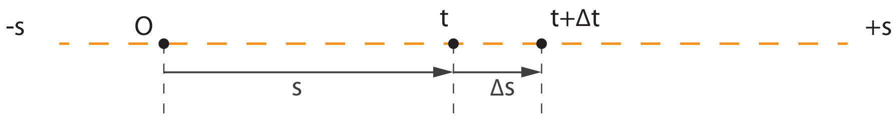
Handout 3 – Kinematics and Dynamics of Particles

Meriam & Kraige, Dynamics: 2/2–2/6, 3/2–3/4, 3/13

In order to study the **dynamics** of a particle under applied forces, first we must consider the **kinematics** of planar motion, and derive expressions for velocity and accelertions in different coordinate systems.

3.1 Kinematics: Rectilinear Motion

Consider a particle moving in a straight line, where its position at time t is described by its distance s from an origin O. Its position at time $t + \Delta t$ is $s + \Delta s$.



The instantaneous *velocity* v and *acceleration* a are defined as:

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{s} \tag{3.1}$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = \ddot{s} \tag{3.2}$$

An alternative formulation can be found via calculus:

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v = \frac{d}{ds}\left(\frac{v^2}{2}\right) \qquad \to \qquad a\,ds = v\,dv \tag{3.3}$$

Often the accelerations will be known, and the velocities and positions are found through integration.

constant acceleration In general acceleration is *not* constant, but the special case is useful (for example motion under gravity, where the acceleration $a = g$):

$$\frac{dv}{dt} = a \qquad \to \qquad dv = a\,dt$$

$$\int_{v_0}^{v} dv = a \int_0^t dt \qquad \to \qquad v = v_0 + at \tag{3.4}$$

where v_0 is v at $t = 0$.

$$\frac{ds}{dt} = v = v_0 + at \qquad \to \qquad ds = (v_0 + at)\,dt$$

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + at)\,dt \qquad \to \qquad s = s_0 + v_0 t + \frac{1}{2}at^2 \tag{3.5}$$

where s_0 is s at $t = 0$.

$$v\frac{dv}{ds} = a \qquad \rightarrow \qquad v\,dv = a\,ds$$

$$\int_{v_0}^{v} v dv = a\int_{s_0}^{s} ds \qquad \rightarrow \qquad v^2 = v_0^2 + 2a\,(s - s_0) \tag{3.6}$$

NB: these 'suvat' equations only apply to the case of a constant acceleration, and it is a common mistake to use them for other situation!

non-constant acceleration We consider three cases of non-constant acceleration:

- acceleration varies as *function of time*: $a = f(t) = dv/dt$

$$\int_{v_0}^{v} dv = \int_{0}^{t} f(t)dt \qquad \rightarrow \qquad v = v_0 + \int_{0}^{t} f(t)dt$$

$$\int_{s_0}^{s} ds = \int_{0}^{t} v(t)dt \qquad \rightarrow \qquad s = s_0 + \int_{0}^{t} v(t)dt$$

- acceleration varies as *function of velocity*: $a = f(v) = dv/dt$

$$t = \int_{0}^{t} dt = \int_{v_0}^{v} \frac{dv}{f(v)}$$

  which would require solving for v as a function of t before integrating to find s as a function of t.

  Alternatively, use $v\,dv = a\,ds = f(v)ds$ to find:

$$\int_{v_0}^{v} \frac{v}{f(v)} dv = \int_{s_0}^{s} ds \qquad \rightarrow \qquad s = s_0 + \int_{v_0}^{v} \frac{v}{f(v)} dv$$

  without explicit reference to t.

- acceleration varies as *function of displacement*: $a = f(s)$, using $v\,dv = a\,ds = f(s)ds$ to find:

$$\int_{v_0}^{v} v dv = \int_{s_0}^{s} f(s)ds \qquad \rightarrow \qquad v^2 = v_0^2 + 2\int_{s_0}^{s} f(s)ds$$

  Solve for v as function of s, with $v = g(s)$ and integrate:

$$\int_{s_0}^{s} \frac{ds}{g(s)} = \int_{0}^{t} dt \qquad \rightarrow \qquad t = \int_{s_0}^{s} \frac{ds}{g(s)}$$

In cases where accelerations are a function of more than one variable, the solution is often found numerically as analytical solutions become intractable; see Example 2.

Example 3.1 – Aircraft take-off

Q: On take-off, an aircraft starts from rest and accelerates according to $a = a_0 - kv^2$ where a_0 is the constant acceleration resulting from engine thrust and $-kv^2$ is acceleration due to aerodynamic drag. If $a_0 = 2$ m/s^2, $k = 4 \cdot 10^{-5}$ m^{-1}, determine the length of runway required to reach take-off speed of 250 km/h.

A: Acceleration is a function of velocity, and therefore:

$$v\,dv = \left(a_0 - kv^2\right) ds$$

which is integrated as

$$\int_0^v \frac{v}{a_0 - kv^2} dv = \int_0^s ds$$

to find

$$s = -\frac{1}{2k} \left[\ln\left(a_0 - kv^2\right) - \ln\left(a_0\right)\right] = 1266 \text{ m}$$

A simple sanity check is provided by assuming a constant acceleration, thereby neglecting the drag term:

$$s = s_0 + \frac{1}{2a}\left(v^2 - v_0^2\right) = 1204 \text{ m}$$

Example 3.2 – Numerical Integration

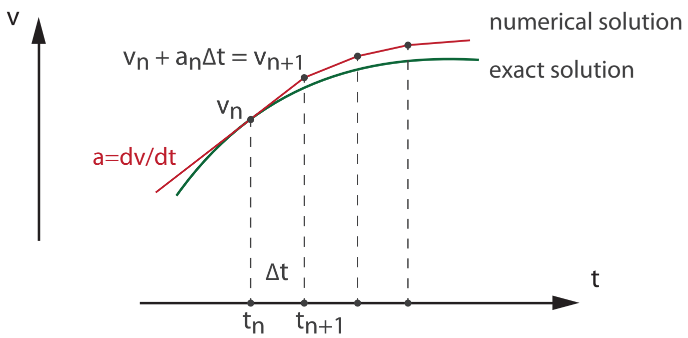
For cases where the acceleration is a more complicated function, with no known analytical solution, numerical integration methods are required to find the position and velocity as a function of time. In numerical methods, the solutions are evaluated at discrete points in time, t_n, and the solutions at the next time step, t_{n+1}, are *approximated* from the current acceleration a_n and velocity v_n.

The simplest numerical integration technique is known as the *Forward Euler* method. Starting from an initial position x_0 and velocity v_0, the subsequent values can be calculated as follows:

$$v_{n+1} = v_n + a_n \Delta t$$

$$x_{n+1} = x_n + v_n \Delta t$$

where $t_{n+1} = t_n + \Delta t$, and the acceleration $a_n = f\left(x_n, v_n, t_n\right)$. This is then repeated, to find the velocities and positions as a function of time.

