Tom Rendall

thomas.rendall@bristol.ac.uk

Aerodynamics 3

Introduction to Course

Compressible flow (chapter 2 in notes)

University of BRISTOL

Overview:

- The primary aim of this year is to introduce to you the way in which the ideas you have been taught in the first two years are applied to real problems
- This will necessarily mean getting involved with some difficult concepts and tricky mathematics. Keep your eye on the bigger picture, i.e. why we are doing what we are doing
- If there's something you don't understand, come and speak to me!

Expectations - derivations

- Do you need to know derivations? **Yes**
- Do you need to memorise derivations? **I don't know, it depends on how you learn**
- **The expectation is that given one step in a derivation, you would be able to get to the next** (one 'step' might require a few pieces of algebra). I would not expect you to be able to reproduce the while working from memory
- This means 'derivation' questions will contain intermediate pieces of information to help you – take a look at past papers to see this in practice
- Can I give you a complete list of everything you need to know for the exam? Yes – during lectures!
- Can I tell you everything you need to know to get 100%? Yes and no. I can tell you the facts and explanations, but you must work out how to understand them, and then judge how well you have understood. Learning is, partly, a solo journey. That doesn't mean I won't help but it does mean you have to let me know how to help

University of BRISTOL

Course Structure

- **Compressible flow (2 lect)**

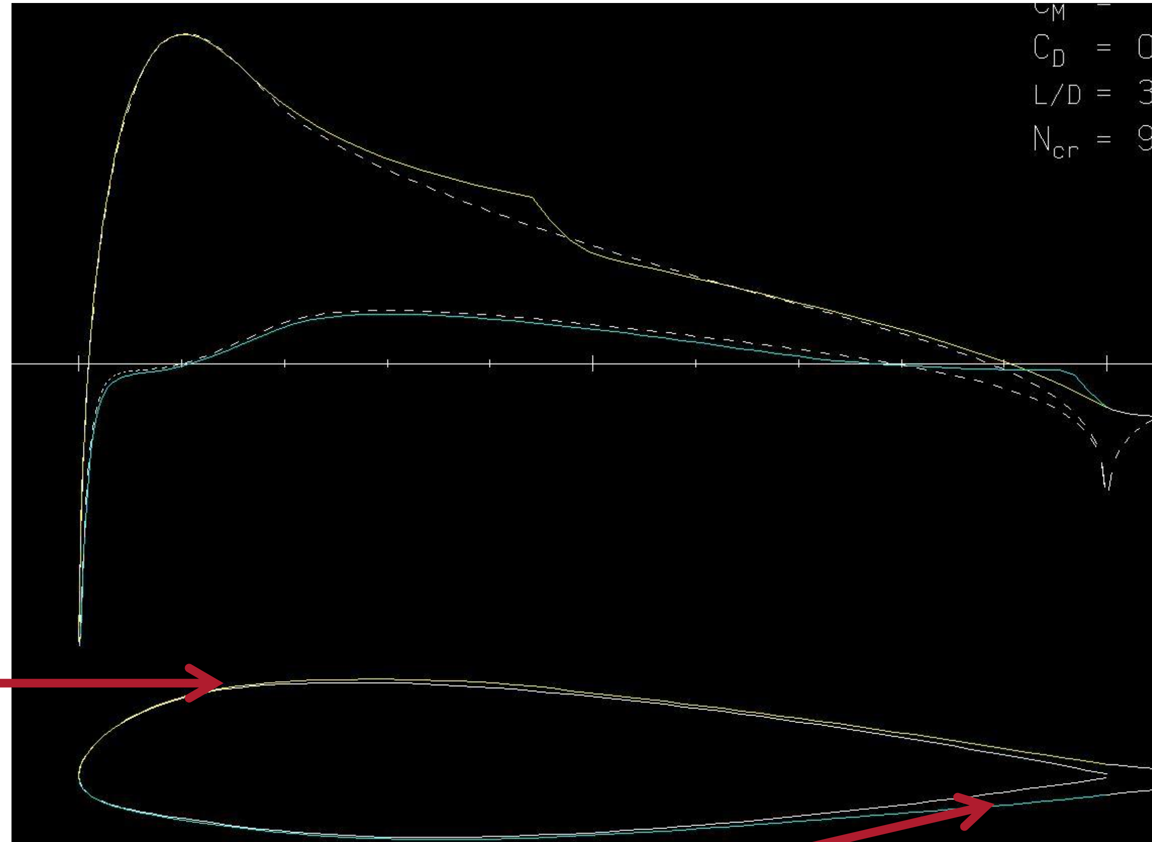
Some topics revisited from aero 2, but more on 2D transonics and linearised 3D to equip you for the group design project next year

- **Joukowsky transformation and the complex potential (6 lect)**
- **Laminar and turbulent boundary layers (12 lect)**

A single piece of coursework using the compressible and incompressible tools. I will plan to set this after roughly the 3rd or 4th lecture. We don't want this to overlap with other coursework, so the plan is to start **early!** This is in your interests – semesterisation.

Why this structure – divide and conquer!

- Aerodynamicists like to break things in to manageable chunks that we can understand in separation
- Compressible lectures will describe the effects of transonics and how you can estimate them
- The Joukowsky transformation/complex potential allows you understand the inviscid external flow around an aerofoil
- The boundary layers can then be linked to the compressible and incompressible inviscid models to provide a complete understanding

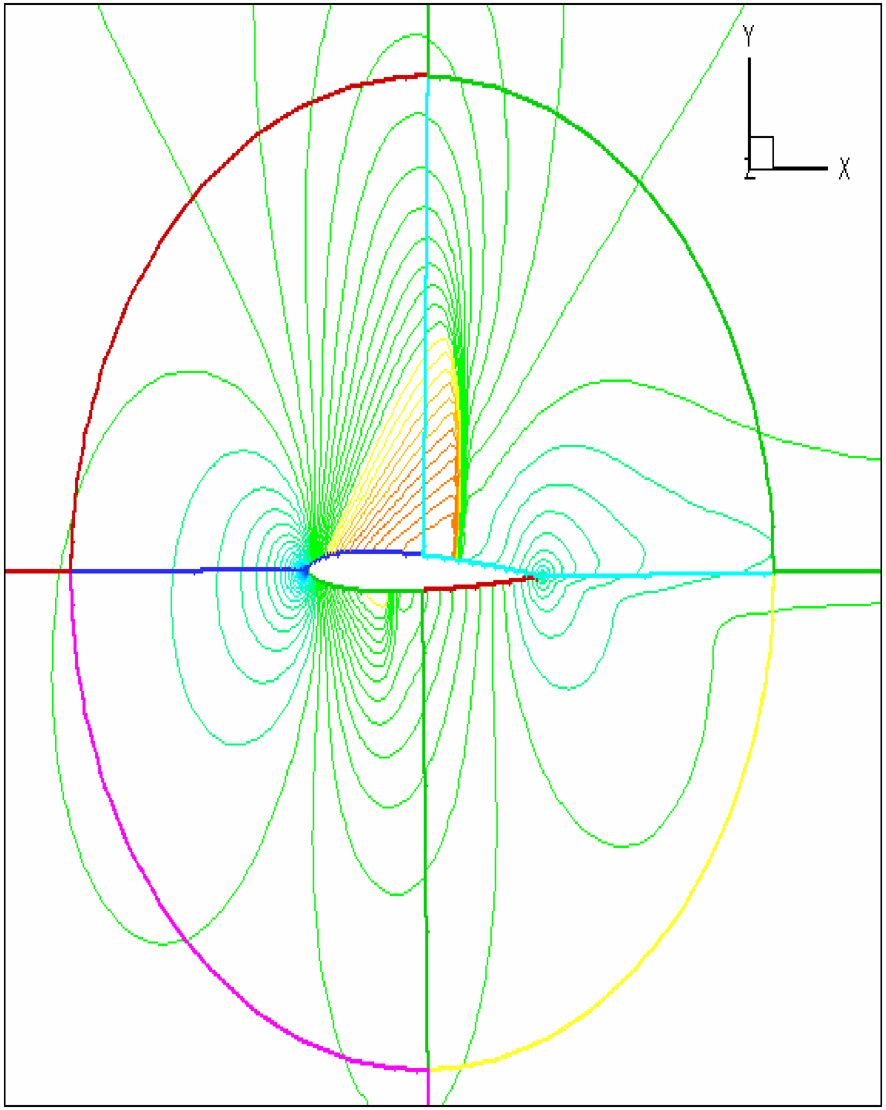


One model here (compressible or incompressible)

Another model here!

Recap: The role of the Aerodynamicist

- What an aerodynamicist does:
  - Predicts body forces and moments about a shape
  - To do this we need to predict the flow about a shape
  - This immediately raises questions
    - *How much of the flow?*
    - *How accurately?*



Aerodynamics Approaches

- Flow is governed by the NAVIER-STOKES equations
  - Unsteady, compressible, viscous 3d.
  - Non-linear partial differential equations -> insoluble analytically
- Hence, we must use physical insight to reduce the equations. There are two main ways:
  - Classical Aerodynamics – Exact solutions of approximate equations
  - Computational Aerodynamics- Approximate solutions of exact equations

Today

- Why operate at transonic speed?
- Breguet range (briefly, again!)
- Ideas relating to wave drag in 2D
- A wonderful Breguet-compressible link
- Supercritical foils and the Korn equation in 2D and 3D
- CFD for judging aerofoils
- Bookmark https://www.polleverywhere.com/vote

University of BRISTOL

Why Mach 0.8?

- For personal transport you drive at the most economical speed. At least, I try to. This means we get from A to B burning a minimum of fuel

BUT

vs

?

Per trip, not metered

When paid per trip, drive more quickly to operate more trips in the same amount of time. Increased fares outweigh increased fuel costs.

University of BRISTOL

Airlines want…profit

- Aircraft A flies a journey in half the time of aircraft B, but aircraft A burns more fuel. Assuming crew and maintenance costs are the same for both, aircraft A is preferable, providing the fuel cost difference is not enormous, as you can sell twice as many tickets.
- Fuel cost is just one component of operating costs

Profit is difficult to estimate

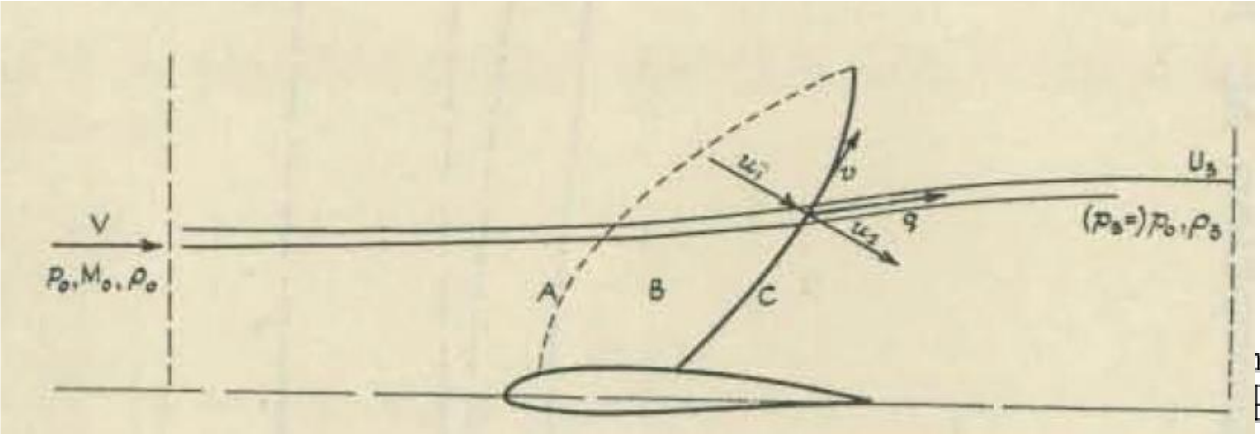
- So many factors influence the profit (fuel, crew costs, crew pension scheme, loan rates, landing fees…) that making an estimate is best done by accountants, not engineers!
- However, Breguet range depends on ML/D, and this is a good indicator for the aircraft. L/D would be too restrictive, only covering fuel burn. Including M goes some way to allowing for `profit’ rather than fuel. It also means you can fly to more destinations!

University of BRISTOL

Lock's semi-empirical Wave drag estimate (2D)

$$C_{d_w} = K(M - M_{crit})^4$$

Not easy to explain, but based on sound normal shock theory. Drag of a shock per depth varies with the cube of M, and the height is about proportional to M, so this gives us a fourth power. Roughly.



So the Breguet question is

$$f = \frac{ML}{D} = \frac{MC_L}{K(M - M_{crit})^4 + C_{d_0} + \frac{C_L^2}{\pi A_R}}$$

$$\frac{df}{dM} = \frac{C_L(K(M - M_{crit})^4 + C_{d_{tot}}) - 4K(M - M_{crit})^3 MC_L}{(K(M - M_{crit})^4 + C_{d_{tot}})^2}$$

Some generic total drag

$$K(M - M_{crit})^4 + C_{d_{tot}} - 4K(M - M_{crit})^3 M = 0$$

Small

Bigger

If Cdtot=0, then M=Mcrit

If Cdtot>0, the \`bigger' term dominates, and M>Mcrit

Thus, **transonic!**

But I prefer my polynomials solved

- Can use the iteration (K=20 not a bad guess – more on this later)

$$M_{i+1} = \left(\frac{20(M_i - M_{crit})^4 + C_{d_{tot}}}{80M_i} \right)^{\frac{1}{3}} + M_{crit}$$

(spreadsheet available on BB)

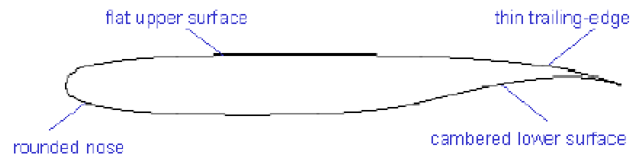
RAE 2822 @ Cl=0.6 (about), Mcrit=0.69, Mcruise=0.765

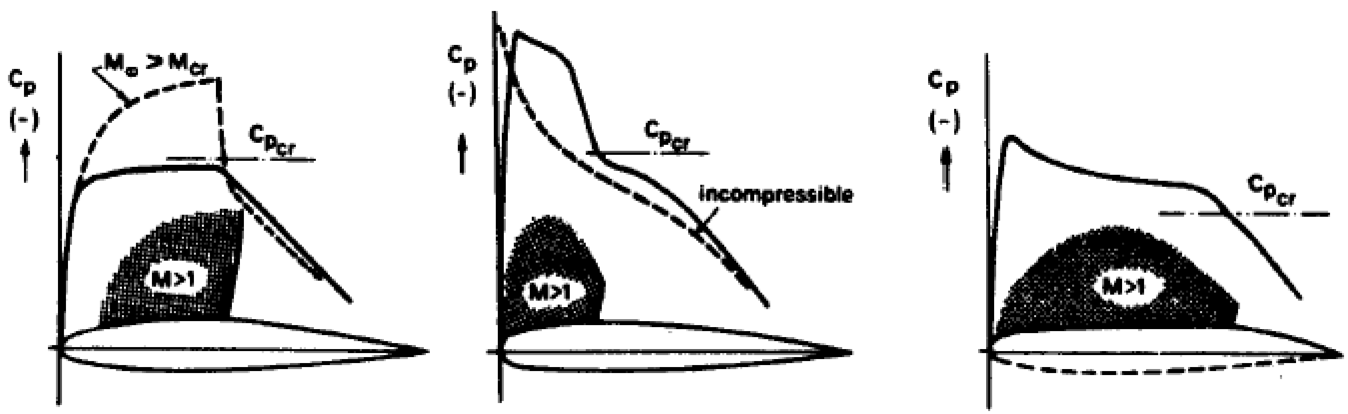
Increases to 0.8 for Mcrit= 0.73

Or to about 0.83 for Cl=0.6 with 25deg sweep

Supercritical sections

- Now we know M, can design specifically for transonic flight
- Supercritical = beyond critical M





a. Conventional section with roof-top pressure distribution

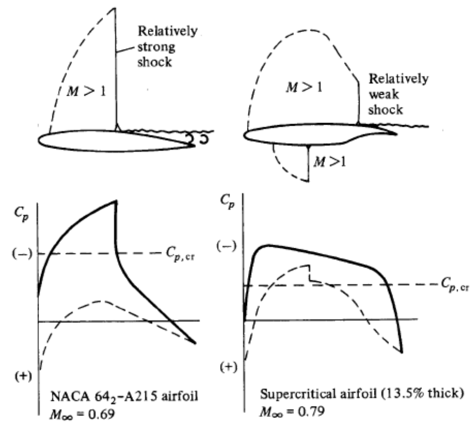
b. Peaky upper surface pressure distribution

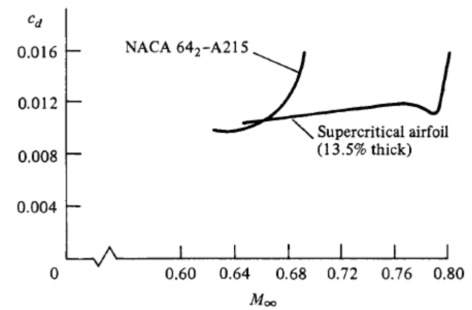
c. Supercritical upper

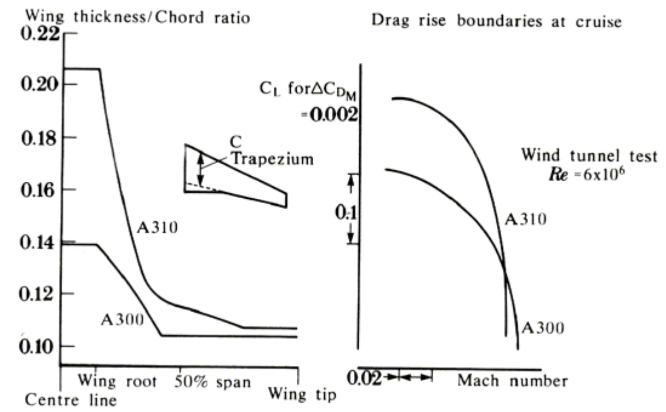
Supercritical sections

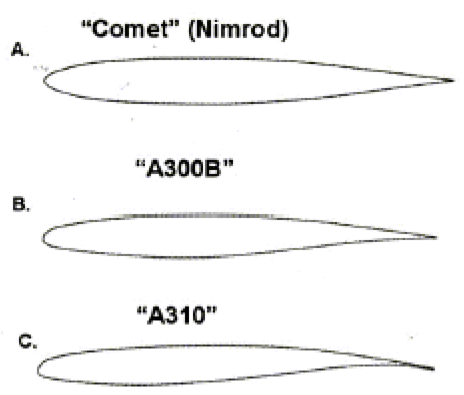
Pro – thicker, lower drag, higher lift

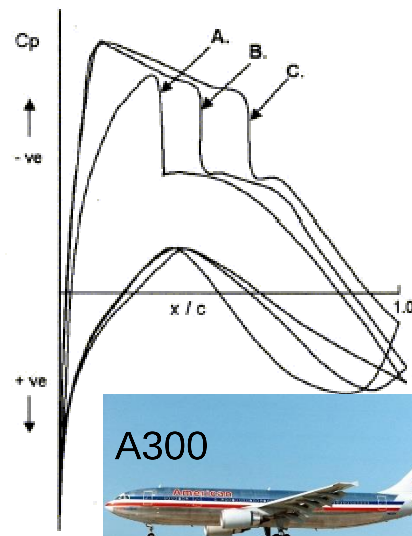
Con – pitching moment











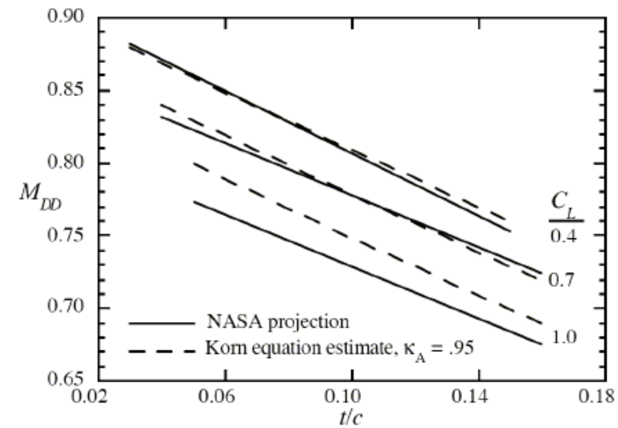
Final thoughts

- Mcrit depends on thickness and Cl. We can trade one for the other if we want to
- Korn equation helps

$$M_{dd} + \left(\frac{t}{c}\right) + \frac{C_L}{10} \approx \kappa$$

Aerofoil \`technology' factor

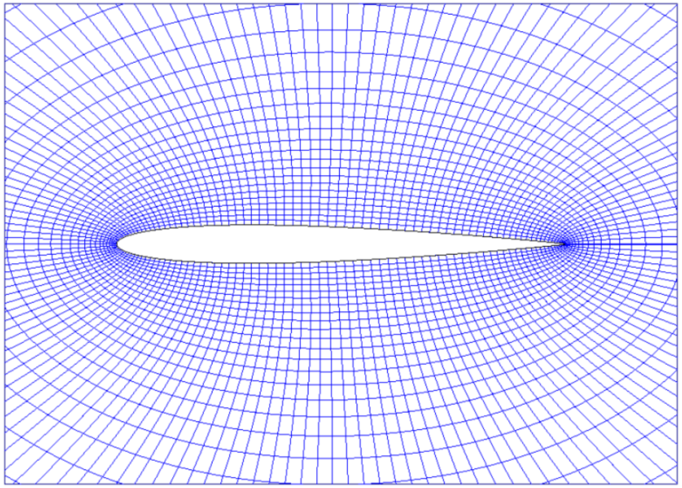
- This factor is roughly constant for a particular section
- Mdd reached when $\frac{\partial C_D}{\partial M} = 0.1$



The mystery factor

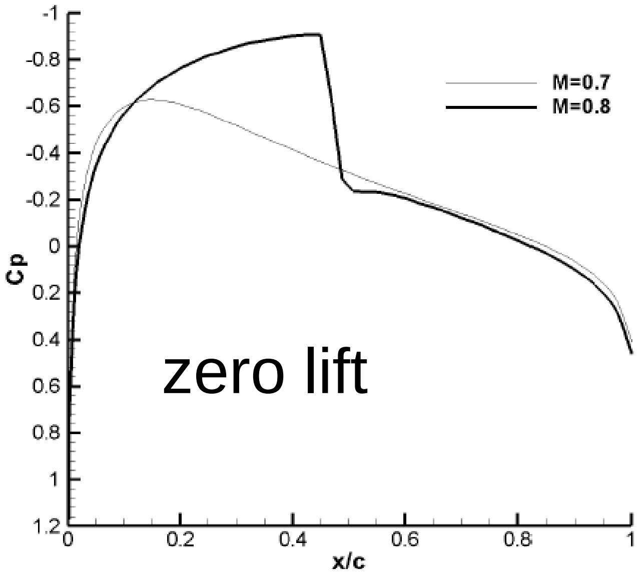
- The tech factor might be found experimentally. However, transonic tests are expensive, although this would give the best accuracy results
- For comparative purposes computational fluid dynamics (CFD) can help!

VGK – viscous Garabedian Korn

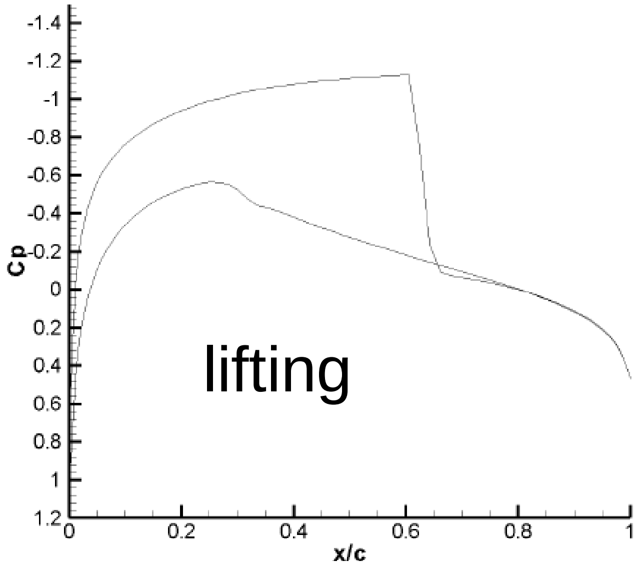


A mesh – in this case for a finite difference solver for the full potential equation

0012



(a) M=0.7 and 0.8, $\alpha = 0^o$



(b) M=0.8, $\alpha = 1.25^o$

Approach

M	$C_{D_{VGK}}$	$C_{D_{Lock}}$	$\frac{\partial C_D}{\partial M}$
0.7400	-0.0009	0.0018	N/A
0.7500	-0.0009	0.0026	N/A
0.7600	-0.0007	0.0037	N/A
0.7700	0.0001	0.0052	0.1200 (for M=0.775)
0.7800	0.0013	0.0070	0.1800
0.7900	0.0037	0.0093	0.2850
0.8000	0.0070	0.0121	0.3700
0.8100	0.0111	0.0155	0.4500
0.8200	0.0160	0.0195	0.5600
0.8300	0.0223	0.0244	0.6500
0.8400	0.0290	0.0300	0.7300
0.8500	0.0369	0.0366	0.8000
0.8600	0.0450	0.0442	0.8700
0.8700	0.0543	0.0529	0.9350
0.8800	0.0637	0.0629	0.9300
0.8900	0.0729	0.0742	0.9400
0.9000	0.0825	0.0870	0.9400
0.9100	0.0917	0.1013	0.7400
0.9200	0.0973	0.1174	N/A

Cdlock=20*(M-(Mcrit+0.1))^4

The 0.1 is observed by experience

Agreement reasonable

$$\frac{\partial C_D}{\partial M} = \frac{C_D(M+\delta) - C_D(M-\delta)}{2\delta} + O(\delta^2)$$

Mdd near 0.77

Hence kappa ~0.89

This is optimistic due to inviscid method

Poll

University of BRISTOL

Know

- Why we operate M>Mcrit
- The consequences of this, and pros/cons of supercritical sections
- Korn equation and what it means
- How to use VGK (download input file for 0012 from BB and try yourself!)

Tom Rendall
thomas.rendall@bristol.ac.uk

Aerodynamics 3

3D Considerations for Compressible flow

(chapter 3 in notes)

University of BRISTOL

Today

- Goethert’s Rule
- Goethert’s Rule applied to lift curve gradients
- Approximate effect of sweep on range
- Area ruling

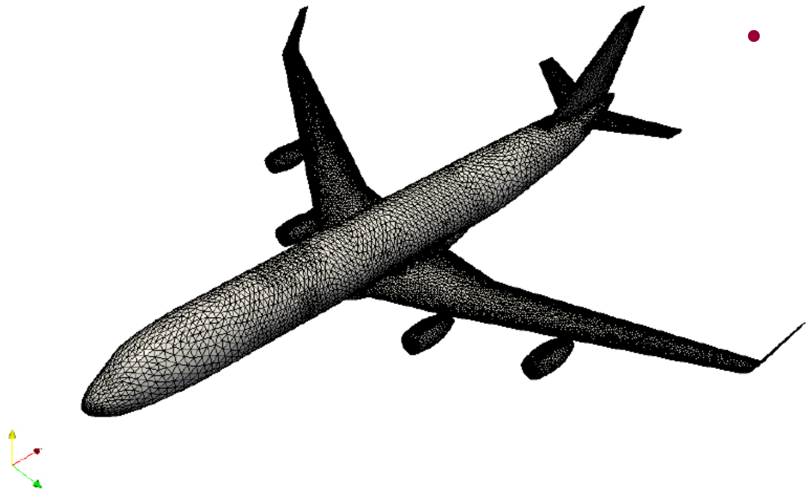
Small disturbance potential equation

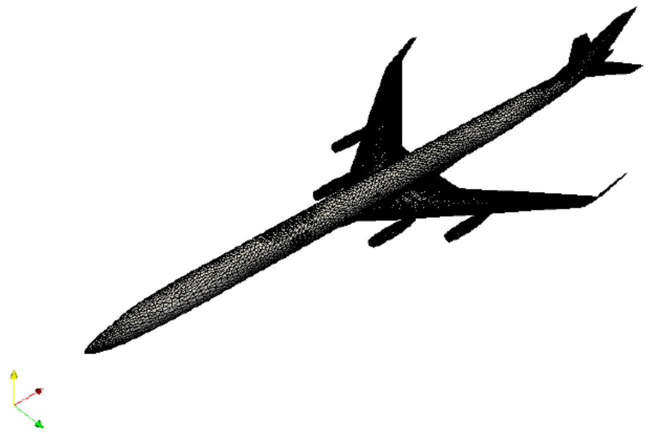
$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x_c^2} + \frac{\partial^2 \phi}{\partial y_c^2} + \frac{\partial^2 \phi}{\partial z_c^2} = 0$$

Last year we looked at 2D

3D is slightly different, because an aspect ratio change is involved

- Linear – no shocks!
- Inviscid





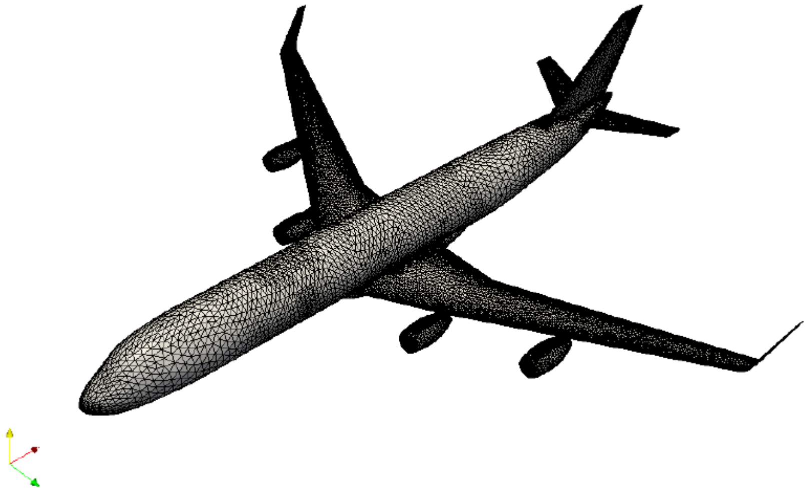
Small disturbance potential equation

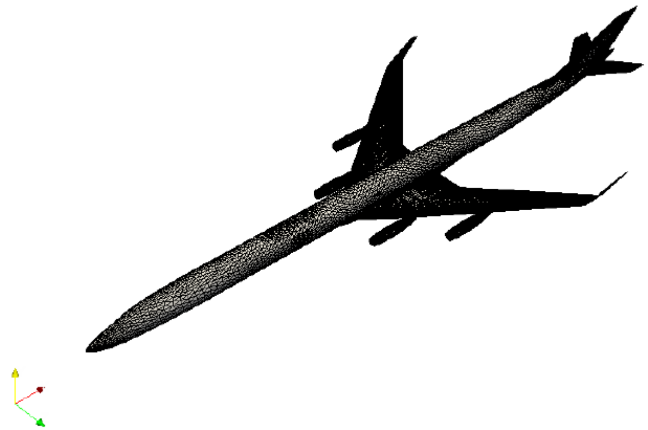
$$\beta = \sqrt{1 - M_\infty^2}$$

$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x_c^2} + \frac{\partial^2 \phi}{\partial y_c^2} + \frac{\partial^2 \phi}{\partial z_c^2} = 0 \longrightarrow \begin{array}{c} x_c = x_{ic} \\ \beta y_c = y_{ic} \\ \beta z_c = z_{ic} \end{array} \longrightarrow \frac{\partial^2 \phi}{\partial x_{ic}^2} + \frac{\partial^2 \phi}{\partial y_{ic}^2} + \frac{\partial^2 \phi}{\partial z_{ic}^2} = 0$$

$$\frac{\partial \phi}{\partial y_c} = \frac{\partial \phi}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial y_c} + \frac{\partial \phi}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial y_c} + \frac{\partial \phi}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial y_c} = \frac{\partial \phi}{\partial y_{ic}}\beta$$

$$\frac{\partial \phi_{y_c}}{\partial y_c} = \frac{\partial \phi_{y_c}}{\partial x_{ic}}\frac{\partial x_{ic}}{\partial y_c} + \frac{\partial \phi_{y_c}}{\partial y_{ic}}\frac{\partial y_{ic}}{\partial y_c} + \frac{\partial \phi_{y_c}}{\partial z_{ic}}\frac{\partial z_{ic}}{\partial z_c} = \frac{\partial \phi_{y_c}}{\partial y_{ic}}\beta = \frac{\partial^2 \phi}{\partial y_{ic}^2}\beta^2$$





Coefficients (Cp, Cl, Cm)

$$\frac{dy_c}{dx_c} = \frac{1}{\beta}\beta\frac{dy_c}{dx_c} = \frac{1}{\beta}\frac{dy_{ic}}{dx_{ic}} = \frac{1}{\beta}\frac{v'_{ic}}{v_{\infty_{ic}}} = \frac{1}{\beta^2}\frac{v'_c}{v_{\infty_{ic}}} = \frac{v'_c}{v_{\infty_c}}$$

$$C_p = -2\frac{u'_c}{V_{\infty_c}} = -2\frac{u'_{ic}}{\beta^2 V_{\infty_{ic}}}$$ (from 2nd year)

Be wary: this is equivalent to the 2D result you saw last year, where the additional beta was cancelled by scaling up the geometry again in the depth direction. Can't do this in 3D. You can no longer link compressible and incompressible flows for any Mach number – but only for one Mach number.

Induced drag? **You can assume nothing changes.** The relation between compressible lift and compressible induced drag is the same as for the incompressible case.

University of BRISTOL

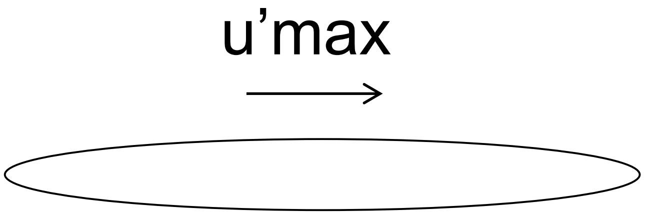
What this means

- Scale y and z (note: must also scale incidence)
- Perform the incompressible calculation
- Divide resulting coefficients by β^2

The key point is we cannot take an incompressible result and 'make it compressible'. We have to perform a particular calculation for *every* Mach number.

2D Example – ellipse maximum velocity

$$\frac{u'_{max}}{V_\infty} = \frac{t}{c}$$



$$\frac{u'_{max_c}}{V_{\infty_c}} = \frac{1}{\beta^2}\beta\frac{t}{c} = \frac{1}{\beta}\frac{t}{c}$$

Which is the 2D scaling you already know

Wing Example

- Take any lift curve gradient, eg $$C_{L_\alpha} = \frac{5.65}{1 + \frac{1.8}{A_R}}$$
- Reduce aspect ratio by beta
- Divide by beta^2
- Multiply by beta (incidence change)
- Gives… $$C_{L_c} = \frac{1}{\beta^2} \frac{5.65}{1 + \frac{1.8}{\beta A_R}} \beta\alpha = \frac{5.65 A_R}{1.8 + \beta A_R}$$ *a
- Notice for large aspect ratios we recover the 2D Prandtl-Glauert rule
- **But for low aspect ratios the result is different – even nil if it is small enough**

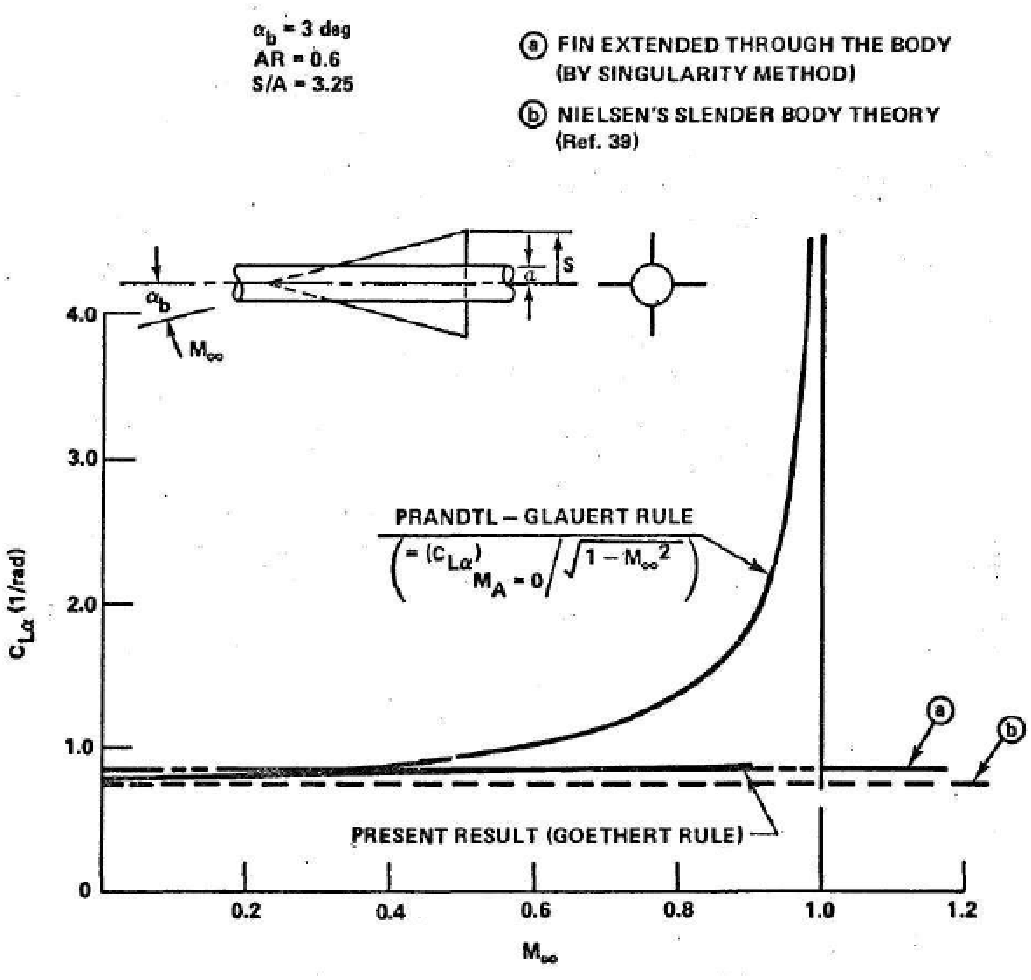
$$\frac{C_{L_c}}{C_{L_{ic}}} = \frac{1.8 + A_R}{1.8 + \beta A_R}$$

University of BRISTOL

Application

- Think about the wing and the tailplane. The wing has a high aspect ratio, so the result is closer to 2D PG.
- The tailplane is always quite stubby, so the compressibility effect here will be smaller. The lift gradient will not increase as much as for the wing
- Remember for next year's design project!
- Poll

Application



An extreme case for AR=0.6 (missile fin). A 2D PG scaling badly overpredicts the compressibility effect

Effect of sweep

- Transport aircraft use sweep
- We've used a Breguet result ML/D to gauge how 'good' an aeroplane is
- Now apply a similar argument to sweep

Effect of sweep

$$\frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + KC_L^2}$$

$$C_{D_0} + KC_L^2 - 2KC_L^2 = 0$$

$$C_L = \sqrt{\frac{C_{D_0}}{K}}$$

$$\frac{C_L}{C_D}_{max} = \sqrt{\frac{C_{D_0}}{K}}\frac{1}{2C_{D_0}} = \frac{1}{2\sqrt{C_{D_0}K}}$$

$$M_{cruise} = \frac{M_0}{\cos(\Lambda)}$$

$$A_R = A_{R_0}\cos^2(\Lambda)$$

K=k/(pi*AR)

$$\left(\frac{L}{D}\right)_{cruise} = n\left(\frac{L}{D}\right)_{max} = n\frac{1}{2\sqrt{C_{D_0}K}} = \frac{n}{2}\sqrt{\frac{\pi A_R}{kC_{D_0}}}$$

$$\left(\frac{L}{D}\right) = \left(\frac{L}{D}\right)_0\sqrt{\frac{A_R}{A_{R_0}}} = \left(\frac{L}{D}\right)_0\cos(\Lambda) = \left(\frac{L}{D}\right)_0\frac{M_0}{M}$$

and hence

$$M\frac{L}{D} = M\frac{M_0}{M}\left(\frac{L}{D}\right)_0 = M_0\left(\frac{L}{D}\right)_0 = constant$$

University of BRISTOL

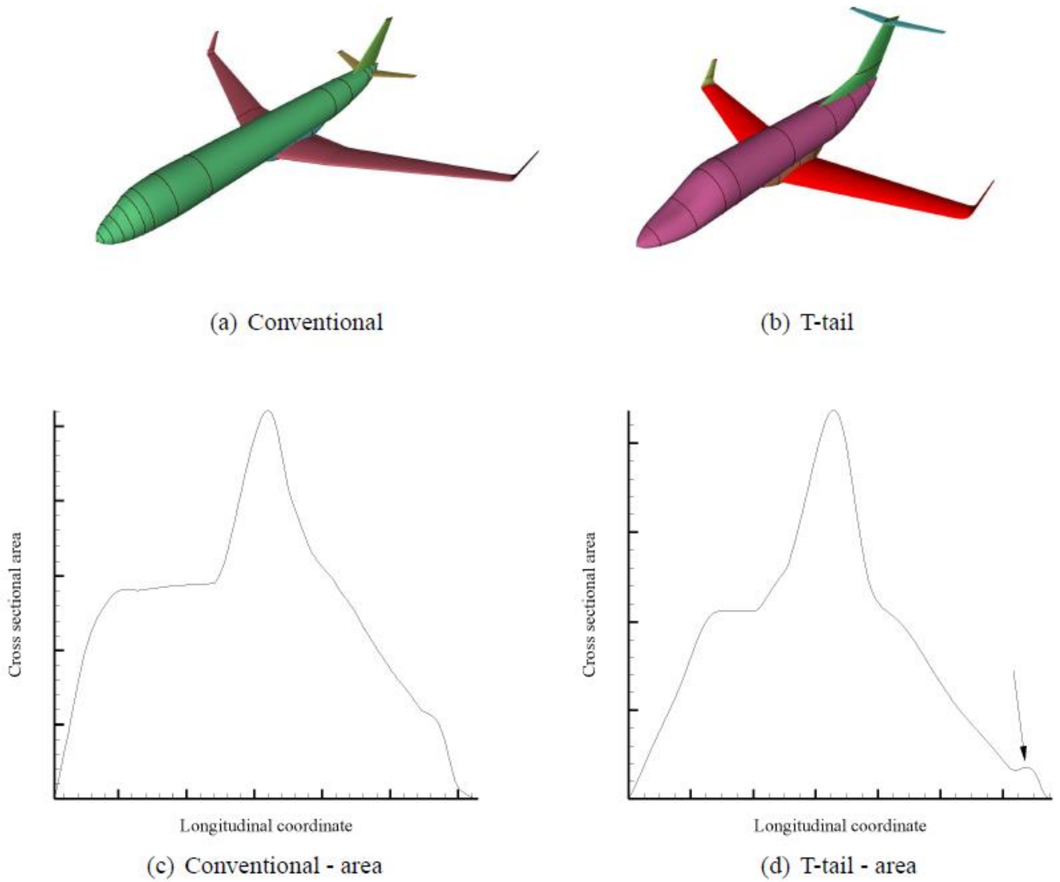
So finally….

- Sweep ‘doesn’t’ change range that much, but it does allow you to arrive sooner!
- Many simplifications above means this is not precisely true (eg 3D compressible effects)

Area ruling

How could you area rule in practice next year?

Example from Sumo…



Know

- How to use Goethert’s rule to adjust 3D results for compressibility (I won’t ask you to derive the rule unless I give intermediate information, so no need to memorise directly)
- Understand the simplified effect of sweep on range
- How to use Sumo to generate area plots for next year

Tom Rendall
thomas.rendall@bristol.ac.uk

Aerodynamics 3

Vortex Lattice Methods (chapter 4 in notes)

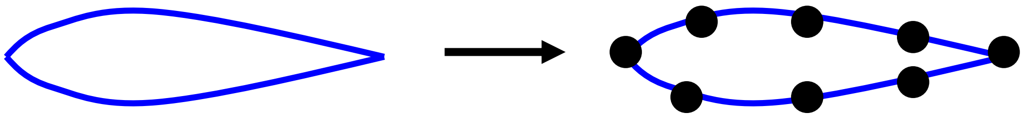
University of BRISTOL

Inviscid Methods

- In the last lecture, we discussed the disadvantages of the methods you have already learned for inviscid analysis:
  - *Thin Aerofoil Theory* (last year)
  - *The Joukowski transformation* (this year)
- The principle problems were that realistic three dimensional shapes could not be analysed
  - *only small incidence and thin aerofoil shapes*
  - *or no control on shape a-priori, only 2D*

The Panel Method:

- As an alternative, we introduced the *Panel Method*, which allows the flow on the surface of complex three dimensional objects to be calculated:



- The process requires
  - Discretisation
  - Calculation of Influence Coefficients for normal and tangential velocities
  - Matrix Solution
  - Velocity, Pressure, Force Calculation

Advantages and Disadvantages:

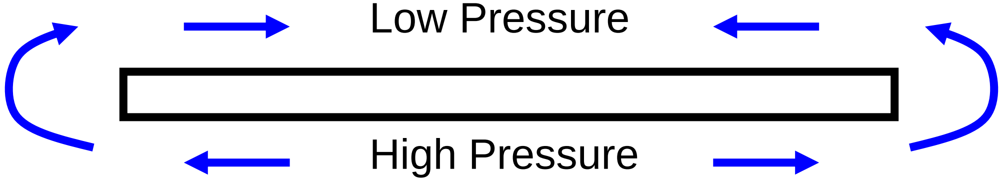
- More general than thin aerofoil and Joukowski
  - Can do 3D shapes
- Faster than full CFD
  - Only need surface mesh – less solution points, less effort
  - No need for time marching
- But
  - Only for low speed (strictly incompressible, but can be corrected)
  - No shocks
  - Sometimes hard/laborious to impose Kutta condition for an arbitrary set of shapes

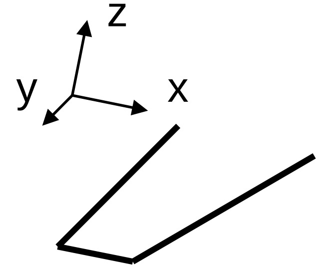
The Vortex Lattice Method

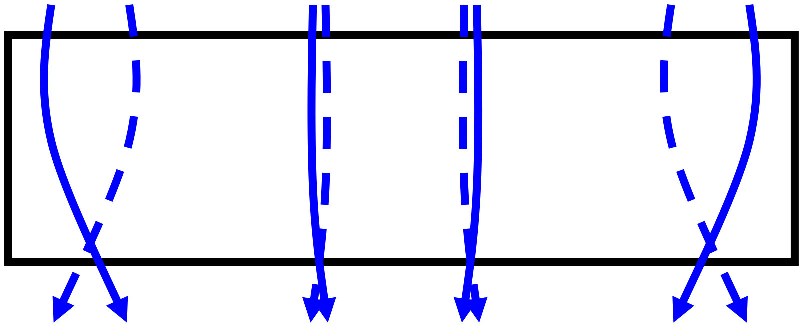
- Widely used in aerospace industry (see later)
- Derives from the lifting line analysis you were introduced to last year
- However, results in a solution form very similar to a panel method
  - In fact, can be thought of as a simple version of the panel method
- Has some unique advantages/disadvantages
- First developed ~1930-50
- First use on digital computer ~1965

Effects of Finite Span

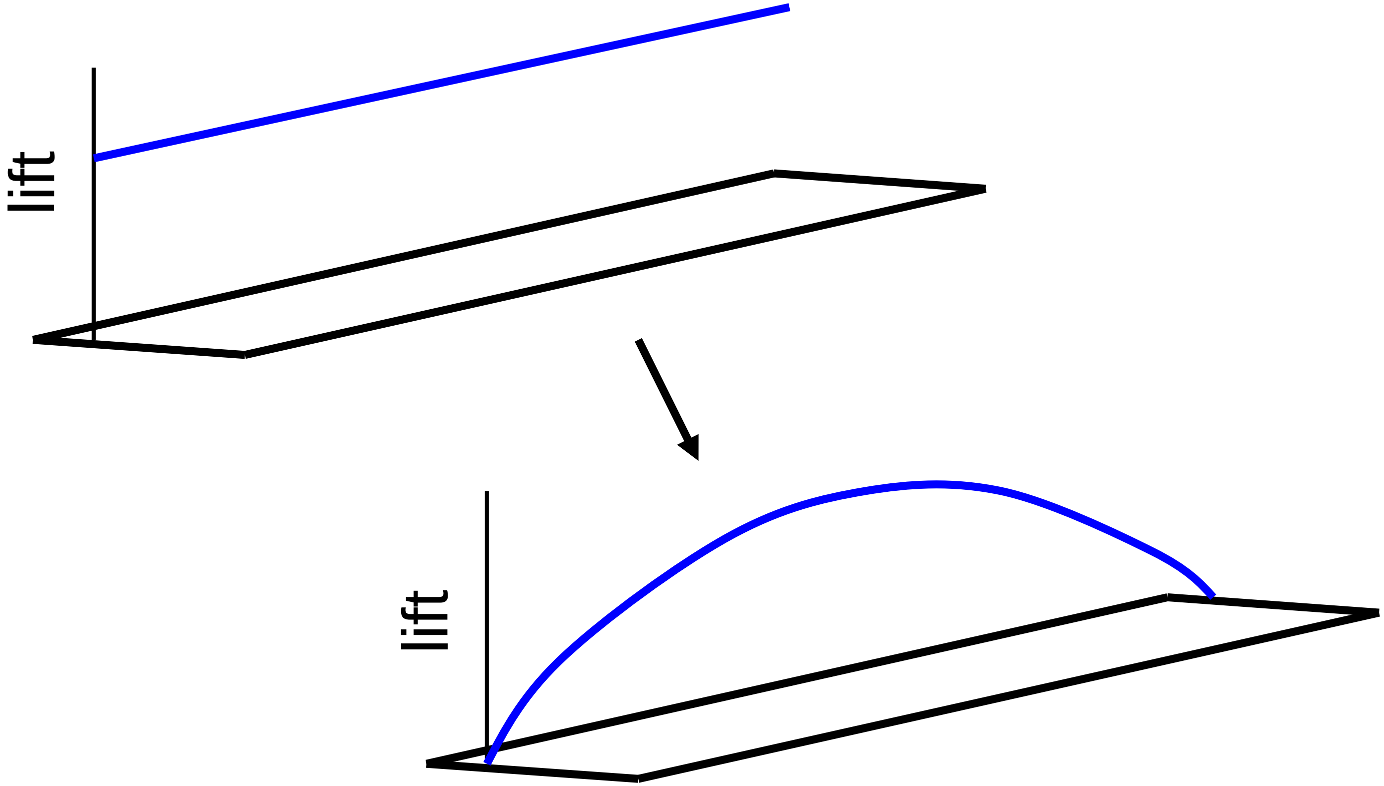
- Consider for now an isolated three dimensional wing.
- Because of the high pressure below, and low above, get a rotation in the *(y,z)* plane:





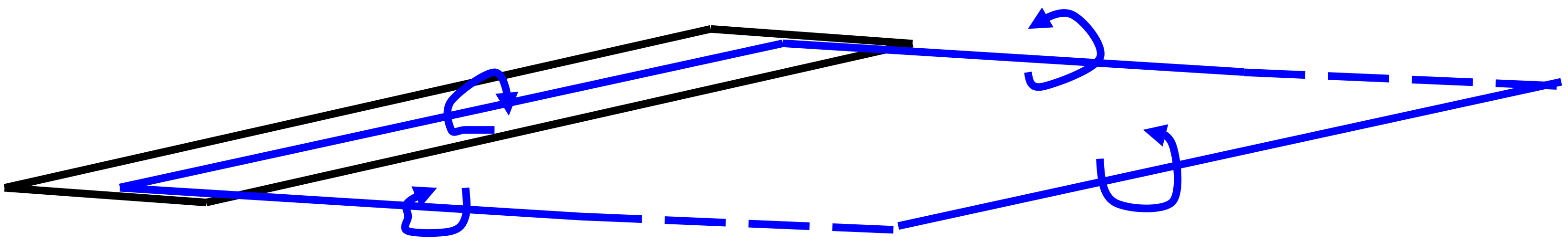


This leads to a change in the lift distribution across the span from a constant value equal to that predicted by 2D theory, to an `elliptical' (or other) variation:



Lift Distribution Models:

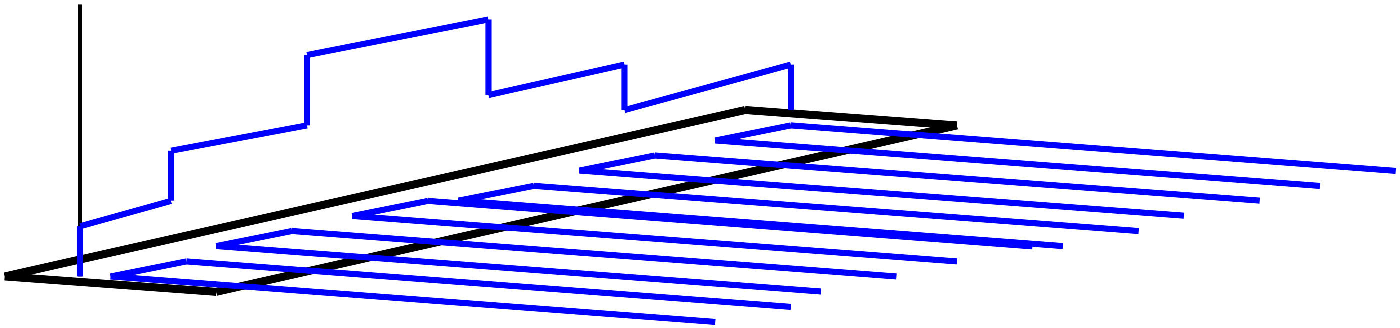
- Simplest is the Horseshoe Vortex:



- Vortex cannot end in fluid (Helmholtz)
  - turns through 90 degrees at tips, and joins with starting vortex
  - Strength constant
- Same as first case on previous slide

Lifting Line:

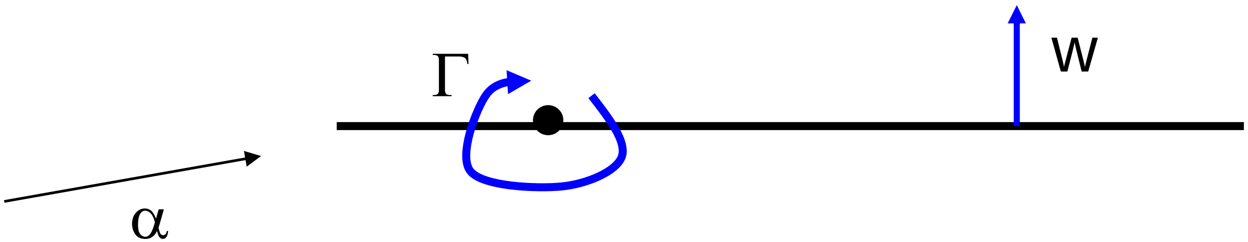
- Improve the previous model by including lots of small vortices:



- Can be solved numerically with lots of vortices
  - Fast
  - But…not good for low aspect ratio or swept wings

2D Aerofoil theory – Lumped Vortex

- Last year, your learned about thin aerofoil theory
- Simplest approximation you used was a *Lumped Vortex:*

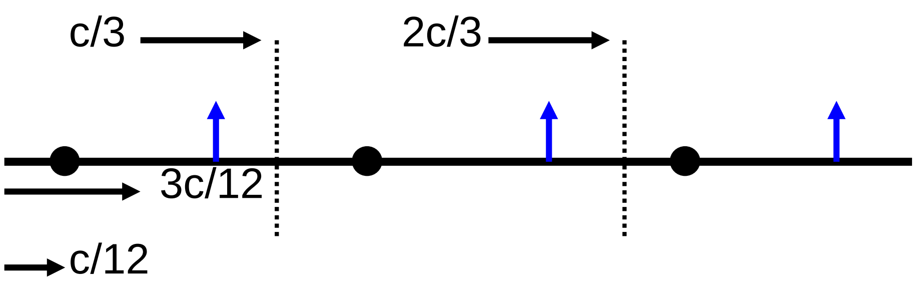


- Where the whole aerofoil is represented by a vortex at 25% chord, and a control point (where normal velocity is zero) at 75 % chord.
- This 25/75 placement satisfies the Kutta condition

- Amazingly – 25/75 rule dates back to 1937! Shown by Enrico Pistolesi

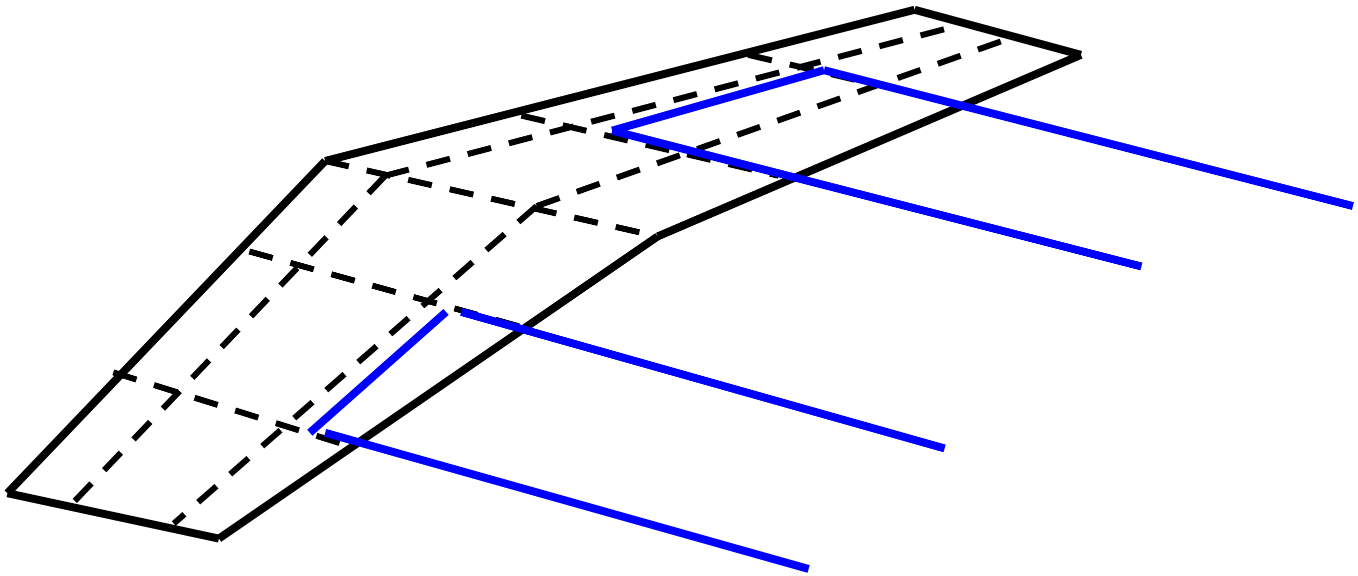
University of BRISTOL

This is a very simple, and somewhat crude approximation. However, like the horseshoe vortex, we can improve accuracy by splitting the chord into several sections. If the vortex is placed at 25% of each section, and the control point at 75%, the method still works:



Kutta condition still satisfied for complete shape providing 25/75 placement used for all panels

Combining these two concepts gives the Vortex Lattice Method:



Again, Kutta condition still satisfied for complete shape providing 25/75 placement used for all panels – very convenient

Steps of the method:

- Represent the wing/aircraft as a flat plate
- Split the plate into cells of equal % of span and local chord
- Give each panel a vortex strength Γ_i per unit length
- Use this to calculate the induced velocity, $w_{i,j}$ caused by vortex i on panel j (this can be done geometrically via the *Biot Savart Law* you were introduced to last year).

University of BRISTOL

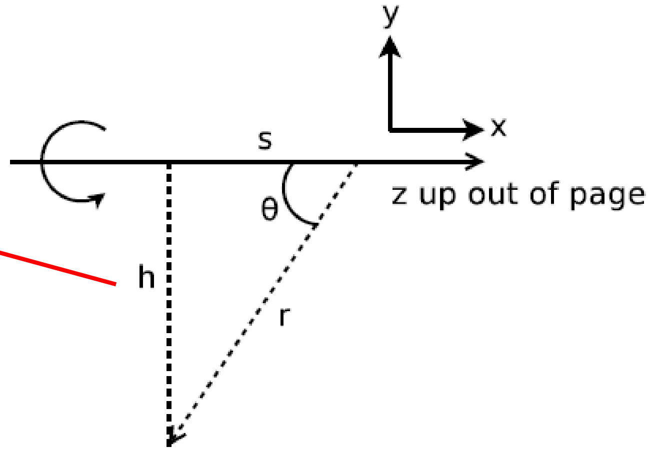
As with the panel method, this results in a matrix of the form

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & . & . \\ \vdots & . & . & . \\ a_{N,1} & . & . & a_{N,N} \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_N \end{bmatrix} = -U_\infty \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix}$$

Which is influence coefficients x vortex strength + normal velocity =0 (no flow through surface). The matrix can be inverted in a similar way, and hence vortex strength of each panel found.

$$h = r\sin(\theta)$$

$$\mathbf{V} = \frac{\Gamma}{4\pi}\int \frac{\mathbf{ds}\times\mathbf{r}}{|\mathbf{r}|^3}$$



$$s = \frac{h}{\tan(\theta)}$$

$$ds = -\frac{h}{\sin^2(\theta)}d\theta$$

$$\mathbf{ds}\times\mathbf{r} = \begin{pmatrix} \frac{-h}{\sin^2(\theta)}d\theta \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -s \\ -h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{h^2}{\sin^2(\theta)}d\theta \end{pmatrix}$$

$$V_z = \frac{\Gamma}{4\pi}\int \frac{h^2}{\sin^2(\theta)}\frac{\sin^3(\theta)}{h^3}d\theta = \frac{\Gamma}{4\pi h}\int \sin(\theta)d\theta$$

$$V_z = \frac{\Gamma}{4\pi h}(\cos(\theta_A) - \cos(\theta_B))$$

What if circulation varies?

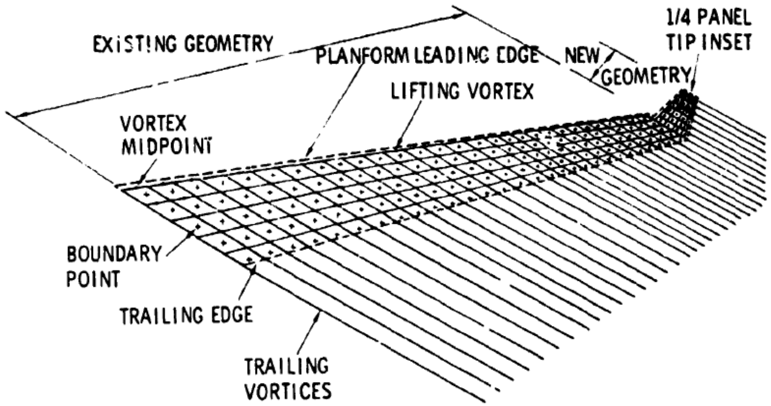
- There are several obvious similarities to a panel method
  - steps 1 – 3 are almost identical – discretisation, geometry based influence coefficients, matrix solution
- However, as vortex strengths are found, lift and (induced) drag may be calculated directly via

$$\sin(\alpha) \approx \tan(\alpha) \approx \alpha \approx \frac{w_i}{U_\infty}$$

$$Lift = \sum_{1}^{N} \rho \Delta y_i U_\infty \Gamma_i \qquad Drag = \sum_{1}^{N} \rho \Delta y_i w_i \Gamma_i$$

- i.e. no need to calculate pressures and integrate,
- no need to use wake panels or wake relaxation (**unless unsteady)**

University of BRISTOL



OPTIMIZATION AND DESIGN OF THREE-DIMENSIONAL AERODYNAMIC CONFIGURATIONS OF ARBITRARY SHAPE BY A VORTEX LATTICE METHOD

Winfried M. Feifel
The Boeing Company

(1976)

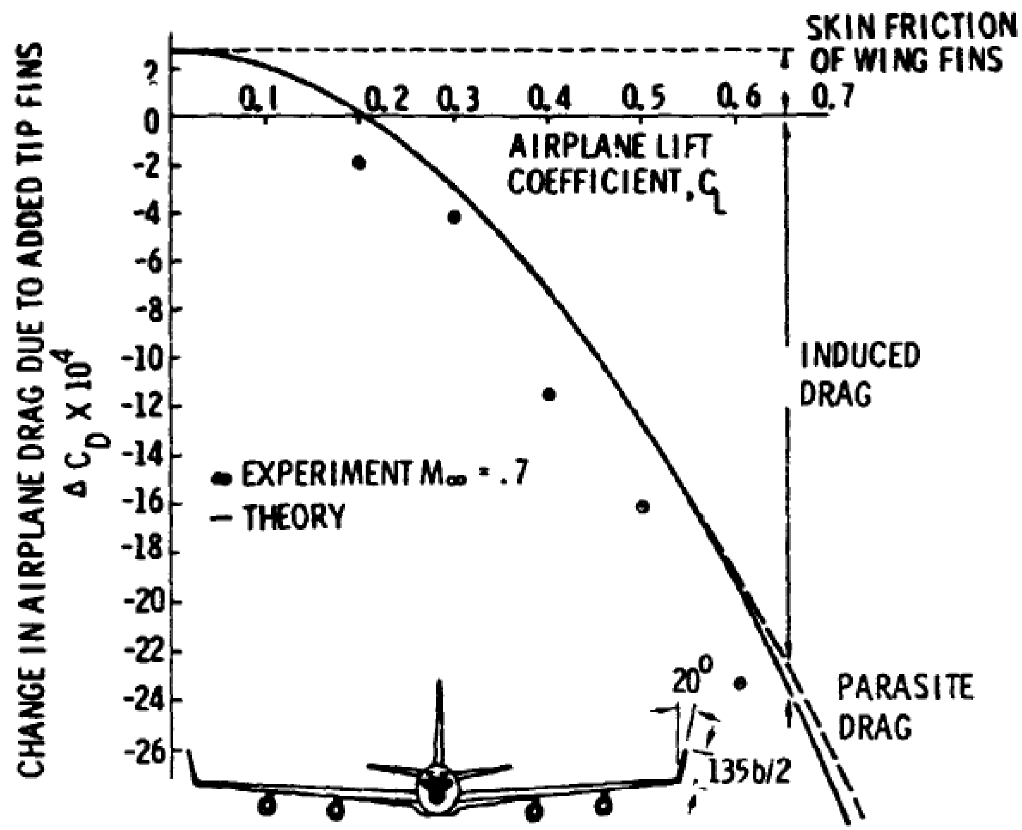
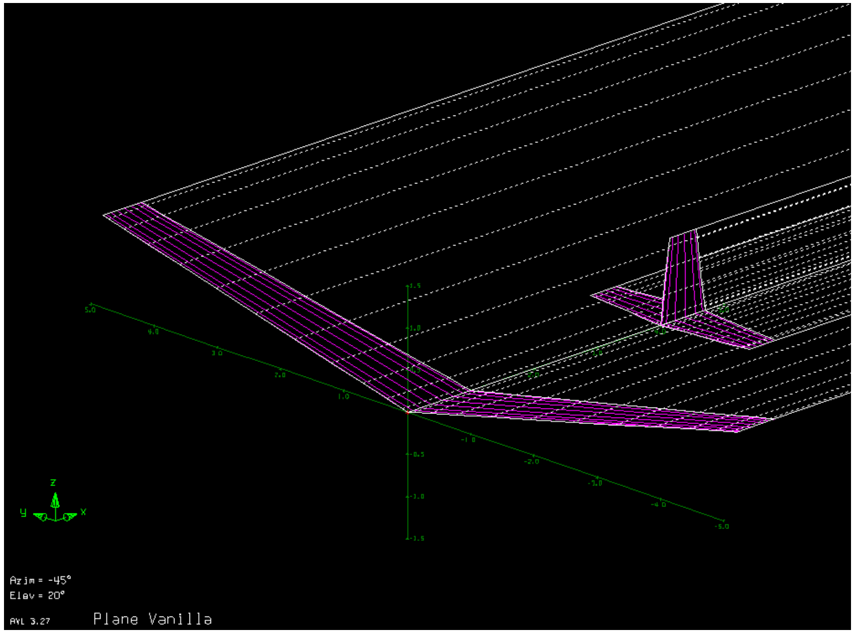
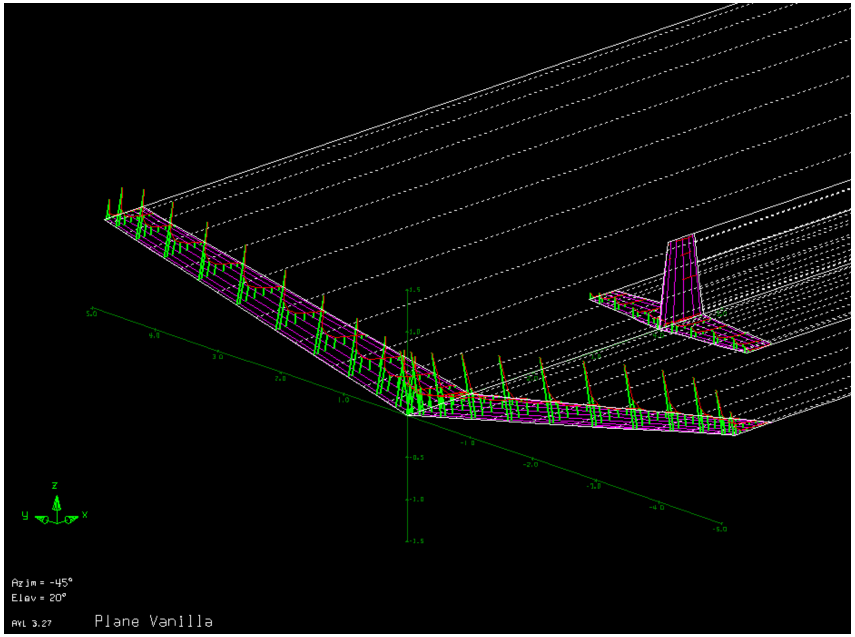
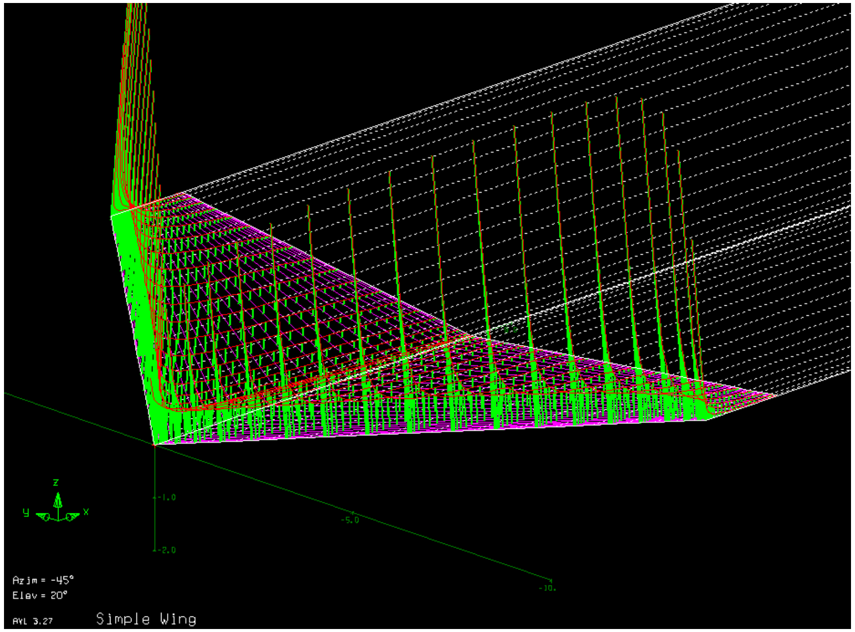


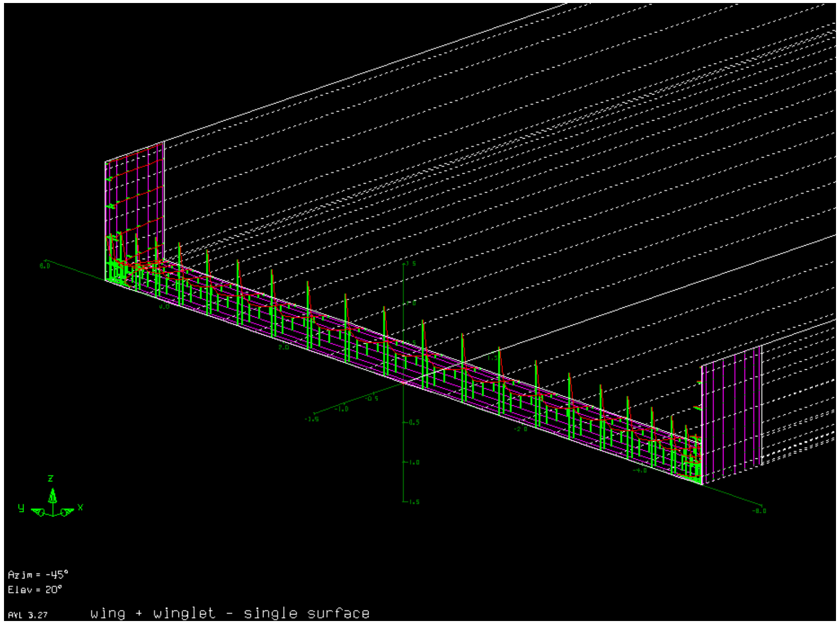
Figure 11.- Measured and predicted drag change of Boeing KC-135 with tip fins.

