Advanced Bending and Torsion
Shearing of Closed Thin-Walled Sections

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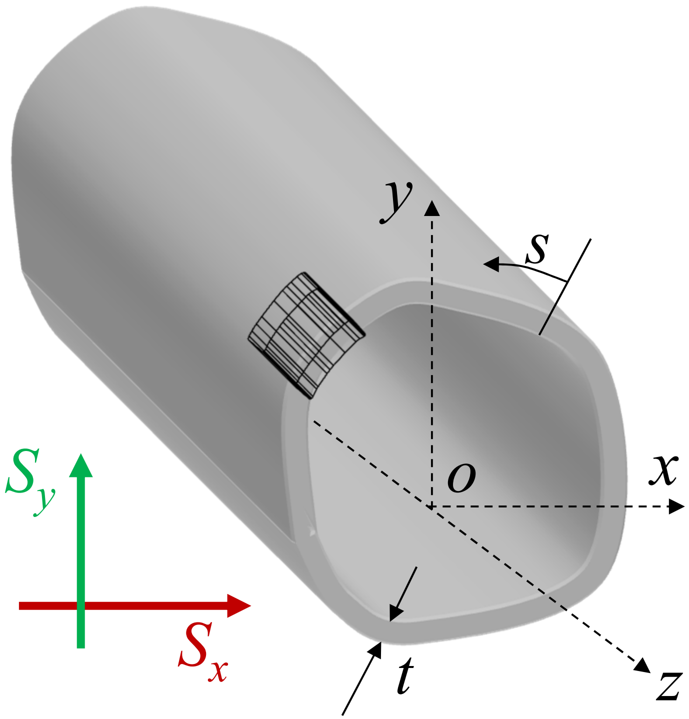
Shear of Closed Sections

- Closed sections do not have ‘free edges’, so when picking an origin for the arc length s we cannot guarantee that $q_{s=0} = 0$

- Therefore, if we pick an arbitrary origin for s and apply the ‘open section’ solution we will end up with a constant of integration:



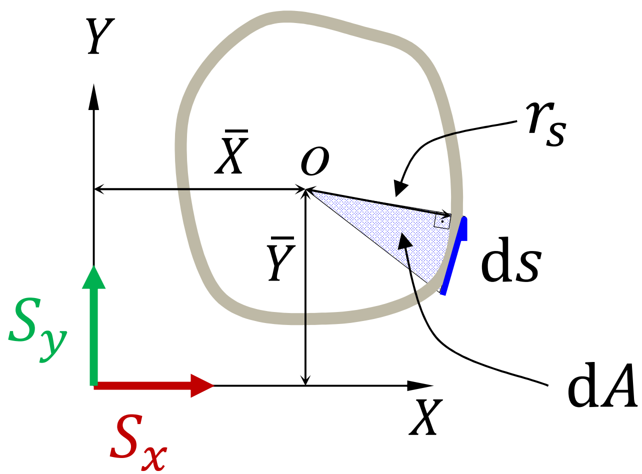
$$q_s^{\text{closed}} = q_s^{\text{open}} + q_0$$

- Balancing moments about z:

$$S_x\bar{Y} - S_y\bar{X} = \oint q_s^{\text{closed}}\, r_s\, \mathrm{d}s$$

$$q_s^{\text{closed}} = q_s^{\text{open}} + q_0$$

$$S_x\bar{Y} - S_y\bar{X} = \oint q_s^{\text{open}}\, r_s\, \mathrm{d}s + q_0 \oint r_s\, \mathrm{d}s$$



$$\mathrm{d}A = \frac{1}{2} r_s\, \mathrm{d}s$$

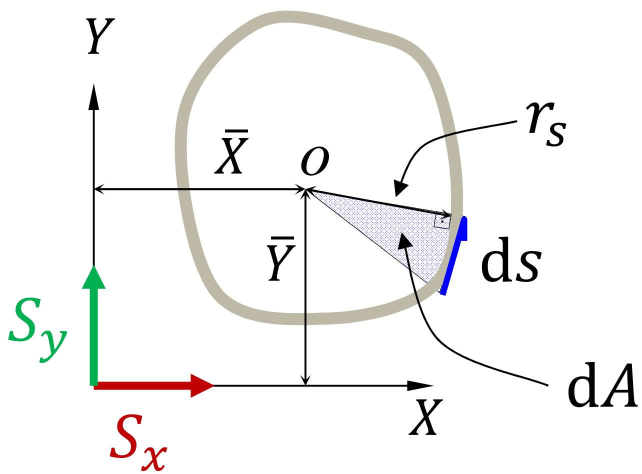
$$\boxed{\oint r_s\, \mathrm{d}s = 2\,A}$$

$$\boxed{S_x\bar{Y} - S_y\bar{X} = \oint q_s^{\text{open}}\, r_s\, \mathrm{d}s + 2\,A\,q_0}$$

Shear of Closed Sections

- So knowing the magnitude of shear forces and their point of application (*i.e.* $\bar{X}, \bar{Y}$ in the example) we get:

$$q_0 = \frac{\left(S_x \bar{Y} - S_y \bar{X}\right) - \oint q_s^{\mathrm{open}}\, r_s\, \mathrm{d}s}{2\,A}$$



And when loading at the shear centre:

$$q_0 = -\frac{\oint q_s^{\mathrm{open}}\, r_s\, \mathrm{d}s}{2\,A}$$

Advanced Bending and Torsion

Torsion of Solid Sections

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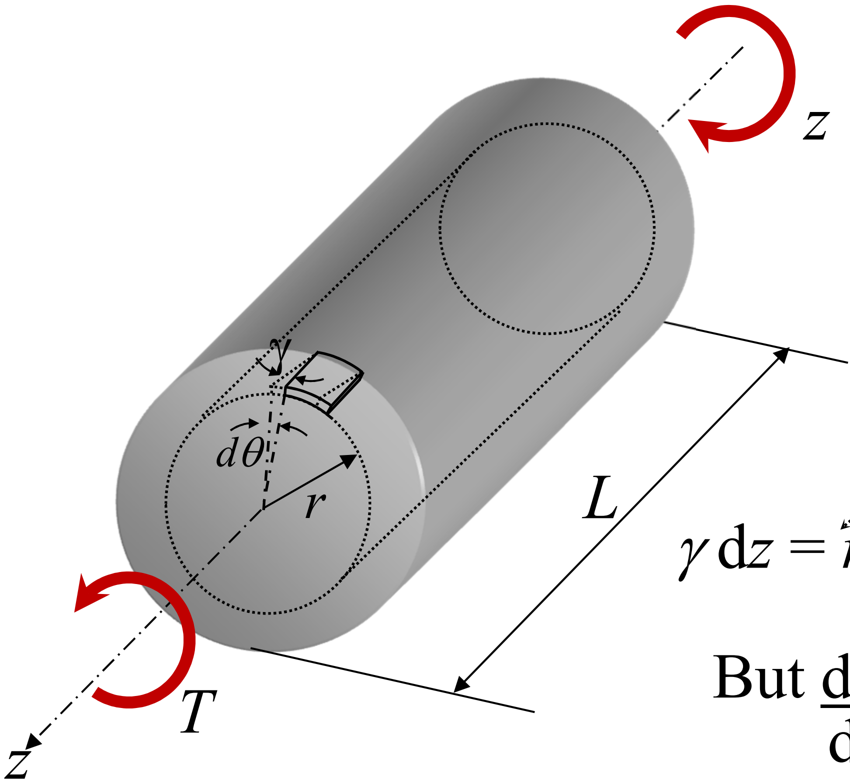
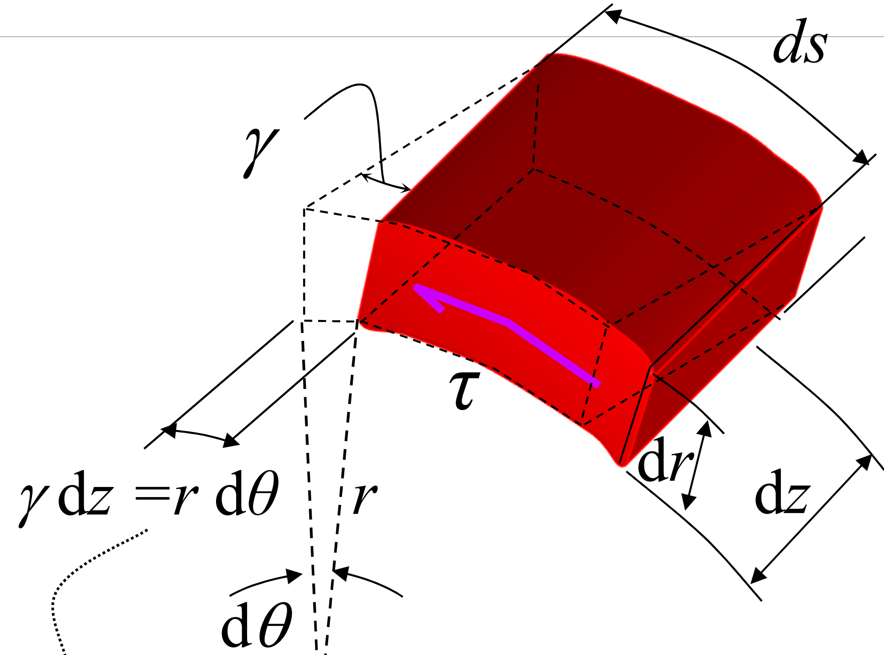
Torsion of solid sections





