Linear homogeneous ODEs

Lectures 4-5: Linear homogeneous ODEs

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Introduction

The story so far ...

Things to try (in this order)

- **Solve by inspection**, if you can ...
- ODEs that don't explicitly depend on the dependent variable (and do not contain derivatives of different orders)
  Solve by direct integration
- Linear homogeneous first-order ODEs
  Solve by separation of variables
- Nonlinear first-order ODEs
  Try a clever substitution

Linear or non-linear?

Linearity is an important property: linear equations are generally much easier to solve than non-linear ones.

An ODE is *linear* if the *dependent variable* and *its derivatives* do not appear as *products*, *raised to powers*, or as *part of nonlinear functions* (sin, cos, exponents, etc).

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + 5y = \cos(t) \qquad \text{Linear}$$

$$\frac{\mathrm{d}^2\,y}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - y = 0 \qquad \text{Linear}$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + 5y + \boxed{\cos(y)} = 0 \qquad \text{Non-linear}$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,t} - \boxed{5y\frac{\mathrm{d}\,y}{\mathrm{d}\,t}} = 2t \qquad \text{Non-linear}$$

Superposition theorem

Consider the ODE

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + x = 0 \tag{1}$$

We have two solutions $x_1 = \sin t$ and $x_2 = \cos t$. For example

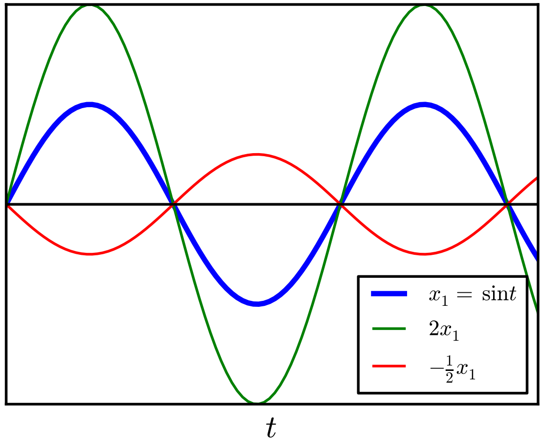
$$\frac{\mathrm{d}^2\,x_1}{\mathrm{d}\,t^2} + x_1 = -\sin t + \sin t = 0.$$

The *general solution* of (**??**) is

$$x_3 = Ax_1 + Bx_2 = A\sin t + B\cos t.$$

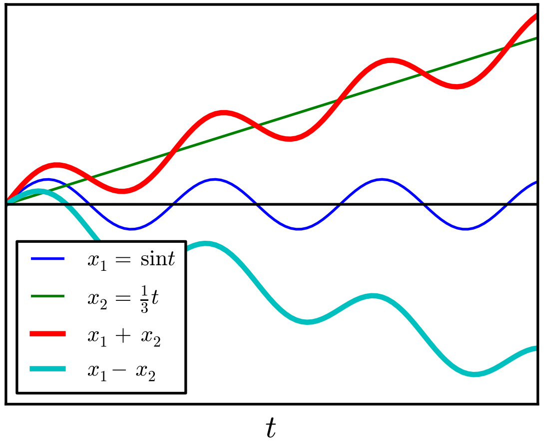
Linearity in pictures

Any multiple of a solution is a solution (picture not referred to eq. (??))



Linearity in pictures

Any sum of two solutions is a solution (picture not referred to eq. (??))



How does linearity work?

We need to define the concept of a **linear operator**.

If $\mathbf{M}$ is a matrix, then for any vector $\mathbf{x}$ and scalar a

$$\mathbf{M}(a\mathbf{x}) = a\mathbf{M}\mathbf{x}$$

Also for any two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\mathbf{M}(\mathbf{x}_1 + \mathbf{x_2}) = \mathbf{M}\mathbf{x}_1 + \mathbf{M}\mathbf{x_2}.$$

These two properties define linearity

Linearity of matrix multiplication

We can combine both properties into an equivalent single rule:

For any two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and scalars a and b

$$\mathbf{M}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{M}\mathbf{x}_1 + b\mathbf{M}\mathbf{x}_2$$

This single property defines linearity

Linear operators

The differential $\frac{\mathrm{d}}{\mathrm{d}\,t}$ is a linear operator on functions.