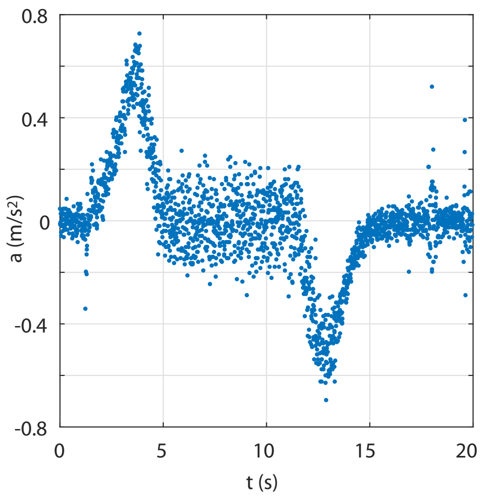


Note that this numerical integration method (Forward Euler) is highly sensitive to the chosen step size Δt, and the solution is only accurate for very small time steps. More sophisticated numerical integration methods overcome this issue, but are the topic of engineering mathematics later this year.

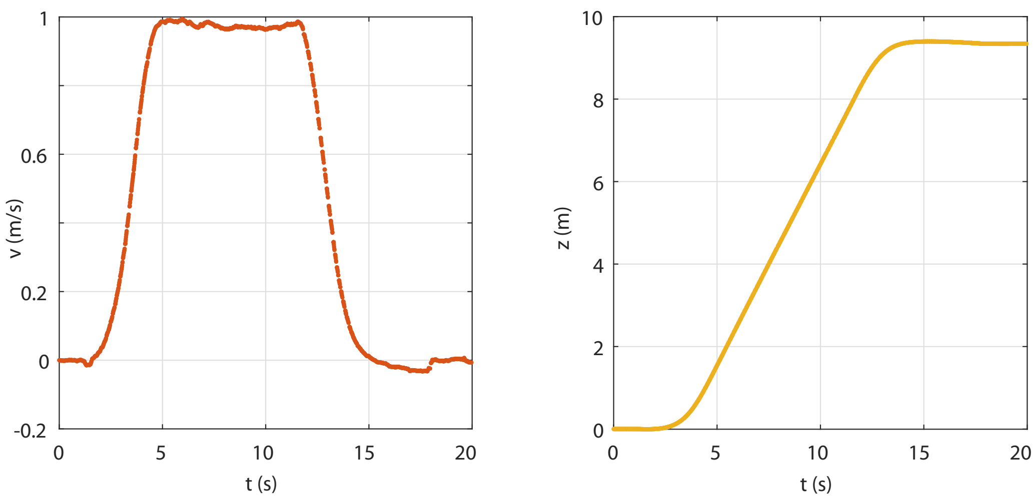
Example 3.3 – Numerical Integration – Height of Queen's Building

Q: How can you calculate the height of Queen's Building, using your mobile phone and some knowledge of mechanics?

The built-in accelerometers provide the acceleration of the lift as a function of time:



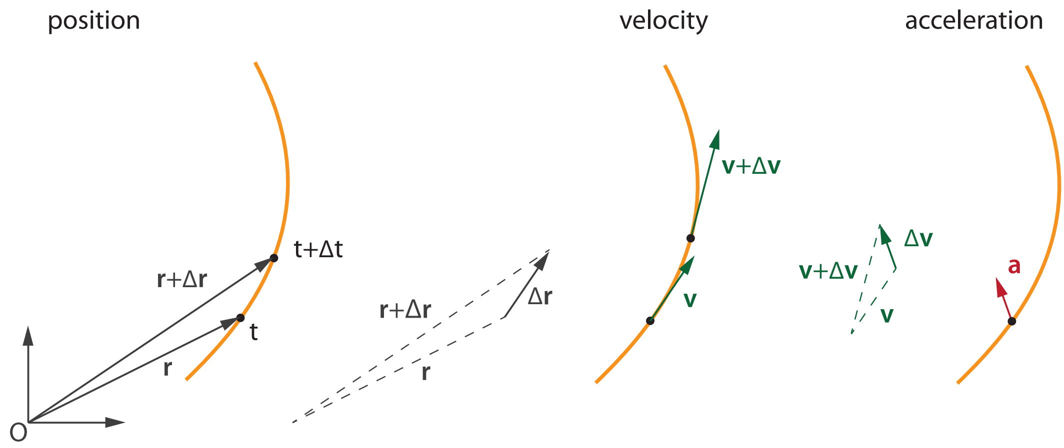
Although the signal is noisy, it can be integrated to find the velocity and displacement of the lift:



Thus, the distance between ground and second floor in Queen's Building is approximately 9.5 metres.

3.2 Kinematics: Planar Curvilinear Motion

Consider a particle moving along a curved 2D path, whose position at time t can be described by a *vector* $\boldsymbol{r}$ from the origin O. At time $t + \Delta t$ the new position of the particle is given by $\boldsymbol{r} + \Delta \boldsymbol{r}$.



Following the same approach as for rectilinear motion, the *velocity vector* $\boldsymbol{v}$ is:

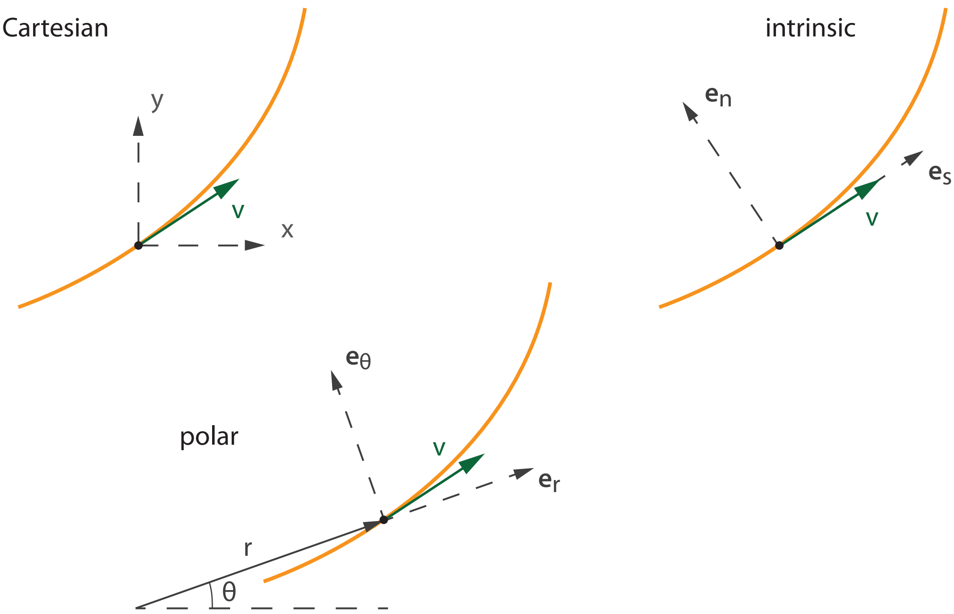
$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} \tag{3.7}$$

which will therefore be *tangent to the curve*. The *acceleration vector* $\boldsymbol{a}$ is given by:

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt} \tag{3.8}$$

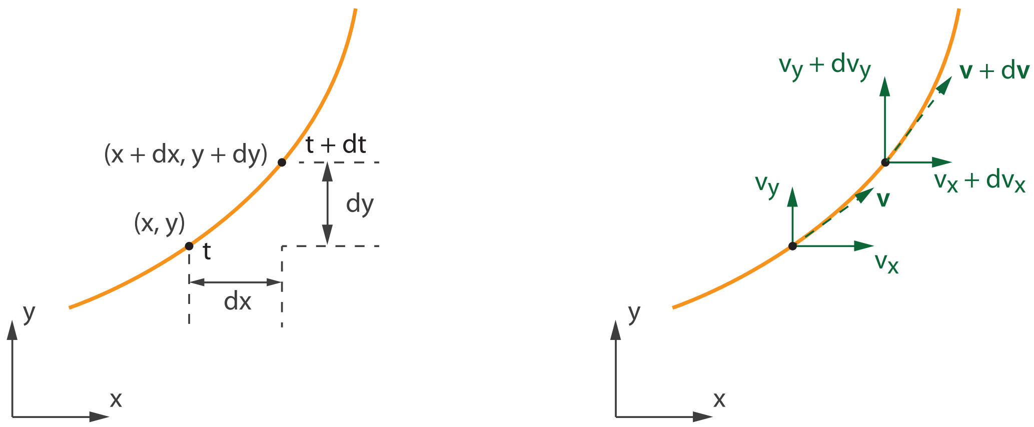
Note that that the acceleration vector has components normal and tangent to the path, as it consists of both a change in velocity along the path and a change in direction of the velocity vector. However, the normal component of the acceleration must always point towards the local centre of curvature of the path.

The planar motion of a particle can be expressed in different coordinate systems: **Cartesian**, **intrinsic** and **polar**. The choice of coordinate system may depend on the type of forces being applied, the kinematic constraints or demands of specific applications. Note that they all describe the same motion, and velocities and accelerations can therefore be converted between coordinate systems.



3.2.1 Cartesian Coordinates

Let the position of the particle along its path be described by the Cartesian coordinates x and y with respect to a fixed origin.



The velocities and accelerations are:

$$v_x = \frac{(x + dx) - x}{dt} = \frac{dx}{dt} = \dot{x}$$
$$v_y = \frac{(y + dy) - y}{dt} = \frac{dy}{dt} = \dot{y}$$

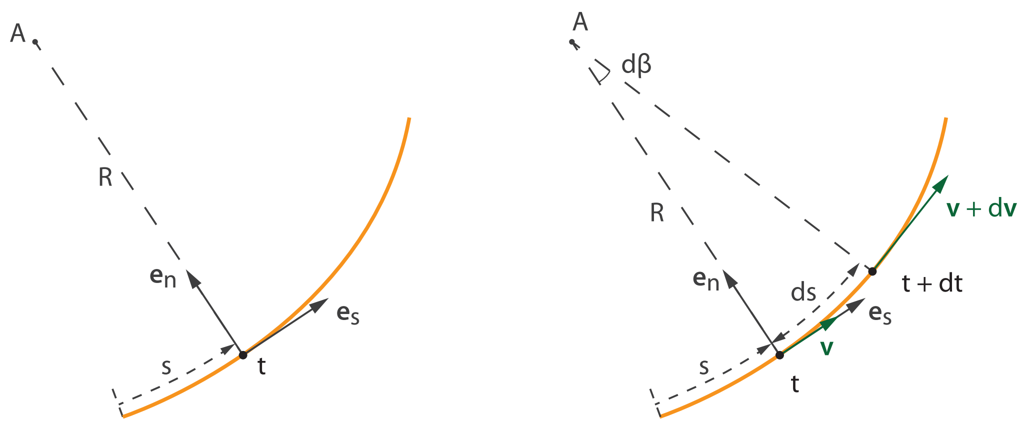
and accelerations:

$$a_x = \frac{(v_x + dv_x) - v_x}{dt} = \frac{dv_x}{dt} = \ddot{x}$$
$$a_y = \frac{(v_y + dv_y) - v_y}{dt} = \frac{dv_y}{dt} = \ddot{y}$$

Note that superimposing a uniform velocity in any direction has no effect on the accelerations.

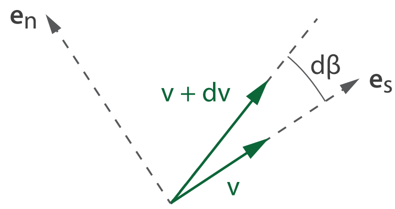
3.2.2 Intrinsic Coordinates

In the intrinsic (or, natural) coordinate system, let the position of the particle be described by coordinate s measured along the path. The motion can then be described in terms of a coordinate system defined by unit vectors $\boldsymbol{e}_s$, a local tangent to the path, and $\boldsymbol{e}_n$, a local normal to the path (and positive towards the local centre of curvature, A, with radius of curvature R).



The velocity v_s is along (or tangent) to the path and there is no velocity component normal to the path.

$$v_s = \frac{ds}{dt} = \dot{s}$$
$$v_n = 0$$



The acceleration in the s direction is the change in velocity projected onto the $\boldsymbol{e}_s$ vector:

$$a_s = \frac{(v+dv)\cos d\beta - v}{dt} \approx \frac{dv}{dt} = \dot{v} = \ddot{s}$$

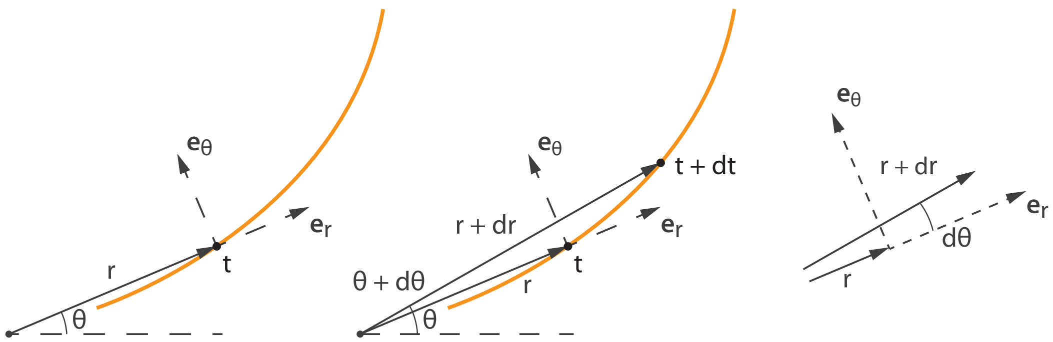
where use is made of small angle approximation, $\cos d\beta \approx 1$. Similarly for acceleration in n direction:

$$a_n = \frac{(v+dv)\sin d\beta - 0}{dt} \approx \frac{v d\beta}{dt} = \frac{v}{dt}\frac{ds}{R} = \frac{v}{R}\frac{ds}{dt} = \frac{v^2}{R}$$

where the small-angle approximation gives $\sin d\beta \approx d\beta$, higher-order terms $(d\beta\, dv)$ are neglected, and geometry gives $ds = R\, d\beta$. This acceleration term a_n is called *centripetal acceleration*, and is positive in the direction of the centre of curvature. The resultant of the two accelerations, normal and tangential, must therefore always lie 'inside' the curve.