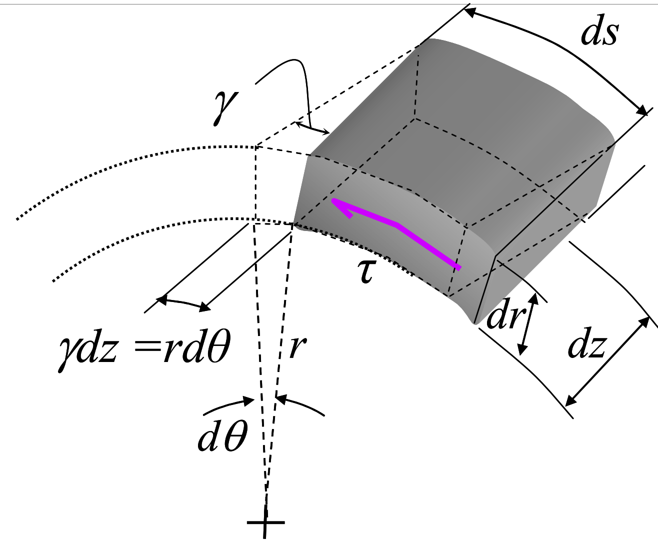
$$\gamma\, \mathrm{d}z = r\, \mathrm{d}\theta \;\Rightarrow\; \gamma = r\,\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

But $\dfrac{\mathrm{d}\theta}{\mathrm{d}z}$ = constant = θ/L

$$\Rightarrow\; \gamma = r\theta/L$$

And elasticity: $\gamma = \tau/G$ $\;\Rightarrow\;$ $\tau/r = G\theta/L$

Equilibrium



Considering annular ring of radius r, thickness dr

$$T = \int_0^R \underbrace{\tau}_{\text{4.1.1 [a] : } Gr\theta/L}\overbrace{(2\pi r dr)}r \quad = \frac{G\theta}{L}\underbrace{\int_0^R 2\,\pi r^3 dr}_{J}$$

"Polar second mmt of area"

Note for a solid circular section:
$J = \pi R^4/2$ or $\pi D^4/32$

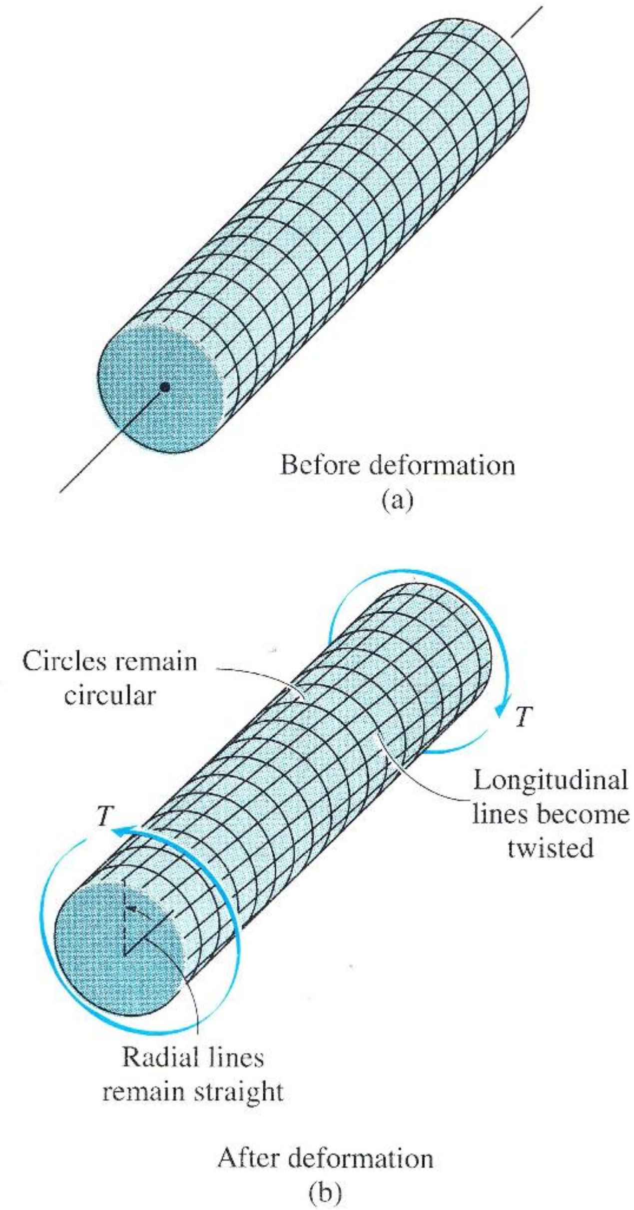
$$\Longrightarrow \qquad T/J = G\theta/L \qquad \text{[a]}$$

4.1.1 [a] $\Longrightarrow$

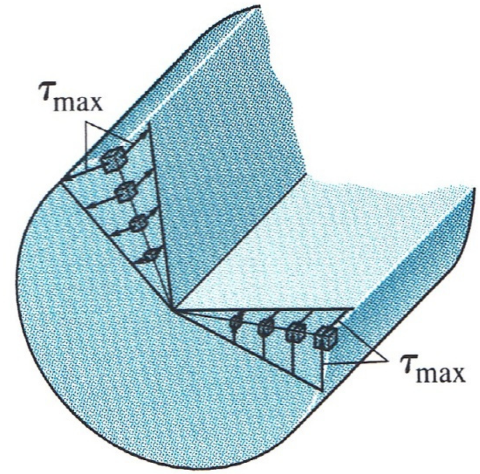
$$\boxed{\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}}$$

[b]* Torsion formula

Remember the Torsion of Circular Bars?

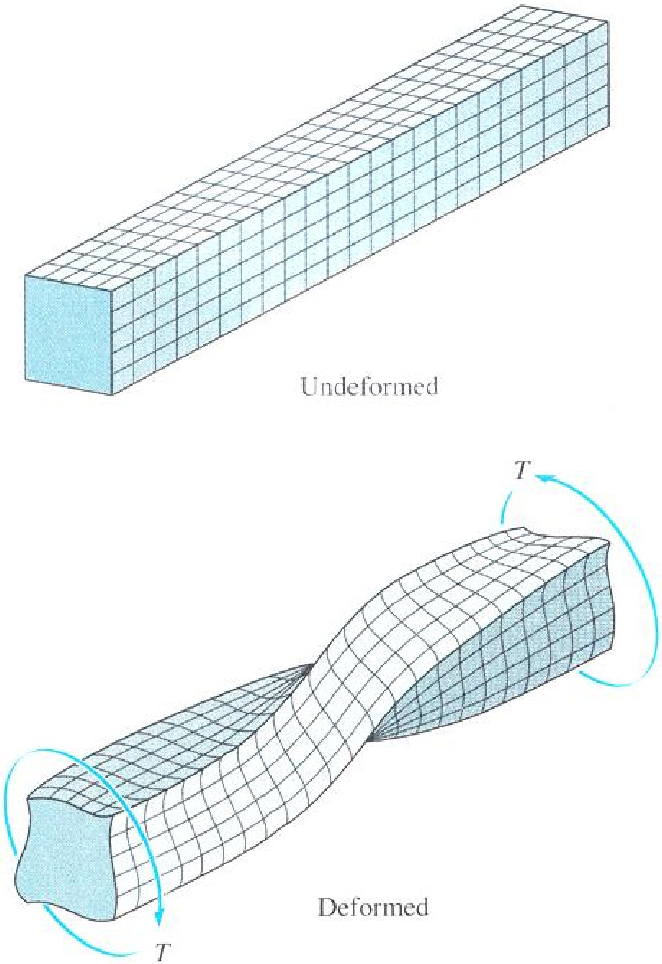


- Circles: remain circular
- Longitudinal lines: twisted
- Radial lines: straight
- Shear stresses: vary linearly from centre (0) to surface (max)

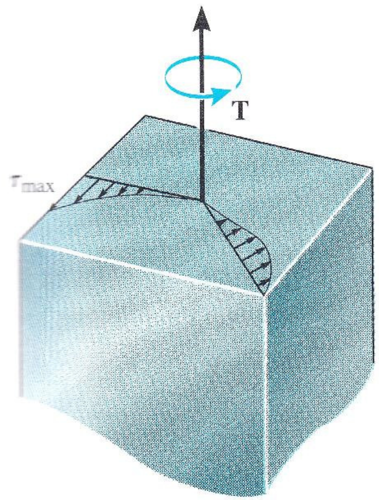


$$\tau_{\max} = \frac{T\, r_o}{J} \qquad J = \frac{\pi}{2} r_o^{\ 4}$$

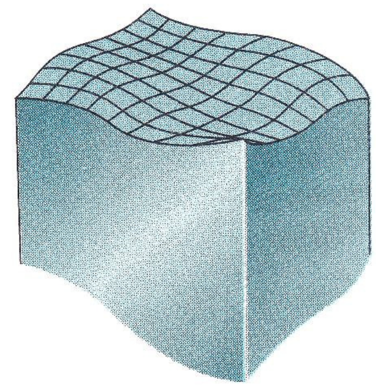
Torsion of Non-Circular Bars



- Stress distribution no longer linear, but dependent on cross section shape
- Cross-sections **warp** (*i.e.* deform longitudinally)

