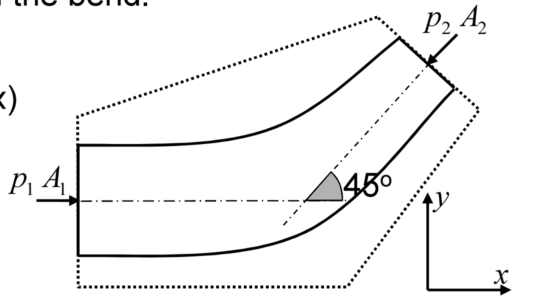
Example 2: Turning pipe

- A 45^o reducing pipe-bend (in a horizontal plane) tapers from a 600mm diameter inlet to a 300mm diameter outlet. The gauge pressure at inlet is 140kPa and the rate of flow of water through the bend is 0.425m^3/s. Neglecting friction, calculate the net resultant horizontal force exerted by the water on the bend.

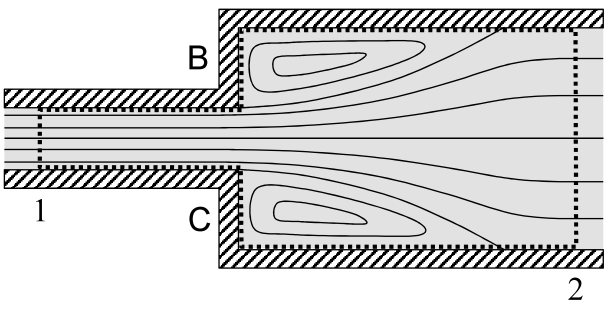


- Assumptions: Frictionless & incompressible
  Straight streamlines at inlet & outlet (1D approx)
  Horizontal so no hydrostatic terms
  Only pressure forces acting on pipe walls
  p_a acts on entire CV (so no net force)
- Continuity: $V_1 A_1 = V_2 A_2$
  $V_1 = Q/A_1 = 0.425/\left(\pi \times 0.3^2\right) = 1.503 ms^{-1} \quad V_2 = 6.01 ms^{-1}$
- Bernoulli's equation between 1 & 2: $p_2 = p_1 + \frac{1}{2}\rho\left(V_1^2 - V_2^2\right)$
  $p_2 = 1.4\times10^5 + \frac{1}{2}\times1000\times\left(1.503^2 - 6.01^2\right) = 1.231\times10^5 \text{ pa (gauge)}$
- Steady Flow momentum in x-direction: $p_1 A_1 - p_2 A_2 \cos 45^o + F_x = \rho Q\left(V_2 \cos 45^o - V_1\right)$
  $F_x = -1.4\times10^5\times\pi\times0.3^2 + 1.231\times10^5\times\pi\times0.15^2\cos 45^o + 1000\times0.425\left(6.01\cos 45^o - 1.503\right) = -32264\,N$
- Steady flow momentum in y-direction: $-p_2 A_2 \sin 45^o + F_y = \rho Q\left(V_2 \sin 45^o - 0\right)$
  $F_y = 1.231\times10^5\times\pi\times0.3^2\sin 45^o + 1000\times0.425\times6.01\sin 45^o = 7959\,N$

$$F = \sqrt{F_x^2 + F_y^2} = 33231\,N \qquad \tan\theta = \frac{F_x}{F_y} \rightarrow \theta = 180^o - 13.86^o$$

Application 1: Loss at an abrupt enlargement

- Assumptions: Straight streamlines at inlet & outlet so uniform conditions at 1 & 2
  Steady Frictionless & incompressible
  $p_C = p_B \approx p_1$



- Continuity: $Q = A_1 V_1 = A_2 V_2$
- Bernoulli's equation between 1 & 2:
  $p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 + \Delta p_{loss}$
- Steady Flow momentum in x-direction:
  $p_1 A_1 + p_1\left(A_2 - A_1\right) - p_2 A_2 = \rho Q\left(V_2 - V_1\right)$
- Rearranging these 3 equations

$$\Delta p_{loss} = p_1 - p_2 + \frac{1}{2}\rho\left(V_1^2 - V_2^2\right)$$
$$\Delta p_{loss} = \rho V_1^2 \frac{A_1}{A_2}\left(\frac{A_1}{A_2} - 1\right) + \frac{1}{2}\rho V_1^2\left(1 - \left(\frac{A_1}{A_2}\right)^2\right)$$
$$\Delta p_{loss} = \frac{1}{2}\rho V_1^2\left(1 - \frac{A_1}{A_2}\right)^2$$

