10, is flying at Mach 2 at an angle of attack of 3°. Work out everything.

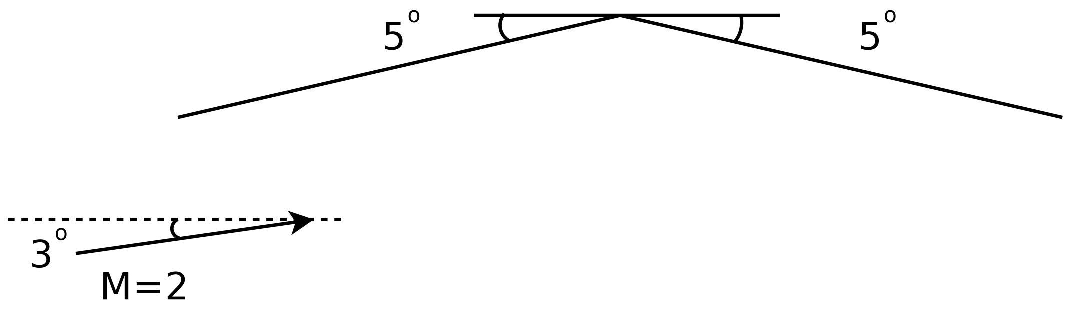

Figure 10: Aerofoil geometry

12.1 Picture

Drawn in all shocks and expansions and label with the correct deflection angles. Note that we don't know exactly what happens at the trailing edge in terms of angle, but linear theory suggests a shock on top and expansion underneath.

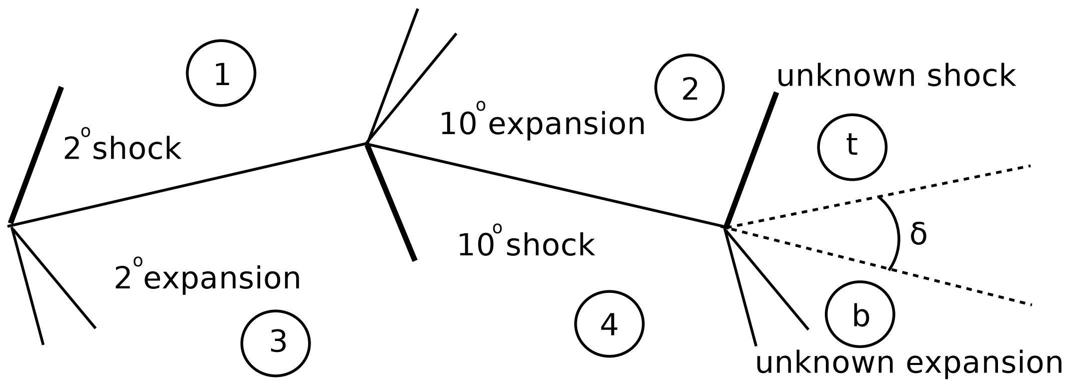

Figure 11: Shock-expansion sketch

12.2 Strategise

Surface 1 requires only $\frac{p_1}{p_\infty}$.

It is always easy to get C_p after an oblique shock in to ambient air. All you need is the pressure ratio across the shock, which is in the tables. It is hardly a calculation at all!

Surface 2 requires $\frac{p_2}{p_\infty} = \frac{p_2}{p_{02}} \frac{p_{01}}{p_1} \frac{p_1}{p_\infty}$

Once you involve expansion fans, you need to start using the total pressure ratios in the regions, so it is a bit harder.

Surface 3 requires only $\frac{p_3}{p_\infty} = \frac{p_3}{p_{03}} \frac{p_{0\infty}}{p_\infty}$

Again, total pressure ratios are needed here.

Surface 4 requires $\frac{p_4}{p_\infty} = \frac{p_4}{p_3} \frac{p_3}{p_{03}} \frac{p_{0\infty}}{p_\infty}$

This surface comes after an expansion and a shock.

It seems sensible to attack surfaces 1 and 3, then 2 and 4. Surfaces 1 and 3 are the easier ones that require fewer ratios.

12.3 Compute

12.3.1 Surface 1

Straight in to a simple oblique shock on surface 1, with a 2^{o} deflection.

$$C_{p_1} = \frac{2}{\gamma M_\infty^2} (1.118 - 1) = 0.0421 \tag{102}$$

12.3.2 Surface 3

Moving to surface 3, which is an expansion of 2^{o}, and working in pressure ratio

$$\nu(2) = 26.32 \tag{103}$$

$$\nu^{-1}(28.32) = \mathcal{L}\,(8.596, 8.868, 27.95, 28.49, 28.32) = 8.782 \tag{104}$$

and we already know $\frac{p_{0\infty}}{p_\infty} = 7.830$ so

$$C_{p_3} = \frac{2}{\gamma M_\infty^2} \left(\frac{7.830}{8.782} - 1 \right) = -0.0386 \tag{105}$$

We now need the Mach number on surface 3, which is

$$\nu^{-1}(26.320 + 2) = \mathcal{L}\,(2.06, 2.08, 27.95, 28.49, 28.32) = 2.0737 \tag{106}$$

Note that you won't always need to find this Mach number, but we need it here for the slip line calculation. We're now halfway with just a shock and an expansion left.

12.3.3 Surface 2

For surface 2, we need ν for the flow from surface 1 (where M=1.928 downstream of the shock). We also need the total pressure ratio for surface 1

$$\mathcal{L}\,(6.916, 7.134, 1.92, 1.94, 1.928) = 7.0032 \tag{107}$$

now ν for surface 1

$$\nu(1.928) = \mathcal{L}\,(24.090, 24.650, 1.92, 1.94, 1.928) = 24.314 \tag{108}$$

What we actually want is the total pressure ratio, not the Mach number, so (remember angle is 2$\times$5, so 10^{o} is added to 24.314)

$$\nu^{-1}(34.314) = \mathcal{L}\,(12.504, 12.901, 36.690, 34.314) = 12.6025 \tag{109}$$

Finally

$$C_{p_2} = \frac{2}{\gamma M_\infty^2}\left(1.118\frac{7.0032}{12.6025} - 1\right) = -0.1350 \tag{110}$$

we can work out the Mach number now, but this is only necessary for the slip line calculation, and requires another interpolation, giving

$$\nu^{-1}(34.314) = \mathcal{L}\left(2.30, 2.32, 34.190, 36.690, 34.314\right) = 2.3049 \tag{111}$$

12.3.4 Surface 4

For surface 4 there is a shock through 10°. Interpolating in terms of Mach number for the shock for pressure ratio

$$\mathcal{L}\left(1.708, 1.736, 2.0, 2.1, 2.0737\right) = 1.7286 \tag{112}$$

and for downstream Mach

$$\mathcal{L}\left(1.64, 1.731, 2.0, 2.1, 2.0737\right) = 1.7071 \tag{113}$$

We will need the total pressure ratio here later for the slip line, so

$$\mathcal{L}\left(4.941, 5.093, 1.70, 1.72, 1.7071\right) = 4.9950 \tag{114}$$

Finally

$$C_{p_2} = \frac{2}{\gamma M_\infty^2}\left(1.7286\frac{7.830}{8.782} - 1\right) = 0.1929 \tag{115}$$

Figure 12 shows the CFD solution for this case, with $C_L = 0.1225018$, $C_D = 2.4307463E - 02$ and $C_M = -0.1124979$, very close to the exact shock-expansion results of $C_L = 0.1225304$, $C_D = 2.4322711E - 02$ and $C_M = -0.1125095$ (moments +ve nose up about about leading edge).

12.4 Integrate

Using equations (79) and (80) gives

$$C_X = C_{p_1}0.5\tan(5) + C_{p_3}0.5\tan(5) + C_{p_2}0.5\tan(5) + C_{p_4}0.5\tan(5) \tag{116}$$

but at this point it is sensible to avoid being a robot and look at what is going on. It is clear that every surface must have a positive contribution to C_X (forward faces have positive C_p, rearwards have negative C_p), so

$$C_X = 0.5\tan(5)(0.0421 + 0.0386 + 0.1350 + 0.1929) = 0.01787 \tag{117}$$

Now in the vertical direction - noting that positive C_p on surface 1 lowers the lift (as does negative C_p on surface 3)

$$C_Y = 0.5(-0.0421 - 0.0386 + 0.1350 + 0.1929) = 0.1236 \tag{118}$$

Now we need to rotate through the angle of attack. The signage here is mostly obvious (C_Y has a positive influence on lift and drag, while C_D has a negative influence on lift and positive on drag), giving

$$C_L = C_Y\cos(3) - C_X\sin(3) = 0.1225 \tag{119}$$

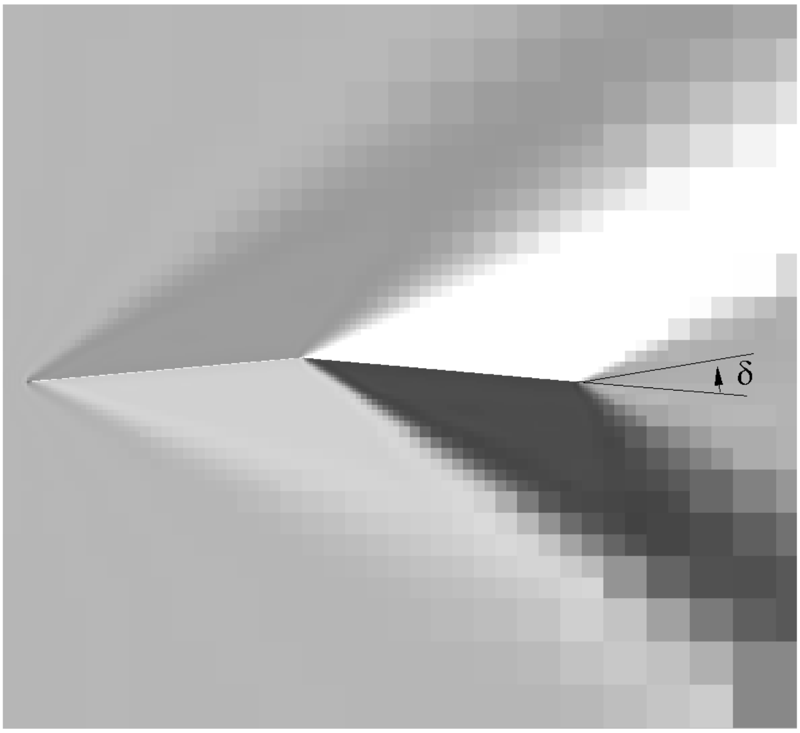

Figure 12: CFD solution (greyscale shows M)