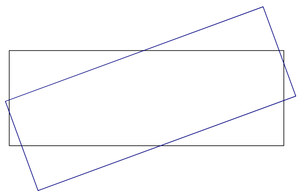


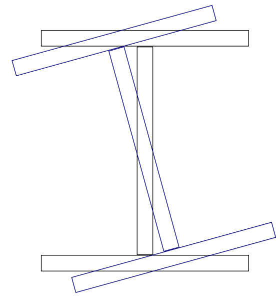
Shear stress distribution along two radial lines

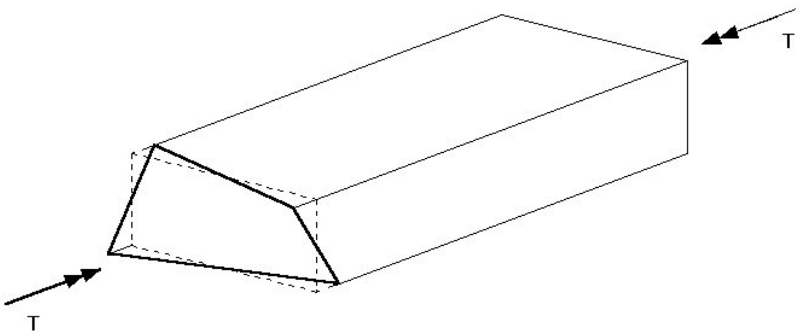


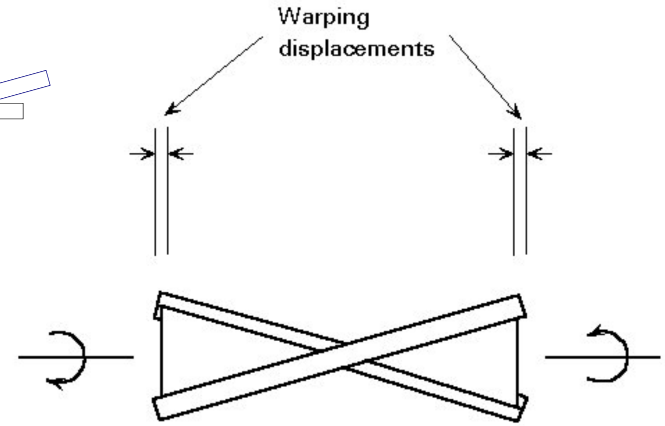
Warping of cross-sectional area

Examples of Warping



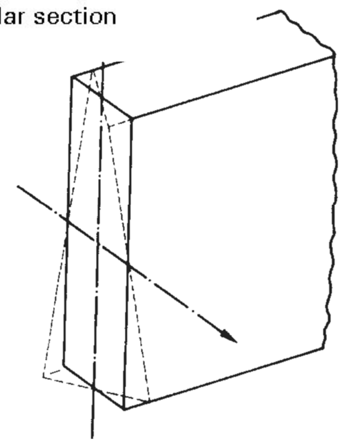


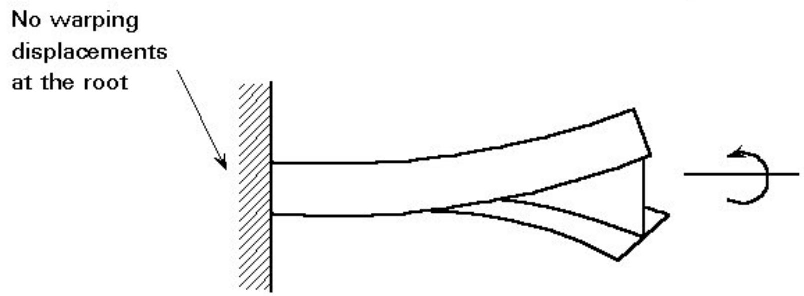




(a) Torsion with unrestrained warping

Warping of a rectangular section
due to pure torsion





(b) Torsion with restrained warping

St Venant's Theory of Torsion

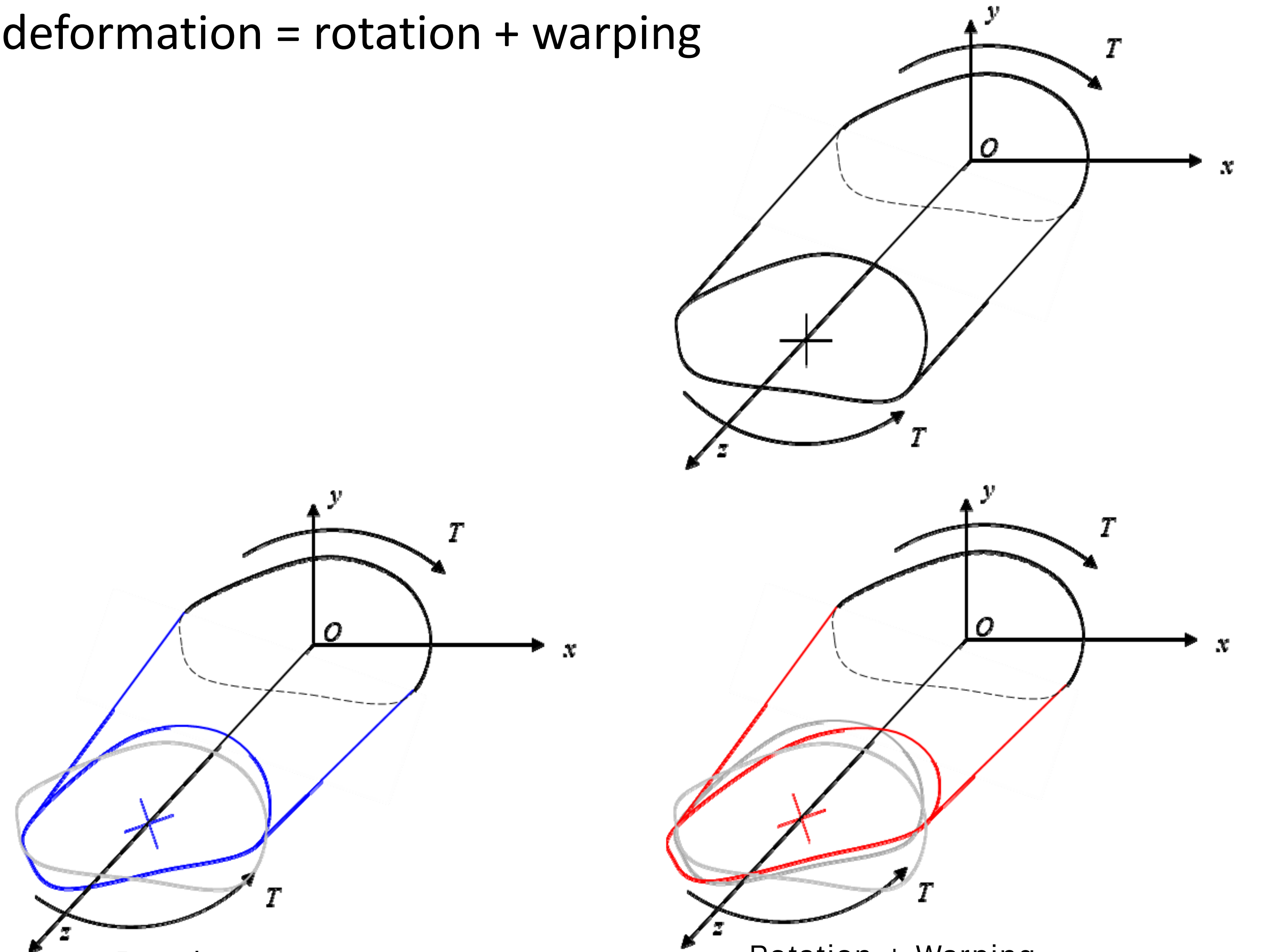
(Not to be mixed with St Venant's Principle: "the stresses and strains in a body at points that are sufficiently remote from points of application of load depends only on the static resultant of the loads and not on the distribution of loads")

St Venant's Torsion:

- Provides exact solutions to the shear stress distributions and warping deformation for various non-circular sections
- It is a semi-inverse method:
  - Direct methods – find stress function that satisfies geometry & BCs
  - Inverse methods – find geometry & BCs that satisfy assumed stress function
  - Semi-inverse methods – make assumptions regarding deformations and stresses, then find coefficients for a 'special' stress function
- Has direct analogy with other physical phenomena (*e.g.* soap film technique) hence the method can easily be 'visualised'

St Venant's Torsion - Assumptions

- Total deformation = rotation + warping



Rotation

Rotation + Warping

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St Venant's Torsion – Solution Procedure

- Find an appropriate torsion stress function

$$\eta(x, y)$$

- Find the stress function coefficients by solving Poisson's equation

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = -2G\theta = F$$

- Determine the $T(\theta')$ relationship via the area integral

$$T = 2\iint \eta \, dx \, dy$$

- Calculate shear stresses

$$\tau_{zx} = \frac{\partial \eta}{\partial y} \qquad \tau_{zy} = -\frac{\partial \eta}{\partial x}$$