- Define $\text{Loss Coefficient} = \dfrac{\Delta p_{loss}}{\frac{1}{2}\rho V_1^2} = \left(1 - \frac{A_1}{A_2}\right)^2$

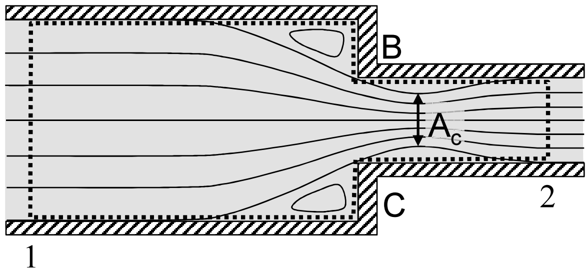
Application 2: Loss at an abrupt contraction

- Assumptions: Straight streamlines at inlet & outlet so uniform conditions at 1 & 2

  Steady Frictionless & incompressible

  But unknown pressures p_C , p_B

  Analysis as before but applied between the Vena-Contracta & 2



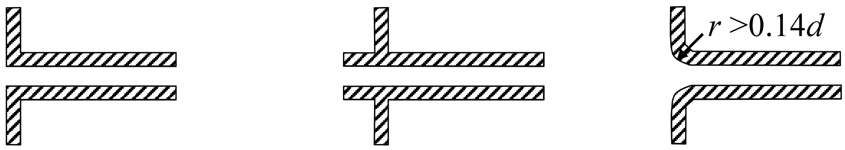
- So

$$\Delta p_{loss} = \tfrac{1}{2}\rho V_2^2\left(1-\tfrac{A_2}{A_c}\right)^2 = \tfrac{1}{2}\rho V_2^2 k$$

- A_c and therefore the loss coefficient k, are functions of the area ratio A_2/A_1
- Must find these from experiment, for Circular ducts with diameters d_1 & d_2

d_1/d_2	0	0.2	0.4	0.6	0.8	1
k	0.5	0.45	0.38	0.28	0.14	0

- Entry loss dependant on geometry



Sharp edged k=0.5 | Protruded k=1.0 | Rounded k=0

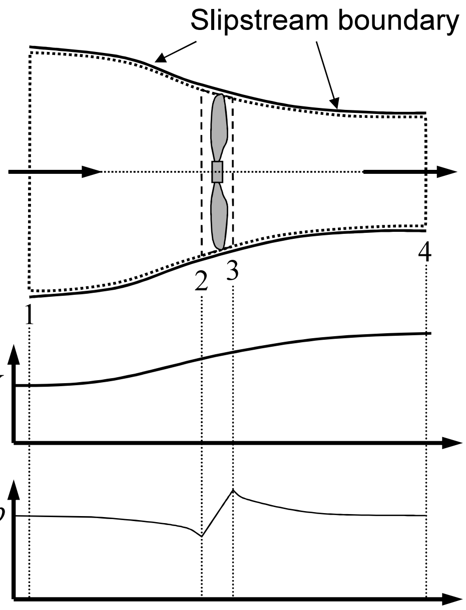
Application 3: Actuator Disc Theory: Propeller

- Propeller does work on the fluid (shaft work) and the increase in momentum gives thrust to the disc.
- Assumptions: Frictionless & incompressible

  Steady 1D flow (neglect rotation and variation across the disc radius)