AVL


Azim = -45°
Elev = 20°
AVL 3.27
Plane Vanilla


Azim = -45°
Elev = 20°
AVL 3.27
Plane Vanilla


Azim = -45°
Elev = 20°
AVL 3.27
Simple Wing

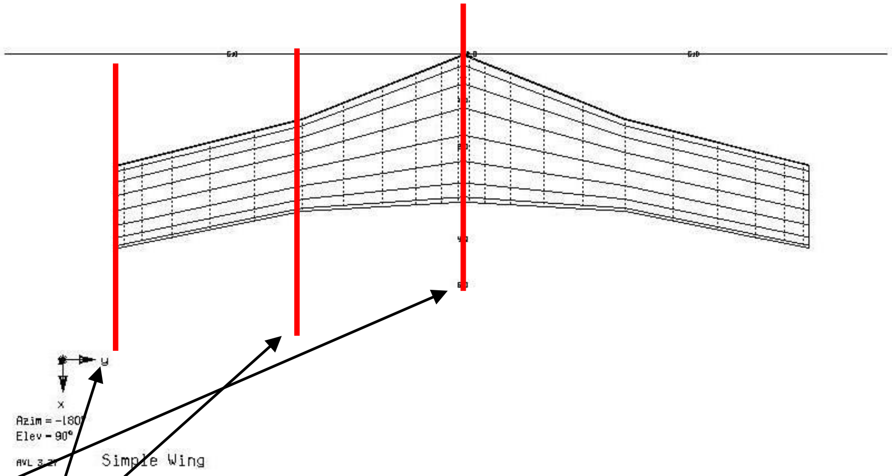

Azim = -45°
Elev = 20°
AVL 3.27
wing + winglet - single surface

Sample AVL Input

```
Simple Wing
#Mach
 0.0
#IYsym   IZsym   Zsym
 0       0       0.0
#Sref    Cref    Bref
33.4     2.0     15.0
#Xref    Yref    Zref
0.50     0.0     0.0
#
#
#==================================
SURFACE
Wing
#Nchordwise  Cspace   Nspanwise   Sspace
8            1.0      12          1.0
#
YDUPLICATE
0.0
#
ANGLE
0.0
#-------------------------------------------------------------
SECTION
#Xle    Yle    Zle     Chord   Ainc   Nspanwise   Sspace
0.      0.     0.      3.2     0.0    0           0
#-------------------------------------------------------------
SECTION
#Xle    Yle    Zle     Chord   Ainc   Nspanwise   Sspace
1.4     3.5    0.0     2.0     0.0    0           0
#-------------------------------------------------------------
SECTION
#Xle    Yle    Zle     Chord   Ainc   Nspanwise   Sspace
2.4     7.5    0.0     1.8     0.0    0           0
```

Ref areas – care!

Panels



Sections

Useful outputs – Cl, Cdi, lift curve slope, aerodynamic centre/neutral point

University of BRISTOL

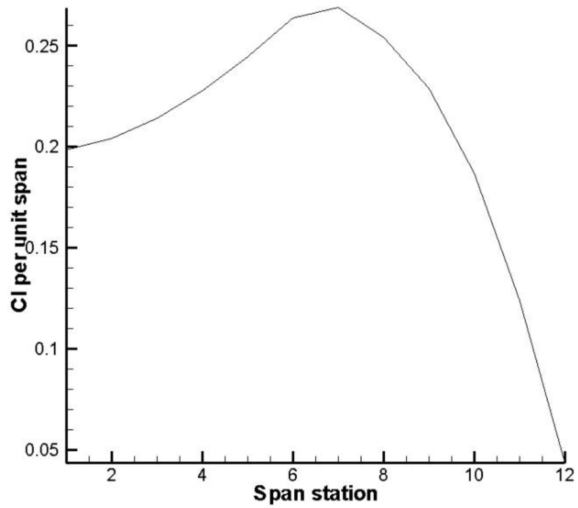
Results

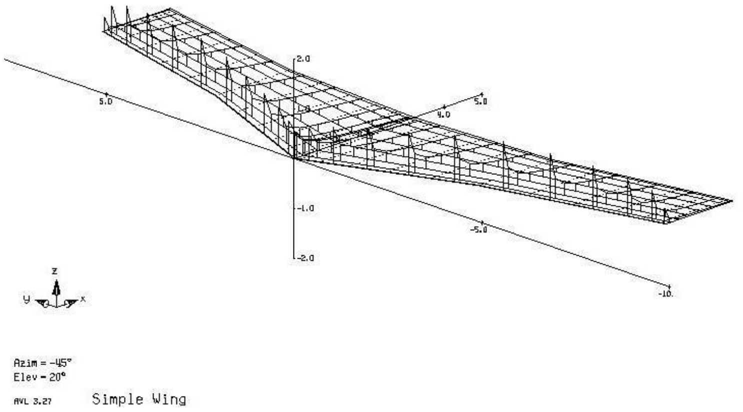
AoA=3deg

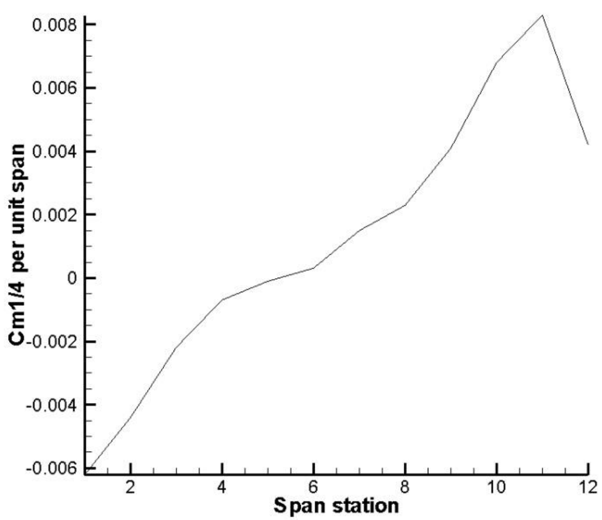
Cl=0.22845   Cdi=0.00272

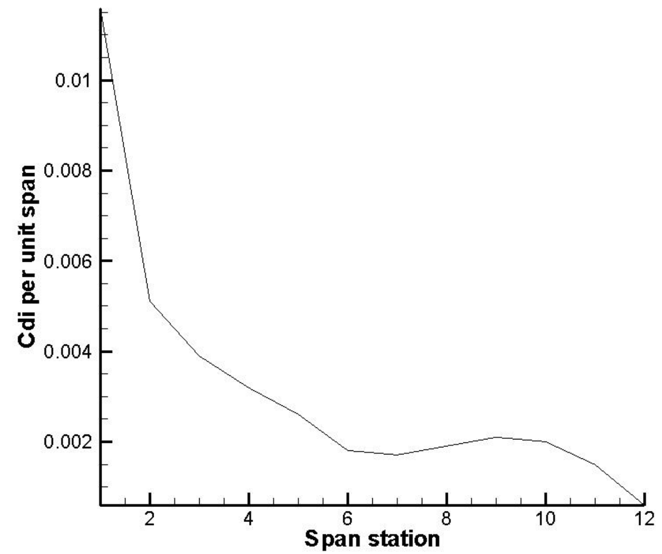
CLa=4.357513 ~ 70% of 2π

Xac =   1.760115

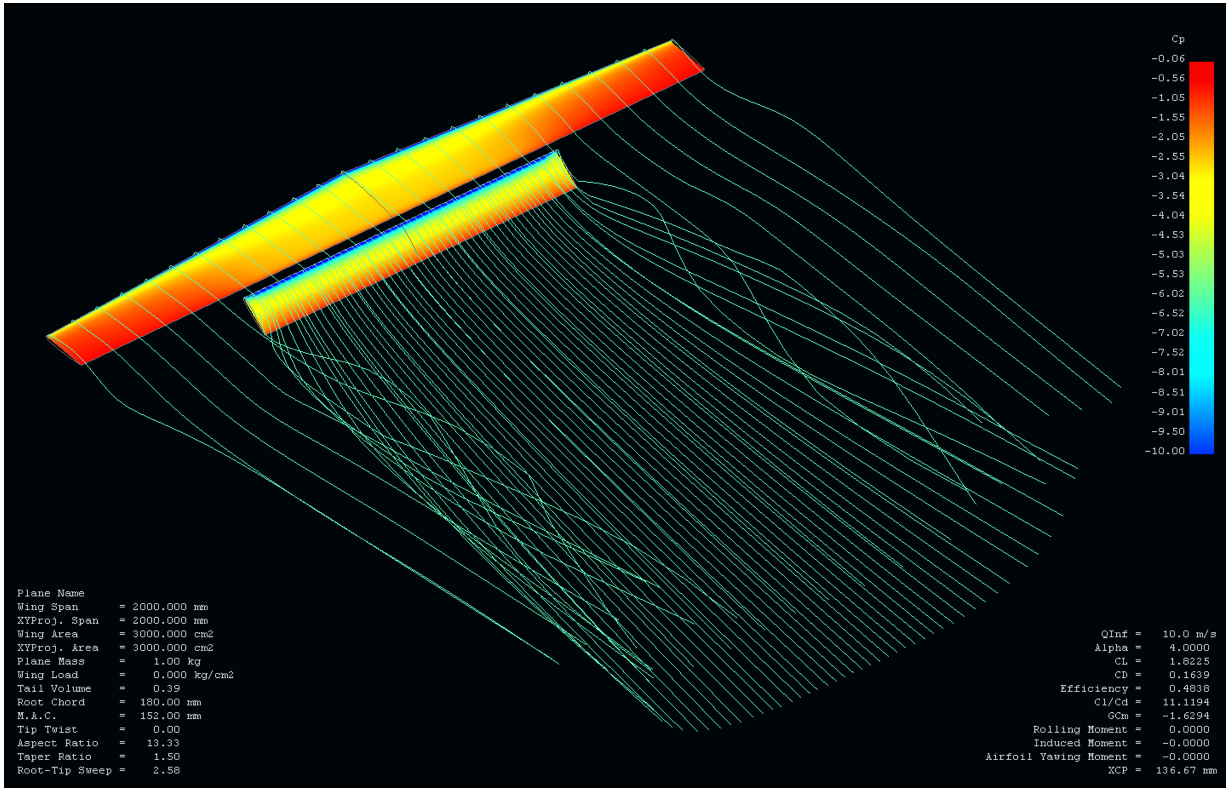




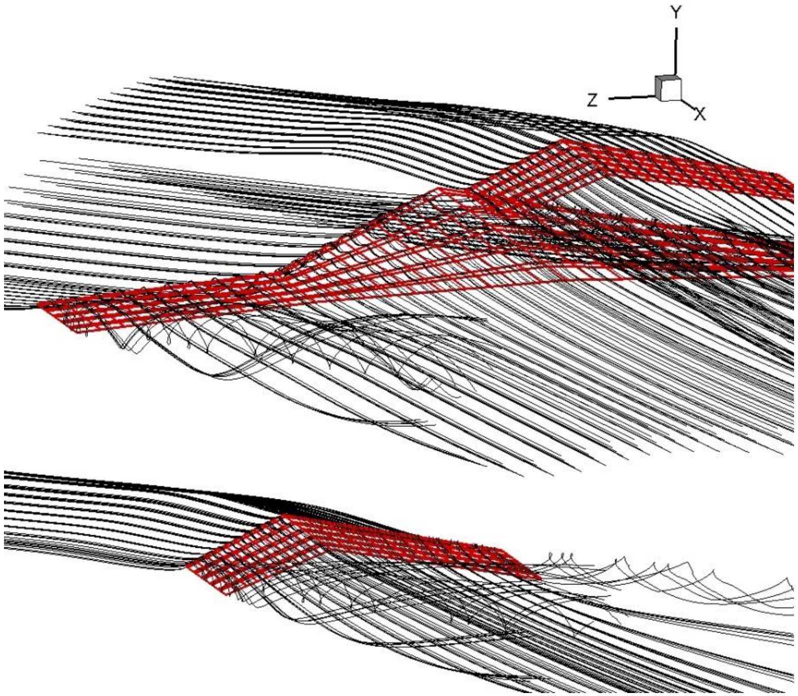


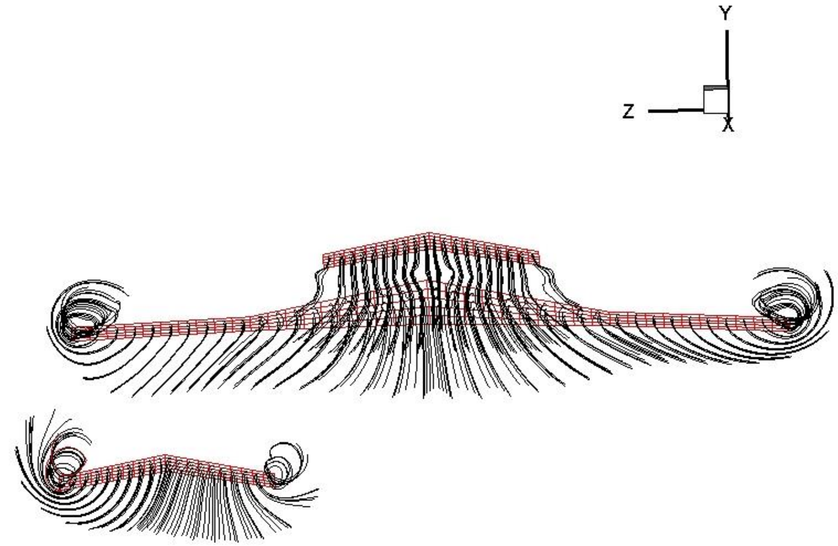


XFLR5

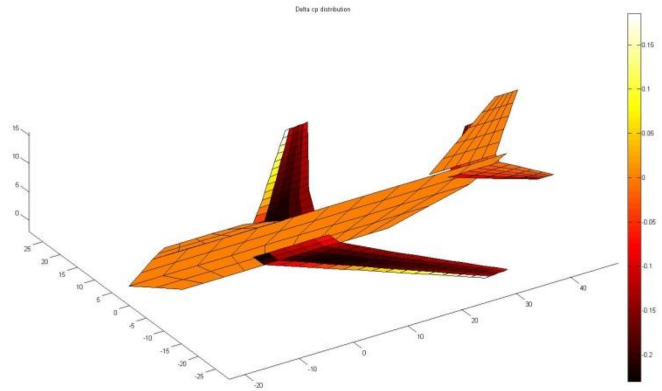

Cp
-0.06
-0.56
-1.05
-1.55
-2.05
-2.55
-3.04
-3.54
-4.04
-4.53
-5.03
-5.53
-6.02
-6.52
-7.02
-7.52
-8.01
-8.51
-9.01
-9.50
-10.00
Plane Name
Wing Span = 2000.000 mm
XYProj. Span = 2000.000 mm
Wing Area = 3000.000 cm2
XYProj. Area = 3000.000 cm2
Plane Mass = 1.00 kg
Wing Load = 0.000 kg/cm2
Tail Volume = 0.39
Root Chord = 180.00 mm
M.A.C. = 152.00 mm
Tip Twist = 0.00
Aspect Ratio = 13.33
Taper Ratio = 1.50
Root-Tip Sweep = 2.58
QInf = 10.0 m/s
Alpha = 4.0000
CL = 1.8225
CD = 0.1639
Efficiency = 0.4838
Cl/Cd = 11.1194
GCm = -1.6294
Rolling Moment = 0.0000
Induced Moment = -0.0000
Airfoil Yawing Moment = -0.0000
XCP = 136.67 mm

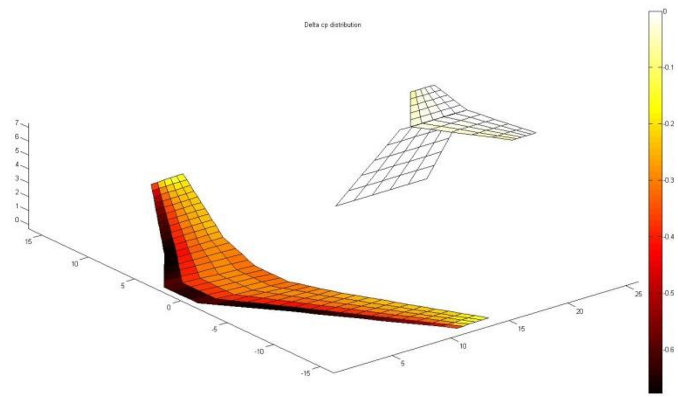
Multiple surfaces

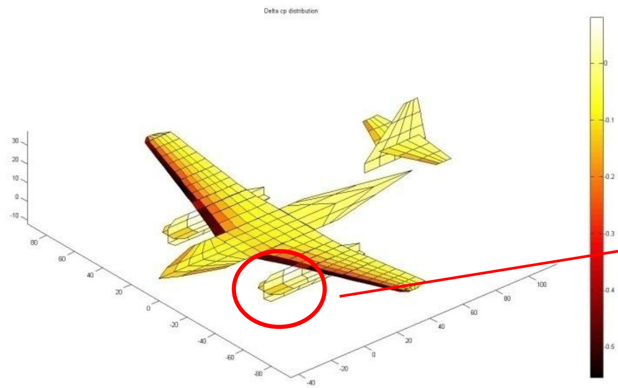


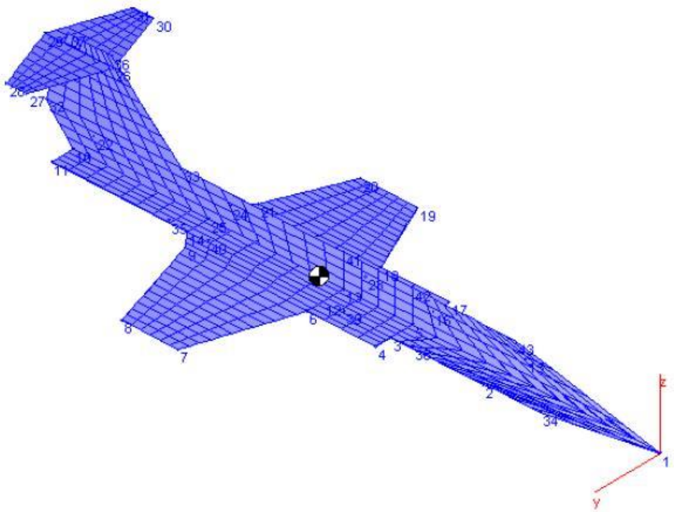


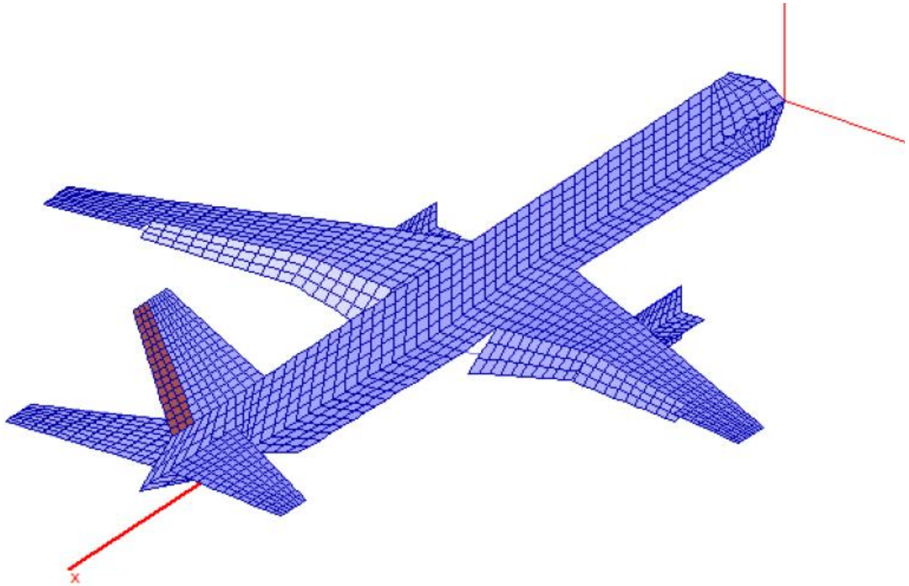
Tornado (MatLab)











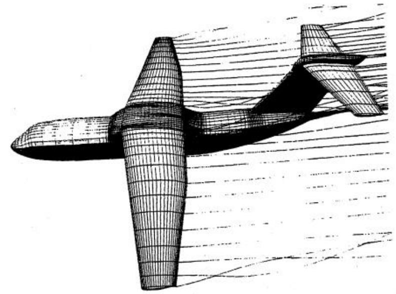
(from ‘Surfaces’ tool)

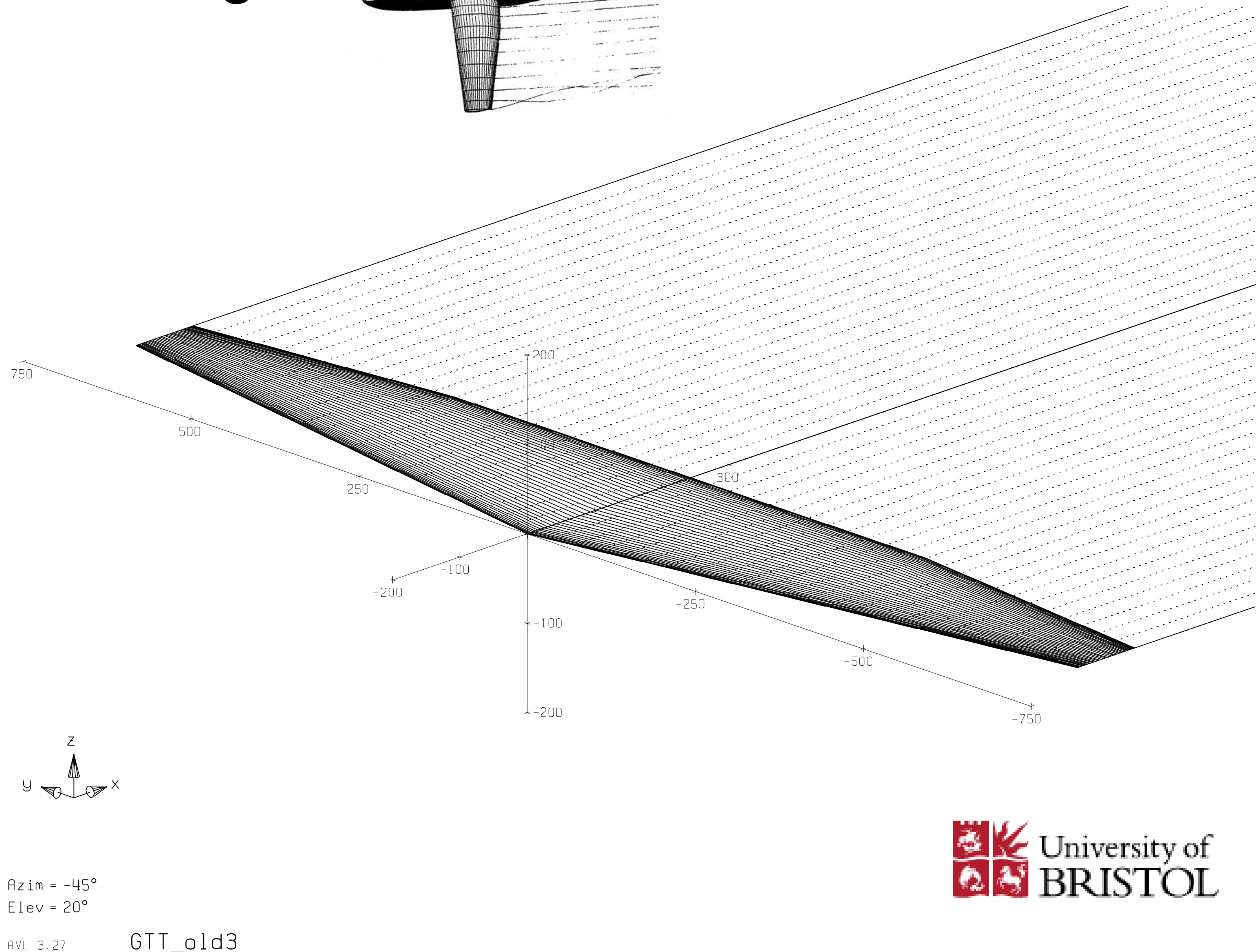
University of BRISTOL

Advantages/Disadvantages of VLM

- Quicker, simpler than full panel method.
- Easy to do multi-element aerofoils – wakes better behaved and easier to use
- But has similar limitations to thin aerofoil theory
  - thin shapes at zero incidence
  - not very good at stagnation
  - Incompressible – not valid for transonic problems, unless calibrated correction included
- These problems are not too important if the vehicle to be analysed is an aircraft at low speeds dominated (aerodynamically) by the wing

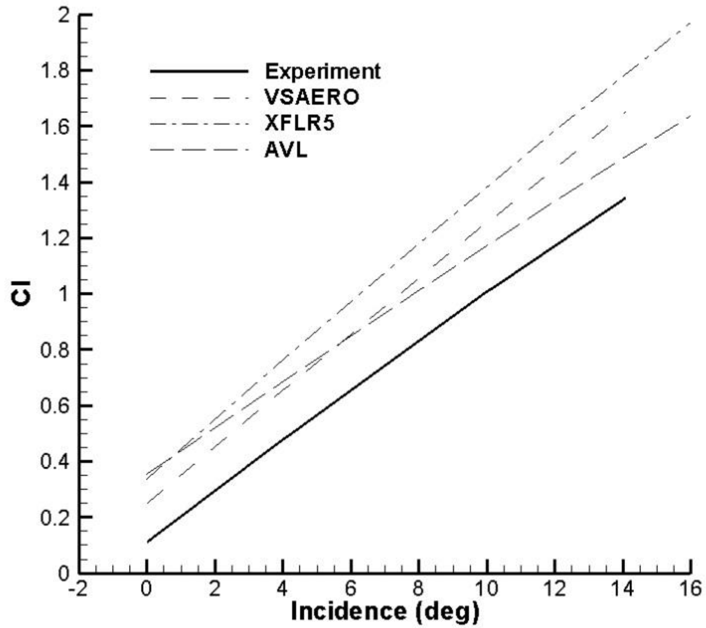
GTT Wing

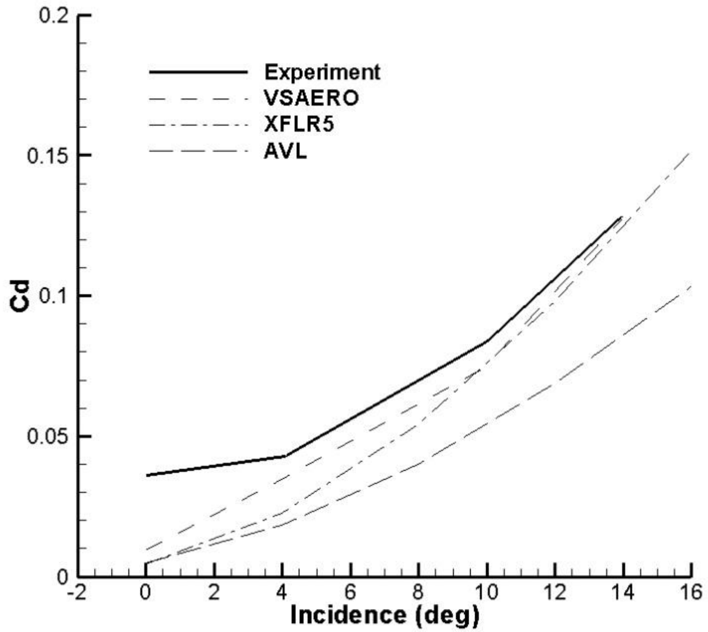




University of BRISTOL

Forces

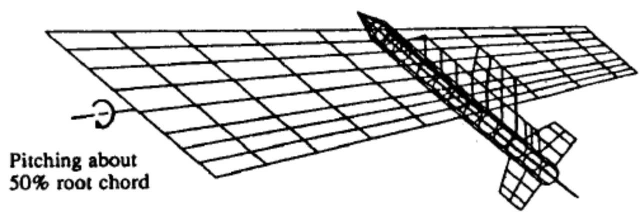

Experiment
VSAERO
XFLR5
AVL
Cl
Incidence (deg)

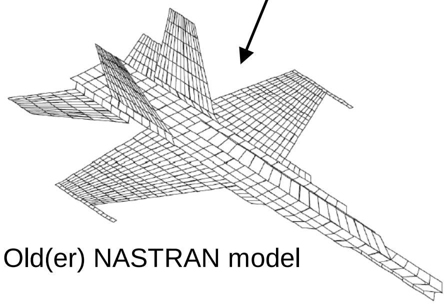

Experiment
VSAERO
XFLR5
AVL
Cd
Incidence (deg)

Aeroelastic analysis

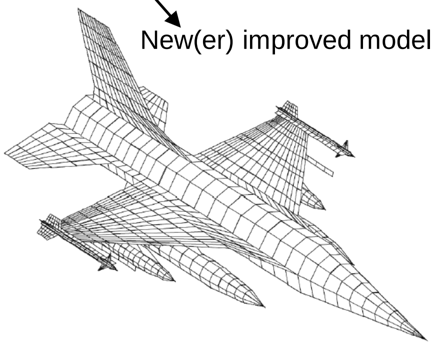
- Perhaps biggest advantage/widest application is for aeroelastic work:
  - Structural representation of a wing is often as flat panels – can be coupled directly to vortex lattice method for static calculations
  - Fast solution of linear aerodynamic equations
  - Can use transonic small perturbation equations to examine transonic region (care must be taken here)
  - Unsteady version called **doublet lattice method**, and works well in **frequency domain** - ideal for flutter stability calculations

$$\Gamma(t) = \tilde{\Gamma} e^{j\omega t}$$



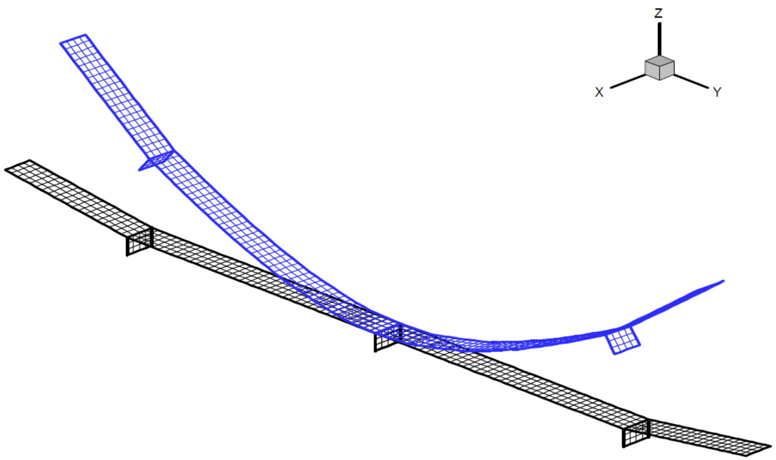


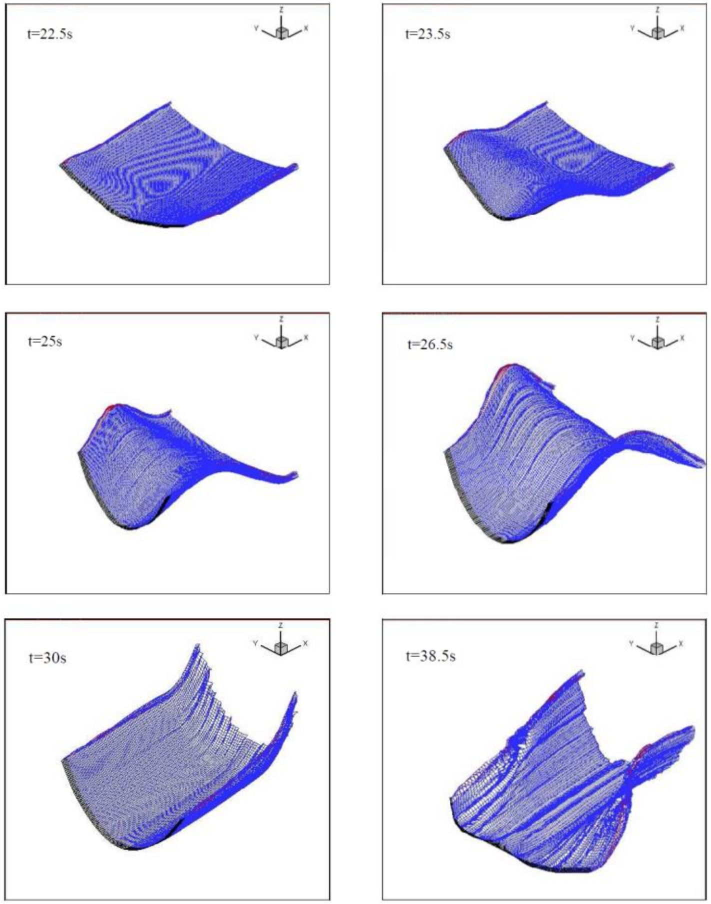
Old(er) NASTRAN model

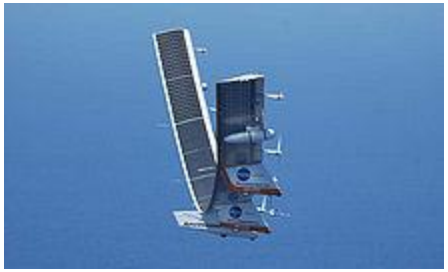


New(er) improved model

Aeroelastics







(Helios)

(modified to account to large, nonlinear deflections, and here applied in the **time domain**)

University of BRISTOL

. . . . .

Coursework Briefing

The coursework isn't supposed to be trivial. I'm here to help, but I also won't hand you the solutions (if I told you the 'answers', where is the scope for you to learn, or for any variation in marks?)

You're a year and a half away from being paid to solve problems and write reports, so you should be getting good at it. If you're not, now is the time to practice!

Part 1

- Add calculations to output the total pitching moment (about the root quarter chord point) and total induced drag coefficient, and include the new lines of code in your report, with comment lines to explain what you are doing. The lines to modify can be found between the comments START MODS—— and END MODS——. **If you are in doubt, please refer to the equations in the course notes, section 4.4.1, page 44.**
- Compare AVL to the LLM
- This is not intended to be tremendously tough

Part 2

- VGK can be used to find C_L, but if you want $\frac{\partial C_L}{\partial \alpha}$, how do you do this? Once answer is a finite difference, so

$$\frac{\partial C_L}{\partial \alpha} = \frac{C_L(\alpha+\Delta\alpha) - C_L(\alpha-\Delta\alpha)}{2\Delta\alpha}$$

But what size to use for $\Delta\alpha$?

- Find Mcrit – as per aero 2
- Find drag as a function of Mach number, and compare to Lock's result. Explain discrepancies – **think carefully about what the question asks**
- Find drag divergence Mach number
- Suggest ways to use a tool like VGK and LLM together

University of BRISTOL

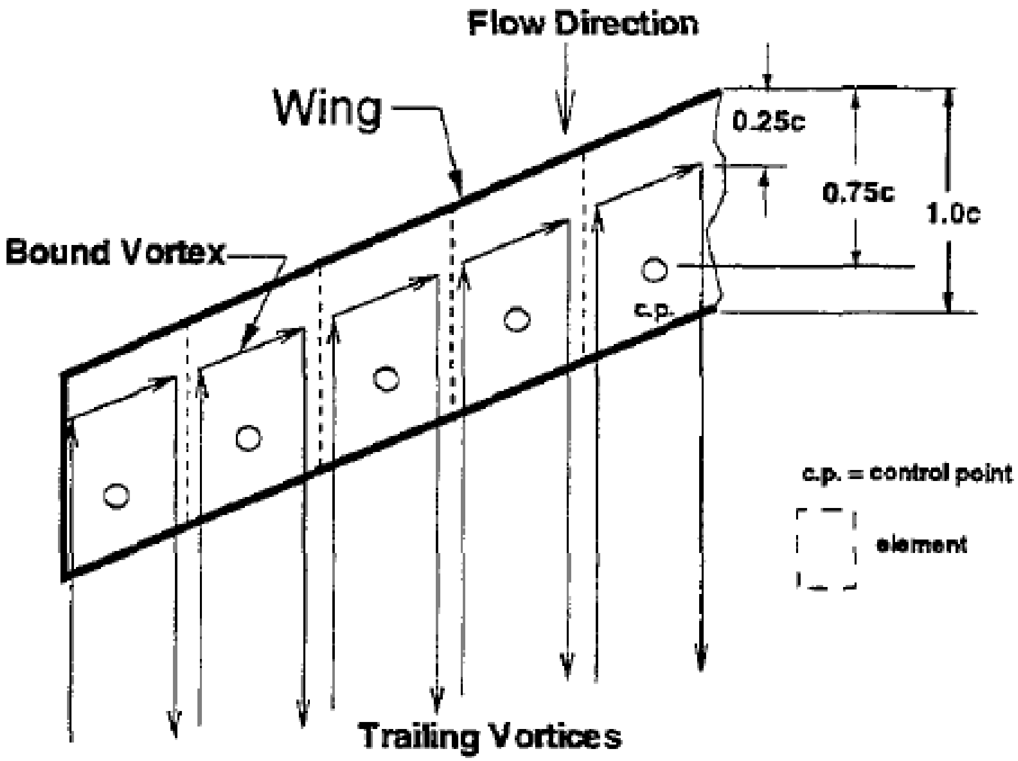
Tips…

Discrete lifting line (aka Weissinger method + variants)

AIAA 98-0597

WEISSINGER'S MODEL OF THE NONLINEAR LIFTING-LINE METHOD FOR AIRCRAFT DESIGN

D. Bruce Owens*
National Research Council
NASA Langley Research Center
Hampton, VA 23681-0001



As sketched, equivalent to VLM.

However, can use a different surface boundary condition…

Tips…

Discrete lifting line (aka Weissinger method + variants)

AIAA 98-0597

WEISSINGER'S MODEL OF THE NONLINEAR LIFTING-LINE METHOD FOR AIRCRAFT DESIGN

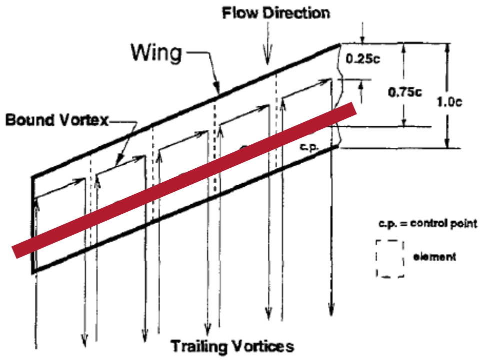
D. Bruce Owens*
National Research Council
NASA Langley Research Center
Hampton, VA 23681-0001

$$C_l = \frac{\rho V_\infty \Gamma}{\frac{1}{2}\rho V^2 c} = \frac{2\Gamma}{V_\infty c} = \frac{\partial C_l}{\partial \alpha}(\alpha_i - \alpha_0)$$

Induced angle
Is a function of vortex strengths

$$\alpha_i = tan^{-1}\left(\frac{w_i}{U_\infty}\right)$$

$$\left(a_{ii} - \frac{2\cos(\alpha_\infty)}{\frac{\partial C_l}{\partial \alpha}c}\right)\Gamma_i + \sum_{j=1,j\neq i}^{j=N} a_{ij}\Gamma_j = \cos(\alpha_\infty)(\alpha_0 - \alpha_t) - \sin(\alpha_\infty)$$



"the sectional lift is consistent with the AoA produced by the sectional lift distribution and the 2D lift curve slope"

University of BRISTOL

Tips…

- Refer to the screencasts and helpsheet for AVL/VGK assistance. Remember you can type`?’ in AVL for a list of options
- **Watch the AVL screencast to get you up and running**
- VGK is *relatively* intuitive. Remember to save the setup file so you can load things up more quickly after a session
- **Watch the VGK screencast to get you up and running**

Tips…

Protocol

- **This text is in red because it's very mean**
- **No extensions will be granted for IT failures (personal or University). Back up your work. If you have a failure and ask for an extension I will refer you to this statement. Not reading this statement or not being present for this lecture will not create an exception to this rule**
- **Always take a receipt (print screen) to confirm your report has been uploaded to BlackBoard. I will not consider extensions based on BlackBoard submission problems unless supported by a receipt clearly showing the date and time of submission**
- **You have been warned**

University of BRISTOL

Tom Rendall
thomas.rendall@bristol.ac.uk

Aerodynamics 3

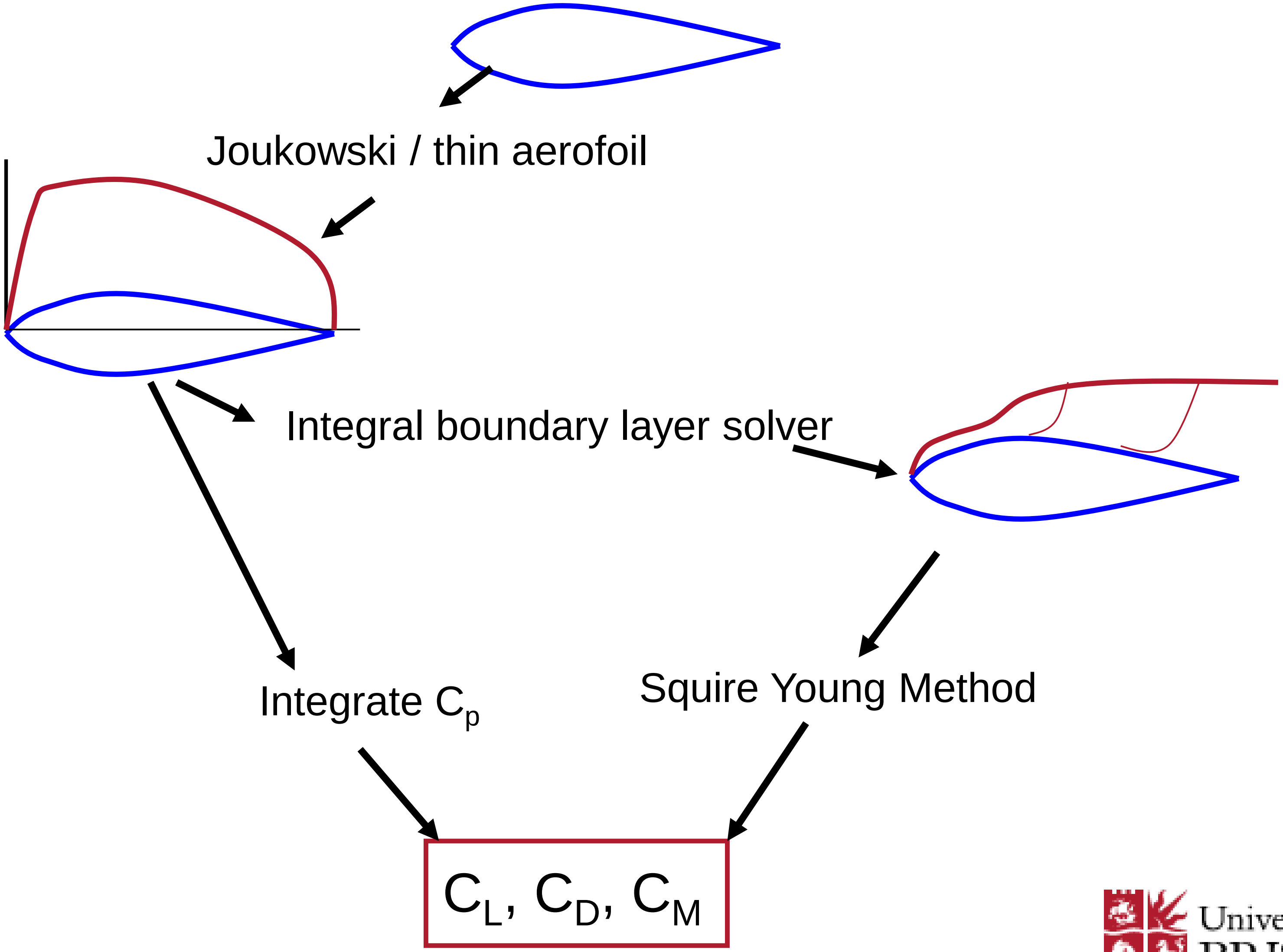
Panel Methods (chapter 5 in notes)

University of BRISTOL

A Quick Re-cap:

- So far, we have examined how the forces on a body may be found by splitting the flow about it into two regions, one assumed inviscid, the other the boundary layer
- The inviscid flow could be solved using
  - *Thin Aerofoil Theory* (last year)
  - *The Joukowski transformation* (this year)
- The boundary layer could be solved by
  - *Integral methods*
  - *Differential methods*

i.e.



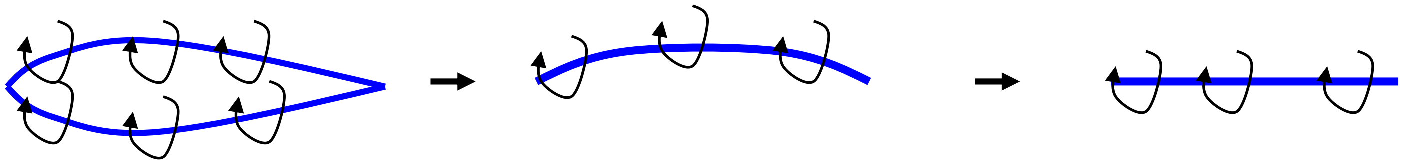
Joukowski / thin aerofoil
Integral boundary layer solver
Integrate C_p
Squire Young Method
C_L, C_D, C_M

However!

- The inviscid analysis methods thus far introduced have some disadvantages
- Over the next lectures, we shall look at methods widely used in industry
- More up to date, more flexible, give better results
- However, much more complex, so we shall only examine them superficially, beginning today with the *Panel Method*
- But first we shall re-examine thin aerofoil and Joukowski theory to see what the problems are

Thin aerofoil theory:

- Aerofoil is represented as a vortex sheet
  - Thickness and camber is assumed small, so sheet collapses to a line
  - Fact that there is no flow through the aerofoil is used to determine vortex strengths



End up with something like:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

University of BRISTOL

Limitations

- Using thin aerofoil theory means that:
  - Thickness distribution is ignored
  - Small incidence and surface gradients are assumed
- This means that
  - Cannot do thick aerofoils
  - Cannot do true three dimensional shapes
  - Cannot do multi-element aerofoils
  - Accuracy near stagnation is poor…afterall, thin objects don't have stagnation points!
  - Can't get Cps, only local load per unit chord

Joukowski Transformation

- Discussed extensively this year
  - Thick aerofoils can be examined
  - Solution is exact, even at stagnation
- BUT
  - Problems at trailing edge
    - Zero thickness
  - Inverse method (shape for flow, not flow about a shape). Method can't do arbitrary aerofoils
  - Most importantly, limited to 2D

Alternative Methods – The Panel Method

- Developed in the 60's as one of the first computational flow solution method
- Still widely used for some low speed applications
  - Landing configurations
  - F1
- Makes use of linear flows (sources, sinks, etc.), but uses them in such a way that flows with *no* analytical solution may be examined

University of BRISTOL

Brief history of panel methods

Earliest non-lifting method ~ 1952

Hess and Smith at Douglas, 1962

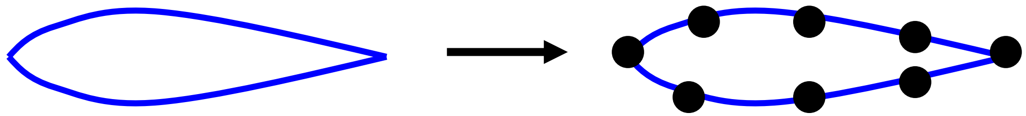
Rubbert and Saaris at Boeing, 1968

Roberts and Rundle, BAC, 1973

Finalised by about 1980 in various forms

Basic idea:

- Split the surface into a series of flat panels:



- On each one, put a source:

$$\phi_s = \frac{\Lambda}{4\pi}\frac{1}{r}$$

Source in 3D

$$\phi_s = \frac{\Lambda}{2\pi}\ln r$$

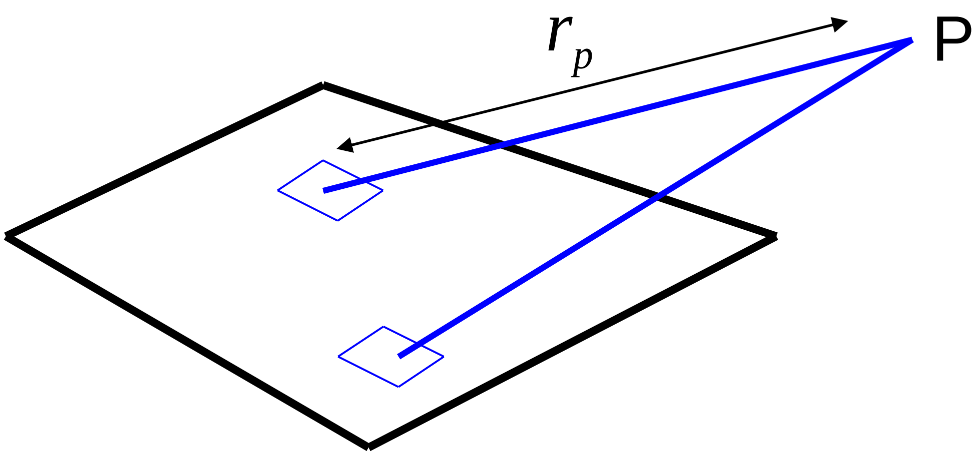
Source in 2D

University of BRISTOL

But source on the surface gives a singularity, so `smear out' source on each panel (area ΔS):

$$\Lambda = \sigma \Delta S$$

Then at any point P above the panel:



$$\phi_P(\Delta S) = \frac{\sigma}{4\pi} \iint_{\Delta S} \frac{dS}{r_p}$$

Linear/quadratic variations also sometimes used – tricky integrals!

Express in terms of x,y and dx,dy then integrate

University of BRISTOL

- Can generate an *influence coefficient* from this equation for the normal velocity U^n produced on any panel on a shape from any other panel on that shape:

$$(U^n{}_j)_i = a_{j,i}\sigma_i$$

- Where σ_i is the source strength on panel *i,* and $a_{j,i}$ is the influence coefficient of panel *i* on panel *j*.

- Hence

$$U^n{}_j = \sum_{i=1}^{i=n} a_{j,i}\sigma_i$$

Add up all panels

- And assuming we have a solid surface and N panels:

$$U^n{}_{total,j} = 0 \longrightarrow \sum_{i=1}^{i=N} a_{j,i}\sigma_i + U^n{}_{\infty\, j} = 0$$

$\longrightarrow$ Linear solve

- We end up with N equations for N unknowns (the source strengths), which we can put into a matrix:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & . & . \\ \vdots & . & . & . \\ a_{N,1} & . & . & a_{N,N} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{bmatrix} = -U_\infty \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix}$$

- This is then solved numerically, usually by using an error vector approach (much quicker than Gaussian elimination for large *N)*
- Cp usually found from Bernoulli's eqn after tangential velocities found
- Cd=0 in 2D - d'Alembert's paradox – in 3D can get the induced drag this way

University of BRISTOL

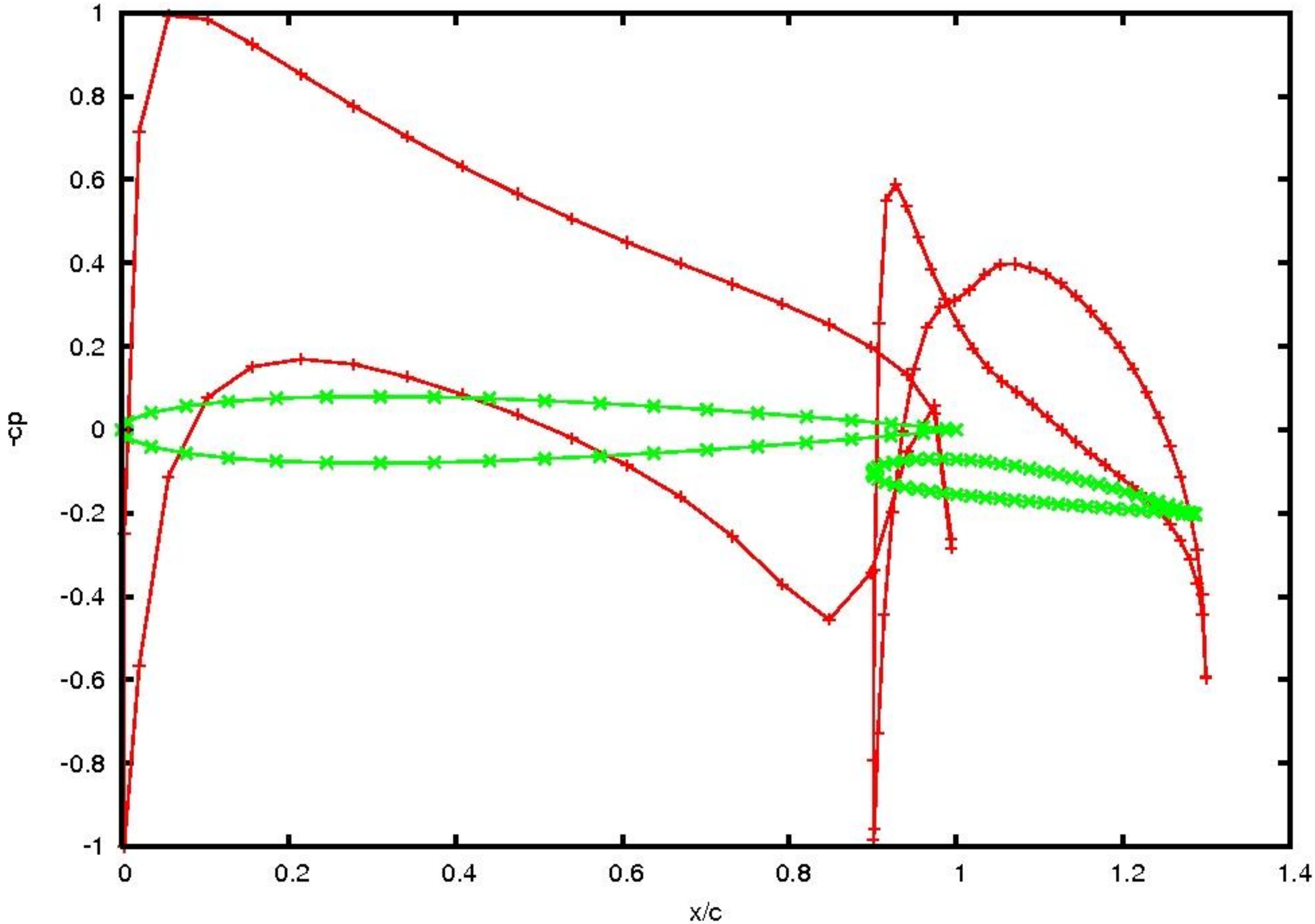
The 4 stages of a Panel Method:

- Discretisation
  - Breaking the surface into panels
- Calculation of Influence coefficients
  - Only depends on Geometry
- Solution and inversion of matrix
  - Find source strengths
- Use sources to calculate surface velocities
  - From which come pressures, etc.

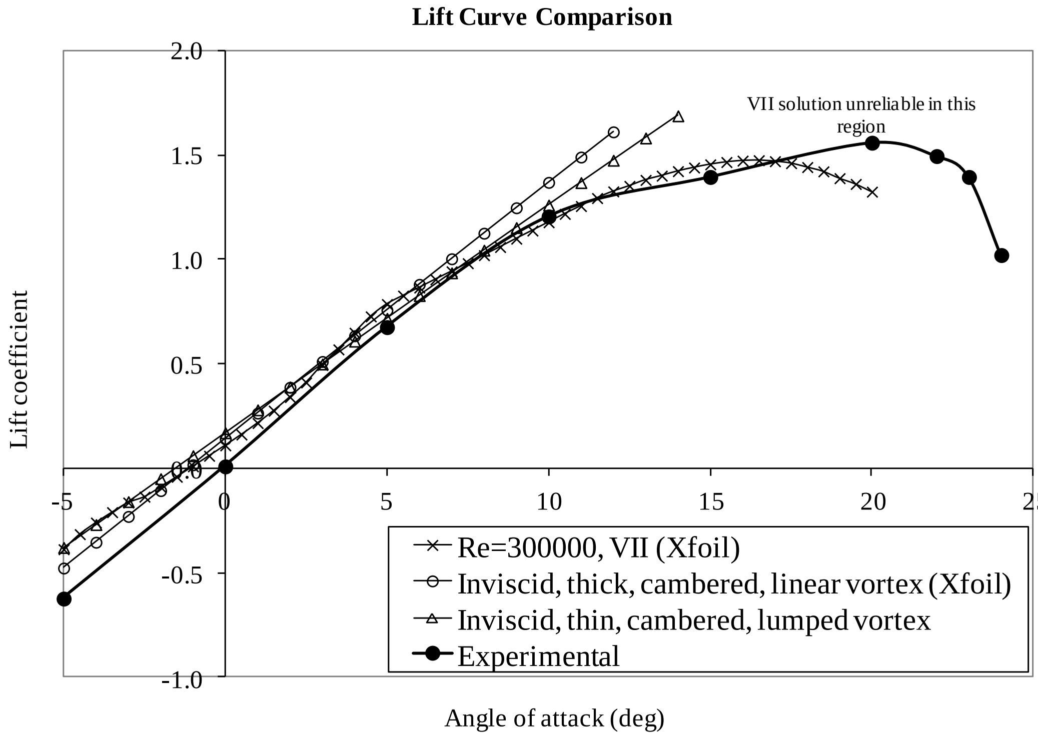
Advantages and Disadvantages:

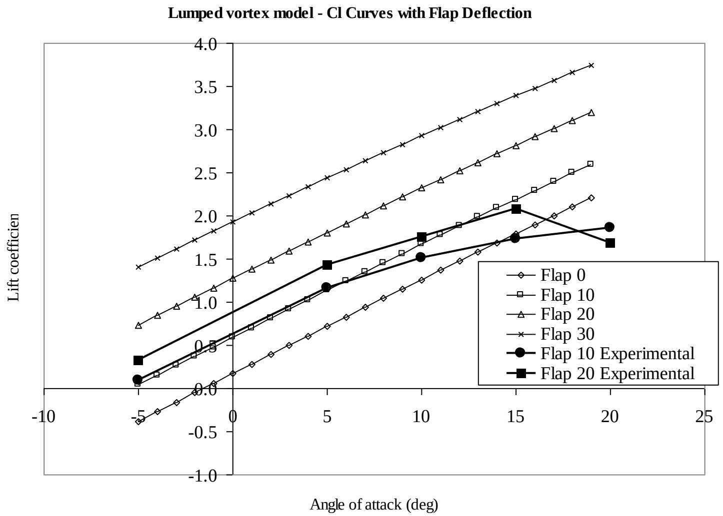
- More general than thin aerofoil and Joukowski
  - Can do 3D shapes
- Faster than full CFD
  - Only need surface mesh – fewer solution points, less effort
  - No need for time marching
- But
  - Best for low speed (linearly compressible if ‘corrected’ via Prandtl-Glauert), so no shocks, etc.
  - Gets complicated with arbitrary shapes! - esp if they don’t have sharp trailing edges. Sometimes makes method quite user intensive

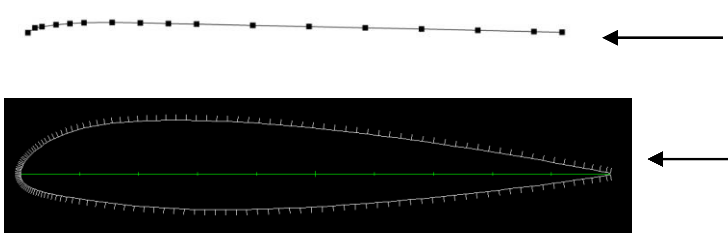
University of BRISTOL


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-cp
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x/c

Comparison to experiment (A1 lab 2nd year)



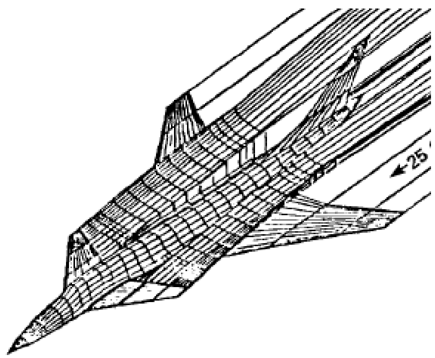


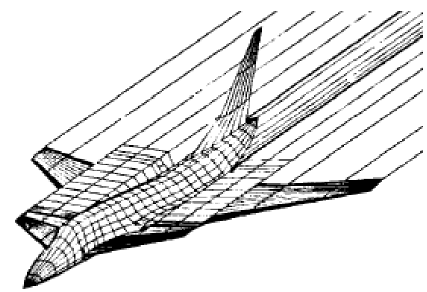


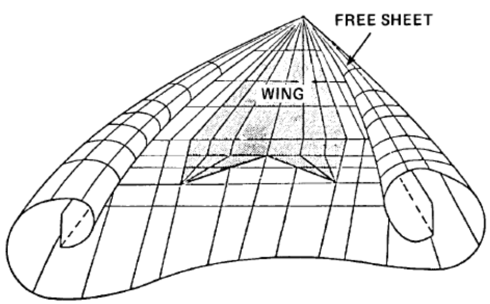
Lumped

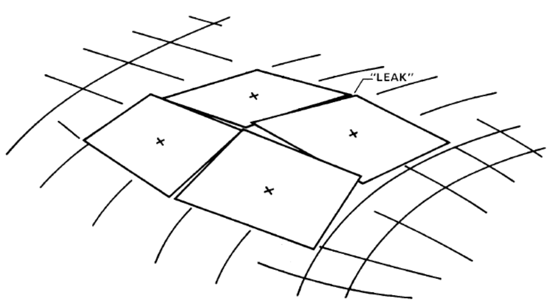
Xfoil

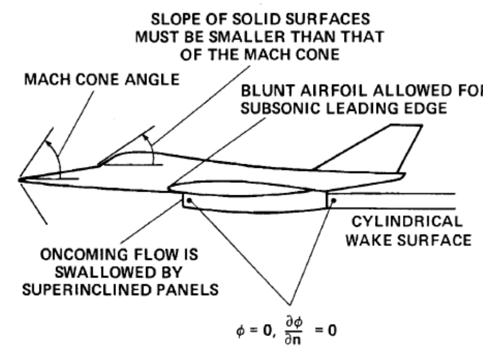
- The preceding is true for a basic method, but current methods can
  - calculate lifting flows and induced drag using wake relaxation
  - calculate off surface flows for aerodynamic interactions - vortex shedding
  - include other linear flow solutions (doublets, vortices, etc)
  - be coupled to boundary layer solvers, aeroelastic representations, thermal effects, etc.
  - Calculate supersonic flows (this requires quite smooth strength distributions)
  - Panel methods can be mathematically **very** elaborate! However, computationally, they are normally small problems



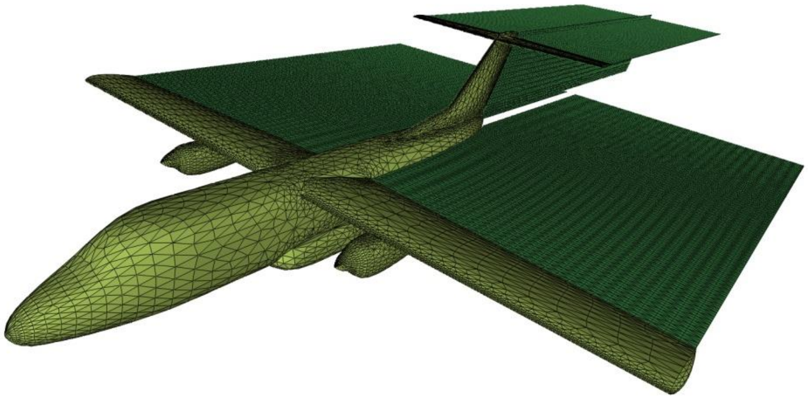
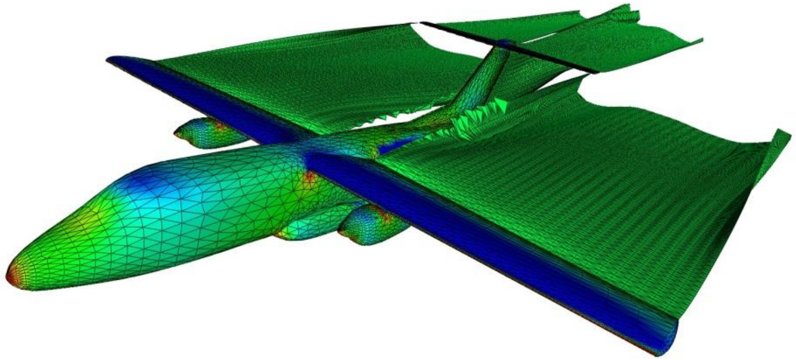
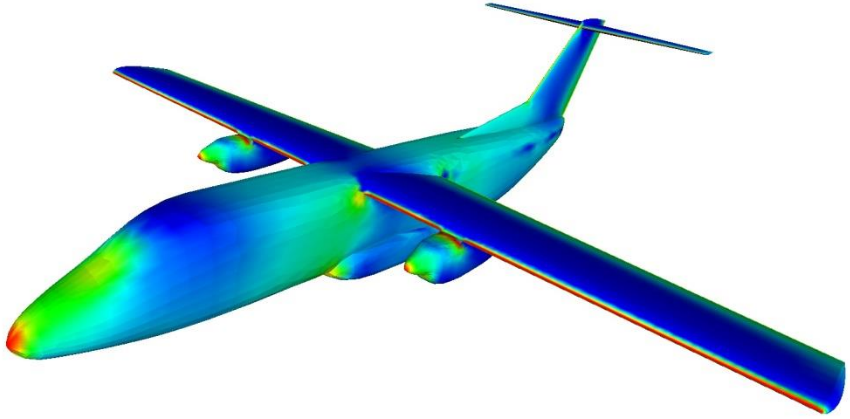






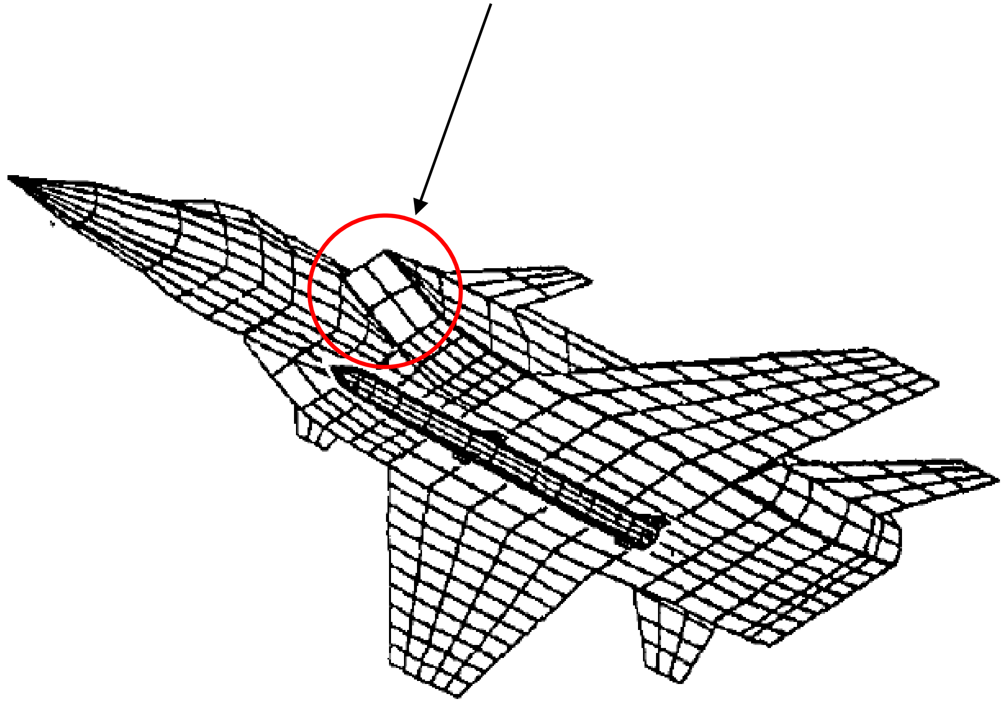


Wakes (vorticity sheets)!