$$C_D = C_Y \sin(3) + C_X \cos(3) = 0.0243 \tag{120}$$

For moments, use the moments of the vertical component of C_p, then the horizontal component (which is tiny)

$$C_M = 0.5(0.0421 \times 0.25 + 0.0386 \times 0.25 - 0.1350 \times 0.75 - 0.1929 \times 0.75) + \tag{121}$$

$$0.5^3 \tan^2(5)(0.0421 + 0.0386 + 0.1350 - 0.1929) = -0.1129 \tag{122}$$

Note that it is 0.5^3 because the projected vertical surface length is $0.5\tan(5)$ and the moment position for this surface is $\frac{0.5\tan(5)}{2}$ (the last 0.5 is the $\frac{1}{2}$!)

12.5 Slip line

Before starting it is worth saying that slip lines are tricky. It is the hardest type of calculation you might encounter in shock expansion theory. I'm including this section to show how it can be done. There are many other parts of the compressible course that are more important, so if you find it hard (it is) then focus on the other bits and return to it later.

Another reason for not getting too worried over slip lines is that there are only a few situations where I can imagine they would matter. The first is if you are unusually interested in trailing edges. The second is if you are designing an intake with two wedges (in this case a slip line exists after the intersection of two unequal oblique shocks). For this second case, however, there is a good reason not to design an intake in this way; slip lines represent shear, and engine designers hate non-uniform inflows, as they produce oscillatory loads on fan/compressor blades. The third, and most likely, situation is that you are looking at a CFD solution and wondering what that discontinuity is called, or why it is there (in this case of course, the calculation is already done - hence my focus on understanding rather than calculation).

Anyhow, it is a tour de force of compressible flow, so let's see one in action.

We need the top and bottom pressures across the slip line to be equal. This is the overarching idea, and in many ways the only one that really matters, ie.

$$\frac{p_t}{p_\infty} = \frac{p_b}{p_\infty} \tag{123}$$

This may be expanded as

$$\frac{p_t}{p_2}\frac{p_2}{p_{02}}\frac{p_{01}}{p_1}\frac{p_1}{p_\infty} = \frac{p_b}{p_{0b}}\frac{p_{04}}{p_4}\frac{p_4}{p_3}\frac{p_3}{p_{03}}\frac{p_{0\infty}}{p_\infty} \tag{124}$$

$$\frac{p_t}{p_2}\frac{1}{12.6025} \times 7.0032 \times 1.118 = \frac{p_b}{p_{0b}} 4.9950 \times 1.7286 \times \frac{1}{8.782} \times 7.83 \tag{125}$$

For a shock from surface 2 to the top of the slip line (we can guess the slip line angle must be near 8^o from linear theory). We need to interpolate the shock pressure ratios to find the limits for a subsequent interpolation in deflection angle, so

Lower limit (interpolating in M)

$$\mathcal{L}(1.435, 1.452, 2.3, 2.4, 2.3049) = 1.4354 \tag{126}$$

Upper limit (interpolating in M)

$$\mathcal{L}(1.608, 1.633, 2.3, 2.4, 2.3049) = 1.6092 \tag{127}$$

The linearised pressure ratio must therefore be (assuming the angle lies in the 6 to 8 range) (this is effectively the interpolation you usually do in terms of angle, but written algebraically because we don't yet know what the angle is)

$$\frac{p_t}{p_2} = \frac{1.6092 - 1.4354}{2}(\delta - 6) + 1.4354 \tag{128}$$

Remember we use $(\delta - 6)$ because the interval starts at 6º (ie. if the angle was 6, we would want to recover 1.4354 exactly)

$$\nu^{-1}(34.314) = \mathcal{L}\,(2.30, 2.32, 34.190, 36.690, 34.314) = 2.3049 \tag{129}$$

Mach number on surface 4 is

$$\mathcal{L}\,(1.64, 1.731, 2.0, 2.1, 2.0737) = 1.7071 \tag{130}$$

$$\nu(1.7071) = \mathcal{L}\,(17.77, 18.360, 1.7, 1.72, 1.7071) = 17.8067 \tag{131}$$

so (we know angle must be around the 8º value, which means ν must be about 25.8067, which puts us in the interval from Mach 1.98 to 2.0, with corresponding ν values of 25.77 and 26.32)

$$\frac{p_{0b}}{p_b} = \frac{7.83 - 7.59}{26.32 - 25.77}(\delta + 17.8067 - 25.77) + 7.59 \tag{132}$$

From this we can return to equation (125) and insert our two linearisations from equations (128) and (132)

$$\frac{1.6092 - 1.4354}{2}(\delta - 6) + 1.4354 = \frac{12.3913}{\frac{7.83-7.59}{26.32-25.77}(\delta - 7.9633) + 7.59} \tag{133}$$

$$0.0869(\delta - 6) + 1.4354 = \frac{12.3913}{0.4364(\delta - 7.9633) + 7.59} \tag{134}$$

$$0.0869\delta + 0.914 = \frac{12.3913}{0.4364\delta + 4.1148} \tag{135}$$

$$(0.0869\delta + 0.9152)(0.4364\delta + 4.1148) = 12.3913 \tag{136}$$

$$0.0379\delta^2 + 0.7570\delta - 8.6254 = 0 \tag{137}$$

The relevant root of which is 8.11º. This is the angle change from the last surface, ie. it represents up relative to the horizontal by 3.11º and up relative to freestream by 0.11º.

An exact calculation which solves the problem iteratively gives 7.93º, while linear theory suggests 8º. The exact Mach numbers across the slip line are 1.996 (top) and 1.985 (bottom). It is a trifle upsetting that 8.11º is just outside our assumed interval from 6 to 8. You could recalculate using the next interval in tables, or you could call it a day, remembering that the tables are not really accurate to tenths of a degree, and my rounding along the way might also have had an influence.

12.6 Exact result

The following gives the exact shock-expansion results for the case above (note - you should follow the method above - this is for information and comparison only). I've included it here so you can compare the hand calculation above with the automated output below. All the examples in section 15 were solved automatically.

More decimal places than sensible are displayed in order to aid any comparisons - I suggest you usually stick to 4dp and quote results to 3dp.

Moments are taken about 0,0.
Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.000000000000000E+000 0.000000000000000E+000
End point: 0.500000000000000 4.374430000000000E-002
Mid point: 0.250000000000000 2.187215000000000E-002
Found a shock
The wedge angle is (deg): 1.99999638788087

$$\frac{p_0}{p\ _1} = 7.82950823755808 \tag{138}$$

$$\frac{p_0}{p\ _2} = 7.00059089415193 \tag{139}$$

$$\frac{p_2}{p_1} = 1.11824798923050 \tag{140}$$

$$\frac{p_{02}}{p_{01}} = 0.999858031090384 \tag{141}$$

$$\frac{p}{p_\infty} = 0.142845084810679 \times 0.999858031090384 \times 7.82950823755808 \tag{142}$$

$$M_2 = 1.92781823529980 \tag{143}$$

$$C_{p_2} = 4.214112232020803E - 002 \tag{144}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 4.214112232020803E - 002 \times 4.374430000000000E - 002 \tag{145}$$

$$\Delta C_Y = 4.214112232020803E - 002 \times -0.500000000000000 \tag{146}$$

$$\Delta C_M = 4.214112232020803E - 002 \times 0.250000000000000 \times 0.500000000000000 \tag{147}$$
$$+4.214112232020803E - 002 \times 2.187215000000000E - 002 \times 4.374430000000000E - 002$$

Surface number 2
Start point: 0.500000000000000 4.374430000000000E-002

End point: 1.00000000000000 0.000000000000000E+000
Mid point: 0.750000000000000 2.187215000000000E-002
Found an expansion
The expansion angle is (deg): 9.99999277576163

$$\nu\left(M_1\right) = 24.3140044366989 \tag{148}$$

$$\nu\left(M_2\right) = 24.3140044366989 + 9.99999277576163 \tag{149}$$

$$\nu\left(M_2\right) = 34.3139972124605 \tag{150}$$

$$\frac{p_0}{p}_1 = 7.00059089415193 \tag{151}$$

$$\frac{p_0}{p}_2 = 12.6001326869578 \tag{152}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{153}$$
$$7.936424360317144E-002 \times 0.999858031090384 \times 7.82950823755808$$

$$M_2 = 2.30491104425588 \tag{154}$$

$$C_{p_2} = -0.134962657878914 \tag{155}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.134962657878914 \times -4.374430000000000E-002 \tag{156}$$

$$\Delta C_Y = -0.134962657878914 \times -0.500000000000000 \tag{157}$$

$$\Delta C_M = -0.134962657878914 \times 0.750000000000000 \times 0.500000000000000 \tag{158}$$
$$+ -0.134962657878914 \times 2.187215000000000E-002 \times -4.374430000000000E-002$$