  Actuator disc is thin so $A_2=A_3=A_d$ & $V_2=V_3=V_d$

  $p=p_a$ at all points on slipstream boundary & 1 & 4



- Continuity: $Q = V_d A_d$
- Bernoulli's equation for CV 1-2 & CV 3-4

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_d^2$$
$$p_3 + \tfrac{1}{2}\rho V_d^2 = p_4 + \tfrac{1}{2}\rho V_4^2 \quad \rightarrow \quad p_3 - p_2 = \tfrac{1}{2}\rho\left(V_4^2 - V_1^2\right)$$

- Steady Flow momentum for CV 1-4:

$$0 + F = \rho Q\left(V_4 - V_1\right)$$

- Steady Flow momentum for CV 2-3: $(p_2 - p_3)A_d + F = \rho Q(V_d - V_d) = 0 \rightarrow F = (p_3 - p_2)A_d$
- From momentum & continuity $(p_3 - p_2) = \rho V_d (V_4 - V_1)$
- Eliminating p_3-p_2 using Bernoulli's equation above $V_d = \tfrac{1}{2}(V_1 + V_4)$

Application 3: Actuator Disc Theory: Propeller(2)

- For a stationary rotor (fixed fan or helicopter in hover)

$$V_1 = 0 \quad \rightarrow \quad V_4 = 2V_d$$

$$F = \rho A_d V_d (2V_d - 0) = 2\rho A_d V_d^2$$

$$V_d = \sqrt{F/2\rho A_d}$$

- For a propeller with a forward velocity v into still air, and a final air velocity relative to the disc of V_4 we have

$$V_1 = v \quad \rightarrow \quad V_d = \tfrac{1}{2}(V_4 + v)$$

$$F = \rho A_d V_d (V_4 - v) = \tfrac{1}{2}\rho A_d (V_4^2 - v^2)$$

- The power supplied to the disc to produce the thrust (“ideal” power input) is

$$FV_d = \rho Q(V_4 - V_1)V_d = \tfrac{1}{2}\rho Q(V_4 - V_1)(V_4 + V_1) = \tfrac{1}{2}\rho Q(V_4 - V_1)(V_4 - V_1) + \rho Q(V_4 - V_1)V_1$$

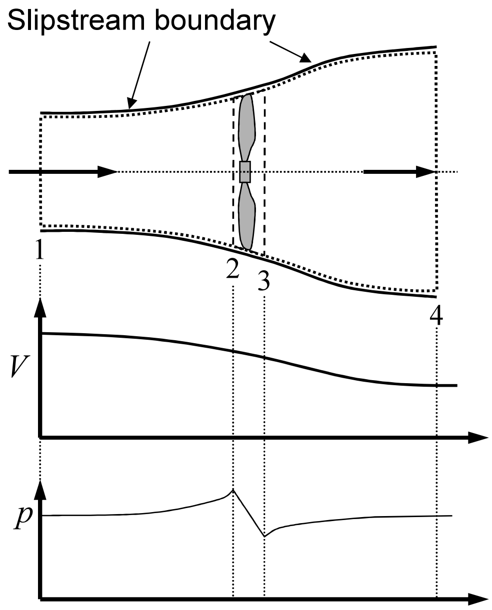
- For a hovering rotor $FV_d = F\sqrt{F/2\rho A_d} = F^{\frac{3}{2}}\big/\sqrt{2\rho A_d}$
- The power put into the air (effective power output) is given by

$$FV_1 = \rho Q(V_4 - V_1)V_1$$

- Propulsive efficiency (for forward flight) defined as the ratio of power input & output

$$\eta = \frac{\rho Q(V_4 - V_1)}{\rho Q(V_4 - V_1)} \frac{V_1}{\tfrac{1}{2}(V_4 - V_1) + V_1} = \frac{2V_1}{V_4 + V_1}$$