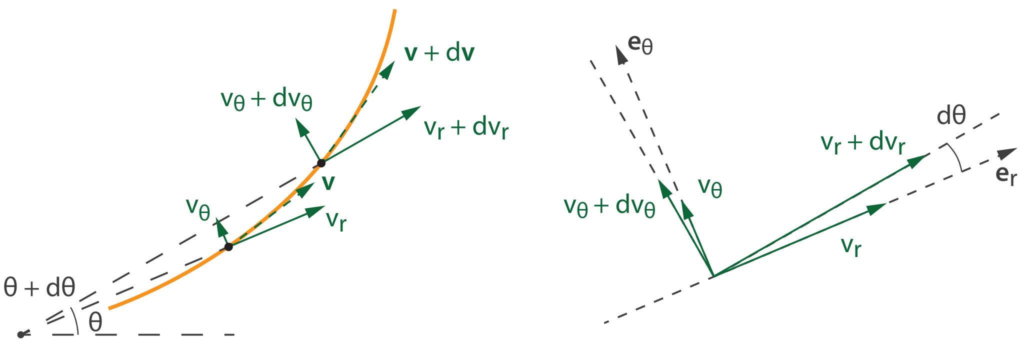
3.2.3 Polar Coordinates

Lastly, consider the polar coordinate system $(\mathbf{e}_r, \mathbf{e}_\theta)$ which is located at a distance r along a line at an angle θ with respect to a fixed line of reference.



The velocity in polar coordinate is given as:

$$v_r = \frac{(r+dr)\cos d\theta - r}{dt} \approx \frac{dr}{dt} = \dot{r}$$
$$v_\theta = \frac{(r+dr)\sin d\theta}{dt} \approx \frac{r d\theta}{dt} = r\dot{\theta}$$



For the acceleration a_r consider projection of $v_r + dv_r$ and $v_\theta + dv_\theta$ along the r direction, using small angle assumptions and neglecting higher-order terms.

$$
\begin{aligned}
a_r &= \frac{(v_r + dv_r)\cos d\theta - (v_\theta + dv_\theta)\sin d\theta - v_r}{dt} \approx \frac{dv_r - v_\theta d\theta}{dt} \\
&= \frac{dv_r}{dt} - v_\theta \frac{d\theta}{dt} = \frac{d^2 r}{dt^2} - r\frac{d\theta}{dt}\frac{d\theta}{dt} \\
&= \ddot{r} - r\dot{\theta}^2
\end{aligned}
$$

and similarly for a_θ we find

$$
\begin{aligned}
a_\theta &= \frac{(v_r + dv_r)\sin d\theta + (v_\theta + dv_\theta)\cos d\theta - v_\theta}{dt} \approx \frac{dv_\theta + v_r d\theta}{dt} \\
&= \frac{dv_\theta}{dt} + v_r\frac{d\theta}{dt} = \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) + \frac{dr}{dt}\frac{d\theta}{dt} = r\frac{d^2\theta}{dt^2} + \frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt} \\
&= r\ddot{\theta} + 2\dot{r}\dot{\theta}
\end{aligned}
$$

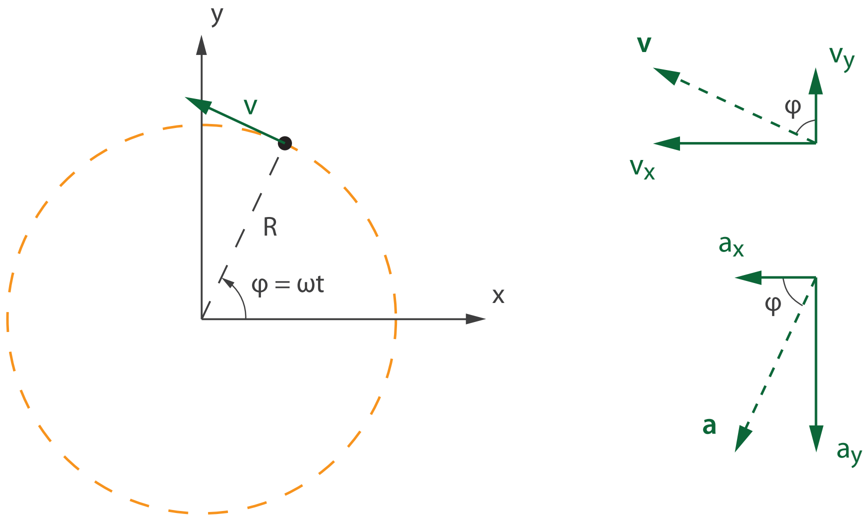
The additional terms $2\dot{r}\dot{\theta}$, known as the Coriolis terms, are due to the rotation of the coordinate system.

Coordinate System	**Velocities**	**Accelerations**
Cartesian (x, y)	$v_x = \dot{x}$ $v_y = \dot{y}$	$a_x = \ddot{x}$ $a_y = \ddot{y}$
Intrinsic (s, n)	$v_s = \dot{s}$ $v_n = 0$	$a_s = \ddot{s}$ $a_n = \frac{v_s^2}{R}$
Polar (r, θ)	$v_r = \dot{r}$ $v_\theta = r\dot{\theta}$	$a_r = \ddot{r} - r\dot{\theta}^2$ $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

The velocity vector $\boldsymbol{v}$ and acceleration vector $\boldsymbol{a}$ at each point along the path can be expressed in each of these coordinate systems, by resolving into the local components.

Example 3.4 – Circular Motion

Different coordinate systems are appropriate for different problems. Consider the uniform motion of a particle around a circle, with a constant speed $v = \omega R$.



The angle φ varies as a function of time $\varphi = \omega t$. The velocity vector $\boldsymbol{v}$ is resolved into **Cartesian** components:

$$\dot{x} = -v \sin \omega t$$
$$\dot{y} = v \cos \omega t$$

which leads to accelerations:

$$\ddot{x} = -v\omega \cos \omega t$$
$$\ddot{y} = -v\omega \sin \omega t$$

Both components are negative, and the ratio of the two magnitudes is $\ddot{y}/\ddot{x} = \sin \omega t / \cos \omega t = \tan \varphi$. This means that the resultant acceleration points towards the origin O with magnitude:

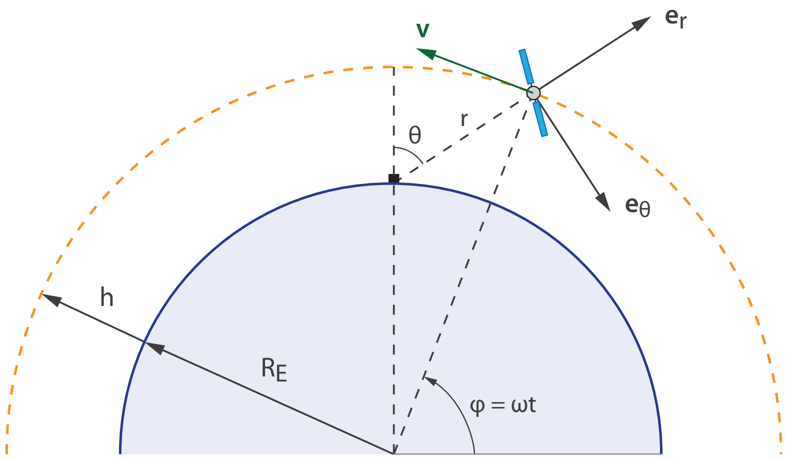
$$a = \sqrt{(-v\omega \cos \omega t)^2 + (-v\omega \sin \omega t)^2} = v\omega \sqrt{\cos^2 \omega t + \sin^2 \omega t}$$
$$= v\omega = \frac{v^2}{R} = \omega^2 R$$

Approaching the problem from an **intrinsic** coordinate system, we instantly note that the tangential acceleration is zero, $a_s = 0$, as the tangential velocity is constant. The centripetal acceleration a_n is:

$$a_n = \frac{v^2}{R} = \omega^2 R$$

Example 3.5 – Satellite Groundstation

A satellite in orbit transmits data down to a ground station. In order to receive the best signal, the ground station needs to continuously point at the satellite. In solving the problem, we assume a circular orbit at an altitude $h = 600$ km, planar motion (the satellite passes directly overhead of the target) and that the Earth is stationary. The question is to find the angular acceleration $\ddot{\theta}$ required for the ground station antennas to track the satellite.



The velocity of the spacecraft is given as (and will be derived later in this handout):

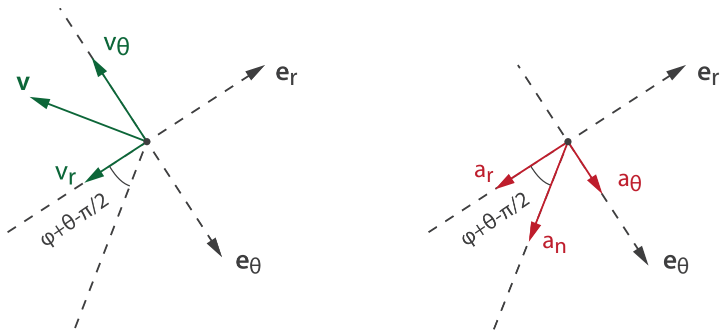
$$v = \sqrt{\frac{GM}{R_E + h}} \approx 7.56 \text{ km/s}$$

The zenith angle θ is measured with respect to the local nadir vector. The distance r between the satellite and ground station can be determined from geometry:

$$r = (R_e + h)\,\frac{\cos\varphi}{\sin\theta}$$

where θ is a function of φ (and thus of time t) as follows:

$$\tan\theta = \frac{\cos\varphi\,(R_E + h)}{\sin\varphi\,(R_E + h) - R_E}$$



Using a polar coordinate system, and resolving the velocity v onto the $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ unit vectors,

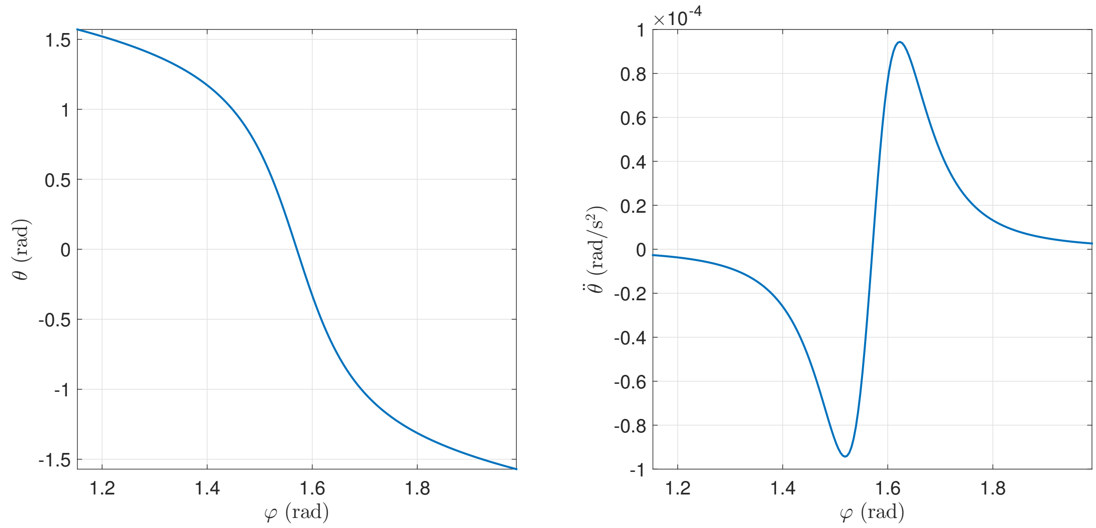
$$v_\theta = -\cos(\varphi + \theta - \pi/2)\,v = r\dot{\theta}$$
$$v_r = -\sin(\varphi + \theta - \pi/2)\,v = \dot{r}$$

it becomes possible to calculate the angular velocity $\dot{\theta}$ and $\dot{r}$ as a function of time.

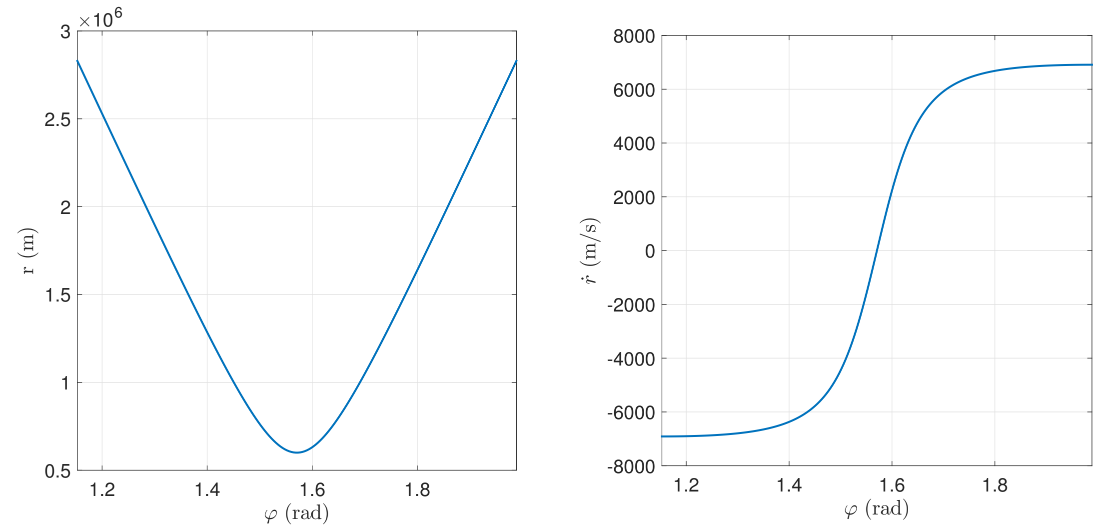
Next, resolving the centripetal acceleration a_n in the polar coordinates

$$\begin{aligned} a_\theta &= \sin\left(\varphi + \theta - \pi/2\right) a_n = \sin\left(\varphi + \theta - \pi/2\right) \frac{v^2}{R_E + h} \\ &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{aligned}$$

allows us to calculate the angular acceleration $\ddot{\theta}$ required for the antenna pointing mechanism.