Working on the lower surface

Surface number 1
Start point: 0.000000000000000E+000 0.000000000000000E+000
End point: 0.500000000000000 4.374430000000000E-002
Mid point: 0.250000000000000 2.187215000000000E-002
Found an expansion
The expansion angle is (deg): 1.99999638788087

$$\nu\left(M_1\right) = 26.3167317934844 \tag{159}$$

$$\nu\left(M_2\right) = 26.3167317934844 + 1.99999638788087 \tag{160}$$

$$\nu\left(M_2\right) = 28.3167281813653 \tag{161}$$

$$\frac{p_0}{p}_1 = 7.82950823755808 \tag{162}$$

$$\frac{p_0}{p}_2 = 8.77931036604423 \tag{163}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{164}$$
$$0.113904163118290 \times 1.00000000000000 \times 7.82950823755808$$

$$M_2 = 2.07356565236176 \tag{165}$$

$$C_{p_2} = -3.855538723206896E-002 \tag{166}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -3.855538723206896E-002 \times -4.374430000000000E-002 \tag{167}$$

$$\Delta C_Y = -3.855538723206896E-002 \times 0.500000000000000 \tag{168}$$

$$\Delta C_M = -3.855538723206896E-002 \times 0.250000000000000 \times -0.500000000000000 \tag{169}$$
$$+-3.855538723206896E-002 \times 2.187215000000000E-002 \times -4.374430000000000E-002$$

Surface number 2
Start point: 0.500000000000000 4.374430000000000E-002
End point: 1.00000000000000 0.000000000000000E+000
Mid point: 0.750000000000000 2.187215000000000E-002
Found a shock
The wedge angle is (deg): 9.99999277576163

$$\frac{p_0}{p_{\ 1}} = 8.77931036604423 \tag{170}$$

$$\frac{p_0}{p_{\ 2}} = 4.99553273996902 \tag{171}$$

$$\frac{p_2}{p_1} = 1.72857427339839 \tag{172}$$

$$\frac{p_{02}}{p_{01}} = 0.983579463100885 \tag{173}$$

$$\frac{p}{p_\infty} = 0.200178850195305 \times 0.983579463100885 \times 7.82950823755808 \tag{174}$$

$$M_2 = 1.70724638657217 \tag{175}$$

$$C_{p_2} = 0.193002857084398 \tag{176}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 0.193002857084398 \times 4.374430000000000E-002 \tag{177}$$

$$\Delta C_Y = 0.193002857084398 \times 0.500000000000000 \tag{178}$$

$$\Delta C_M = 0.193002857084398 \times 0.750000000000000 \times -0.500000000000000 \tag{179}$$
$$+0.193002857084398 \times 2.187215000000000E-002 \times 4.374430000000000E-002$$

$$C_L = C_Y\cos(\alpha) - C_X\sin(\alpha) \tag{180}$$

$$C_L = 0.122529475171213 \tag{181}$$

$$C_D = C_Y\sin(\alpha) + C_X\cos(\alpha) \tag{182}$$

$$C_D = 2.432266481685639E-002 \tag{183}$$

$$C_M = -0.112509003992511 \tag{184}$$

13 Linear Interpolation - Anatomy of $\mathcal{L}$

An entire section on linear interpolation? You *cannot* be serious?

If you're entirely happy that you can linearly interpolate at a rapid pace then don't read this section.

I've included it because:

1. In the past I have seen students attempt to fit a line between two points by solving the simultaneous equations from scratch. Fun though this is, it is not the fastest or least error prone method

2. A significant % of students are in denial that a straight line must pass through both its start point and its end point. What I mean here is that if you have, say, $\frac{p_0}{p}_a$ and $\frac{p_0}{p}_b$ from your tables, together with M_a and M_b, and you want to know $\frac{p_0}{p}_{\mathcal{L}}$ for $M_{\mathcal{L}}$, then *we must have* $\frac{p_0}{p}_a < \frac{p_0}{p}_{\mathcal{L}} < \frac{p_0}{p}_b$ or $\frac{p_0}{p}_b < \frac{p_0}{p}_{\mathcal{L}} < \frac{p_0}{p}_a$.

   In other words, *your linearly interpolated result must be bracketed by the two data points you used to calculate it.*

   Error check: ensure the number you get from your interpolation is bounded by the points you are using from the tables. Anything else is simply madness.

   Let's be honest, we have all made this mistake, and whilst tapping away at your calculator you will probably make it again in the future. Just make sure that when that happens *you notice*.

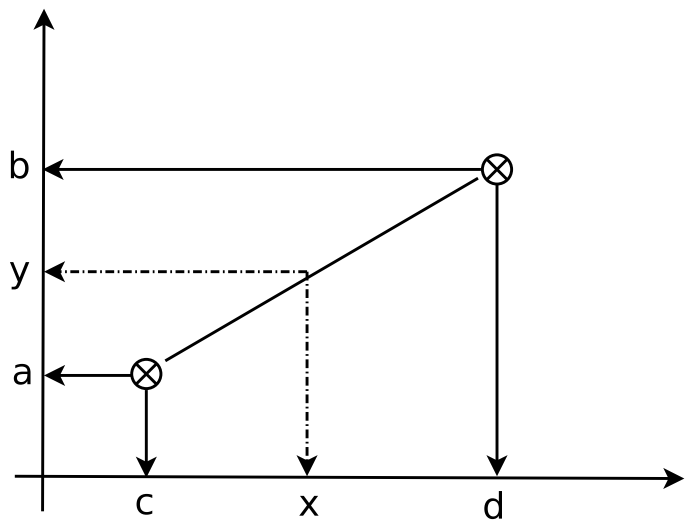

Figure 13: A straight line

So let's define our linear interpolation procedure as

$$y = \mathcal{L}\left(a, b, c, d, x\right) = \frac{b-a}{d-c} \times (x-c) + a \tag{185}$$

You'll notice that when $x = c$ we have $y = a$, and when $x = d$ we see $y = b$.

Finally, this is *interpolation*. Sometimes you will need to perform *extrapolation*. For this x is no longer between c and d, and similarly y will no longer be between a and d, but the method is identical.

So, where might you *ever* need to extrapolate information? The idea is repugnant, afterall. Extrapolation is needed when 'there is nothing on the other side', and this happens sometimes if you are trying to estimate θ_{max} in the oblique shock tables for a Mach number that is not in the tables, and which is below the smallest Mach number listed. If there *are* Mach numbers above and below the one you need then standard interpolation applies, but if there is no θ_{max} for a lower Mach number then you *must* extrapolate θ_{max} using the two Mach numbers above your point of interest. It's shaky, but there is no other option.

Physically this situation *will eventually arise* if there are oblique shocks reflecting down a channel. Each time the oblique shock reflects the Mach number drops, but the required deflection remains the same (if the channel has straight walls). This state of affairs can only ever end with a normal shock, and just before this happens you will need to extrapolate θ_{max} for the penultimate reflection.

14 Linearising the Full Potential Equation

Note: this section is for reference only; it is not examinable, but it does make for some good vector calculus practice. You can find equivalent working in 'Fundamentals of Aerodynamics', but I've rewritten it here to be as compact and as quick as possible to follow.

Conservation of mass gives

$$\nabla \cdot \rho \mathbf{u} = 0 \tag{186}$$

By the definition of velocity potential we have

$$\mathbf{u} = \nabla \phi \tag{187}$$

Expanding equation (186) gives

$$\mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \tag{188}$$

$$\mathbf{u} \cdot \frac{\nabla \rho}{\rho} + \nabla \cdot \mathbf{u} = 0 \tag{189}$$

$$\nabla \phi \cdot \frac{\nabla \rho}{\rho} + \nabla \cdot \nabla \phi = 0 \tag{190}$$

This is great but we need to remove the ρ terms. Return to the 1D Euler equation

$$dp = -\rho V dV \tag{191}$$

Using $dp = a^2 d\rho$ gives

$$a^2 d\rho = -\rho V dV = -\frac{1}{2}\rho d(V^2) \tag{192}$$

$$a^2 \frac{d\rho}{\rho} = -V dV = -\frac{1}{2} d(V^2) \tag{193}$$

$$d\rho = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \nabla \rho \cdot d\mathbf{s} \tag{194}$$

Also

$$d(|V|^2) = \frac{\partial(|V|^2)}{\partial x}dx + \frac{\partial(|V|^2)}{\partial y}dy = \nabla(|V|^2)\cdot d\mathbf{s} \tag{195}$$

So it must also be true that

$$\frac{\nabla\rho}{\rho} = -\frac{1}{2a^2}\nabla(|V|^2) \tag{196}$$

This can now be substituted in to eliminate ρ and arrive at

$$-\nabla\phi\cdot\frac{1}{2a^2}\nabla\cdot(|V|^2) + \nabla\cdot\nabla\phi = 0 \tag{197}$$

Which is great, but we want ϕ not $\mathbf{V}$. However since

$$\nabla(\mathbf{V}\cdot\mathbf{V}) = 2\mathbf{V}\times(\nabla\times\mathbf{V}) + 2(\nabla\mathbf{V})\mathbf{V} \tag{198}$$

and remembering that the flow is irrotational, so $\nabla\times\mathbf{V} = \mathbf{0}$ then

$$\nabla(|V|^2) = \nabla(\mathbf{V}\cdot\mathbf{V}) = 2(\nabla\mathbf{V})\mathbf{V} = 2(\nabla\nabla\phi)\nabla\phi \tag{199}$$

Finally we get

$$\nabla\cdot\nabla\phi - \frac{1}{2a^2}\nabla\phi\cdot(2(\nabla\nabla\phi)\nabla\phi) = 0 \tag{200}$$

$$\nabla\cdot\nabla\phi - \frac{1}{a^2}\nabla\phi\cdot((\nabla\nabla\phi)\nabla\phi) = 0 \tag{201}$$

This is quite a tidy compact way to write this, but remember that $\nabla\nabla\phi$ is a matrix (the gradient operator applied to a scalar gives a vector, and applied to a vector it gives a matrix), and that we need to find the matrix-vector multiplication for the second term on the left hand side before finding the dot product in that term. This does eventually give a scalar, consistent with the rest of the equation.