A502 – PANAIR – aircraft with stores
(1980)

Boundary condition?



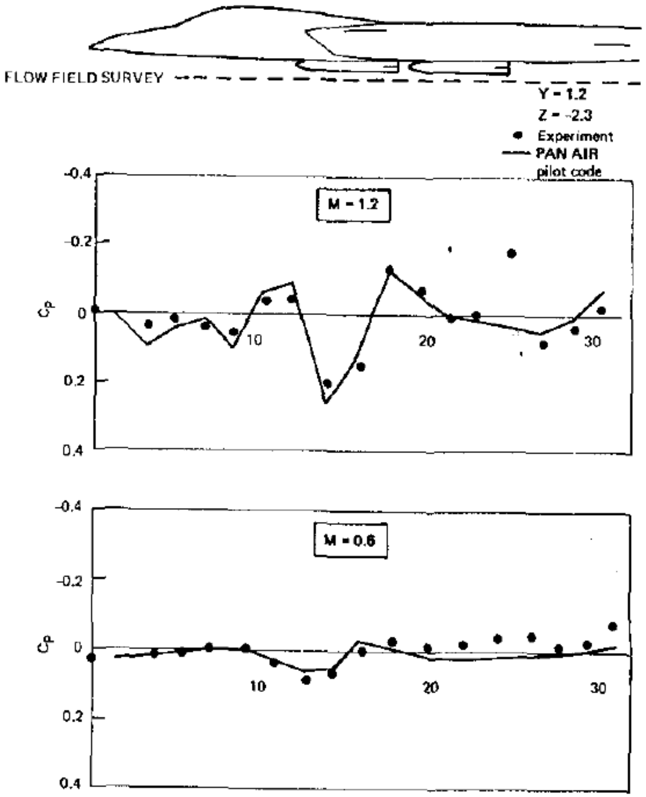


Figure 15. Off-Body Pressures - Aircraft with Stores

A502 - PANAIR 737 configuration
(1986)

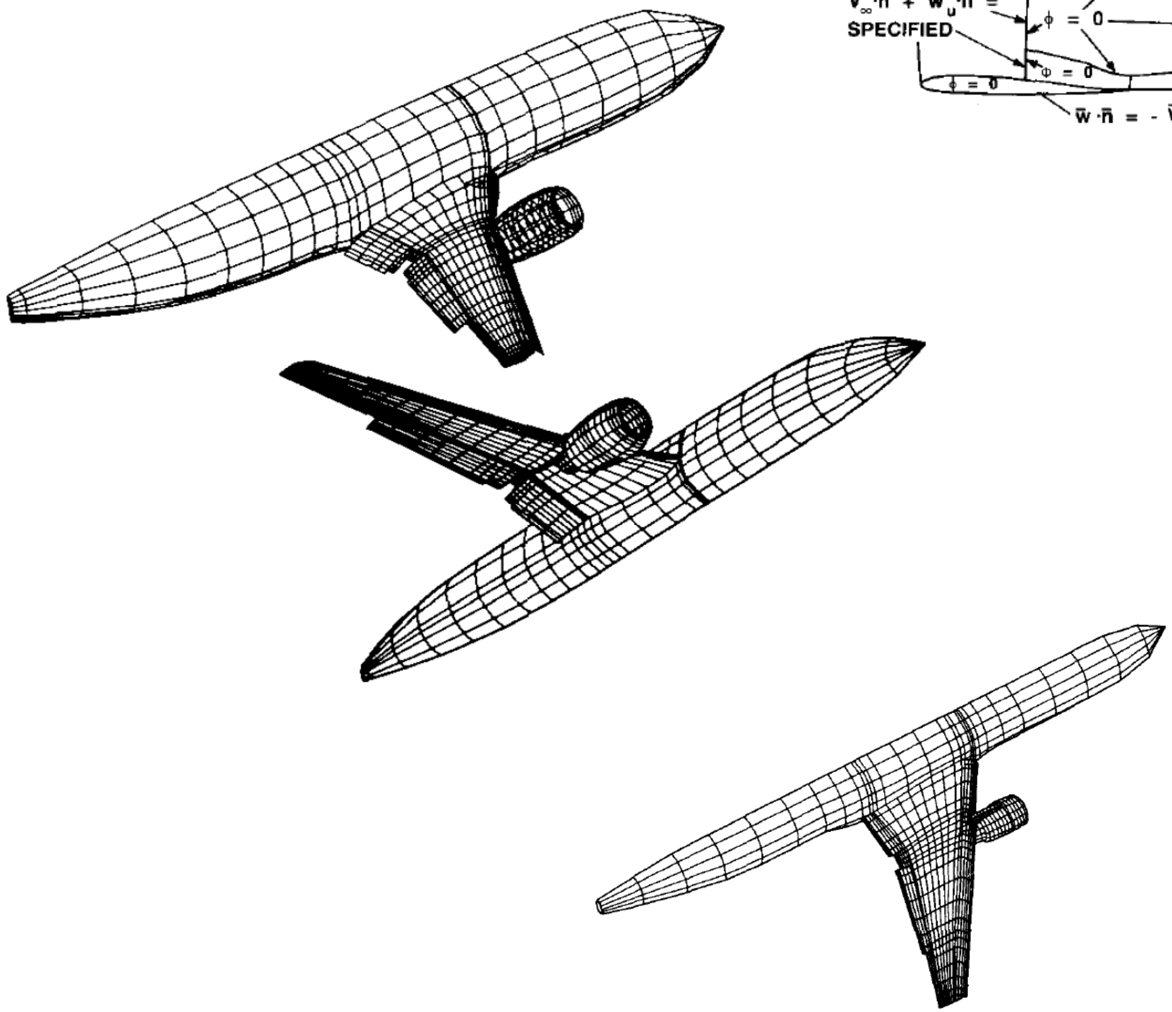
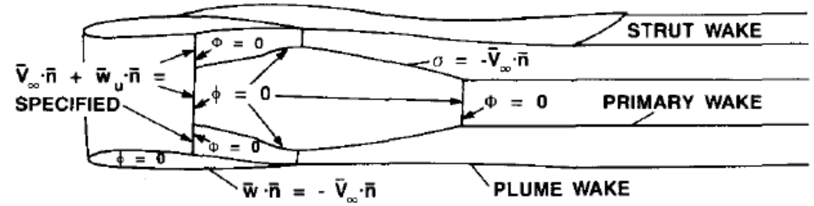


Figure 20. 767-300 Paneling



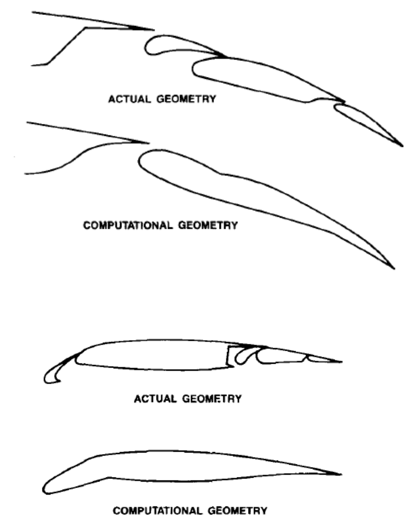


Figure 4. Merged Flaps 1 Geometry—Wing Section

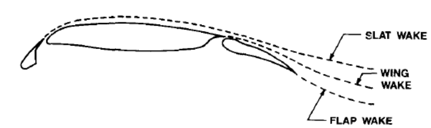


Figure 5. Flaps 15 Geometry and Wakes

University of BRISTOL

A502 - PANAIR 737 configuration

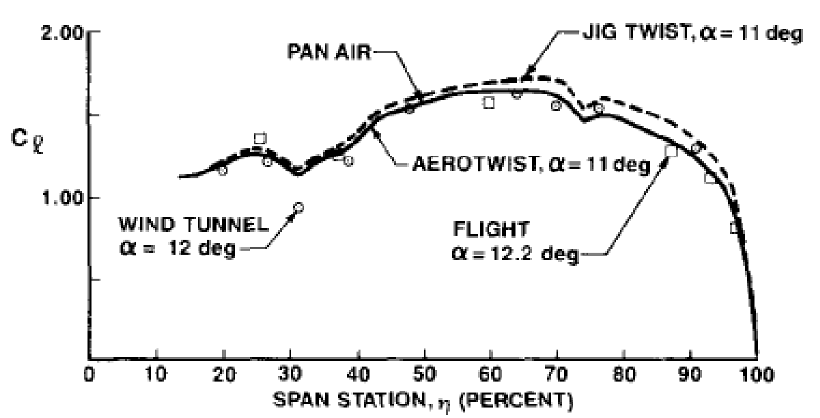


Figure 11. 737-300 Flaps 1 Wing Spanload

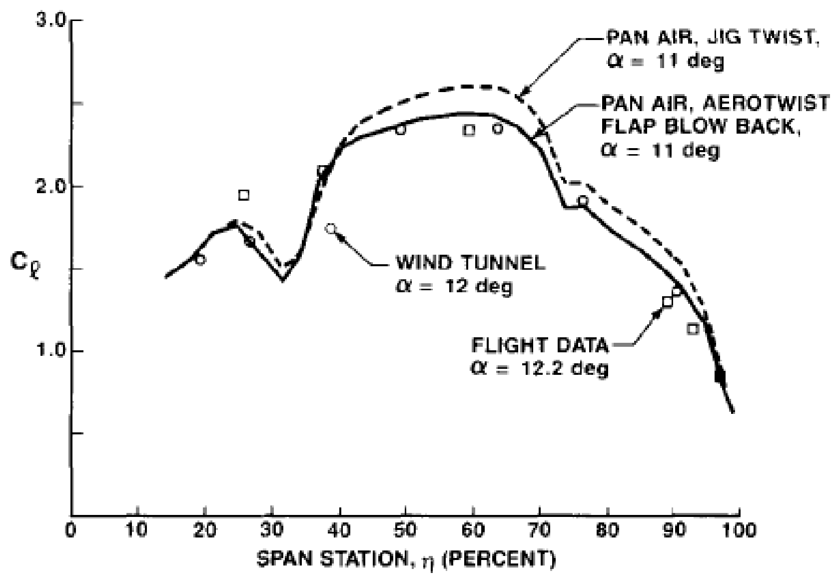
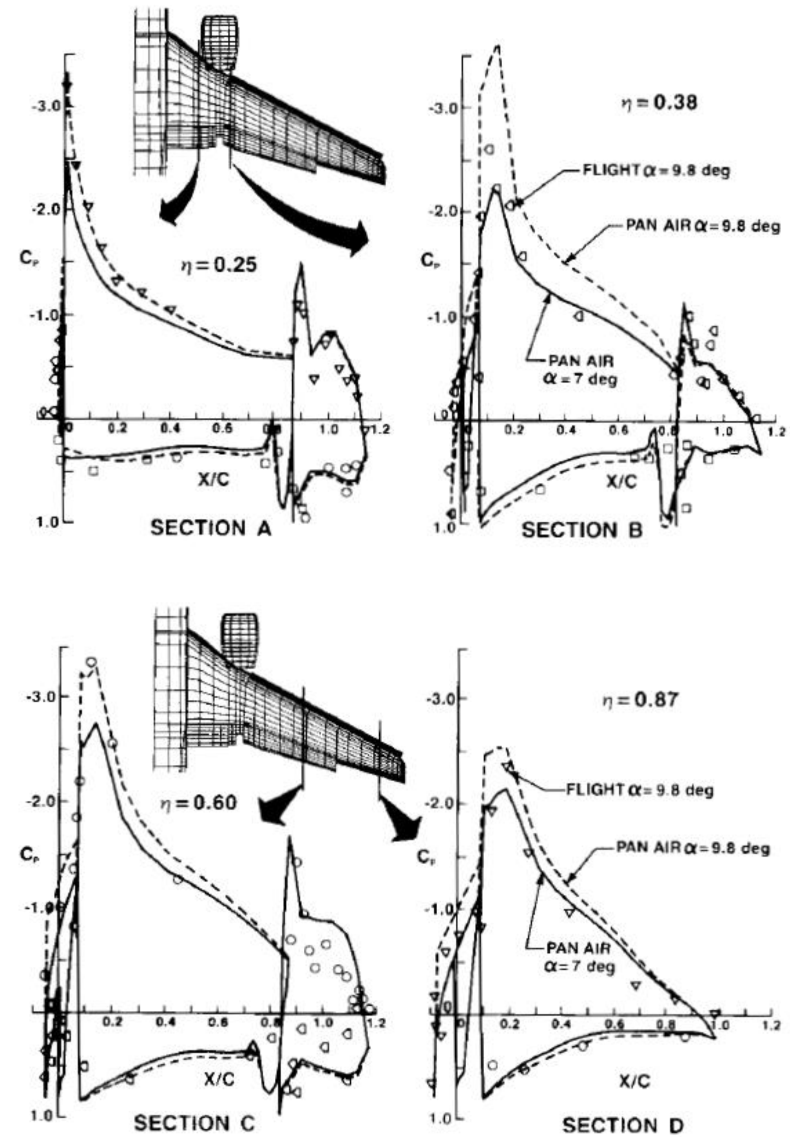
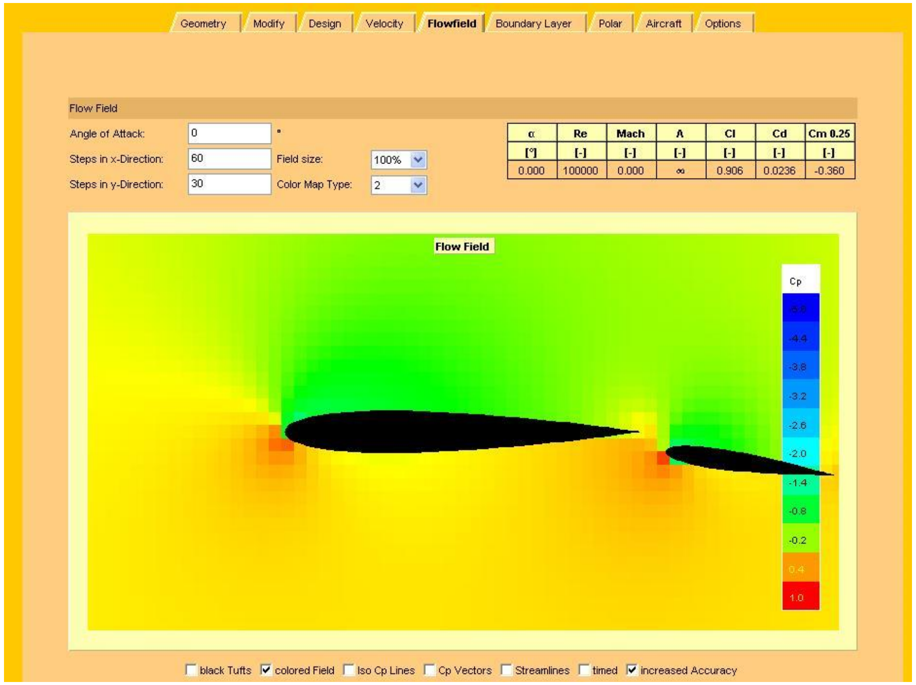


Figure 12. 737-300 Flaps 15 Wing Spanload



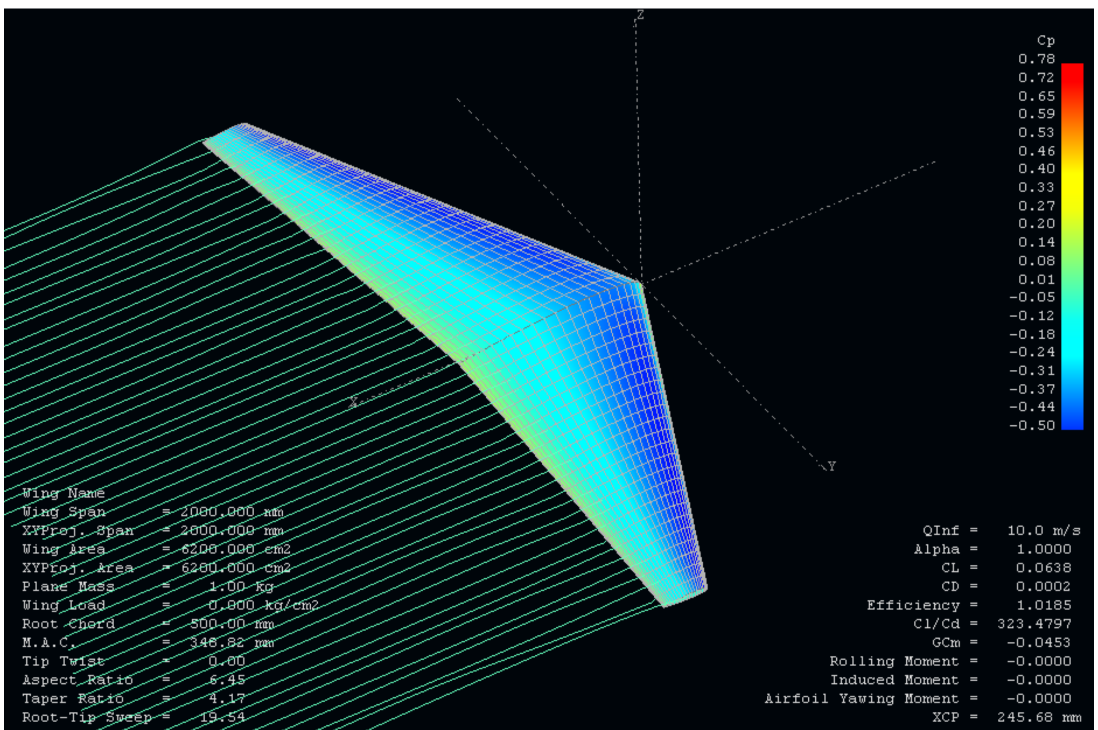
Javafoil

- 2D panel method, works for multi-elements



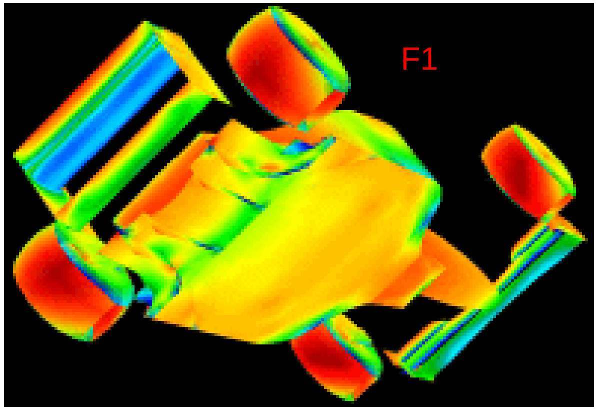
XFLR5

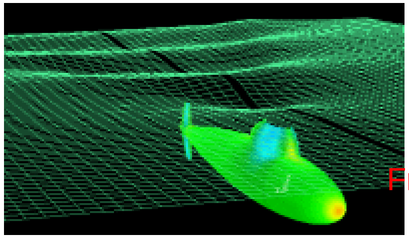
- 3D (fairly) easy to use full 3D panel code
- **With full thickness treatment**, only does single element wings in isolation. So for example winglets can be calculated, but not flaps
- Good for finding Cpmin – eg for critical Mach number

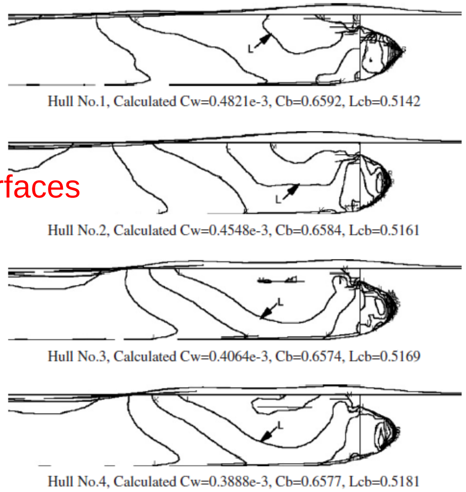


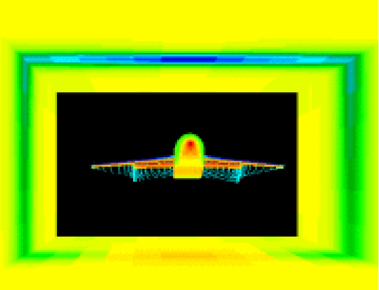
University of BRISTOL

Further examples – mostly NEWPAN

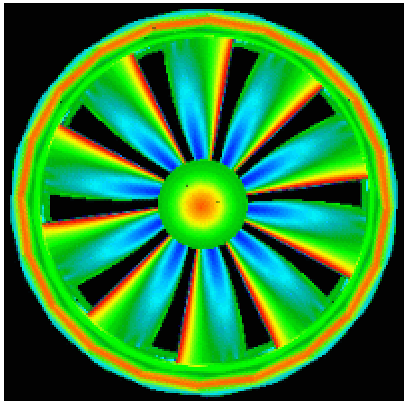


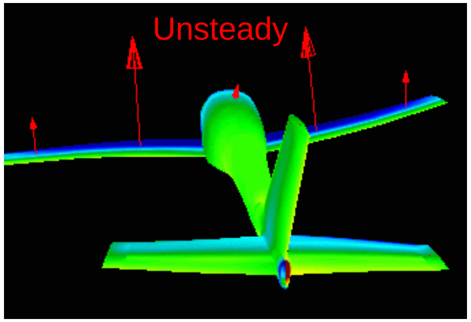


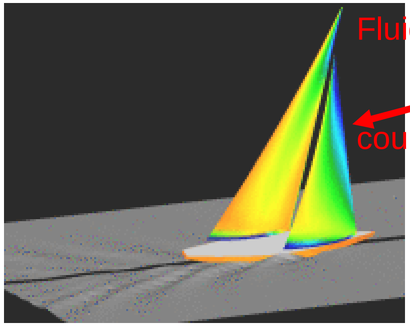


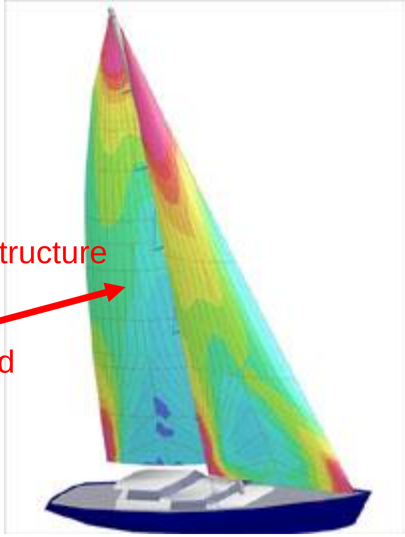


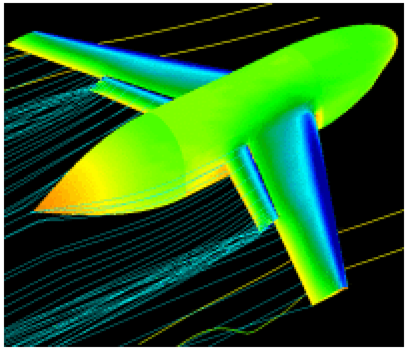
Wind tunnel corrections







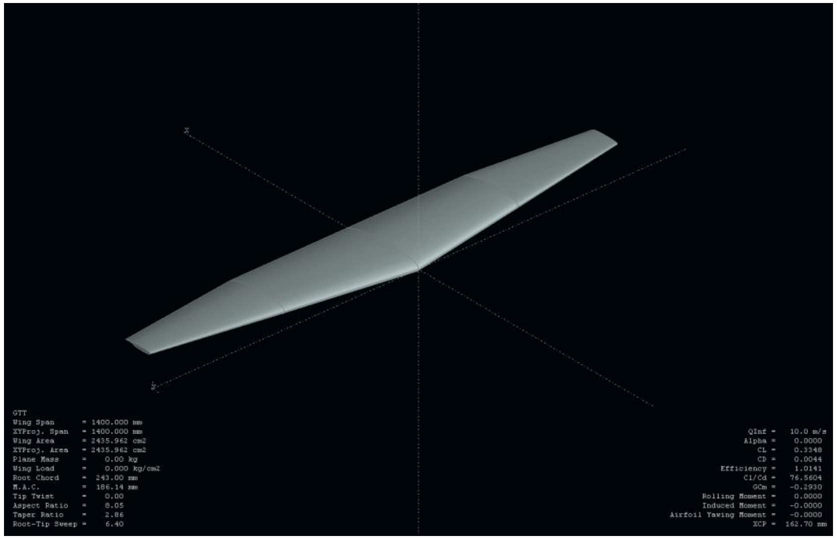


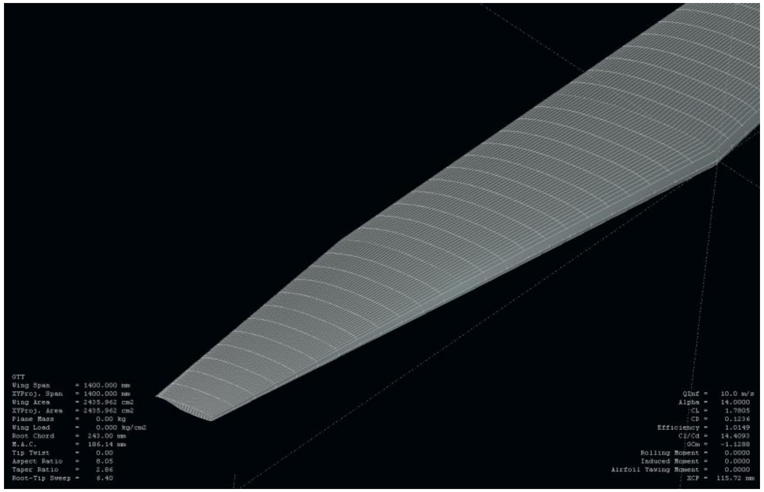


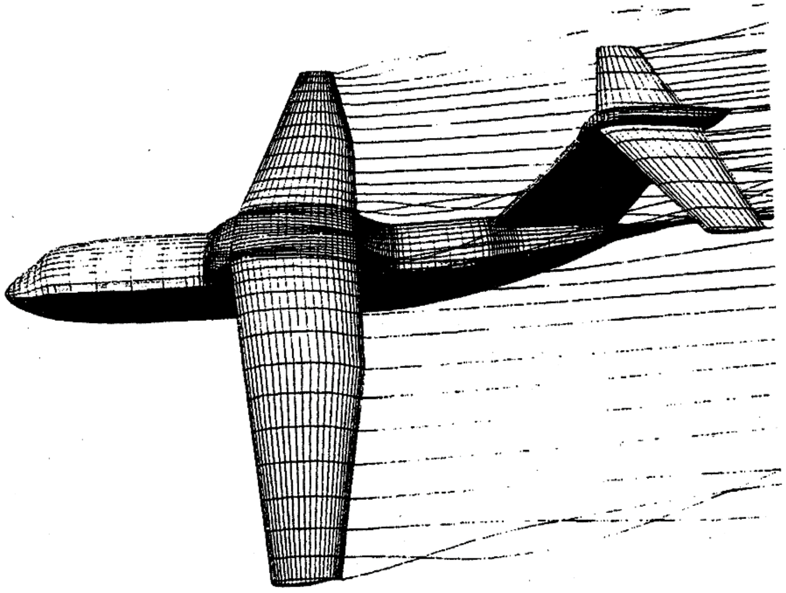
Low speed, high lift

University of BRISTOL

XFLR5 Example – Generic Turboprop Transport (GTT) wing





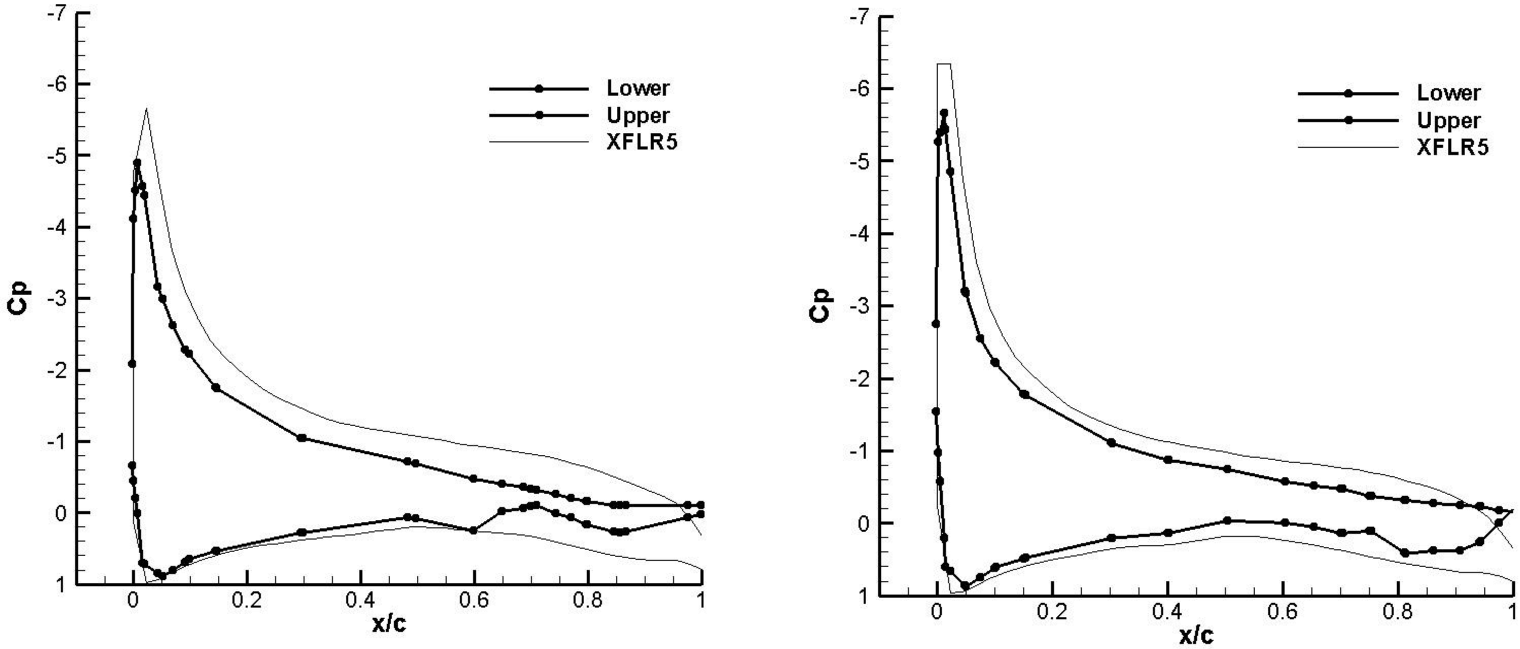


L Troeger and G Selby, *Computation of the aerodynamic characteristics of a subsonic transport*, Journal of Aircraft 35(2), p183-190

PDF Available on blackboard

University of BRISTOL

Cp Comparison



31% span

84% span

Cp Comparison

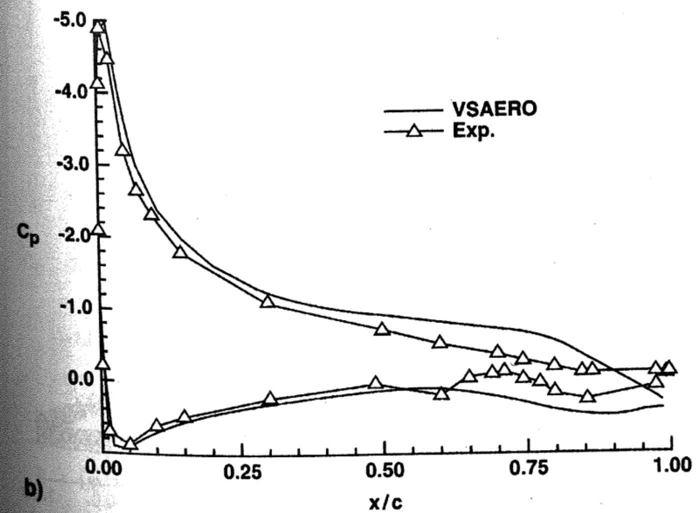


Fig. 13 Computational and experimental pressure distributions for the *WBFT* configuration for q = 40 psf at y = 22.0 in. α = a) 0 and b) 14 deg.

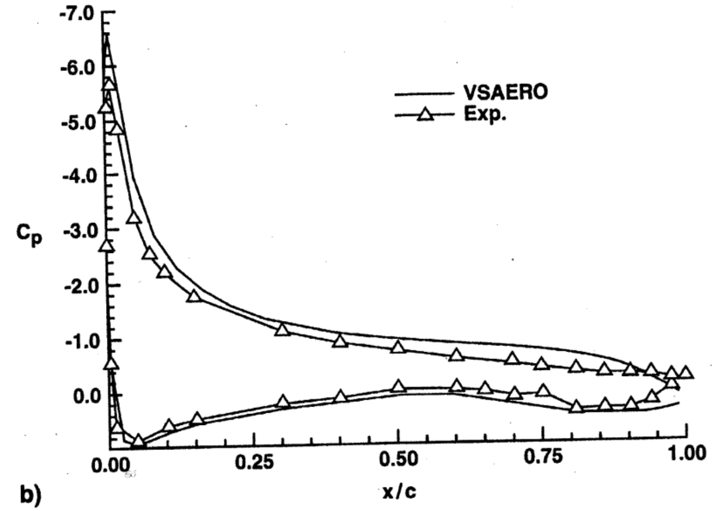
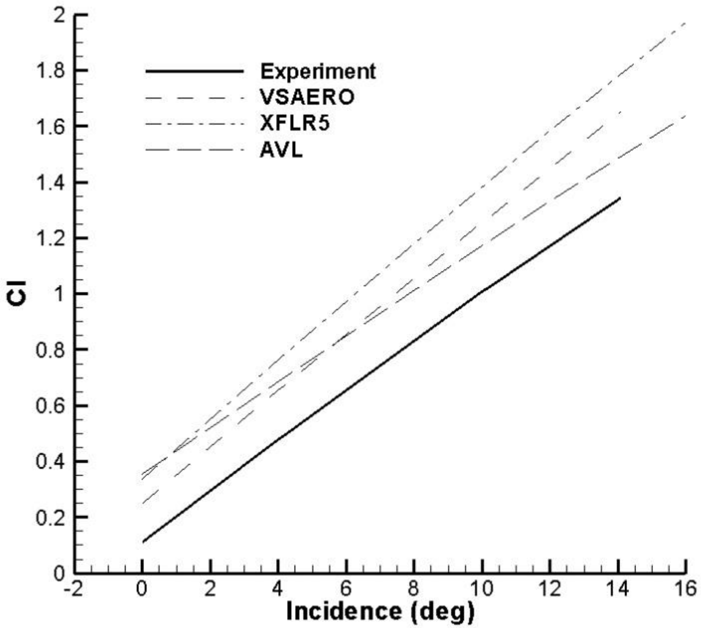


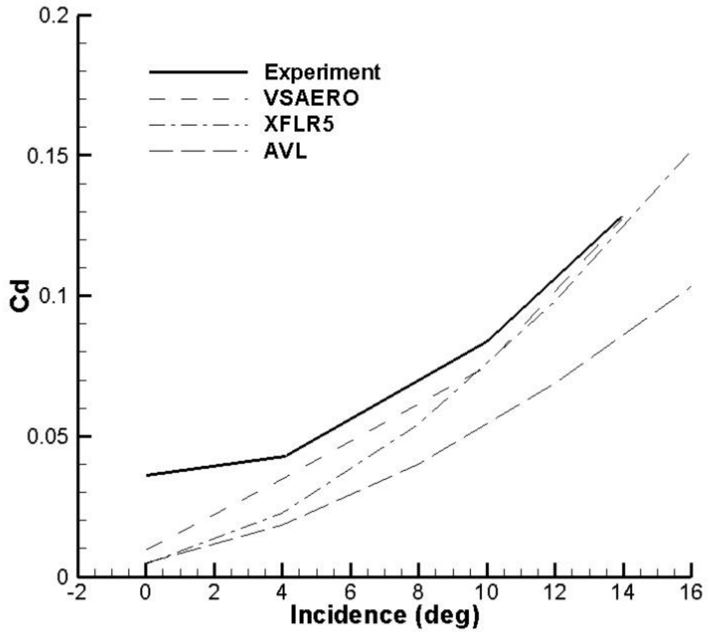
Fig. 15 Computational and experimental pressure distributions for the *WBFT* configuration for q = 40 psf at y = 59.0 in. α = a) 0 and b) 14 deg.

31% span

84% span

Forces


Cl
Incidence (deg)
Experiment
VSAERO
XFLR5
AVL


Cd
Incidence (deg)
Experiment
VSAERO
XFLR5
AVL

Forces

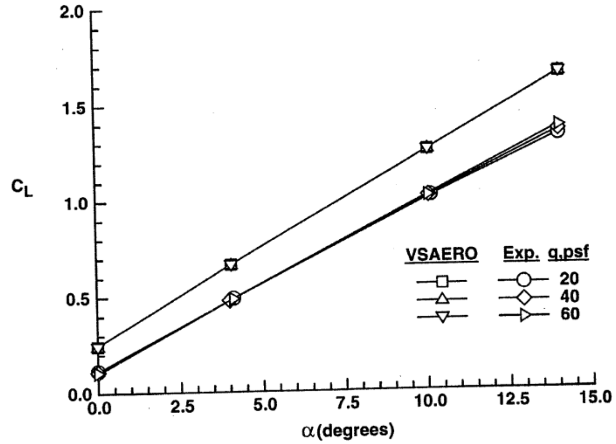


Fig. 16 Computational and experimental lift coefficients vs angle of attack for the *WBFT* configuration for all *q* values.

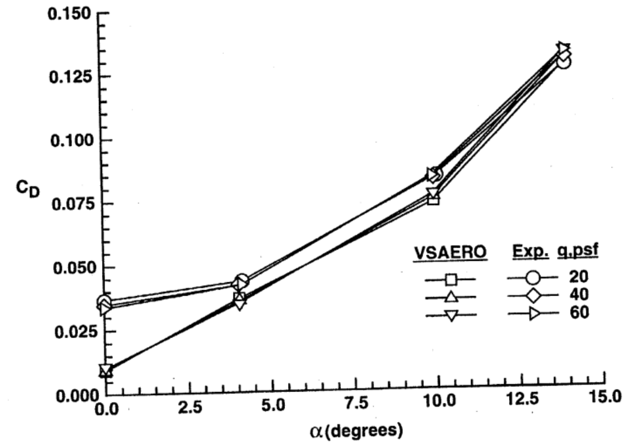


Fig. 17 Computational and experimental drag coefficients vs angle of attack for the *WBFT* configuration for all *q* values.

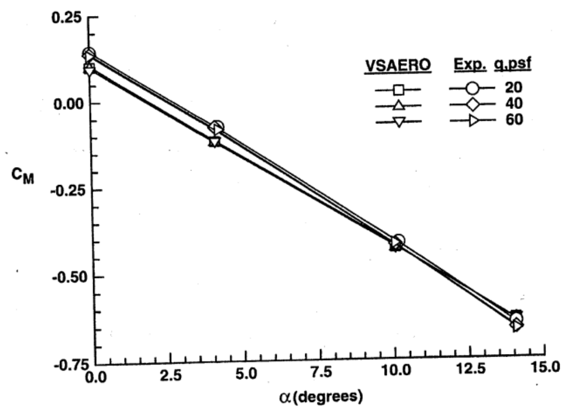


Fig. 18 Computational and experimental pitching moment coefficients vs angle of attack for the *WBFT* configuration for all *q* values.

University of BRISTOL

Exam Q

Q3 The velocity induced by a vortex is given by the Biot-Savart law $\mathbf{v} = \frac{1}{4\pi} \int \frac{\Gamma \mathbf{ds} \times \mathbf{r}}{|\mathbf{r}|^3}$.

(a) Integrate the Biot-Savart Law using the nomenclature in figure Q3(a) to find the velocity induced by a semi-infinite vortex at a distance h from the start point of the vortex.

[4 marks]

Now consider a simple untwisted untapered rectangular wing of semispan b (total span $2b$) as shown in figure Q3(b). The wing is modelled using a single bound vortex along the quarter chord and a trailing semi-infinite vortex from each wingtip.

(b) Using the induced velocity result found in part **Q3**a, and by applying the 2D sectional lift relation $C_{l_{2D}} = C_{l_{\alpha_{2D}}}(\alpha_\infty - \alpha_i)$ at the centre span and $\frac{1}{4}$ chord position of the wing (labelled 'control point location' in figure Q3(b)), find the sectional lift coefficient for the wing.

[6 marks]

(c) Show that the wing lift coefficient is equal to the sectional lift coefficient.

[2 marks]

(d) By applying Göthert's rule, find the ratio between the compressible and incompressible lift coefficients for the wing, with $\beta = \sqrt{1 - M_\infty^2}$

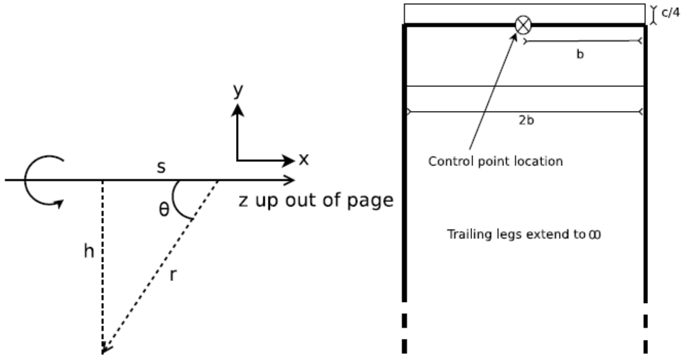
[5 marks]

(e) Explain the steps in the process for applying Göthert's rule to a vortex lattice method for calculation of the influence of compressibility.

[3 marks]

Exam Q

(a)



Line vortex

(b) Wing arrangement

$$\mathbf{V} = \frac{\Gamma}{4\pi}\int \frac{\mathbf{ds}\times\mathbf{r}}{|\mathbf{r}|^3}$$

Introduce

$$h = r\sin(\theta)$$

$$s = \frac{h}{\tan(\theta)}$$

$$ds = -\frac{h}{\sin^2(\theta)}d\theta$$

$$\mathbf{ds}\times\mathbf{r} = \begin{pmatrix} \frac{-h}{\sin^2(\theta)}d\theta \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -s \\ -h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{h^2}{\sin^2(\theta)}d\theta \end{pmatrix}$$

Integral becomes

$$V_z = \frac{\Gamma}{4\pi}\int \frac{h^2}{\sin^2(\theta)}\frac{\sin^3(\theta)}{h^3}d\theta = \frac{\Gamma}{4\pi h}\int \sin(\theta)d\theta$$

$$V_z = \frac{\Gamma}{4\pi h}(\cos(\theta_A) - \cos(\theta_B)) = \frac{\Gamma}{4\pi h}$$

University of BRISTOL

(b)

$$C_l = \frac{2\Gamma}{Uc}$$

$$\alpha_i = \frac{\frac{2UcC_l}{2}}{4\pi bU} = \frac{C_l c}{4\pi b}$$

Can now write sectional lift as

$$C_l = C_{l_{a2D}}(\alpha - \frac{C_l c}{4\pi b})$$

which gives

$$C_l = \frac{\alpha}{\frac{1}{C_{l_{a2D}}} + \frac{c}{4\pi b}}$$

$$C_l = \frac{C_{l_{a2D}}\alpha}{1 + \frac{cC_{l_{a2D}}}{2b2\pi}} = \frac{C_{l_{a2D}}\alpha}{1 + \frac{C_{l_{a2D}}}{A_R 2\pi}}$$

(c)

$$C_L = \frac{\rho U\Gamma 2b}{\frac{1}{2}\rho U^2 2bc} = \frac{2\Gamma}{Uc} = C_l$$

(d)

$$C_{L_c} = \frac{1}{\beta^2}\frac{C_{l_{a2D}}\alpha\beta}{1 + \frac{C_{l_{a2D}}}{\beta A_R 2\pi}} = \frac{C_{l_{a2D}}\alpha}{\beta + \frac{C_{l_{a2D}}}{A_R 2\pi}}$$

Hence

$$\frac{C_{L_c}}{C_L} = \frac{1 + \frac{C_{l_{a2D}}}{A_R 2\pi}}{\beta + \frac{C_{l_{a2D}}}{A_R 2\pi}} = \frac{A_R 2\pi + C_{l_{a2D}}}{\beta A_R 2\pi + C_{l_{a2D}}}$$

(e)

So for $A_R = 0$ no changes, and for large A_R tends to $\frac{1}{\beta}$.

Scale down in height and span by factor of β (this includes AoA). Perform calculations. Divide resulting coefficients by β^2 (for lift/moment).

Aerodynamics 3

Introduction to Complex Potential

(chapter 6 in notes)

University of BRISTOL

Overview:

- The lecture this week will
  - Introduce the concept of complex potential
  - Show how it can represent some flow-fields
  - Discuss limitations on the types of functions that can be used for complex potential analysis
- This is the first step towards being able to use conformal mapping and the Joukowsky transformation.
- We shall begin, however, with a revision of some important concepts introduced last year, which allow simplification of the flow

Navier-Stokes Equations

The NS Eqns. describe the flow of a fluid. They are:

- 5 coupled, non-linear, partial differential equations in 6 variables, plus the Equation of State.
- Insoluble, very complex; e.g., the x-momentum equation below:

$$\frac{\partial \rho u}{\partial t}+\frac{\partial \rho u^2}{\partial x}+\frac{\partial \rho uv}{\partial y}+\frac{\partial \rho uw}{\partial z}=\frac{-\partial P}{\partial x}+\mu\nabla^2 u+\mu\frac{\partial \Delta/3}{\partial x}-2\Delta\frac{\partial \mu/3}{\partial x}$$
$$+2\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

Where:

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \qquad \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

University of BRISTOL

Navier-Stokes interpretation

The maths of NS can also be written in integral form in words...(note - the previous was in differential form)

Rate change of mass inside a box=(mass flow in)-(mass flow out)

Rate change of momentum in a box=(momentum flux in)-(momentum flux out)+pressure forces+viscous forces

Rate change of energy=(energy flux in)-(energy flux out)

I prefer this form!

University of BRISTOL

Inviscid Flow

- Assuming inviscid flow would allow us to remove nearly all the RHS of the previous equation.
- This can be assumed iff the *effects* of viscosity are negligible, i.e. viscosity itself does need not be (and generally is not) zero.
- How do we know when this assumption can be used, and what limitations does this have?

Inviscid Flow

•Assuming inviscid flow would allow us to remove nearly all the RHS of the previous equation.

•This can be assumed iff the *effects* of viscosity are negligible, i.e. viscosity itself does not need be (and generally is not) zero.

•How do we know when this assumption can be used, and what limitations does this have?

•*The answer is when the* **Reynolds Number** *is sufficiently large, generally anything bigger than* $\sim 10^5$

Reynolds Number:

Defined as the ratio between inertial and viscous forces:

$$\mathrm{Re} = \frac{\rho U l}{\mu}$$

Where; ρ is density, U representative velocity, l representative length, μ the coefficient of viscosity

e.g. for a typical small aircraft at sea level, say

Reynolds Number:

Defined as the ratio between inertial and viscous forces:

$$\mathrm{Re} = \frac{\rho U l}{\mu}$$

Where; ρ is density, *U* representative velocity, *l* representative length, μ the coefficient of viscosity

e.g. for a typical small aircraft at sea level, say

$$\rho \approx 1.225 kgm^{-3}, M \approx 0.3 \Rightarrow U \approx 100 ms^{-2}$$

$$l \approx 5, \mu \approx 1.8 \times 10^{-5}$$

And hence

Reynolds Number:

Defined as the ratio between inertial and viscous forces:

$$\mathrm{Re} = \frac{\rho U l}{\mu}$$

Where; ρ is density, *U* representative velocity, *l* representative length, μ the coefficient of viscosity

e.g. for a typical small aircraft at sea level, say

$$\rho \approx 1.225 kgm^{-3}, M \approx 0.3 \Rightarrow U \approx 100 ms^{-2}$$

$$l \approx 5, \mu \approx 1.8 \times 10^{-5}$$

And hence

$$\mathrm{Re} = \frac{1.225 \times 100 \times 5}{1.8 \times 10^{-5}} \approx 3.6 \times 10^{7} >> 10^{5}$$

University of BRISTOL

Boundary layers and Re

Boundary layer thickness is inversely proportional to Re (we will show this in coming lectures)

If Re goes up, boundary layers get thinner

Don’t forget this!

This is why high Re attached flow can be thought of as inviscid - the boundary layers are so thin they can’t change the external flow

Euler Equations

The inviscid assumption allows us to reduce the Navier-Stokes to the Euler Equations. Again, using the x-momentum equation as an example, we have

$$\frac{\partial \rho u}{\partial t}+\frac{\partial \rho u^2}{\partial x}+\frac{\partial \rho uv}{\partial y}+\frac{\partial \rho uw}{\partial z}=\frac{-\partial P}{\partial x}$$

But, the flow must be:

- of high enough Re
- about a streamlined shape
- outside the boundary layers (we will return to this later in the course)

Further Assumptions:

- Steady Flow - rules out manoeuvre and aeroelastics, but removes all time dependency

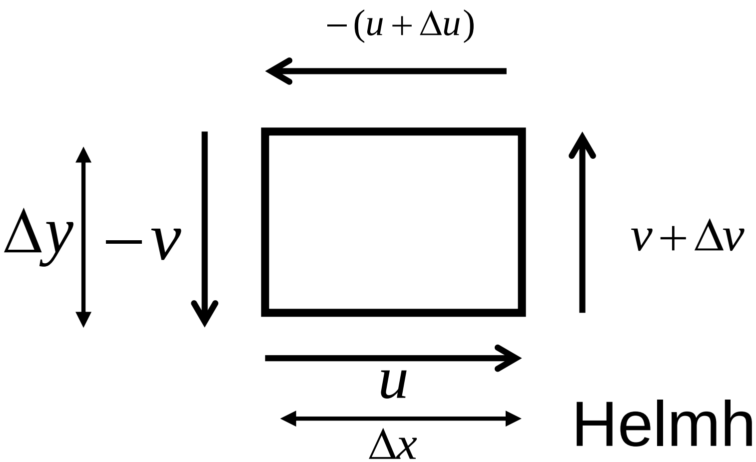
$$\frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = \frac{-\partial P}{\partial x}$$

- 2D Flow - reduces number of variables, but limits applications

$$\frac{\partial \rho u^2}{\partial x} + \frac{\partial uv}{\partial y} = \frac{-\partial P}{\partial x}$$

Irrotational Flow:

- Vorticity of a flow is the velocity integral about a closed area, divided by the area (the Stokes Theorem):



$$velocity = \nabla \phi$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}; \Rightarrow \frac{\partial \phi}{\partial x} = u; \frac{\partial \phi}{\partial y} = v$$

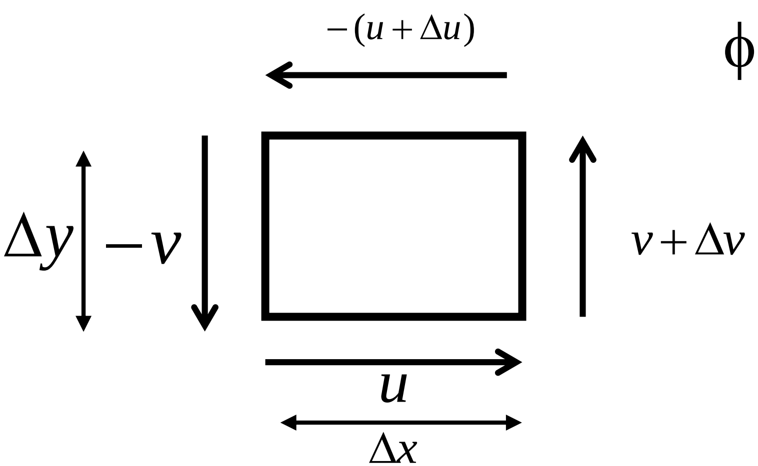
Helmholtz theorem $V = \nabla \phi + \nabla \times A$

$$\nabla \times V = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Only an off-centre force can impart rotation, but in inviscid flows away from surfaces, only pressure acts – which acts through the centre, so *most* inviscid flows are irrotational

Irrotational Flow:

- Vorticity of a flow is the velocity integral about a closed area, divided by the area:



ϕ = Velocity Potential

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}; \Rightarrow \frac{\partial \phi}{\partial x} = u; \frac{\partial \phi}{\partial y} = v$$

$$\frac{\partial \varphi}{\partial x} = -v; \frac{\partial \varphi}{\partial y} = u$$

Only an off-centre force can impart rotation, but in inviscid flows away from surfaces, only pressure acts – which acts through the centre, so *most* inviscid flows are irrotational

Incompressible flow

- In low speed flows, density changes very slowly, and hence can be approximated as constant. This decouples the energy equation, and as ρ is fixed, this equation need not be calculated.
- The range of applicability is approx M=0 to M=0.3, although simple compressibility corrections (e.g. Prandtl-Glauert you met last year) can extend this towards Mcrit

Laplace's Equation

If the flow is: **2D, steady, inviscid, incompressible, irrotational**, the complete Navier-Stokes equations may be represented by **Laplace's Equation**, in either ϕ or φ, i.e.

$$\nabla \cdot \nabla \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{or} \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

This is because *P* is now a function of *u,v* only, ρ is constant, *T* independent. Hence if we know ϕ or φ, we know *u,v,P*

Complex Numbers:

The imaginary number *i* is defined as

$$i = \sqrt{-1}$$

A point (coordinate) in the complex plane, *Z* is thus

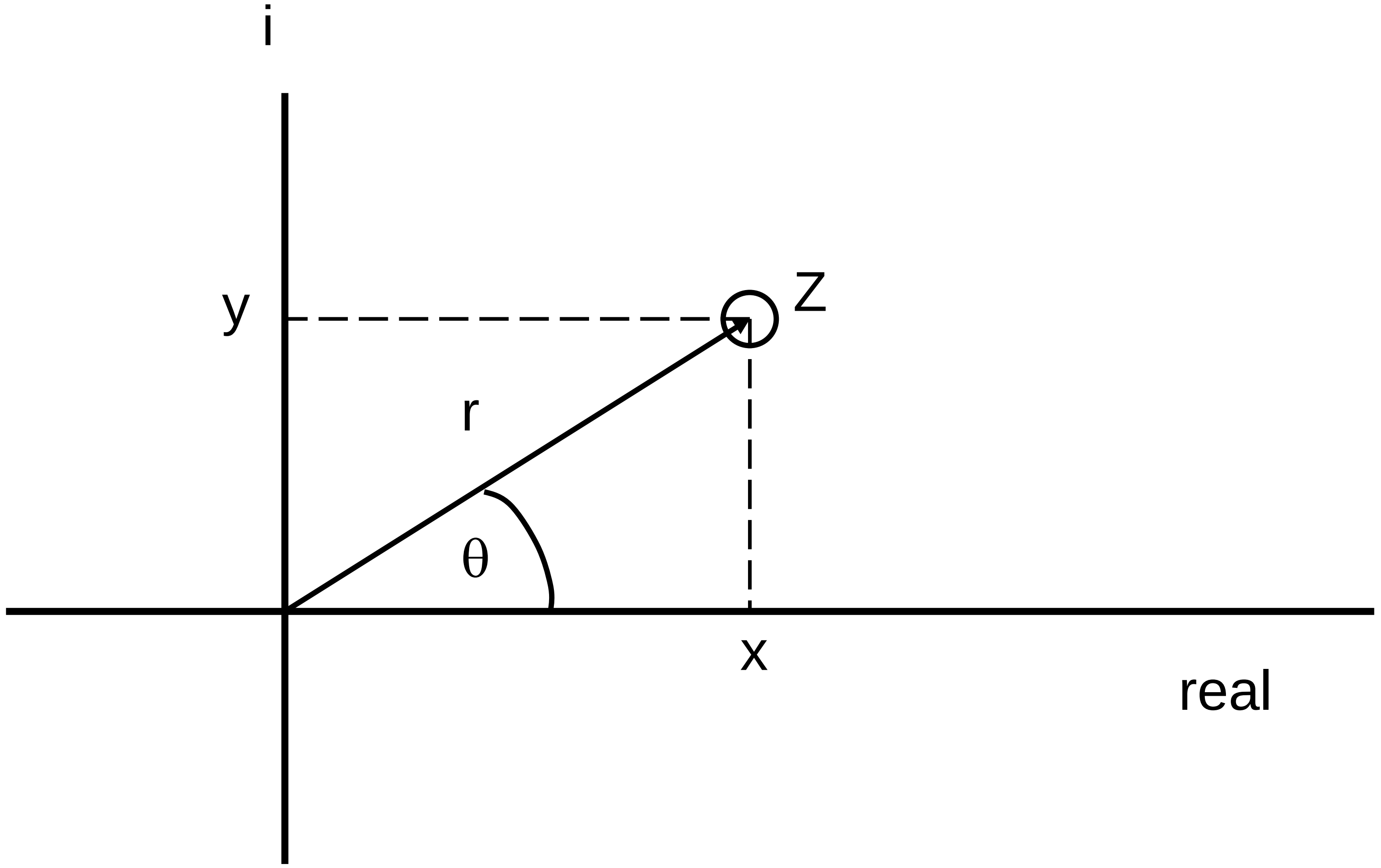
$$Z = x + iy$$

Such a point (x,y) may also be expressed in (r,θ):

$$x = r\cos\theta, y = r\sin\theta$$

i.e.

$$Z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$


i
y
Z
r
θ
x
real

We can also have Complex Functions:

i.e. $$\Phi = W(x+iy) = W(Z)$$

Where *W(Z)* is only a function of *Z*, i.e. its location in the complex plane, e.g.

$$W(Z) = Z^3, W(Z) = Z - Z^2$$

The function will itself be complex, and hence described as

$$W(Z) = f(x,y) + ig(x,y)$$

For instance in the case of $W(Z) = Z^3$

$$f(x,y) = x^3 - 3xy^2$$

$$g(x,y) = 3x^2y - y^3$$

University of BRISTOL

Is Φ a solution of Laplace?

This would be handy, as we know ϕ is. So;

$$\frac{\partial W(Z)}{\partial x} = \frac{\partial W}{\partial Z}\frac{\partial Z}{\partial x}$$

But we know (we defined it that way)

$$\frac{\partial W}{\partial Z} = \frac{dW}{dZ}$$

And

$$\frac{\partial Z}{\partial x} = \frac{\partial\left(x+iy\right)}{\partial x} = 1$$

So

$$\frac{\partial W(Z)}{\partial x} = \frac{dW}{dZ}$$

University of BRISTOL

The second derivative is thus

$$\frac{\partial^2 W(Z)}{\partial x^2} = \frac{\partial}{\partial x}\frac{dW}{dZ} = \frac{\partial Z}{\partial x}\frac{\partial}{\partial Z}\frac{dW}{dZ} = \frac{d^2W}{dZ^2}$$

If we look at the y derivatives:

$$\frac{\partial W(Z)}{\partial y} = \frac{\partial W}{\partial Z}\frac{\partial Z}{\partial y} = \frac{dW}{dZ}\frac{\partial\left(x+iy\right)}{\partial y} = i\frac{dW}{dZ}$$

and hence

$$\frac{\partial^2 W(Z)}{\partial y^2} = \frac{\partial}{\partial y} i\frac{dW}{dZ} = \frac{\partial Z}{\partial y}\frac{\partial}{\partial Z} i\frac{dW}{dZ} = i^2\frac{d^2W}{dZ^2} = -\frac{d^2W}{dZ^2}$$

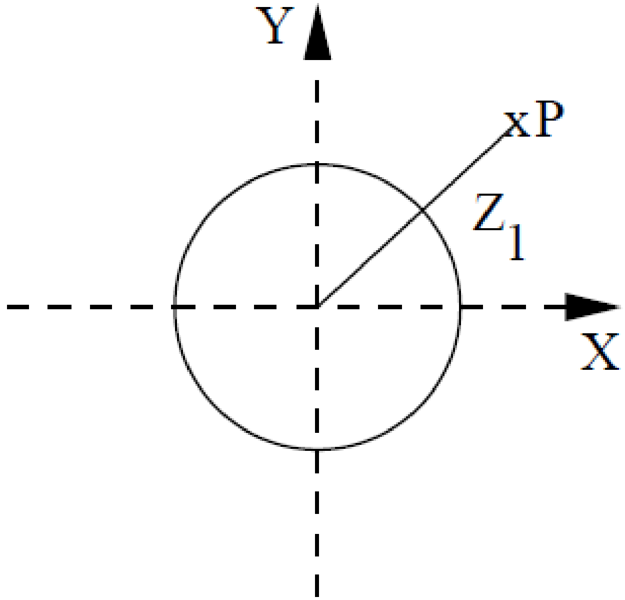
Finally

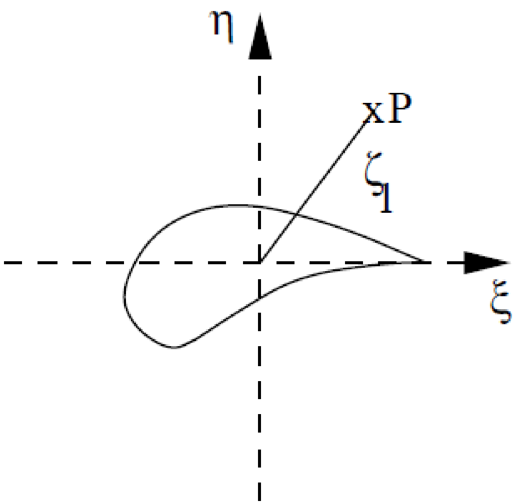
$$\frac{\partial^2 W(Z)}{\partial x^2} + \frac{\partial^2 W(Z)}{\partial y^2} = \frac{d^2W}{dZ^2} - \frac{d^2W}{dZ^2} = 0$$

Which demonstrates Laplace's equation is indeed satisfied,

which is what we wanted

University of BRISTOL





From this...

...to this

If Laplace is solved on the left, it is also solved on the right!

The tool to do this shape change is coming up in the next lectures

Limitations

However, there is one important limitation on the type of functions we can use. It must be the case that the derivative with respect to Z, defined as

$$\frac{dW(Z)}{dZ} = \lim_{\Delta z \to 0} \frac{W(Z+\Delta Z) - W(Z)}{\Delta Z}$$

Where

$$(Z+\Delta Z) = ((x+\Delta x) + i(y+\Delta y))$$

Gives the same answer regardless of how Δx, Δy tend to 0

(We could take the limit as either the real or imaginary parts tend to zero, but we need the same result from both)

University of BRISTOL

For this to be the case, (this is proven formally in your notes),

$$W(Z) = f + ig \qquad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}; \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

These are the **Cauchy-Riemann Conditions.** If a function satisfies

these, and is

- Single valued (has one value at a particular location)
- Bounded (not growing to infinity)

Then we can use it. But we know from earlier that

$$\frac{\partial \phi}{\partial x} = u = \frac{\partial \varphi}{\partial y}; \frac{\partial \phi}{\partial y} = v = -\frac{\partial \varphi}{\partial x}$$

University of BRISTOL

Hence we may write $$\frac{\partial W(Z)}{\partial Z} = \frac{\partial \phi}{\partial x} + i\frac{\partial \varphi}{\partial x} = \frac{1}{i}\frac{\partial \phi}{\partial y} + \frac{i}{i}\frac{\partial \varphi}{\partial y} = u - iv$$

$$W(Z) = \phi(x,y) + i\varphi(x,y)$$

This means that

- A known flow-field expressed in terms of its potential and stream functions may be expressed by one variable (W) in the complex plane
- If W(Z) is known the process may be reversed and the associated flow (ϕ,φ) produced

Finally,

$$x = Z - jy$$

$$y = \frac{Z - x}{j}$$

$$\frac{dW(Z)}{dZ} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial Z} + j\frac{\partial \psi}{\partial x}\frac{\partial x}{\partial Z} = u - jv$$

$$\frac{dW(Z)}{dZ} = \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial Z} + j\frac{\partial \psi}{\partial y}\frac{\partial y}{\partial Z} = v\frac{1}{j} + ju\frac{1}{j} = u - jv$$

University of BRISTOL

Next

- How we can construct complex potentials from ones we know
- How we can work with them to get pressures and velocities in practice
- How we can transform them and take advantage of the geometric flexibility they give us

Aerodynamics 3

Examples of Complex Potential Functions

(chapter 7 in notes)

University of BRISTOL

Overview:

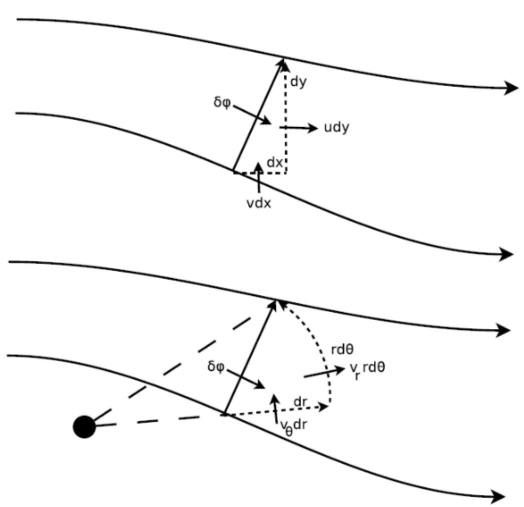
- This weeks lecture will proceed from the concept of complex functions introduced last week to give
  - Specific examples of simple potential flows
  - Demonstrate how to derive the flow around a circle from summations of these simple flows
  - Introduce lifting flows
- The simple flows thus derived will then be used in conjunction with Conformal Mapping, to be introduced next week

Useful to know

- Check the revision of potential flow section at end of notes if unsure

$$\delta\psi = -v\Delta x + u\Delta y$$

$$\delta\psi = \frac{\partial\psi}{\partial x}\Delta x + \frac{\partial\psi}{\partial y}\Delta y$$



$$\frac{\partial\phi}{\partial x} = u = \frac{\partial\psi}{\partial y}$$

$$\frac{\partial\phi}{\partial y} = v = -\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\phi}{\partial r} = v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

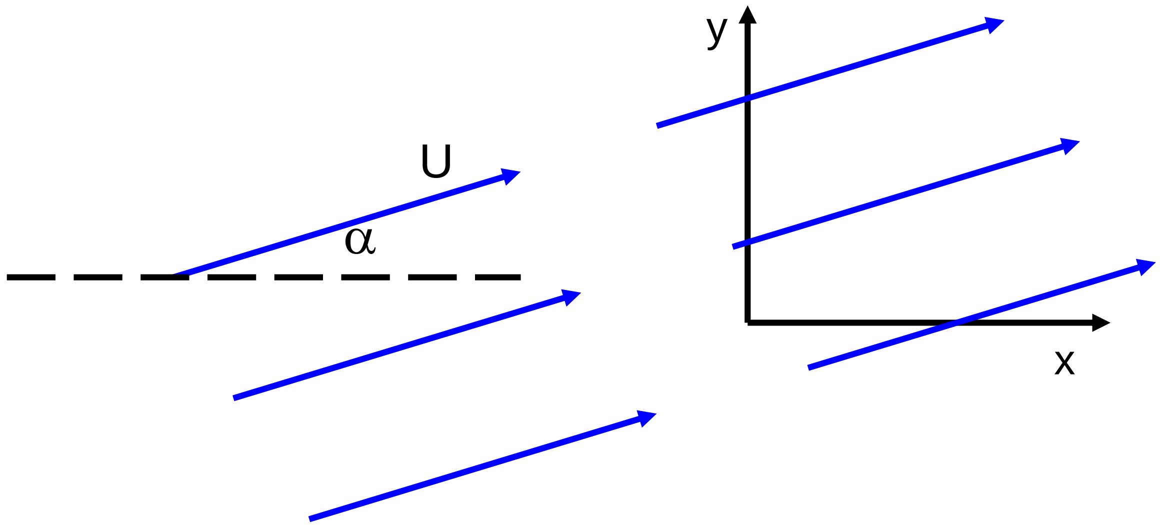
$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = v_\theta = -\frac{\partial\psi}{\partial r}$$

$$\delta\psi = -v_\theta\Delta r - rV_r\Delta(-\theta) = -v_\theta\Delta r + rv_r\Delta\theta$$

$$\delta\psi = \frac{\partial\psi}{\partial r}\Delta r + \frac{\partial\psi}{\partial\theta}\Delta\theta$$

University of BRISTOL

Uniform Onset Flow:



Flow of velocity U,

at an incidence α

What type of function do we want?