Can you think of a reason why the distance r and $\dot{r}$ might be relevant for radio communications?



3.3 Dynamics: Newton's Laws of Motion

In his *Philosophiae Naturalis Principia Mathematica* (1687), Sir Isaac Newton outlined three laws of motion, which are here translated from the original (translation: Andrew Motte):

Law I *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*

Law II *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

Law III *To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

In a more modern formulation, using linear momentum $\boldsymbol{p} = m\boldsymbol{v}$, Newton's second law reads:

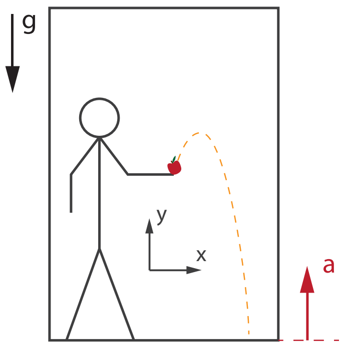
$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d\,(m\boldsymbol{v})}{dt} = m\,\boldsymbol{a} \tag{3.9}$$

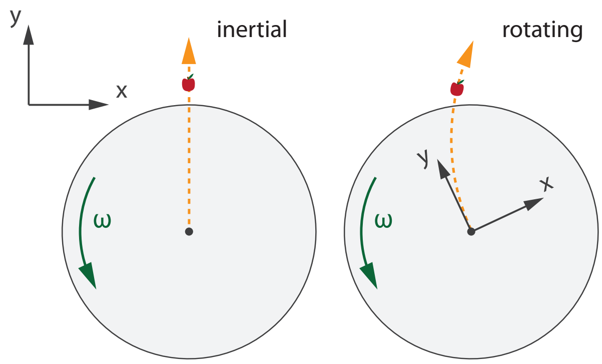
This equation forms the basis of what is known as Newtonian mechanics.

Inertial reference frame It is crucial to note that Newtonian mechanics is only valid when velocities are much smaller than the speed of light ($c = 3 \cdot 10^8$ m/s) and with respect to an inertial reference frame. An inertial reference frame is a set of axes assumed to have no translation or rotation in space. Any reference frame that is in uniform motion with respect to an inertial frame is also an inertial frame.

Consider two examples of non-inertial reference frames:

- A person standing in a lift, moving upwards with constant acceleration a, throws an apple. Viewed in an inertial reference frame it falls as expected, but for the observer it seems to experience a greater acceleration of $g + a$. In order to account for the observed behaviour, the observer requires the existence of an additional, *fictitious* force, which is a result of the accelerating reference frame.
- A person standing at the centre of a rotating platform, throws an apple radially outward. For an observer in an inertial reference frame, the apple will simply travel in a straight line. However, for the rotating observer, the apple will appear to curve away, as a result of the rotating reference frame. The rotating observer would again require the existence of fictitious forces (the Coriolis and centrifugal forces) in order to account for the observation.





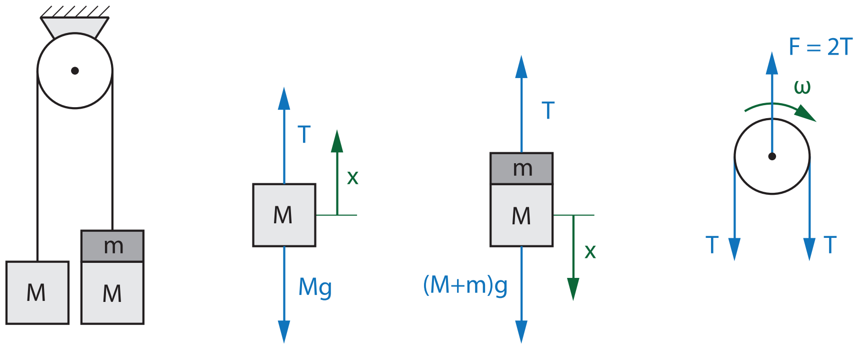
In this unit we shall focus only on mechanics problems in an inertial reference frame[1], and not consider the **fictitious forces** required in moving and rotating reference frames to account for observed effects.

[1]Note that an Earth-fixed coordinate system, as we use for most of our analyses, is merely an *approximation*. Consider the centripetal acceleration of an object on the equator at a radius $R = 6.371 \cdot 10^6$ m. The rotational period $T = 24 \cdot 3600$ s, giving a rotation rate of $\omega = \frac{2\pi}{T} \approx 7.27 \cdot 10^{-5}$ rad/s. The centripetal acceleration is $a_n = \omega^2 R = 0.0337$ m/s^2. Comparing this number to the gravity acceleration, $g = 9.81$ m/s^2, this effect can generally be neglected.

Example 3.6 – Atwood Machine & Balance

The Atwood Machine is a classic experiment deviced by George Atwood (1746-1807), a tutor at Trinity College Cambridge, and has been taught to many generations of students. A mass M and a combined mass $M+m$ are connected by a string and slung over a massless and frictionless pulley.

Q: What is the acceleration of the system of masses?



Drawing the FBD of the masses isolates the unknown tension T in the string, which is constant along its length as the pulley is massless and frictionless. From the FBDs, tension T is bounded by:

$$Mg < T < (M+m)g$$

The tension T is found by considering the equations of motion of the two sets of masses

$$\sum F_x : \qquad T - Mg = M\ddot{x}$$

$$\sum F_x : \qquad (M+m)\,g - T = (M+m)\,\ddot{x}$$

which are combined to solve for T and $\ddot{x}$.

However, the problem can be understood more intuitively. The net force on the combined system of masses is due to the additional mass m. A single equation of motion can therefore be written as:

$$\sum F_x : \qquad mg = (2M+m)\,\ddot{x}$$

to give the acceleration as:

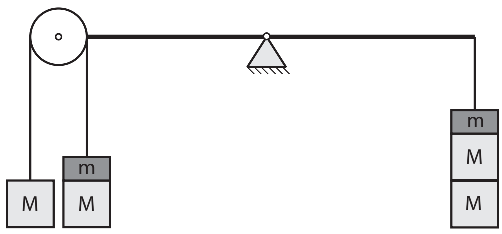
$$\ddot{x} = \frac{mg}{2M+m}$$

and tension T as:

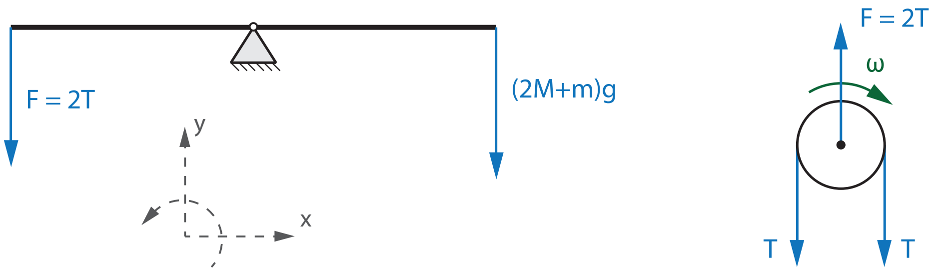
$$T = \frac{2Mg\,(M+m)}{2M+m}$$

Balance The Atwood balance is an elegant variation on the original device. A balance supports an Atwood machine on the left-hand side, and a total mass of $2M + m$ on the right hand. If the pulley on the Atwood machine is prevented from rotating, the balance is in static equilibrium.