If we want to expand to be written in terms of partial derivatives we just need to observe that

$$\nabla\nabla\phi = \begin{pmatrix} \frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial x\partial y} \\ \frac{\partial^2\phi}{\partial y\partial x} & \frac{\partial^2\phi}{\partial y^2} \end{pmatrix} \tag{202}$$

Now define a perturbation velocity potential so that $\mathbf{V}' = \nabla\phi'$ and $\mathbf{V} = \mathbf{V}_\infty + \mathbf{V}'$, i.e. $\nabla\phi = \mathbf{V}_\infty + \nabla\phi'$. We can substitute this into equation (201) to find

$$\nabla\cdot(\mathbf{V}_\infty + \nabla\phi') - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla\phi')\cdot((\nabla(\mathbf{V}_\infty + \nabla\phi'))(\mathbf{V}_\infty + \nabla\phi')) = 0 \tag{203}$$

$\mathbf{V}_\infty$ is a constant so any derivatives of this vanish

$$\nabla\cdot\nabla\phi' - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla\phi')\cdot((\nabla\nabla\phi')(\mathbf{V}_\infty + \nabla\phi')) = 0 \tag{204}$$

Then expanding

$$\nabla\cdot\nabla\phi' - \frac{1}{a^2}(\mathbf{V}_\infty + \nabla\phi')\cdot((\nabla\nabla\phi')\mathbf{V}_\infty + (\nabla\nabla\phi')\nabla\phi') = 0 \tag{205}$$

Ignore products of derivatives to give

$$\nabla\cdot\nabla\phi' - \frac{1}{a^2}\mathbf{V}_\infty^T(\nabla\nabla\phi')\mathbf{V}_\infty = 0 \tag{206}$$

Now deal with the product term

$$\mathbf{V}_\infty^T \left(\nabla\nabla\phi'\right)\mathbf{V}_\infty = \left(\begin{array}{cc} u_\infty & v_\infty \end{array} \right) \left(\begin{array}{cc} \frac{\partial^2\phi'}{\partial x^2} & \frac{\partial^2\phi'}{\partial x\partial y} \\ \frac{\partial^2\phi'}{\partial y\partial x} & \frac{\partial^2\phi'}{\partial y^2} \end{array} \right) \left(\begin{array}{c} u_\infty \\ v_\infty \end{array} \right) \tag{207}$$

Given that $v_\infty = 0$ this becomes

$$\mathbf{V}_\infty^T \left(\nabla\nabla\phi'\right)\mathbf{V}_\infty = u_\infty^2 \frac{\partial^2\phi'}{\partial x^2} \tag{208}$$

Which leads to

$$\left(a^2 - u_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2} + a^2\frac{\partial^2\phi'}{\partial y^2} + a^2\frac{\partial^2\phi'}{\partial z^2} = 0 \tag{209}$$

The energy equation states

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left(\left(\mathbf{V}_\infty + \nabla\phi'\right)\cdot\left(\mathbf{V}_\infty + \nabla\phi'\right)\right) \tag{210}$$

Which we can expand as (neglecting the products of derivatives and noting $a_\infty^2 = a_0^2 - \frac{\gamma-1}{2}(u_\infty^2 + u_\infty^2)$)

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2} 2u_\infty \frac{\partial\phi'}{\partial x} \tag{211}$$

So when we substitute in, the term $2u_\infty\frac{\partial\phi'}{\partial x}$ will be multiplied by $\frac{\partial^2\phi'}{\partial x^2}$ and can be ingored. Thus we arrive at

$$\left(1 - M_\infty^2\right)\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} + \frac{\partial^2\phi'}{\partial z^2} = 0 \tag{212}$$

Clearly there are an enormous number of different models we could construct of higher accuracy than this, but only with some effort!

An interesting footnote too good to miss is a way to neatly get Bernoulli's equation from the steady vector momentum equation. This is only of interest for the incompressible part of this course; it cannot be used for the compressible case.

$$\mathbf{V}\cdot\nabla\mathbf{V} = -\frac{\nabla p}{\rho} \tag{213}$$

Again use

$$\nabla(\mathbf{V}\cdot\mathbf{V}) = 2\mathbf{V}\times(\nabla\times\mathbf{V}) + 2\mathbf{V}\cdot\nabla\mathbf{V} \tag{214}$$

but if the flow is irrotational, and the curl term is zero, this can be used in the momentum equation to get

$$\frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) = -\frac{\nabla p}{\rho} \tag{215}$$

If ρ is constant, we put this inside the gradient and get

$$\nabla\left(\frac{1}{2}\rho\mathbf{V}\cdot\mathbf{V} + p\right) = 0 \tag{216}$$

which if course implies that $\frac{1}{2}\rho\mathbf{V}\cdot\mathbf{V} + p$ is constant, for an incompressible (otherwise ρ cannot go inside the gradient) irrotational (otherwise we can't use the vector identity in the same way) steady (otherwise a time derivative exists in the momentum equation) fluid!

14.1 1D Euler equation

We can exend the working above if we wish. The momentum equation for steady 1D compressible flow is

$$\rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p \tag{217}$$

a useful vector calculus identity is

$$\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) = \mathbf{V} \cdot \nabla \mathbf{V} + \mathbf{V} \times \nabla \times \mathbf{V} \tag{218}$$

There are two alternatives at this point. Either $\nabla \times \mathbf{V} = 0$ (irrotational), or we can take a dot product with $\mathbf{ds}$ for both sides. Assuming irrotationality gives

$$\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) = \mathbf{V} \cdot \nabla \mathbf{V} \tag{219}$$

Taking the dot product gives

$$\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) \cdot \mathbf{ds} = (\mathbf{V} \cdot \nabla \mathbf{V}) \cdot \mathbf{ds} + (\mathbf{V} \times \nabla \times \mathbf{V}) \cdot \mathbf{ds} \tag{220}$$

Now, if $\mathbf{ds}$ is a differential element along a streamline we can simplify further. Remember that $\nabla \times \mathbf{V}$ is a vector 'out of the page' for a 2D flow. This means $\mathbf{V} \times \nabla \times \mathbf{V}$ is a vector perpendicular to both a vector out of the page and the velocity vector - which means it must be a vector 'in the page' at right angles to the streamline (which is aligned to the velocity vector). That means $(\mathbf{V} \times \nabla \times \mathbf{V}) \cdot \mathbf{ds}$ must be zero, because $(\mathbf{V} \times \nabla \times \mathbf{V})$ and $\mathbf{ds}$ are by definition perpendicular.

This is an interesting point - it means that $\frac{1}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) = \mathbf{V} \cdot \nabla \mathbf{V}$ is true everywhere if the flow is irrotational, or just along a streamline if it is rotational.

We can now take

$$(\rho \mathbf{V} \cdot \nabla \mathbf{V}) \cdot \mathbf{ds} = -\nabla p \cdot \mathbf{ds} \tag{221}$$

to arrive at

$$\frac{\rho}{2}\nabla(\mathbf{V} \cdot \mathbf{V}) \cdot \mathbf{ds} = -\nabla p \cdot \mathbf{ds} \tag{222}$$

which is the 1D Euler equation we set out to find (you can expand the V^2 term if you like, of course).

15 Examples

These examples were solved using an automatic program, which also produced the LaTeX source code. This means the results are hopefully free from errors and typos; if you find any, email me.

The results have also been calculated using exact shocks and expansions, ie without linear interpolations. Where appropriate iterations are used to find these exact results (such as the downstream Mach number behind an oblique shock, or $\nu^{-1}(M)$). *This is not how you are intended to work* - you must always use linear interpolation from tables. I've done it here because it is easier to write a program that way, and because each of you will probably work to a slightly different accuracy, so it is best if you have the exact answer to compare to. I'd recommend you work to three decimal places (note that the results below are unrounded for your convenience, but you should always quote results to an appropriate level of accuracy), and if you do this, you should find your results are very close to the ones here.

In case you are wondering, a decidedly average netbook solves these problems instantaneously. You might want to ponder the cost comparison to the Navier-Stokes equations, where 10-20 'reasonable' CPUs might solve a viscous aircraft case on a moderately sized finite volume mesh overnight.

A note on pressure ratios: the program works by calculating the total pressure relative to the freestream total pressure after each shock or expansion. When moving through a shock, this of course drops. When moving through an expansion, total pressure remains unchanged. However, the way the calculations are done here keeps a 'moving' ratio relative to freestream, so if an expansion comes after a shock (often the case) then there will be a lower total pressure due to the shock that already occurred upstream. Keep this in mind when you read the solutions.

15.1 5% thick wedge, M=2, AoA=0^o

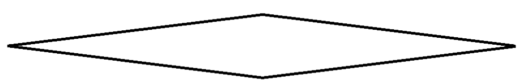

Figure 14: The plain old wedge aerofoil. Not to scale

Moments are taken about 0,0.

Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 2.5000000E-02
Mid point: 0.2500000 1.2500000E-02
Found a shock
The wedge angle is (deg): 2.862386

$$\frac{p_0}{p}_1 = 7.829508 \tag{223}$$

$$\frac{p_0}{p}_2 = 6.674739 \tag{224}$$

$$\frac{p_2}{p_1} = 1.172525 \tag{225}$$

$$\frac{p_{02}}{p_{01}} = 0.9995904 \tag{226}$$

$$\frac{p}{p_\infty} = 0.1498186 \times 0.9995904 \times 7.829508 \tag{227}$$

$$M_2 = 1.896962 \tag{228}$$

$$C_{p_2} = 6.1484423E-02 \tag{229}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 6.1484423E-02 \times 2.5000000E-02 \tag{230}$$

$$\Delta C_Y = 6.1484423E-02 \times -0.5000000 \tag{231}$$

$$\begin{aligned} \Delta C_M = 6.1484423E-02 \times 0.2500000 \times 0.5000000 \\ +6.1484423E-02 \times 1.2500000E-02 \times 2.5000000E-02 \end{aligned} \tag{232}$$

Surface number 2
Start point: 0.5000000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 23.44641 \tag{233}$$

$$\nu\left(M_2\right) = 23.44641 + 5.724814 \tag{234}$$

$$\nu\left(M_2\right) = 29.17122 \tag{235}$$

$$\frac{p_0}{p}_1 = 6.674739 \tag{236}$$

$$\frac{p_0}{p}_2 = 9.227528 \tag{237}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_\infty} = \tag{238}$$
$$0.1083714 \times 0.9995904 \times 7.829508$$

$$M_2 = 2.105494 \tag{239}$$

$$C_{p_2} = -5.4117214E - 02 \tag{240}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -5.4117214E - 02 \times -2.5000000E - 02 \tag{241}$$

$$\Delta C_Y = -5.4117214E - 02 \times -0.5000000 \tag{242}$$

$$\Delta C_M = -5.4117214E - 02 \times 0.7500000 \times 0.5000000 \tag{243}$$
$$+ - 5.4117214E - 02 \times 1.2500000E - 02 \times -2.5000000E - 02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 -2.5000000E-02
Mid point: 0.2500000 -1.2500000E-02
Found a shock
The wedge angle is (deg): 2.862386

$$\frac{p_0}{p}_1 = 7.829508 \tag{244}$$

$$\frac{p_0}{p}_2 = 6.674739 \tag{245}$$

$$\frac{p_2}{p_1} = 1.172525 \tag{246}$$

$$\frac{p_{02}}{p_{01}} = 0.9995904 \tag{247}$$

$$\frac{p}{p_\infty} = 0.1498186 \times 0.9995904 \times 7.829508 \tag{248}$$

$$M_2 = 1.896962 \tag{249}$$

$$C_{p_2} = 6.1484423E - 02 \tag{250}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 6.1484423E - 02 \times 2.5000000E - 02 \tag{251}$$

$$\Delta C_Y = 6.1484423E - 02 \times 0.5000000 \tag{252}$$

$$\Delta C_M = 6.1484423E - 02 \times 0.2500000 \times -0.5000000 \tag{253}$$
$$+6.1484423E - 02 \times -1.2500000E - 02 \times 2.5000000E - 02$$

Surface number 2
Start point: 0.5000000 -2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 -1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 23.44641 \tag{254}$$

$$\nu\left(M_2\right) = 23.44641 + 5.724814 \tag{255}$$

$$\nu\left(M_2\right) = 29.17122 \tag{256}$$

$$\frac{p_0}{p}_1 = 6.674739 \tag{257}$$

$$\frac{p_0}{p}_2 = 9.227528 \tag{258}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{259}$$
$$0.1083714 \times 0.9995904 \times 7.829508$$

$$M_2 = 2.105494 \tag{260}$$

$$C_{p_2} = -5.4117214E - 02 \tag{261}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -5.4117214E - 02 \times -2.5000000E - 02 \tag{262}$$

$$\Delta C_Y = -5.4117214E - 02 \times 0.5000000 \tag{263}$$

$$\Delta C_M = -5.4117214E - 02 \times 0.7500000 \times -0.5000000 \tag{264}$$
$$+ - 5.4117214E - 02 \times -1.2500000E - 02 \times -2.5000000E - 02$$

$$C_L = C_Y\cos(\alpha) - C_X\sin(\alpha) \tag{265}$$

$$C_L = 0.0000000E + 00 \tag{266}$$

$$C_D = C_Y\sin(\alpha) + C_X\cos(\alpha) \tag{267}$$

$$C_D = 5.7800817E - 03 \tag{268}$$

$$C_M = -2.0888744E - 10 \tag{269}$$

15.2 5% thick wedge, M=2, AoA=2^o

Moments are taken about 0,0.
Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 2.5000000E-02
Mid point: 0.2500000 1.2500000E-02
Found a shock
The wedge angle is (deg): 0.8623004

$$\frac{p_0}{p}_1 = 7.829508 \tag{270}$$

$$\frac{p_0}{p}_2 = 7.458668 \tag{271}$$

$$\frac{p_2}{p_1} = 1.049707 \tag{272}$$

$$\frac{p_{02}}{p_{01}} = 0.9999884 \tag{273}$$

$$\frac{p}{p_\infty} = 0.1340722 \times 0.9999884 \times 7.829508 \tag{274}$$

$$M_2 = 1.968742 \tag{275}$$

$$C_{p_2} = 1.7714560E-02 \tag{276}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 1.7714560E-02 \times 2.5000000E-02 \tag{277}$$

$$\Delta C_Y = 1.7714560E-02 \times -0.5000000 \tag{278}$$

$$\Delta C_M = 1.7714560E-02 \times 0.2500000 \times 0.5000000 \tag{279}$$
$$+1.7714560E-02 \times 1.2500000E-02 \times 2.5000000E-02$$

Surface number 2
Start point: 0.5000000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 25.45420 \tag{280}$$

$$\nu\left(M_2\right) = 25.45421 + 5.724814 \tag{281}$$

$$\nu\left(M_2\right) = 31.17902 \tag{282}$$

$$\frac{p_0}{p_{\ 1}} = 7.458668 \tag{283}$$

$$\frac{p_0}{p_{\ 2}} = 10.39515 \tag{284}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{285}$$
$$9.6198723E-02 \times 0.9999884 \times 7.829508$$

$$M_2 = 2.181797 \tag{286}$$

$$C_{p_2} = -8.7961547E-02 \tag{287}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -8.7961547E-02 \times -2.5000000E-02 \tag{288}$$

$$\Delta C_Y = -8.7961547E-02 \times -0.5000000 \tag{289}$$

$$\Delta C_M = -8.7961547E-02 \times 0.7500000 \times 0.5000000 \tag{290}$$
$$+ -8.7961547E-02 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 -2.5000000E-02
Mid point: 0.2500000 -1.2500000E-02
Found a shock
The wedge angle is (deg): 4.862403

$$\frac{p_0}{p_{\ 1}} = 7.829508 \tag{291}$$

$$\frac{p_0}{p_{\ 2}} = 5.981108 \tag{292}$$

$$\frac{p_2}{p_1} = 1.306502 \tag{293}$$

$$\frac{p_{02}}{p_{01}} = 0.9980611 \tag{294}$$

$$\frac{p}{p_\infty} = 0.1671931 \times 0.9980611 \times 7.829508 \tag{295}$$

$$M_2 = 1.825608 \tag{296}$$

$$C_{p_2} = 0.1092308 \tag{297}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 0.1092308 \times 2.5000000E-02 \tag{298}$$

$$\Delta C_Y = 0.1092308 \times 0.5000000 \tag{299}$$

$$\Delta C_M = 0.1092308 \times 0.2500000 \times -0.5000000 \\ +0.1092308 \times -1.2500000E-02 \times 2.5000000E-02 \tag{300}$$

Surface number 2
Start point: 0.5000000 -2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 -1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 21.41599 \tag{301}$$

$$\nu\left(M_2\right) = 21.41599 + 5.724814 \tag{302}$$

$$\nu\left(M_2\right) = 27.14080 \tag{303}$$

$$\frac{p_0}{p}_1 = 5.981108 \tag{304}$$

$$\frac{p_0}{p}_2 = 8.204935 \tag{305}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \\ 0.1218779 \times 0.9980611 \times 7.829508 \tag{306}$$

$$M_2 = 2.030122 \tag{307}$$

$$C_{p_2} = -1.6965954E-02 \tag{308}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -1.6965954E-02 \times -2.5000000E-02 \tag{309}$$

$$\Delta C_Y = -1.6965954E-02 \times 0.5000000 \tag{310}$$

$$\Delta C_M = -1.6965954E-02 \times 0.7500000 \times -0.5000000 \\ + -1.6965954E-02 \times -1.2500000E-02 \times -2.5000000E-02 \tag{311}$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{312}$$

$$C_L = 8.1004091E-02 \tag{313}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{314}$$

$$C_D = 8.6290799E-03 \tag{315}$$

$$C_M = -3.8069282E-02 \tag{316}$$

15.3 5% thick wedge, M=2, AoA=4^o

Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 2.5000000E-02
Mid point: 0.2500000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 1.137632

$$\nu\left(M_1\right) = 26.31672 \tag{317}$$

$$\nu\left(M_2\right) = 26.31672 + 1.137632 \tag{318}$$

$$\nu\left(M_2\right) = 27.45436 \tag{319}$$

$$\frac{p_0}{p}_1 = 7.829508 \tag{320}$$

$$\frac{p_0}{p}_2 = 8.353504 \tag{321}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_\infty} = \\ 0.1197102 \times 1.000000 \times 7.829508 \tag{322}$$

$$M_2 = 2.041651 \tag{323}$$

$$C_{p_2} = -2.2354864E-02 \tag{324}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -2.2354864E-02 \times 2.5000000E-02 \tag{325}$$

$$\Delta C_Y = -2.2354864E-02 \times -0.5000000 \tag{326}$$

$$\Delta C_M = -2.2354864E-02 \times 0.2500000 \times 0.5000000 \\ + -2.2354864E-02 \times 1.2500000E-02 \times 2.5000000E-02 \tag{327}$$