For any two functions $x_1(t)$ and $x_2(t)$ and constant a

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(ax_1(t)) = a\frac{\mathrm{d}\,x_1(t)}{\mathrm{d}\,t}$$
$$\frac{\mathrm{d}}{\mathrm{d}\,t}(x_1(t) + x_2(t)) = \frac{\mathrm{d}\,x_1(t)}{\mathrm{d}\,t} + \frac{\mathrm{d}\,x_2(t)}{\mathrm{d}\,t}$$

Other operators: expectation and variance

The *expected value* of a random variable X is written $\mathrm{E}(X)$.
E is a linear operator on random variables: for any constant a and random variables X_1 and X_2

$$\mathrm{E}(aX_1) = a\mathrm{E}(X_1), \qquad \mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2).$$

On the other hand Var is *nonlinear* since

$$\mathrm{Var}(aX_1) = \boxed{a^2}\mathrm{Var}(X_1)$$

and also

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \boxed{2\mathrm{Cov}(X_1, X_2)}$$

Other operators: integration

The integral operator I is a linear operator on functions where

$$\mathrm{I}(x(t)) = \int_a^b x(t)\mathrm{d}t.$$

We can see that it is linear since for any constants c and d and functions $x_1(t)$ and $x_2(t)$ we have

$$\begin{aligned}
\mathrm{I}(cx_1(t) + dx_2(t)) &= \int_a^b (cx_1(t) + dx_2(t))\mathrm{d}t \\
&= \int_a^b cx_1(t)\mathrm{d}t + \int_a^b dx_2(t)\mathrm{d}t \\
&= c\int_a^b x_1(t)\mathrm{d}t + d\int_a^b x_2(t)\mathrm{d}t \\
&= c\,\mathrm{I}x_1(t) + d\,\mathrm{I}x_2(t).
\end{aligned}$$

ODEs and linear operators

We can write ODEs in terms of differential operators. For example

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + x = 0$$

can be written as

$$\mathrm{L}x = 0$$

where the operator L is defined by

$$\mathrm{L}x = \left(\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + 1\right)x = \frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + x.$$

We can write

$$\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + 1$$

Showing linearity

We can show that $\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + 1$ is linear since

$$\begin{aligned}
\mathrm{L}(ax_1 + bx_2) &= \left(\frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + 1\right)(ax_1 + bx_2) \\
&= \frac{\mathrm{d}^2}{\mathrm{d}\,t^2}(ax_1 + bx_2) + ax_1 + bx_2 \\
&= a\frac{\mathrm{d}^2\,x_1}{\mathrm{d}\,t^2} + b\frac{\mathrm{d}^2\,x_2}{\mathrm{d}\,t^2} + ax_1 + bx_2 \\
&= a\left(\frac{\mathrm{d}^2\,x_1}{\mathrm{d}\,t^2} + x_1\right) + b\left(\frac{\mathrm{d}^2\,x_2}{\mathrm{d}\,t^2} + x_2\right) \\
&= a\mathrm{L}x_1 + b\mathrm{L}x_2
\end{aligned}$$

Linear ODEs - formal definition

Linear: An ODE for an unknown function $x(t)$ is *linear* if it can be written in the form

$$\mathrm{L}x(t) = f(t)$$

where $f(t)$ is a given function and L is a linear differential operator.

Homogeneous: If the function $f(t) = 0$ (for all t) then we say that the linear ODE is *homogeneous*. Otherwise it is *non-homogeneous*.

General form of a *linear homogeneous* ODE:

$$\mathrm{L}x = 0$$

Exercise: classification of ODEs

Using the idea of linear operators L classify the following ODEs as linear homogeneous, linear non-homogeneous or nonlinear.