- Find the warping function ψ

$$\tau_{zx} = G\,\theta' \left(\frac{\partial \psi}{\partial x} - y \right)$$

- Calculate warping displacements

$$w = \theta' \psi(x, y)$$

- Calculate rotational displacements u and v

$$u = -\theta' \, z \, y$$

$$v = \theta' \, z \, x$$

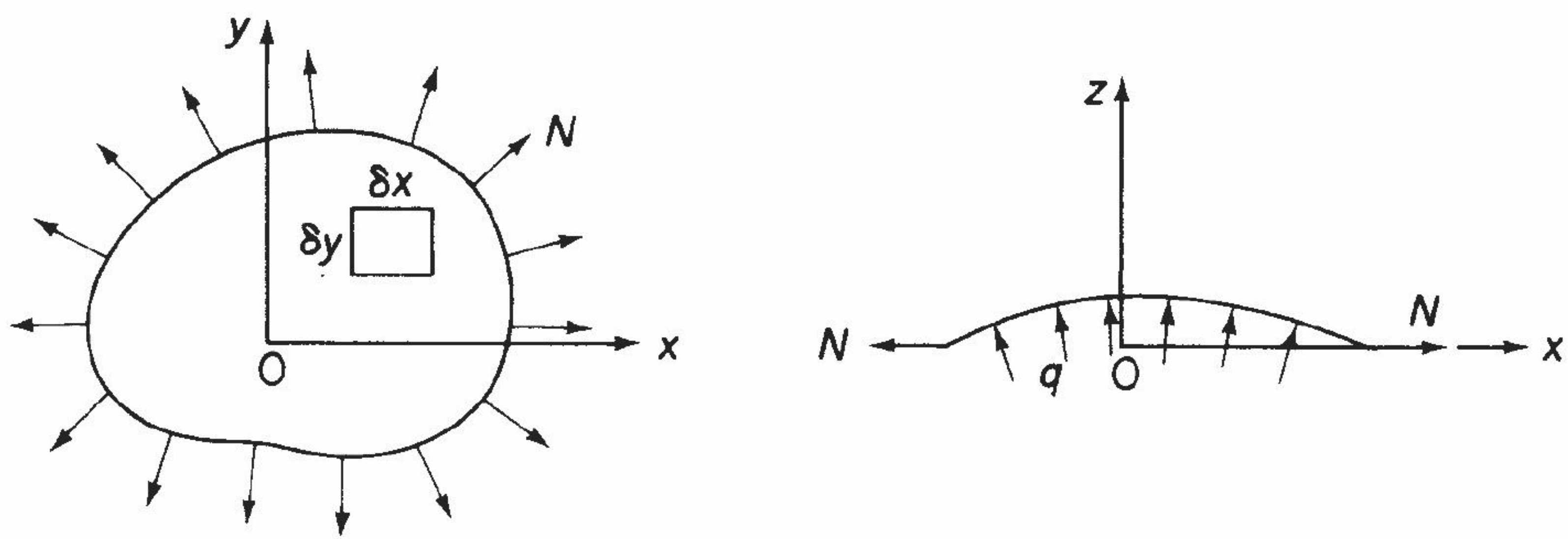
St Venant's Torsion – The $\eta(x, y)$ function

The function $\eta(x, y)$:

- Must atisfy Poisson's equation: $$\boxed{\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = -2G\theta = F}$$
- Must be constant (usually zero) along boundaries
- Defines a volume which is proportional to the applied torque
- Has as spatial derivatives the local shear stresses

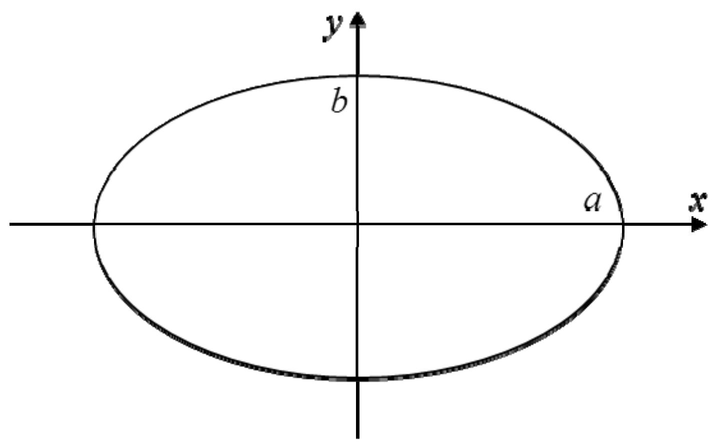
The Membrane Analogy (Soap Film Method)

- The function h can be visualised as the out-of-plane deflection of a membrane under constant pressure

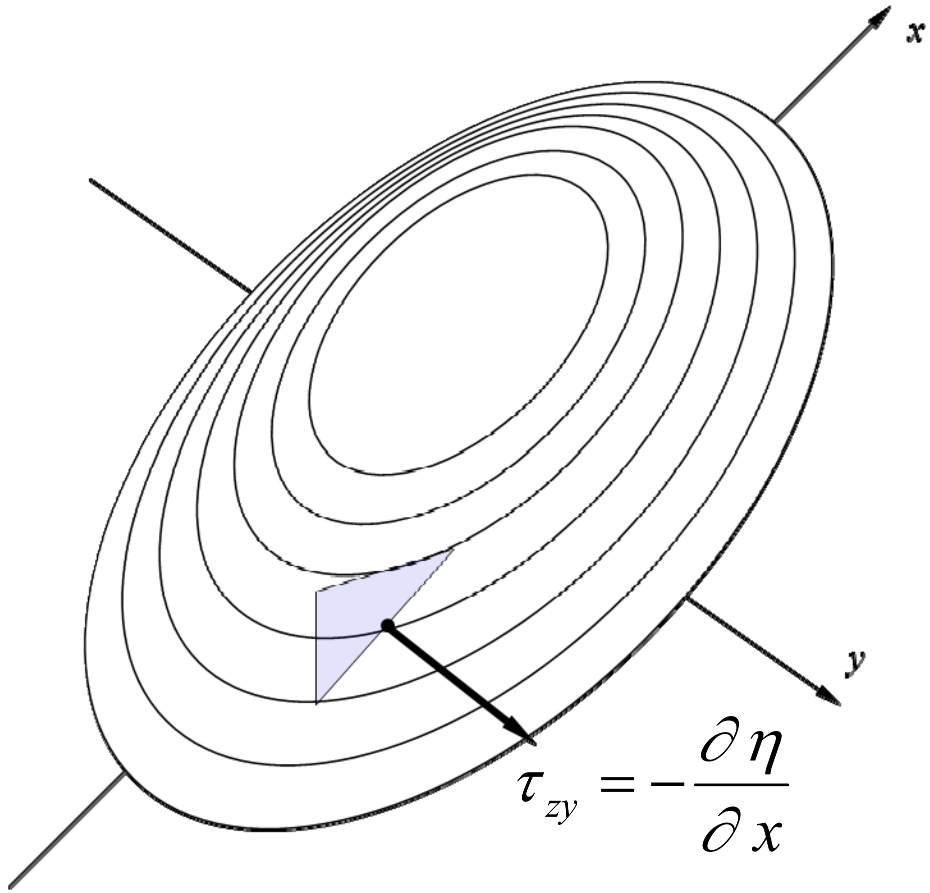


Membrane Analogy – Elliptical Section Example

- Assume a solid bar of elliptical cross section:



- The local ‘slope’ of the ‘torsion hill’ along one direction gives the shear stress in the orthogonal direction



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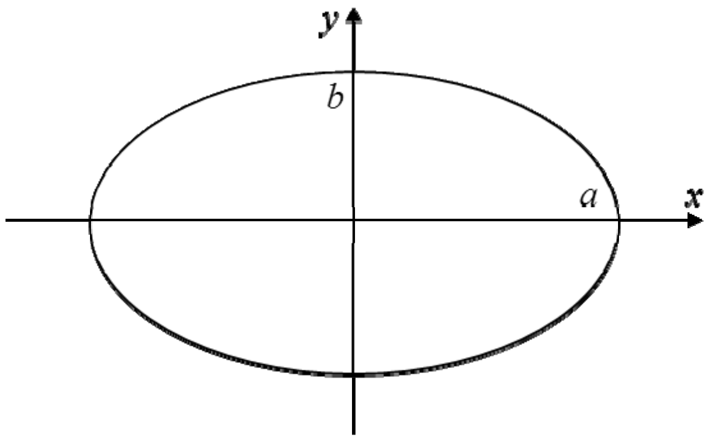