- U,α are constant everywhere, and hence so are components, *u,v*.
- From last week, we know

$$\frac{\partial W(Z)}{\partial Z} = u - iv \qquad \Rightarrow W(Z) = Z \times const.$$

Uniform Flow Continued:

Consider $W(Z)=Ue^{-i\alpha}Z$:

Obviously, $Ue^{-i\alpha}$ is a constant. From deMoivres theorem

$$e^{-i\alpha} = \cos\alpha - i\sin\alpha$$

and thus

$$W(Z) = U(\cos\alpha - i\sin\alpha)(x+iy)$$

This is a single valued function, and can be represented as a sum of ϕ and φ:

$$W(Z) = \underbrace{U(x\cos\alpha + y\sin\alpha)}_{\phi} + i\underbrace{U(y\cos\alpha - x\sin\alpha)}_{\varphi}$$

So:

$$\phi = U(x\cos\alpha + y\sin\alpha)$$

$$\varphi = U(y\cos\alpha - x\sin\alpha)$$

And hence

$$\frac{\partial\phi}{\partial x} = u = U\cos\alpha$$

$$\frac{\partial\phi}{\partial y} = v = U\sin\alpha$$

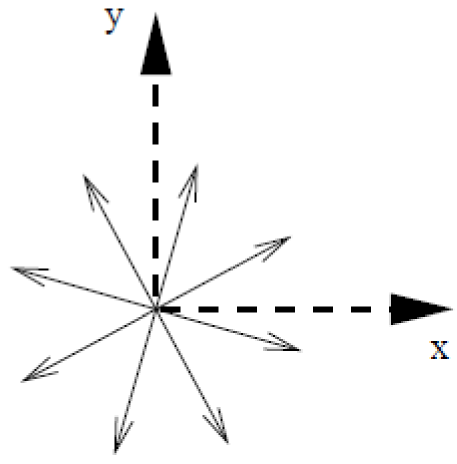
Also

$$\frac{dW(Z)}{dZ} = \frac{dUe^{-i\alpha}Z}{dZ} = Ue^{-i\alpha} = U\cos\alpha - iU\sin\alpha = u - iv$$

As required

Source Flow:

Source Strength Λ



Source

$$u = v_r \cos(\theta) = \frac{\Lambda}{2\pi r}\frac{x}{r} = \frac{\Lambda x}{2\pi r^2}$$

$$v = v_r \sin(\theta) = \frac{\Lambda}{2\pi r}\frac{y}{r} = \frac{\Lambda y}{2\pi r^2}$$

Flow emanates from a single point (in this case the origin), in all directions equally. In 2D, at a point P, due to Conservation of Mass:

$$u = \frac{\Lambda x}{2\pi r^2} \qquad v = \frac{\Lambda y}{2\pi r^2}$$

University of BRISTOL

Source Flow Continued:

Consider $W(Z) = \frac{\Lambda}{2\pi}\ln(Z):$

Differentiating *W(Z)* w.r.t. *Z*:

$$\frac{dW(Z)}{dZ} = \frac{\Lambda}{2\pi Z} = \frac{\Lambda}{2\pi\left(x+iy\right)}$$

Can multiply top and bottom by complex conjugate:

$$\frac{dW(Z)}{dZ} = \frac{\Lambda\left(x-iy\right)}{2\pi\left(x+iy\right)\left(x-iy\right)} = \frac{\Lambda\left(x-iy\right)}{2\pi\left(x^2+y^2\right)}$$

i.e.

$$u = \frac{\Lambda x}{2\pi r^2} \qquad v = \frac{\Lambda y}{2\pi r^2}$$

University of BRISTOL

Source Flow in Polar Coordinates:

$$Z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

but

$$\cos\theta + i\sin\theta = e^{i\theta}$$

hence

$$Z = re^{i\theta}$$

This means that

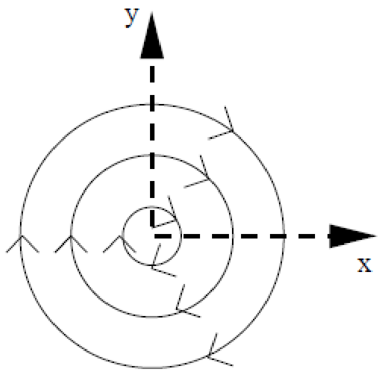
$$W(Z) = \frac{\Lambda}{2\pi}\ln(Z) = \frac{\Lambda}{2\pi}\ln(re^{i\theta}) = \frac{\Lambda}{2\pi}\ln(r) + i\frac{\Lambda\theta}{2\pi}$$

i.e.

$$\phi = \frac{\Lambda}{2\pi}\ln(r) \qquad \varphi = \frac{\Lambda\theta}{2\pi}$$

Vortex

$$2\pi r v_\theta = \Gamma \longrightarrow v_\theta = \frac{\Gamma}{2\pi r} = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$



Point Vortex

$$\phi = \frac{\Gamma\theta}{2\pi}$$

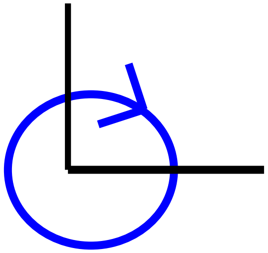
$$\psi = -\frac{\Gamma}{2\pi}\ln(r)$$

$$W = \phi + j\psi = \frac{\Gamma}{2\pi}(\theta - j\ln(r)) = \frac{\Gamma}{2\pi j}(\theta j + \ln(r)) = \frac{\Gamma}{2\pi j}\ln(Z)$$

University of BRISTOL

$$\ln(Z) = \ln(re^{j\theta}) = \ln(r) + \ln(e^{j\theta}) = \ln(r) + j\theta$$

Vortex



- Point Vortex (Γ +ve clockwise)

$$W(Z) = \frac{-\Gamma}{2\pi i}\ln(Z)$$

$$\Rightarrow \phi = -\frac{\Gamma\theta}{2\pi}, \quad \varphi = \frac{\Gamma}{2\pi}\ln(r)$$

(note the similarity between ϕ and φ between source flow and vortex flow – this is because the velocities induced are normal to each other)

Source: $\phi = \frac{\Lambda}{2\pi}\ln(r) \qquad \varphi = \frac{\Lambda\theta}{2\pi}$

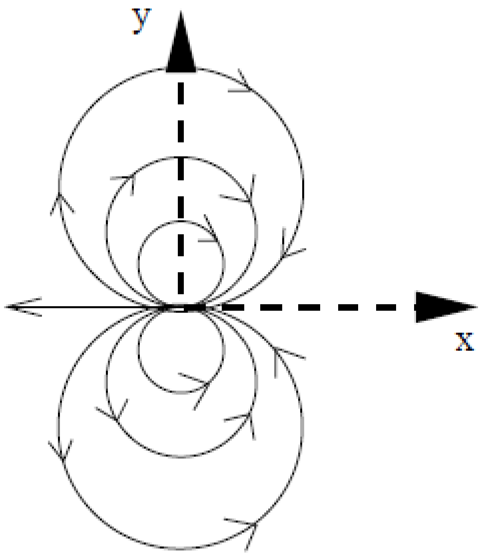
Doublet

$$\phi = \frac{\kappa \cos\theta}{2\pi r}, \qquad \varphi = -\frac{\kappa \sin\theta}{2\pi r}$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$W(Z) = \frac{\kappa}{2\pi Z}$$

These come from taking the limit of a source on top of a sink



Doublet

Doublet as a limit

$$W = \frac{\Lambda}{2\pi}\left(\ln(Z+a) - \ln(Z-a)\right)$$

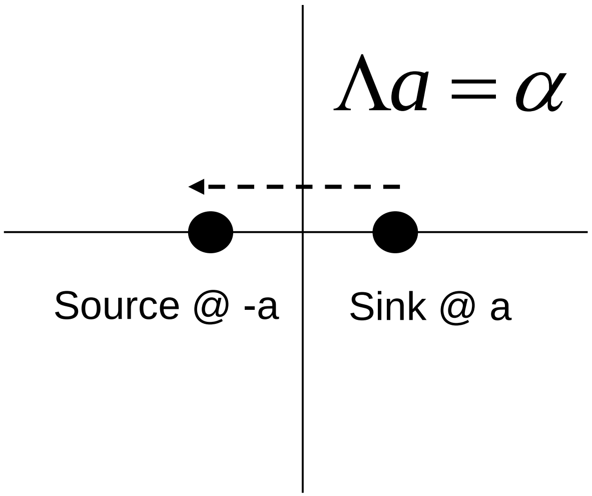
$$W = \frac{\alpha}{2\pi a}\left(\ln(Z) + \ln(1+\frac{a}{Z}) - \ln(Z) - \ln(1-\frac{a}{Z})\right)$$

$$W = \frac{\alpha}{2\pi a}\left(\ln(1+\frac{a}{Z}) - \ln(1-\frac{a}{Z})\right)$$

$$\ln(1+Z) = Z - \frac{Z^2}{2} + \ldots$$

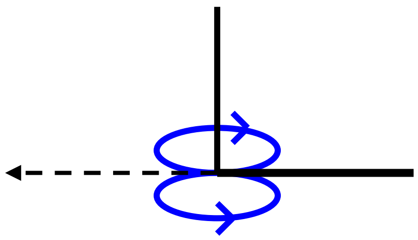
$$W = \frac{\alpha}{2\pi a}\left(\frac{a}{Z} - \frac{a^2}{2Z^2} - \left(-\frac{a}{Z} - \frac{a^2}{2Z^2}\right)\right)$$

$$W = \frac{\alpha}{2\pi a}\left(\frac{2a}{Z}\right) = \frac{\alpha}{\pi Z}$$



Doublet

- Doublet (with axis in –x direction):



$$W(Z) = \frac{\kappa}{2\pi Z} \quad \Rightarrow \phi = \frac{\kappa x}{2\pi r^2}, \quad \varphi = -\frac{\kappa y}{2\pi r^2}$$

- more generally $W(Z) = -\frac{\kappa e^{i\alpha_d}}{2\pi Z}$
- where α_d is the angle of the doublet from the x-axis (in the above = π, and hence:

$$e^{i\alpha_d} = e^{i\pi} = \cos\pi - i\sin\pi = -1$$

Composite Solutions:

- We now have a `tool kit' of solutions, same as last year, but a different way of expressing them
- As they are linear solutions of Laplace, can sum them to produce solutions to more complex flows:

Flow about a Circle at Zero Incidence:

- Flow is produced by the sum of *Uniform onset flow* at zero incidence, and a *Doublet* pointing into the flow (α_d = π, as on the previous slide) then:

$$W(Z) = UZe^{-i\alpha} - \frac{\kappa e^{i(\pi+\alpha)}}{2\pi Z} = \phi + i\varphi$$

$$W(Z) = UZe^{0i} + \frac{\kappa e^{0i}}{2\pi Z} = \phi + i\varphi$$

$$= U(x+iy) + \frac{\kappa(x-iy)}{2\pi(x^2+y^2)}$$

or

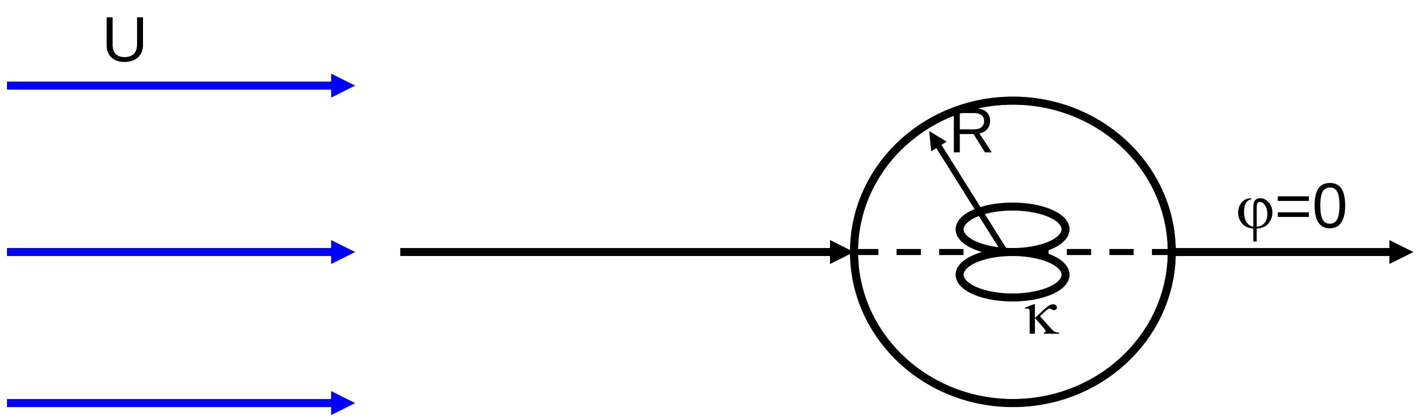
$$W(Z) = \phi + i\varphi = Ux + \frac{\kappa x}{2\pi\left(x^2+y^2\right)} + iU\left(y - \frac{\kappa y}{2\pi r^2 U}\right)$$

Remember that as the stream function, φ, is constant on a streamline, the zero streamline is given by

$$Uy\left(1-\frac{\kappa}{2U\pi r^2}\right)=0$$

i.e. either *y*=0 or $2U\pi r^2=\kappa \Rightarrow r^2=\dfrac{\kappa}{2\pi U}$

Set κ=2UπR^2, then:



If we put this expression for κ back into our expression for W(Z), we get

$$W(Z) = UZ + \frac{\kappa}{2\pi Z} = UZ + \frac{UR^2}{Z}$$

or

$$W(Z) = U\left(Z + \frac{R^2}{Z}\right)$$

This gives the complex potential equation for flow about a circle of radius R directly (no need to calculate κ), and is of fundamental importance

University of BRISTOL

Velocities:

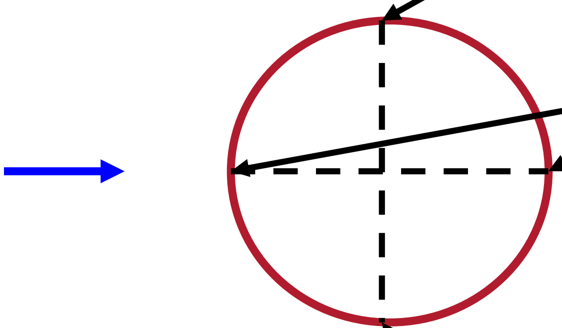
We know that $\dfrac{dW(Z)}{dZ} = u - iv$

i.e

$$u - iv = \frac{d\left(U\left(Z + R^2/Z\right)\right)}{dZ} = U\left(1 - \frac{R^2}{Z^2}\right)$$

$$Z = iR \Rightarrow Z^2 = -R^2, \Rightarrow u = 2U, v = 0$$

$$Z = \pm R \Rightarrow Z^2 = R^2 \Rightarrow u = 0, v = 0$$



$$Z = -iR \Rightarrow Z^2 = -R^2, \Rightarrow u = 2U, v = 0$$

Lifting Flows:

Including a vortex at the centre of the circle allows lift to be produced:

Complex potential is now

$$W(Z) = U\left(Z + \frac{R^2}{Z}\right) + \frac{-\Gamma}{2\pi i}\ln(Z)$$

Velocities in a Lifting Flow:

Again, differentiate w.r.t. Z:

$$\frac{dW(Z)}{dZ} = U\left(1 - \frac{R^2}{Z^2}\right) - \frac{\Gamma}{2\pi i Z} = u - iv$$

Stagnation points still occur when u and v are both = 0, but this is now when

$$U\left(1 - \frac{R^2}{Z^2}\right) - \frac{\Gamma}{2\pi i Z} = 0 = Z^2 - \frac{\Gamma}{2U\pi i} Z - R^2$$

i.e. a *Quadratic in Z*. This means that if Γ is not zero, the stagnation points have an *i* component, i.e. are no longer on centreline

Lift curve slope:

- Can use the Kutta-Joukowski Theorem to calculate the lift curve slope in an inviscid, irrotational, incompressible flow:

$$\frac{dC_L}{d\alpha} = 2\pi$$

- Next week, we shall show how to transform the flow about a circle to one about an aerofoil
- This is in your handout, but paraphrasing…

Lifting circle

$$\frac{dW(Z)}{dZ} = U\left(e^{-i\alpha} - e^{i\alpha}\frac{R^2}{Z^2}\right) - \frac{\Gamma}{2\pi i Z} \xrightarrow{\text{Polar}} \frac{dW(Z)}{dZ} = U\left(e^{-i\alpha} - e^{i\alpha}\frac{R^2}{r^2 e^{2i\theta}}\right) - \frac{\Gamma}{2\pi i r e^{i\theta}}$$

$$\frac{dW(Z)}{dZ} = U\Big(\cos\alpha - i\sin\alpha - \frac{R^2}{r^2}(\cos\alpha + i\sin\alpha)(\cos 2\theta - i\sin 2\theta)\Big) + \frac{\Gamma i}{2\pi r}(\cos\theta - i\sin\theta)$$

$$\frac{dW(Z)}{dZ} = U\cos\alpha - U\frac{R^2}{r^2}\cos 2\theta\cos\alpha - U\frac{R^2}{r^2}\sin 2\theta\sin\alpha + \frac{\Gamma}{2\pi r}\sin\theta$$

$$+i\Big(-U\sin\alpha + U\frac{R^2}{r^2}\sin 2\theta\cos\alpha - U\frac{R^2}{r^2}\cos 2\theta\sin\alpha + \frac{\Gamma}{2\pi r}\cos\theta\Big)$$

$$-U\sin\alpha + U\frac{R^2}{R^2}\sin 2\theta\cos\alpha - U\frac{R^2}{R^2}\cos 2\theta\sin\alpha + \frac{\Gamma}{2\pi R}\cos\theta = 0$$

But for trailing edge theta=0

$$2U\sin\alpha = \frac{\Gamma}{2\pi R}$$

$$\Gamma = U\sin\alpha 4\pi R$$

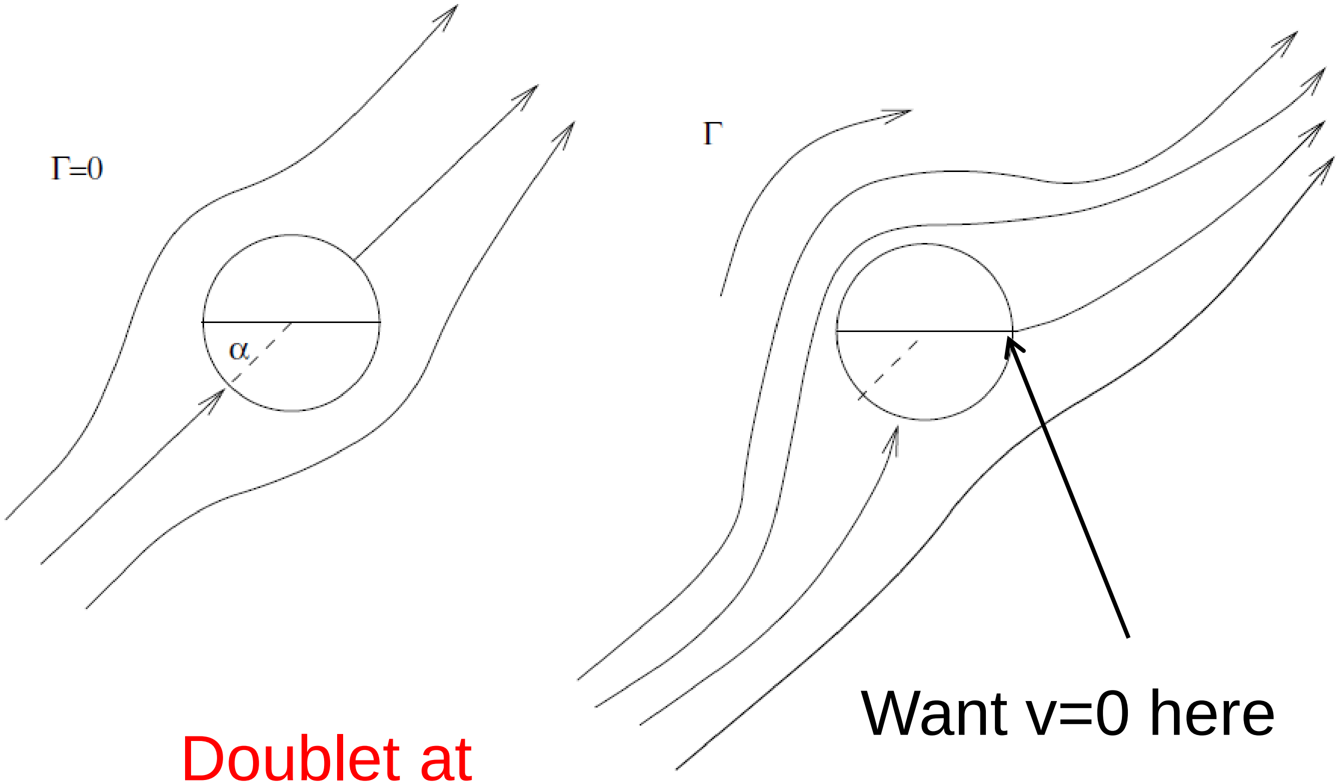
$$C_L = \frac{Lift}{\frac{1}{2}\rho U^2 S} = \frac{\rho U(U\alpha 4\pi R)}{\frac{1}{2}\rho U^2 S}$$

$$C_L = \frac{4\rho U^2 R\alpha\pi}{\frac{1}{2}4\rho U^2 R}$$

$$= 2\pi\alpha$$

University of BRISTOL

Why this value for the vortex?



Want v=0 here

(Kutta-Joukowski)

Doublet at incidence

$$W(Z) = U\left(e^{-i\alpha}Z + \frac{e^{i\alpha}R^2}{Z}\right) - \frac{\Gamma}{2\pi i}\ln(Z)$$

Vortex

University of BRISTOL

Aerodynamics 3

Conformal Mapping

(chapter 8 in notes)

Revision 1

- Last week, we demonstrated how simple solutions of the Laplace equation could be represented as Complex Potential Functions:

Uniform Onset Flow, $$W(Z) = Ue^{-i\alpha} Z$$

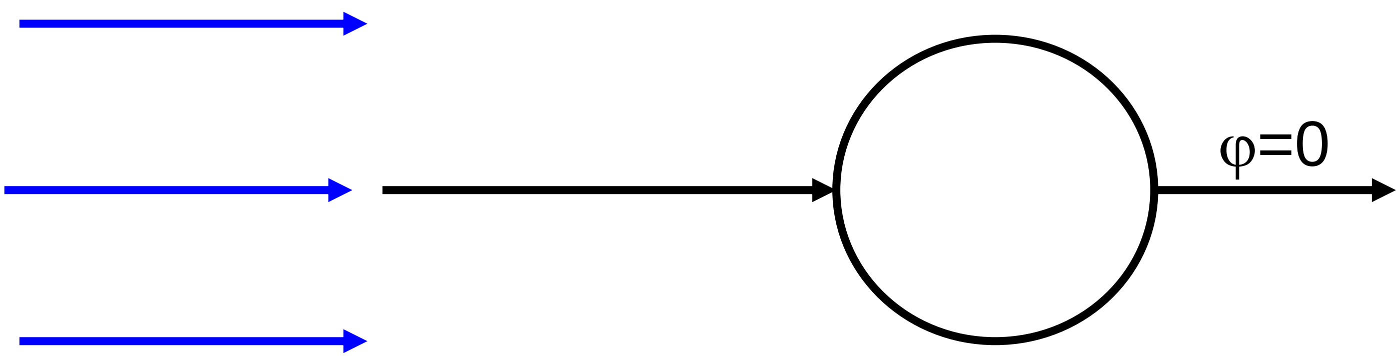
Source, $$W(Z) = \frac{\Lambda}{2\pi} \ln(Z):$$

Point Vortex, $$W(Z) = \frac{-\Gamma}{2\pi i} \ln(Z)$$

Doublet, $$W(Z) = -\frac{\kappa e^{i\alpha_d}}{2\pi Z}$$

Revision 2

By combining uniform onflow and a doublet at the origin, we could describe the potential flow about a circle in the same way as 2nd Year:



To get the flow about a circle radius R, need $\kappa=2U\pi R^2$

If this is the case then:

$$W(Z)=U\left(Z+\frac{R^2}{Z}\right)$$

University of BRISTOL

Revision 3

We can get velocities from:

$$\frac{dW(Z)}{dZ} = u - iv$$

Which results in

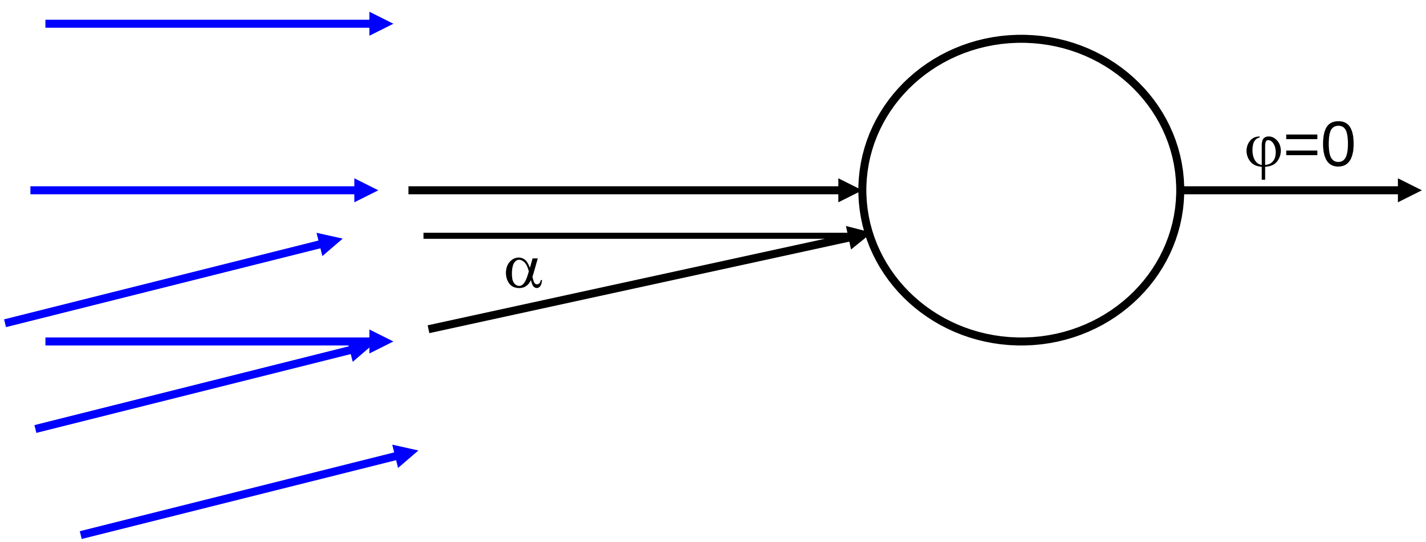
$$u - iv = U\left(1 - \frac{R^2}{Z^2}\right)$$

Including circulation gives:

$$W(Z) = U\left(Z + \frac{R^2}{Z}\right) + \frac{-\Gamma}{2\pi i}\ln(Z)$$

$$\frac{dW(Z)}{dZ} = U\left(1 - \frac{R^2}{Z^2}\right) - \frac{\Gamma}{2\pi i Z} = u - iv$$

Also, it was shown in your handout that if we rotate the flow to some angle α, whilst adding enough circulation to keep the trailing stagnation point at (R,0):

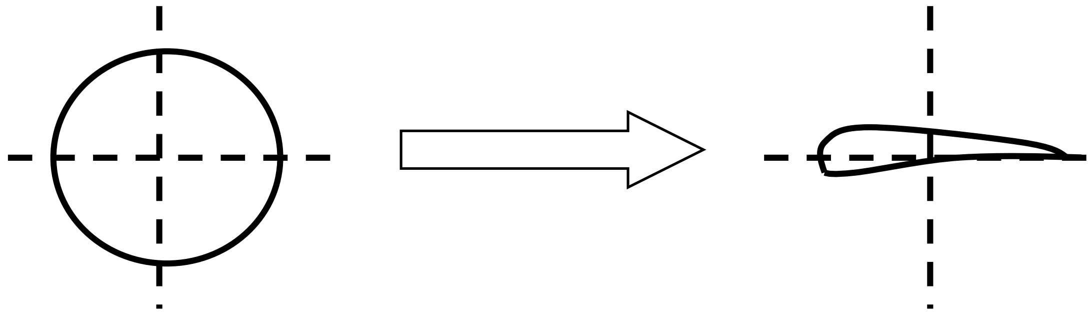


We can show that $$\frac{dC_L}{d\alpha} = 2\pi$$

Conformal Mapping

What are we trying to achieve:

- Convert simple solution about a circle:
- To solution about aerofoil shape in another plane:



Z plane, (*x*,*y*) ζ plane, (ξ,η)

We want some function *f(Z)* to give position in the new plane, or expressed mathematically,

$$\zeta = f(Z)$$

i.e.

$$\zeta(\xi + i\eta) = f(Z(x + iy))$$

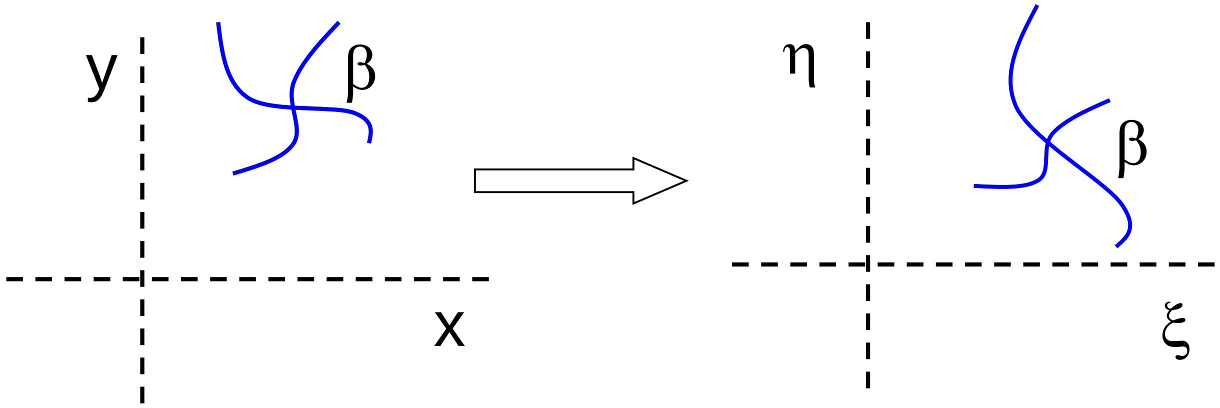
The idea is that we can solve the flow about a circle relatively easily (we have already done this!).

If we can find a simple way to transfer this solution to a more complex shape, the job is done.

University of BRISTOL

First, an important property of the transformation:

- Lines that cross at an angle β in the *Z* plane cross at the same angle β in the ζ plane



Why? Consider the angle of attack – we would like the same angle of attack in both planes.

The Joukowsky Transformation:

The Joukowsky (sometimes Zhukovsky, sometimes Kutta-Joukowsky) transformation is

$$\zeta = Z + \frac{b^2}{Z}$$

Similarity to potential for doublet flow coincidental

where *b* is some constant. Notice that for large values of Z, the mapping leaves uniform flow unchanged.

On the surface of a circle in the Z plane, radius R

$$x = R\cos\theta \qquad y = R\sin\theta$$

University of BRISTOL

In the transformed ζ plane, the surface moves to

$$\zeta = \xi + i\eta = Z + \frac{b^2}{Z} = \mathrm{R}e^{i\theta} + \frac{b^2}{R}e^{-i\theta}$$

$$= R\cos\theta + Ri\sin\theta + \frac{b^2}{R}(\cos\theta - i\sin\theta)$$

And hence

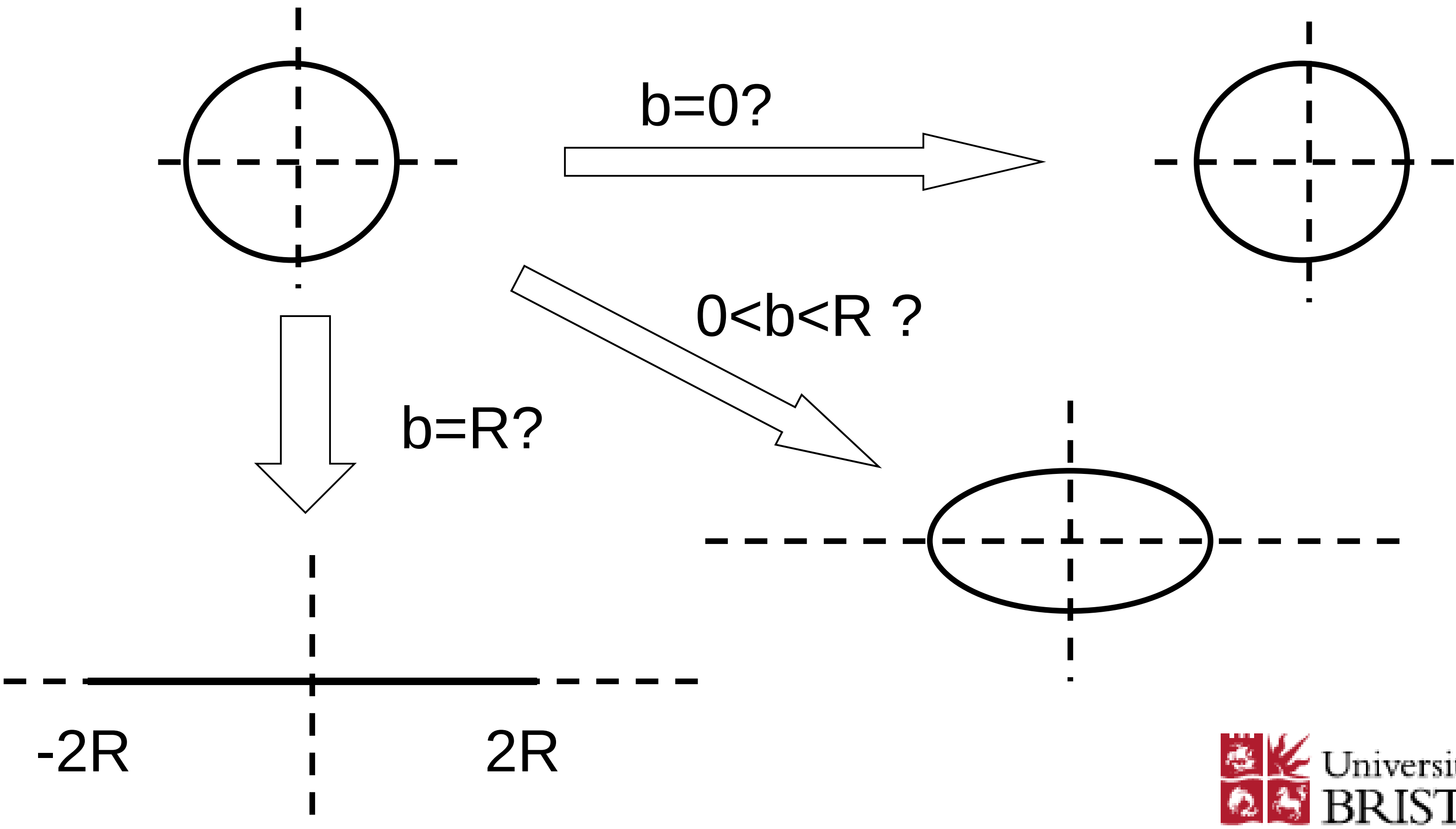
$$\xi = \left(R + \frac{b^2}{R}\right)\cos\theta, \quad \eta = \left(R - \frac{b^2}{R}\right)\sin\theta$$

Which gives the surface in the ζ plane

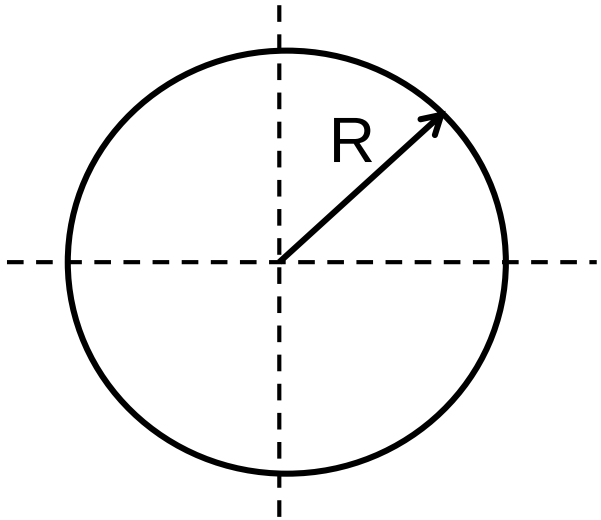
Effect of *b:*

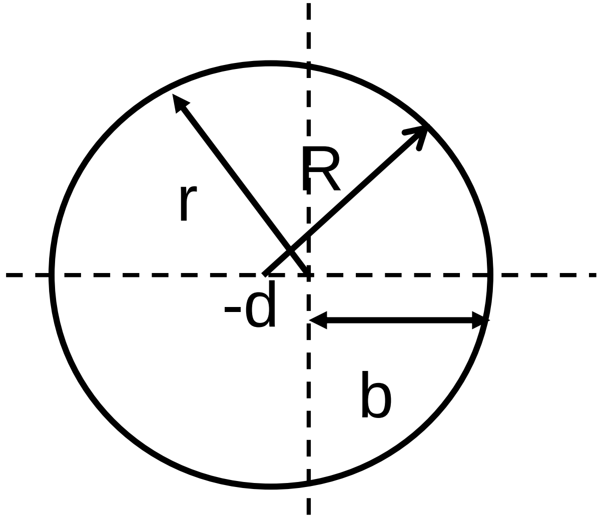
$$\xi = \left(R + \frac{b^2}{R} \right) \cos\theta, \qquad \eta = \left(R - \frac{b^2}{R} \right) \sin\theta$$

Zplane:



What happens if we move the circle?



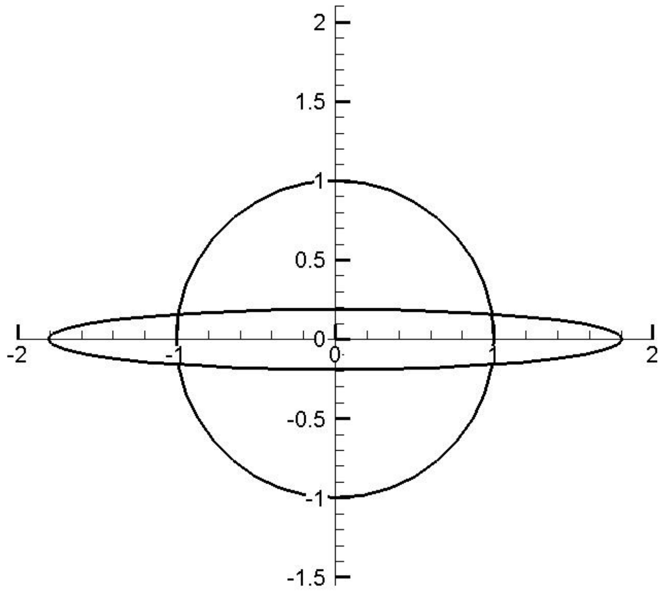


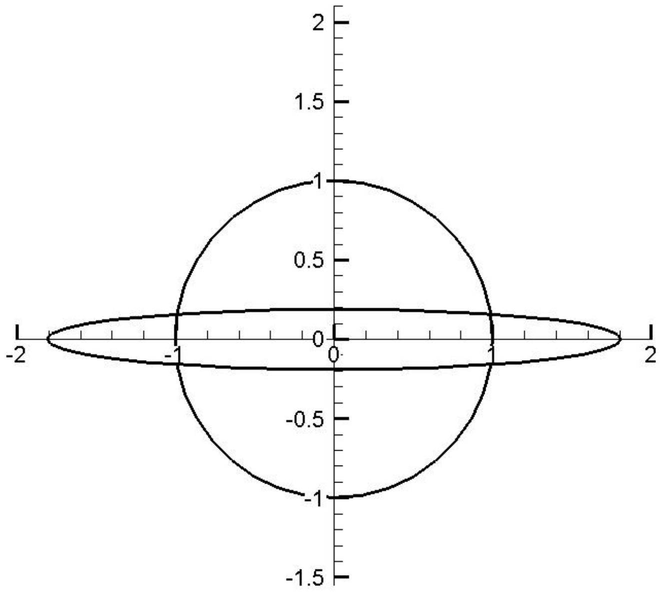
Where

- +ve d moves circle left
- d << b
- R = b+d

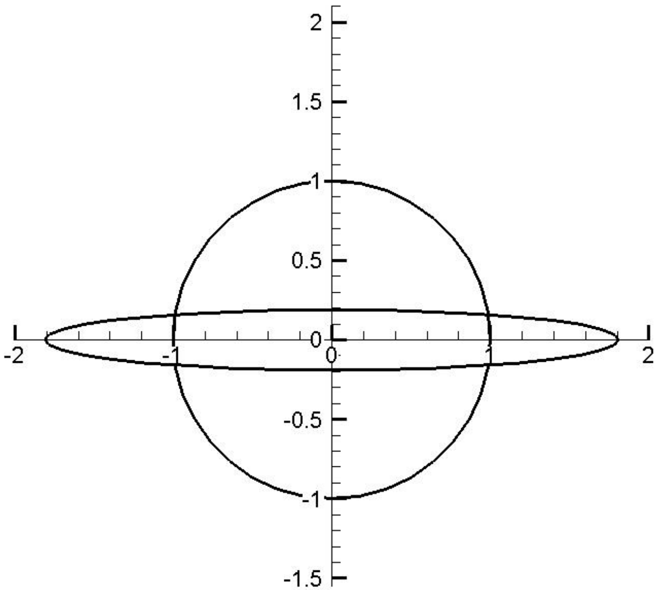
University of BRISTOL

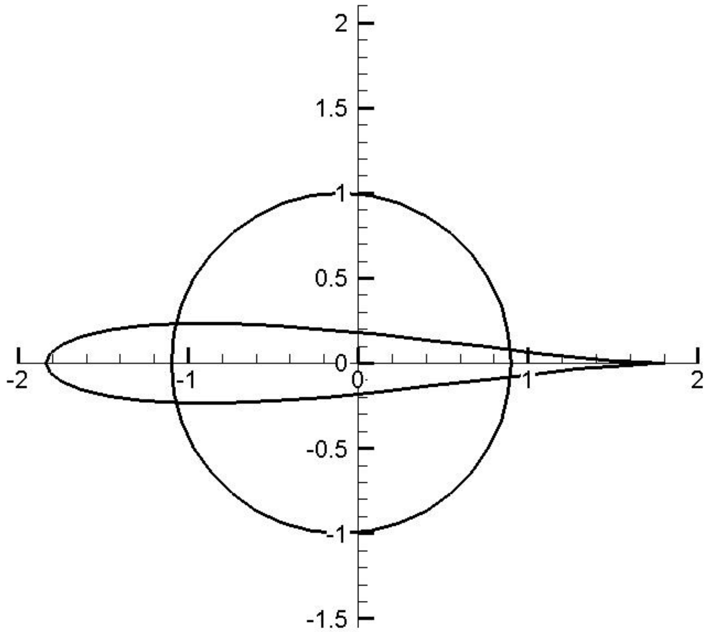
Base case





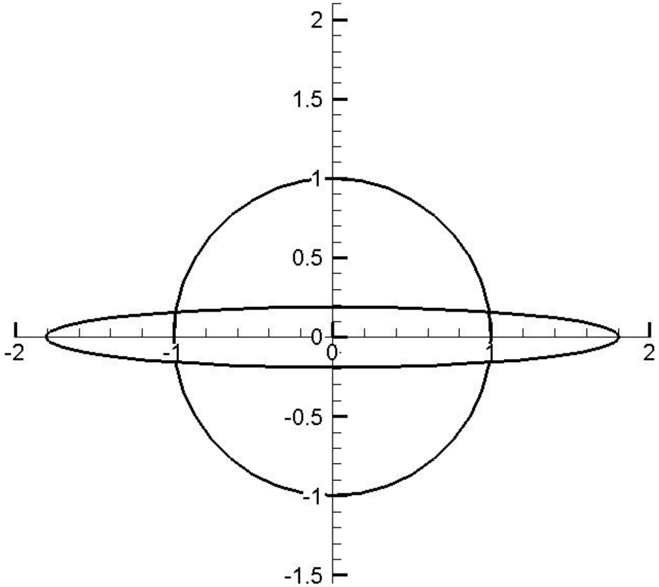
Move left

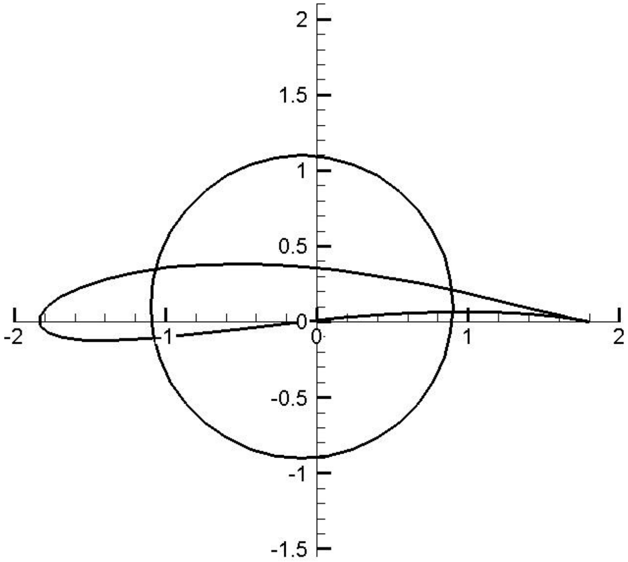




(today)

Move left and up





(next time)

Task

We would prefer a solution in terms of angle and radius

Need to examine the algebra…

$$\zeta = \xi + i\eta = Z + \frac{b^2}{Z} = re^{i\theta} + \frac{b^2}{r}e^{-i\theta}$$

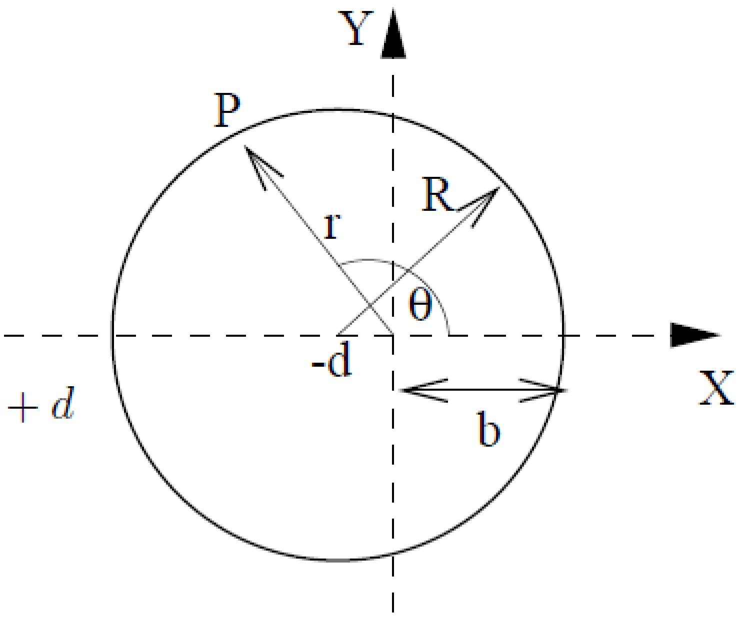
$$= r(\cos\theta + i\sin\theta) + \frac{b^2}{r}(\cos\theta - i\sin\theta)$$

$$= b\left(\frac{r}{b} + \frac{b}{r}\right)\cos\theta + ib\left(\frac{r}{b} - \frac{b}{r}\right)\sin\theta$$

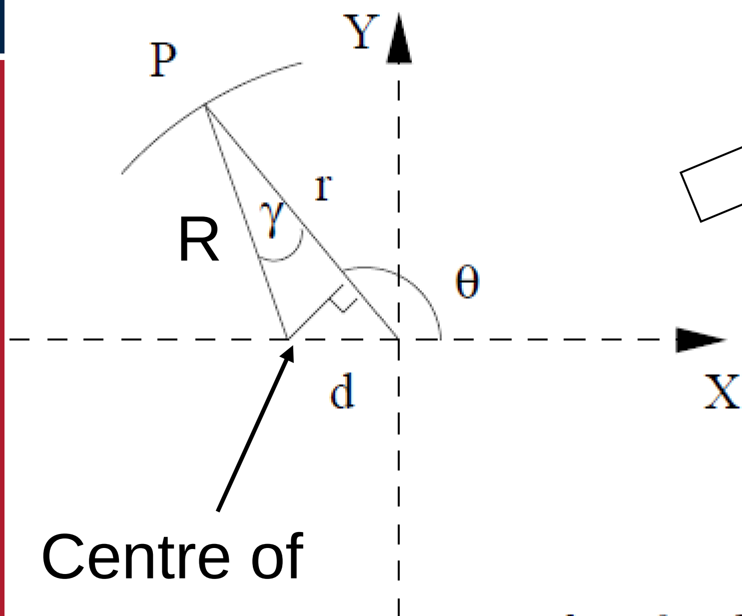
r/b=?

b/r=?

$R = b + d$



The surface we want to transform is no longer $Z=Re^{i\theta}$, but $Z=re^{i\theta}$, where r is dependent on θ:



Centre of Circle

$$r = -d\cos\theta + R\cos\gamma$$

But d<<R, hence γ is small, so $\cos\gamma$ is approximately 1. setting e=d/b,

$$r = -d\cos\theta + R = -d\cos\theta + b + d \longrightarrow \frac{r}{b} = 1 + e - e\cos\theta$$

$$R = b + d$$

We can also show through binomial expansion that:

$$\frac{b}{r} = 1 - e + e\cos\theta$$

University of BRISTOL

Now,

$$\zeta = \xi + i\eta = Z + \frac{b^2}{Z} = re^{i\theta} + \frac{b^2}{r}e^{-i\theta}$$

Which, after some manipulation and gathering of real and imaginary terms gives us

$$\frac{r}{b} = 1 + e - e\cos\theta$$

$$b\left(\frac{r}{b} + \frac{b}{r}\right)\cos\theta + ib\left(\frac{r}{b} - \frac{b}{r}\right)\sin\theta$$

$$\frac{b}{r} = 1 - e + e\cos\theta$$

Substituting in the previously derived expressions for the terms in the brackets:

$$\xi = 2b\cos\theta \qquad \eta = 2be(1 - \cos\theta)\sin\theta$$

University of BRISTOL

So this gives us the shape that the circle turns into in the ζ plane. The question is, what does it look like?

$$\xi = 2b\cos\theta:$$

As θ varies from 0 to 2π, $\cos\theta$ varies from 1 to -1 back to 1 again. Hence shape extends from -2b to 2b on the x axis.

$$\eta = 2be\left(1 - \cos\theta\right)\sin\theta:$$

as $(1-\cos\theta)$ +ve, and $\sin(\theta+\pi)=-\sin(\theta)$, $\eta(\theta+\pi)=-\eta(\theta)$, and hence shape is symmetric about ξ axis

It can be shown that (see handout for details):

$$\frac{d\eta}{d\theta} = (2\cos\theta + 1)(-\cos\theta + 1)$$

When this is 0, surface gradient is zero, i.e. when

$\theta = 2\pi/3$, or $\theta = 0$:

Infinitely sharp t.e.

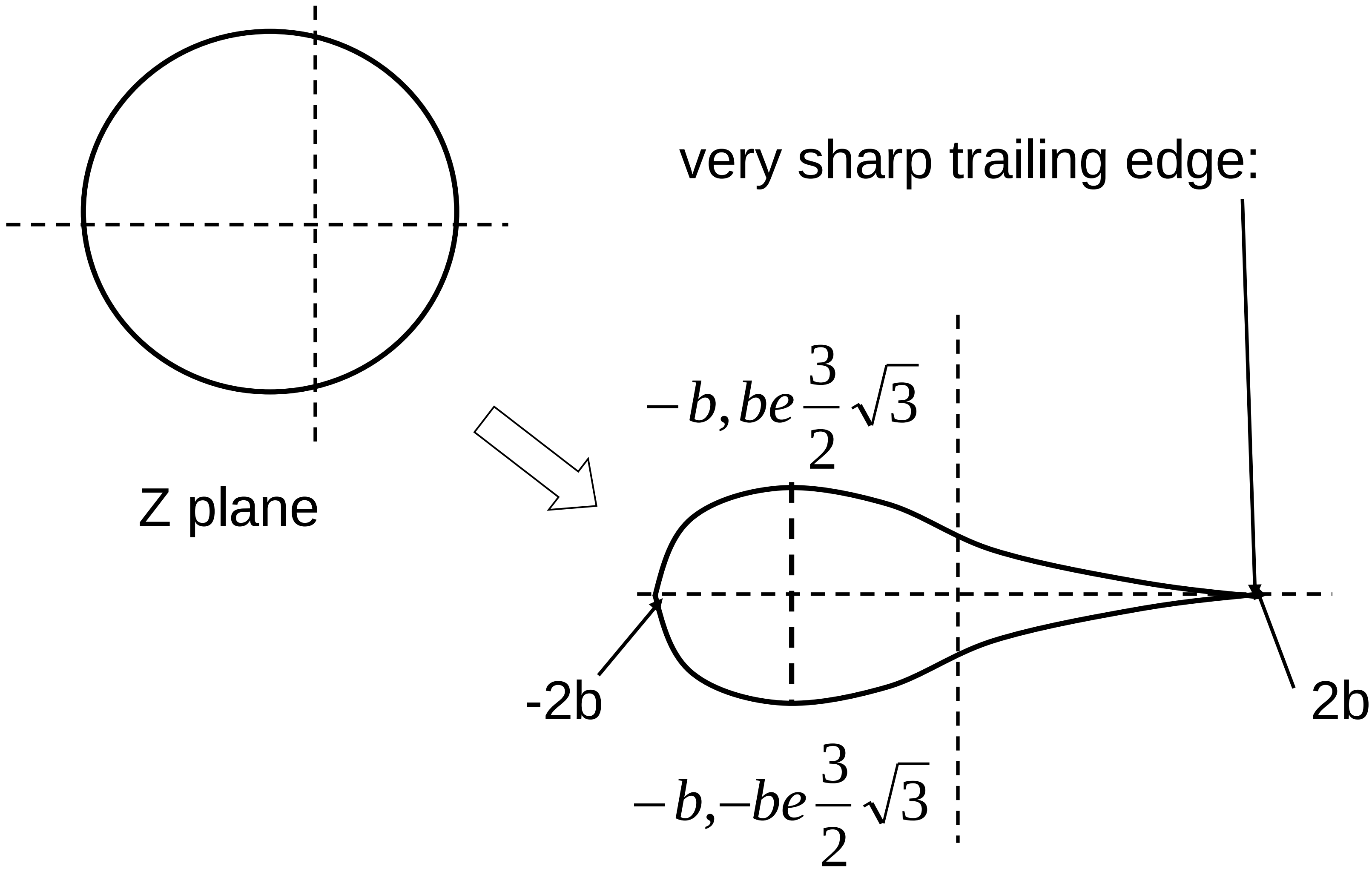
Gives point of maximum thickness. We can also show that:

$$\xi_{\max\eta} = -b$$

$$\eta_{\max} = be\frac{3}{2}\sqrt{3}$$

$$\frac{t}{c} = \frac{2be\frac{3}{2}\sqrt{3}}{4b} \approx 1.3e$$

i.e.


Z plane
very sharp trailing edge:
$-b, be\frac{3}{2}\sqrt{3}$
-2b
2b
$-b, -be\frac{3}{2}\sqrt{3}$
ζ plane

Other transformations possible

$$\zeta = Z + \frac{b^2}{Z}$$

$$\left(\frac{Z+2b}{Z-2b}\right) = \left(\frac{\zeta+b}{\zeta-b}\right)^n$$

Karman-Trefftz

$\downarrow$

Relieves problems with a sharp trailing edge

Aerodynamics 3

Conformal Mapping 2 – Cambered Aerofoils and Velocity Transformations

(chapter 9 in notes)

University of BRISTOL

Conformal Mapping - Overview

Last week, we introduced the Joukowsky Transformation:

$$\zeta = Z + \frac{b^2}{Z}$$

Conformal Mapping - Overview

Last week, we introduced the Joukowsky Transformation:

$$\zeta = Z + \frac{b^2}{Z}$$

- The idea being to allow a simple flow in one plane to map to an aerofoil shape in another:

University of BRISTOL

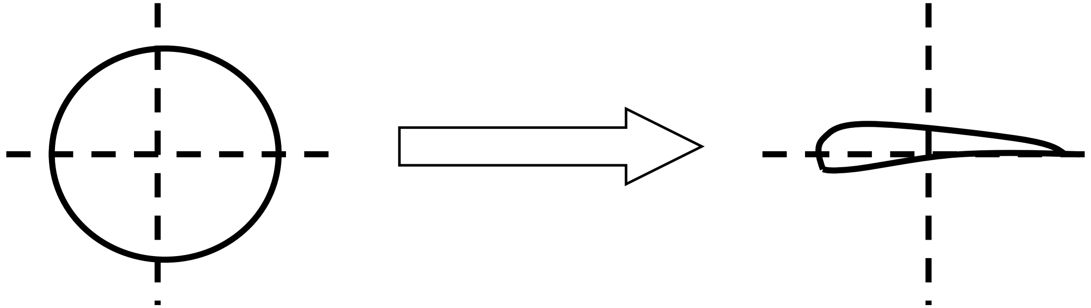
Conformal Mapping - Overview

Last week, we introduced the Joukowsky

Transformation:

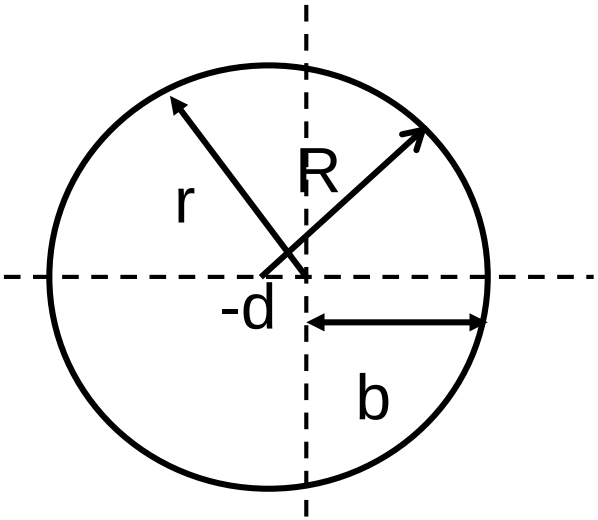
$$\zeta = Z + \frac{b^2}{Z}$$

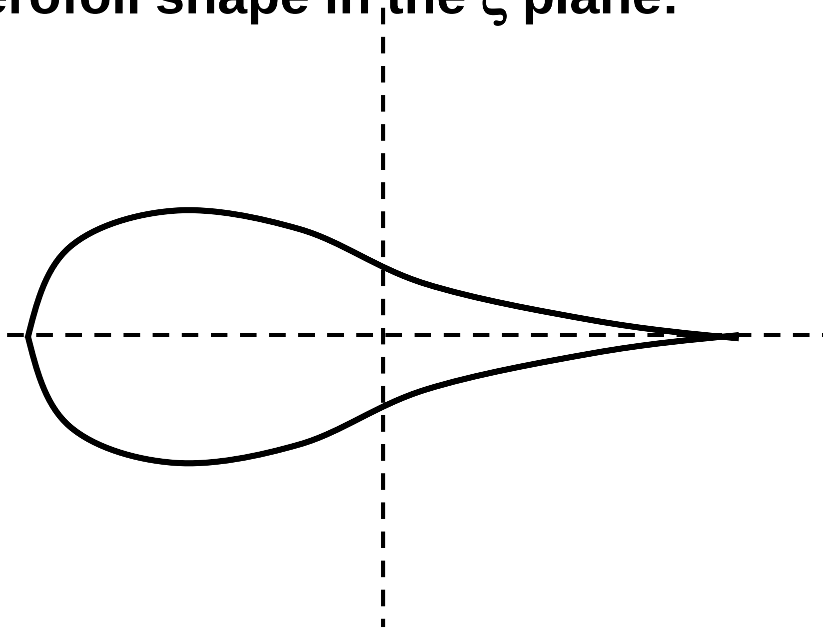
- The idea being to allow a simple flow in one plane to map to an aerofoil shape in another:



Z plane, (x,y) ζ plane, (ξ,η)

We showed that if we move the circle to the left in the Z plane, we get a symmetrical aerofoil shape in the ζ plane:





where the chord length is 4b, and

$$\xi_{\max\,\eta} = -b$$

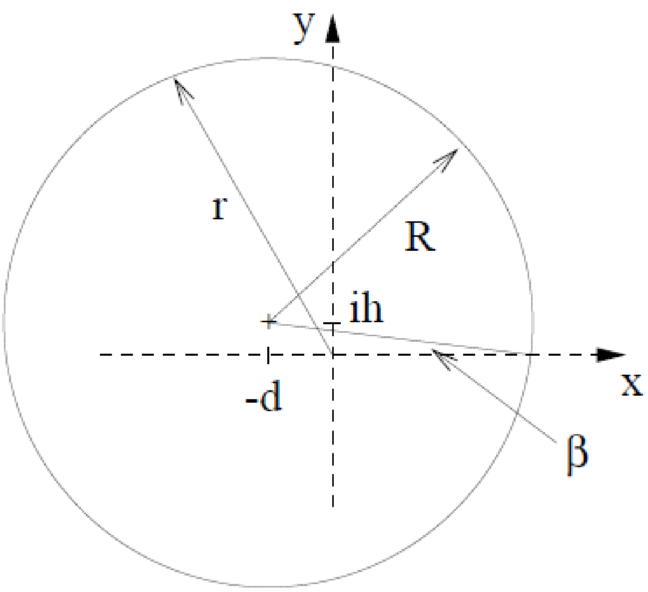
$$\eta_{\max} = be\frac{3}{2}\sqrt{3}$$

$$\frac{t}{c} = \frac{2be\frac{3}{2}\sqrt{3}}{4b} \approx 1.3e$$

University of BRISTOL

Adding Camber:

If we then take the circle and move it a distance h, we have:



where

$$h = R\sin\beta = (b+d)\sin\beta$$

But d and β are <<R, so

$$h \approx b\beta$$

Surface Defined by $Z = re^{i\theta}$:

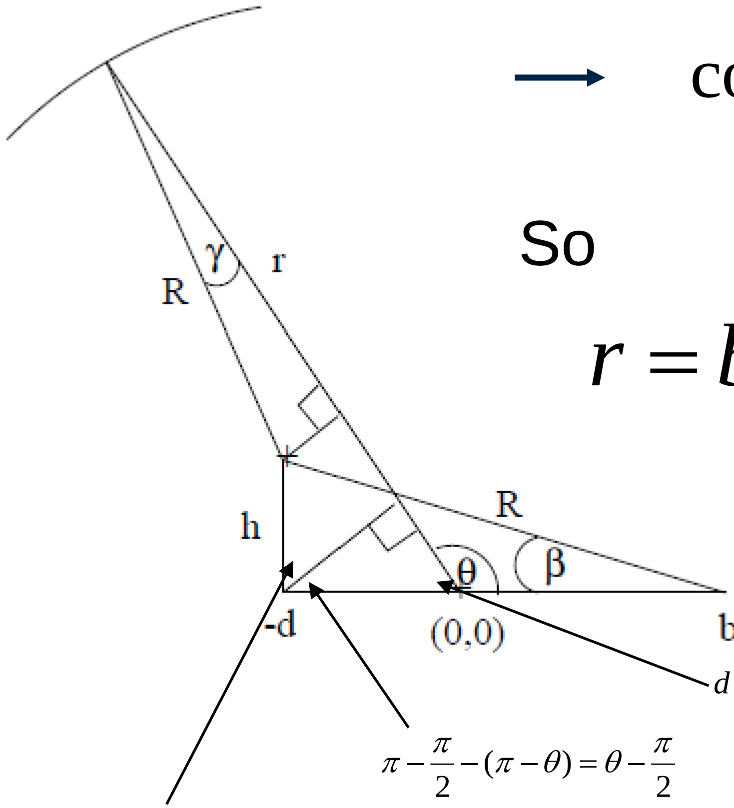
θ varies from 0 to 2π. What about r?

$\longrightarrow \quad r = R\cos\gamma + h\sin\theta - d\cos\theta$

$\longrightarrow \quad \cos\gamma \to 1 \qquad R = b + d$

So

$$r = b + d + b\beta\sin\theta - d\cos\theta$$



$d\cos(\pi-\theta) = d\cos(\pi)\cos(\theta) + d\sin(\pi)\sin(\theta) = -d\cos(\theta)$

$\pi - \frac{\pi}{2} - (\pi - \theta) = \theta - \frac{\pi}{2}$

$h\sin(\pi-\theta) = h\sin(\pi)\cos(\theta) - h\cos(\pi)\sin(\theta) = h\sin(\theta)$

Using the binomial expansion as last week, we get:

$$\frac{r}{b} = 1 + e - e\cos\theta + \beta \sin\,\theta$$

And

$e\ (= \frac{d}{b})$

$$\frac{b}{r} = (1 + (e - e\cos\theta + \beta \sin\,\theta))^{-1}$$

$$\approx 1 + (-1)(e - e\cos\theta + \beta \sin\,\theta)$$

$$\frac{b}{r} = 1 - e + e\cos\theta - \beta \sin\,\theta$$

$$\zeta = \xi + i\eta = Z + \frac{b^2}{Z} = re^{i\theta} + \frac{b^2}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{b^2}{r}(\cos\theta - i\sin\theta)$$

$$= b\left(\frac{r}{b} + \frac{b}{r}\right)\cos\theta + ib\left(\frac{r}{b} - \frac{b}{r}\right)\sin\theta$$

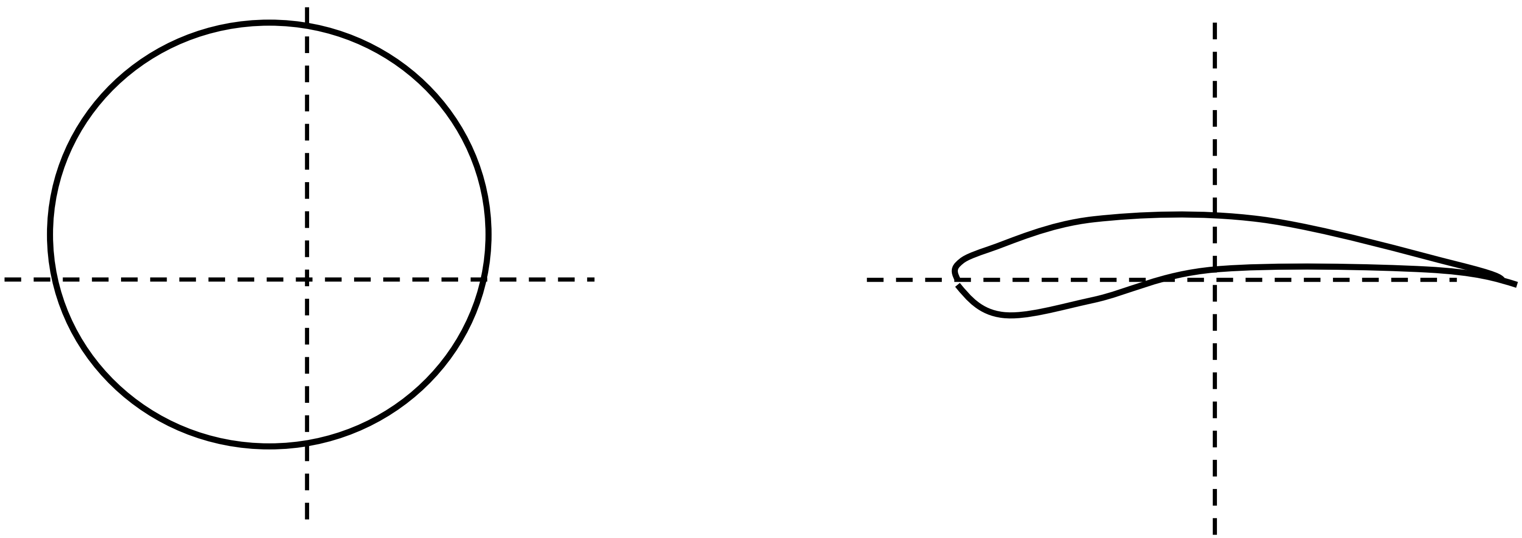
$$\Rightarrow \zeta = \xi + i\eta = 2b\cos\theta + i2b(e - e\cos\theta + \beta\sin\theta)\sin\theta$$

$$\Rightarrow \xi = 2b\cos\theta, \qquad \eta = 2be(1 - \cos\theta)\sin\theta + \underline{2b\beta\sin^2\theta}$$

Comparing to the result last week for zero h,

$$\xi = 2b\cos\theta \qquad \eta = 2be(1-\cos\theta)\sin\theta$$

we see that the only difference is the introduction of the term $2b\beta\sin^2\theta$ to the η coordinate. This is always positive, and hence adds camber, i.e.



Velocity Transformations:

- We now have an aerofoil shape, but to be of any use, we need to know what velocities occur in the flow around it, and particularly on the surface, in order to calculate cp’s, and thence lift, drag, etc.