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + tx = 0$$

$$\frac{\mathrm{d}^3\,y}{\mathrm{d}\,x^3} + x^2 = 0$$

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x^2 = 0$$

$$\frac{1}{x}\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = 4$$

(Identify the operator L and check if it is linear)

Exercise: classification of ODEs

Superposition of solutions

Theorem: Given a linear homogeneous ODE

$$\mathrm{L}x = 0$$

any linear combination of solutions is also a solution.

Proof: Suppose x_1 and x_2 are solutions. Then $\mathrm{L}x_1 = 0$ and $\mathrm{L}x_2 = 0$.
If $x_3 = ax_1 + bx_2$ then by linearity of L

$$\mathrm{L}x_3 = \mathrm{L}(ax_1 + bx_2) = a\mathrm{L}x_1 + b\mathrm{L}x_2 = 0.$$

Hence x_3 is also a solution.

Superposition exercise

Find the particular solution of the linear homogeneous ODE

$$t^2\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} - 2t\frac{\mathrm{d}\, x}{\mathrm{d}\, t} + 2x = 0$$

with initial conditions $x(1) = -1$ and $\dot{x}(1) = 0$.

Steps:

- Show that $x_1 = t$ and $x_2 = t^2$ are both solutions
- Hence find the general solution
- Use the initial conditions to find the particular solution

Superposition exercise

So what does this all mean?

For an nth order ODE if we have linearly independent solutions $x_1, x_2, \ldots, x_n$ then the general solution is

$$x = A_1x_1 + A_2x_2 + \cdots + A_nx_n.$$

But how do we find the linearly independent solutions?

There is no general method except that. . .

Constant coefficients

Variable coefficients:

$$a(t)\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2}+b(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t}+c(t)x=0$$

Constant coefficients:

$$a\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2}+b\frac{\mathrm{d}\,x}{\mathrm{d}\,t}+cx=0$$

Linear homogeneous ODEs with constant coefficients must always have at least one solution of the form

$$x=\mathrm{e}^{mt}$$

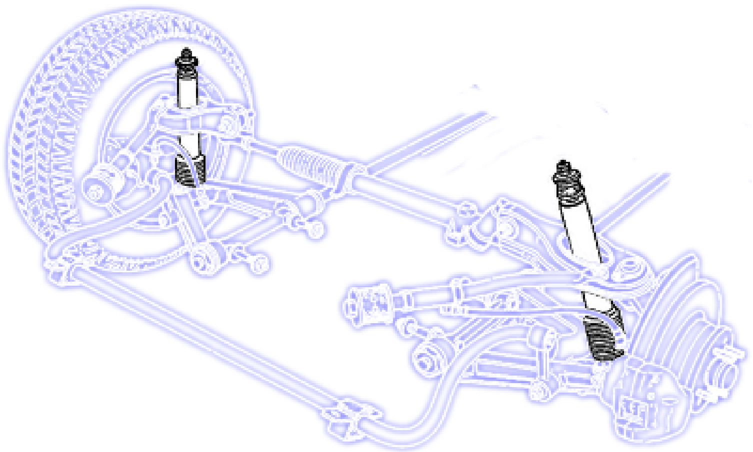
where m is a constant.

Linear ODEs with constant coefficients

Introductory Example

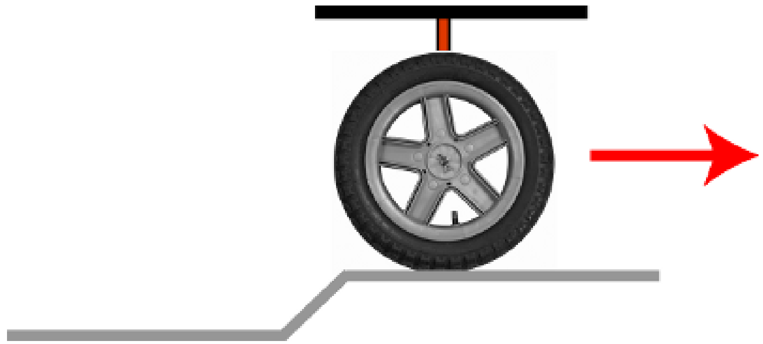
Higher-order linear homogeneous ODEs with constant coefficients appear in many engineering applications ...