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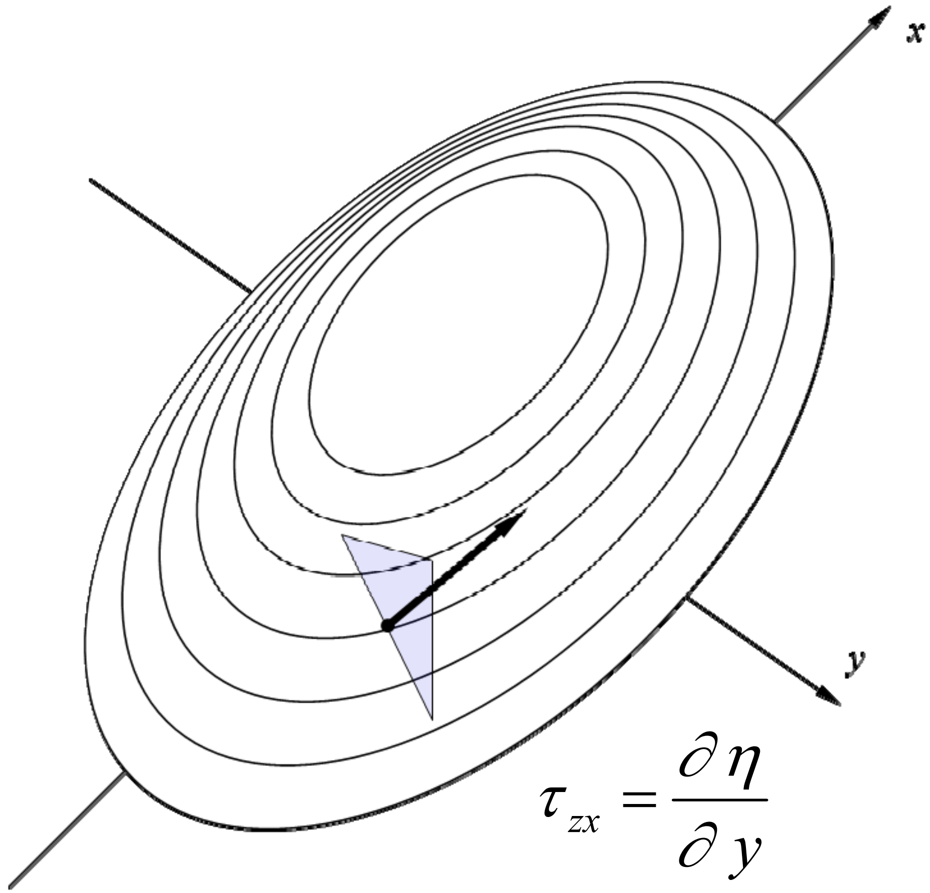

$$\tau_{zx} = \frac{\partial \eta}{\partial y}$$

Membrane Analogy – Elliptical Section Example

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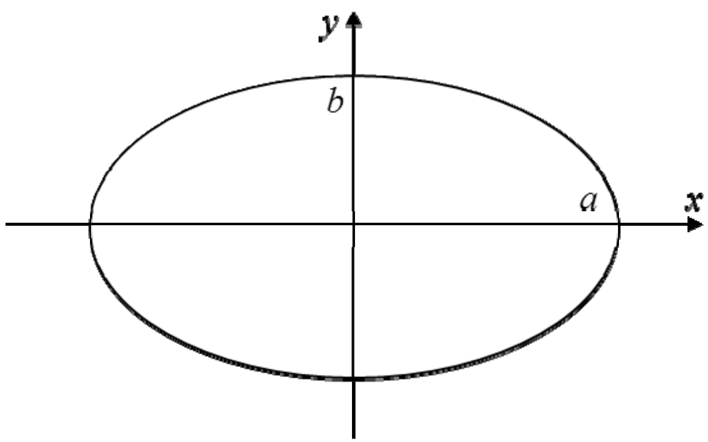
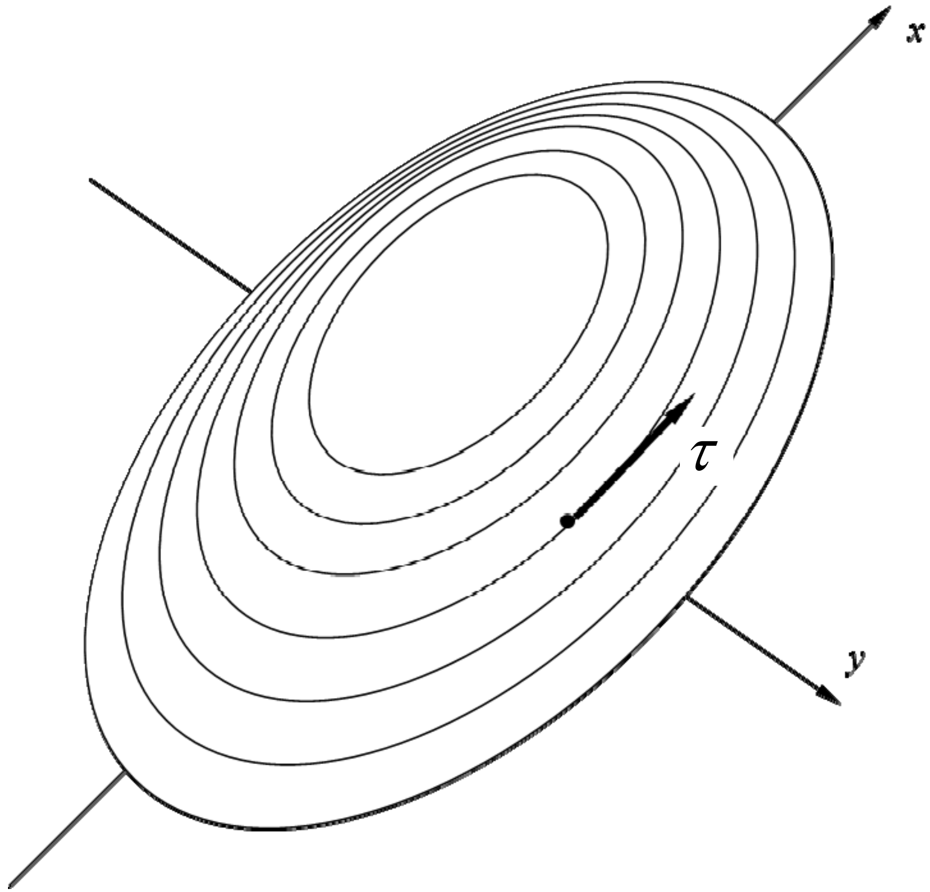


- The resultant shear is always parallel to the 'contour lines' of the 'torsion hill'



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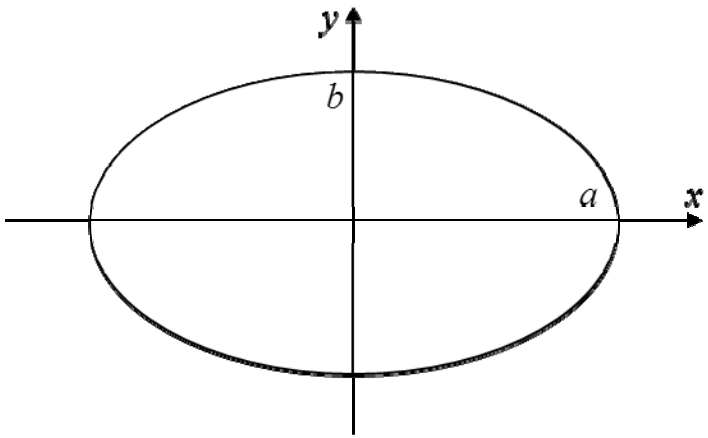
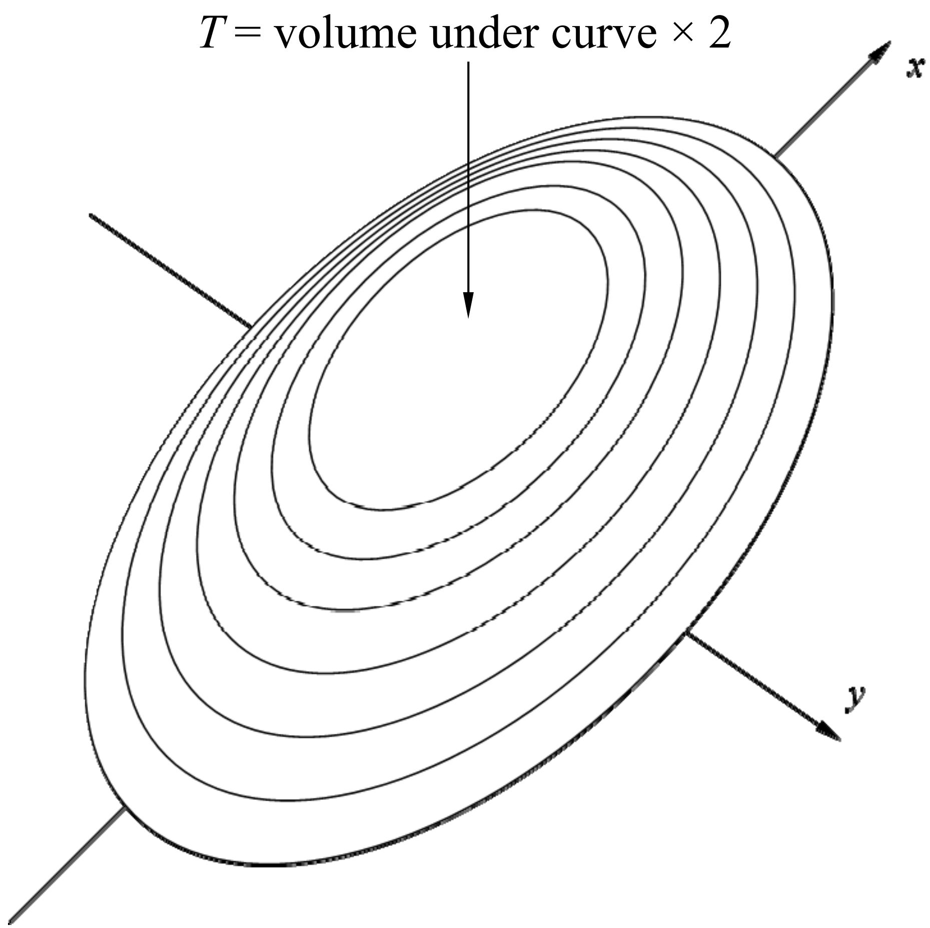