Surface number 2
Start point: 0.5000000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 27.45435 \tag{328}$$

$$\nu\left(M_2\right) = 27.45435 + 5.724814 \tag{329}$$

$$\nu\left(M_2\right) = 33.17916 \tag{330}$$

$$\frac{p_0}{p_{\ 1}} = 8.353504 \tag{331}$$

$$\frac{p_0}{p_{\ 2}} = 11.74177 \tag{332}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{333}$$
$$8.5166045E-02 \times 1.000000 \times 7.829508$$

$$M_2 = 2.259749 \tag{334}$$

$$C_{p_2} = -0.1187426 \tag{335}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.1187426 \times -2.5000000E-02 \tag{336}$$

$$\Delta C_Y = -0.1187426 \times -0.5000000 \tag{337}$$

$$\Delta C_M = -0.1187426 \times 0.7500000 \times 0.5000000 \tag{338}$$
$$+-0.1187426 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 -2.5000000E-02
Mid point: 0.2500000 -1.2500000E-02
Found a shock
The wedge angle is (deg): 6.862413

$$\frac{p_0}{p_{\ 1}} = 7.829508 \tag{339}$$

$$\frac{p_0}{p_{\ 2}} = 5.361930 \tag{340}$$

$$\frac{p_2}{p_1} = 1.452522 \tag{341}$$

$$\frac{p_{02}}{p_{01}} = 0.9947394 \tag{342}$$

$$\frac{p}{p_\infty} = 0.1865000 \times 0.9947394 \times 7.829508 \tag{343}$$

$$M_2 = 1.753991 \tag{344}$$

$$C_{p_2} = 0.1612693 \tag{345}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 0.1612693 \times 2.5000000E-02 \tag{346}$$

$$\Delta C_Y = 0.1612693 \times 0.5000000 \tag{347}$$

$$\begin{aligned} \Delta C_M = 0.1612693 \times 0.2500000 \times -0.5000000 \\ +0.1612693 \times -1.2500000E-02 \times 2.5000000E-02 \end{aligned} \tag{348}$$

Surface number 2
Start point: 0.5000000 -2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 -1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 19.34809 \tag{349}$$

$$\nu\left(M_2\right) = 19.34809 + 5.724814 \tag{350}$$

$$\nu\left(M_2\right) = 25.07291 \tag{351}$$

$$\frac{p_0}{p_{\ 1}} = 5.361930 \tag{352}$$

$$\frac{p_0}{p_{\ 2}} = 7.301541 \tag{353}$$

$$\begin{aligned} \frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \\ 0.1369574 \times 0.9947394 \times 7.829508 \end{aligned} \tag{354}$$

$$M_2 = 1.955008 \tag{355}$$

$$C_{p_2} = 2.3759058E-02 \tag{356}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 2.3759058E-02 \times -2.5000000E-02 \tag{357}$$

$$\Delta C_Y = 2.3759058E-02 \times 0.5000000 \tag{358}$$

$$\begin{aligned} \Delta C_M = 2.3759058E-02 \times 0.7500000 \times -0.5000000 \\ +2.3759058E-02 \times -1.2500000E-02 \times -2.5000000E-02 \end{aligned} \tag{359}$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{360}$$

$$C_L = 0.1622578 \tag{361}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{362}$$

$$C_D = 1.7207898E-02 \tag{363}$$

$$C_M = -7.6403998E-02 \tag{364}$$

15.4 5% thick wedge, M=2, AoA=6^o

Moments are taken about 0,0.
Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 2.5000000E-02
Mid point: 0.2500000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 3.137624

$$\nu\left(M_1\right) = 26.31672 \tag{365}$$

$$\nu\left(M_2\right) = 26.31672 + 3.137624 \tag{366}$$

$$\nu\left(M_2\right) = 29.45435 \tag{367}$$

$$\frac{p_0}{p}_1 = 7.829508 \tag{368}$$

$$\frac{p_0}{p}_2 = 9.382145 \tag{369}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{370}$$
$$0.1065854 \times 1.000000 \times 7.829508$$

$$M_2 = 2.116143 \tag{371}$$

$$C_{p_2} = -5.8976650E - 02 \tag{372}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -5.8976650E - 02 \times 2.5000000E - 02 \tag{373}$$

$$\Delta C_Y = -5.8976650E - 02 \times -0.5000000 \tag{374}$$

$$\Delta C_M = -5.8976650E - 02 \times 0.2500000 \times 0.5000000 \tag{375}$$
$$+ -5.8976650E - 02 \times 1.2500000E - 02 \times 2.5000000E - 02$$

Surface number 2
Start point: 0.5000000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 29.45434 \tag{376}$$

$$\nu\left(M_2\right) = 29.45434 + 5.724814 \tag{377}$$

$$\nu\left(M_2\right) = 35.17916 \tag{378}$$

$$\frac{p_0}{p_{\ 1}} = 9.382145 \tag{379}$$

$$\frac{p_0}{p_{\ 2}} = 13.30606 \tag{380}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{381}$$
$$7.5153746E-02 \times 1.000000 \times 7.829508$$

$$M_2 = 2.339823 \tag{382}$$

$$C_{p_2} = -0.1466797 \tag{383}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.1466797 \times -2.5000000E-02 \tag{384}$$

$$\Delta C_Y = -0.1466797 \times -0.5000000 \tag{385}$$

$$\Delta C_M = -0.1466797 \times 0.7500000 \times 0.5000000 \tag{386}$$
$$+ -0.1466797 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.5000000 -2.5000000E-02
Mid point: 0.2500000 -1.2500000E-02
Found a shock
The wedge angle is (deg): 8.862396

$$\frac{p_0}{p_{\ 1}} = 7.829508 \tag{387}$$

$$\frac{p_0}{p_{\ 2}} = 4.805032 \tag{388}$$

$$\frac{p_2}{p_1} = 1.611618 \tag{389}$$

$$\frac{p_{02}}{p_{01}} = 0.9890631 \tag{390}$$

$$\frac{p}{p_\infty} = 0.2081152 \times 0.9890631 \times 7.829508 \tag{391}$$

$$M_2 = 1.681429 \tag{392}$$

$$C_{p_2} = 0.2179681 \tag{393}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 0.2179681 \times 2.5000000E-02 \tag{394}$$

$$\Delta C_Y = 0.2179681 \times 0.5000000 \quad (395)$$

$$\Delta C_M = 0.2179681 \times 0.2500000 \times -0.5000000 \quad (396)$$
$$+0.2179681 \times -1.2500000E-02 \times 2.5000000E-02$$

Surface number 2
Start point: 0.5000000 -2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.7500000 -1.2500000E-02
Found an expansion
The expansion angle is (deg): 5.724814

$$\nu\left(M_1\right) = 17.22848 \quad (397)$$

$$\nu\left(M_2\right) = 17.22848 + 5.724814 \quad (398)$$

$$\nu\left(M_2\right) = 22.95329 \quad (399)$$

$$\frac{p_0}{p}_1 = 4.805032 \quad (400)$$

$$\frac{p_0}{p}_2 = 6.497720 \quad (401)$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_\infty} = \quad (402)$$
$$0.1539001 \times 0.9890631 \times 7.829508$$

$$M_2 = 1.879526 \quad (403)$$

$$C_{p_2} = 6.8347752E-02 \quad (404)$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 6.8347752E-02 \times -2.5000000E-02 \quad (405)$$

$$\Delta C_Y = 6.8347752E-02 \times 0.5000000 \quad (406)$$

$$\Delta C_M = 6.8347752E-02 \times 0.7500000 \times -0.5000000 \quad (407)$$
$$+6.8347752E-02 \times -1.2500000E-02 \times -2.5000000E-02$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \quad (408)$$

$$C_L = 0.2440183 \quad (409)$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \quad (410)$$

$$C_D = 3.1613126E-02 \quad (411)$$

$$C_M = -0.1152727 \quad (412)$$

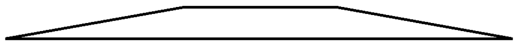

Figure 15: The exotic aerofoil (many variations). Not to scale

15.5 5% thick exotic, M=2, AoA=0^o

Moments are taken about 0,0.
Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.3500000 2.5000000E-02
Mid point: 0.1750000 1.2500000E-02
Found a shock
The wedge angle is (deg): 4.085619

$$\frac{p_0}{p}_1 = 7.829508 \tag{413}$$

$$\frac{p_0}{p}_2 = 6.240887 \tag{414}$$

$$\frac{p_2}{p_1} = 1.253087 \tag{415}$$

$$\frac{p_{02}}{p_{01}} = 0.9988337 \tag{416}$$

$$\frac{p}{p_\infty} = 0.1602336 \times 0.9988337 \times 7.829508 \tag{417}$$

$$M_2 = 1.853315 \tag{418}$$

$$C_{p_2} = 9.0195082E - 02 \tag{419}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 9.0195082E - 02 \times 2.5000000E - 02 \tag{420}$$

$$\Delta C_Y = 9.0195082E - 02 \times -0.3500000 \tag{421}$$

$$\begin{aligned}\Delta C_M = 9.0195082E - 02 \times 0.1750000 \times 0.3500000 \\ +9.0195082E - 02 \times 1.2500000E - 02 \times 2.5000000E - 02\end{aligned} \tag{422}$$