Mass on a spring: suspension



Example: Car Suspension

Mass on a spring: suspension



What happens just after the car has driven up a bump?

The gas cylinder in the suspension is now compressed and starts to relax, lifting the car.

Mass on a spring: suspension

Modelling

The underlying equation is Newton's second law

$$F = ma$$

where F are the forces, m is the mass resting on the wheel and a is the (upward) acceleration of the car's body. We can write this as

$$-kx - b\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = m\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2}$$

where x is the elongation of the cylinder, k is the spring constant and b is the damping constant. We can rewrite this as ($c = \frac{b}{m}$ and $\omega^2 = \frac{k}{m}$)

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + c\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \omega^2 x = 0.$$

Linear Homogeneous Constant-Coefficient ODEs

Solving ODEs:
Linear Homogeneous ODEs with Constant Coefficients

In this relatively simple class of equations the order does not complicate the solution much

Linear Homogeneous Constant-Coefficient ODEs

First-order Example

Constant coefficient ODEs tend to have exponential solutions.

For example

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = mx$$

has the general solution

$$x(t) = A\mathrm{e}^{mt}$$

where A is an arbitrary factor.

Second order example

Let's solve

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} - 4x = 0$$

with $x(0) = 1$ and $\dot{x}(0) = 0$.

We now try the Ansatz

$$x(t) = \mathrm{e}^{mt} \quad \rightarrow \quad \frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = m^2\mathrm{e}^{mt}$$

In the ODE this gives the **characteristic equation**

$$m^2\mathrm{e}^{mt} - 4\mathrm{e}^{mt} = (m^2 - 4)\mathrm{e}^{mt} = 0 \quad \rightarrow \quad m^2 - 4 = 0$$

so $m = \pm 2$.

Why two solutions for m?

We tried the Ansatz $x = \mathrm{e}^{mt}$ and found that it works if $m = \pm 2$. So which value of m is correct?

They're both correct. Each gives a different solution:

$$x_1 = \mathrm{e}^{m_1 t} = \mathrm{e}^{2t}, \quad x_2 = \mathrm{e}^{m_2 t} = \mathrm{e}^{-2t}$$

We can easily check that each is a solution since

$$\frac{\mathrm{d}^2\left(\mathrm{e}^{2t}\right)}{\mathrm{d}\,t^2} = 4\mathrm{e}^{2t}, \qquad \frac{\mathrm{d}^2\left(\mathrm{e}^{-2t}\right)}{\mathrm{d}\,t^2} = 4\mathrm{e}^{-2t}$$

so e.g.

$$\frac{\mathrm{d}^2\,x_1}{\mathrm{d}\,t^2} - 4x_1 = \frac{\mathrm{d}^2\left(\mathrm{e}^{2t}\right)}{\mathrm{d}\,t^2} - 4\mathrm{e}^{2t} = 4\mathrm{e}^{2t} - 4\mathrm{e}^{2t} = 0.$$

The general solution

So we have two solutions

$$x_1 = \mathrm{e}^{m_1 t} = \mathrm{e}^{2t}, \quad x_2 = \mathrm{e}^{m_2 t} = \mathrm{e}^{-2t}$$

The general solution is a linear combination

$$x_3 = Ax_1 + Bx_2 = A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}$$

This is also a solution since

$$\begin{aligned}\frac{\mathrm{d}^2\,x_3}{\mathrm{d}\,t^2} - 4x_3 &= \frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\left(A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}\right) - 4(A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}) \\ &= 4A\mathrm{e}^{2t} + 4B\mathrm{e}^{-2t} - 4(A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}) \\ &= 0\end{aligned}$$

And the initial conditions...