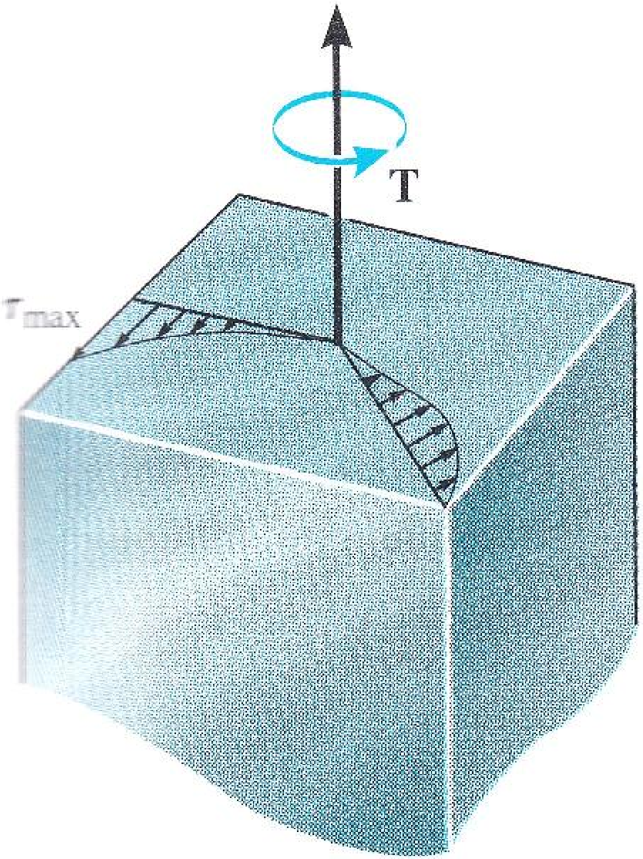


- The total applied torque (or couple) is equal to twice the volume under the 'torsion hill'

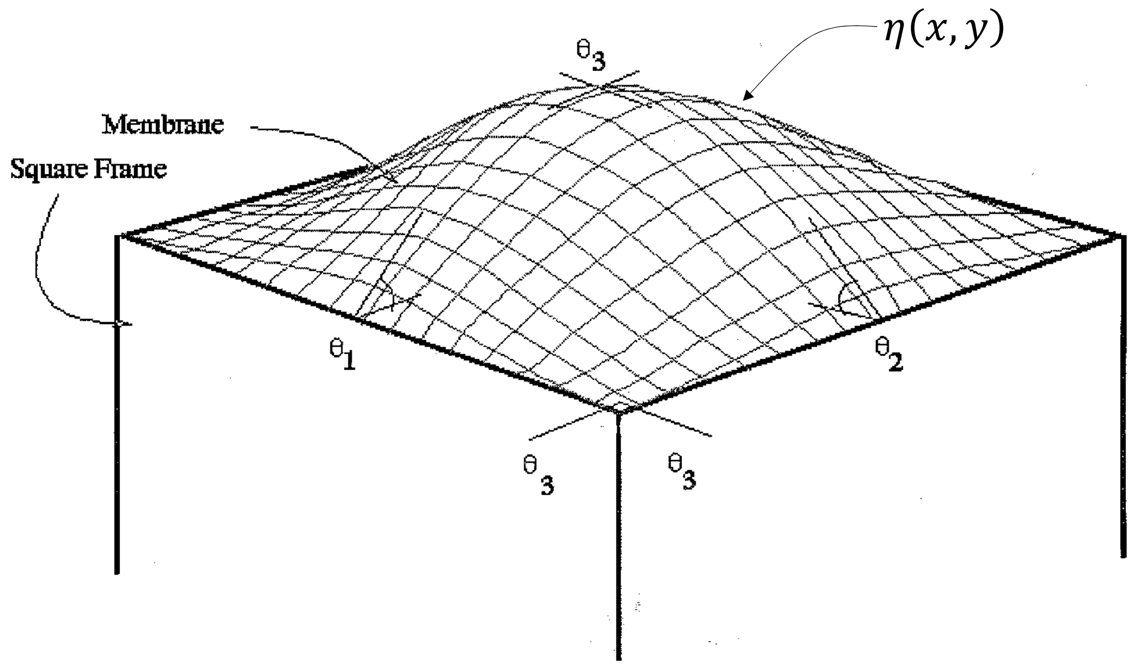
$$T = 2\iint \eta \, \mathrm{d}x \, \mathrm{d}y$$

Membrane Analogy – Square Section Example

- Compare the slopes of the 'torsion hill' (right) with the shear stress distribution (left)

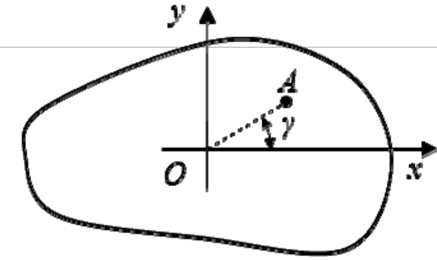


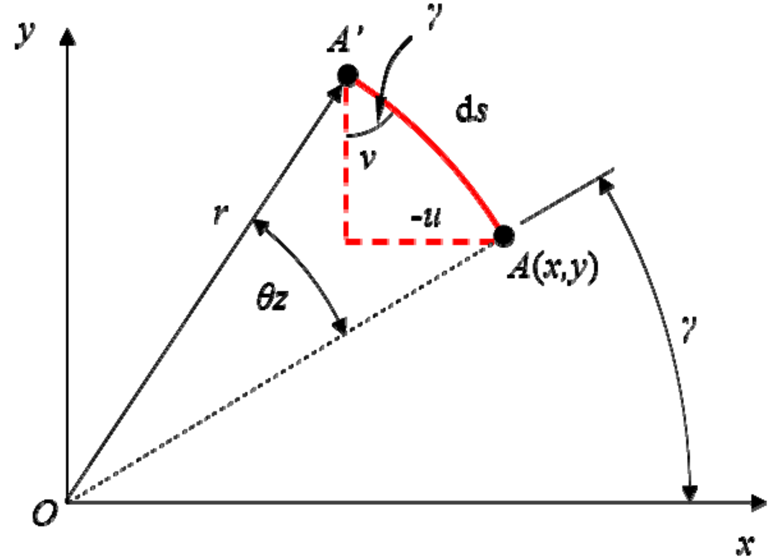
Shear stress distribution along two radial lines



Rotation

- Consider a point A on the cross-section
- After rotation the point will become A'
- The angle of twist per unit length is $\theta' = \dfrac{\mathrm{d}\theta}{\mathrm{d}z}$
- By trigonometry we can find the two in-plane displacements:





$$u = -\mathrm{d}s \sin\gamma \qquad v = \mathrm{d}s \cos\gamma$$

$$\sin\gamma = \frac{y}{r} \qquad \cos\gamma = \frac{x}{r}$$

$$\boxed{u = -\theta' \, z \; y} \qquad \boxed{v = \theta' \, z \; x}$$

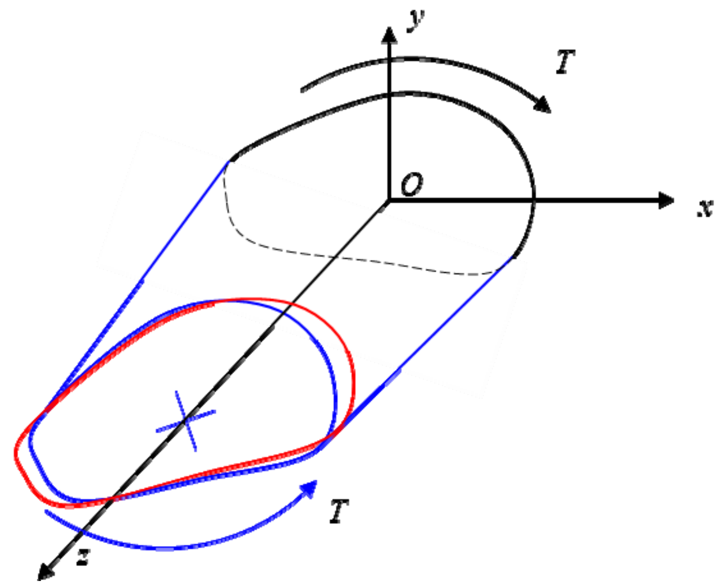
Warping

- Warping is the distortion of the cross section in the z-direction

- We must find the warping function $\psi(x, y)$

- The deflections in the z-direction are then proportional to the angle of twist:

$$w = \theta' \psi(x, y)$$



Strains

- St Venant's theory assumes zero in-plane strains, therefore:

$$\varepsilon_x = \varepsilon_x = \gamma_{xy} = 0$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

- The out-of-plane shear strains are:

$$\boxed{\begin{aligned} \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta'\left(\frac{\partial \psi}{\partial x} - y\right) \\ \gamma_{zy} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \theta'\left(\frac{\partial \psi}{\partial y} + x\right) \end{aligned}}$$

Stresses

- Assume an isotropic material with linear-elastic behaviour

- Using the shear modulus we can calculate stresses from the strains:

$$\boxed{\sigma_x = \sigma_x = \tau_{xy} = 0}$$

$$\tau_{zx} = G\,\theta'\left(\frac{\partial\psi}{\partial x} - y\right)$$

$$\tau_{zy} = G\,\theta'\left(\frac{\partial\psi}{\partial y} + x\right)$$

Warping Function

- Consider the basic equilibrium equations as discussed earlier:

$$\boxed{\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} &= 0 \end{aligned}}$$