Velocity Transformations:

- We now have an aerofoil shape, but to be of any use, we need to know what velocities occur in the flow around it, and particularly on the surface, in order to calculate cp's, and thence lift, drag, etc.
- We can do this, because the Joukowsky transformation consists of a stretch and a rotation – we now need to know the stretch

In other words, if we use q to represent velocity vector, then we need to know -> $\left|\frac{q_\zeta}{q_z}\right|$

In other words, if we use q to represent velocity vector, then we need to know ->

$$\left|\frac{q_\zeta}{q_z}\right|$$

Remembering that

$$u - iv = \frac{dW(Z)}{dZ}$$ and also $$\bar{u} - i\bar{v} = \frac{dW(\zeta)}{d\zeta}$$

In other words, if we use q to represent velocity vector, then we need to know -> $\left|\dfrac{q_\zeta}{q_z}\right|$

Remembering that

$$u - iv = \frac{dW(Z)}{dZ} \quad \text{and also} \quad \bar{u} - i\bar{v} = \frac{dW(\zeta)}{d\zeta}$$

we can use the fact that $\dfrac{dW}{dZ} = \dfrac{dW}{d\zeta}\dfrac{d\zeta}{dZ}$

$$\frac{\left|\dfrac{dW(\zeta)}{d\zeta}\right|}{\left|\dfrac{dW(Z)}{dZ}\right|} = \left|\frac{dZ}{d\zeta}\right|$$

As

$$\zeta = Z + \frac{b^2}{Z}, \Rightarrow \frac{d\zeta}{dZ} = 1 - \frac{b^2}{Z^2}$$

$$\frac{d\zeta}{dZ} = 1 - \frac{b^2}{r^2} e^{-i2\theta} = 1 - \frac{b^2}{r^2}(\cos 2\theta - i\sin 2\theta)$$

So $$\left|\frac{d\zeta}{dZ}\right| = \left(\left(1 - \frac{b^2}{r^2}\cos 2\theta\right)^2 + \frac{b^4}{r^4}\sin^2 2\theta\right)^{\frac{1}{2}}$$

$$= \left(1 - \frac{2b^2}{r^2}\cos 2\theta + \frac{b^4}{r^4}\cos^2 2\theta + \frac{b^4}{r^4}\sin^2 2\theta\right)^{\frac{1}{2}}$$

$$\left|\frac{dZ}{d\zeta}\right| = \left(1 - \frac{2b^2}{r^2}\cos 2\theta + \frac{b^4}{r^4}\right)^{-\frac{1}{2}}$$

Returning now to our complex potential expressions for flows about a circle, we had for zero incidence:

$$W(Z) = U\left(Z + \frac{R^2}{Z}\right)$$

allowing for incidence:

$$W(Z) = U\left(e^{-i\alpha} Z - \frac{R^2 e^{i\alpha}}{Z}\right)$$

And including circulation by adding a point vortex:

$$W(Z) = U\left(e^{-i\alpha} Z - \frac{R^2 e^{i\alpha}}{Z}\right) - \frac{\Gamma}{2\pi i}\ln(Z)$$

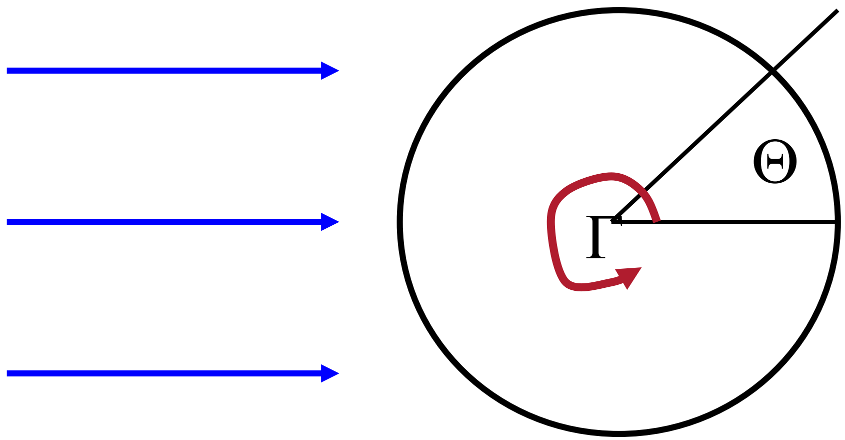
University of BRISTOL

It is important to note that the velocity on the circle is unaffected by moving the circle, so the solution is the same, so long as we use a coordinate system relative to the centre of the circle. This is *not* the same system as the Joukowsky transformation.

Secondly, the equation at the bottom of the last slide is undefined, as Γ is unknown. We need to fix the vorticity, and we do this the same way as last year, by fixing the stagnation points

Last year, it was shown to you that the circumferential velocity about a rotating cylinder is given by

$$V_\theta = -\left(1 + \frac{R^2}{r^2}\right) U \sin\Theta - \frac{\Gamma}{2\pi r}$$



Using complex potential…

$$\frac{dW}{dZ} = U\left(1 - \frac{R^2}{r^2} e^{-2j\theta}\right) - \frac{\Gamma}{2\pi j r} e^{-j\theta}$$

$$\mathbf{a} = (A_{real}, A_{imag})^T$$
$$\mathbf{b} = (B_{real}, B_{imag})^T$$

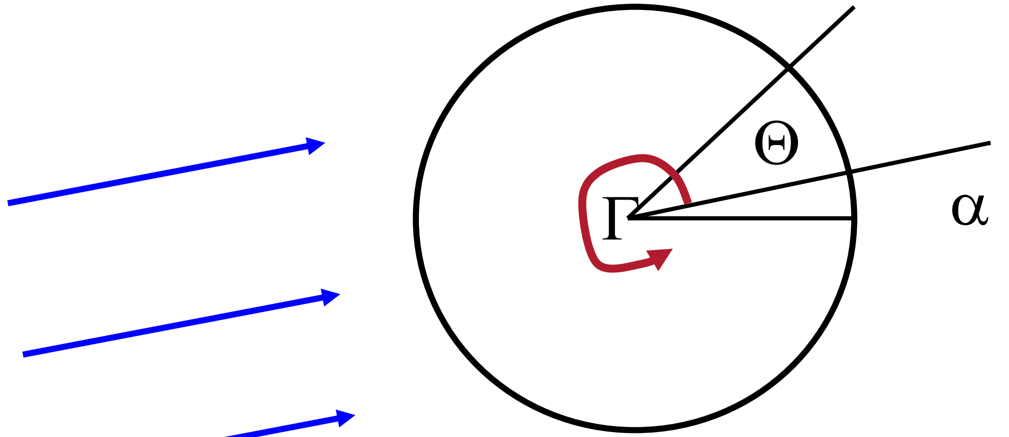
$$\overline{A}B = \mathbf{a} \cdot \mathbf{b} + |\mathbf{a} \times \mathbf{b}|\, j$$

$$\frac{dW}{dZ} j e^{j\theta} = U\left(j e^{j\theta} - j\frac{R^2}{r^2} e^{-j\theta}\right) - \frac{\Gamma}{2\pi r}$$

$$v_{tangential} = U\left(-\sin(\theta) - \frac{R^2}{r^2}\sin(\theta)\right) - \frac{\Gamma}{2\pi r}$$

University of BRISTOL

In our case, we have an incidence α, hence



$$V_\theta = -\left(1+\frac{R^2}{r^2}\right)U\sin\Theta_f - \frac{\Gamma}{2\pi r}$$

$\Theta_f = \Theta - \alpha$ is the angle relative to the freestream direction

And hence the velocity on the circle is

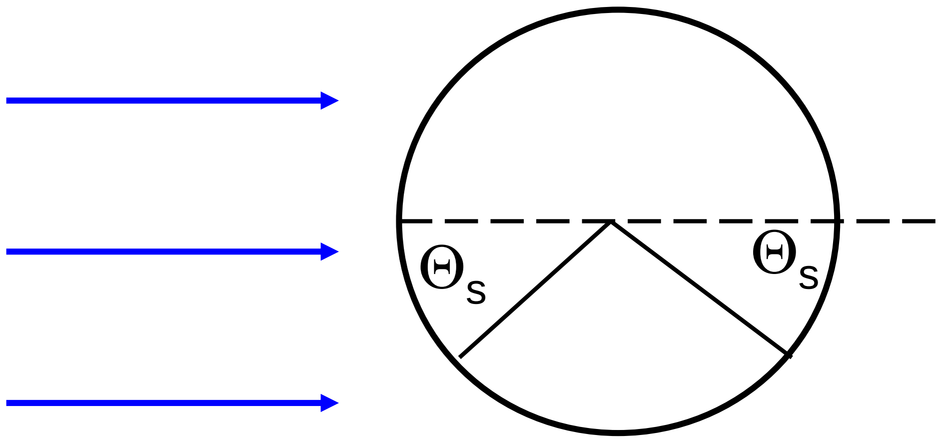
$$q_{circle} = -2U\sin\Theta_f - \frac{\Gamma}{2\pi R}$$

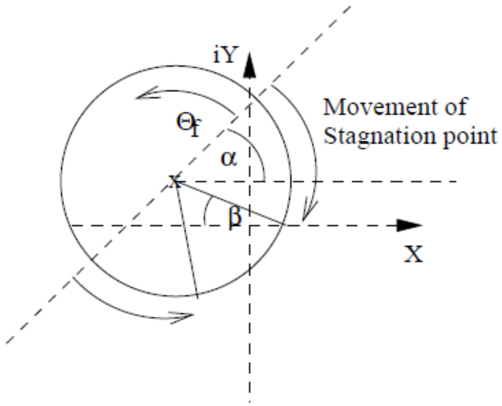
University of BRISTOL

However, the vorticity still needs to be fixed. Last year it was shown that

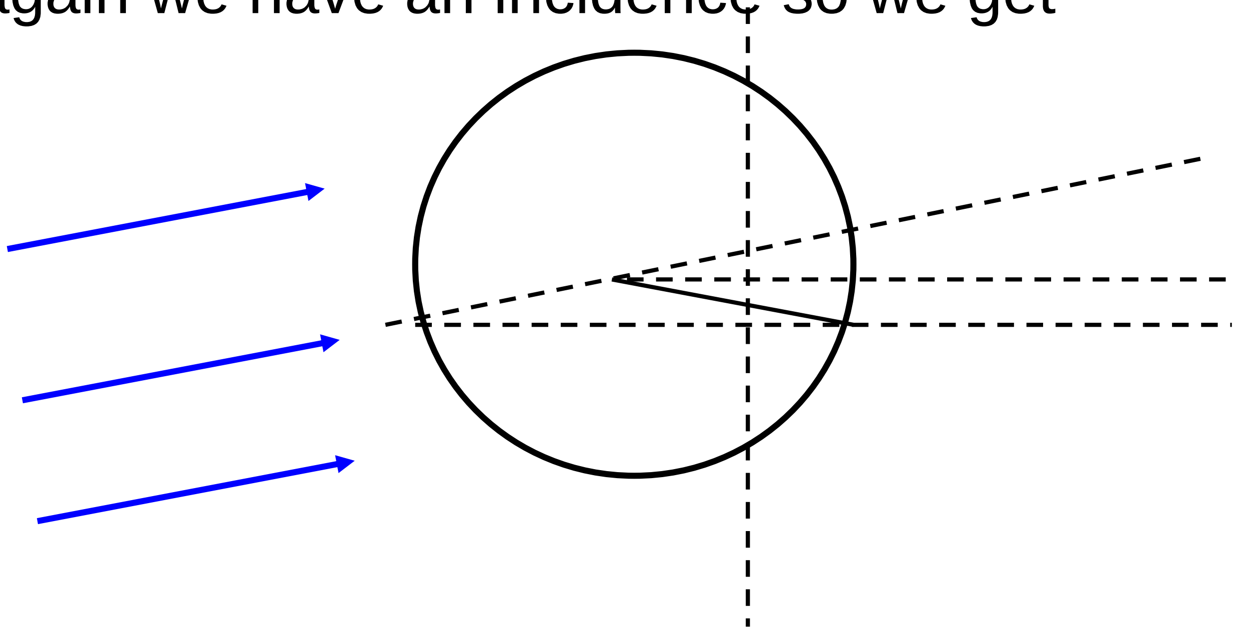
$$\Gamma = 4\pi RU \sin\Theta_s$$

where





But again we have an incidence so we get



$$\Gamma = 4\pi RU \sin(\alpha + \beta)$$

finally, we have an expression for velocity:

$$q_{circle} = -2U(\sin \Theta_f + k \sin(\alpha + \beta))$$

So, now we have an expression for the velocity in the Z plane, but want to know the velocity in the ζ plane. This is achieved by using the transformation derived earlier, i.e.

$$q_{aerofoil} = q_{circle}\left|\frac{dZ}{d\zeta}\right|$$

and from this the pressures

$$C_p = 1 - \left(\frac{q_{aerofoil}}{U}\right)^2$$

This is normally done in Cartesian coordinates, and is covered in detail in your handout.

Finally…

$$\boxed{q_{aerofoil} = q_{circle}\left|\frac{dZ}{d\zeta}\right|}$$

$$\boxed{q_{circle} = -2U\left(\sin\Theta_f + k\sin(\alpha+\beta)\right)}$$

$$\zeta = Z + \frac{b^2}{Z}$$

$$\xi + i\eta = x + iy + \frac{b^2}{x+iy} = x + iy + \frac{b^2(x-iy)}{x^2+y^2}$$

$$\Rightarrow \xi = x\left(1 + \frac{b^2}{x^2+y^2}\right), \qquad \eta = y\left(1 - \frac{b^2}{x^2+y^2}\right)$$

$$\frac{d\zeta}{dZ} = 1 - \frac{b^2}{Z^2} = 1 - \frac{b^2}{x^2-y^2+2ixy}$$

$$= 1 - \frac{b^2(x^2-y^2-2ixy)}{(x^2-y^2+2ixy)(x^2-y^2-2ixy)}$$

$$= 1 - \frac{b^2(x^2-y^2)}{(x^2+y^2)^2} + i\frac{2b^2xy}{(x^2+y^2)^2} = S_r + iS_i$$

$$\left|\frac{d\zeta}{dZ}\right| = \sqrt{S_R^2 + S_i^2}$$

$$\frac{q_{aerofoil}}{U} = \frac{-2(\sin\Theta_f + k\sin(\alpha+\beta))}{\sqrt{S_R^2+S_i^2}}$$

$$\frac{q_{aerofoil}}{U} = \frac{-2(\sin(\Theta-\alpha) + k\sin(\alpha+\beta))}{\sqrt{S_R^2+S_i^2}}$$

$$C_P = 1 - \left(\frac{q_{aerofoil}}{U}\right)^2$$

University of BRISTOL

$$\frac{\partial^2 W(Z)}{\partial x^2} + \frac{\partial^2 W(Z)}{\partial y^2} = \frac{d^2 W}{dZ^2} - \frac{d^2 W}{dZ^2} = 0$$

Laplace is neatly solved in complex plane

$$W(Z) = Ue^{-i\alpha} Z$$

Uniform flow

$$W(Z) = \phi + i\varphi = \frac{\kappa}{2\pi Z}$$

Doublet

$$W(Z) = \frac{-\Gamma}{2\pi i} \ln(Z)$$

Vortex

$$W(Z) = U\left(e^{-i\alpha} Z + \frac{e^{i\alpha} R^2}{Z}\right) - \frac{\Gamma}{2\pi i} \ln(Z) \leftarrow \Gamma = 4\pi RU \sin \Theta_s$$

$$\frac{\left|\frac{dW}{d\zeta}\right|}{\left|\frac{dW}{dZ}\right|} = \left|\frac{dZ}{d\zeta}\right| \qquad q_{aerofoil} = q_{circle}\left|\frac{dZ}{d\zeta}\right| \qquad \frac{d\zeta}{dZ} = 1 - \frac{b^2}{Z^2}$$

$$\frac{q_{aerofoil}}{U} = \frac{-2(\sin(\Theta - \alpha) + k \sin(\alpha + \beta))}{\sqrt{S_R^2 + S_i^2}}$$

Exam Q

(a) If $W = W(z) = \phi + i\psi$ where $Z = x + iy$ show that W is a solution of Laplace's equation. What limitation is imposed on the function W for this to be true?

[3 marks]

(b) Show that $\frac{dW}{dz} = u - iv$.

[2 marks]

(c) The Joukowsky transformation is

$$\zeta = Z + \frac{b^2}{Z} = \xi + i\eta$$

Calculate the transformation of a circle $Z = Re^{i\theta}$ under this mapping (i.e. find ξ and η as functions of θ, b and R). What happens if $b = R$, and what happens if $b = 0$? How would the circle be modified to generate a shape like a symmetric aerofoil, and how would the circle be modified to generate a shape like a cambered aerofoil?

[4 marks]

(d) Using $2\pi r v_r = \Lambda$ for a source, $2\pi r v_\theta = \Gamma$ for a vortex, together with $v_r = \frac{\Lambda}{2\pi r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}$ and $v_\theta = \frac{\Gamma}{2\pi r} = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$ find the complex potential of a source and a vortex in terms of Z. By what factor do the two potentials differ, and what does this mean for the velocity fields they produce?

[8 marks]

(e) What are the drawbacks to using the Joukowsky transformation compared to a numerical approach such as CFD or a panel method? What improvements can be made to the transformation?

[3 marks]

(a)

$$\frac{dw}{dz} = \frac{\partial w}{\partial z}$$
$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz}$$
$$\frac{\partial^2 w}{\partial x^2} = \frac{d^2 w}{dz^2}\frac{\partial z}{\partial x} = \frac{d^2 w}{dz^2}$$
$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = \frac{dw}{dz}j$$
$$\frac{\partial^2 w}{\partial y^2} = \frac{d^2 w}{dz^2}j\frac{\partial z}{\partial y} = \frac{d^2 w}{dz^2}jj = -\frac{d^2 w}{dz^2}$$

(b)

$$w = \phi + j\psi$$

Velocity is

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial z} + j\frac{\partial \psi}{\partial x}\frac{\partial x}{\partial z} = u - jv$$

Alternatively

$$\frac{dw}{dz} = \phi + j\psi = \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial z} + j\frac{\partial \psi}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial \phi}{\partial y}\frac{1}{j} + j\frac{\partial \psi}{\partial y}\frac{1}{j} = u + \frac{\partial \phi}{\partial y}\frac{j}{jj} = u - jv$$

Since Cauchy-Riemann holds, both these are correct and equal.

(c)

$$\zeta = Re^{j\theta} + \frac{b^2}{R}e^{-j\theta} = R(\cos(\theta) + j\sin(\theta)) + \frac{b^2}{R}(\cos(-\theta) + j\sin(-\theta))$$
$$\zeta = Re^{j\theta} + \frac{b^2}{R}e^{-j\theta} = R(\cos(\theta) + j\sin(\theta)) + \frac{b^2}{R}(\cos(\theta) - j\sin(\theta))$$
$$\zeta = R\cos(\theta) + \frac{b^2}{R}\cos(\theta) + j\left(R\sin(\theta) - \frac{b^2}{R}\sin(\theta)\right)$$

University of BRISTOL

(d)

$$2\pi r v_r = \Lambda$$

$$v_r = \frac{\Lambda}{2\pi r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}$$

$v_\theta = 0$ implies no integration function $f(r)$, just a constant of integration, which may be set to zero.

$$\phi = \frac{\Lambda}{2\pi}\ln(r)$$

$$\psi = \frac{\Lambda\theta}{2\pi}$$

Notice that

$$\ln(Z) = \ln(re^{j\theta}) = \ln(r) + \ln(e^{j\theta}) = \ln(r) + j\theta$$

So

$$W = \phi + j\psi = \frac{\Lambda}{2\pi}(\ln(r) + j\theta) = \frac{\Lambda}{2\pi}\ln(Z)$$

$$2\pi r v_\theta = \Gamma$$

$$v_\theta = \frac{\Gamma}{2\pi r} = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$v_r = 0$ implies no integration function $f(\theta)$, just a constant of integration, which may be set to zero.

$$\phi = \frac{\Gamma\theta}{2\pi}$$

$$\psi = -\frac{\Gamma}{2\pi}\ln(r)$$

Notice that

$$\ln(Z) = \ln(re^{j\theta}) = \ln(r) + \ln(e^{j\theta}) = \ln(r) + j\theta$$

So

$$W = \phi + j\psi = \frac{\Gamma}{2\pi}(\theta - j\ln(r)) = \frac{\Gamma}{2\pi j}(\theta j + \ln(r)) = \frac{\Gamma}{2\pi j}\ln(Z)$$

Differing by a factor of $\frac{1}{j} = -j$ directly implies the velocities produced are orthogonal (source is radial, vortex circumferential).

(e) Can only model Joukowsky sections, not general sections. Trailing edge is very sharp and tough to build. Could use a different transformation, such as Karma-Trefftz, or could just switch to a panel code to allow analysis of any aerofoil section.

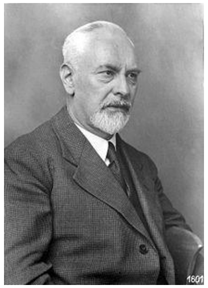
Aerodynamics 3

Introduction to Boundary Layers

(chapters 10 and 11 in notes)

What is a Boundary Layer?

- ~1900, aerodynamics consisted of 2 separate fields
  - hydrodynamics – inviscid fluids, good results far field, but no drag, lift through circulation
  - Hydraulics – empirical, but gave good results for pipe flows, etc.
- But in 1904 Prandtl (the 'grandfather' figure in fluid mechanics) united them through Boundary Layer Theory



University of BRISTOL

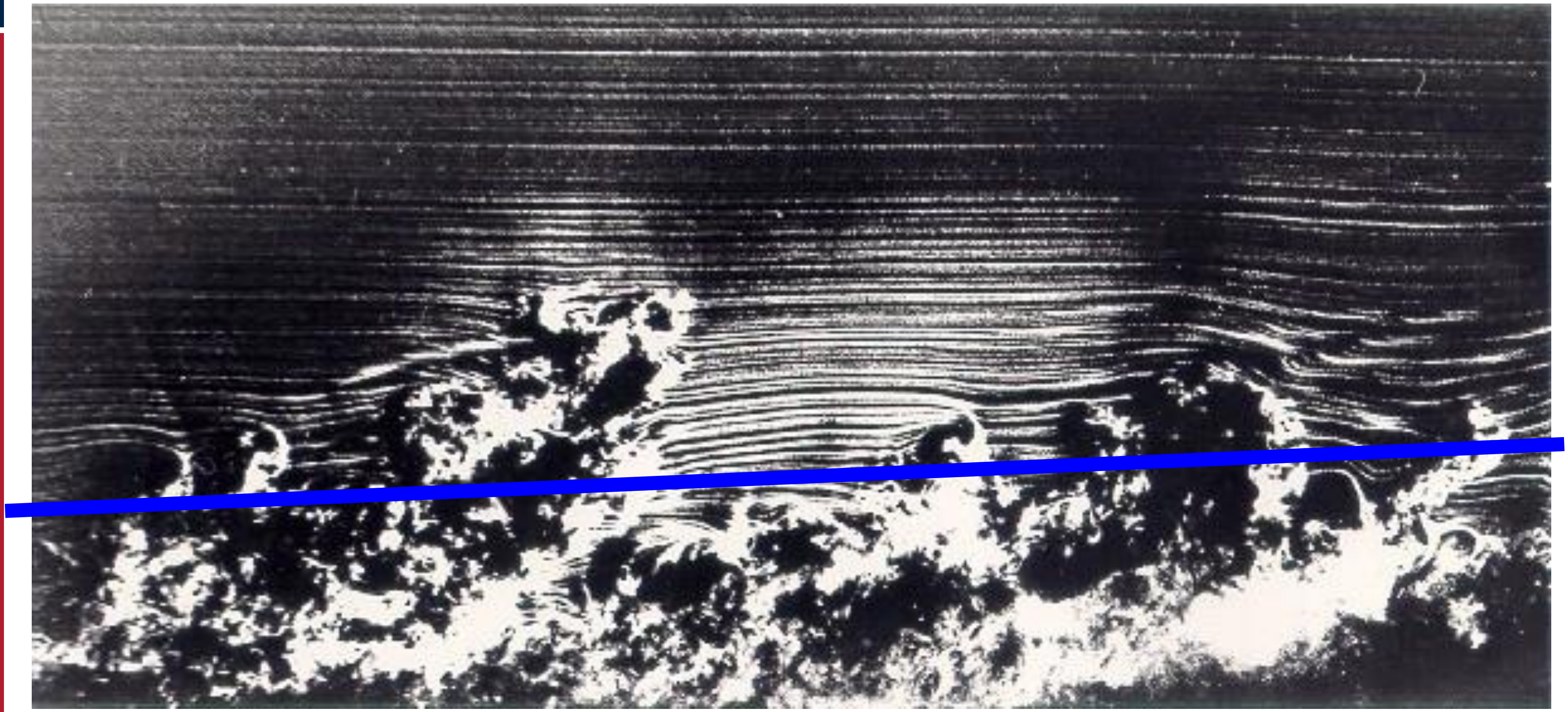
What is a Boundary Layer?

- Key to this idea was the realisation that even though viscosity is very small, it cannot always be neglected. Specifically, as in a Newtonian Fluid

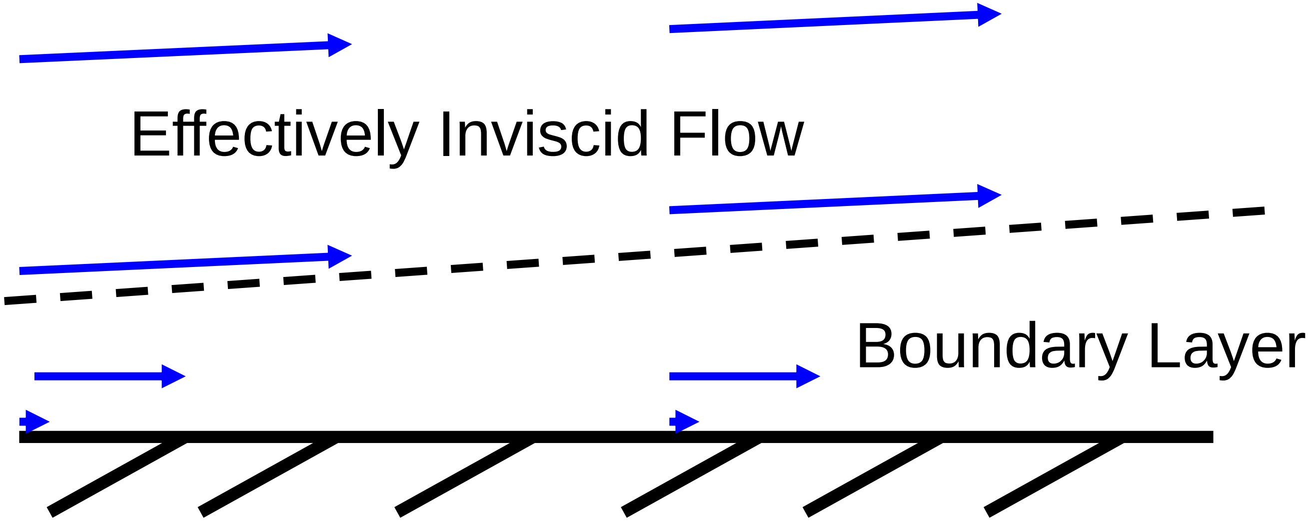
$$\tau = \mu \frac{\partial U}{\partial y}$$

  - Wherever velocity changes rapidly in the normal direction, μ can cause significant τ.
  - We have a no slip condition at the wall, i.e. U=0. If the flow reaches freestream only a small distance from the wall, then this is the case.

University of BRISTOL



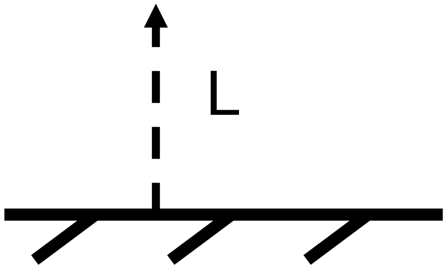
So we have..



- The boundary layer is *very* thin
- The use of the boundary layer concept depends on large Reynolds number for main flow: $>10^4$

$$\left(\mathrm{Re} = \frac{\rho U l}{\mu}\right)$$

Alternatively:



$$\mathrm{Re} = \frac{\rho U l}{\mu}$$

- Re is the ratio of Inertial to viscous forces.
- Usually L is defined in streamwise direction, but imagine for now it is distance normal to surface. Very near surface, L <<1, hence Re is small.
- However, if $\rho U >> \mu$, Re quickly rises with increasing L, and hence viscous forces are only important very near the surface.

University of BRISTOL

Advantages of Boundary Layer Theory

- We can calculate the vast majority of the flow field using inviscid methods – e.g. Joukowski Transformation, full potential, Euler.
- The effects of the boundary layer can be added afterwards, as a correction.
- The small thickness of the boundary layer means we can make significant reductions in the complexity of the Navier-Stokes Equations in this region.

Types of Boundary Layer

- The flow in a boundary layer may be Laminar, Transitional, or Turbulent
  - Laminar flows characterised by smooth layering
  - Turbulent flows by turbulent eddies and large amounts of mixing
  - Transitional flows by *intermittency*, i.e. the flow is sometimes laminar, sometimes turbulent
- Flows generally begin Laminar, and pass through Transition to Turbulent. For this and the next few lectures, we shall be looking at Laminar Flows

Turbulence is not separation

- Often confused: turbulence and separation
- Consider low Re (<100) flow around a circle with recirculating regions. This flow is separated but **still laminar**
- Consider high Re flow around an aerofoil. This flow is turbulent (the boundary layer is turbulent) but it is **still attached**
- **Separation and turbulence are not the same thing**

University of BRISTOL

Boundary Layer Equations

- We will not be deriving the Navier Stokes Equations, but will be using them. It is recommended you read through their derivation at some point (any text book dealing with viscous flow)
- Before looking at thin layer assumptions specifically, we assume:
  - 2D flow
  - Steady Flow
  - Incompressible Flow

University of BRISTOL

Effect of Incompressible Flow:

- By definition, ρ is constant:μ,ν are also constant
  - Have one less variable -> need one less Eqn.
- In fact, this assumption de-couples the energy equation
  - we don't need to solve it unless heat transfer is important
  - This makes life a lot simpler, but *remember, only applies to incompressible flows*

University of BRISTOL

Given these assumptions, the Navier Stokes Eqns. become:

Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

x-momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

y-momentum

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Thin Layer Assumption

- If Re is large, the boundary layer thickness δ is very small compared to any chordwise length. ie.

$$\frac{\delta}{x} << 1$$

- In turn this means that

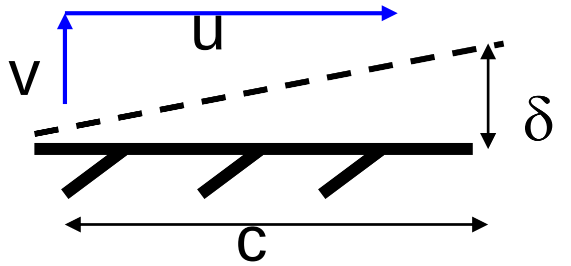
$$\frac{\partial}{\partial y} >> \frac{\partial}{\partial x}$$

- What effect does this have on the equations?

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

- It may initially seem that the first term may be ignored. However:



$$\frac{v}{u_e}=O\left(\frac{\delta}{c}\right)<<1$$

- Hence first term is a small differential of a large number, second is a large differential of small number, and hence are of similar magnitude. Neither can be neglected

x-momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

- partial derivative w.r.t x is small, so second derivative is neglected:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

University of BRISTOL

y-momentum: $$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- first term: v small, d/dx small -> first term neglected
- 2nd term: v small, dv/dy also small, neglect
- For similar reasons to above, can also neglect 2nd derivatives on RHS. Hence

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}\approx 0$$

- Means pressure is approximately constant through a boundary layer. If we assume this, do not need to solve y-momentum equation.

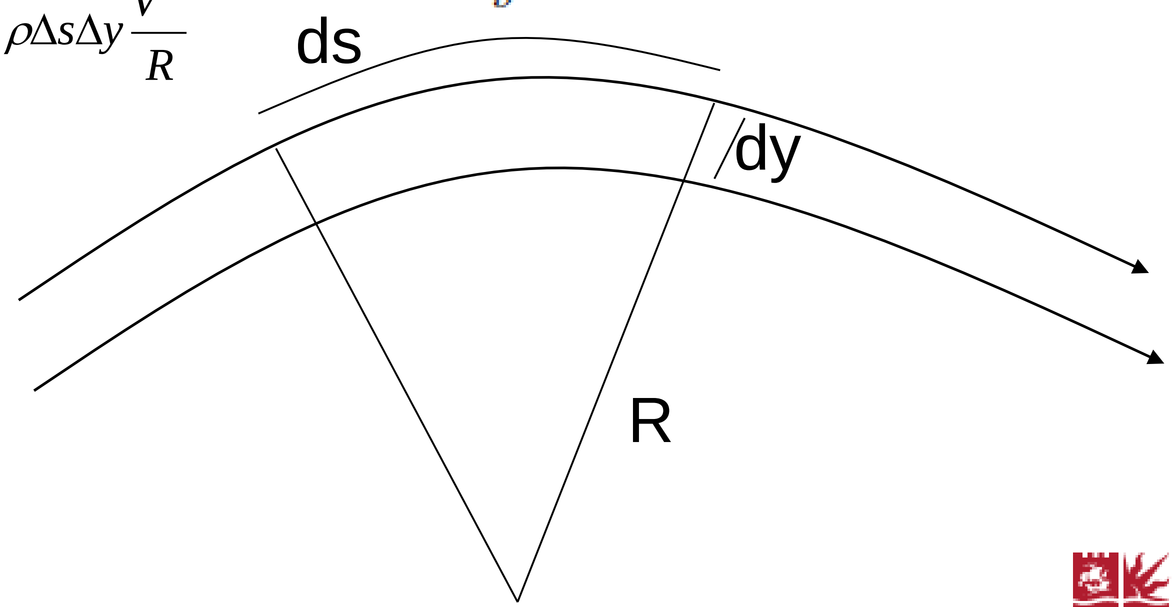
When would this not be true?

A strongly curving streamline requires a centripetal force – so pressure gradient in normal direction can not be zero, otherwise the air will not turn the corner

$$\frac{\partial p}{\partial y} = \frac{\rho u^2}{R}$$

$$(p_t - p_b)\Delta s = \rho \Delta s \Delta y \frac{V^2}{R}$$

$$\frac{dp}{dy} = \rho \frac{V^2}{R}$$



So, equations to solve are just:

- Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

- x-Momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

- We also need to assume small curvature:

$$\frac{\delta}{R}<<1$$

- This is generally the case for aerodynamic shapes

Integral Methods

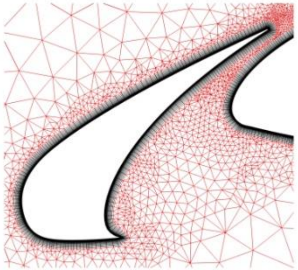
- Equations governing the flow in a steady, incompressible, 2D boundary layer are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

- These equations can be solved:
  – Numerically, on high speed computers
  – In integral fashion – semi-analytical

Integral vs. Differential



- Differential methods solve the differential equations:
  - need high speed computers
  - defined grids
- Integral methods solve analytical integrals of flow properties along the boundary layer
  - Involve more assumptions
  - Less versatile, BUT
  - Much simpler, can be done by hand/in simple processes
  - Easier to calibrate accurately

Boundary layer thickness

- The boundary layer thickness is denoted by δ, and is defined as the distance from the wall where the boundary layer flow is indistinguishable from the external flow, i.e.

$$u = u_e,\ \rho = \rho_{e,}\ \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \ldots = 0$$

- Technically this can only happen an infinite distance from the wall, however, δ is usually defined where $u = 0.995u_e$, or some similar value

Boundary layer growth and Re

$$u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} = \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{U^2}{c} \qquad\qquad \frac{\mu}{\rho}\frac{U}{\delta^2}$$

'flat plate'

$$\frac{\delta^2}{c} \sim \frac{\mu}{\rho U}$$

$$\frac{\delta^2}{c^2} \sim \frac{\mu}{\rho U c} = \frac{1}{\text{Re}}$$

$$\frac{\delta}{c} \sim \frac{1}{\sqrt{\text{Re}}}$$

Somewhat dubious – but valid from a dimensional point of view!

How thick is a boundary layer?

- *Very* thin. It was shown last week that δ grows proportional to $\mathrm{Re}^{-1/2}$. In fact, for a laminar B.L.

$$\frac{\delta}{x} \approx 5\,\mathrm{Re}^{-1/2}$$

- So if on the t.e. of an a/c Re = 10^6,

$$\frac{\delta}{c} \approx 0.005$$

- i.e if the chord is say 3 metres, the boundary layer at the t.e. is 1.5 *centimetres* thick

Note – the boundary layer might transition to a turbulent state, which is **thicker**

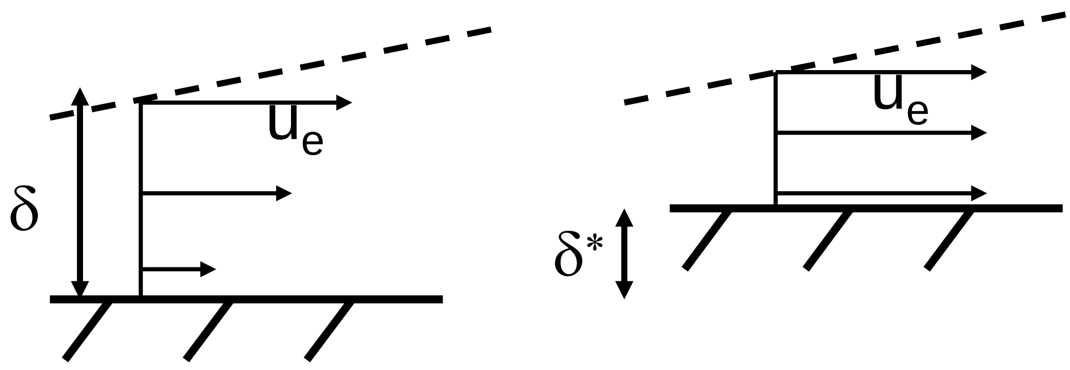
University of BRISTOL

Displacement Thickness, δ*

- Boundary layer thickness is a bit inexact (depends on how close we set *u* to u_e). Instead we can use *displacement thickness*, defined as

$$\rho_e u_e \delta^* = \int_0^h (\rho_e u_e - \rho u)\,dy \qquad \delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

- Physically, this is the distance the wall would move to give an equivalent inviscid mass flow:



University of BRISTOL

Momentum Thickness, θ

- The *Momentum Thickness* tells us what the loss in the momentum of the flow is, compared to what would be the case were there no boundary layer. It is defined as

$$\rho_e u_e{}^2 \theta = \int_0^h \rho u (u_e - u) dy \qquad \theta = \int_0^h \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$

- As a change in momentum is equivalent to a force, this gives us the drag created by the boundary layer, including both *skin friction* and *pressure drags.*

University of BRISTOL

Wall Boundary Conditions:

- Start with steady 2D flow over a flat plate in the *x* direction
  - at the wall, *y=u=v =0*, and hence:

$$\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

- So

$$\frac{dp}{dx}=\mu\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall}$$

University of BRISTOL

Edge Boundary Conditions:

- Again steady 2D flow over a flat plate in the *x* direction, at the boundary layer edge
  - $u = u_e$, derivatives w.r.t $y = 0$ (by definition of boundary layer edge),

$$u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{dp}{dx} + \cancel{\nu\frac{\partial^2 u}{\partial y^2}}$$

- So

$$\frac{dp}{dx} = -\rho_e u_e \frac{du_e}{dx} = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall}$$

Flat Plate with no Pressure Gradient

- In uniform flow over a flat plate there is no pressure gradient, i.e.

$$\frac{dp}{dx} = 0$$

- and hence

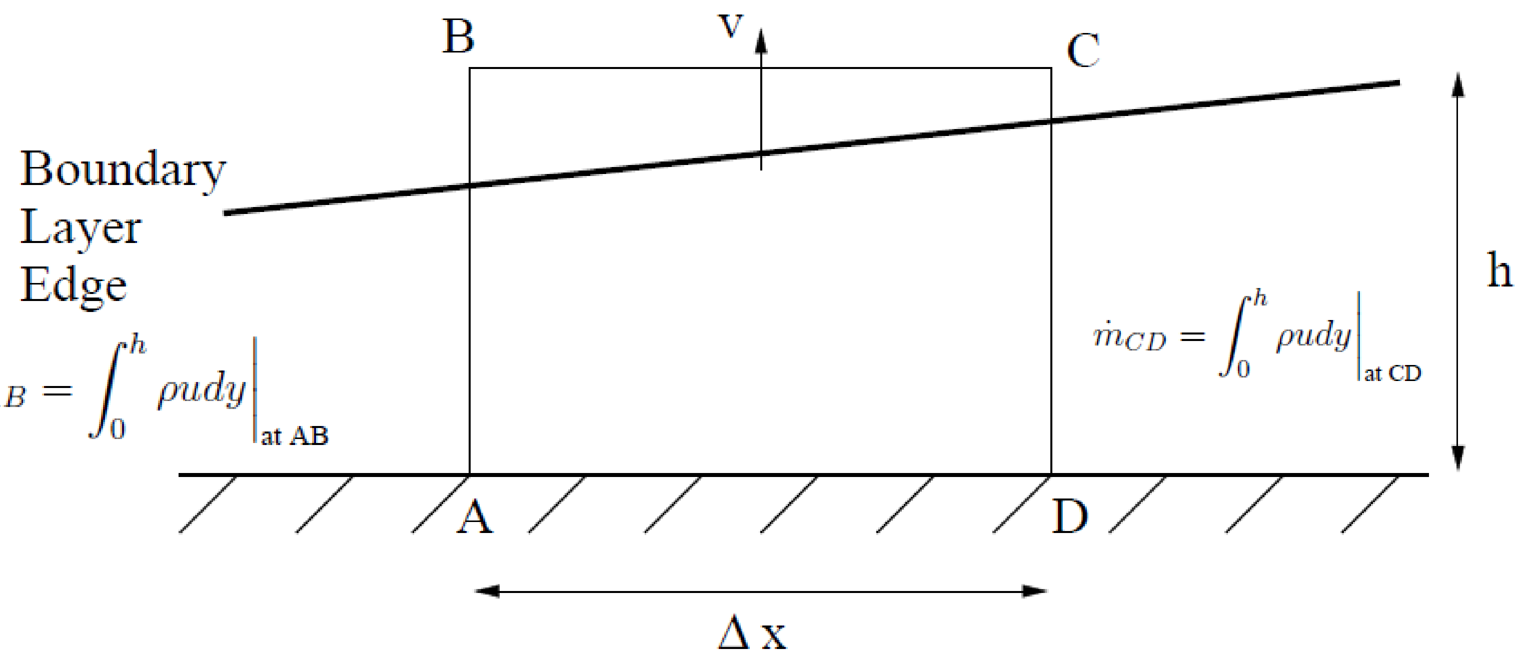
$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} = 0$$

- This makes it a special case, reducing the number of boundary conditions to be solved – allows an accurate solution to be derived

University of BRISTOL

Mass conservation

0=(mass in)-(mass out)



$$\dot{m}_{AB} = \int_0^h \rho u dy \Big|_{\text{at AB}}$$

$$\dot{m}_{CD} = \int_0^h \rho u dy \Big|_{\text{at CD}}$$

$$\int_0^h \rho u dy \Big|_{\text{at AB}} - \int_0^h \rho u dy \Big|_{\text{at CD}} = \rho_e v_h \Delta x$$

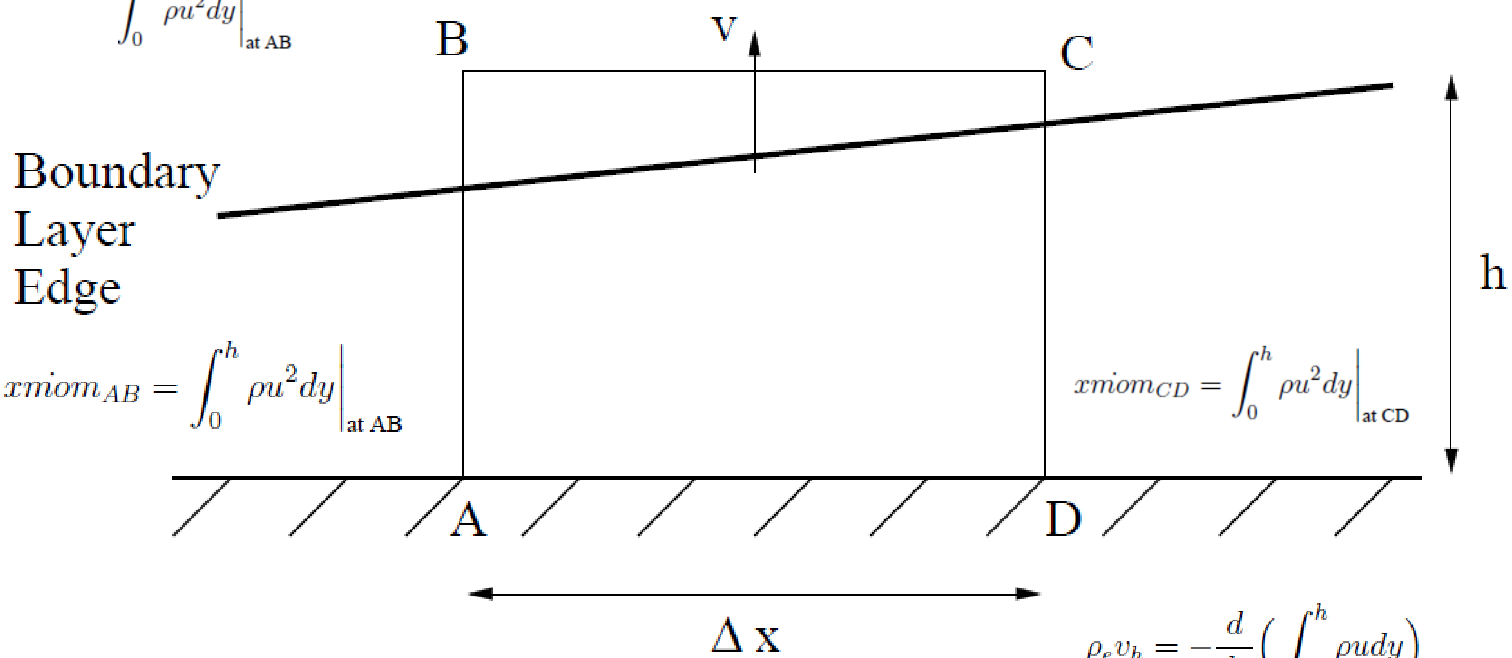
$$\rho_e v_h = -\frac{d}{dx}\left(\int_0^h \rho u dy \right)$$

Balance of momentum

$$\int_0^h \rho u^2 dy\Big|_{\text{at CD}} \qquad -(p_{CD} - p_{AB})h \qquad -\tau_w \Delta x$$

0=(momentum in)-(momentum out)+(pressure force L->R)+(shear force L->R)

$$\int_0^h \rho u^2 dy\Big|_{\text{at AB}}$$



$$\rho_e v_h = -\frac{d}{dx}\Big(\int_0^h \rho u dy\Big)$$

$$-\int_0^h \rho u^2 dy\Big|_{\text{at AB}} + \int_0^h \rho u^2 dy\Big|_{\text{at CD}} + \rho_e u_e v_h \Delta x = -(p_{CD} - p_{AB})h - \tau_w \Delta x$$

$$\frac{d}{dx}\Big(\int_0^h \rho u^2 dy\Big) - u_e \frac{d}{dx}\Big(\int_0^h \rho u dy\Big) = -h\frac{dp}{dx} - \tau_w$$

$$\frac{d}{dx}\left(\int_0^h \rho u^2 dy\right) - u_e \frac{d}{dx}\left(\int_0^h \rho u dy\right) = -h\frac{dp}{dx} - \tau_w$$

$$\frac{d}{dx}\left(u_e \int_0^h \rho u dy\right) = u_e \frac{d}{dx}\left(\int_0^h \rho u dy\right) + \frac{du_e}{dx}\int_0^h \rho u dy$$

$$u_e \frac{d}{dx}\left(\int_0^h \rho u dy\right) = \frac{d}{dx}\left(u_e \int_0^h \rho u dy\right) - \frac{du_e}{dx}\int_0^h \rho u dy$$

Replace using product rule

$$u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{dp}{dx} + \cancel{\nu\frac{\partial^2 u}{\partial y^2}}$$

$$-\frac{dp}{dx} = \rho_e u_e \frac{du_e}{d_x}$$

(for the external flow)

$$\frac{d}{dx}\left[\int_0^h \rho u(u-u_e)dy\right] + \frac{du_e}{dx}\left[\int_0^h \rho u dy\right] = h\rho_e u_e \frac{du_e}{dx} - \tau_w$$

$$\delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_e u_e}\right)dy$$

$$\theta = \int_0^h \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right)dy$$

$$h\rho_e u_e \frac{du_e}{dx} = \frac{du_e}{dx}\int_0^h \rho_e u_e dy$$

$$\boxed{\frac{d}{dx}(\rho_e u_e^2\theta) + \frac{du_e}{dx}\rho_e u_e \delta^* = \tau_w}$$

University of BRISTOL

$$\boxed{\frac{d}{dx}(\rho_e u_e^2\theta) + \frac{du_e}{dx}\rho_e u_e \delta^* = \tau_w}$$

$$\rho_e u_e^2 \frac{d\theta}{dx} + \theta\left(u_e^2 \frac{d\rho_e}{dx} + \rho_e 2u_e \frac{du_e}{dx}\right) + \frac{du_e}{dx}\rho_e u_e \delta^* = \tau_w$$

Expanding derivative

$$\frac{d\theta}{dx} + \theta\left(\frac{1}{\rho_e}\frac{d\rho_e}{dx} + \frac{2}{u_e}\frac{du_e}{dx}\right) + \frac{1}{u_e}\frac{du_e}{dx}\delta^* = \frac{\tau_w}{\rho_e u_e^2}$$

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) + \frac{\theta}{\rho_e}\frac{d\rho_e}{dx} = \frac{\tau_w}{\rho_e u_e^2}$$

Define ‘shape factor’ $H = \dfrac{\delta^*}{\theta}$

$$\boxed{\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{\tau_w}{\rho_e u_e^2} = \frac{\frac{1}{2}\tau_w}{\frac{1}{2}\rho_e u_e^2} = \frac{c_f}{2}}$$

Incompressible (no density derivatives)

Momentum integral equation

MIE

- The Momentum Integral Equation is a key part of integral methods, and we will be using it many times over the next few weeks
  - read the derivation, and make sure you **understand** the steps. Don't memorise
- There are other integral equations, e.g. *kinetic energy integral equation,* and *total energy integral equation*
- These are used in compressible analysis, heat transfer problems, etc

University of BRISTOL

Use of The MIE, worked example

Worked Example: Calculate the drag coefficient of a flat plate at zero incidence with steady flow on one side only, length 1 metre, in a sea level flow at 40 m/s. Assume incompressible, laminar flow, and a linear velocity profile within the boundary layer.

MIE -> $$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{c_f}{2}$$

no pressure gradients, so no velocity gradients in external flow

$$\frac{d\theta}{dx} = \frac{c_f}{2} = \frac{\tau_{wall}}{\rho u_e^2}$$

assume linear profile given by:

$$\frac{u}{u_e} = \frac{y}{\delta}$$

Integral properties:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy = \left[y - \frac{y^2}{2\delta}\right]_0^\delta = \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

$$\theta = \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

and

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}$$

and

$$u = \frac{y}{\delta} u_e \Rightarrow \frac{\partial u}{\partial y} = \frac{u_e}{\delta}$$

substitute into the MIE: $\dfrac{d\theta}{dx} = \dfrac{c_f}{2} = \dfrac{\tau_{wall}}{\rho u_e^2}$

$$\frac{d\theta}{dx} = \frac{\mu u_e}{\delta \rho_e u_e^2} \Rightarrow \frac{d\theta}{d\delta}\frac{d\delta}{dx} = \frac{\nu}{\delta u_e} \Rightarrow \frac{1}{6}\frac{d\delta}{dx} = \frac{\nu}{\delta u_e} \Rightarrow \delta d\delta = \frac{6\nu}{u_e} dx$$

integrate, and ignore constants of integration as delta = 0 when x = 0:

$$\delta^2 = \frac{12\nu x}{u_e} \Rightarrow \frac{\delta}{x} = \sqrt{\frac{12\nu}{x u_e}} = \sqrt{12} Re^{-\frac{1}{2}}$$

then

$$\frac{\delta^*}{x} = \frac{1}{2}\frac{\delta}{x} = 1.732 Re^{-\frac{1}{2}},$$

$$\frac{\theta}{x} = \frac{1}{6}\frac{\delta}{x} = 0.557 Re^{-\frac{1}{2}},$$

$$H = 3.0$$

and

$$u = \frac{y}{\delta} u_e \Rightarrow \frac{\partial u}{\partial y} = \frac{u_e}{\delta}$$

$$c_f = \frac{2\tau_{wall}}{\rho_e u_e^2} = \frac{2\nu}{u_e \delta} = \frac{2\nu}{\sqrt{12} Re^{-\frac{1}{2}} x u_e} = \frac{2}{\sqrt{12}} \frac{Re^{\frac{1}{2}}}{Re} = \frac{\theta}{x} = 0.577 Re^{-\frac{1}{2}}$$

this is for one side, locally

$$C_D = C_F = \frac{1}{c} \int_0^c c_f dx = \frac{0.577}{c} \int_0^c \sqrt{\frac{\nu}{u_e x}} dx = \frac{0.577}{c} \left[2\sqrt{\frac{x\nu}{u_e}} \right]_0^c$$

$$= \frac{1.155}{c} \sqrt{\frac{\nu c}{u_e}} = 1.155 Re_c^{-\frac{1}{2}}$$

$$Re_c = \frac{40 * 1}{1.461 * 10^{-5}} = 2.738 * 10^6$$

$$\mathrm{C_D} = 0.00070$$

x2 to include both sides (0.00140)

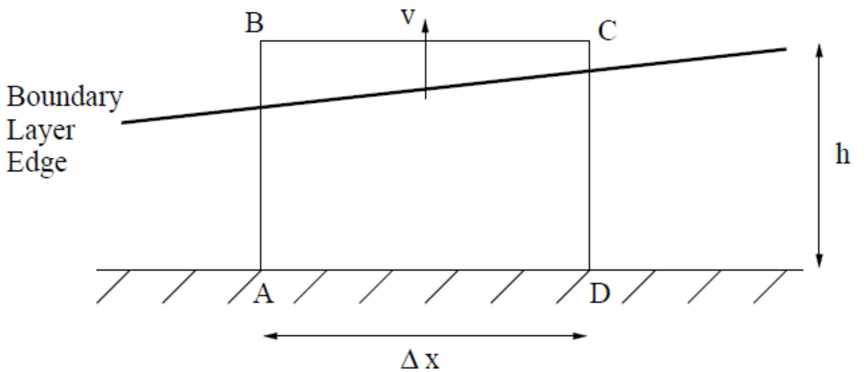


Figure Q3: boundary layer control volume

(a) Given that the mass flux across AB is $\int_0^h \rho u dy\Big|_{AB}$ and the mass flux across DC is $\int_0^h \rho u dy\Big|_{DC}$, write down the steady conservation of mass equation for the control volume in figure Q3, and take the limit as $\Delta x \to 0$. Make sure to include the contribution from face BC.

[3 marks]

(b) A total force (left to right) acts on the control volume given by $p_{AB}h - p_{CD}h - \tau_w \Delta x$. Write down the steady balance of momentum equation for the control volume. Using the result from **Q3**a, eliminate v and take the limit as $\Delta x \to 0$. **Note:** if you have not completed part **Q3**a you may use $\rho v = k$, where k is a constant.

[7 marks]

This analysis may be continued, resulting in the momentum integral equation written in terms of δ^*, θ and H, where $\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy$ and $\theta = \int_0^\delta \left(\frac{u}{u_e}\right)\left(1 - \frac{u}{u_e}\right) dy$ and $H = \frac{\delta^*}{\theta}$.

(c) If $\frac{u}{u_e} = \sin\left(\frac{y}{\delta}\frac{\pi}{2}\right)$ find H, given that $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$.

[7 marks]

(d) Sketch typical velocity profiles in terms of y vs. $\frac{u}{u_e}$ through a laminar and turbulent boundary layer. Also sketch the velocity profile of a boundary layer on the point of separation. Make sure to indicate any differences in total height on each line.

[3 marks]

Aerodynamics 3

Simple Solutions for Laminar Boundary Layers

(chapter 12 in notes)

Introduction

- In the worked example at the end of last week, we calculated the drag per unit depth of a flat plate assuming:
  - Incompressible Flow
  - Steady Flow
  - Laminar Flow with a linear velocity profile
- Whilst the first two assumptions are valid for some flows, the linear profile is incorrect. This week we shall look at more accurate profiles.

Blasius Profile

- For 2D, steady, incompressible flow along a flat plate (no velocity or pressure gradients in external flow), the boundary layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

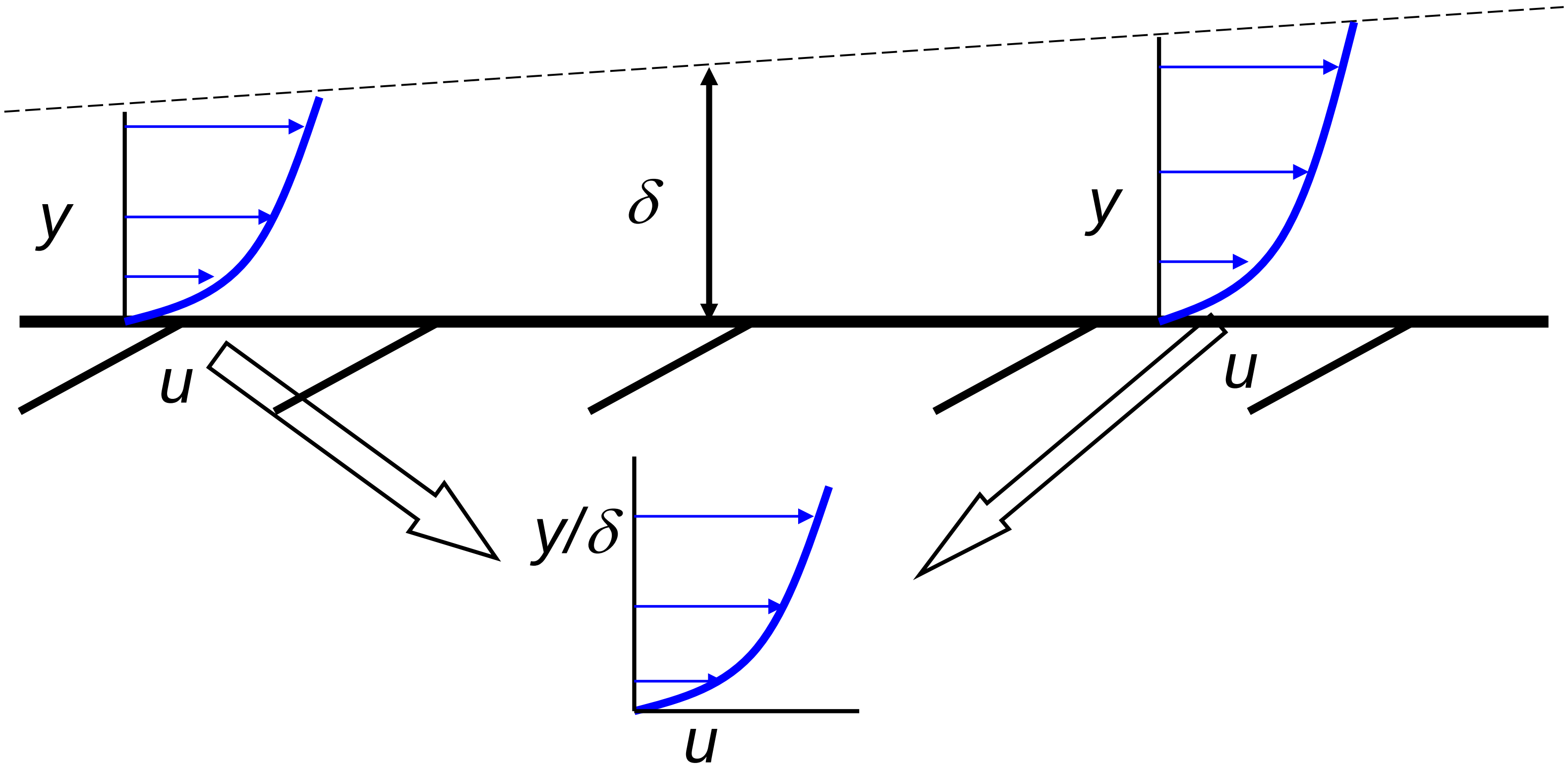
- With Boundary Conditions

$$u,v=0$$ at the wall, and

$$u=u_e,\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}=\ldots=0$$ at the boundary layer edge

University of BRISTOL

Also, no external gradients, so flow is *similar:*



i.e. the velocity profiles are identical everywhere in terms of a fraction of boundary layer thickness

- This is because there are no gradients in *x* direction – No way for changes to occur, apart from b.l. thickness.
- If we introduce a new coordinate η which scales with boundary layer height, then *u* for constant η is constant.
- What choice for η?
  - Know from dimensional analysis (lecture 6) that

$$\delta \propto \frac{x}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \mathrm{Re} = \frac{\rho u x}{\mu} = \frac{u x}{\nu} \qquad \frac{x}{\mathrm{Re}^{\frac{1}{2}}} = \frac{x \nu^{\frac{1}{2}}}{(u x)^{\frac{1}{2}}}$$

  - So try

$$\eta = \frac{1}{2}\frac{y}{\delta} = \frac{1}{2}\frac{y}{\dfrac{x}{\mathrm{Re}^{\frac{1}{2}}}} = \frac{1}{2}\frac{y\,\mathrm{Re}^{\frac{1}{2}}}{x} = \frac{1}{2}\frac{y}{x}\left(\frac{u_e x}{\nu}\right)^{1/2} = \frac{1}{2}y\left(\frac{u_e}{x\nu}\right)^{1/2}$$

Also, as we have incompressible flow, we know that the stream function exists (from continuity), i.e.

$$u = \frac{\partial \varphi}{\partial y}, \qquad v = -\frac{\partial \varphi}{\partial x}$$

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \equiv 0$$

-Means we only need to find one variable, not two.

$$\boxed{u = \frac{\partial \varphi}{\partial y}}$$

With these two ideas, we set

$$\varphi = (u_e x \nu)^{\frac{1}{2}}.F(\eta),$$

Where *F* is some function. Then

$$\Rightarrow \frac{\partial \varphi}{\partial y} = (u_e x \nu)^{\frac{1}{2}}.F'(\eta)\frac{\partial \eta}{\partial y} + 0.F(\eta)$$

$$\eta = \frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} \Rightarrow \frac{\partial \eta}{\partial y} = \frac{1}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Hence $$u = (u_e x \nu)^{\frac{1}{2}} \frac{F'}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} = \frac{u_e F'}{2} \Rightarrow \frac{u}{u_e} = \frac{F'}{2}$$

University of BRISTOL

$$v = -\frac{\partial \varphi}{\partial x}$$

Similarly,

$$\varphi = (u_e x \nu)^{\frac{1}{2}}.F(\eta)$$

Where *F* is some function. Then

$$\Rightarrow -\frac{\partial \varphi}{\partial x} = -(u_e x \nu)^{\frac{1}{2}}.F'(\eta)\frac{\partial \eta}{\partial x} - \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}.F(\eta)$$

$$\eta = \frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} \Rightarrow \frac{\partial \eta}{\partial x} = -\frac{1}{2x}\frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} = -\frac{\eta}{2x}$$

Hence

$$v = (u_e x \nu)^{\frac{1}{2}}\frac{\eta}{2x}F' - \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}.F$$

$$\Rightarrow v = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}(F'\eta - F)$$

In the same way, differentiating u w.r.t, x, y & y again:

$$\frac{\partial u}{\partial x} = -\frac{u_e}{2} F'' \frac{y}{4x}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} \qquad \frac{\partial u}{\partial y} = u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_e^2 F'''}{8\nu x}$$

We can then insert these into the x momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

University of BRISTOL

One term at a time:

$$u = \frac{u_e F'}{2}, \qquad \frac{\partial u}{\partial x} = -\frac{u_e}{2} F'' \frac{y}{4x}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Hence

$$u\frac{\partial u}{\partial x} = -\frac{u_e F'}{2}\frac{u_e F''}{2}\frac{y}{4x}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

$$= -\frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}\left(\frac{u_e^2 F' F''}{8x}\right)$$

$$= -\left(\frac{\eta u_e^2}{8x}\right)F' F''$$

2nd Term

$$v = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}} (F'\eta - F) \qquad \frac{\partial u}{\partial y} = u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

Hence

$$v\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}} (F'\eta - F) u_e \frac{F''}{4}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

$$= \frac{u_e^2 \eta}{8x} F'F'' - \frac{u_e^2}{8x} FF''$$

$$= \left(\frac{u_e^2 F''}{8x}\right)(F'\eta - F)$$

University of BRISTOL

and the RHS Directly is

$$\nu \frac{\partial^2 u}{\partial y^2} = \nu \frac{u_e^2 F'''}{8\nu x} = \frac{u_e^2 F'''}{8x}$$

All the terms derived thus far have a common factor $u_e^2/8x$, so we divide by this, and add up the terms, giving:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow -\eta F' F'' + \eta F' F'' - FF'' = F'''$$

i.e.

$$FF'' + F''' = 0$$

$$\boxed{FF'' + F''' = 0}$$

i.e. we want a function that when multiplied by its second derivative w.r.t. η is the negative of its third derivative.

This equation must be solved numerically, but this is quite simple on today's computers, and can even be done by hand. The first to do so was Blasius (a student of Prandtl's), hence it is named after him. It may be considered *exact* for incompressible, steady, 2D flow over a flat plate

University of BRISTOL

How to solve?

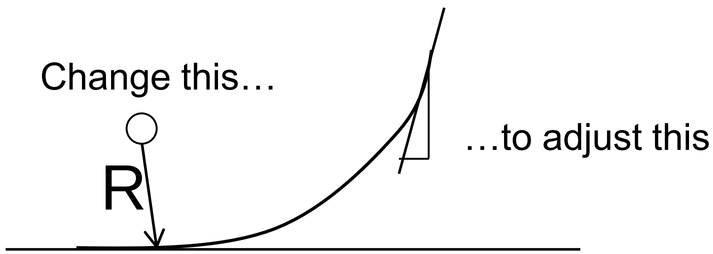
- Boundary conditions are key – at surface, u=0, v=0, so F'(0)=0 (u equation), then F(0) must also = 0 (from v equation)

$$\frac{u}{u_e} = \frac{F'}{2} \qquad v = \frac{1}{2}\left(\frac{u_e \nu}{x}\right)^{\frac{1}{2}}(F'\eta - F)$$

- At edge of boundary layer, u=ue, so F'=2. This makes things tricky because we have boundary conditions **at both ends.** Life is simpler when we have only **initialising** boundary conditions

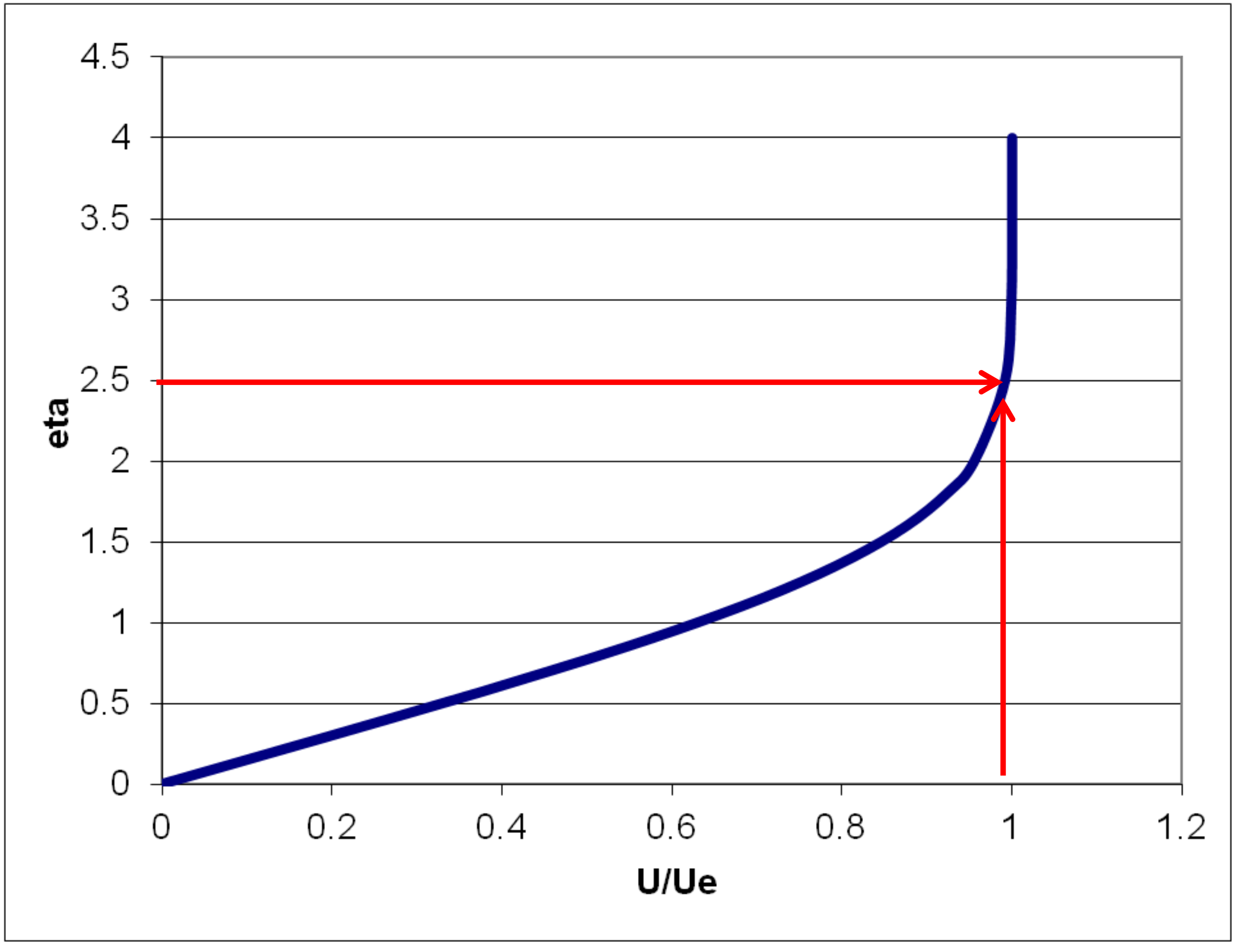
Shooting method

- Awkward to satisfy gradient condition at edge of boundary layer.
- Avoid this by guessing F''(0) and then marching up the layer to find F' at the edge.
- Check the value of F' at the edge and adjust F''
- Repeat process with new value of F''(0)
- This is equivalent to iterating on the wall shear stress



$$\frac{\partial u}{\partial y} = u_e \frac{F''}{4} \left(\frac{u_e}{\nu x} \right)^{\frac{1}{2}}$$

University of BRISTOL



$$\frac{u}{u_e} = \frac{F'}{2}$$

$$2.5 = \frac{1}{2}\delta\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}} = \frac{1}{2}\frac{\delta}{x}\left(\frac{u_e x}{\nu}\right)^{\frac{1}{2}}$$

From last time… $\frac{\delta}{x} = 5\left(\frac{u_e x}{\nu}\right)^{-\frac{1}{2}}$

University of BRISTOL

This profile can be numerically integrated using definitions of momentum and displacement thickness to give:

$$c_f = \frac{0.664}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\delta^*}{x} = \frac{1.721}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\theta}{x} = \frac{0.664}{\mathrm{Re}_x^{\frac{1}{2}}}$$

c.f. those calculated last week assuming a linear profile:

$$c_f = \frac{0.557}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\delta^*}{x} = \frac{1.732}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\theta}{x} = \frac{0.557}{\mathrm{Re}_x^{\frac{1}{2}}}$$

University of BRISTOL

(a) In SI units, what are the dimensions of the streamfunction ψ?