The general solution is

$$x = A\,\mathrm{e}^{2t} + B\,\mathrm{e}^{-2t}.$$

We have initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$.
Since $x(0) = 1$ we have

$$x(0) = A\,\mathrm{e}^{0} + B\,\mathrm{e}^{0} = A + B = 1$$

For the other condition we need to find $\dot{x}$ which is

$$\dot{x}(t) = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = 2A\,\mathrm{e}^{2t} - 2B\,\mathrm{e}^{-2t}.$$

So $\dot{x}(0) = 0$ gives

$$\dot{x}(0) = 2A\,\mathrm{e}^{0} - 2B\,\mathrm{e}^{0} = 2A - 2B = 0$$

And the initial conditions...

The general solution is

$$x = A\,\mathrm{e}^{2t} + B\,\mathrm{e}^{-2t}.$$

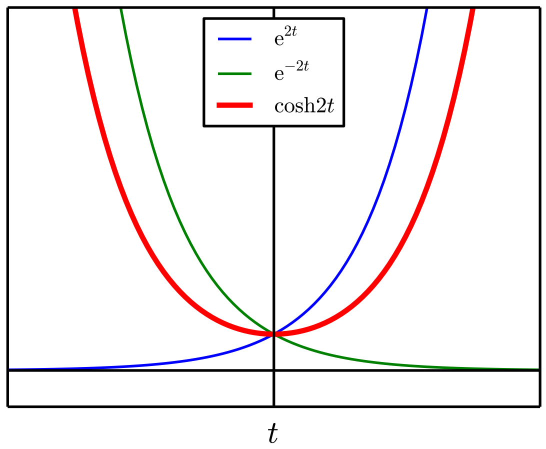
We have two equations for the constants A and B since our initial conditions have given

$$A + B = 1, \qquad 2A - 2B = 0$$

Solving these gives $A = B = \frac{1}{2}$ and

$$x = \frac{1}{2}\,\mathrm{e}^{2t} + \frac{1}{2}\,\mathrm{e}^{-2t} \quad (= \cosh 2t)$$

Particular solution



Linear Homogeneous Constant-Coefficient ODEs

Algorithm

The general procedure for solving a linear homogeneous ODE n-th order ODE with constant coefficients is

- Use Ansatz $x = \mathrm{e}^{mt}$ to find the characteristic equation for m
- Find the roots of the characteristic equation.
- Use the roots to obtain n linearly independent solutions.
- Write the general solution as a linear combination of particular solutions.

Undamped oscillations

Consider the suspension example with $c = 0$ (no damping) and $\omega^2 = 4$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + 4x = 0.$$

We try the Ansatz $x = \mathrm{e}^{mt}$ to find the characteristic equation

$$m^2 \mathrm{e}^{mt} + 4 \mathrm{e}^{mt} = 0 \quad \rightarrow \quad m^2 = -4$$

so $m = \pm 2j$ and

$$x_1 = \mathrm{e}^{m_1 t} = \mathrm{e}^{+2jt}, \qquad x_2 = \mathrm{e}^{m_2 t} = \mathrm{e}^{-2jt}$$

which gives

$$x = A\mathrm{e}^{+2jt} + B\mathrm{e}^{-2jt}$$

Why is the solution complex?

We had a *real* ODE but our solution is

$$x = A\,\mathrm{e}^{2jt} + B\,\mathrm{e}^{-2jt}$$

Use Euler's equation

$$\mathrm{e}^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$x = A(\cos 2t + j\sin 2t) + B(\cos 2t - j\sin 2t)$$

$$x = (A + B)\cos 2t + j(A - B)\sin 2t$$

$$x = C\cos 2t + D\sin 2t$$

Note that C and D can both be *real*.

Final simplification

We can always rewrite

$$x = C\cos 2t + D\sin 2t$$

as

$$x = E\cos(2t+\phi)$$

with new constants E and ϕ.