- The first two equilibrium equations are trivial since:

$$\sigma_x = \sigma_x = \tau_{xy} = 0$$

- For the third equation we substitute the values of shear stress:

$$\left\{\begin{aligned} \tau_{zx} &= G\,\theta\left(\frac{\partial\psi}{\partial x} - y\right) \\ \tau_{zy} &= G\,\theta\left(\frac{\partial\psi}{\partial y} + x\right) \\ \frac{\partial\sigma_z}{\partial z} &+ \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0 \end{aligned}\right.$$

- This gives the Laplace equation which the warping function must satisfy:

$$\boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0}$$

$$\therefore \nabla^2\psi = 0$$

Stress Function

- We now introduce a stress function η:

$$\boxed{\eta(x, y)}$$

- Which is defined in terms of the non-zero shear stresses:

$$\boxed{\tau_{zx} = \frac{\partial \eta}{\partial y} \qquad \tau_{zy} = -\frac{\partial \eta}{\partial x}}$$

- Combining:

$$\begin{cases} \tau_{zx} = \dfrac{\partial \eta}{\partial y} \qquad \tau_{zy} = -\dfrac{\partial \eta}{\partial x} \\ \tau_{zx} = G\,\theta' \left(\dfrac{\partial \psi}{\partial x} - y \right) \\ \tau_{zy} = G\,\theta' \left(\dfrac{\partial \psi}{\partial y} + x \right) \end{cases}$$

- Yields:

$$\boxed{\begin{aligned} \frac{\partial \eta}{\partial y} &= G\,\theta' \left(\frac{\partial \psi}{\partial x} - y \right) \\ -\frac{\partial \eta}{\partial x} &= G\,\theta' \left(\frac{\partial \psi}{\partial y} + x \right) \end{aligned}}$$

- We substitute these terms in the Laplace equation for ψ and re-arrange:

$$\begin{cases} \dfrac{\partial \eta}{\partial y} = G\theta'\left(\dfrac{\partial \psi}{\partial x} - y\right) \\ -\dfrac{\partial \eta}{\partial x} = G\theta'\left(\dfrac{\partial \psi}{\partial y} + x\right) \\ \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} = 0 \end{cases}$$

- And we finally obtain the Poisson's equation for the torsion stress function h:

$$\boxed{\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = -2G\theta'}$$

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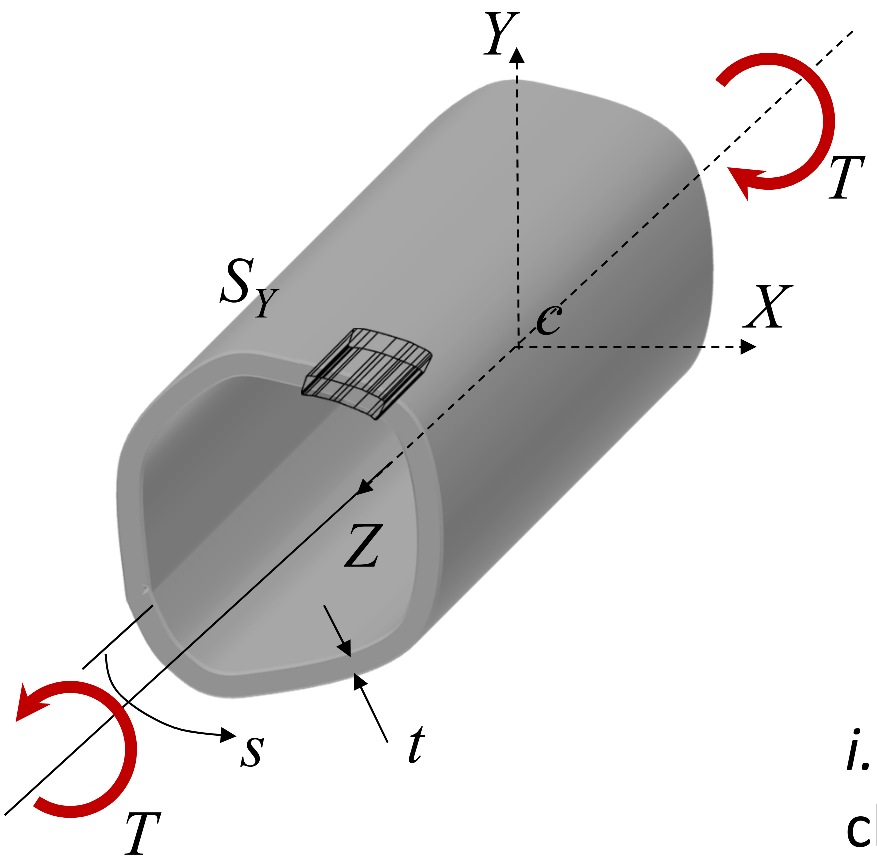
Torsion of closed thin-walled sections

- Pure free torsion assumes no axial constraint, *i.e.* zero direct stress:



$$\sigma_z = 0$$

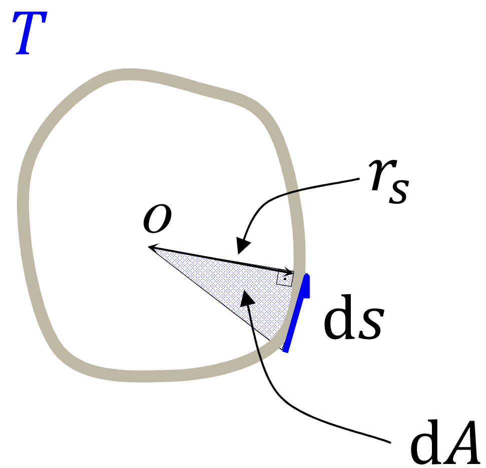
$$\cancelto{0}{t\,\frac{\mathrm{d}\sigma_z}{\mathrm{d}z}} + \frac{\mathrm{d}q_s}{\mathrm{d}s} = 0$$

$$\frac{\mathrm{d}q_s}{\mathrm{d}s} = 0$$

i.e. the shear flow is <u>**constant**</u> around closed sections in pure free torsion

Torsional Equilibrium

- For any arbitrary point O:



$$T = \oint q_s \, r_s \, \mathrm{d}s$$

$$T = \bar{q} \oint r_s \, \mathrm{d}s$$

$$T = \bar{q} \, (2 \, A)$$

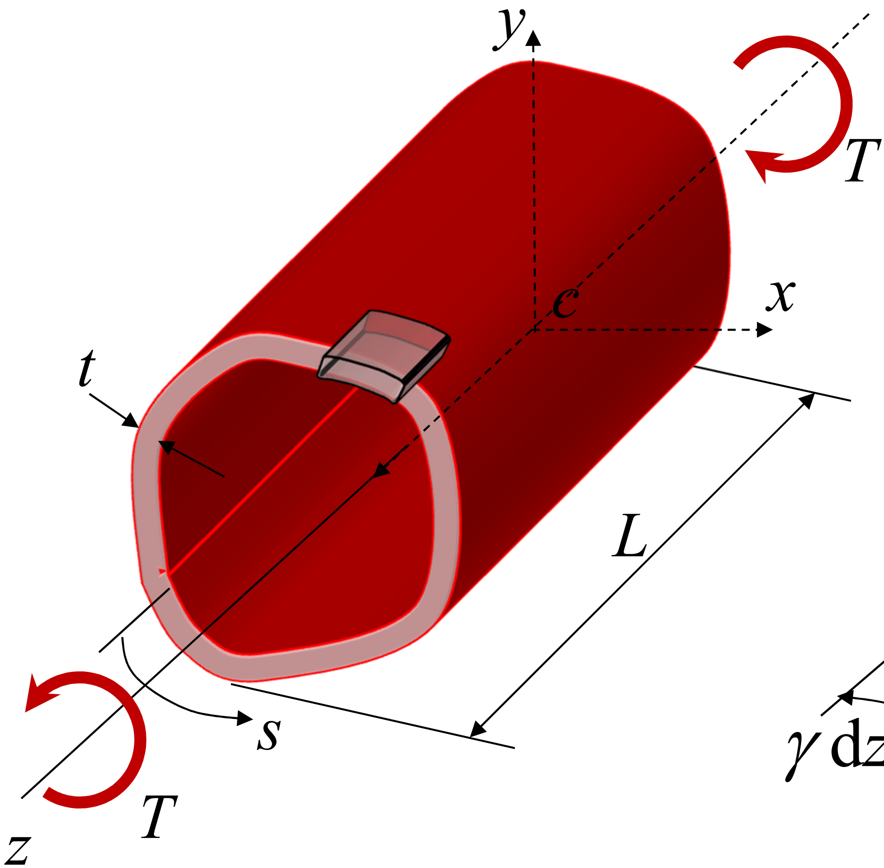
$$\boxed{\bar{q} = \frac{T}{2 \, A}}$$

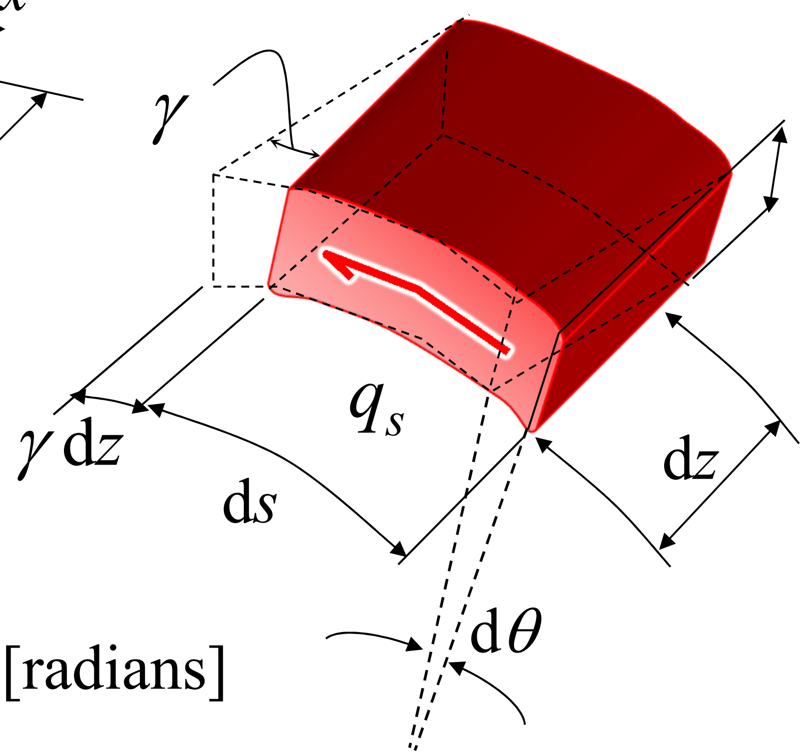
- This is known as the **Bredt-Batho** shear flow equation for torsion of closed thin-walled sections
- Note that torsional equilibrium must exist about any point, even points outside section

