Ordinary Differential Equations

Lecture 1-2: Introduction and separation of variables

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Introduction

Why do we need differential equations?

Science and engineering use mathematics to predict the behaviour of systems.

So, what's easier predicting the weather tomorrow, or predicting the weather a year from now?

Predictions get easier if we predict the state of a system in the near future. So lets take the limit and predict what the system does in the very next instant.

If we are capable of predicting the next instant we can use mathematics to concatenate tiny instantaneous changes to obtain a prediction for a given time in the future.

Introduction

Differential Equation Hall of Fame

Differential equations arise in every area of engineering

Aerospace	Navier-Stokes equations (fluid flow)
Civil	Euler-Bernoulli equation (bending stresses in a beam)
Electrical	Maxwell equations (electromagnetic radiation)
Mechanical	Newton's laws of motion (most things. . .)

Engineering Mathematics deals with all of these and a lot more!

Introduction

Ordinary Differential Equations (ODEs)

Ordinary differential equations (ODEs) describe the rate-of-change of a quantity, x, over time t

$$\underbrace{\frac{\mathrm{d}\,x}{\mathrm{d}\,t}}_{\text{rate-of-change of } x} = \underbrace{f(x,t)}_{\text{some function of } x \text{ and } t}$$

We say

- x is the dependent variable
- t is the independent variable.

Introduction

Dependent and Independent Variables

The variables will not always be named x and t, but ...

- Independent variables do not depend on any other variables. They usually appear at the bottom of differentials.
- Dependent variables depend on other variables. They usually appear on top of differentials.

Our aim is typically to determine the dependence of the dependent variables as a function of the independent variables (e.g. $x(t)$).

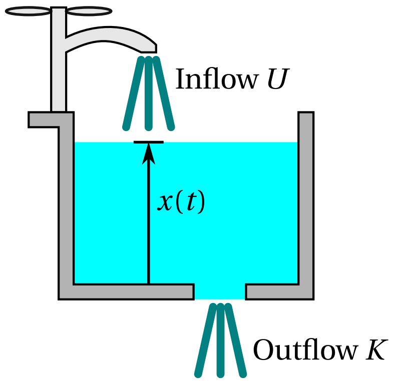
Introductory Examples

Introductory Examples

Lets look at some introductory examples.

Introductory Examples

Example 1: Water flowing into a tank



How do we predict the height of the water as time progresses?

Introductory Examples

Suppose the rate of flow is constant in time.

If we wanted to know the change in the water lever δx in a given time period δt, we could just write this as

$$\delta x = (U - K)\delta t$$

or equivalently

$$\frac{\delta x}{\delta t} = (U - K)$$

where the U is the inflow and K is the outflow.

Introductory Examples

Of course in reality the outflow does depend on the water level and even the inflow might depend on time.

However, for a very short time δt the equation

$$\frac{\delta x}{\delta t} = (U - K)$$

and in the limit $\delta t \to 0$ it becomes

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = (U - K)$$

a differential equation!

Introductory Examples

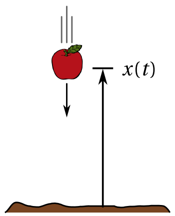
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = (U - K)$$

Note the Convention:

Typically we write differential equations such that the derivative is on the left of the equation, whereas everything else is on the right.

Introductory Examples

Example 2: Newton's Laws of Motion Consider an apple of mass m falling from the sky...



Using Newton's laws of motion we find

$$ma = -mg \qquad \Rightarrow \qquad \frac{\mathrm{d}\,v}{\mathrm{d}\,t} = -g \qquad \Rightarrow \qquad \frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = -g$$

Introductory Examples

Example 3: Newton's Law of Cooling

The rate of change of temperature is proportional to the temperature difference

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -\alpha(x - x_s)$$

x is the temperature of the object, x_s is the temperature of the surroundings and α is a material constant.

Solving ODEs: Direct Integration

Solving Differential Equations:
Direct Integration

Lets start thinking about how we solve differential equations.
The simplest equations we can integrate directly...