General solutions for undamped SHM

Given the ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + \omega^2 x = 0$$

we can write the general solution in three ways

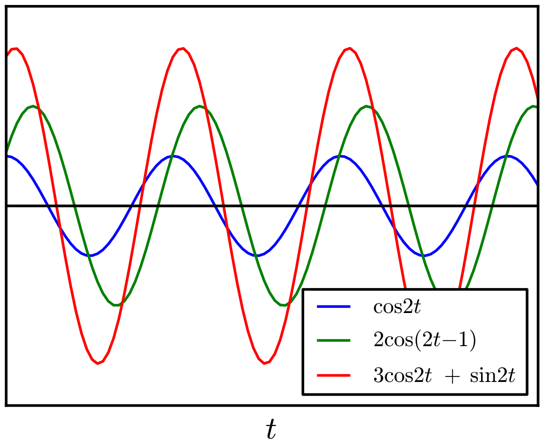
$$x = A\,\mathrm{e}^{+j\omega t} + B\,\mathrm{e}^{-j\omega t}$$

$$x = C\cos\omega t + D\sin\omega t$$

$$x = E\cos(\omega t + \phi)$$

All solutions are translations and scalings of $\cos\omega t$ (or $\sin\omega t$): the car bounces up and down *forever*!

Particular solutions for undamped SHM



Can we bounce forever?

We assumed that the suspension is *undamped* i.e. that $c = 0$.

With damping we have in general

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + c\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \omega^2 x = 0$$

where $c \neq 0$.

This makes a **big difference** to the solutions.

Exercise: Damped SHM

Suppose that $\omega^2 = 4$ as before but now $c = 5$. Solve

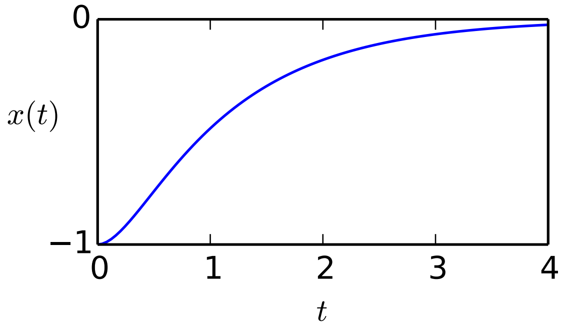
$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + 5\frac{\mathrm{d}\, x}{\mathrm{d}\, t} + 4x = 0$$

with $x(0) = -1$ and $\dot{x}(0) = 0$.

Exercise: Damped SHM

Exercise: Damped SHM

$$x(t) = \tfrac{1}{3}\mathrm{e}^{-4t} - \tfrac{4}{3}\mathrm{e}^{-t}$$



The car is slowly lifted to the neutral position.
Could we make it go there faster by reducing damping?

Underdamped SHM

Engineers reduce the damping on the suspension to $c = 2$. The new ODE is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + 2\frac{\mathrm{d}\, x}{\mathrm{d}\, t} + 4x = 0$$

How much of a difference does that make?

Now the Ansatz $x = \mathrm{e}^{mt}$ gives

$$m^2 + 2m + 4 = 0$$

We can use the quadratic formula to find

$$m_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \times 4}}{2} = -1 \pm \frac{\sqrt{-12}}{2} = -1 \pm \sqrt{3}j$$

Underdamped SHM

We found that $m_{1,2} = -1 \pm \sqrt{3}j$ so we get two solutions

$$x_1 = \mathrm{e}^{(-1+\sqrt{3}j)t}, \qquad x_2 = \mathrm{e}^{(-1-\sqrt{3}j)t}$$

The general solution is

$$x = Ax_1 + Bx_2 = A\,\mathrm{e}^{(-1+\sqrt{3}j)t} + B\,\mathrm{e}^{(-1-\sqrt{3}j)t}$$

We can simplify this

$$\begin{aligned} x &= A\,\mathrm{e}^{(-1+\sqrt{3}j)t} + B\,\mathrm{e}^{(-1-\sqrt{3}j)t} \\ &= A\,\mathrm{e}^{-t+\sqrt{3}jt} + B\,\mathrm{e}^{-t-\sqrt{3}jt} \\ &= A\,\mathrm{e}^{-t}\,\mathrm{e}^{+\sqrt{3}jt} + B\,\mathrm{e}^{-t}\,\mathrm{e}^{-\sqrt{3}jt} \\ &= \mathrm{e}^{-t}(A\,\mathrm{e}^{+\sqrt{3}jt} + B\,\mathrm{e}^{-\sqrt{3}jt}) \end{aligned}$$

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Final form

But wait our solution

$$x = \mathrm{e}^{-t}(A\,\mathrm{e}^{+\sqrt{3}jt} + B\,\mathrm{e}^{-\sqrt{3}jt})$$

is just like the solution in the undamped case but multiplied by e^{-t}.