Surface number 2
Start point: 0.3500000 2.5000000E-02
End point: 0.6500000 2.5000000E-02
Mid point: 0.5000000 2.5000000E-02
Found an expansion
The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 22.20823 \tag{423}$$

$$\nu\left(M_2\right) = 22.20823 + 4.085619 \tag{424}$$

$$\nu\left(M_2\right) = 26.29385 \tag{425}$$

$$\frac{p_0}{p}_1 = 6.240887 \tag{426}$$

$$\frac{p_0}{p}_2 = 7.819379 \tag{427}$$

$$\begin{aligned}\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \\ 0.1278874 \times 0.9988337 \times 7.829508\end{aligned} \tag{428}$$

$$M_2 = 1.999167 \tag{429}$$

$$C_{p_2} = 4.5457571E - 05 \tag{430}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 4.5457571E - 05 \times 0.0000000E + 00 \tag{431}$$

$$\Delta C_Y = 4.5457571E - 05 \times -0.3000000 \tag{432}$$

$$\begin{aligned}\Delta C_M = 4.5457571E - 05 \times 0.5000000 \times 0.3000000 \\ +4.5457571E - 05 \times 2.5000000E - 02 \times 0.0000000E + 00\end{aligned} \tag{433}$$

Surface number 3
Start point: 0.6500000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.8250000 1.2500000E-02
Found an expansion

The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 26.29384 \tag{434}$$
$$\nu\left(M_2\right) = 26.29384 + 4.085619 \tag{435}$$
$$\nu\left(M_2\right) = 30.37946 \tag{436}$$
$$\frac{p_0}{p}_1 = 7.819379 \tag{437}$$
$$\frac{p_0}{p}_2 = 9.909796 \tag{438}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_\infty} = \tag{439}$$
$$0.1009103 \times 0.9988337 \times 7.829508$$

$$M_2 = 2.151188 \tag{440}$$
$$C_{p_2} = -7.5140364E-02 \tag{441}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -7.5140364E-02 \times -2.5000000E-02 \tag{442}$$

$$\Delta C_Y = -7.5140364E-02 \times -0.3500000 \tag{443}$$

$$\Delta C_M = -7.5140364E-02 \times 0.8250000 \times 0.3500000 \tag{444}$$
$$+ -7.5140364E-02 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 1.000000 0.0000000E+00
Mid point: 0.5000000 0.0000000E+00
The expansion angle is (deg): 0.0000000E+00

$$\nu\left(M_1\right) = 26.31672 \tag{445}$$
$$\nu\left(M_2\right) = 26.31672 + 0.0000000E+00 \tag{446}$$
$$\nu\left(M_2\right) = 26.31672 \tag{447}$$
$$\frac{p_0}{p}_1 = 7.829508 \tag{448}$$
$$\frac{p_0}{p}_2 = 7.829504 \tag{449}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_\infty} = \tag{450}$$
$$0.1277220 \times 1.000000 \times 7.829508$$

$$M_2 = 2.000000 \tag{451}$$

$$C_{p_2} = 1.6993484E-07 \tag{452}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 1.6993484E-07 \times 0.0000000E+00 \tag{453}$$

$$\Delta C_Y = 1.6993484E-07 \times 1.000000 \tag{454}$$

$$\begin{aligned}\Delta C_M &= 1.6993484E-07 \times 0.5000000 \times -1.000000 \\ &+1.6993484E-07 \times 0.0000000E+00 \times 0.0000000E+00\end{aligned} \tag{455}$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{456}$$

$$C_L = -5.2826167E-03 \tag{457}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{458}$$

$$C_D = 4.1333861E-03 \tag{459}$$

$$C_M = -1.6113933E-02 \tag{460}$$

15.6 5% thick exotic, M=2, AoA=2^o

Moments are taken about 0,0.
Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.3500000 2.5000000E-02
Mid point: 0.1750000 1.2500000E-02
Found a shock
The wedge angle is (deg): 2.085628

$$\frac{p_0}{p}_1 = 7.829508 \tag{461}$$

$$\frac{p_0}{p}_2 = 6.967444 \tag{462}$$

$$\frac{p_2}{p_1} = 1.123547 \tag{463}$$

$$\frac{p_{02}}{p_{01}} = 0.9998392 \tag{464}$$

$$\frac{p}{p_\infty} = 0.1435246 \times 0.9998392 \times 7.829508 \tag{465}$$

$$M_2 = 1.924749 \tag{466}$$

$$C_{p_2} = 4.4029478E-02 \tag{467}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 4.4029478E-02 \times 2.5000000E-02 \tag{468}$$

$$\Delta C_Y = 4.4029478E-02 \times -0.3500000 \tag{469}$$

$$\begin{aligned}\Delta C_M = 4.4029478E-02 \times 0.1750000 \times 0.3500000 \\ +4.4029478E-02 \times 1.2500000E-02 \times 2.5000000E-02\end{aligned} \tag{470}$$

Surface number 2
Start point: 0.3500000 2.5000000E-02
End point: 0.6500000 2.5000000E-02
Mid point: 0.5000000 2.5000000E-02
Found an expansion
The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 24.22801 \tag{471}$$

$$\nu\left(M_2\right) = 24.22801 + 4.085619 \tag{472}$$

$$\nu\left(M_2\right) = 28.31363 \tag{473}$$

$$\frac{p_0}{p}_1 = 6.967444 \tag{474}$$

$$\frac{p_0}{p}_2 = 8.777732 \tag{475}$$

$$\begin{aligned}\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \\ 0.1139246 \times 0.9998392 \times 7.829508\end{aligned} \tag{476}$$

$$M_2 = 2.073450 \tag{477}$$

$$C_{p_2} = -3.8549356E-02 \tag{478}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -3.8549356E-02 \times 0.0000000E+00 \tag{479}$$

$$\Delta C_Y = -3.8549356E-02 \times -0.3000000 \tag{480}$$

$$\begin{aligned}\Delta C_M = -3.8549356E-02 \times 0.5000000 \times 0.3000000 \\ + -3.8549356E-02 \times 2.5000000E-02 \times 0.0000000E+00\end{aligned} \tag{481}$$

Surface number 3
Start point: 0.6500000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.8250000 1.2500000E-02
Found an expansion

The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 28.31363 \tag{482}$$

$$\nu\left(M_2\right) = 28.31363 + 4.085619 \tag{483}$$

$$\nu\left(M_2\right) = 32.39925 \tag{484}$$

$$\frac{p_0}{p_{\ 1}} = 8.777732 \tag{485}$$

$$\frac{p_0}{p_{\ 2}} = 11.19284 \tag{486}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{487}$$
$$8.9342855E-02 \times 0.9998392 \times 7.829508$$

$$M_2 = 2.229111 \tag{488}$$

$$C_{p_2} = -0.1071282 \tag{489}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.1071282 \times -2.5000000E-02 \tag{490}$$

$$\Delta C_Y = -0.1071282 \times -0.3500000 \tag{491}$$

$$\Delta C_M = -0.1071282 \times 0.8250000 \times 0.3500000 \tag{492}$$
$$+ -0.1071282 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 1.000000 0.0000000E+00
Mid point: 0.5000000 0.0000000E+00
Found a shock
The wedge angle is (deg): 1.999978

$$\frac{p_0}{p_{\ 1}} = 7.829508 \tag{493}$$

$$\frac{p_0}{p_{\ 2}} = 7.000600 \tag{494}$$

$$\frac{p_2}{p_1} = 1.118247 \tag{495}$$

$$\frac{p_{02}}{p_{01}} = 0.9998584 \tag{496}$$

$$\frac{p}{p_\infty} = 0.1428449 \times 0.9998584 \times 7.829508 \tag{497}$$

$$M_2 = 1.927819 \tag{498}$$

$$C_{p_2} = 4.2140696E-02 \tag{499}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 4.2140696E-02 \times 0.0000000E+00 \tag{500}$$

$$\Delta C_Y = 4.2140696E-02 \times 1.000000 \tag{501}$$

$$\begin{aligned}\Delta C_M = 4.2140696E-02 \times 0.5000000 \times -1.000000 \\ +4.2140696E-02 \times 0.0000000E+00 \times 0.0000000E+00\end{aligned} \tag{502}$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{503}$$

$$C_L = 7.5612016E-02 \tag{504}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{505}$$

$$C_D = 6.4216764E-03 \tag{506}$$

$$C_M = -5.5041991E-02 \tag{507}$$

15.7 5% thick exotic, M=2, AoA=4^o

Moments are taken about 0,0.
Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.3500000 2.5000000E-02
Mid point: 0.1750000 1.2500000E-02
Found a shock
The wedge angle is (deg): 8.6229227E-02

$$\frac{p_0}{p}_1 = 7.829508 \tag{508}$$

$$\frac{p_0}{p}_2 = 7.791443 \tag{509}$$

$$\frac{p_2}{p_1} = 1.004886 \tag{510}$$

$$\frac{p_{02}}{p_{01}} = 1.000000 \tag{511}$$

$$\frac{p}{p_\infty} = 0.1283459 \times 1.000000 \times 7.829508 \tag{512}$$

$$M_2 = 1.996863 \tag{513}$$

$$C_{p_2} = 1.7411099E - 03 \tag{514}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 1.7411099E - 03 \times 2.5000000E - 02 \tag{515}$$

$$\Delta C_Y = 1.7411099E - 03 \times -0.3500000 \tag{516}$$

$$\Delta C_M = 1.7411099E - 03 \times 0.1750000 \times 0.3500000 \tag{517}$$
$$+1.7411099E - 03 \times 1.2500000E - 02 \times 2.5000000E - 02$$