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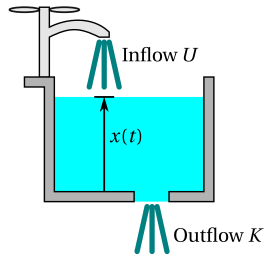
Solving ODEs: Direct Integration

Example 1: Water Tank



Lets consider the (simplified) case of a water tank with constant in and outflow.

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = U - K$$

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Solving ODEs: Direct Integration

We can solve this by direct integration.

$$\begin{aligned}
\frac{\mathrm{d}\,x}{\mathrm{d}\,t} &= U - K \\
x &= \int (U - K)\mathrm{d}t \\
x &= (U - K)t + C
\end{aligned}$$

To show the time-dependece of x explicitly we can write this as

$$x(t) = (U - K)t + C,$$

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Solving ODEs: Direct Integration

The solution is not surprising

$$x(t) = (U - K)t + C,$$

In time the water level x rises (or falls) linearly.

... but wait why is there still an arbitrary constant of integration in the equation?

By considering the solution at $t = 0$, we find

$$x(0) = C$$

and realize that C is actually the initial water level in the tank.

Solving ODEs: Direct Integration

General Solution and Initial Value Problem

Solutions that still contain constants of integration such as

$$x(t) = (U - K)t + C,$$

are called *general solutions*, they describe all possible solutions of a differential equation for arbitrary *initial conditions*.

If we are interested in the solution for particular initial conditions (say, $x(0) = 5$) then we can use these to determine the constants (here, $C = 5$) and we obtain a *particular solution*, such as

$$x(t) = (U - K)t + 5,$$

When we do this we say we solve the *initial value problem*.

Solving ODEs: Direct Integration

Exampercise

Lets look at a tank where we increase the inflow linearly in time

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = rt - K$$

with constant r, K.

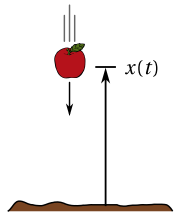
To Do

- Integrate both sides
- Write the solution $x(t) = \dots$

Solving ODEs: Direct Integration

Solving ODEs: Direct Integration

Example 2: Newton's Law of Motion



Lets solve the Newton's laws of motion for a free-falling object

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = -g$$

Solving ODEs: Direct Integration

In this case we need to integrate twice

$$\begin{aligned} \frac{d^2 x}{d t^2} &= -g \\ \frac{d x}{d t} &= -gt + C \\ x &= -\frac{gt^2}{2} + Ct + D \end{aligned}$$

The *general solution* is $x(t) = -\frac{gt^2}{2} + Ct + D$.

Solving ODEs: Direct Integration

In the solution there are two constants of integration C, D. That's good because we have 2 initial conditions to accommodate: the initial position and the initial velocity.

We found

$$x(t) = -\frac{gt^2}{2} + Ct + D.$$

Therefore,

$$x(0) = D,$$

that's the initial position.
Furthermore, we found $x'(t) = -gt + C$.
Thus, $x'(0) = C$. So, that's the initial velocity.

Solving ODEs: Direct Integration

Beware Notation!

On the last slide we used an alternative notation for the derivative

$$x'(0) = C.$$

You may see derivatives denoted like this x', x'', x''' or like this $\dot{x}, \ddot{x}, \dddot{x}$ or even like this x_t, x_{tt}, x_{ttt} but generally it is good to stick to one notation.
To indicate that the derivative is evaluated at 0 we could have written

$$\left.\frac{d x}{d t}\right|_{t=0} = \left.\frac{d x}{d t}\right|_0 = x'(0)$$

Solving ODEs: Solution by Inspection

Solving Differential Equations:
Solution by Inspection

Perhaps you could have guessed the solutions for some of the ODEs from the previous section straight away. We call this solving the ODE by inspection.

Solving ODEs: Solution by Inspection

Solution by inspection typically requires a great deal of intuition and experience.