Q: What happens if the pulley is released? Does the Atwood balance: (a) stay level, (b) rotate clockwise, (c) rotate counterclockwise?



Looking at the FBD of the pulley, it exerts a downward force $F = 2T$ on the balance.

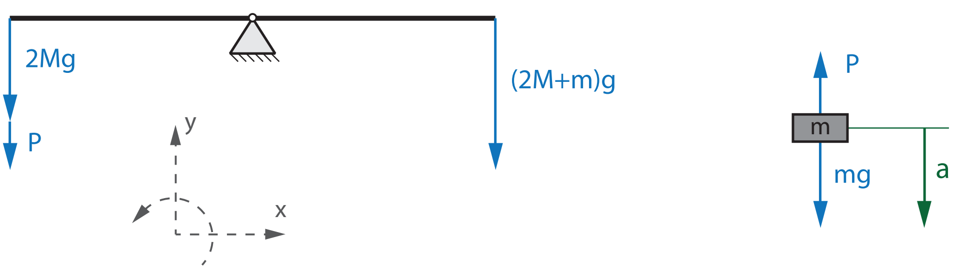


Consider the moments around the pivot:

$$\sum M: \qquad F\frac{L}{2} - (2M+m)\,g\frac{L}{2} = \frac{L}{2}\left(\frac{4Mg\,(M+m)}{2M+m} - (2M+m)\,g\right)$$
$$= -\frac{L}{2}\frac{m^2 g}{2M+m}$$

This means there is a resulting CW torque, and the left-hand side will move upwards.

A more intuitive interpretation can again be formulated. As the masses in the Atwood machine move, the combined centre of mass of the two masses M will remain stationary. It is therefore the motion of mass m that changes the location of the centre of mass.



The upward force on the pulley is therefore *reduced* by:

$$\Delta F = m\,\ddot{x} = m\,\frac{mg}{2M+m}$$

and therefore causes the balance to rotate clockwise.

Example 3.7 – Centripetal Forces

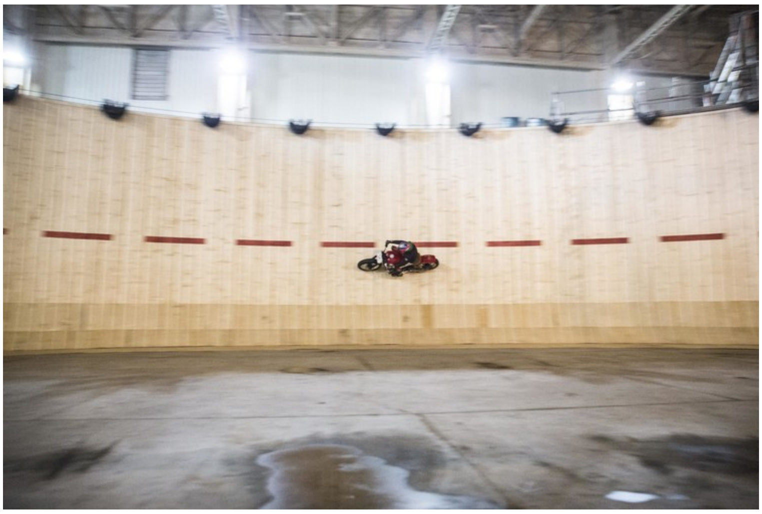
Consider a particle with mass m attached to a string of length R, and the particle moves in a circular motion with constant velocity v. Using an intrinsic coordinate system, the centripetal acceleration is found as:

$$a_n = \frac{v^2}{R}$$

A force is required to achieve achieve this centripetal acceleration, and is equal to the tension T in the string:

$$T = m\, a_n = m\frac{v^2}{R}$$

This centripetal force can be surprisingly large.



In April 2016, the racing driver Guy Martin drove a motor bike at almost 80 mph around a 37 m diameter 'Wall of Death'. During this world record attempt, he experienced a centripetal acceleration of approximately:

$$a_n = \frac{v^2}{R} = \frac{(80 \cdot 1.61/3.6)^2}{37/2} \approx 69 \text{ m/s}^2 \approx 7g$$

The biggest risk was therefore of him blacking out due to the high G-forces as he sped around the wall!

Problems in dynamics The two examples highlight two general types of problems in dynamics of particles:

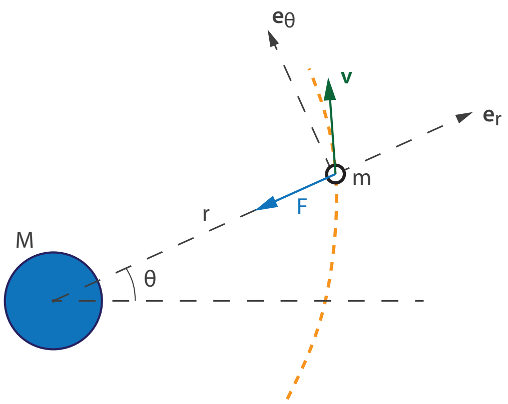
- the applied forces are known, and the equations of motion are formulated (and integrated);
- the resulting motion is known, and the required forces are calculated.

3.4 Orbital Mechanics

Consider a satellite orbiting the Earth under the force of the central gravitational attraction:

$$F = G\frac{Mm}{r^2} \tag{3.10}$$

where G is the universal gravitational constant, M the mass of the Earth, m the mass of the satellite, and r the distance between the centres of mass of the two bodies. It is assumed that the Earth is fixed in space (providing us with an inertial reference frame).



Using polar coordinates:

$$\sum F_r : \qquad -G\frac{mM}{r^2} = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$\sum F_\theta : \qquad 0 = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

we find the governing differential equations for an orbit. The solutions will be discussed in detail in your second year Space Systems course, and are described by conic sections (circle, ellipse, parabola, hyperbola).