We can rewrite this in the same three ways as before so

$$x = \mathrm{e}^{-t}(A\,\mathrm{e}^{+j\sqrt{3}t} + B\,\mathrm{e}^{-j\sqrt{3}t})$$

$$x = \mathrm{e}^{-t}(C\cos\sqrt{3}t + D\sin\sqrt{3}t)$$

$$x = E\,\mathrm{e}^{-t}\cos(\sqrt{3}t + \phi).$$

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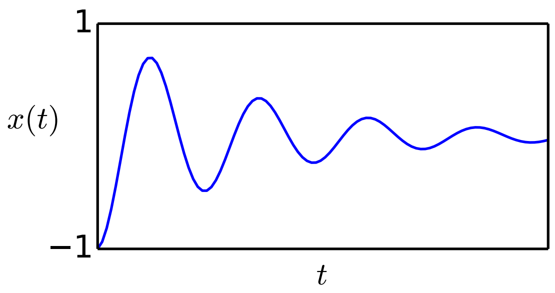
Linear Homogeneous Constant-Coefficient ODEs

$$-\,\mathrm{e}^{-at}\cos bt$$



Now we overshoot. Not good!

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Linear Homogeneous Constant-Coefficient ODEs

Aperiodic limit

In the real world engineering problem we want to find the lowest value of damping that still prevents an overshoot.

By considering the ODE with a damping parameter

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0$$

one can find the characteristic equation

$$m^2 + cm + 4 = 0$$

which has the solutions

$$m = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - 4}$$

So the smallest positive value of c that does not lead to complex roots is $c = 4$.

Linear Homogeneous Constant-Coefficient ODEs

Multiple roots: A Complication

In the aperiodic limit

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + 4\frac{\mathrm{d}\, x}{\mathrm{d}\, t} + 4x = 0$$

we encounter a problem:
The characteristic equation has two identical roots

$$m_1 = m_2 = -2$$

and thus

$$x_1(t) = x_2(t) = \mathrm{e}^{-2t}$$

so we don't find a second linearly independent solutions that we need to write the general solution.

Linear Homogeneous Constant-Coefficient ODEs

Multiple roots: Solution

Whenever the characteristic equation of an ODE has multiple identical roots, we can find additional linearly independent solutions by multiplying one of the duplicate solutions by t.

For example, for

$$m_1 = m_2 = -2$$

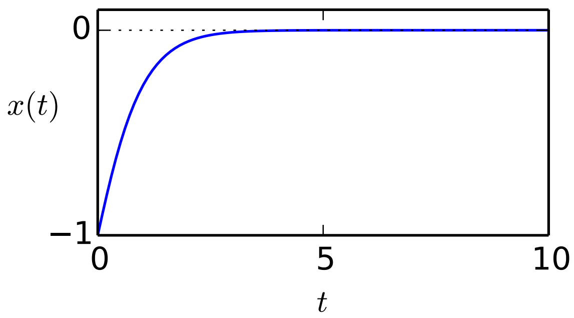
we get the solutions

$$x_1(t) = \mathrm{e}^{-2t} \qquad x_2(t) = t\mathrm{e}^{-2t}$$

Hence the general solution is

$$x(t) = A\mathrm{e}^{-2t} + Bt\mathrm{e}^{-2t}$$

Linear Homogeneous Constant-Coefficient ODEs



The car returns to the neutral position as quickly as it is possible without overshooting.

Homework

James 5th edition
Read section 10.9.1
solve exercises from 10.9.2

James 4th edition
Read section 10.9.1
solve exercises from 10.9.2