We can sometimes help intuition along if we can "translate" what the function wants to tell us about itself. For example

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{3}{t}x$$

can be translated as

> *$x(t)$ says: when you differentiate me I lose one power of t but grow by a factor 3.*

The functions that lose a power when differentiated are polynomials.
Lets give it a try ...

Solving ODEs: Solution by Inspection

We try

$$x(t) = At^k$$

Substituting into

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{3}{t}x$$

yields

$$Akt^{k-1} = 3At^k/t \qquad \Rightarrow \qquad k = 3$$

Thus our guessed solution solves the ODE if $k = 3$, hence

$$x(t) = At^3$$

Solving ODEs: Solution by Inspection

Exampercise

Radioactive decay follows differential equations of the form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -3x$$

Find the general solution by inspection.

To Do:

- Translate what the function is trying to say
- What type of functions obey this rule
- Make a guess and see if it works

Solving ODEs: Solution by Inspection

Solving ODEs: Separation of Variables

Solving Differential Equations:
Separation of Variables

Actually, we could not have solved
the past two examples by direct integration.
But a closely related technique works: separation of variables.

Solving ODEs: Separation of Variables

Separation of Variables

Lets consider the previous example again

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -3x$$

If we try to integrate it directly we get

$$x(t) = -3\int x\mathrm{d}t$$

which we cannot solve because x is a function of t.
Instead we use a different trick ...

Solving ODEs: Separation of Variables

Start with

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -3x$$

Bring all x's to the left side all t's to the right: $\frac{1}{x}\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -3$.
Split up the differential operator and form two integrals

$$\int \frac{1}{x}\mathrm{d}x = \int -3\mathrm{d}t$$

Solve

$$\begin{aligned} \ln(x) + B &= -3t + C \\ x &= \mathrm{e}^{-3t-B+C} \\ x(t) &= A\mathrm{e}^{-3t} \end{aligned}$$

which gives us the desired result (after we defined $A = \mathrm{e}^{C-B}$).

Solving ODEs: Separation of Variables

So in summary ...

1. Separate the variables (the dependent variable wants to be with the differential operator)
2. Split the differential operator to form two integrals.
3. Solve.

(Note: Separation of variables always results in two integrals, which yield two constants of integration, but we can always drop one of them without loss of generality. We will use this from now on.)

Solving ODEs: Separation of Variables

Exampercise

Find the general solution for Newton's law of cooling

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -\alpha(x - x_s)$$

To Do:

1. Separate the variables
2. Split the operator
3. Solve.

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Solving ODEs: Separation of Variables

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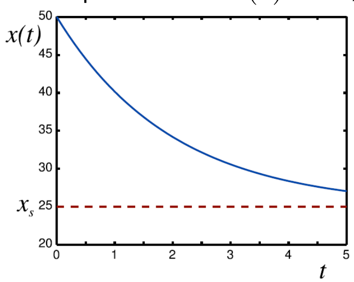
Solving ODEs: Separation of Variables

Solution of an initial value problem with $x(0) = 50$, $x_s = 25$, $\alpha = 0.5$



Verify for yourself.

(Also: Note the convention, we plot the independent variable on the horizontal axis and the dependent variable on the vertical axis.)

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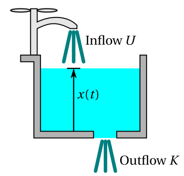
Revision

Bonus Level: Water flowing from a tank



Actually the outflow from the tank is not constant. By Torricelli's law or Bernoulli's principle one finds

$$K = A\sqrt{2gx}$$

So the differential equation (without inflow) is

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -A\sqrt{2gx}$$

Solve this equation by separation of variables.

Revision

Bonus Level 2: Geometric equation

We solved

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{3}{t}x$$

by direct inspection.

Can you also solve it by separation of variables?

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Revision

James 5th Edition

Read sections 10.1, 10.2, 10.4.1-2, 10.5.3

Try exercises 3, 4 and 6 from 10.4.5 and exercise 31 from 10.5.11

James 4th Edition

Read sections 10.1, 10.2, 10.4.1-2, 10.5.3

Try exercises 3, 4 and 6 from 10.4.5 and exercise 31 from 10.5.11