Circular Orbit For the specific case of a circular orbit (r is constant):

$$\dot{\theta}^2 = \frac{GM}{r^3}$$

where with $\dot{\theta} = v/r$ we find:

$$v = \sqrt{\frac{GM}{r}} \tag{3.11}$$

Note that as r increases, the velocity v decreases! The orbital period is given as:

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM}}$$

Micro-gravity Contrary to popular conception, an object in orbit is not weightless. The weight due to gravity is given as:

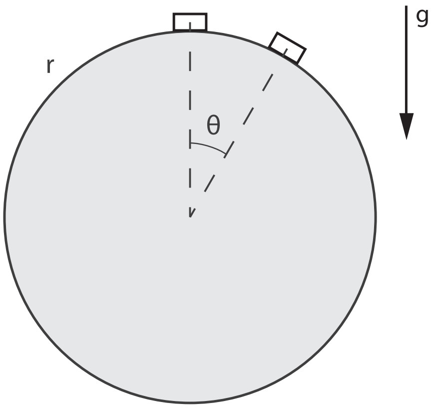
$$a = \frac{GM}{r^2} = \frac{GM}{(R+h)^2}$$

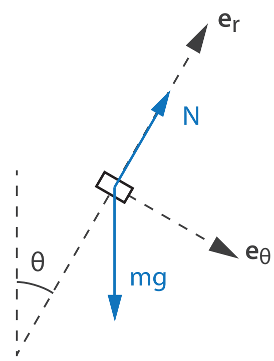
where R is the radius of the Earth, and h is the orbit altitude. The International Space Station (ISS) is in a near-circular orbit of 410 km altitude. The gravity constant at this altitude, as a percentage of g at the Earth surface, is:

$$\frac{a}{g} = \left(\frac{R}{R+h}\right)^2 = 88\%$$

Example 3.8 – Particle on Cylinder

A particle slides (frictionless) from the top of a cylinder. At what angle will it lose contact with the surface?





Using a polar coordinate system, consider the dynamics in radial and tangential direction:

$$\sum F_r : \qquad N - mg\cos\theta = m\left(\ddot{r} - \dot{\theta}^2 r\right)$$

$$\sum F_\theta : \qquad mg\sin\theta = m\left(\ddot{\theta} r + 2\dot{r}\dot{\theta}\right)$$

As long as the particle remains in contact with the cylinder, the distance r remains constant ($\ddot{r} = \dot{r} = 0$).

Let us integrate the angular acceleration:

$$\ddot{\theta}\, d\theta = \dot{\theta}\, d\dot{\theta}$$

$$\frac{g\sin\theta}{r}\, d\theta = \dot{\theta}\, d\dot{\theta}$$

$$\frac{g}{r}\int_0^\theta \sin\theta\, d\theta = \int_0^{\dot{\theta}} \dot{\theta}\, d\dot{\theta}$$

$$\frac{2g}{r}(1-\cos\theta) = \dot{\theta}^2$$

to find the angular velocity as a function of the angular position of the particle (as long as it remains in contact).

At a certain angle θ the normal force N will become zero, as the radial gravity component is no longer sufficient to maintain the centripetal acceleration. From the original equations of motion, we find that at the point of loss of contact:

$$g\cos\theta = \dot{\theta}^2 r$$

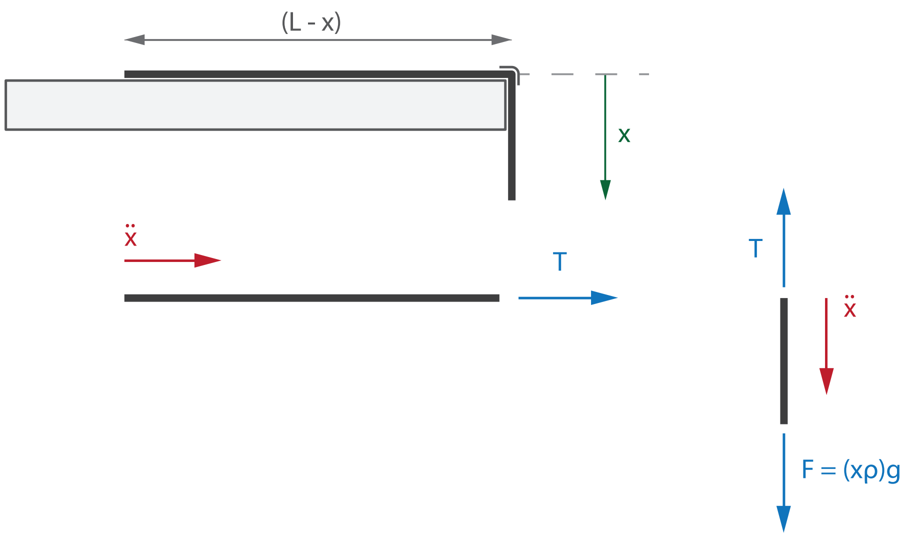
Substituting this result into the expression for angular velocity

$$\frac{2g}{r}(1-\cos\theta) = \frac{g\cos\theta}{r}$$

we find that at $\cos\theta = 2/3$, *i.e.* $\theta \approx 48.2°$, the particle loses contact with the cylinder surface.

Example 3.9 – Sliding String

Consider a string of length L, and mass per unit length ρ, dangling of a fricionless table. What is the acceleration of the string?



The equation of motion in x for each of the parts in the FBD are written as

$$T = (L - x)\,\rho\ddot{x}$$
$$x\rho g - T = x\rho\ddot{x}$$

which can be combined into a single equation by eliminating T to yield the equation of motion:

$$\ddot{x} - \frac{g}{L}x = 0$$

The general solution takes the form:

$$x = C_1 e^{\sqrt{g/L}\,t} + C_2 e^{-\sqrt{g/L}\,t}$$

with initial conditions that at $t = 0$, $x = x_0$ and $\dot{x} = 0$ gives $C_1 = C_2 = x_0/2$. Therefore, the final solution is

$$x = x_0 \frac{e^{\sqrt{g/L}\,t} + e^{-\sqrt{g/L}\,t}}{2} = x_0 \cosh\left(\sqrt{g/L}\,t\right)$$

While this problem has a (non-trivial) analytical solution to the differential equation, in many dynamics problem the solution to the differential equation is found numerically; see Example 2.

Revision Objectives Handout 3:

Planar Kinematics of Particles

- express the relationships between displacement s, velocity v, acceleration a (including $a\,ds = v\,dv$)
- solve for velocity and displacement in cases of constant and non-constant acceleration
- stop relying on suvat and learn to love integration
- recognise which coordinate systems are most suited to particular problems
- recall the expressions for velocities and accelerations in Cartesian, intrinsic and polar coordinates
- convert velocities and accelerations between different coordinate systems

Cartesian (x, y)	$v_x = \dot{x}$ $v_y = \dot{y}$	$a_x = \ddot{x}$ $a_y = \ddot{y}$
Intrinsic (s, n)	$v_s = \dot{s}$ $v_n = 0$	$a_s = \ddot{s}$ $a_n = \frac{v_s^2}{R}$
Polar (r, θ)	$v_r = \dot{r}$ $v_\theta = r\dot{\theta}$	$a_r = \ddot{r} - r\dot{\theta}^2$ $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Note: solving for accelerations using numerical integration is **not** examinable.

Dynamics of Particles

- recall Newton's Laws of Motion
- recognise the importance of an inertial reference frame
- set up equations of motion for particles under applied loads (in a suitable coordinate system)
- integrate equations of motion for simple cases
- recall the equation for gravitational force ($F = GMm/r^2$)
- derive the orbital velocity and period for a particle in a circular orbit