# Vibrations 2, Lecture 5
# Friction

Dr Brano Titurus
brano.titurus@bristol.ac.uk



## Lecture 4

Solutions of the characteristic equation

$$s_{1,2} = -\frac{c}{2m} \pm \left( \underline{\frac{c^2}{4m^2} - \frac{k}{m}} \right)^{1/2}$$

Underdamped vibration

$$s_{1,2} = -\zeta\omega_0 \pm i\,\omega_0\sqrt{1-\zeta^2}$$

$$x(t) = C\,e^{-\zeta\omega_0 t}\cos(\underline{\omega_D}\,t - \phi)$$

$$x(t) = e^{-\zeta\omega_0 t}(A\cos(\omega_D\,t) + B\sin(\omega_D\,t))$$

New dynamic parameters

$$\omega_D = \omega_0\sqrt{1-\zeta^2}$$

$$\zeta = \frac{c}{2\sqrt{mk}}$$

Logarithmic decrement and damping

$$\zeta_{\exp} \approx \frac{1}{2\pi N}\ln\left(\frac{x(t_1)}{x(t_1 + NT_D)}\right)$$

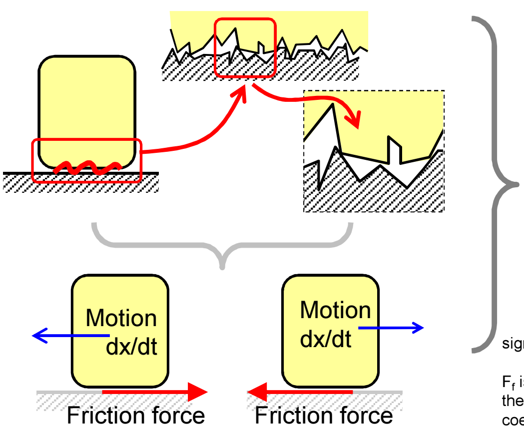
# Lecture 5

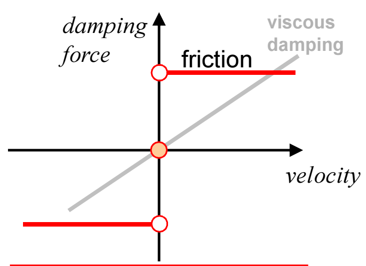
- Coulomb model of dry friction
- Free vibration with friction
- Solved example



# Coulomb model of dry friction

**Coulomb model of dry friction** (Coulomb friction; or simply *friction*) is the simplest friction model. Friction is the resistance to the *relative motion* between two objects due to uneven contacting surfaces. This simple model is represented by the *constant motion-opposing force*.




$$F_{D,Coulomb} = \text{sign}(\dot{x})\, F_f$$

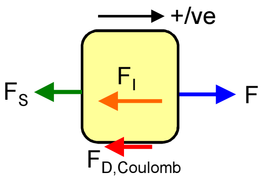
sign(…) is the signum function

$F_f$ is the magnitude of the friction force, where $F_f$ depends on the normal force ($F_f = \mu F_G = \mu mg$, where $\mu$ is the friction coefficient).

# Coulomb friction

We can use alternative ways to indicate the presence of friction in our problems or model sketches:


Equation of motion:



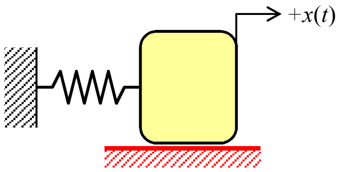

$$F_I + \boxed{F_{D,Coulomb}} + F_S = F$$

$$m\ddot{x} + \boxed{\text{sign}(\dot{x})F_f} + kx = F$$

*Numerical algorithms don't like sign(.) function!*



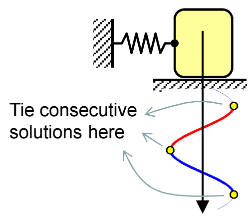
# Free vibration with Coulomb friction




$$m\ddot{x} + \text{sign}(\dot{x})F_f + kx = 0$$

Function **sign** has the three possible *discrete* values -1, 0 and 1. Thus, all three cases have to be considered independently:

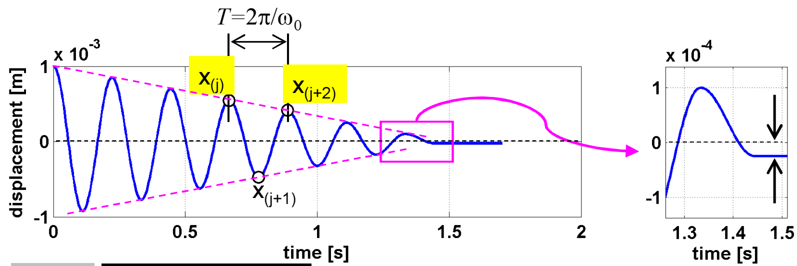
$$\text{sign}(\dot{x}) \begin{cases} \text{sign}(\dot{x}) = -1:\ m\ddot{x} + kx = F_f \\ \text{sign}(\dot{x}) = 0:\ m\ddot{x} + kx = 0 \\ \text{sign}(\dot{x}) = +1:\ m\ddot{x} + kx = -F_f \end{cases}$$