[1 mark]

(b) Using the speed u_e, distance x and kinematic viscosity ν show that $(u_e x \nu)^{\frac{1}{2}}$ has the same dimensions as streamfunction.

[4 marks]

η is now defined as $\eta = \frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$. A streamfunction ψ may now be defined as $\psi = (u_e x \nu)^{\frac{1}{2}} F(\eta)$.

(c) Consider the equations $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$. Show how the streamfunction allows an immediate solution of one of these equations.

[2 marks]

(d) By differentiating the streamfunction ψ appropriately (where $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$), find results for u and v. Also find the value of F' at the outside edge of the boundary layer.

[8 marks]

(e) It is possible to extend this analysis to derive the Blasius result $FF'' + F''' = 0$ for a laminar boundary layer. What are the limitations of this model, and to what other similar model can it be extended? Suggest a technique for satisfying the boundary condition on F' found in part (**Q2**d).

[5 marks]

University of BRISTOL

Aerodynamics 3

Application of Integral Methods

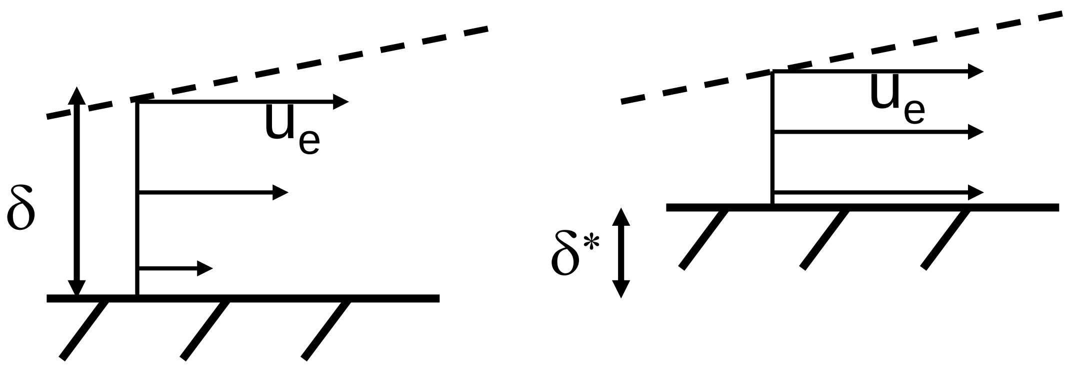
(chapter 13 in notes)

University of BRISTOL

Recap:

- In the previous lectures, we introduced important integral parameters:

displacement thickness: $$\delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$



momentum thickness:

$$\theta = \int_0^h \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$

Shape Factor:

$$H = \frac{\delta^*}{\theta}$$

Momentum Integral Equation:

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{c_f}{2}$$

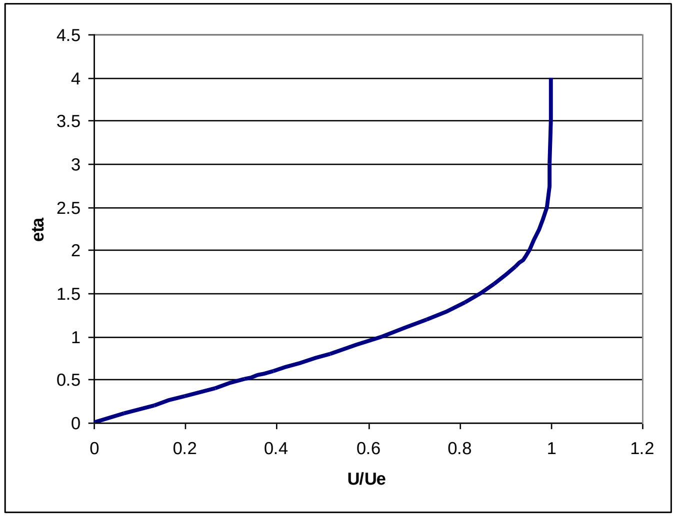
We used these concepts, along with the simplifying assumption that we had uniform flow over a flat plate to derive an exact expression for this flow known as the Blasius Equation:

$$\boxed{FF''+F'''=0}$$

where F is some function of the non-dimensional boundary layer thickness:

$$\eta = \frac{y}{2}\left(\frac{u_e}{\nu x}\right)^{\frac{1}{2}}$$

This could be numerically integrated to give a velocity profile:



Which produces the following results:

$$c_f = \frac{0.664}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\delta^*}{x} = \frac{1.721}{\mathrm{Re}_x^{\frac{1}{2}}} \qquad \frac{\theta}{x} = \frac{0.664}{\mathrm{Re}_x^{\frac{1}{2}}}$$

Other Flows with Similar Solutions

- The previous provides solution for a flat plate
- It would be useful to know whether other solutions exist for more complex shapes which have similar solutions
  - i.e. solutions where one solution fits all points, provided it is scaled by the change in thickness of the boundary layer
- This leads naturally to the question as to whether such flows exist

The answer to this is yes, but only for a limited number of special cases. By far the most important of these is when the external velocity of a flow may be expressed as

$$u_e = K.x^n$$

Where K is a constant.

The boundary condition this imposes on the freestream velocity can be used to generate a similar equation for the velocity function η

- This is done in your handouts
  - *the Falkner-Skan derivation will not be examined, it is for completeness only*
- The process results in

$$F''' + (n+1)FF'' - 2n(F')^2 + 8n = 0$$

- The bonus here is we can now consider pressure gradients in a limited kind of way

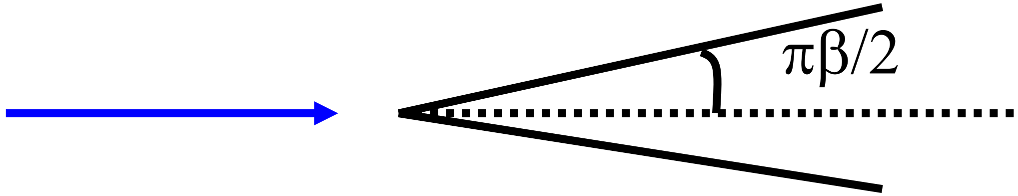
University of BRISTOL

This is the Falkner-Skan equation, and again must be solved numerically. The solution process produces a series of different velocity profiles for different values of n, as would be expected. These velocity profiles can then be integrated in the usual manner to give expressions for c_f, δ^*, θ, etc.

$$\delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \qquad \theta = \int_0^h \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$

The practical use of these solution derives from the fact that subsonic flow over a wedge may be described as

$$u_e = Kx^{\beta/(2-\beta)} = Kx^n$$



where the wedge semi-angle is given by $\pi\beta/2$. Even more importantly, flow at a stagnation point on a bluff body (e.g. an aerofoil), is given by this equation when β=1.

University of BRISTOL

However, the flows thus described are only a small subset of those possible, and obviously do not include, for instance, those about an entire aerofoil.

So the question is, how to proceed? we can either:

- Use numerical methods, retaining flow detail but requiring lots of time and computer power
- Or make use of empirical results (and the exact solutions previously discussed) to provide and approximate integral method.

We shall be doing the latter.

University of BRISTOL

Thwaites Method

Thwaites method is the most widely used known integral method for 2D, laminar, incompressible boundary layers. It is generally reasonably accurate, provided it is used within a certain range of conditions (discussed later)

In a 2D, steady, incompressible boundary layer, the equations of motion are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

At the wall, u,v,=0, so

$$0=-\frac{1}{\rho}\frac{dp}{dx}+\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{wall}$$

In the freestream, $u=u_e$,v=0, so

$$u_e\frac{\partial u_e}{\partial x}=-\frac{1}{\rho_e}\frac{dp}{dx}$$

University of BRISTOL

As the flow is incompressible, $\rho=\rho_e$, and hence we can substitute in to give

$$\nu \left.\frac{\partial^2 u}{\partial y^2}\right|_{wall} = -u_e \frac{du_e}{dx}$$

We now introduce two parameters l and m, which are related to the wall stress and gradient of stress at the wall respectively:

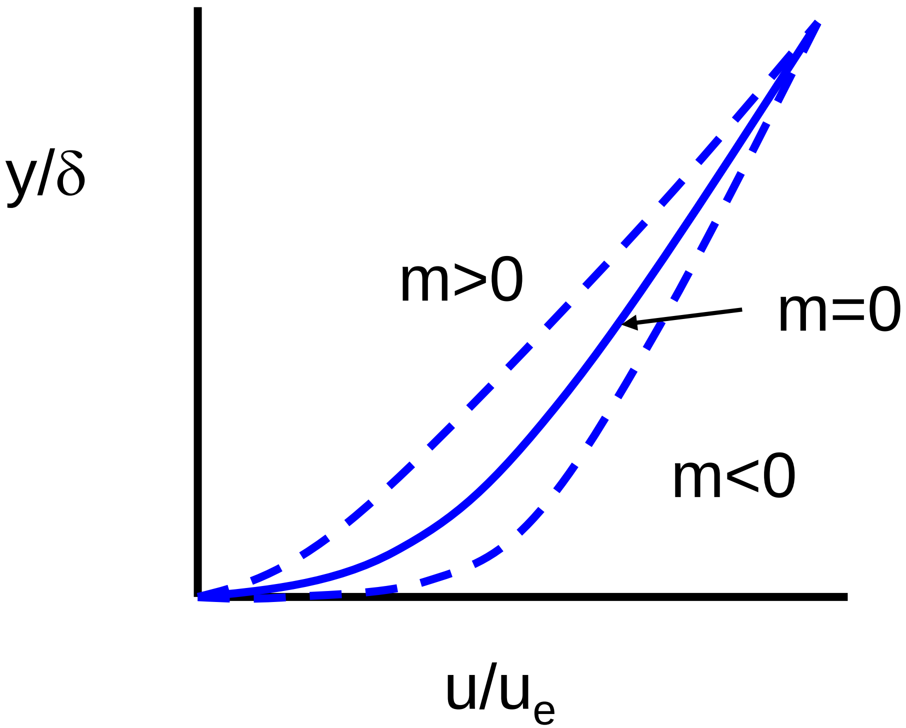
$$l = \frac{\theta}{u_e}\left(\frac{\partial u}{\partial y}\right)_{wall} \left(= \frac{\theta \tau_{wall}}{u_e \mu} \right)$$

$$m = \frac{\theta^2}{u_e}\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} = \frac{\theta^2 \nu}{u_e \nu}\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} = \frac{\theta^2 (-u_e)}{u_e \nu}\left(\frac{du_e}{dx}\right)$$

$$= -\frac{\theta^2}{\nu}\left(\frac{du_e}{dx}\right)$$

University of BRISTOL

So, l depends on wall stress, but m depends on freestream velocity gradient. If we assume that the family of velocity profiles are uni-parametric (i.e. depend on only one variable), and we pick m.



This assumption means that:

- Whatever values θ, ν and du_e/dx have, if their product is the same in two different situations, then the velocity profiles (scaled with local δ and u_e) are the same .

- H and l are functions of m only.

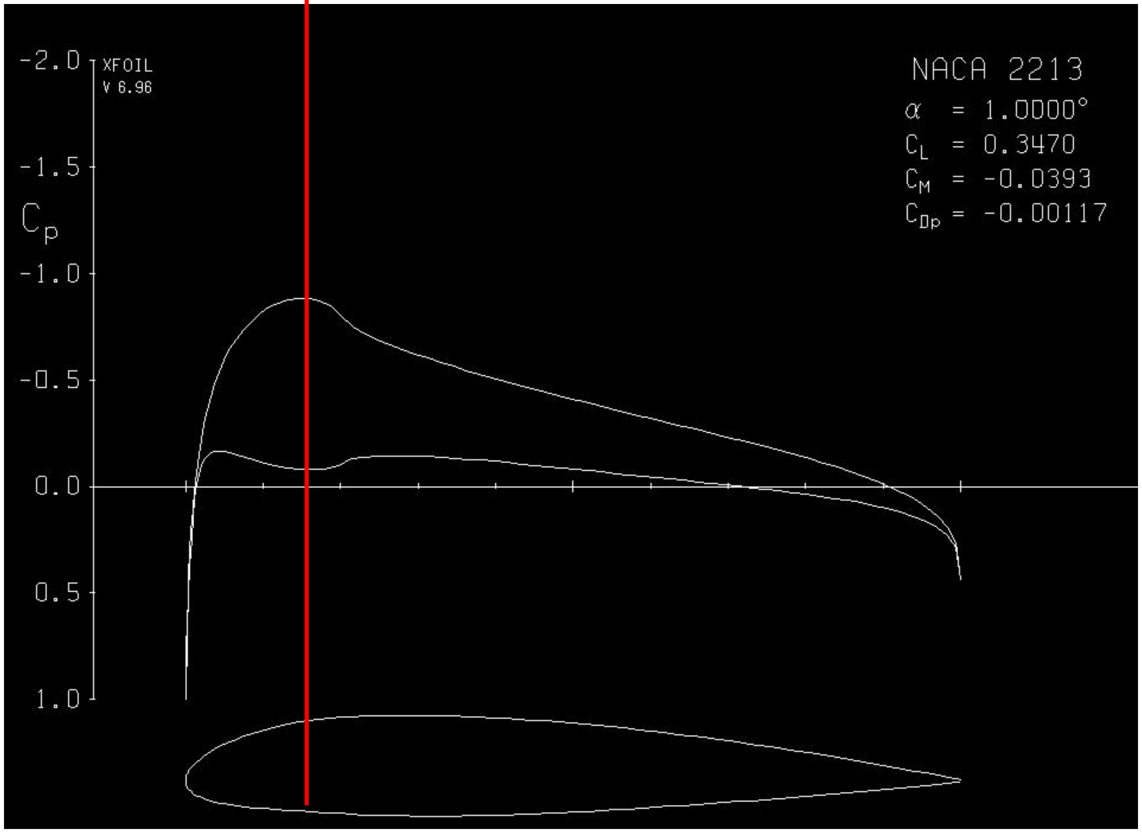
- As θ,ν are positive, if m = 0, due/dx must be zero, i.e. no pressure gradient -> Blasius solution.

- if m –ve, due/dx is +ve -> **favourable pressure gradient**

- if m +ve, -> **adverse pressure gradient**

University of BRISTOL

Favourable | Adverse



Returning to the MIE:

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{c_f}{2} = \frac{\tau_{wall}}{\rho u_e^2}$$

Multiply through by θu_e:

$$\theta u_e \frac{d\theta}{dx} + \theta^2 \frac{du_e}{dx}(H+2) = \frac{\theta \tau_{wall}}{\rho u_e}$$

But

$$m = -\frac{\theta^2}{\nu}\left(\frac{du_e}{dx}\right) \qquad l = \frac{\theta \tau_{wall}}{u_e \mu}$$

and so we have:

$$\theta u_e \frac{d\theta}{dx} - \nu m(H+2) = \frac{\mu}{\rho} l$$

re-arranging and using the fact that $\nu=\mu/\rho$

$$\theta u_e \frac{d\theta}{dx} = \nu (m(H+2)+l)$$

Rearrange:

$$u_e \frac{1}{2} \frac{d\theta^2}{dx} = \nu (m(H+2)+l)$$

So

$$u_e \frac{d\theta^2}{dx} = 2\nu(m(H+2)+l) = \nu L(m)$$

where

$$L(m) = 2(m(H+2)+l)$$

After considering a large amount of empirical and analytical data, Thwaites settled on the formula

$$L(m) = 0.45 + 6m \qquad m = -\frac{\theta^2}{\nu}\left(\frac{du_e}{dx}\right)$$

$$L(m) = 0.45 + 6m$$

$$u_e \frac{d\theta^2}{dx} = \nu L(m)$$

$$m = -\frac{\theta^2}{\nu}\left(\frac{du_e}{dx}\right)$$

$$u_e \frac{d(\theta^2)}{dx} = 0.45\nu - 6\theta^2 \frac{du_e}{dx}$$

$$u_e \frac{d(\theta^2)}{dx} + 6\theta^2 \frac{du_e}{dx} = 0.45\nu$$

$$u_e^6 \frac{d(\theta^2)}{dx} + 6u_e^5\theta^2 \frac{du_e}{dx} = 0.45u_e^5\nu$$

$$\frac{d}{dx}\left(u_e^6\theta^2\right) = 0.45\nu u_e^5$$

If this expression is used it can be shown that (this is done in your notes)

$$\theta_{x1}^{2} = 0.45\frac{\nu}{u_e^6}\int_0^{x1} u_e^5 dx$$

Whilst this may appear to mean that if x=0, $\theta=0$, this is not necessarily the case. Consider a stagnation point, where as previously discussed, u_e = Kx. Then

$$\int u_e^5 dx = \int k^5 x^5 dx = \tfrac{1}{6} k^5 x^6 ; \Rightarrow \theta^2 = \tfrac{0.45\nu}{6k}$$

This then gives us a method for calculation of momentum thickness over an arbitrary shape. Further to this, from m values of l and H can be derived, allowing τ,δ and δ^* to be calculated (tables of values and approximate equations are given in your notes

However, it should be noted that the method breaks down **if the adverse pressure gradient is too high** (i.e. m is large and +ve, say above ~0.09. This is because separation/transition is imminent, neither of which can be predicted easily with this approach

$$m = -\frac{\theta^2}{\nu}\left(\frac{du_e}{dx}\right)$$

University of BRISTOL

Numerical Example:

Consider the flow along a flat plate with a mildly favourable pressure distribution at sea level (ν=1.461*10^{-3}) such that the flow velocity is

$$u_e = 30(1 + 0.2x)$$

Calculate the momentum thickness at 1metre

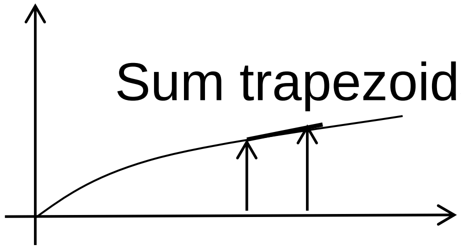
University of BRISTOL

Method

$$\theta_{x1}^2 = 0.45\frac{\nu}{u_e^6}\int_0^{x1} u_e^5\,dx$$

x	u_e	u_e^5	$\sum_0^x u_e^5 \Delta x$	U_e^6	θ^2
0.0	30	$24.3 * 10^6$	0	$729 * 10^6$	0.0
0.2	31.2	$29.56 * 10^6$	$5.386 * 10^6$	$0.922 * 10^9$	$3.839 * 10^{-8}$
0.4	32.4	$35.70 * 10^6$	$11.91 * 10^6$	$1.157 * 10^9$	$6.769 * 10^{-8}$
0.6	33.6	$42.82 * 10^6$	$19.76 * 10^6$	$1.439 * 10^9$	$9.029 * 10^{-8}$
0.8	34.8	$51.38 * 10^6$	$29.18 * 10^6$	$1.776 * 10^9$	$1.080 * 10^{-7}$
1.0	36.0	$60.47 * 10^6$	$40.36 * 10^6$	$2.177 * 10^9$	$1.219 * 10^{-7}$

$$0.5\times(24.3\times10^6 + 29.56\times10^6)\times0.2$$



Note: this problem can also be integrated analytically, but requires setting the constant of integration to achieve theta=0 @ x=0.

The rest

$$H(n) = \begin{cases} 2.61 + 3.75m + 5.24m^2 & \text{for } -0.1 < m < 0.0 \\ 2.088 + \frac{0.0731}{0.14-m} & \text{for } 0.0 < m < 0.1 \end{cases}$$

$$l(n) = \begin{cases} 0.22 - 1.57m - 1.8m^2 & \text{for } -0.1 < m < 0.0 \\ 0.22 - 1.402m - \frac{0.018m}{0.107-m} & \text{for } 0.0 < m < 0.1 \end{cases}$$

<1mm –

But turbulent BLs

grow more quickly!

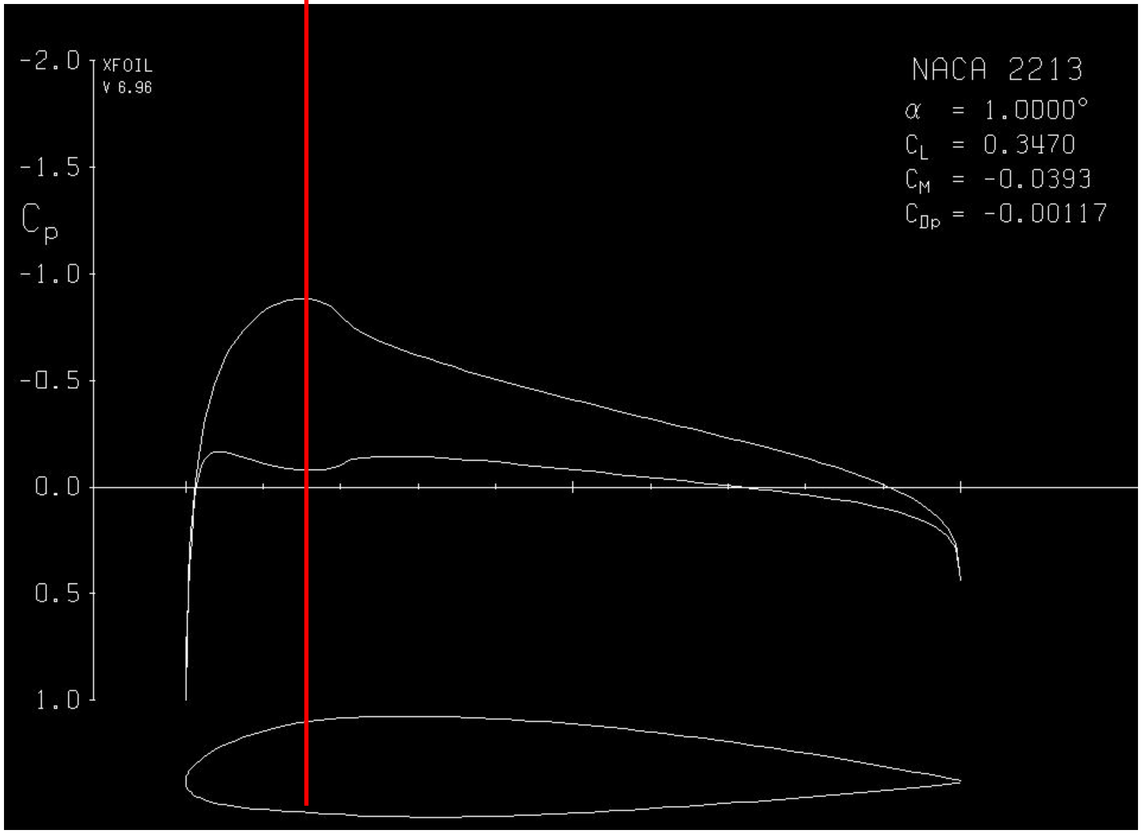
$$H = \frac{\delta^*}{\theta}$$

x	θ	$m(= -\frac{\theta^2}{\nu}\frac{du_e}{dx})$	l	$\tau_{wall}(= \frac{l u_e \mu}{\theta})$	H	δ^*
0.0	0.0	0	0.220	na	2.61	0.0
0.2	$1.959 * 10^{-4}$	-0.016	0.244	0.695	2.55	$5.00 * 10^{-4}$
0.4	$2.602 * 10^{-4}$	-0.028	0.262	0.584	2.51	$6.53 * 10^{-4}$
0.6	$3.005 * 10^{-4}$	-0.037	0.275	0.550	2.47	$7.42 * 10^{-4}$
0.8	$3.286 * 10^{-4}$	-0.044	0.285	0.540	2.45	$8.05 * 10^{-4}$
1.0	$3.491 * 10^{-4}$	-0.050	0.294	0.542	2.43	$8.48 * 10^{-4}$

Why bother?

Can now compute laminar boundary layers in favourable and **mildly** adverse pressure gradients.

Favourable | Adverse



Adverse gradients trigger transition to a turbulent BL, and possible even separation -> more analysis required

The boundary layer momentum equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

(a) Write down the momentum equation at the wall and at the edge of the boundary layer.

[2 marks]

In the laminar boundary layer analysis by Thwaites, the parameters $l = \frac{\theta}{u_e}\left(\frac{\partial u}{\partial y}\right)_w = \frac{\theta\tau_w}{u_e\mu}$ and $m = \frac{\theta^2}{u_e}\left(\frac{\partial^2 u}{\partial y^2}\right)_w = -\frac{\theta^2}{\nu}\frac{du_e}{dx}$ are used. The momentum integral equation is given by

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{\tau_w}{\rho u_e^2}$$

(b) Multiply the momentum integral equation by θu_e and rearrange to show that

$$u_e\frac{d\theta^2}{dx} = 2\nu(m(H+2)+l) = \nu L(m)$$

[7 marks]

(c) Multiply the result from **Q4**b by u_e^5 and use $L(m) = 0.45 + 6m$ to find an integral expression for θ in terms of u_e.

[7 marks]

(d) What are the differences in assumed pressure gradient in the boundary layer models of Blasius, Falkner-Skan and Thwaites? Under what conditions will the Thwaites analysis break down?

[4 marks]

of
)L

Aerodynamics 3

Squire Young Method for Drag Prediction

(chapter 14 in notes)

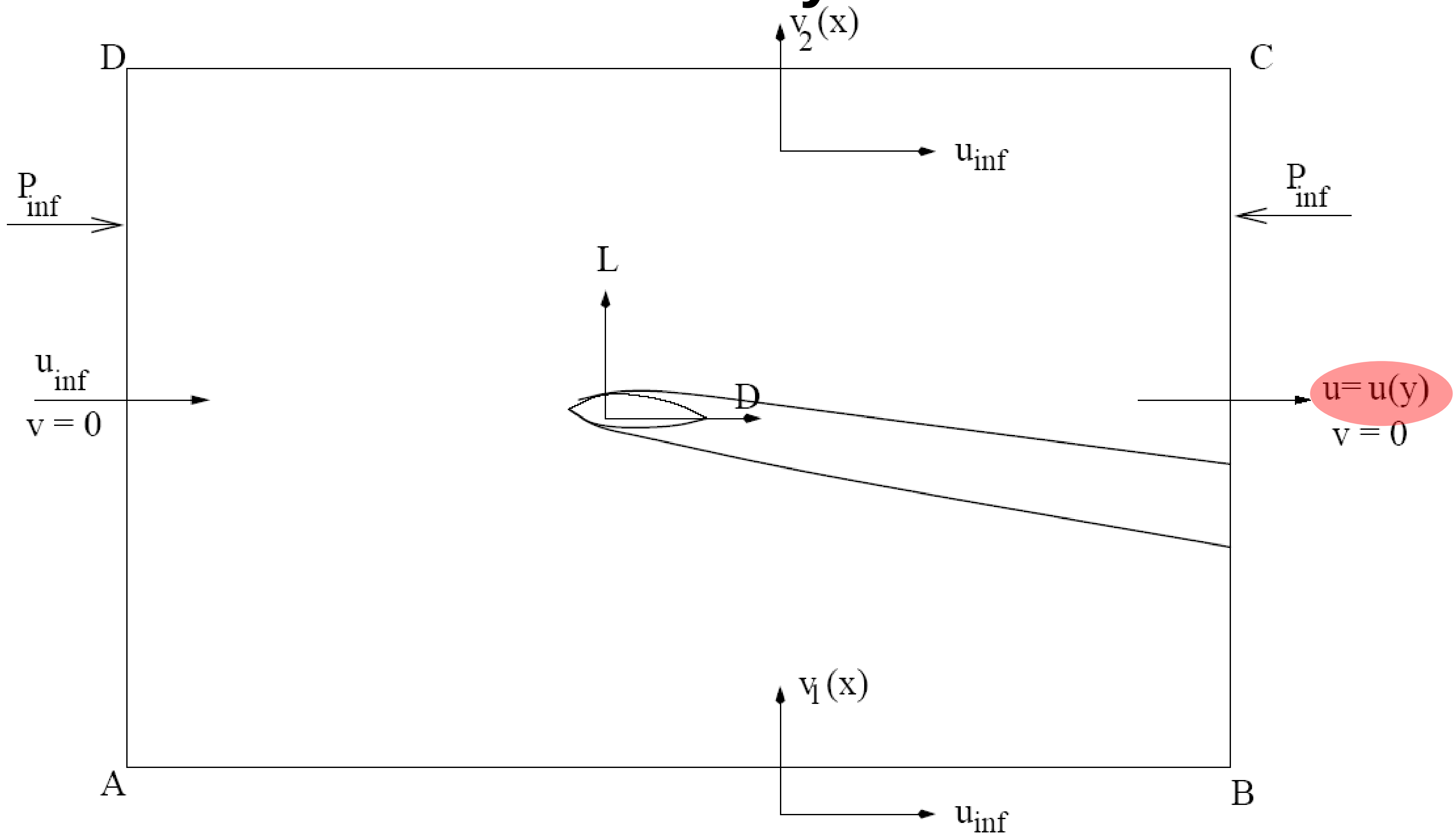
Introduction

- Thwaites method introduced last week allows calculation of integral properties about (fairly) arbitrary shapes
- However, boundary layers contribute two components of drag:
  - skin friction
  - pressure drag
- The last cannot be estimated from integral properties directly – need a method to do this

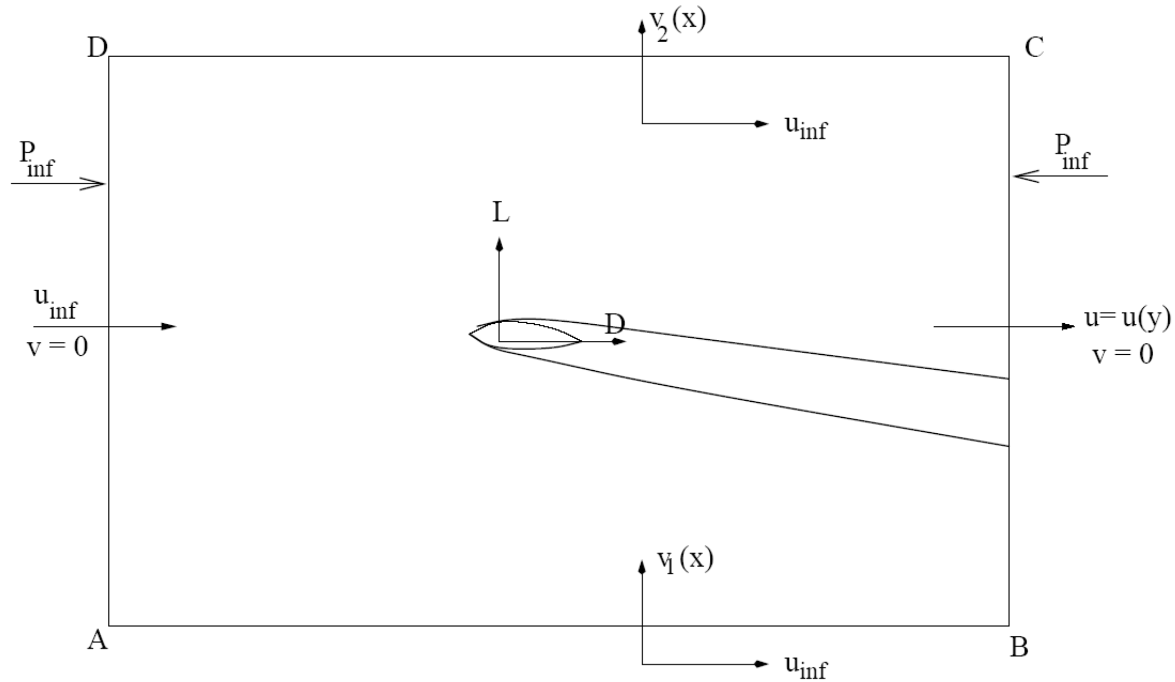
2 Step approach

- Work out how to link the drag to the momentum thickness far behind the aerofoil
- Work out how to link the momentum thickness at the trailing edge of the aerofoil to the momentum thickness far behind the aerofoil

Use control volume analysis:



Sides AB BC CD DA are a large distance from the body



Assuming incompressible flow, conservation of mass gives:

$$\int_A^B V_1(x)dx - \int_B^C U(y)dy + \int_D^C V_2(x)dx + \int_A^D U_\infty dy = 0$$

or

$$\int_A^B V_1(x)dx - \int_D^C V_2(x)dx = \int_B^C U(y)dy - \int_A^D U_\infty dy$$

As no forces act on the boundaries (pressure terms cancel), any momentum imbalance must be equal and opposite to the force on the body, i.e. the drag. Hence

$$\int_A^D \rho_\infty U_\infty^2 dy + \int_A^B \rho_\infty U_\infty V_1(x)dx - \int_D^C \rho_\infty U_\infty V_2(x)dx - \int_B^C \rho_\infty U^2(y)dy = Drag$$

But as ρ_∞, U_∞ are constants,

$$\int_A^B \rho_\infty U_\infty V_1(y)dy - \int_D^C \rho_\infty U_\infty V_2(x)dx = \rho_\infty U_\infty \left[\int_A^B V_1(x)dx - \int_D^C V_2(x)dx\right]$$

and from Equation (1)

$$\int_A^B V_1(x)dx - \int_B^C U(y)dy - \int_D^C V_2(x)dx + \int_A^D U_\infty dy = 0$$

$$= \rho_\infty U_\infty \left[\int_B^C U(y)dy - \int_A^D U_\infty dy\right]$$

Substituting this into the above Eqn. gives:

University of BRISTOL

$$Drag = \int_A^D \cancel{\rho_\infty U_\infty^2 dy} + \rho_\infty U_\infty \int_B^C U(y)dy - \cancel{\rho_\infty U_\infty \int_A^D} U_\infty dy - \int_B^C \rho_\infty U^2(y)dy$$

$$= \rho_\infty U_\infty \int_B^C U(y)dy - \int_B^C \rho_\infty U^2(y)dy$$

$$\times \frac{U_\infty}{U_\infty} \qquad \times \frac{U_\infty^2}{U_\infty^2}$$

$$= \rho_\infty U_\infty^2 \int_B^C \frac{U(y)}{U_\infty}\left(1 - \frac{U(y)}{U_\infty}\right) dy = \rho_\infty U_\infty^2 \theta_\infty$$

and hence

$$Drag = \rho_\infty U_\infty^2 \theta_\infty = C_D \frac{1}{2} \rho_\infty U_\infty^2 c \qquad \text{(per unit span)}$$

$$\Rightarrow C_D = \frac{2\theta_\infty}{c}$$

Thus the drag coefficient depends solely upon the momentum thickness at the far field!

This is good news - but we need to know how the momentum thickness changes between the trailing edge and the farfield

The momentum integral eq’n holds the key here!

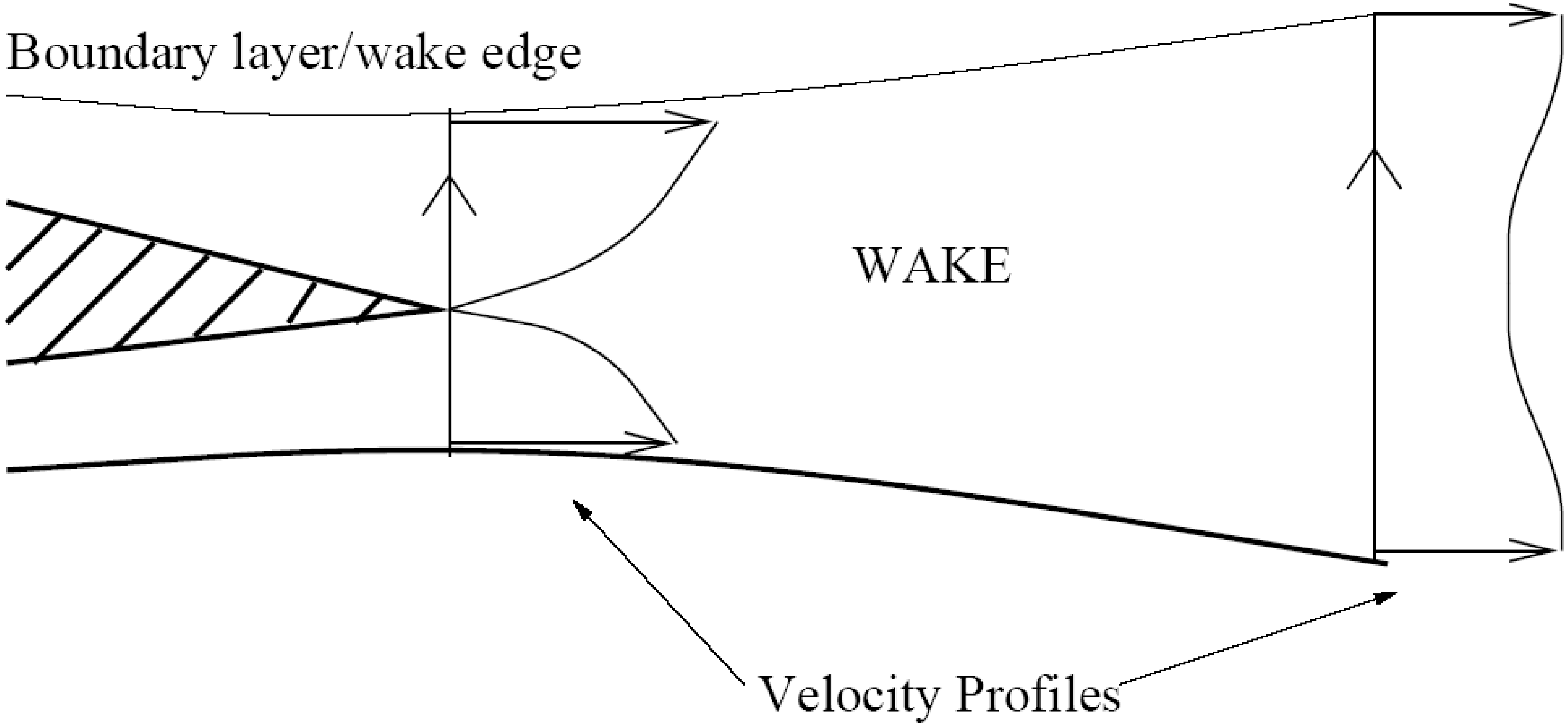
So, the question is, how does the flow behave in the wake?

- There is no solid surface in the wake, so
  - no skin friction !
  - MIE reduces to $$\frac{d\theta}{dx} + (H+2)\frac{\theta}{u_e}\frac{du_e}{dx} = 0$$
- Know properties at t.e. from inviscid analysis (u_e) and B.L. calculations (integral properties).
- Find total values by summing upper and lower surfaces:

$$\theta_{t.e.} = \theta_{upper} + \theta_{lower}$$
$$\delta^*_{t.e.} = \delta^*_{upper} + \delta^*_{lower}$$
$$H_{t.e.} = \frac{\delta^*_{t.e.}}{\theta_{t.e.}}$$

University of BRISTOL

- Far downstream, we have
  - $u_e \longrightarrow U_{inf}$, $\theta \longrightarrow \theta_{inf}$
- To solve the MIE, however, we need to know how H, the shape factor, varies.
- Consider the flow in the wake:



- Near the aerofoil
  - large velocity reduction in wake centre (this is due to no slip condition at t.e.)
- Far downstream, mixing due to viscosity has caused wake to
  - widen
  - have a reduced velocity deficit (mixing of low and high speed flows)

Experimental results show that:

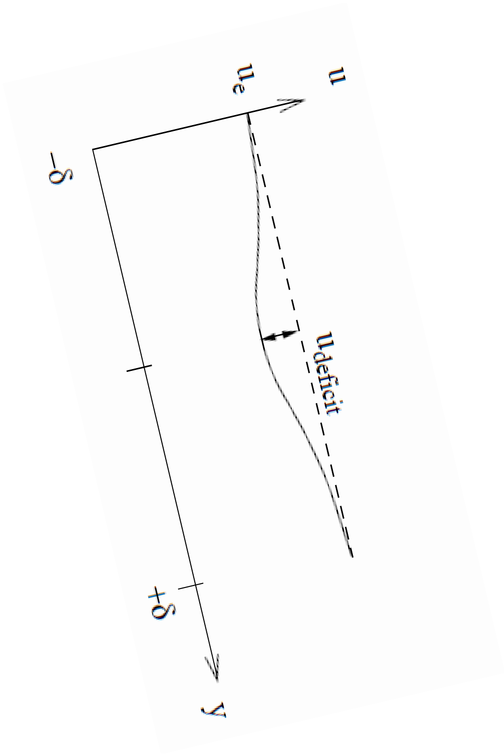
- Velocity profile may be represented by

$$U(y) \approx u_e\left(1 - \frac{U_{deficit}}{u_e}\cos^2\left(\frac{\pi y}{2\delta}\right)\right)$$

- Giving the displacement thickness:

$$\delta^* = \int_{-\delta}^{\delta} 1 - \left(1 - \frac{u_{deficit}}{u_e}\cos^2\left(\frac{\pi y}{2\delta}\right)\right) dy$$

$$\Rightarrow \delta^* = \frac{u_{deficit}}{u_e}\int_{-\delta}^{\delta}\cos^2\left(\frac{\pi y}{2\delta}\right) dy$$



University of BRISTOL

- however

$$\int_{-\delta}^{\delta} \cos^2\left(\frac{\pi y}{2\delta}\right) dy = \text{const.}$$

$$\Rightarrow \delta^* = \frac{u_{deficit}}{u_e} \times \text{const.}$$

velocity profile

δ^*

- and momentum thickness:

$$\theta = \int_{-\delta}^{\delta} \frac{u(y)}{u_e}\boxed{\left(1 - \frac{u(y)}{u_e}\right)} dy = \boxed{\frac{u_{deficit}}{u_e} \int_{-\delta}^{\delta} \cos^2\left(\frac{\pi y}{2\delta}\right)}\left(1 - \frac{u_{deficit}}{u_e}\cos^2\left(\frac{\pi y}{2\delta}\right)\right) dy$$

$$\theta = \frac{u_{deficit}}{u_e} \int_{-\delta}^{\delta} \cos^2\left(\frac{\pi y}{2\delta}\right) dy - \left(\frac{u_{deficit}}{u_e}\right)^2 \int_{-\delta}^{\delta} \cos^4\left(\frac{\pi y}{2\delta}\right) dy$$

$$\Rightarrow \theta = \underbrace{\frac{u_{deficit}}{u_e} \times \text{const.}}_{\delta^*} - \left(\frac{u_{deficit}}{u_e}\right)^2 \times \text{different const.}$$

- but as we go downstream, the deficit reduces so $\left(\frac{u_{deficit}}{u_e}\right)^2$ becomes very small. Hence:
  - displacement thickness approximately equals Momentum thickness
  - **H_{inf} is therefore 1**
- To integrate the MIE we need to know H for all points between trailing edge and downstream boundary.
- Simplest is to assume linear variation, i.e.

$$H_{average} = \frac{1}{2}(H_{t.e.} + H\infty) = \frac{1}{2}(H_{t.e.} + 1)$$

University of BRISTOL

MIE:

$$\frac{d\theta}{dx} + (H_{average} + 2)\frac{\theta}{u_e}\frac{du_e}{dx} = 0$$

Integrate from t.e. to infinity:

$$\int_{t.e.}^{\infty} \frac{d\theta}{\theta} = -(H_{average} + 2)\int_{t.e.}^{\infty} \frac{du_e}{u_e}$$

$$[\ln(\theta)]_{t.e.}^{\infty} = -(H_{average} + 2)[\ln(u_e)]_{t.e.}^{\infty}$$

$$\ln(\theta_\infty) - \ln(\theta_{t.e.}) = -(H_{average} + 2)\Big(\ln(u_\infty) - \ln(u_{t.e.})\Big)$$

$$\ln\left(\frac{\theta_\infty}{\theta_{t.e.}}\right) = -(H_{average} + 2)\ln\left(\frac{u_\infty}{u_{t.e.}}\right)$$

taking the exponent of both sides:

$$\left(\frac{\theta_\infty}{\theta_{t.e.}}\right) = \left(\frac{u_\infty}{u_{t.e.}}\right)^{-(H_{average}+2)}$$

- But

$$H_{average} + 2 = 2 + \frac{1}{2}H_{t.e.} + \frac{1}{2} = \frac{5 + H_{t.e.}}{2}$$

- So

$$\theta_\infty = \theta_{t.e.}\left(\frac{u_{t.e.}}{u_\infty}\right)^{\left(\frac{H_{t.e.}+5}{2}\right)}$$

- Finally

$$C_D = \frac{2\theta_\infty}{c} = \frac{2\theta_{t.e.}}{c}\left(\frac{u_{t.e.}}{u_\infty}\right)^{\left(\frac{H_{t.e.}+5}{2}\right)}$$

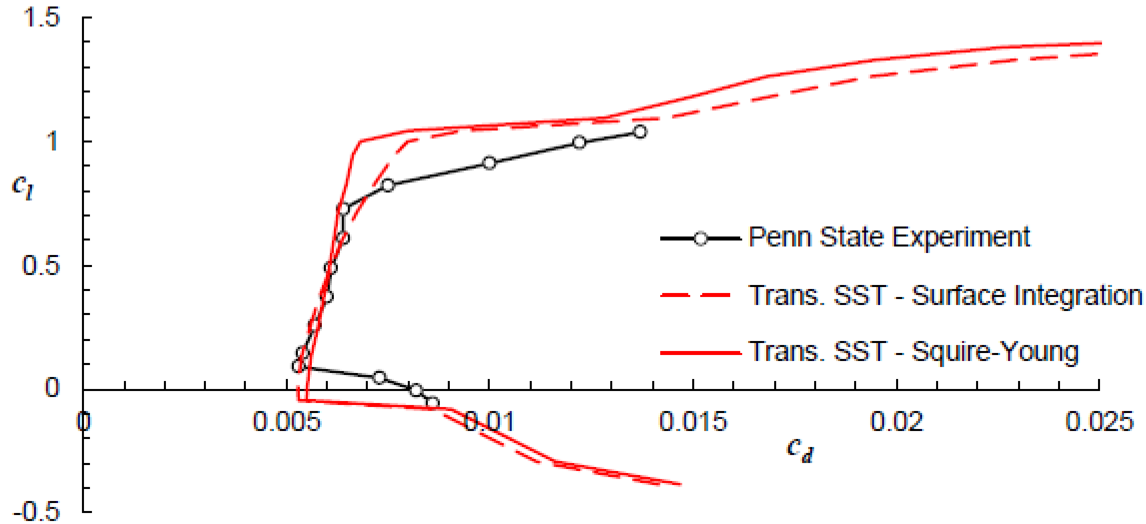
This is a crucial result:

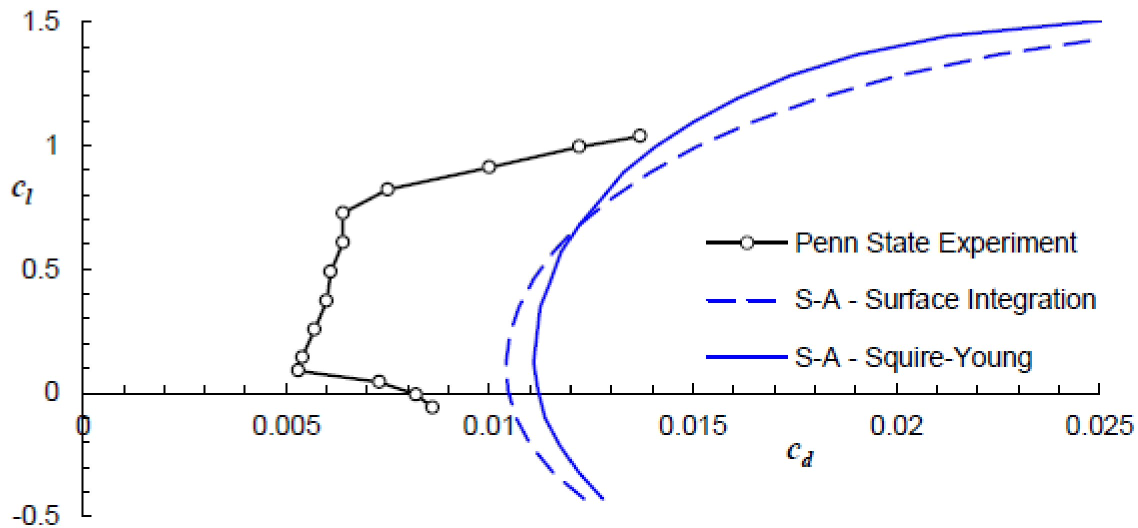
- The drag coefficient depends solely on the properties at the trailing edge
  - no need to calculate the flow in the wake directly
  - no need to attempt to integrate profile or skin friction drags around the aerofoil shape
- Assumes linear H, but this has been shown experimentally to be okay
- Works for small incidence and *no separation!*
- Is independent of flow type
  - i.e. works for *turbulent* or *laminar* B.L. flow

Does it work?

Numerical Validation of the Squire-Young Formula for Profile Drag Prediction

James G. Coder[1] and Mark D. Maughmer[2]
Penn State University, University Park, PA, 16802





(b) Spalart-Allmaras Turbulence Model

University of BRISTOL

Tom Rendall
thomas.rendall@bristol.ac.uk

Aerodynamics 3

Introduction to Turbulent Boundary Layers

(chapter 15 in notes)

Next 8 lectures

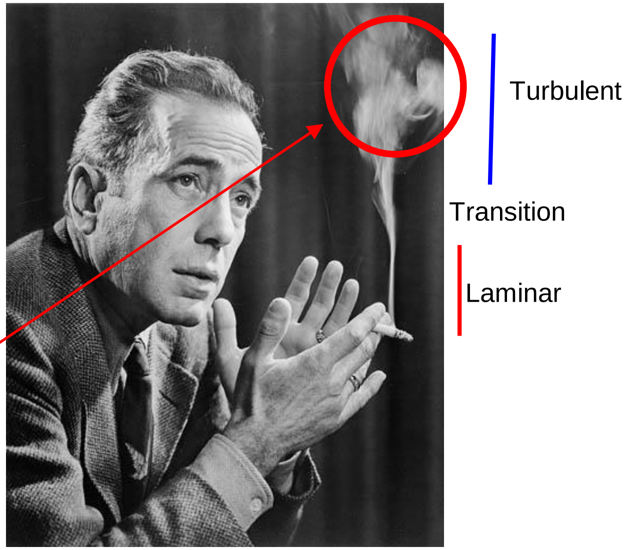
- Turbulent boundary layers + their integral methods
- Transition
- Differential methods
- Practical boundary layer effects

- Panel method (2D) – Xfoil (http://web.mit.edu/drela/Public/web/xfoil/)
- Vortex lattice method – AVL (http://web.mit.edu/drela/Public/web/avl/)
- A round-up of aerodynamic methods

University of BRISTOL

Turbulence

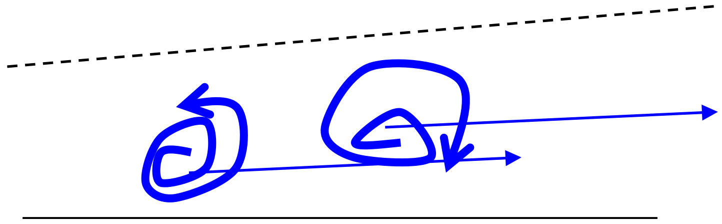
- Turbulence is very complex
  - rapid fluctuations
  - ‘random’ motion
  - **many scales of interactions**



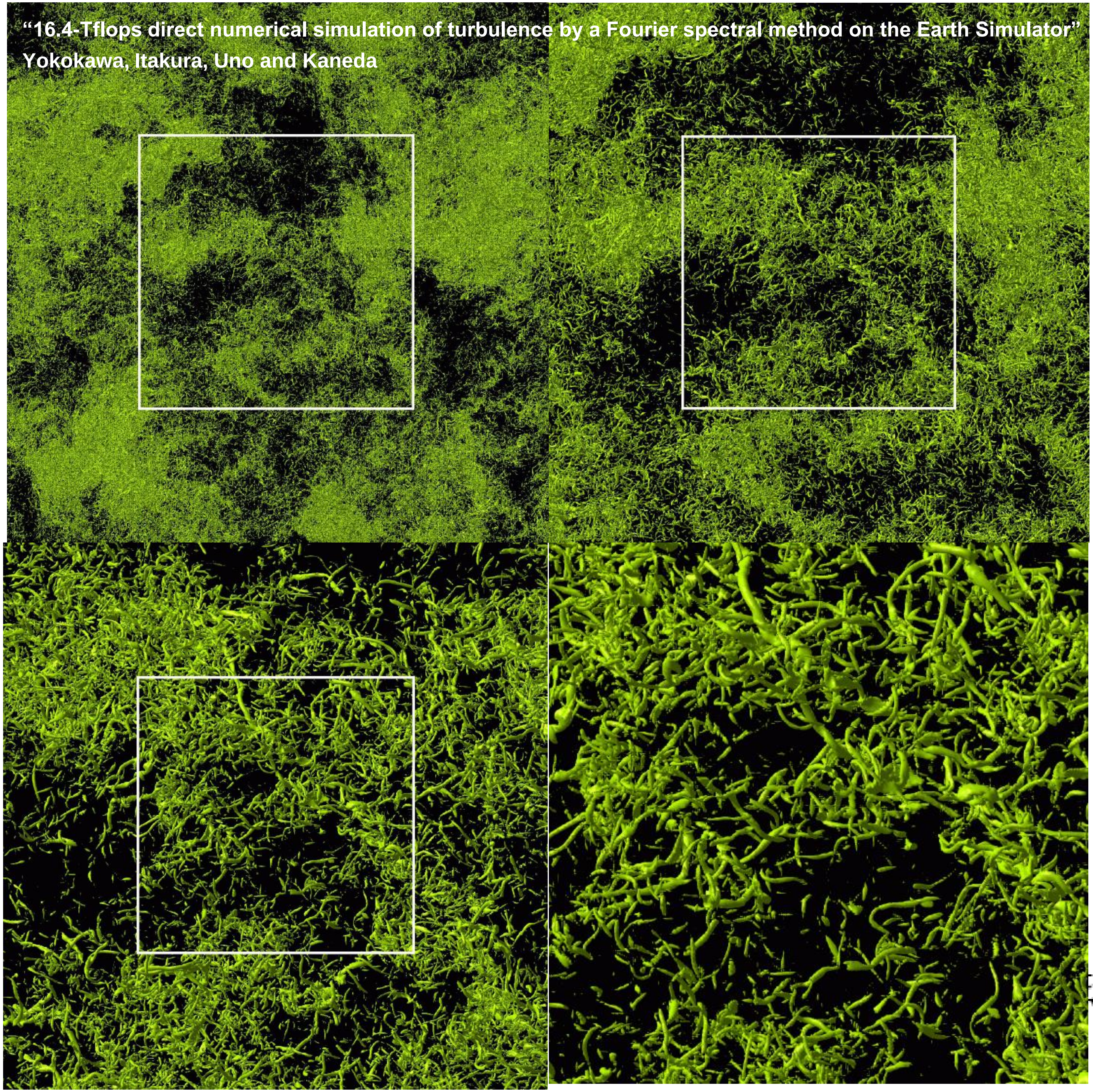
- In turbulent flows there are rapid fluctuations locally even in steady flows
- Last great scientific challenge in aerodynamics? Fully resolving turbulence with CFD at realistic Re unlikely – so the challenge may be more to do with dealing with the uncertainty rather than removing it

What causes turbulence?

- Basically, turbulence is caused by the presence in the flow of *turbulent eddies*



- Fortunately these are cohesive
- Occur when inertial forces dominate viscous dissipation, i.e. high Reynolds numbers. Will return to the process of change between the two (*transition*) later in the course.


"16.4-Tflops direct numerical simulation of turbulence by a Fourier spectral method on the Earth Simulator"
Yokokawa, Itakura, Uno and Kaneda

‘Turbulence’

- Attached flows may have laminar or turbulent boundary layers. Usually, aircraft operate with attached flows and turbulent boundary layers
- Flow separation (such as when stalling) is separate, but it does also result in large scale turbulence. However, **turbulence does not imply separation**!

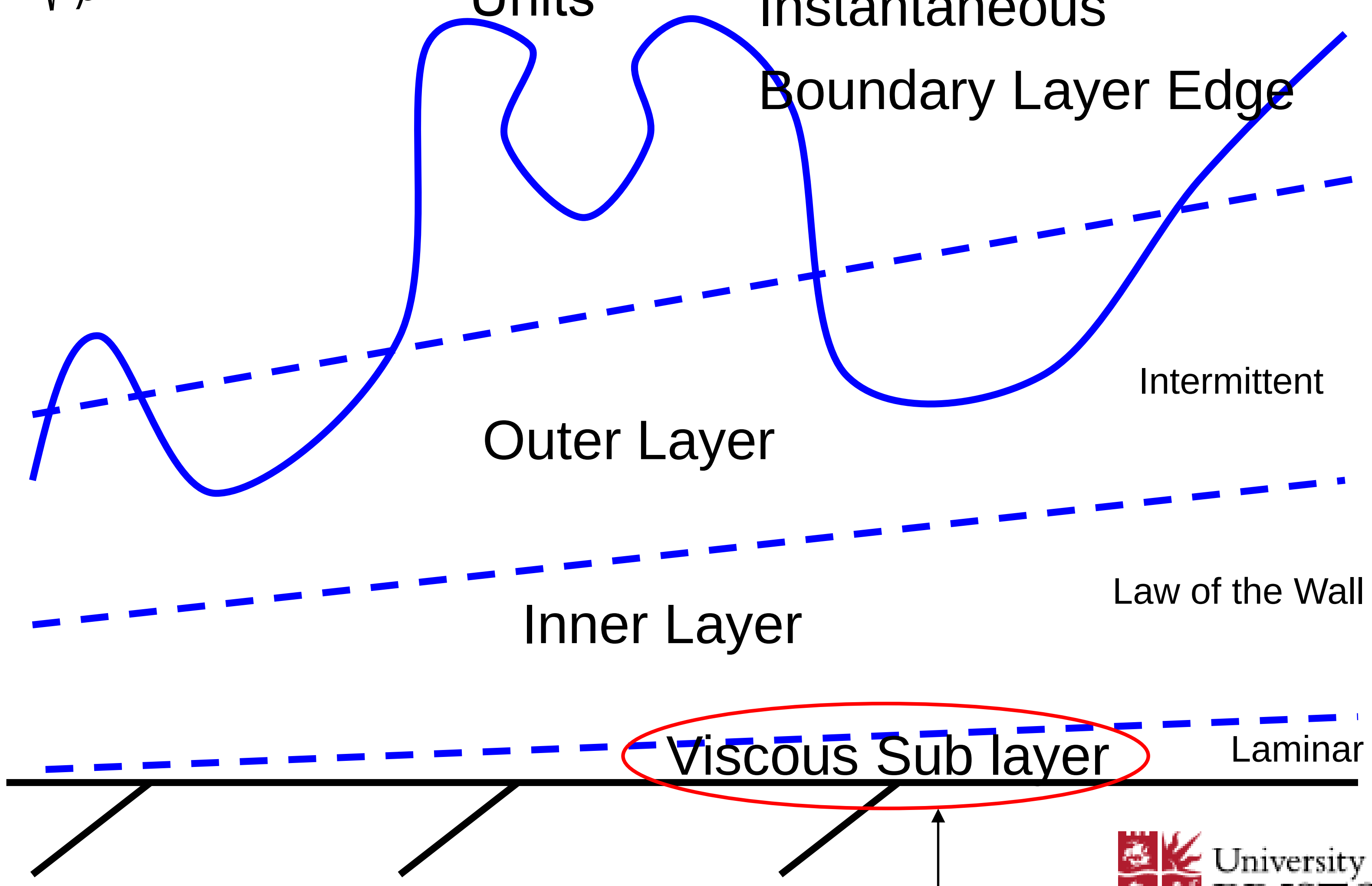
- ‘Turbulence’ and ‘wake turbulence’ are often not really turbulence at all. Atmospheric ‘turbulence’ may consist of large roughly uniform air currents, and ‘wake turbulence’ consists of clearly identifiable vortices. The exception is w the vortices break down, or where there is shear between currents…

University of BRISTOL

Structure of a Turbulent Boundary Layer

$$u_t = \sqrt{\frac{\tau_w}{\rho}} \qquad y^+ = \frac{y u_t}{\nu}$$

'Wall Units'



v=0 and perturbations in v, called v'=0 too

Viscous sub layer:

- Very near the wall velocity oscillations tend to zero, hence turbulent effects not significant, and boundary layer behaves in a laminar fashion
- Very thin region, ~0.01 of bl thickness

Inner Layer:

- Dominated by turbulent eddies, turbulent stress >> viscous stress. Extends to about 0.4 bl thickness

Outer Layer:

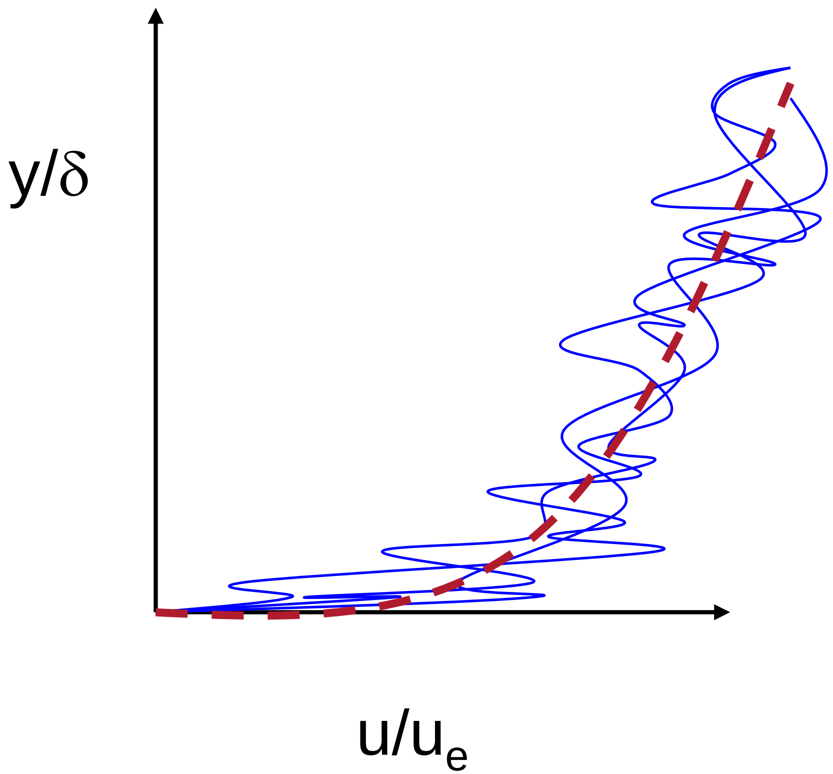
- Contains bl edge, extends from 0.4 to 1.2 bl.
- Characterised by *intermittency*

-'sometimes turbulent, sometimes not'

Time Scales:

- The turbulent eddies introduce very small variations in each velocity component (i.e. *u*,*v*)
- These are constantly changing, but do so much faster than e.g. the time scale of a smooth manoeuvre.
- Hence, we do not need to know exactly what’s going on, only what effect these fluctuations have over a certain time scale.

E.g. A velocity Profile



- At any time t, the velocity profile has some distribution with height
- At t+Δt, it is slightly different
- However, all we need to know is what the average profile is

The Turbulent Boundary Layer Equations

- As with laminar boundary layers, we have

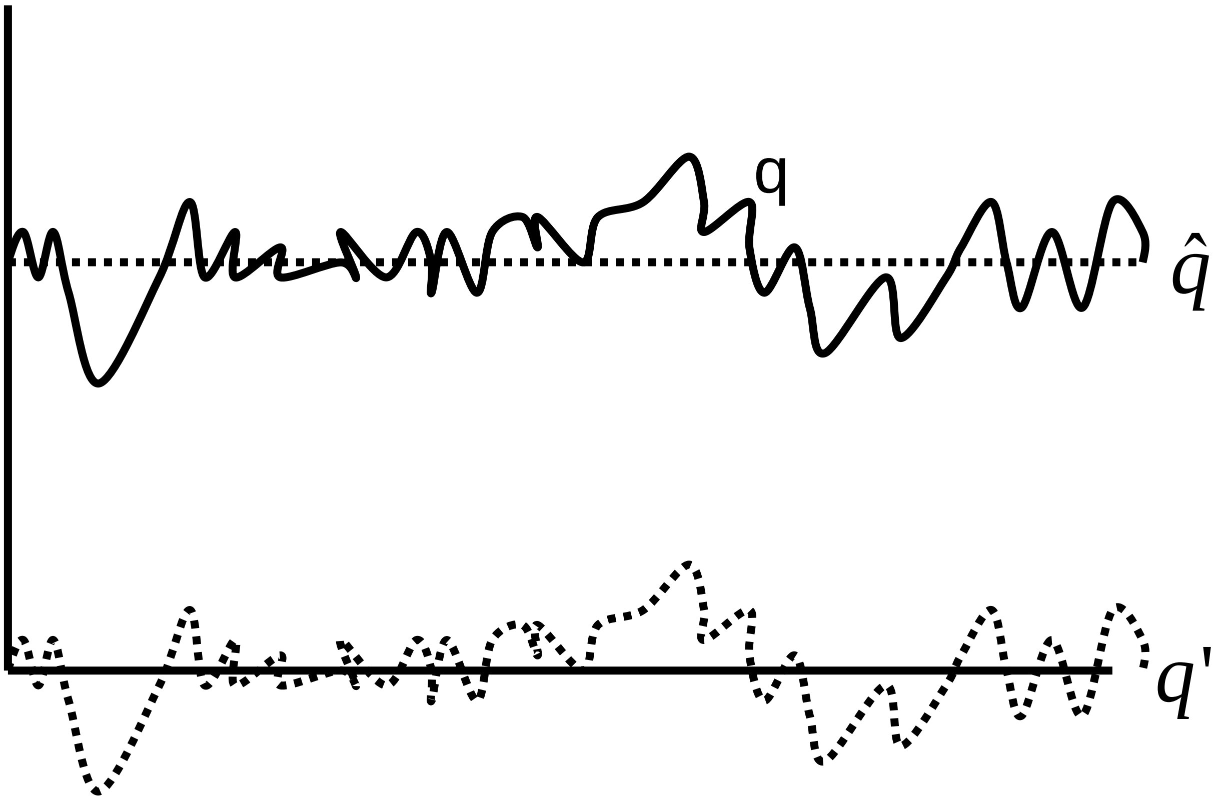
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

- But now we represent the flow properties *u,v* etc. as a mean plus turbulent fluctuation, i.e.

$$u=\hat{u}+u'$$

Mean and Fluctuating Components



integral of q’ over time is 0

Continuity Equation

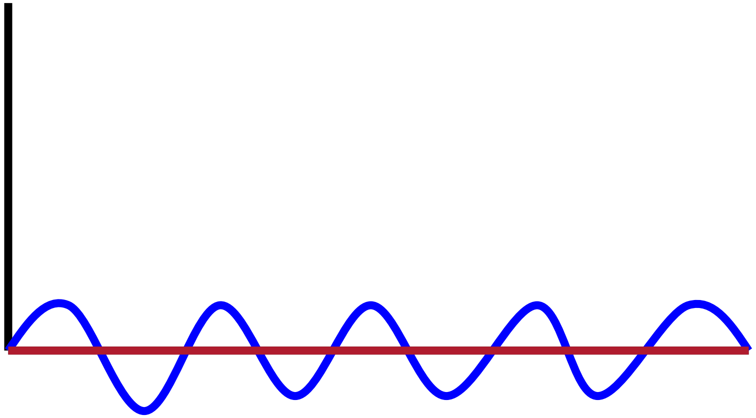
- Substituting in these variables into the continuity equation gives:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial \hat{u}+u'}{\partial x}+\frac{\partial \hat{v}+v'}{\partial y}=0$$

- If we average over some time much longer than the timescale of the fluctuation, but less than that of anything we are interested in (denote this by an overline) then -

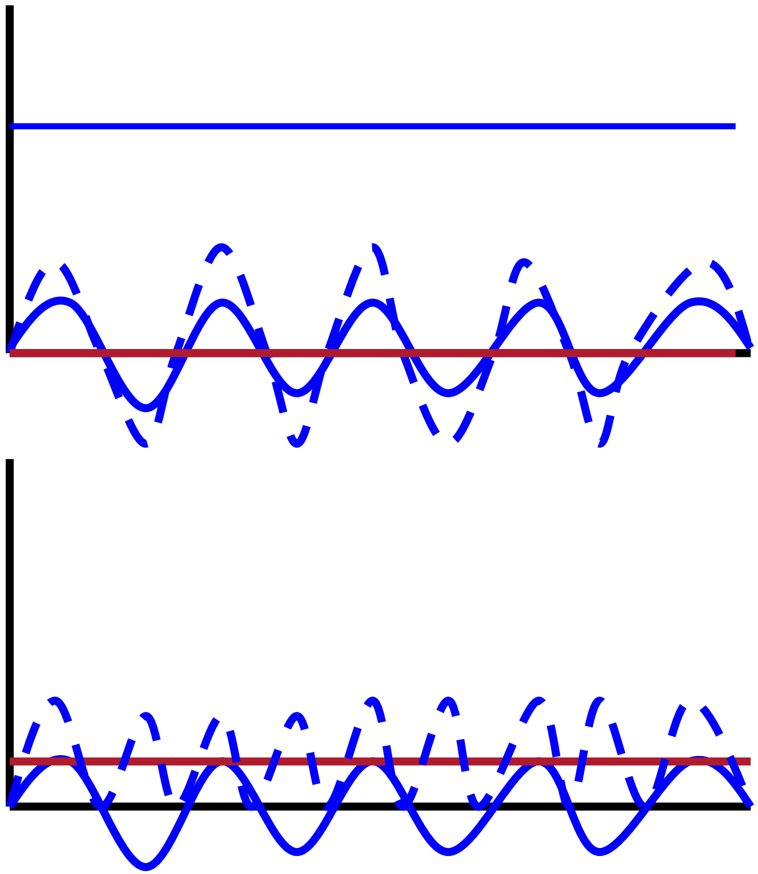
Properties of time averaged variables:

$$\overline{\hat{q}} = \hat{q}$$



$$\overline{q'} = 0$$

Properties of time averaged variables:



$$\overline{\hat{q}s'} = 0$$

$$\overline{q's'} \neq 0$$

Finally

$$\frac{\overline{\partial \hat{q}}}{\partial x} = \frac{\partial \overline{\hat{q}}}{\partial x} = \frac{\partial \hat{q}}{\partial x}$$

Mean of a gradient = gradient of a mean

and

$$\frac{\overline{\partial q'}}{\partial x} = \frac{\partial \overline{q'}}{\partial x} = 0$$

Time averaging and differentiation are commutative

So, returning to the continuity equation, and time averaging:

$$\overline{\frac{\partial \hat{u} + u'}{\partial x}} + \overline{\frac{\partial \hat{v} + v'}{\partial y}} = 0$$

$$\Rightarrow \quad \overline{\frac{\partial \hat{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} + \overline{\frac{\partial \hat{v}}{\partial y}} + \overline{\frac{\partial v'}{\partial y}} = 0$$

but, using our rules, the time average of the fluctuating differentials are zero, hence

$$\Rightarrow \quad \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0$$

University of BRISTOL

Similarly, we can show that the x-momentum equation becomes

$$\hat{u}\frac{\partial \hat{u}}{\partial x}+\hat{v}\frac{\partial \hat{u}}{\partial y}=-\frac{1}{\rho}\frac{d\hat{p}}{dx}+\frac{\partial}{\partial y}\left(\nu\frac{\partial \hat{u}}{\partial y}-\overline{u'v'}\right)-\frac{\partial \overline{u'^2}}{\partial x}$$

where the last two terms are called *Reynolds Stresses*. We sometimes ignore the last one due to the thin layer assumption (=gradients in normal direction dominant).