Surface number 2
Start point: 0.3500000 2.5000000E-02
End point: 0.6500000 2.5000000E-02
Mid point: 0.5000000 2.5000000E-02
Found an expansion
The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 26.23050 \tag{518}$$

$$\nu\left(M_2\right) = 26.23050 + 4.085619 \tag{519}$$

$$\nu\left(M_2\right) = 30.31612 \tag{520}$$

$$\frac{p_0}{p}_1 = 7.791443 \tag{521}$$

$$\frac{p_0}{p}_2 = 9.872542 \tag{522}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{523}$$
$$0.1012910 \times 1.000000 \times 7.829508$$

$$M_2 = 2.148776 \tag{524}$$

$$C_{p_2} = -7.3749468E - 02 \tag{525}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -7.3749468E - 02 \times 0.0000000E + 00 \tag{526}$$

$$\Delta C_Y = -7.3749468E - 02 \times -0.3000000 \tag{527}$$

$$\Delta C_M = -7.3749468E - 02 \times 0.5000000 \times 0.3000000 \tag{528}$$
$$+ - 7.3749468E - 02 \times 2.5000000E - 02 \times 0.0000000E + 00$$

Surface number 3
Start point: 0.6500000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.8250000 1.2500000E-02
Found an expansion

The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 30.31611 \tag{529}$$

$$\nu\left(M_2\right) = 30.31611 + 4.085619 \tag{530}$$

$$\nu\left(M_2\right) = 34.40173 \tag{531}$$

$$\frac{p_0}{p}_1 = 9.872542 \tag{532}$$

$$\frac{p_0}{p}_2 = 12.66961 \tag{533}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{534}$$
$$7.8929037E-02 \times 1.000000 \times 7.829508$$

$$M_2 = 2.308432 \tag{535}$$

$$C_{p_2} = -0.1361456 \tag{536}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.1361456 \times -2.5000000E-02 \tag{537}$$

$$\Delta C_Y = -0.1361456 \times -0.3500000 \tag{538}$$

$$\Delta C_M = -0.1361456 \times 0.8250000 \times 0.3500000 \tag{539}$$
$$+ -0.1361456 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 1.000000 0.0000000E+00
Mid point: 0.5000000 0.0000000E+00
Found a shock
The wedge angle is (deg): 3.999978

$$\frac{p_0}{p}_1 = 7.829508 \tag{540}$$

$$\frac{p_0}{p}_2 = 6.270248 \tag{541}$$

$$\frac{p_2}{p_1} = 1.247307 \tag{542}$$

$$\frac{p_{02}}{p_{01}} = 0.9989039 \tag{543}$$

$$\frac{p}{p_\infty} = 0.1594833 \times 0.9989039 \times 7.829508 \tag{544}$$

$$M_2 = 1.856368 \tag{545}$$

$$C_{p_2} = 8.8135175E - 02 \tag{546}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 8.8135175E - 02 \times 0.0000000E + 00 \tag{547}$$

$$\Delta C_Y = 8.8135175E - 02 \times 1.000000 \tag{548}$$

$$\Delta C_M = 8.8135175E - 02 \times 0.5000000 \times -1.000000$$
$$+8.8135175E - 02 \times 0.0000000E + 00 \times 0.0000000E + 00 \tag{549}$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{550}$$

$$C_L = 0.1566779 \tag{551}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{552}$$

$$C_D = 1.4411573E - 02 \tag{553}$$

$$C_M = -9.4292313E - 02 \tag{554}$$

15.8 5% thick exotic, M=2, AoA=6^o

Surfaces numbered in ascending order.
First along the upper surface, then lower.
For shocks/expansions 1 denotes properties before. 2 denotes properties after.

Working on the upper surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 0.3500000 2.5000000E-02
Mid point: 0.1750000 1.2500000E-02
Found an expansion
The expansion angle is (deg): 1.914382

$$\nu(M_1) = 26.31672 \tag{555}$$

$$\nu(M_2) = 26.31672 + 1.914382 \tag{556}$$

$$\nu(M_2) = 28.23111 \tag{557}$$

$$\frac{p_0}{p}_1 = 7.829508 \tag{558}$$

$$\frac{p_0}{p}_2 = 8.735870 \tag{559}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}} \frac{p_{02}}{p_{01}} \frac{p_{01}}{p_\infty} =$$
$$0.1144706 \times 1.000000 \times 7.829508 \tag{560}$$

$$M_2 = 2.070383 \tag{561}$$

$$C_{p_2} = -3.6974996E - 02 \tag{562}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -3.6974996E - 02 \times 2.5000000E - 02 \tag{563}$$

$$\Delta C_Y = -3.6974996E - 02 \times -0.3500000 \tag{564}$$

$$\Delta C_M = -3.6974996E - 02 \times 0.1750000 \times 0.3500000 \tag{565}$$
$$+ - 3.6974996E - 02 \times 1.2500000E - 02 \times 2.5000000E - 02$$

Surface number 2
Start point: 0.3500000 2.5000000E-02
End point: 0.6500000 2.5000000E-02
Mid point: 0.5000000 2.5000000E-02
Found an expansion
The expansion angle is (deg): 4.085619

$$\nu(M_1) = 28.23111 \tag{566}$$

$$\nu(M_2) = 28.23111 + 4.085619 \tag{567}$$

$$\nu(M_2) = 32.31673 \tag{568}$$

$$\frac{p_0}{p_{\ 1}} = 8.735870 \tag{569}$$

$$\frac{p_0}{p_{\ 2}} = 11.13660 \tag{570}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{571}$$
$$8.9794040E - 02 \times 1.000000 \times 7.829508$$

$$M_2 = 2.225888 \tag{572}$$

$$C_{p_2} = -0.1058292 \tag{573}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.1058292 \times 0.0000000E + 00 \tag{574}$$

$$\Delta C_Y = -0.1058292 \times -0.3000000 \tag{575}$$

$$\Delta C_M = -0.1058292 \times 0.5000000 \times 0.3000000 \tag{576}$$
$$+ - 0.1058292 \times 2.5000000E - 02 \times 0.0000000E + 00$$

Surface number 3
Start point: 0.6500000 2.5000000E-02
End point: 1.000000 0.0000000E+00
Mid point: 0.8250000 1.2500000E-02
Found an expansion

The expansion angle is (deg): 4.085619

$$\nu\left(M_1\right) = 32.31673 \tag{577}$$
$$\nu\left(M_2\right) = 32.31673 + 4.085619 \tag{578}$$
$$\nu\left(M_2\right) = 36.40235 \tag{579}$$
$$\frac{p_0}{p\ _1} = 11.13660 \tag{580}$$
$$\frac{p_0}{p\ _2} = 14.38783 \tag{581}$$

$$\frac{p}{p_\infty} = \frac{p}{p_{02}}\frac{p_{02}}{p_{01}}\frac{p_{01}}{p_\infty} = \tag{582}$$
$$6.9503173E-02 \times 1.000000 \times 7.829508$$

$$M_2 = 2.389930 \tag{583}$$
$$C_{p_2} = -0.1624463 \tag{584}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = -0.1624463 \times -2.5000000E-02 \tag{585}$$
$$\Delta C_Y = -0.1624463 \times -0.3500000 \tag{586}$$

$$\Delta C_M = -0.1624463 \times 0.8250000 \times 0.3500000 \tag{587}$$
$$+ -0.1624463 \times 1.2500000E-02 \times -2.5000000E-02$$

Working on the lower surface

Surface number 1
Start point: 0.0000000E+00 0.0000000E+00
End point: 1.000000 0.0000000E+00
Mid point: 0.5000000 0.0000000E+00
Found a shock
The wedge angle is (deg): 5.999989

$$\frac{p_0}{p\ _1} = 7.829508 \tag{588}$$
$$\frac{p_0}{p\ _2} = 5.620646 \tag{589}$$
$$\frac{p_2}{p_1} = 1.388018 \tag{590}$$
$$\frac{p_{02}}{p_{01}} = 0.9964299 \tag{591}$$
$$\frac{p}{p_\infty} = 0.1779155 \times 0.9964299 \times 7.829508 \tag{592}$$
$$M_2 = 1.784949 \tag{593}$$

$$C_{p_2} = 0.1382814 \tag{594}$$

Contributions to force coefficients from this surface:

$$\Delta C_X = 0.1382814 \times 0.0000000E + 00 \tag{595}$$

$$\Delta C_Y = 0.1382814 \times 1.000000 \tag{596}$$

$$\begin{aligned} \Delta C_M = 0.1382814 \times 0.5000000 \times -1.000000 \\ +0.1382814 \times 0.0000000E + 00 \times 0.0000000E + 00 \end{aligned} \tag{597}$$

$$C_L = C_Y \cos(\alpha) - C_X \sin(\alpha) \tag{598}$$

$$C_L = 0.2381859 \tag{599}$$

$$C_D = C_Y \sin(\alpha) + C_X \cos(\alpha) \tag{600}$$

$$C_D = 2.8188411E - 02 \tag{601}$$

$$C_M = -0.1341470 \tag{602}$$

References

[1] J.D. Anderson. *Fundamentals of Aerodynamics*. McGraw-Hill, 2^{nd} edition, 1991.

[2] J.D. Anderson. *Modern Compressible Flow with Historical Perspective*. McGraw-Hill, 2^{nd} edition, 1990.