# Free vibration with Coulomb friction

Free vibration with Coulomb friction is represented as a *sequence of undamped half-cycles with constant load*. Solution of this problem is discussed in the separate note "*Free vibration decay in 1 DOF systems with Coulomb friction*" (Bb). The key observations are:

- free vibration occurs at the undamped natural frequency $\omega_0$
- free vibration decays linearly based on the relationship $\boxed{x_{(j)}-x_{(j+2)}=4F_f/k}$
- vibration stops when $F_f>kx_n$ (friction forces > elastic forces)



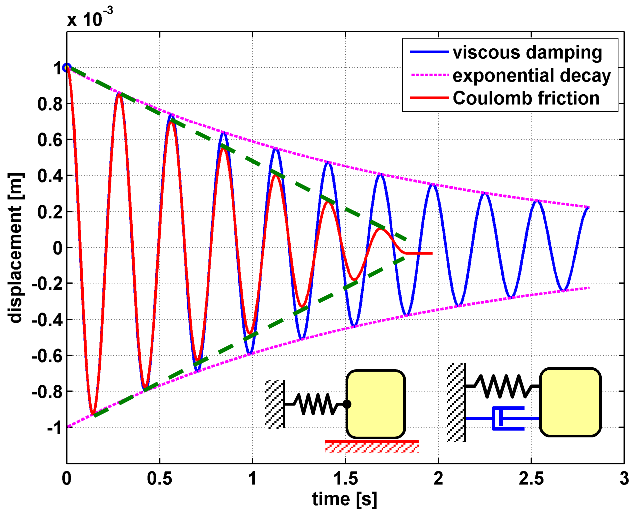

Matlab: » vib2_1dof_freefriction



# Friction versus viscous damping

| | Viscous damping | Coulomb friction |
|---|---|---|
| Natural frequency | $\omega_0\sqrt{1-\zeta^2}$ | $\omega_0$ |
| Vibration decay | exponential<br>$\exp(-\zeta\omega_0 t)$ | linear<br>$-\dfrac{4F_f/k}{2\pi/\omega_0}t=-\dfrac{2F_f\omega_0 t}{\pi k}$ |
| Energy dissipated per one cycle | $\int_0^T f_D\, v\, dt=\pi\omega c X_0^2$ | $\int_0^T f_D\, v\, dt=4F_f X_0$ |
| Theoretical duration of motion | Infinite | Finite |

# Matlab example




Try Matlab: » vib2_1dof_freefriction



# Example: free vibration with friction

The mass block shown in Fig. 1 is displaced 10 mm and released. How many cycles of motion will be executed? Assume friction due to the normal force $F_N$=mg, m=1 kg, g=9.81 m.s$^{-2}$, and the friction coefficient is μ=0.12. Stiffness of the spring is 10 kN/m.

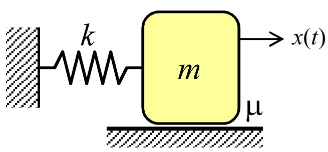



Fig. 1

Equation of motion: $m\ddot{x} + kx = \pm F_f \Rightarrow \ddot{x} + \omega_0^2 x = \pm F_f/m$

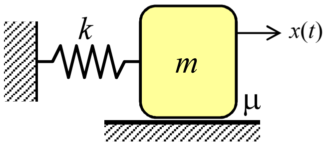
Friction force:

$$F_f = \mu F_N = \mu(mg) = 0.12 \times (1\,kg) \times (9.81\,ms^{-2}) = 1.18\,N$$

Amplitude decrease per one full cycle:

$$x_{(j)} - x_{(j+2)} = 4F_f/k = 4 \times (1.18\,N)/(10000\,N/m) = 0.47\,mm$$

# Example




Motion stops when the spring force cannot overcome the friction force:

$$F_f \geq F_K = k\,x_{stop} \Rightarrow x_{stop} = F_f/k = (1.18\,N)/(10000\,N/m) = 0.118\,mm$$

Condition for the number of cycles from the initial 10 mm to the displacement when the system is "locked":

$$x_0 - N_{cyc}\,(x_{(j)} - x_{(j+2)}) \leq x_{stop} \Rightarrow N_{cyc} \geq \frac{x_0 - x_{stop}}{x_{(j)} - x_{(j+2)}}$$

$$N_{cyc} \geq \frac{x_0 - x_{stop}}{x_{(j)} - x_{(j+2)}} = \frac{10\,mm - 0.118\,mm}{0.47\,mm} = 21.02 \Rightarrow N_{cyc} = 22\,cycles$$



# Summary

- Coulomb friction model
- Free vibration with friction:
  - decay rate and decay trend
  - natural frequency
  - convergence to the equilibrium position
- Comparison with viscous damping