Overall, the continuity and momentum equations are the same in turbulent flow as laminar, as long as we use time averaged values, and include this extra term involving *u'v'.*

We now have the RANS equations (Reynolds Averaged NS).

University of BRISTOL

The Boussinesq Hypothesis:

- The extra stress acts just like a viscous stress, but is a function of local flow variables. This is in analogy with how the momentum transfer caused by the molecular motion in a gas can be described by a molecular viscosity i.e.

$$\tau_t = -\rho\overline{u'v'} = \mu_t \frac{\partial \hat{u}}{\partial y}, \qquad \mu_t = F(u, v, etc.)$$

- In turn this means that the boundary layer equations become (dropping the hat symbol, it is implicit that time averaged values are meant)

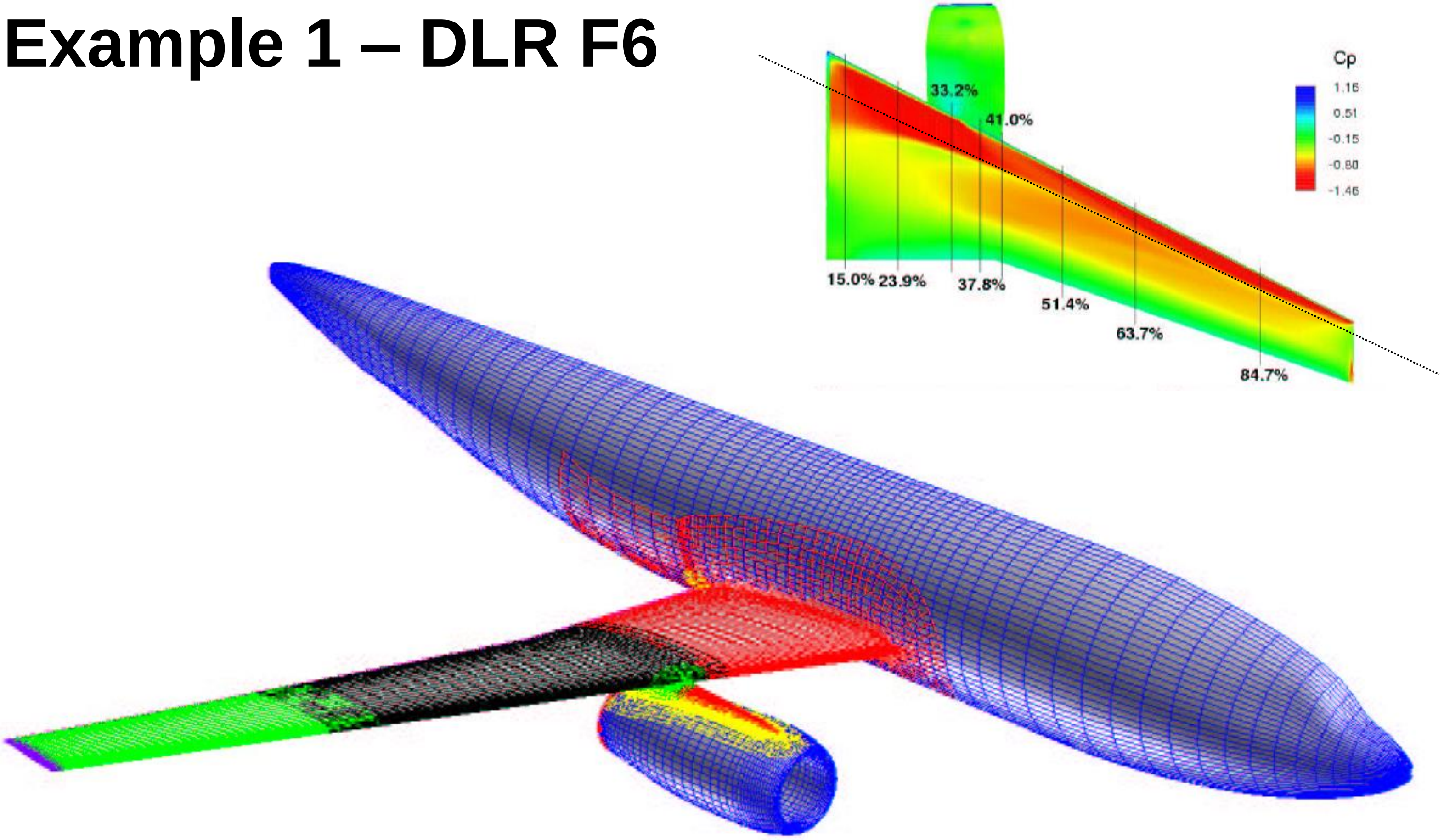
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + (\nu + \nu_t)\frac{\partial^2 u}{\partial y^2}$$

This is almost always how turbulence modelling is done in CFD!

Finally…

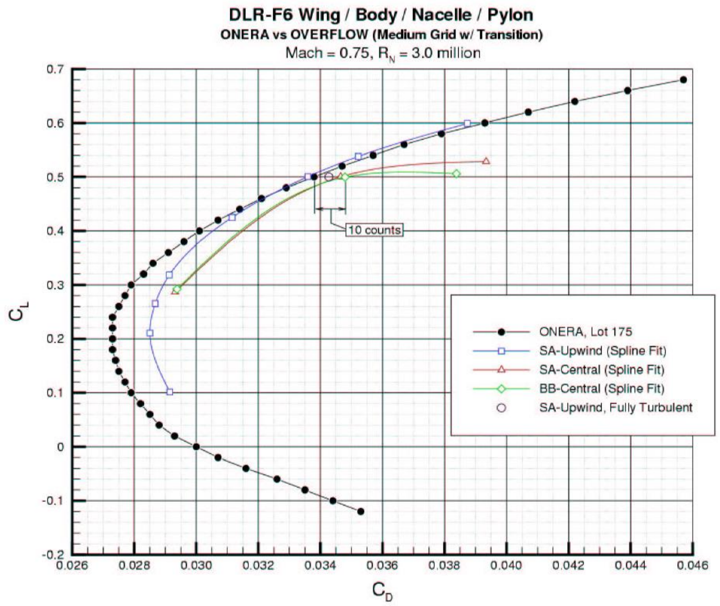
- The problem now is that we have more unknowns than equations. Finding additional closing equations (like $\mu_t = F(u, v, etc.)$) that give 'good' results is the field of turbulence modelling. The additional equations often (but not always) mimic physical ones and are largely empirical. Sometimes further unknowns are introduced to model more complicated effects.
- Next time…use integral boundary layer method for turbulent boundary layers

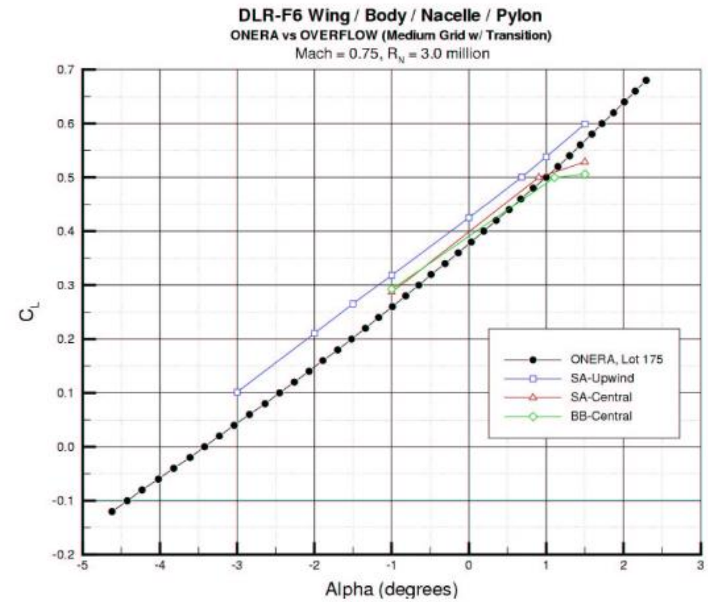
Example 1 – DLR F6

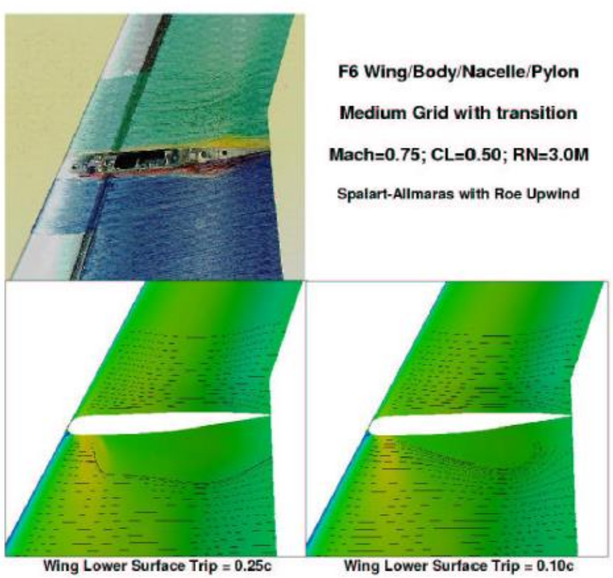


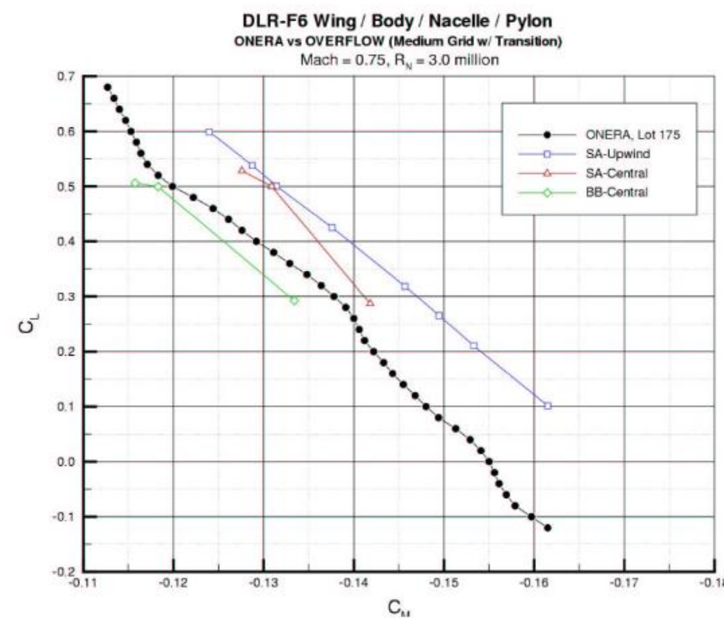
From paper AIAA 2004-393

University of BRISTOL


DLR-F6 Wing / Body / Nacelle / Pylon
ONERA vs OVERFLOW (Medium Grid w/ Transition)
Mach = 0.75, R_N = 3.0 million
10 counts
ONERA, Lot 175
SA-Upwind (Spline Fit)
SA-Central (Spline Fit)
BB-Central (Spline Fit)
SA-Upwind, Fully Turbulent
C_L
C_D


DLR-F6 Wing / Body / Nacelle / Pylon
ONERA vs OVERFLOW (Medium Grid w/ Transition)
Mach = 0.75, R_N = 3.0 million
ONERA, Lot 175
SA-Upwind
SA-Central
BB-Central
C_L
Alpha (degrees)

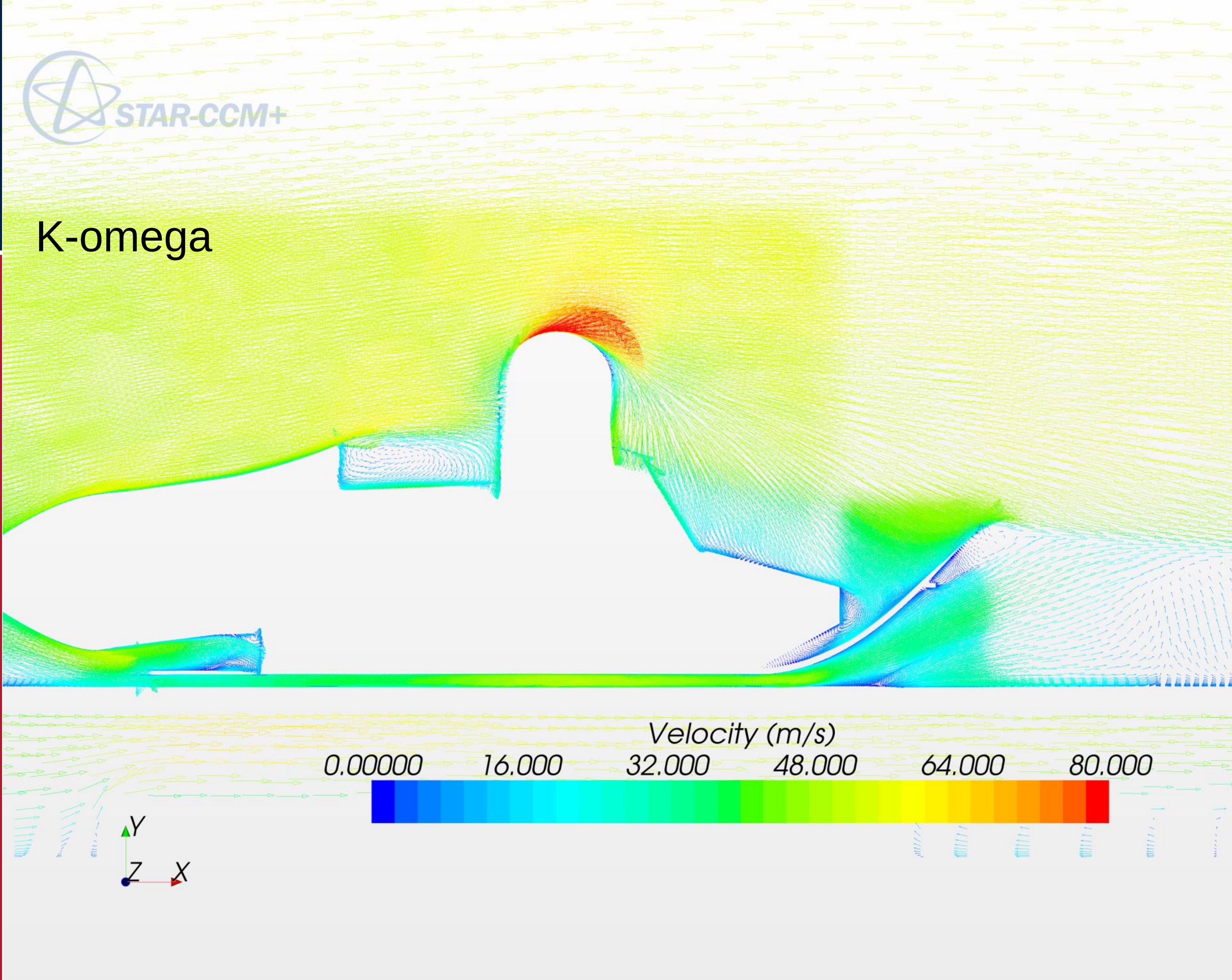

F6 Wing/Body/Nacelle/Pylon
Medium Grid with transition
Mach=0.75; CL=0.50; RN=3.0M
Spalart-Allmaras with Roe Upwind
Wing Lower Surface Trip = 0.25c
Wing Lower Surface Trip = 0.10c


DLR-F6 Wing / Body / Nacelle / Pylon
ONERA vs OVERFLOW (Medium Grid w/ Transition)
Mach = 0.75, R_N = 3.0 million
ONERA, Lot 175
SA-Upwind
SA-Central
BB-Central
C_L
C_M

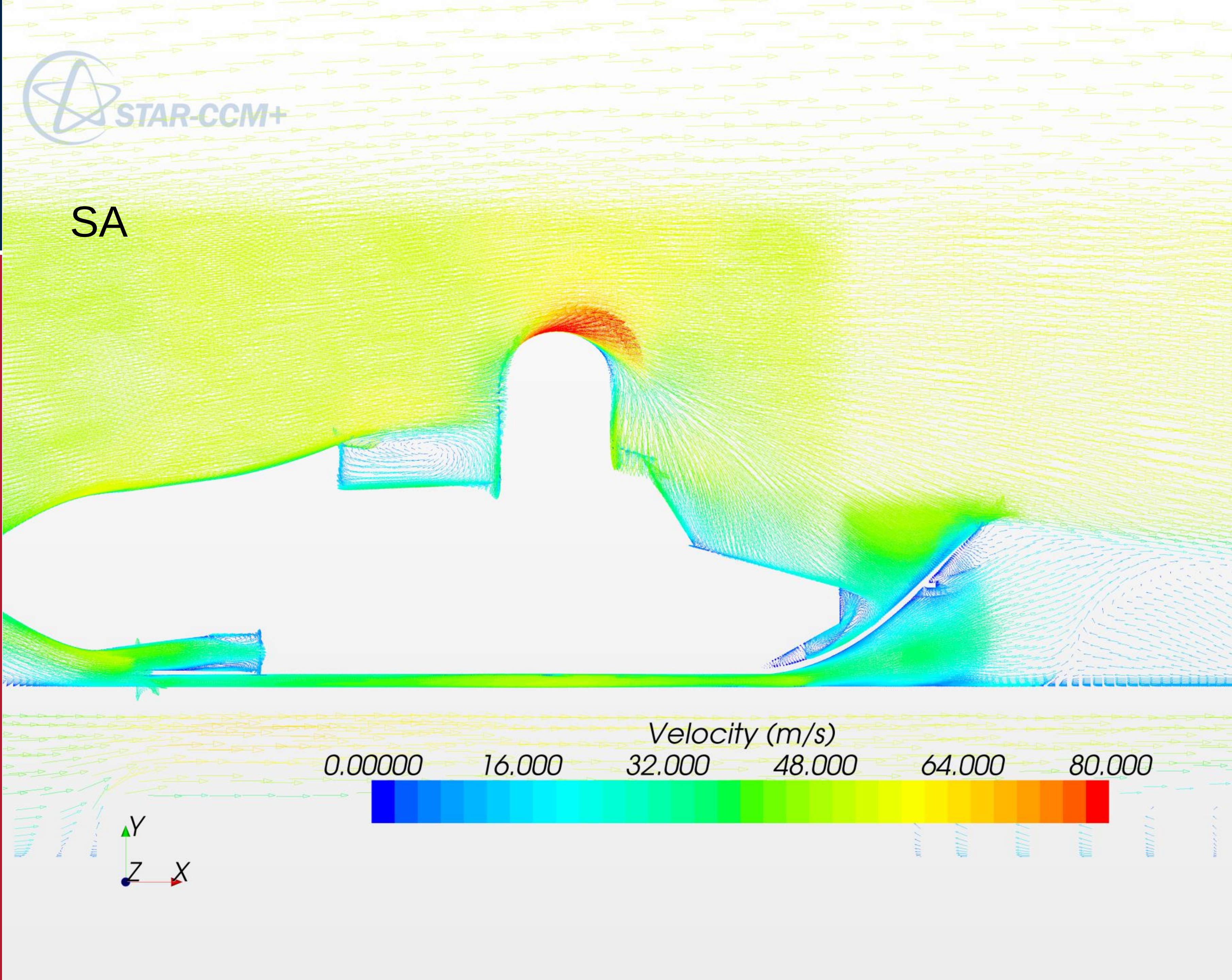
Example 2 – Small Racecar

- Full NS model + turbulence model
- Simulated at wind tunnel Re
- Variations in the following flows are only a result of differing turbulence models
- Similar effects carry through to the forces (eg drag, downforce)

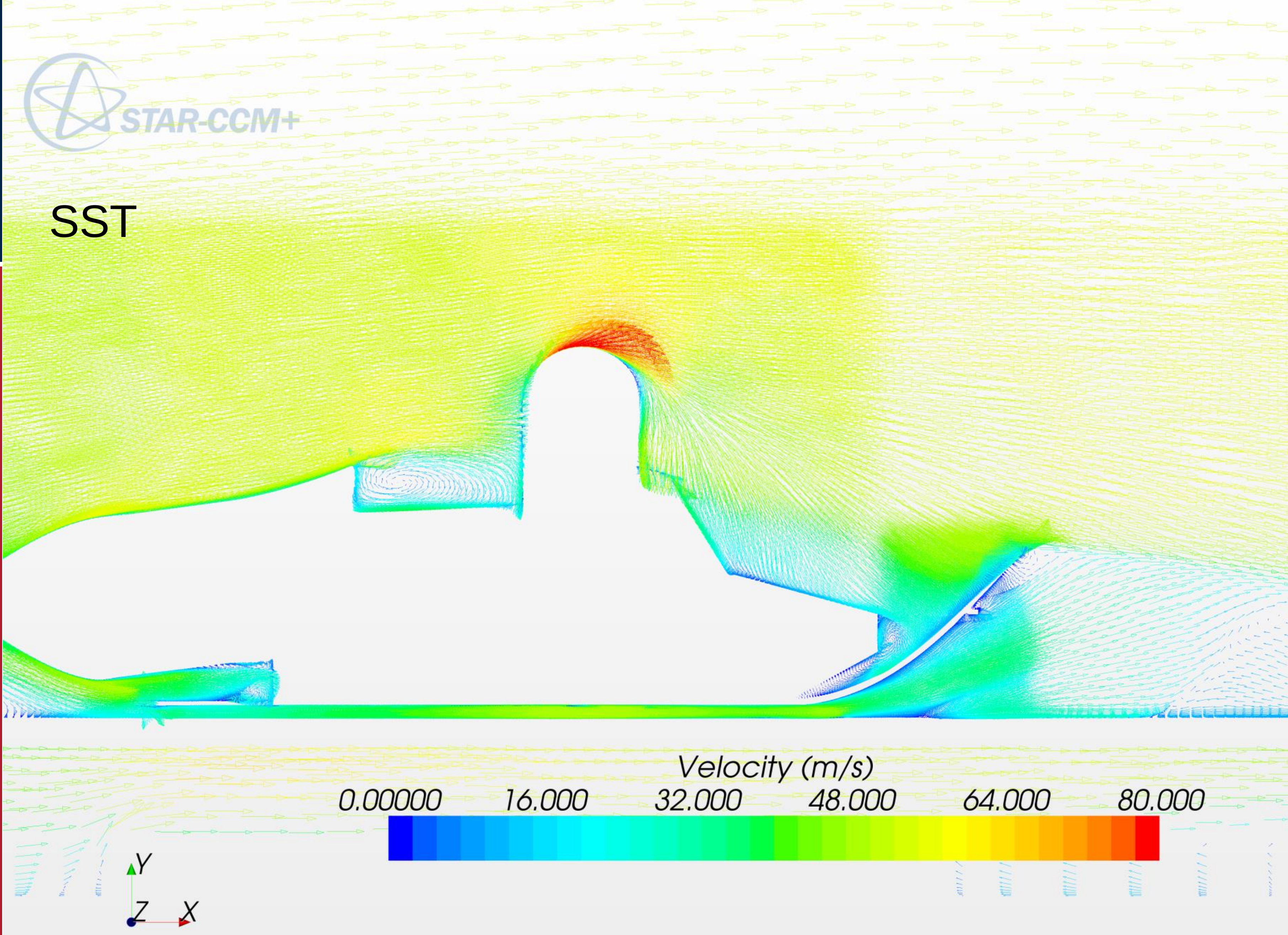
K-omega


STAR-CCM+
Velocity (m/s)
0.00000
16.000
32.000
48.000
64.000
80.000
Y
Z
X

SA


STAR-CCM+
Velocity (m/s)
0.00000
16.000
32.000
48.000
64.000
80.000
Y
Z
X

SST


STAR-CCM+
Velocity (m/s)
0.00000
16.000
32.000
48.000
64.000
80.000
Y
Z
X

Example 3

$$V_{crit} = \sqrt{\frac{2T_{max}y_{eng}}{\rho C_{L_{max}} S_{ref} x_{fin}}}$$

A320 extimates are $T_{max} = 120kN$, $S_{ref} = 22.56\text{m}^2$

$$x_{fin} = 17.79\text{m}, \; y_{eng} = 5.68\text{m}$$

$$V_{crit} = \sqrt{\frac{2 \times 120 \times 10^3}{1.225 \times 1.7 \times 22.56 \times 17.79}} = 40.4\text{ms}^{-1} = 78.5\text{kts}$$

This rises to 43.0ms^{-1} (83.6kts) if $C_{L_{max}}$ drops to 1.5.

University of BRISTOL

Which means…

Crude constant acceleration working…and no crosswind

Takeoff acceleration $\frac{69.4^2}{2\times 2090} = 1.15\text{ms}^{-2}$ $v^2 = u^2 + 2as$

Distance to gain the extra speed $\frac{43.0^2 - 40.4^2}{2\times 1.15} = 94\text{m}$, or about 309ft

Acceleration drops with speed, so the real distance is above this value

0.2 difference? Surely not?

- By the time of 1st flight, Clmax for the fin will be known to an accuracy much better than this – there’s no uncertainty on the day
- The trouble is, during design there is uncertainty, and limited tunnel results. For CFD - which turbulence model to use? Where is transition taking place? These all have a substantial impact
- This means one may end up with an over- or undersized fin. Typically conservatism rules and the fin is somewhat too large, and the aircraft is a little heavier than needed

University of BRISTOL

(a) By plotting y against $\frac{u}{u_e}$ sketch the velocity profile through a typical laminar boundary layer. On the same graph, sketch the time-averaged velocity profile through a typical turbulent boundary layer. Make sure to indicate any difference in the overall thickness of each layer by how you draw each curve. Finally, on a separate graph sketch a set of possible instantaneous velocity profiles through a turbulent boundary layer.

[3 marks]

(b) Consider the equation for x-momentum in an incompressible boundary layer

$$\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

By decomposing the pressure and velocity components into a constant time-averaged part ($\widehat{p}$, $\widehat{u}$ and $\widehat{v}$) and a time varying part (p', u' and v'), so that $p = \widehat{p} + p'$, $u = \widehat{u} + u'$ and $v = \widehat{v} + v'$, find the equivalent time-averaged x-momentum equation. Take special care to indicate which terms vanish and why, as well as which new terms appear. What are these new terms called?

[6 marks]

(c) Now consider the partial differential equation

$$\frac{\partial z}{\partial t} + k\frac{\partial^2\left(z^2\cos(z)\right)}{\partial x^2} = 0$$

where k is a constant. Again, by decomposing z into a constant time-averaged part $\widehat{z}$ and a time varying part z' so that $z = \widehat{z} + z'$, find the equivalent time averaged equation. Take special care to indicate which terms vanish and why.

HINT: You should first use $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and you may then assume that z' is small such that $\cos(z') \approx 1$ and $\sin(z') \approx z'$.

[8 marks]

(d) State what approach is used to determine the extra unknown variables introduced in part (**Q5**b) through the time averaging. Using an analogous idea, suggest a suitable additional equation that would permit solution of the partial differential equation in part (**Q5**c).

DO NOT attempt any integration; just suggest an additional equation.

[3 marks]

Tom Rendall

thomas.rendall@bristol.ac.uk

Aerodynamics 3

Integral Methods for Turbulent Boundary Layers

(chapter 16 in notes)

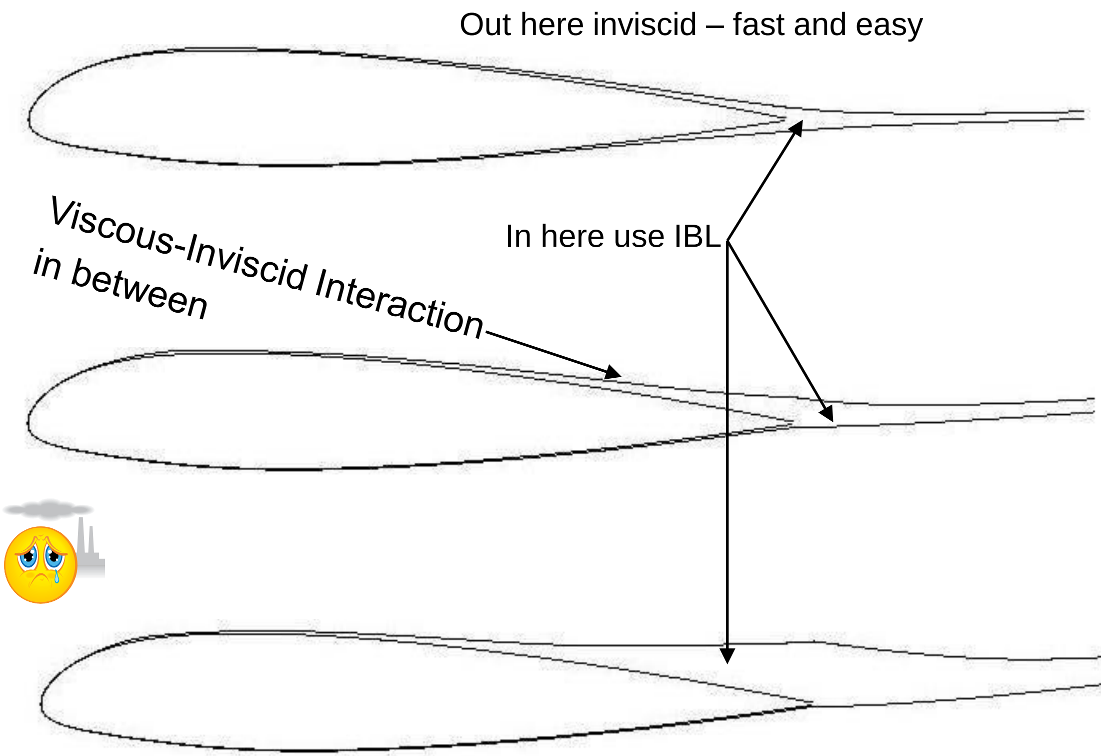
University of BRISTOL

Why integral boundary layer methods? (laminar, turbulent or otherwise!)

Engineers like

methods that are:

1) Fast
2) Accurate
3) Always applicable

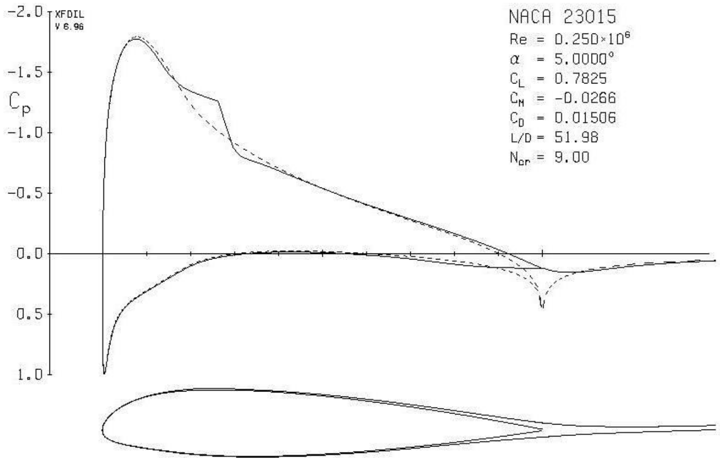


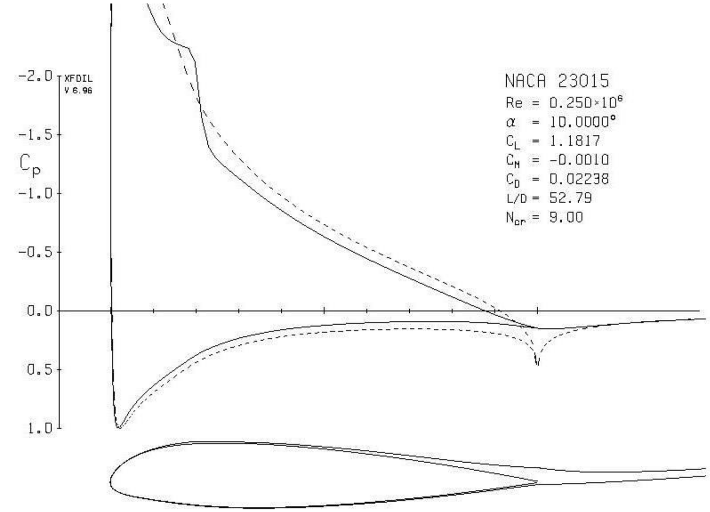
If you couple a simple inviscid panel method to an IBL method, it is fast and accurate, but not always applicable

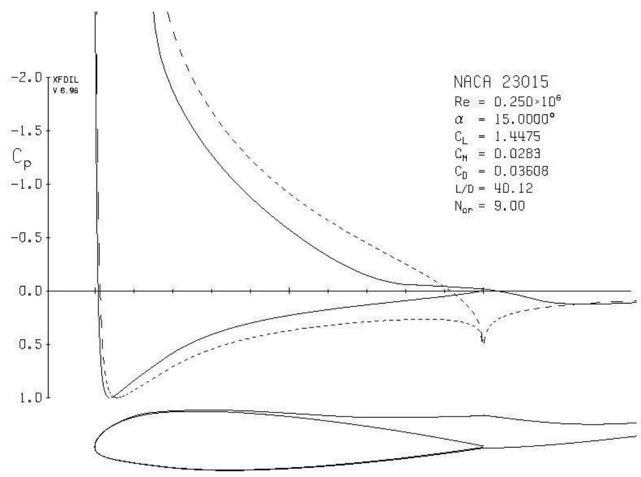
2/3 is not bad! **When it works** IBL beats full CFD easily…

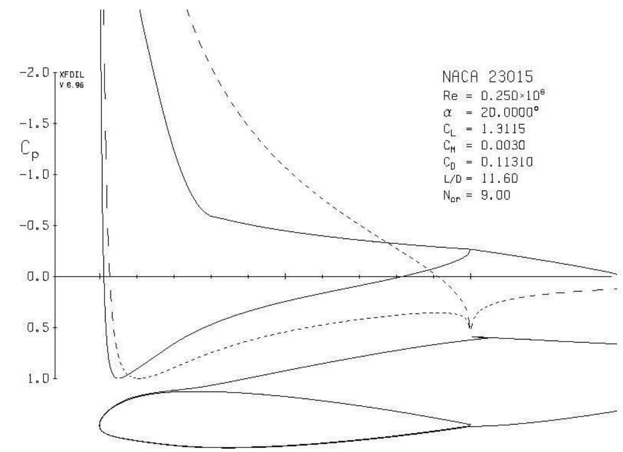
University of BRISTOL

Xfoil – 2D panel method + IBL (Incompressible)



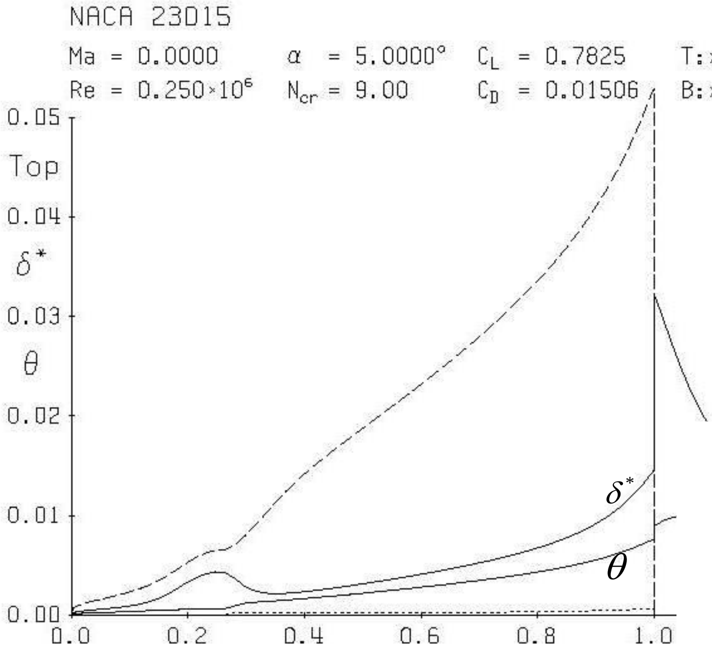


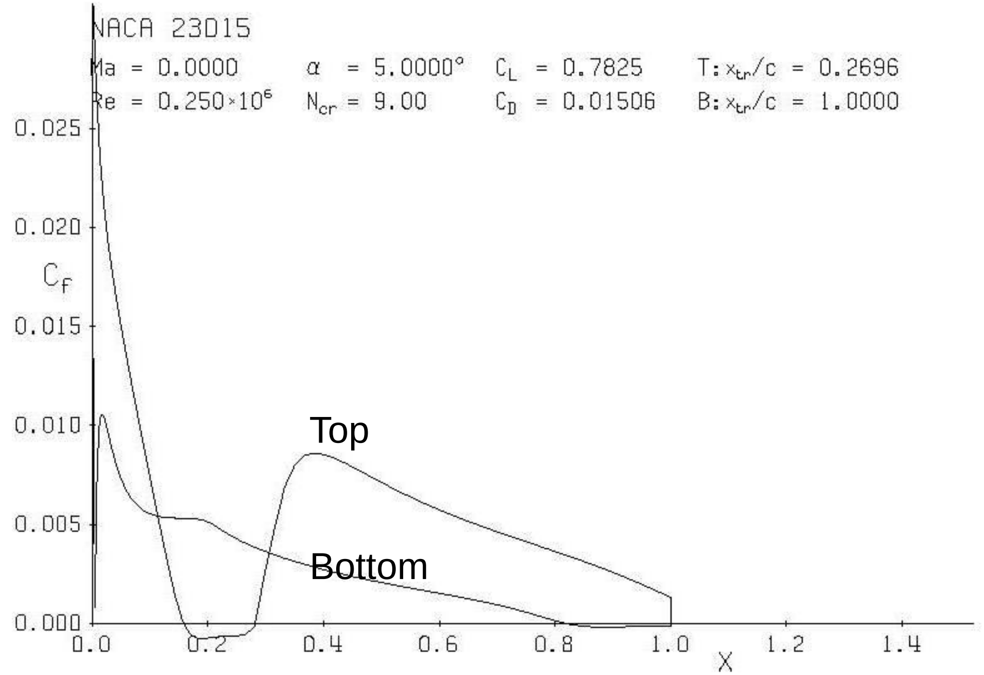




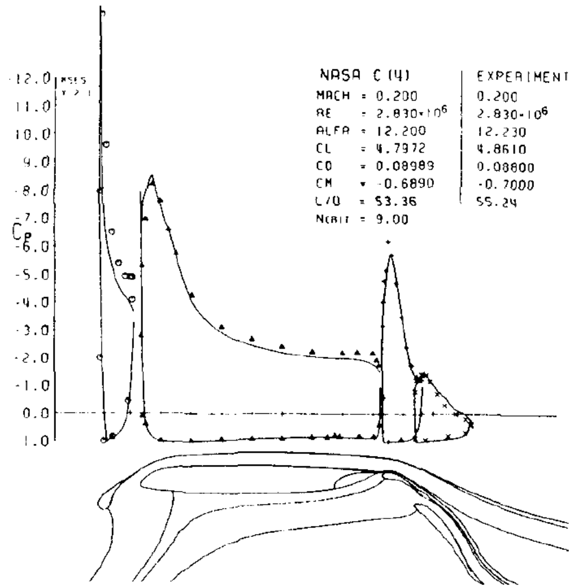
University of BRISTOL

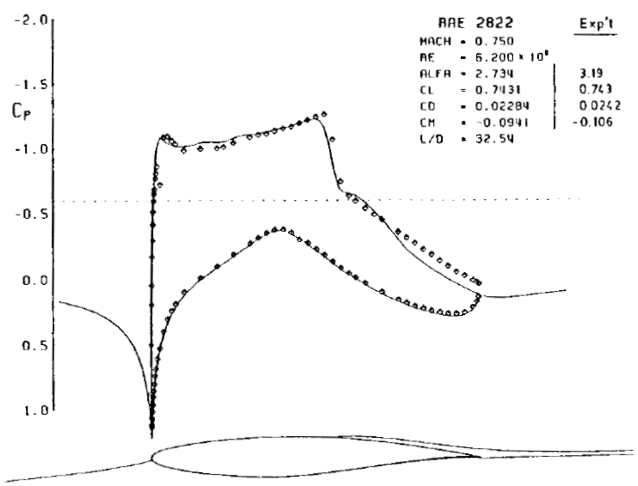
Boundary layer plots

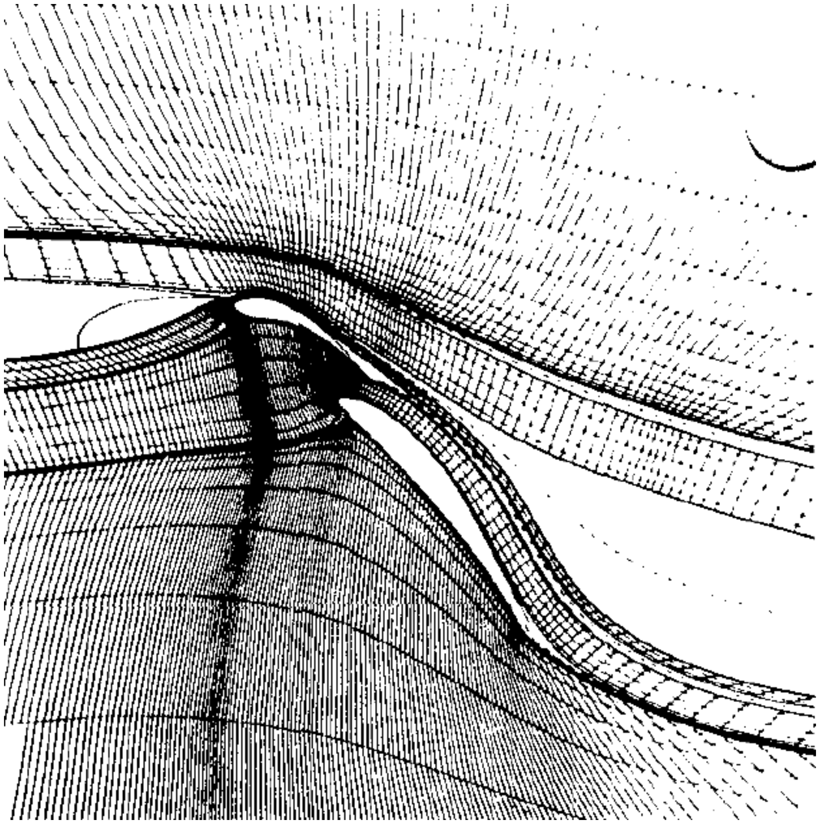




MSES 2D Euler streamtube + IBL (Compressible)



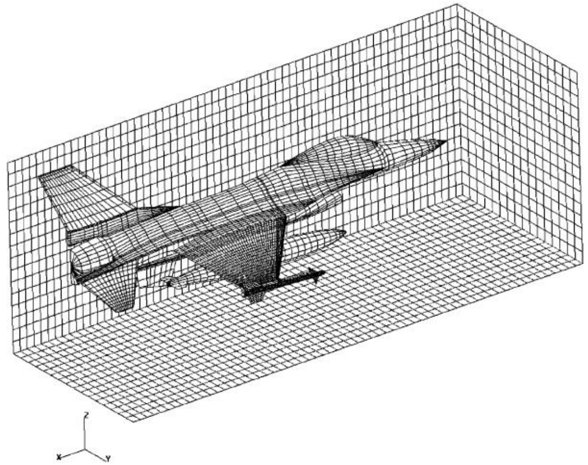


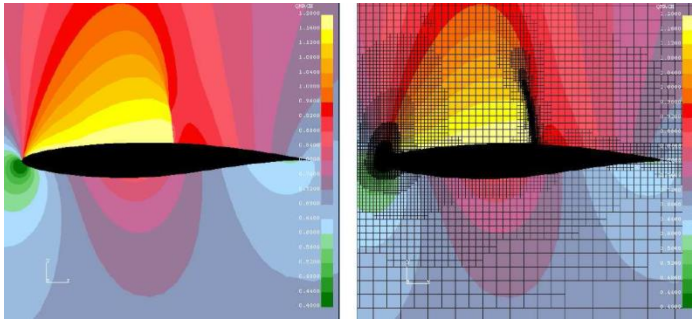


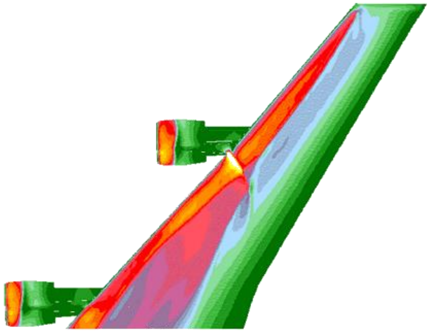
Like Xfoil, but compressible and handles multiple elements

University of BRISTOL

TRANAIR (Compressible 3D full potential + IBL)







Used **very** extensively by Boeing for cruise design

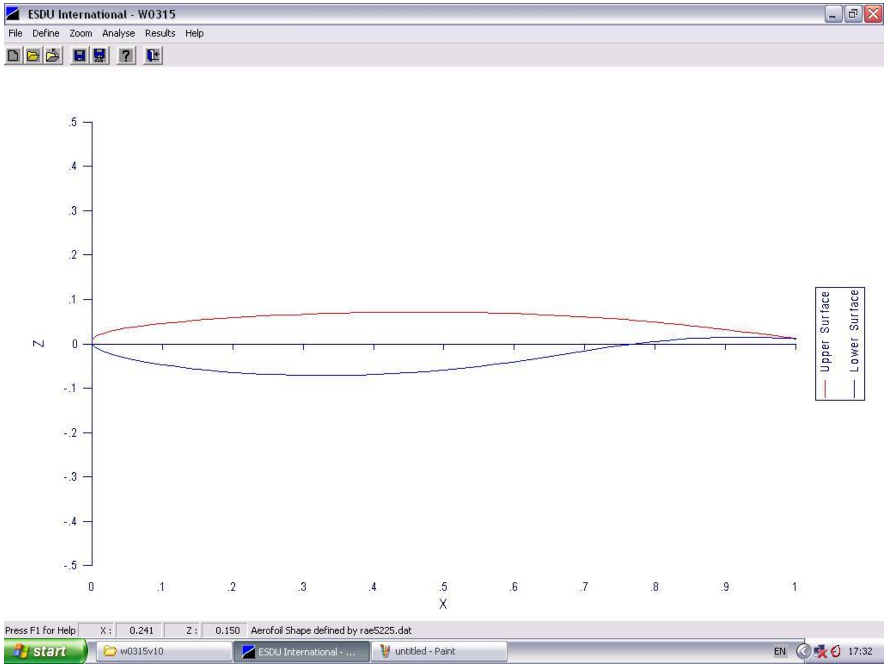
Same BL formulation to Xfoil/MSES, but modified for 3D

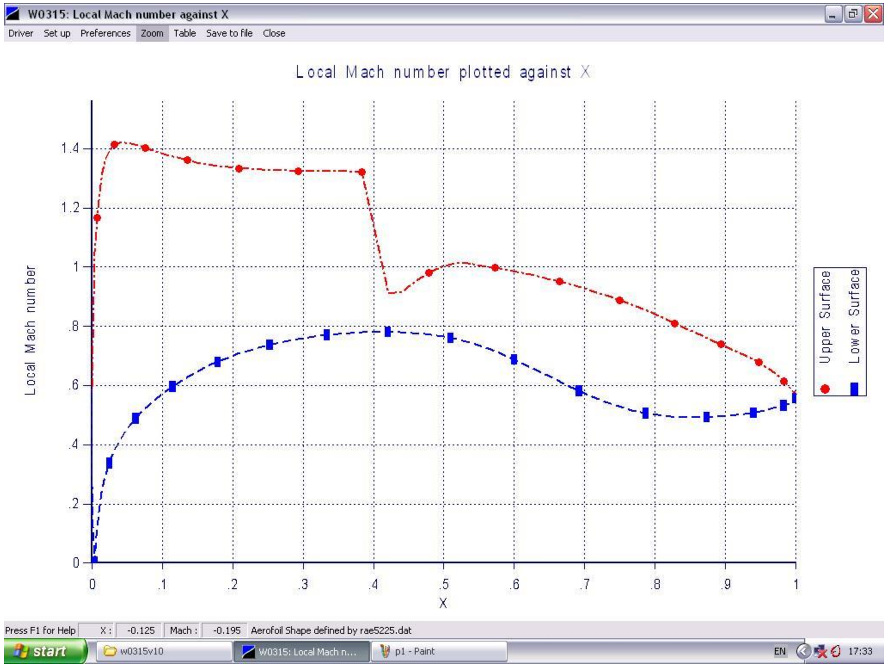
Airbus also has an IBL version of Flite3D (Euler)

3D is a **big** restriction on IBL methods

University of BRISTOL

VGK (2D full potential + IBL, compressible)





Windows .exe available through ESDU sheet 96028

Very fast! ~ immediate answers

Access via UoB Metalib

So…

- If we want a viscous-inviscid interaction (VII) model to work, we need simple-ish rules to find the boundary layer properties – like thickness
- It may be laminar **or** turbulent, so we need a model for **each**, and a model to **choose** one or the other (transition) depending on Re, pressure gradients etc.

Turbulent Boundary Layers

- Last week introduced the concept of turbulence, and looked at how we can realistically model its effect
  - i.e. in terms of mean variables, not the fluctuations
- Demonstrated that provided we deal in these mean variables, we get equations that are remarkably similar to laminar for boundary layer flows

Turbulent Boundary Layers

- Also considered the general structure of a boundary layer
  - i.e. viscous sub layer
  - inner layer
  - outer layer
- However, this just gives us a framework to operate in, it doesn’t tell us anything useful about a *specific* flow
- To do this we need methods to *solve* the equations

University of BRISTOL

Finding Solutions

- As with laminar flows, we have two options, differential or integral. For the usual reasons, we will look at integral first
  - quicker, simpler to use
- Due to the greater complexity of the flow, formal analytical solutions do not generally exist, and we shall be considering mainly empirical methods. In particular, we shall concentrate on Power Law methods

Power Law Methods

- Simplest approximation is to assume a velocity profile.
- from empirical data, it has been found that a *power law* gives a good approximation to a turbulent boundary layer under zero pressure gradient
  - i.e. scale the boundary layer with respect to local height, δ, through the boundary layer variable $\eta = y/\delta$
  - The velocity ratio U/U_e is a power of η

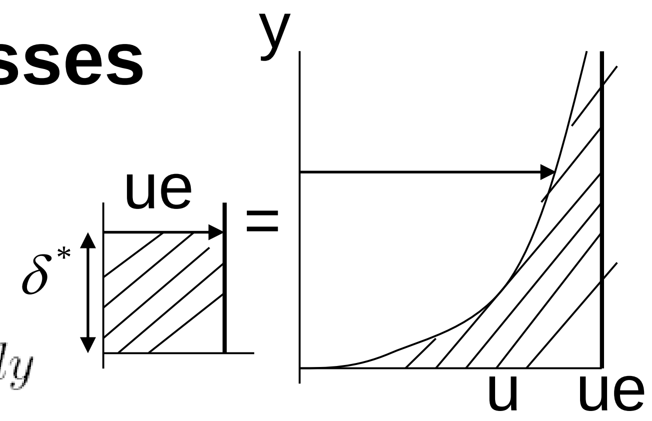
University of BRISTOL

Power Law Methods

- Developed from pipe flow results
  - implicitly assumes velocity profile is *similar* (like Blasius)
  - Hence only really applicable to zero pressure gradient flows
  - Has *no* theoretical underpinnings, i.e. is entirely empirical (unlike Blasius)
- Most common form is $\frac{u}{u_e} = \eta^{\frac{1}{n}}$
- Where
  - $5 \times 10^5 < Re < 10^7, n = 7$
  - $10^6 < Re < 10^8, n = 9$

University of BRISTOL

Power Law Methods - Thicknesses



- Consider usual integral variables:

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

As we have incompressible flow, $\rho = \rho_e$, and further

$$\eta = \frac{y}{\delta} \Rightarrow \frac{d\eta}{dy} = \frac{1}{\delta} \Rightarrow dy = \delta d\eta$$

we can write this as

Displacement thickness

$$\delta^* = \delta \int_0^1 1 - \frac{u}{u_e} d\eta$$

$$\Rightarrow \frac{\delta^*}{\delta} = \int_0^1 1 - \eta^{\frac{1}{n}} d\eta = \left[\eta - \frac{n}{n+1}\eta^{\frac{n+1}{n}}\right]_0^1 = \left(1 - \frac{n}{n+1}\right)$$

Momentum thickness

$$\theta = \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy$$

$$\Rightarrow \frac{\theta}{\delta} = \int_0^1 \eta^{\frac{1}{n}} - \eta^{\frac{2}{n}} d\eta = \left[\frac{n}{n+1}\eta^{\frac{n+1}{n}} - \frac{n}{n+2}\eta^{\frac{n+2}{n}}\right]_0^1$$

$$\Rightarrow \frac{\theta}{\delta} = \frac{n}{n+1} - \frac{n}{n+2} = \frac{n}{(n+1)(n+2)}$$

University of BRISTOL

however, we have a problem with skin friction:

$$c_f = \frac{\tau_{wall}}{\frac{1}{2}\rho u_e^2}$$

$$\tau_{wall} = \mu\frac{\partial u}{\partial y} = \mu\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\mu}{\delta}u_e\frac{1}{n}\eta^{\frac{1-n}{n}} = \frac{\mu u_e}{n\delta}\frac{1}{\eta^{\frac{n-1}{n}}}$$

for any value of n > 1, the second term on the rhs above tends to 1/η, and hence as η is zero at the wall, skin friction is infinite!

Obviously this is a problem with our analysis, not real, and hence skin friction is approximated through empirical relations (both of type $c_f = K_\delta \operatorname{Re}_\delta^{\frac{-2}{n+1}}$):

$$c_f = \frac{0.0468}{Re_\delta^{0.25}} \text{for n} = 7, \qquad c_f = \frac{0.0290}{Re_\delta^{0.2}} \text{for n} = 9$$

University of BRISTOL

Use of the Momentum Integral Equation (MIE)

- The derivation of the MIE made no assumptions about the nature of the flow, so the MIE can be used for turbulent flows

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\cancel{\frac{du_e}{dx}}(H+2) = \frac{c_f}{2}$$

- Consider zero pressure gradient (i.e. no velocity gradients). MIE becomes

$$\frac{d\theta}{dx} = \frac{c_f}{2}$$

- We can solve for integral properties in the following manner:

Assume for now n = 7. Use the empirical Blasius Correction for skin friction:

$$c_f = \frac{0.0468}{\mathrm{Re}_\delta^{0.25}} = \frac{0.0468\nu^{0.25}}{\delta^{0.25} u_e^{0.25}}$$

so

$$2\frac{d\theta}{dx} = \frac{0.0468\nu_e^{0.25}}{u_e^{0.25}\delta^{0.25}}$$

3 slides ago we showed that $\dfrac{\theta}{\delta} = \dfrac{n}{(n+1)(n+2)}$

for n = 7, this gives δ=10.29θ. Substituting this for δ:

$$2\frac{d\theta}{dx} = \frac{0.0261\nu_e^{0.25}}{u_e^{0.25}\theta^{0.25}}$$

$$\Rightarrow 2\theta^{0.25}d\theta = \frac{0.0261\nu_e^{0.25}}{u_e^{0.25}}dx$$

integrate:

$$\Rightarrow 1.6\theta^{1.25} = \frac{0.0261\nu_e^{0.25}}{u_e^{0.25}}x$$

University of BRISTOL

divide both sides by 1.6 and raise to the power of 0.8:

$$\theta = \frac{0.0372\nu_e^{0.2}}{u_e^{0.2}} \frac{x}{x^{0.2}}$$

$$\Rightarrow \frac{\theta}{x} = \frac{0.0372}{Re_x^{0.2}}$$

and once we know one integral property as a function of x, and as a function of δ, all the others follow from relations derived previously. $\frac{\delta}{x} = \frac{\theta}{x}\frac{\delta}{\theta}, \frac{\delta^*}{x} = \frac{\delta}{x}\frac{\delta^*}{\delta}$ $\mathrm{Re}_\delta = \mathrm{Re}_x \frac{\delta}{x}$

Turbulent boundary layer grows proportional to $x^{0.8}$ (from above =x/ $x^{0.2}$) compared to $x^{0.5}$ in a laminar one - so turbulent boundary layers are **thicker**.

University of BRISTOL

More complex Geometries

- The power law is strictly only applicable to zero pressure gradient flows. It can be used for others with a reasonable loss of accuracy.
- If we have a pressure gradient, the MIE is

Comes from some external calculation panel method, or conf. mapping

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{c_f}{2}$$

- Noting that both the Blasius corrections for n=7 and n=9 are in the form of $c_f = K_\delta \mathrm{Re}_\delta^{\frac{-2}{n+1}}$ $c_f = \frac{0.0468}{Re_\delta^{0.25}}$ for n = 7, $c_f = \frac{0.0290}{Re_\delta^{0.2}}$ for n = 9
- And that we have a formula relating δ to θ, we have

$$\frac{\theta}{\delta} = \frac{n}{(n+1)(n+2)}$$

$$K_\theta = K_\delta\left(\frac{\delta}{\theta}\right)^{\frac{-2}{n+1}}$$

$$\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}(H+2) = \frac{K_\theta}{2Re_\theta^{\frac{2}{n+1}}}$$

so $$\theta^{\frac{2}{n+1}}\frac{d\theta}{dx}+\theta^{\frac{n+3}{n+1}}\frac{1}{u_e}\frac{du_e}{dx}(H+2)=\frac{K_\theta}{2}\left(\frac{u_e}{\nu}\right)^{-\frac{2}{n+1}}$$

and on multiplying by the integrating factor u_e^{ϕ}

$$\frac{d}{dx}\left(\frac{n+1}{n+3}\theta^{(n+3)/(n+1)}u_e^{\phi}\right)=\frac{K_\theta}{2}\left(\frac{u_e}{\nu}\right)^{-\frac{2}{n+1}}u_e^{\phi}$$

where $\phi=\frac{(H+2)(n+3)}{n+1}$

integrating w.r.t. x: (remember n,H are not functions of x)

$$(\theta^{\frac{n+3}{n+1}}u_e^{\phi})_x-(\theta^{\frac{n+3}{n+1}}u_e^{\phi})_t=\frac{n+3}{n+1}\frac{K_\theta}{2}\int_{x_t}^{x}u_e^{\phi}\left(\frac{u_e}{\nu}\right)^{-\frac{2}{n+1}}dx$$

where t stands for transition, and

assume H is about 1.4

University of BRISTOL

This equation is similar in form to that we had for Thwaites method

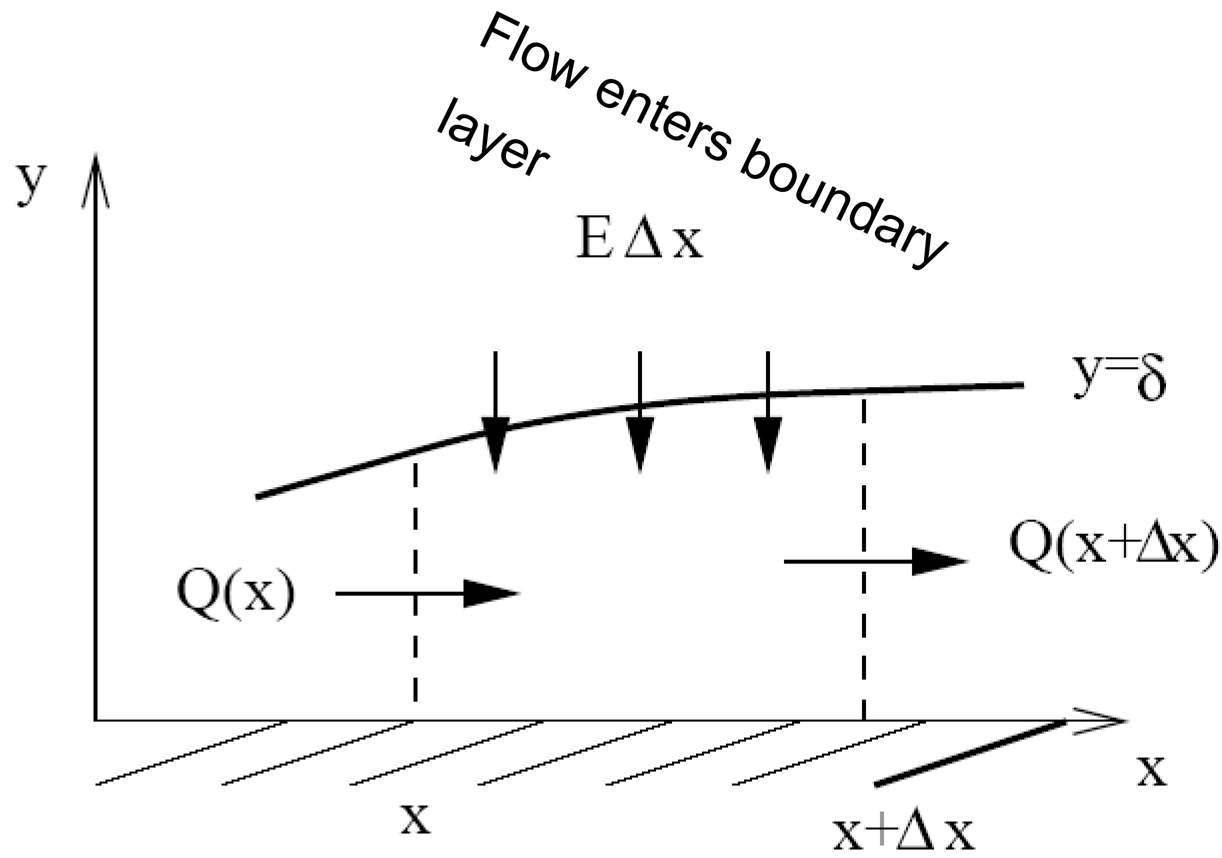
$$\theta_{x1}^2 = 0.45\frac{\nu}{u_e^6}\int_0^{x1} u_e^5 dx$$

and can be solved in a similar manner. Both methods can be used by hand, and are straightforward to use in a computer program.

More Advanced Methods - Entrainment

- The power law profile is of dubious accuracy for flows with pressure gradients, as H cannot vary (but should)
- More advanced methods **allow H to vary**, for instance Heads Entrainment Method, where a relationship is created through the amount of flow going into the boundary layer from the mean stream - *entrainment*

University of BRISTOL


Flow enters boundary layer
y
E Δ x
y=δ
Q(x)
Q(x+Δx)
x
x
x+Δ x

Even More Advanced Methods - Lag

- However, such method do not allow for the lag between changes in streamwise flow properties at the edge of the boundary layer, and the wall
- Lag methods, such as Greens, include this, but are significantly more complex.
  - Use turbulent kinetic integral equation, etc.

Further reading: “Viscous-Inviscid Analysis of Transonic and Low Reynolds Number Airfoils” M. Drela and M. Giles AIAA-1986-1786. Also: AIAA-1987-0424, AIAA-1993-0969

Interaction

- Also need interaction between boundary layer and the external inviscid flow via a transpiration boundary condition

$$\mathbf{Ax} \neq 0, = \mathbf{b}$$

Derived from δ^* to displace the external flow outwards to model the boundary layer influence. It is an outwards blowing velocity

- The entire system of inviscid + boundary layer is then solved simultaneously

Tom Rendall

thomas.rendall@bristol.ac.uk

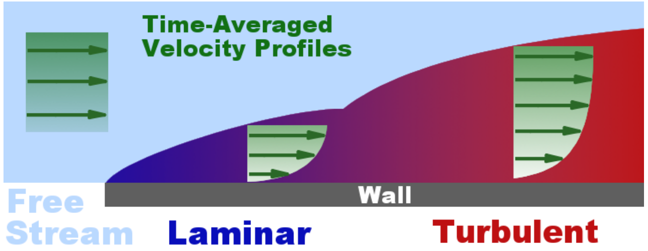
Aerodynamics 3

Boundary Layer Transition

(chapter 17 in notes)

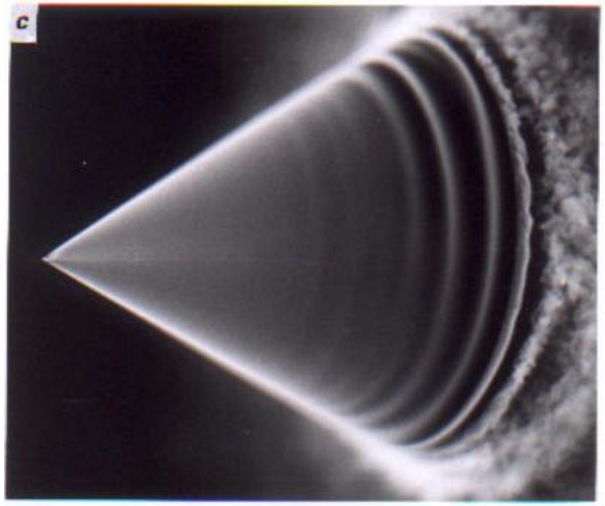
University of Bristol

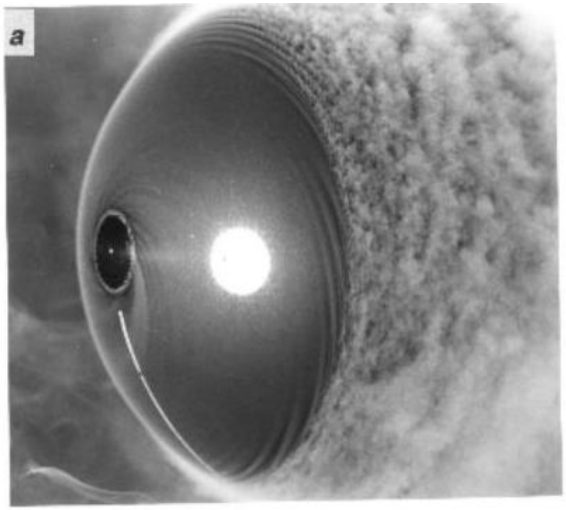
What is Transition?