Application 4: Vertical axis Turbine



- Fluid does work on the turbine (shaft work) and the decrease in momentum gives thrust to the disc.
- Assumptions: Frictionless & incompressible
  Steady 1D flow (neglect rotation and variation across the disc radius)
  Actuator disc is thin so $A_2 = A_3 = A_d$ & $V_2 = V_3 = V_d$
  $p = p_a$ at all points on slipstream boundary & 1 & 4
- Continuity: $Q = V_d A_d$
- Bernoulli’s equation for CV 1-2 & CV 3-4

$$\begin{aligned} p_1 + \tfrac{1}{2}\rho V_1^2 &= p_2 + \tfrac{1}{2}\rho V_d^2 \\ p_3 + \tfrac{1}{2}\rho V_d^2 &= p_4 + \tfrac{1}{2}\rho V_4^2 \end{aligned} \quad \rightarrow \quad p_3 - p_2 = \tfrac{1}{2}\rho(V_4^2 - V_1^2)$$

- Steady Flow momentum for CV 1-4:

$$0 + F = \rho Q(V_4 - V_1)$$

- Steady Flow momentum for CV 2-3: $(p_2 - p_3)A_d + F = \rho Q(V_d - V_d) = 0 \rightarrow F = (p_3 - p_2)A_d$
- From momentum & continuity $(p_3 - p_2) = \rho V_d (V_4 - V_1)$
- Eliminating p_3-p_2 using Bernoulli’s equation above $V_d = \tfrac{1}{2}(V_1 + V_4)$

Application 4: Vertical axis Turbine(2)

- The power drawn from the air by the disc is

$$P_{\text{disc}} = -FV_d = -\rho Q(V_4 - V_1)V_d = \rho A_d V_d (V_1 - V_4)V_d = \tfrac{1}{4}\rho A_d (V_4 + V_1)(V_1^2 - V_4^2)$$

- Power in the wind passing through the disc area, if the disc were not present.

$$P_{\text{wind}} = \tfrac{1}{2}\dot{m}V_1^2 = \tfrac{1}{2}\rho A_d V_1 V_1^2 = \tfrac{1}{2}\rho A_d V_1^3$$

- The efficiency of the turbine is therefore

$$\eta = \frac{P_{\text{disc}}}{P_{\text{wind}}} = \frac{\tfrac{1}{4}\rho A_d (V_4 + V_1)(V_1^2 - V_4^2)}{\tfrac{1}{2}\rho A_d V_1^3} = \frac{(V_4 + V_1)(V_1^2 - V_4^2)}{2V_1^3}$$

- Differentiating w.r.t V_4 and equating to zero defines the minima. From this we find efficiency is a maximum when

$$\frac{V_4}{V_1} = \frac{1}{3}$$

$$\eta_{\max} = \frac{V_1^3}{V_1^3}\frac{\left(\frac{1}{3}+1\right)\left(1-\frac{1}{9}\right)}{2} = 0.59$$

- In reality $\eta \approx 0.15$, efficiency not the design driver

$$\eta \to 0.3$$

Example 3: Windmill

- An ideal windmill, 12m diameter, operates at a theoretical efficiency of 50% in a 14m/s wind. If the air density is 1.235 kg/m^3 determine the thrust on the windmill, the air velocity through the disc, the mean gauge pressures immediately in front of and behind the disc, and the shaft power developed.
- Use previous results for vertical axis turbine & assume same figure labeling

$$\eta = 0.5 = \frac{(V_4 + V_1)(V_1^2 - V_4^2)}{2V_1^3} = \frac{(V_4 + 14)(14^2 - V_4^2)}{2\times 14^3}$$

- Solving (neglect negative root and zero root) $V_4 = 8.65ms^{-1}$

$$V_d = \tfrac{1}{2}(V_1 + V_4) = 11.13ms^{-1}$$

- Steady Flow momentum : $F = \rho Q(V_4 - V_1)$ $\quad F = \rho A_d V_d (V_4 - V_1) = \tfrac{1}{2}\rho A_d (V_4^2 - V_1^2)$

$F = \tfrac{1}{2}1.235\times\pi\times 6^2(8.65^2 - 14^2) = -8463N$ Thrust on windmill =8463N

- Bernoulli's equation for CV 1-2 $\quad p_2 - p_1 = \tfrac{1}{2}\rho(V_1^2 - V_2^2)$

$$p_2 = \tfrac{1}{2}\times 1.235\times(14^2 - 11.33^2) = 41.8\text{pa} \quad (\text{gauge as } p_1 \text{ atmosperic})$$