Angle of Twist

- The angle of twist θ is commonly expressed in rate form as:

$$\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{\theta}{L} = \frac{T}{GJ}$$





$$\gamma = \text{shear strain [radians]}$$

Angle of Twist

- Consider an incremental length dz

- The external work done by the applied torque is:

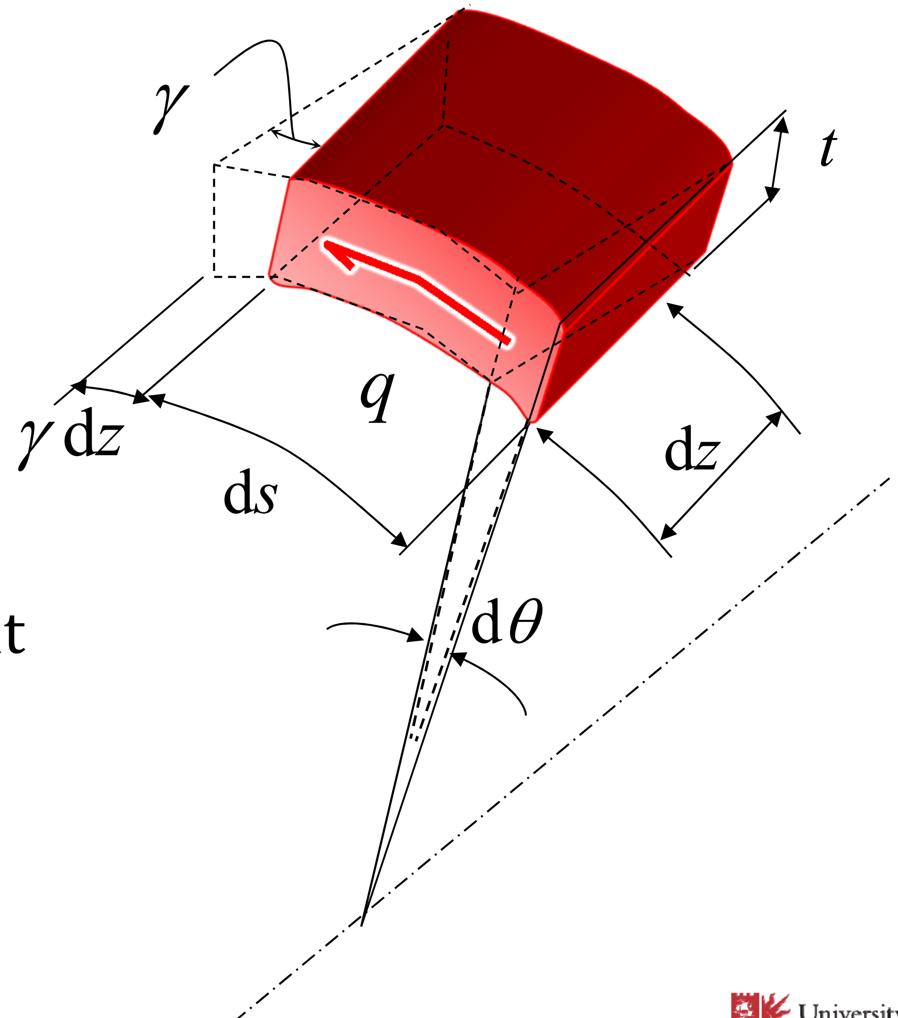
$$U_{\text{ext}} = \frac{1}{2} \times \text{torque } \times \text{twist}$$

$$U_{\text{ext}} = \frac{1}{2} T\,\theta'\,\text{d}z$$

- The internal strain energy is:

$$U_{\text{int}} = \frac{1}{2} \times \text{force } \times \text{displacement}$$

$$\text{d}U_{\text{int}} = \frac{1}{2} (q_s\,\text{d}s)\,(\gamma\,dz)$$



Angle of Twist

- Assuming linear elasticity: $\gamma = \frac{\tau}{G} = \frac{q_s}{G\,t}$

- Global energy balance gives:

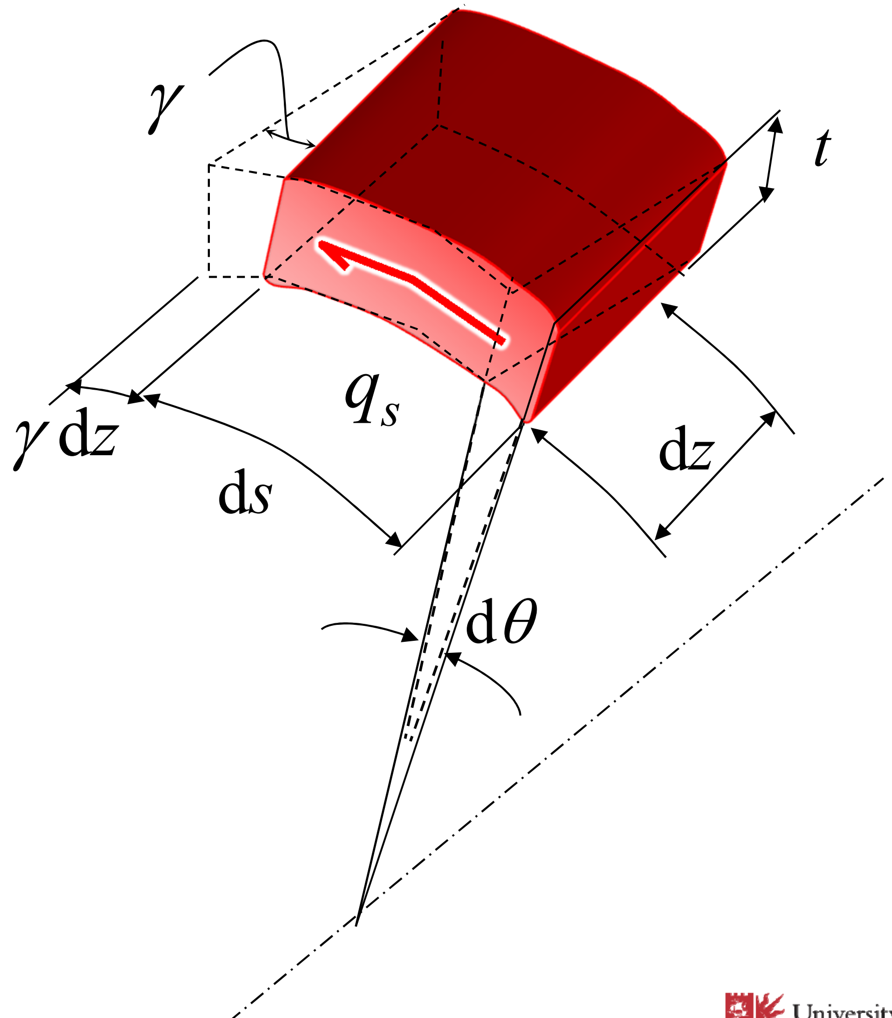
$$U_{\text{ext}} = U_{\text{int}}$$

$$\frac{1}{2} T\,\theta'\,\mathrm{d}z = \oint \frac{1}{2} (q_s\,\mathrm{d}s)\,(\gamma\,dz)$$

$$\frac{1}{2} T\,\theta' = \oint \frac{1}{2} \gamma\,\bar{q}\,\mathrm{d}s$$

$$\frac{1}{2} T\,\theta' = \oint \frac{1}{2} \frac{\bar{q}^2}{G\,t}\,\mathrm{d}s$$

$$\boxed{\bar{q} = \frac{T}{2\,A}}$$



$$\frac{1}{2} T\,\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{4\,A^2} \oint \frac{\mathrm{d}s}{G\,t}$$

Angle of Twist

- For constant G we get:

$$T = \theta' \, G \, \frac{4 \, A^2}{\oint \frac{\mathrm{d}s}{t}}$$

- But remember:

$$T = \theta' \, G \, J$$

- Therefore:

$$J = \frac{4 \, A^2}{\oint \frac{\mathrm{d}s}{t}}$$

- In discrete form:

$$J = \frac{4 \, A^2}{\sum \left(\frac{b_i}{t_i} \right)}$$

Length of wall i

Thickness of wall i