- Bernoulli's equation for CV 3-4 $\quad p_3 - p_4 = \tfrac{1}{2}\rho(V_4^2 - V_3^2)$

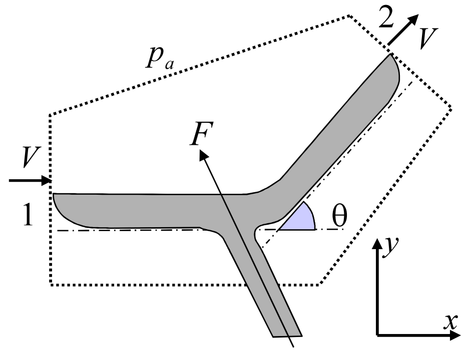
$$p_3 = \tfrac{1}{2}\times 1.235\times(8.65^2 - 11.33^2) = -33.1\text{pa} \quad (\text{gauge as } p_4 \text{ atmosperic})$$

- Power $\quad P_{\text{disc}} = -FV_d = \tfrac{1}{2}\rho A_d V_d^2 (V_1 - V_4)$

$$p_3 = \tfrac{1}{2}\times 1.235\times\pi\times 6^2\times 11.33^2(14^2 - 8.65^2) = 95.9kW$$

Example 4: Turning vane

- A fixed vane turns a water jet of area A through an angle θ without changing the magnitude of its velocity The flow is steady, pressure is p_a everywhere and friction on the vane is negligible. (a) Find the component of force F_x and F_y that the flow applies to the vane (b) Find the expression for the force magnitude F and the angle ϕ between F and the horizontal.



- (a) Continuity $\dot{m} = \rho A V$
- Steady Flow momentum in x :

$$F_x = \dot{m}\left(V\cos\theta - V\right) = \dot{m}V\left(\cos\theta - 1\right)$$

- Steady Flow momentum in y :

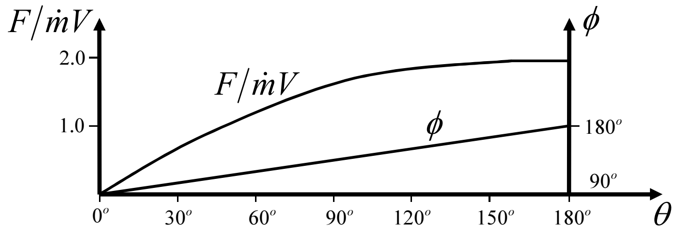
$$F_y = \dot{m}\left(V\sin\theta - 0\right) = \dot{m}V\sin\theta$$

- (b) $F = \sqrt{F_x^2 + F_y^2} = \dot{m}V\left(\sin^2\theta + \left(\cos\theta - 1\right)^2\right)^{\frac{1}{2}}$

$$F = \dot{m}V\left(2 - 2\cos\theta\right)^{\frac{1}{2}} = 2\dot{m}V\sin\tfrac{\theta}{2}$$

$$\phi = \pi - \tan^{-1}\left|\frac{F_y}{F_x}\right| = \pi - \tan^{-1}\left|\frac{\sin\theta}{\cos\theta - 1}\right|$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}$$



Learning Outcomes: "What you should have learnt"

- How to apply the principle of mass conservation to a range of examples
- How to apply the linear momentum conservation principle to a range of examples

  You will need to remember that $f_{\text{tot}_x} = \dot{m}\left(V_{ex} - V_{ix}\right)$ with similar for y & z
- Understanding Bernoulli's equation as a conservation of energy, and how terms representing viscous work, shaft work and heat losses for the CV can be included as pressure loss terms.
- How to apply the conservation of energy to a range of examples.
- You should be able to reproduce the derivations involved in the analysis of: the sudden expansion; the actuator disc theory of propellers and axial turbines.
- You should understand: the nature of flow in a sudden expansion; how to apply CV analysis to this situation and the effects of inlet design on pressure losses.

Remember: Being able to apply these principles to simple systems is the most important part of this section. You do not need to memorise the derivations for energy momentum and mass conservation.