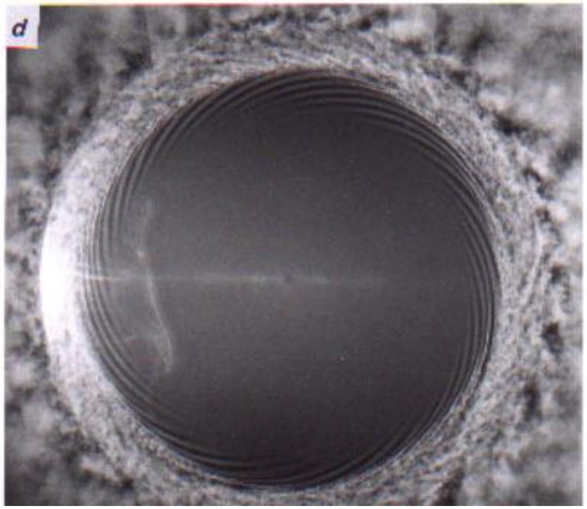


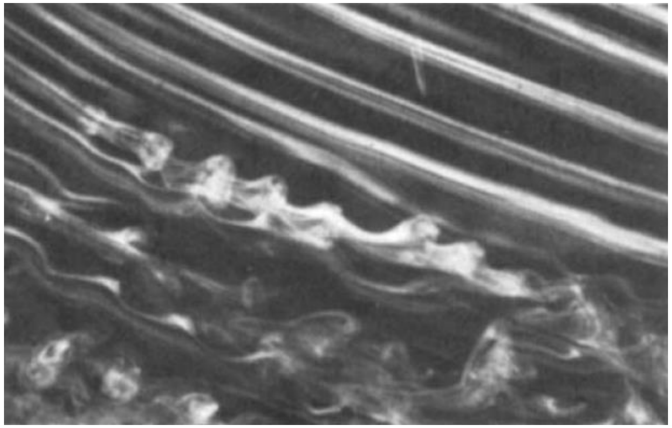
- Nearly all boundary layers begin laminar.
- At some distance downstream nearly all are turbulent.
- How do we get from one to the other?
- Answer is the process called transition. But
  - This process is *very* complex
  - Not very well understood
  - Hence we shall treat it largely empirically and qualitatively

Transition



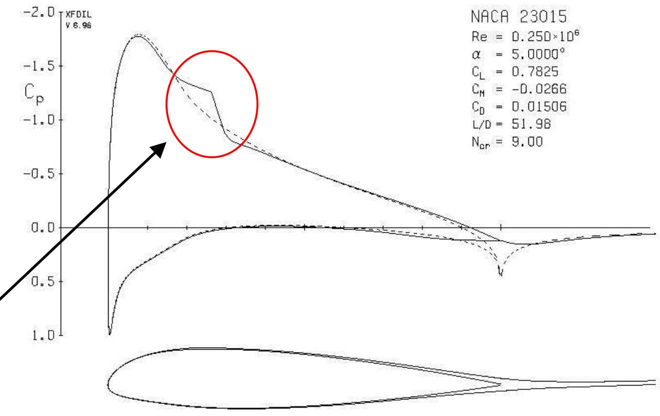


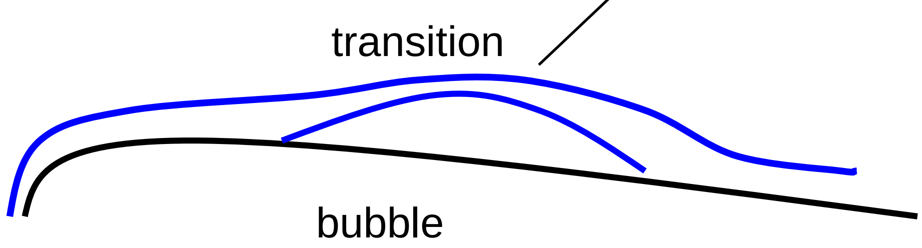




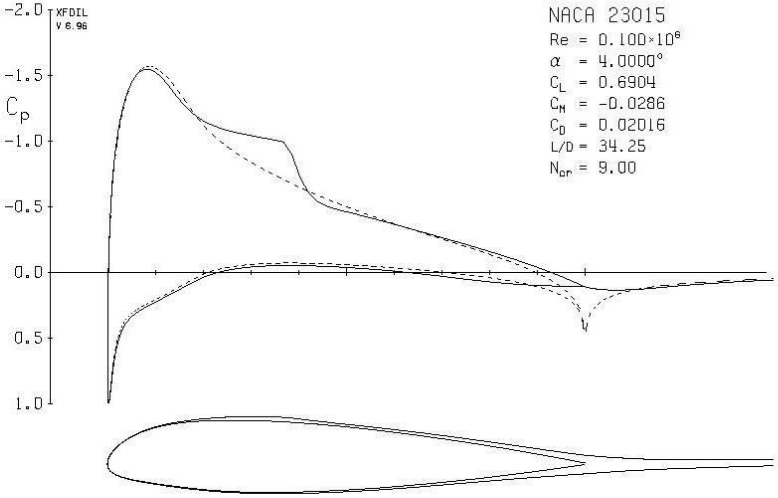
Transition Bubbles

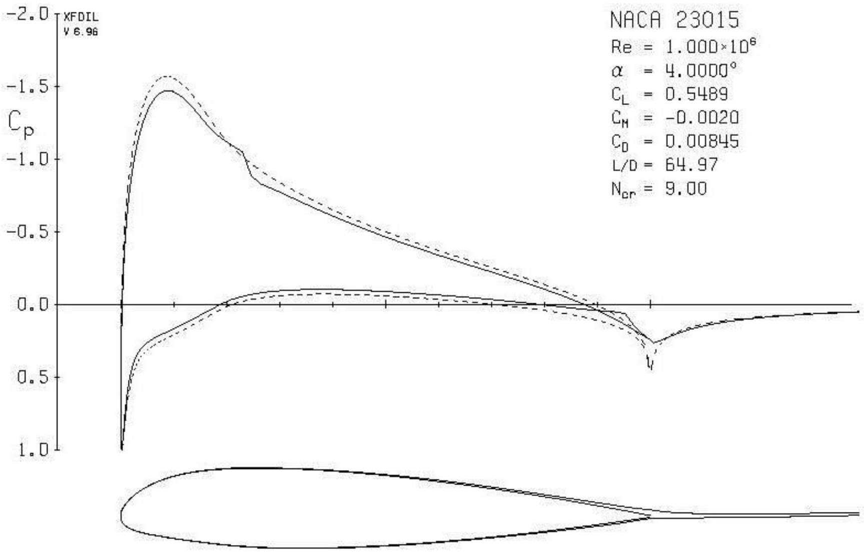
- Laminar Boundary layers cannot tolerate even mildly adverse pressure gradients
  - hence separate
  - but also undergo transition
- Turbulent free shear layers grow much more rapidly than laminar, hence can reattach to surface, forming a separation bubble

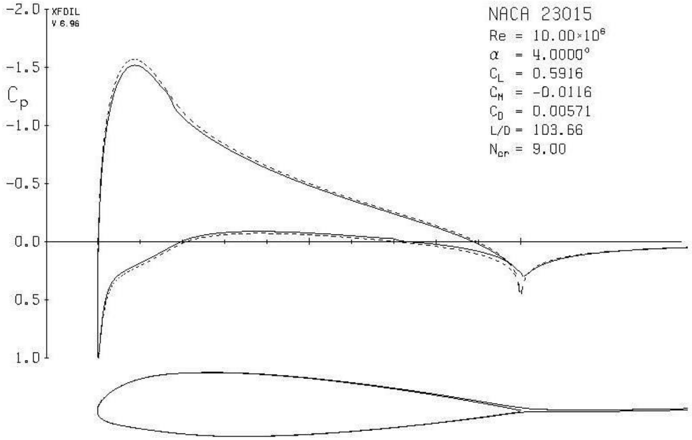




Re Influences Transition



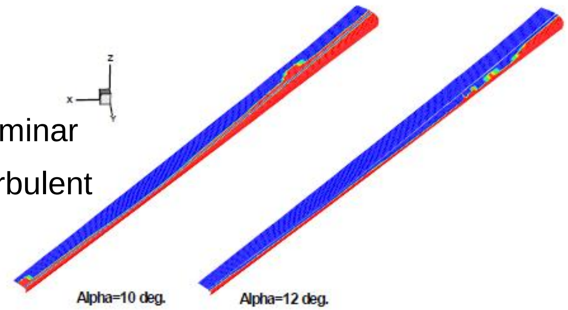




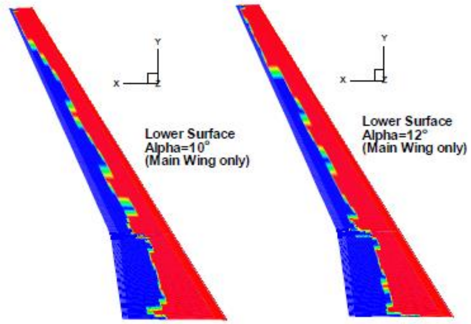
3D Transition examples

Red – laminar

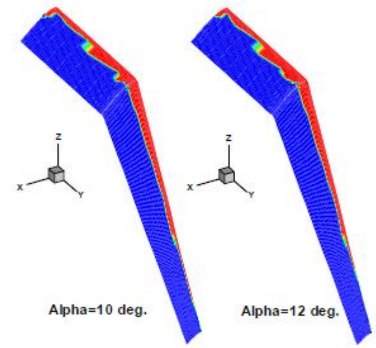
blue - turbulent



Slat upper surface

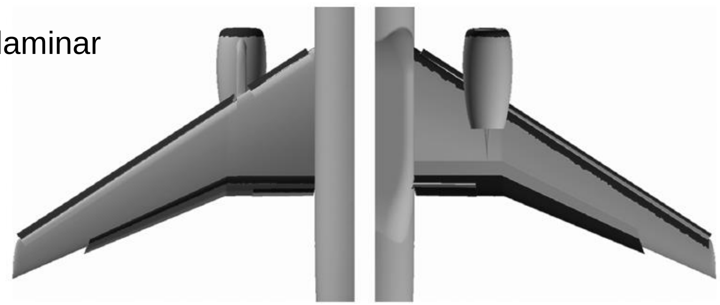


Main wing lower surface



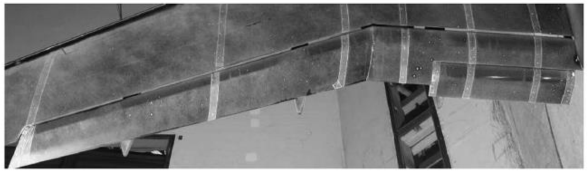
Flap upper surface

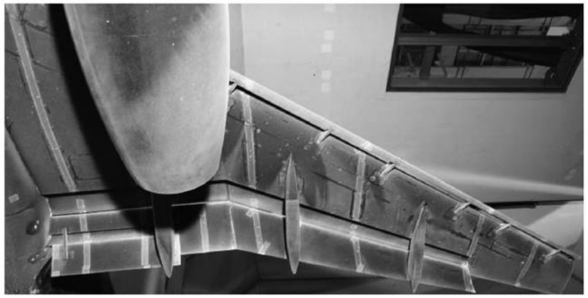
Black - laminar



Glossy – laminar

Matt - turbulent





What causes Transition?

- At all times, tiny disturbances are introduced into the flow within the boundary layer:
  - Noise and vibrations
  - turbulence in the external flow
  - non-uniformities in the surface
- These disturbances travel in a wave like manner, and are called *Tollmien-Schlicting* waves.
- Their direction, amplitude, and growth/decay can be calculated, and shown to be functions of Re and Pressure gradient

100 Years of Transition…

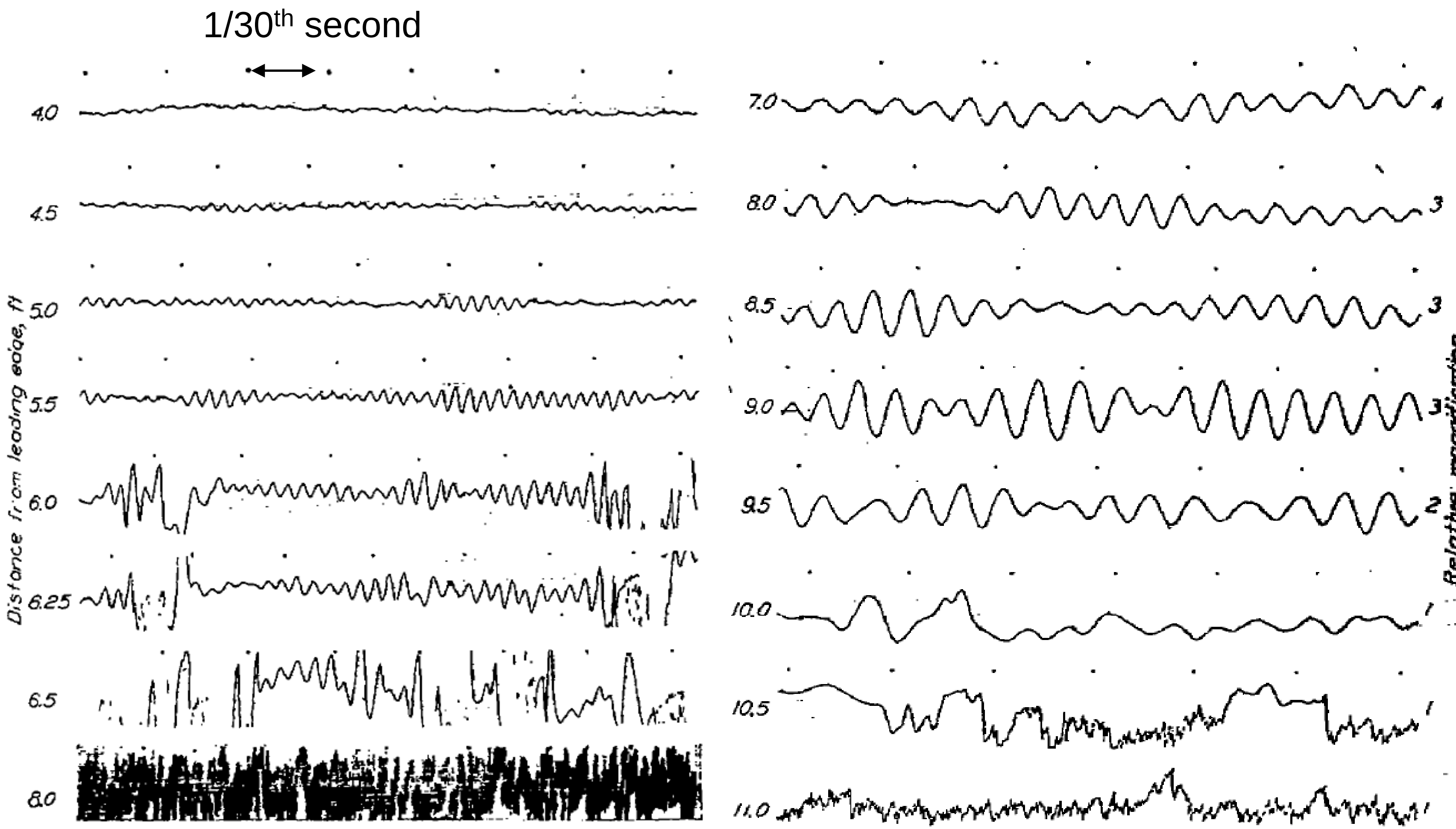
- 1907 – Orr-Sommerfeld stability equation
- 1929 – Tollmien's solution to OS equation, and in 1933 Schlichting's amplification rates
- Experimental confirmation by Schubauer and Skramstad (1943,1947)
- e^n method for predicting transition, Smith and Gamberoni (1956)
- Modern developments relate to trying to find more general models and further understanding of instabilites through numerical experiments

Tollmein-Schlicting Waves

- In a laminar flow, where Re is relatively low, viscous forces damp these waves out
- As Re increases, however:
    - Inertial forces grow relative to viscous
    - There comes a point where the waves are no longer damped
    - And transition begins

Schubauer and Skramstad's Results
(NACA Report No. 909 'Laminar boundary layer oscillations and transition on a flat plate',1943)

Graphs show u'

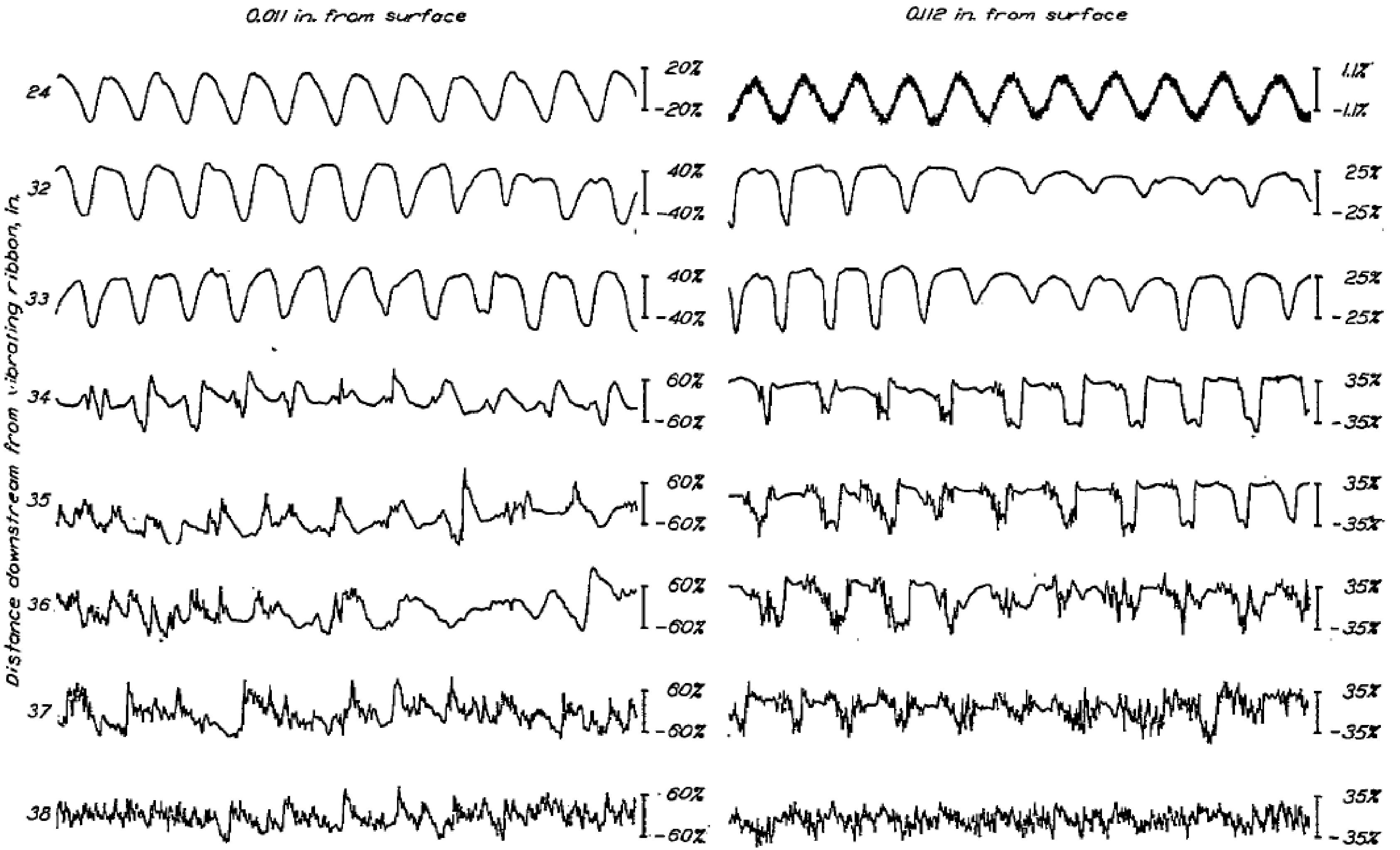


Higher speed

Lower speed

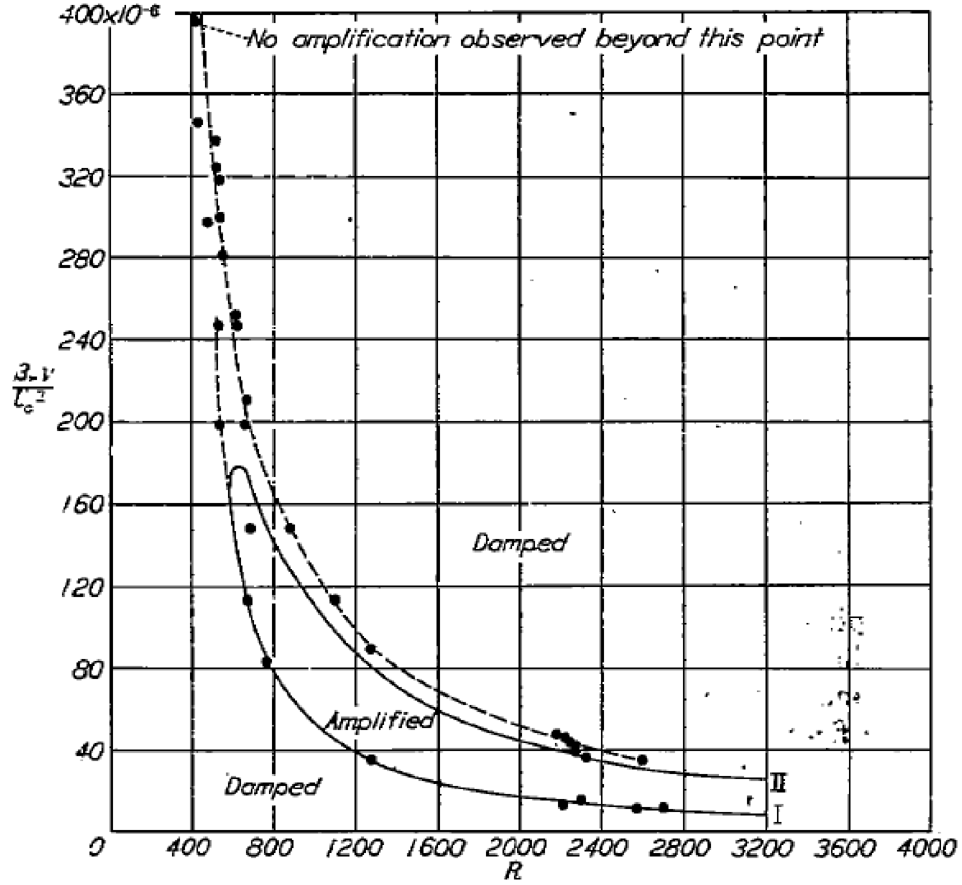
Schubauer and Skramstad's Results

Forcing at 70Hz using an oscillating ribbon



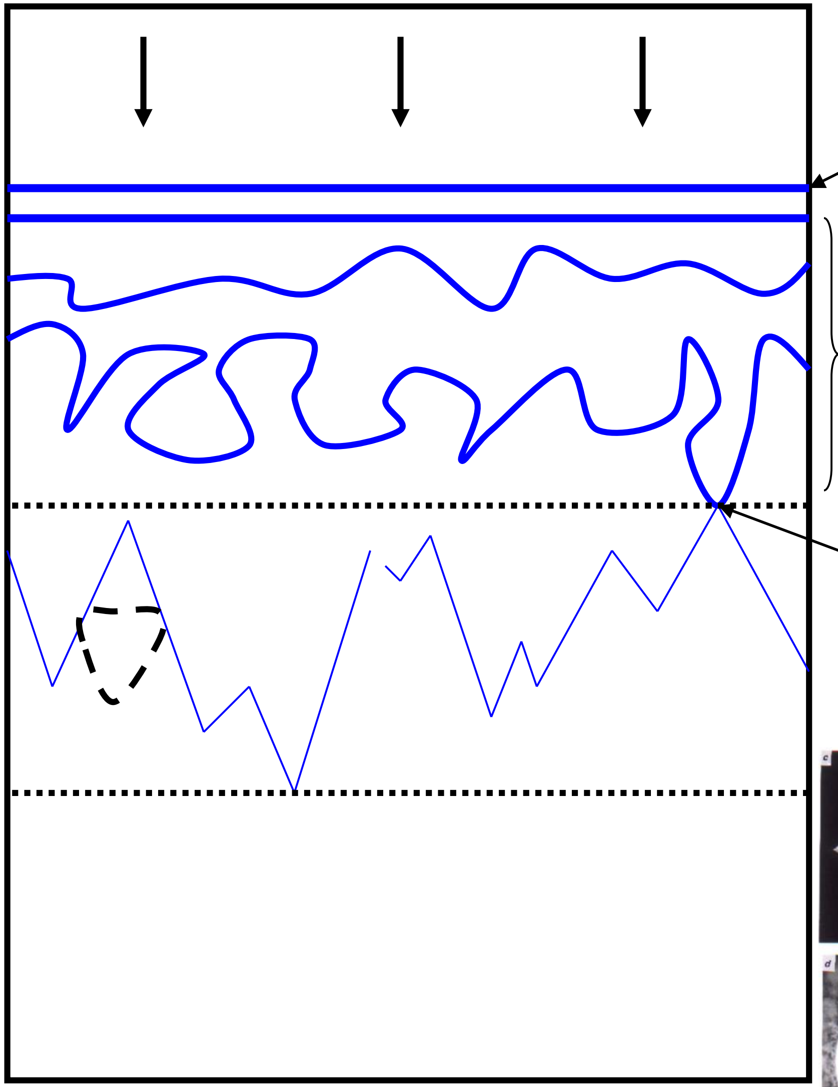
Schubauer and Skramstad's Results

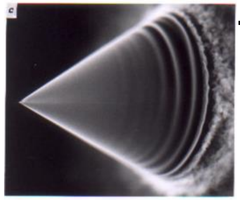
Non-dimensional frequency

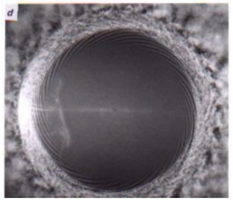


Re_δ

The Transition Process on a Flat Plate (1):

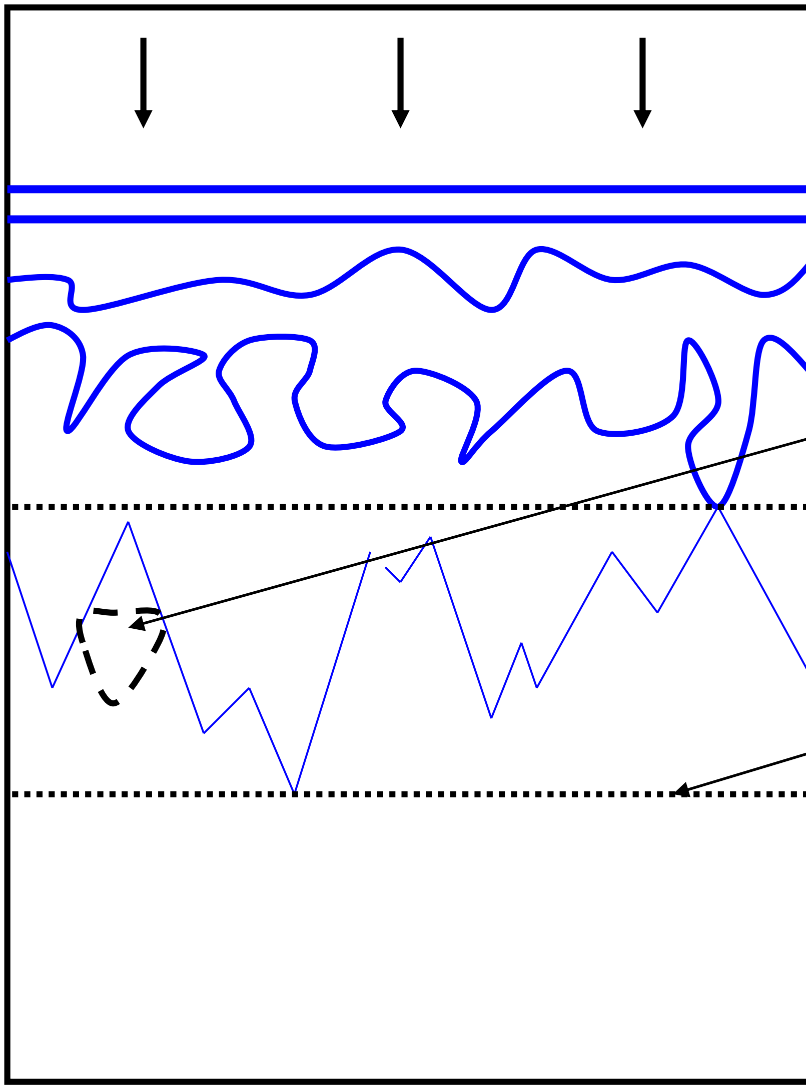




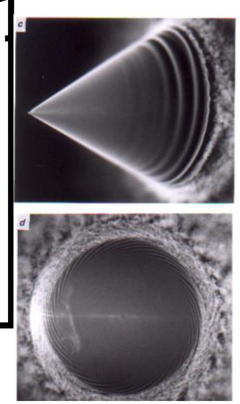


- 2d waves propagate
- Waves become looped due to non-uniformities in the flow
- Loops cause `Turbulent spots’ and transition begins

The Transition Process on a Flat Plate (2):



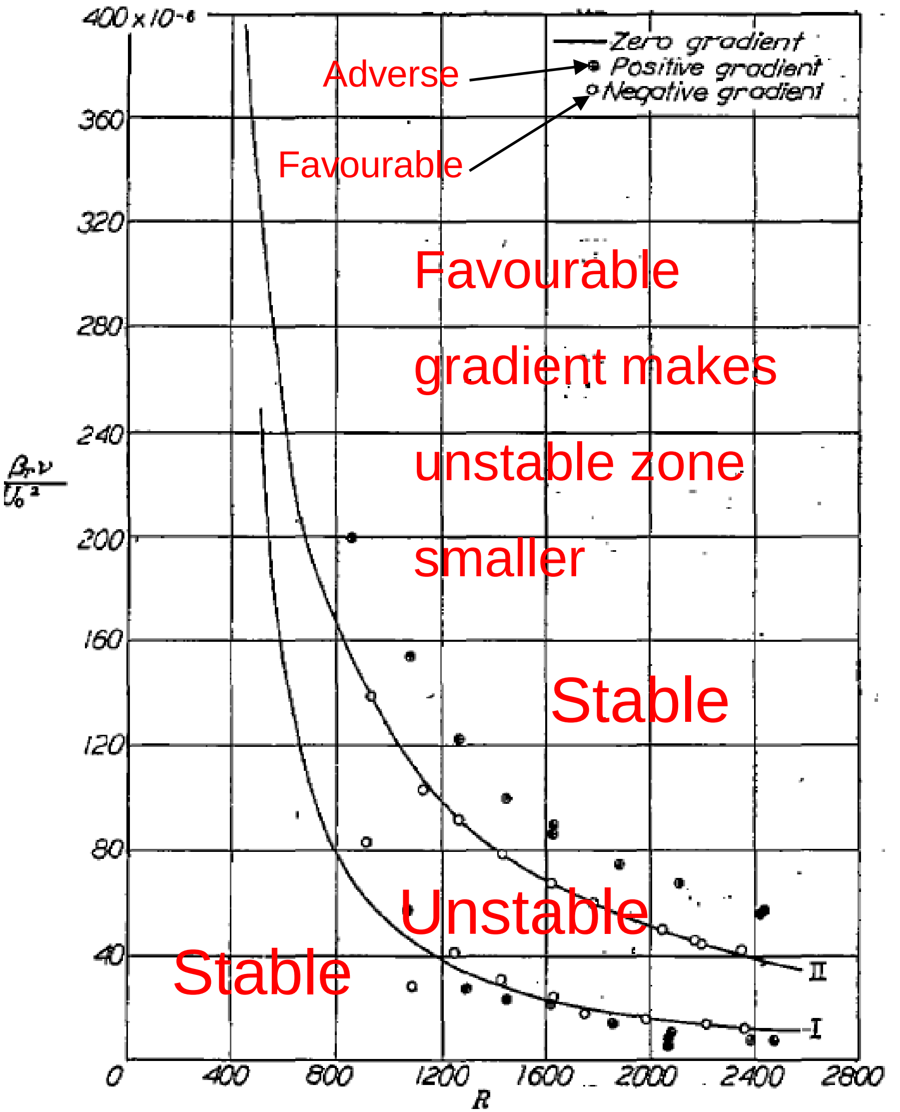
- Formation of turbulent spots is *dynamic*
- `Packet' of turbulent flow moves downstream
- Finally packets merge to form continuous turbulent flow



Transition in other situations

- Process Effected by:
  - dP/dx: +ve speeds transition, -ve delays
  - Roughness: speeds transition by forming turbulent spots directly, as does:
  - high turbulence in the external flow
  - Suction delays transition by removing mass
  - Cooling delays transition by removing energy
- On a flat plate, transition occurs between $Re_x=1.4\times10^5$ and $Re_x = 3.2\times10^6$ depending on the above factors

Effect of dP/dx



Typical Transition Models:

- As discussed, transition is very complex, and difficult to model accurately/reliable in different situations
- Fortunately, for high Re's, generally happens in a very short distance, so we assume a point transition (line in 3D)
- Simplest method (and most common) is to specify a critical Re
- Alternatively, use Pressure gradients (e.g. in your code)
- Or…

Michel has shown for an aerofoil type transition the formula

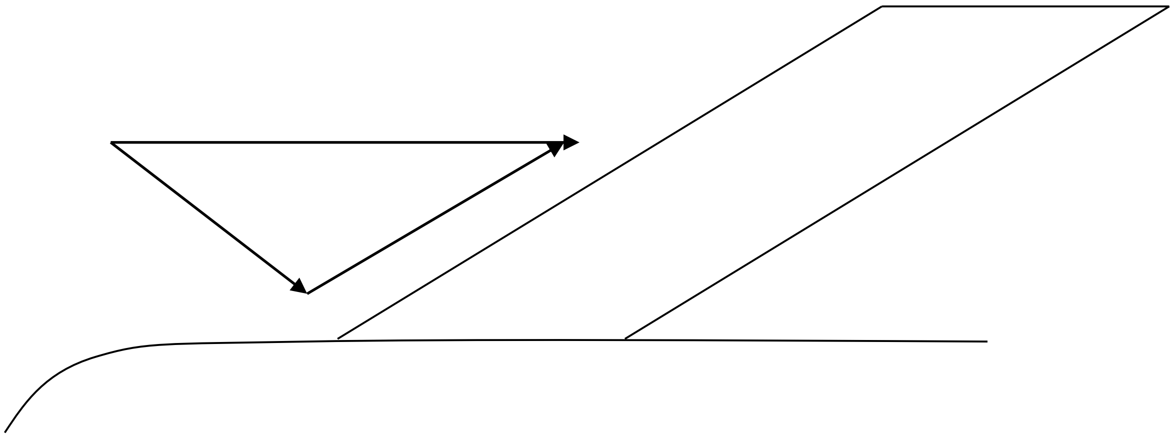
$$Re_\theta = 1.174\left(1 + \frac{22400}{Re_x}\right) Re_x^{0.46}$$

may be used to determine the point of transition

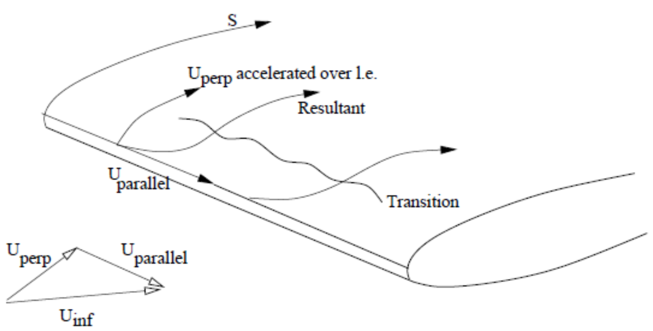
3D effects on swept wings

- Tollmein-Schlicting waves and separation bubbles are 2D effects

- Sweep on a 3D wing introduces two other transition mechanisms, due to the introduction of a parallel and perpendicular velocity component



Cross Flow Instability:



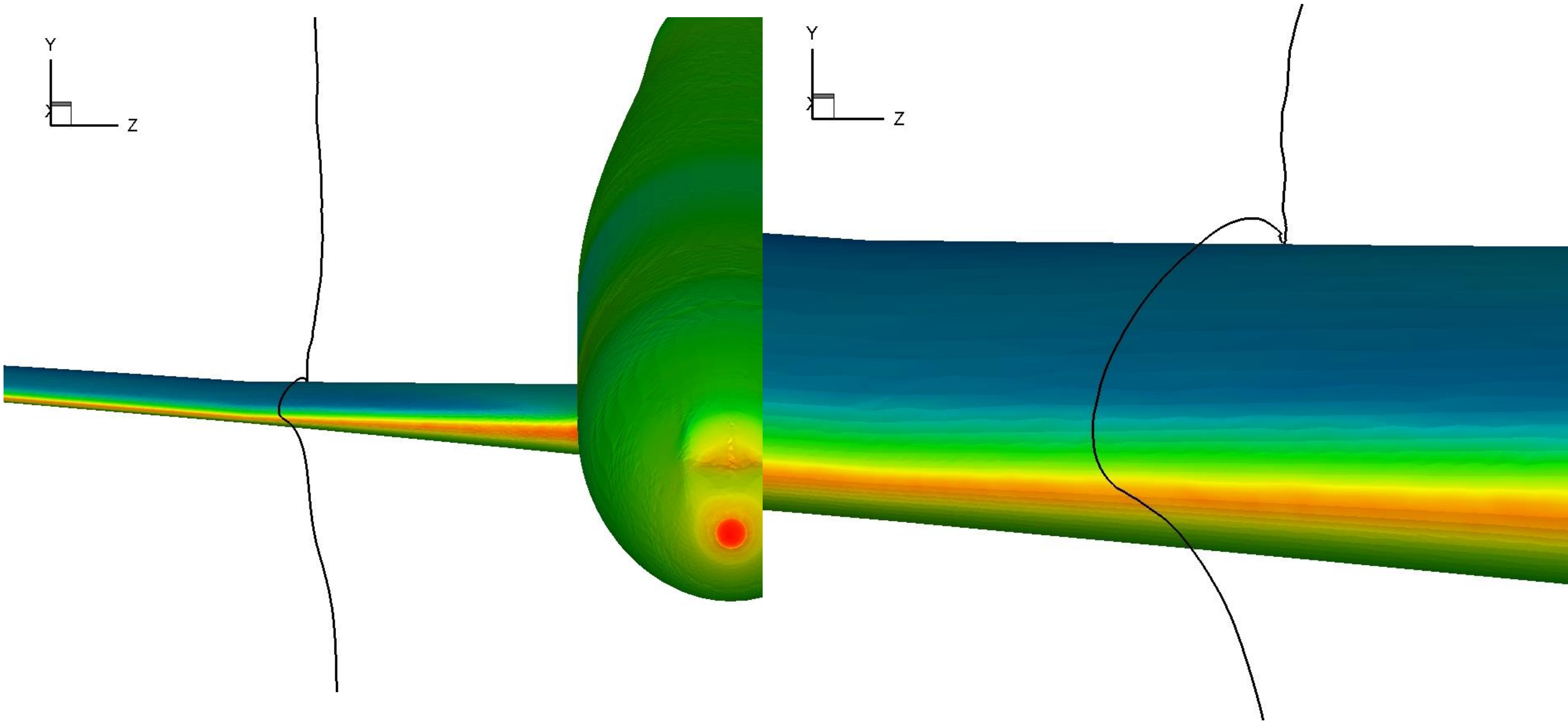
U_{perp} is 0 at the l.e., but grows rapidly along the aerofoil surface

$U_{parallel}$ is approximately constant

These two factors combine to give an inflection in the boundary layer, destabilising it

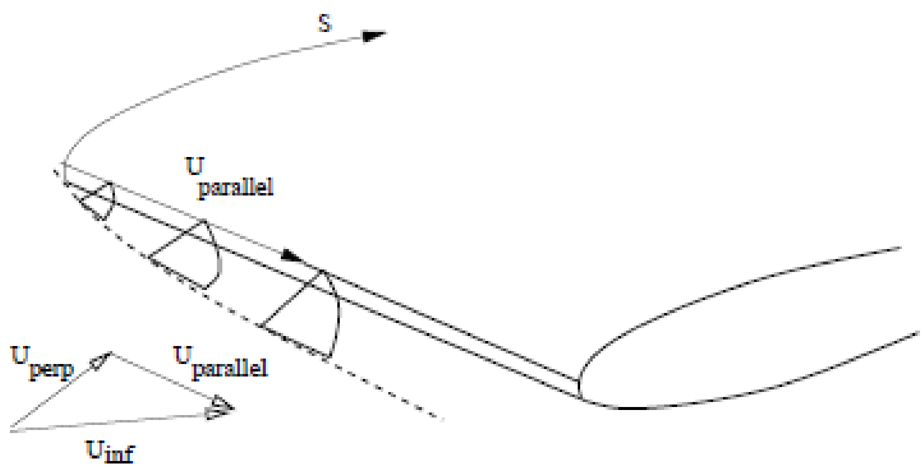
Can avoid by reducing dU_{perp}/ds, i.e. alter aerofoil shape.

Cross Flow Instability – example streamline



Attachment line transition (1)

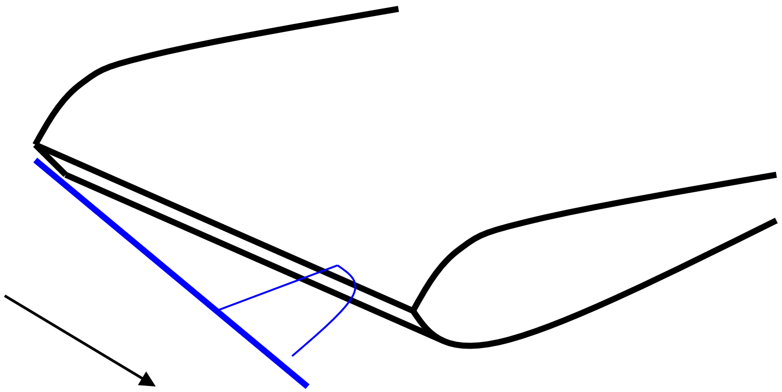
Re_{al} is defined as $$Re_{a.l.} = \frac{U_{parallel}}{\nu}\left(\frac{\nu}{\frac{\partial U_{perp}}{\partial s}}\right)^{\frac{1}{2}}$$



$U_{parallel}$ causes a boundary layer to grow along the attachment line

If the Re_{al} along this is > 800, we get transition, and the *whole wing* is then in a turbulent boundary layer -> unnecessarily high drag

Attachment line transition (2)



- Can be avoided by:
  - Reduce $U_{parallel}$ (i.e. reduce sweep)
  - Increase dU_{perp}/ds, (i.e. opposite of cross flow! )
  - Use suction on L.E. (not very practical – small holes are a maintenance nightmare as they block with insects and rubbish etc.)

Tom Rendall
thomas.rendall@bristol.ac.uk

Aerodynamics 3

Differential Methods For Boundary Layers

(chapter 18 in notes)

Today

- So far in the course we have only mentioned differential methods in passing:
  - alternative to integral methods
  - more time consuming
- But they are are more general and can be more accurate more difficult shapes
- So we shall spend 1 lecture examining them in more detail:

1) How they work
2) The meshes they require
3) Examples and turbulence models

What are differential methods?

- Belong within the realm of computational fluid dynamics (CFD)
  - Wide class of techniques for all sorts of flows
  - CPA option if interested
- *Differential Boundary Layer solutions:*
  - Unusual in that they only solve a small region of the flow
  - Also can be *space marched*.
- As ever, main difference between CFD and analytical methods:
  - CFD *approximate* solution to *exact* Eqns.
  - Analytical *exact* solution to *approximate* Eqns.

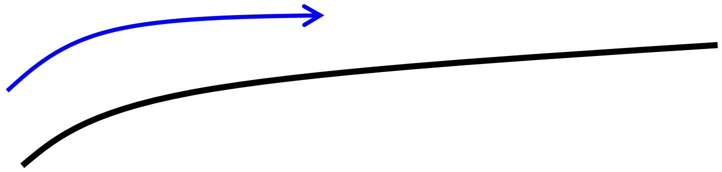
Small Solution region

- Boundary layer idea is to only solve the flow very near the surface, not whole flowfield. Nowadays can also solve whole flowfield via CFD

- Means the solution requires another method to provide velocity profiles etc. imposed on B.L.

- Could be CFD, but could also be Joukowski, panel, vortex lattice, etc. CFD now most common option **by far**

du_e/dx?,
dp/dx?, etc.



Where numerical schemes came from…

Need to model inviscid nonlinearity (eg transonic /w shocks)

Need to model viscous nonlinearity (boundary layers, separation…)

Euler CFD codes

Differential boundary layer methods

Panel methods

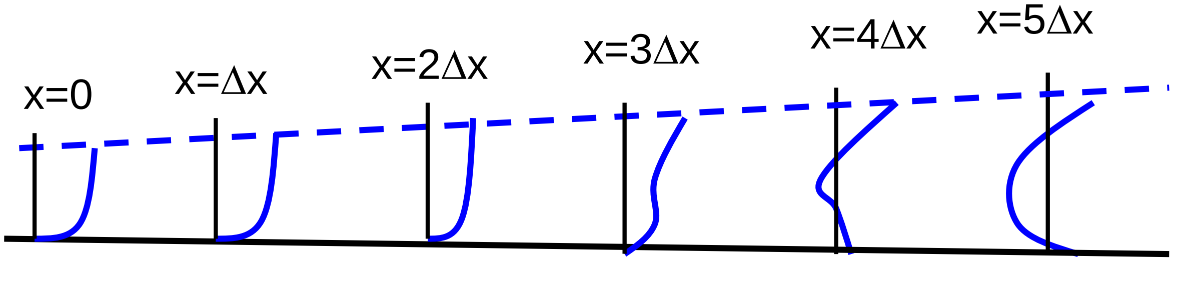
Complete Navier-Stokes model

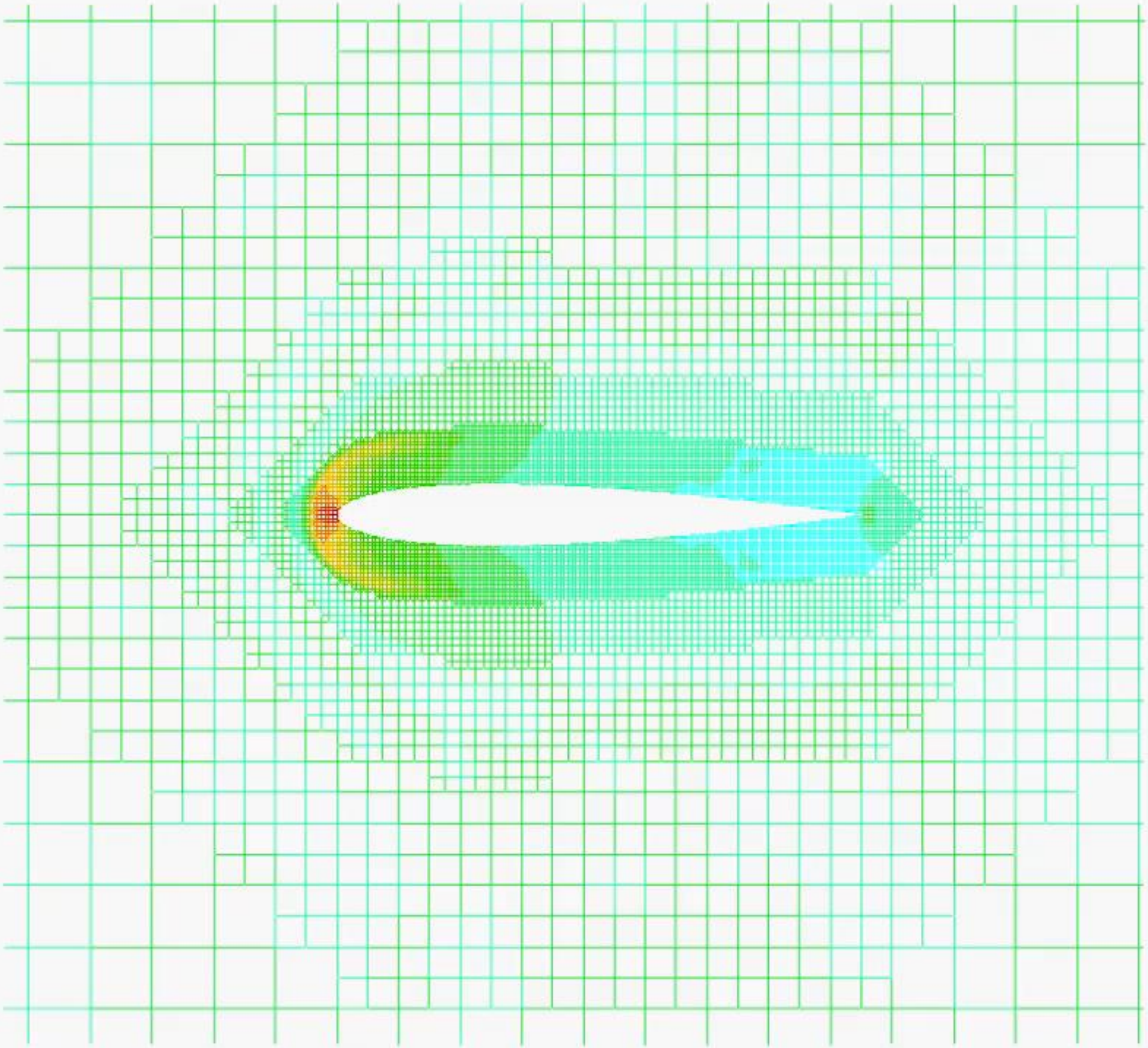
Most development now focussed here

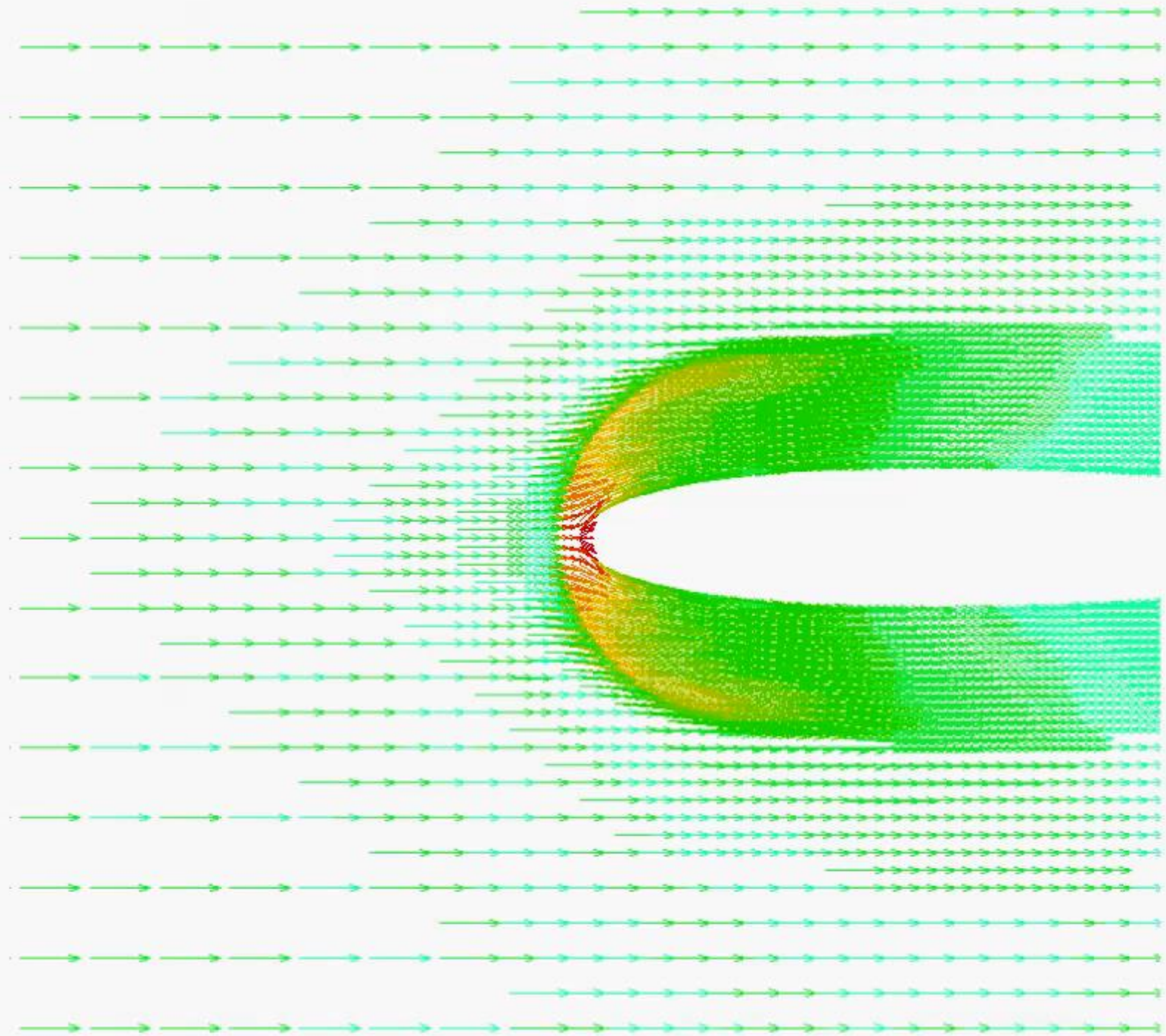
Space Marching vs. Time Marching

- Most CFD method must be time marched to solution, i.e. an initial estimate of the whole flow is used to create another guess, then another, and so on till convergence
  - > time consuming!

Space marching ->

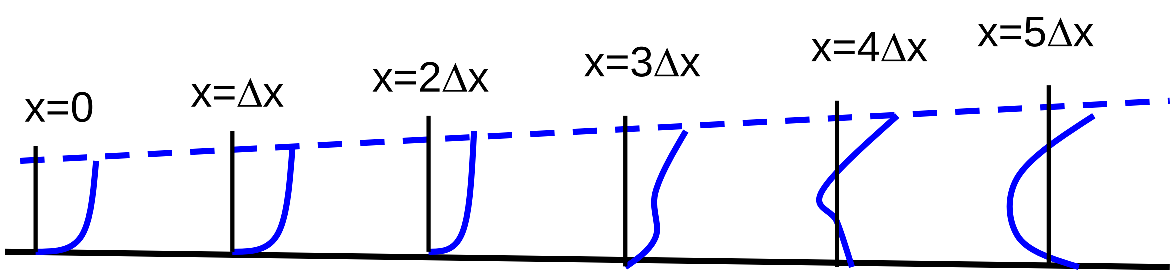






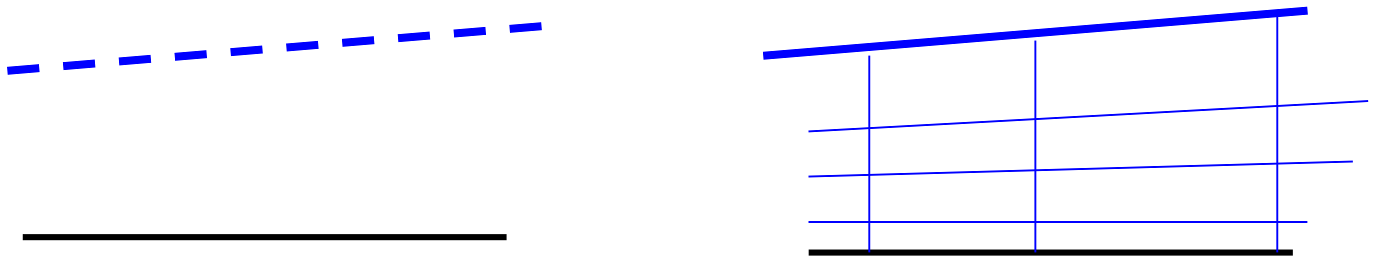
Space Marching

- However, in boundary layers, equations are *parabolic*, not *elliptic*.
- This means that information only travels downstream (it is the same difference as between subsonic and supersonic flows)
- Hence with an accurate input upstream, can quickly move downstream
- This no longer holds if coupled to full CFD



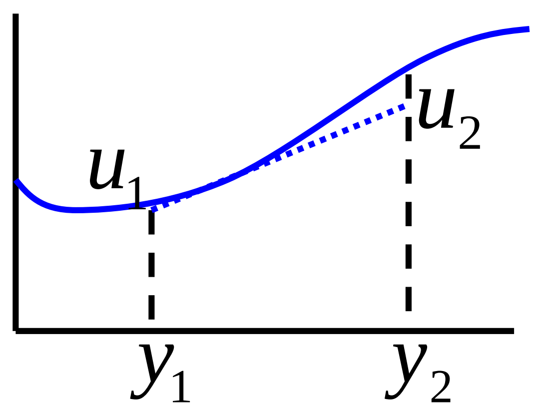
Finite difference formulation

- Break boundary layer into lots of ‘cells’:



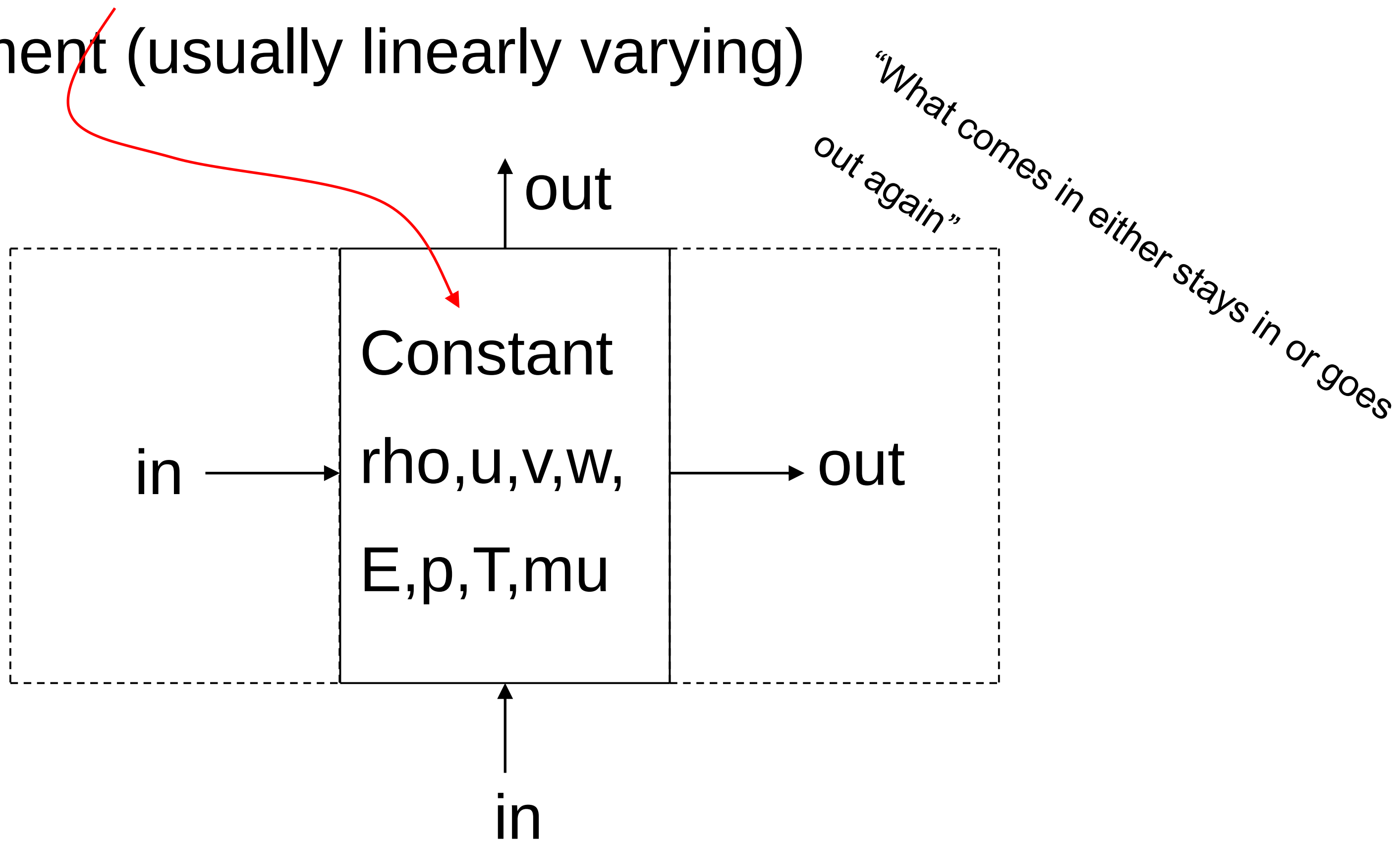
- Approximate Gradients with *Finite Differences*:

$$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y}$$

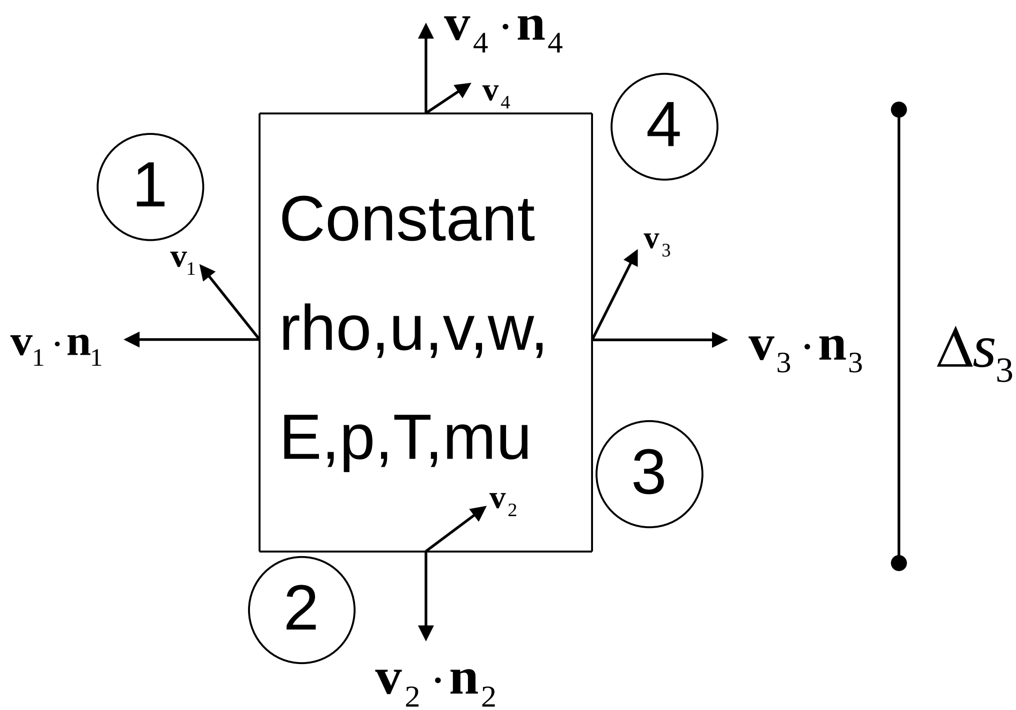


Details

- Formulation using finite differences is somewhat out of date – better approach is to use finite volumes (constant properties within a cell) or finite element (usually linearly varying)



How A Finite Volume Works



All inviscid flow eq'ns can be written as zero divergence:

(viscous eq'ns can be made similar to this too)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{A} = \mathbf{0}$$

$$\mathbf{A} = \begin{pmatrix} \rho u & \rho v & \rho w \end{pmatrix}^T$$

For conservation of mass

$$\int_{volume} \nabla \cdot \mathbf{A} dv = \int_{boundary} \mathbf{A} \cdot \mathbf{n} ds = \int_{boundary} \rho \mathbf{v} \cdot \mathbf{n} ds \approx \sum_{i=1}^{i=4} \underbrace{\rho_i \mathbf{v}_i \cdot \mathbf{n}_i \Delta s_i}_{"\rho u A"} = R$$

=rate change of mass in the volume

(Divergence Theorem)

Explicit Solution Procedure

$$\int_{volume} \frac{d\rho}{dt} dv + \int_{volume} \nabla \cdot \mathbf{A} dv = \frac{d\rho}{dt} V + \int_{boundary} \mathbf{A} \cdot \mathbf{n} ds = \frac{d\rho_i}{dt} V_i + R_i = 0$$

for cell i

$$\frac{d\rho_i}{dt} V_i + R_i(\rho, \mathbf{v}) = 0$$

$$\rho_i^{n+1} = \rho_i^{n} - \frac{\Delta t}{V_i} R_i(\rho, \mathbf{v})$$

Repeat 1000s of times

‘Time marching’ – eventually the idea is that

$$\rho^{n+1} \approx \rho^{n}$$

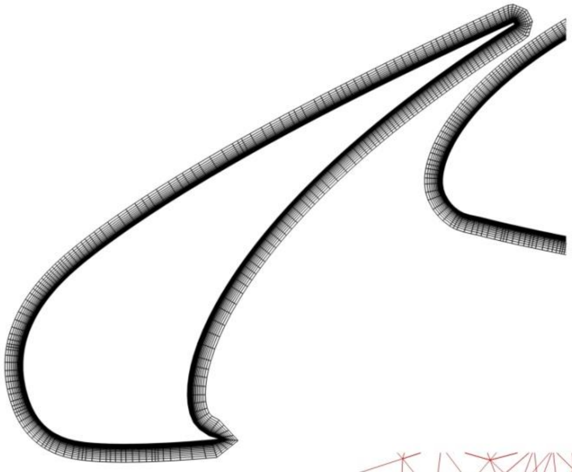
Which is called ‘convergence’

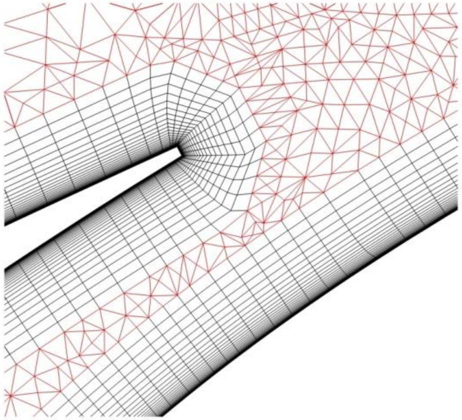
Similar equations are produced for momentum (to update velocity) and energy (to update temperature). There are then as many equations as unknowns.

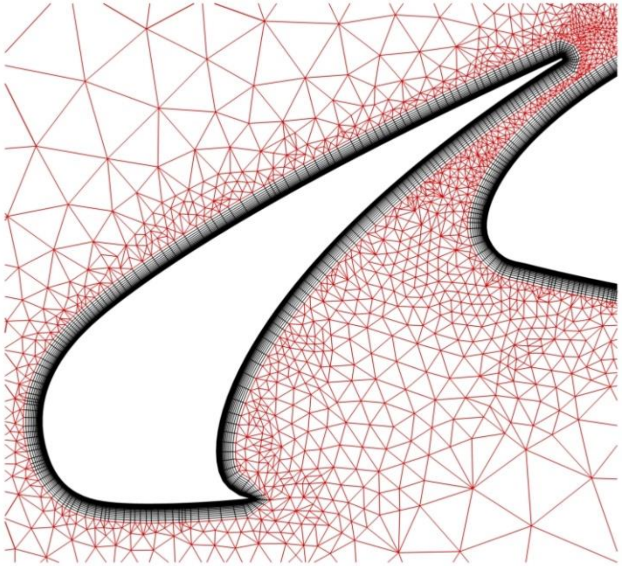
Process

- Formulating the discrete problem is just one part of the solution
- It is also necessary to define the set of finite volumes (control volumes) to use

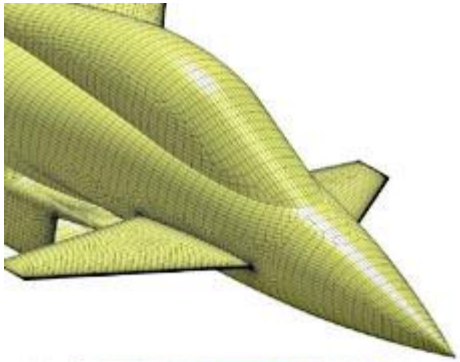
Full CFD Meshes

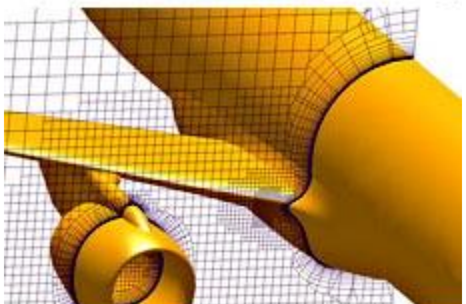


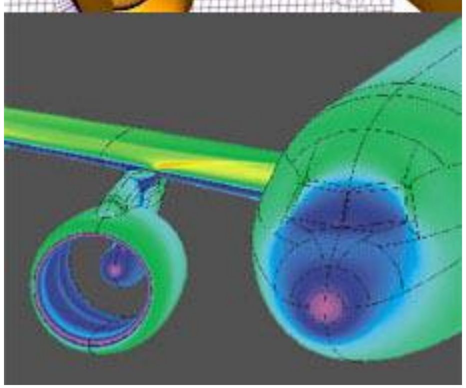


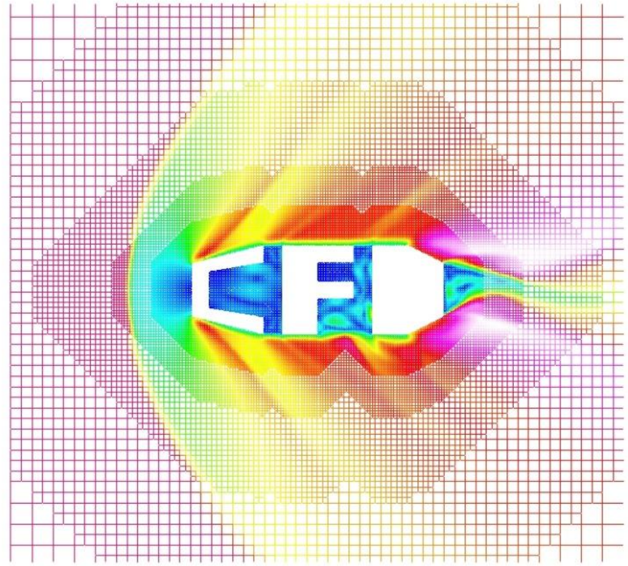


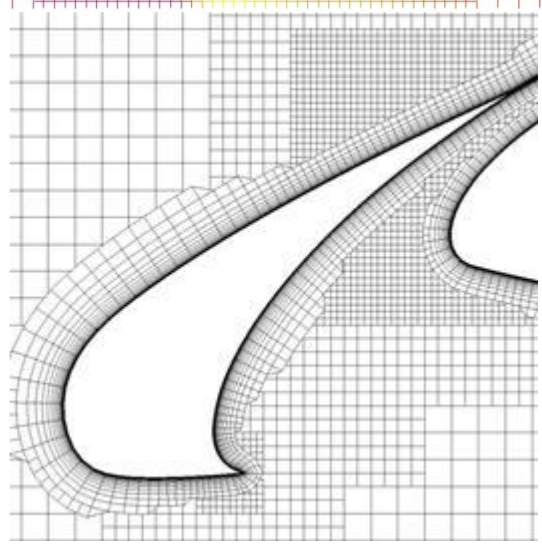
Full CFD Meshes

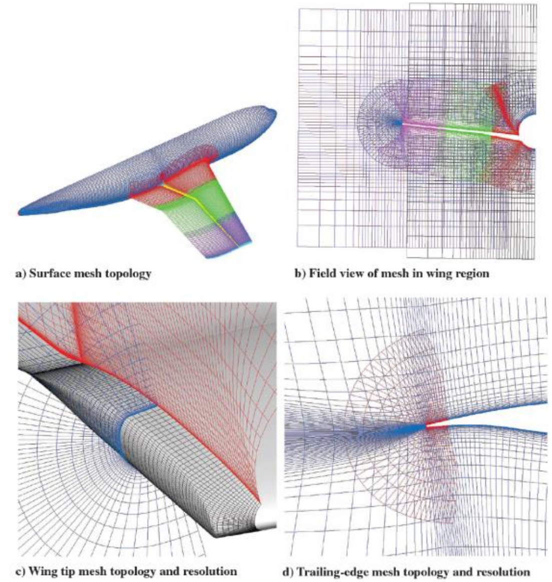


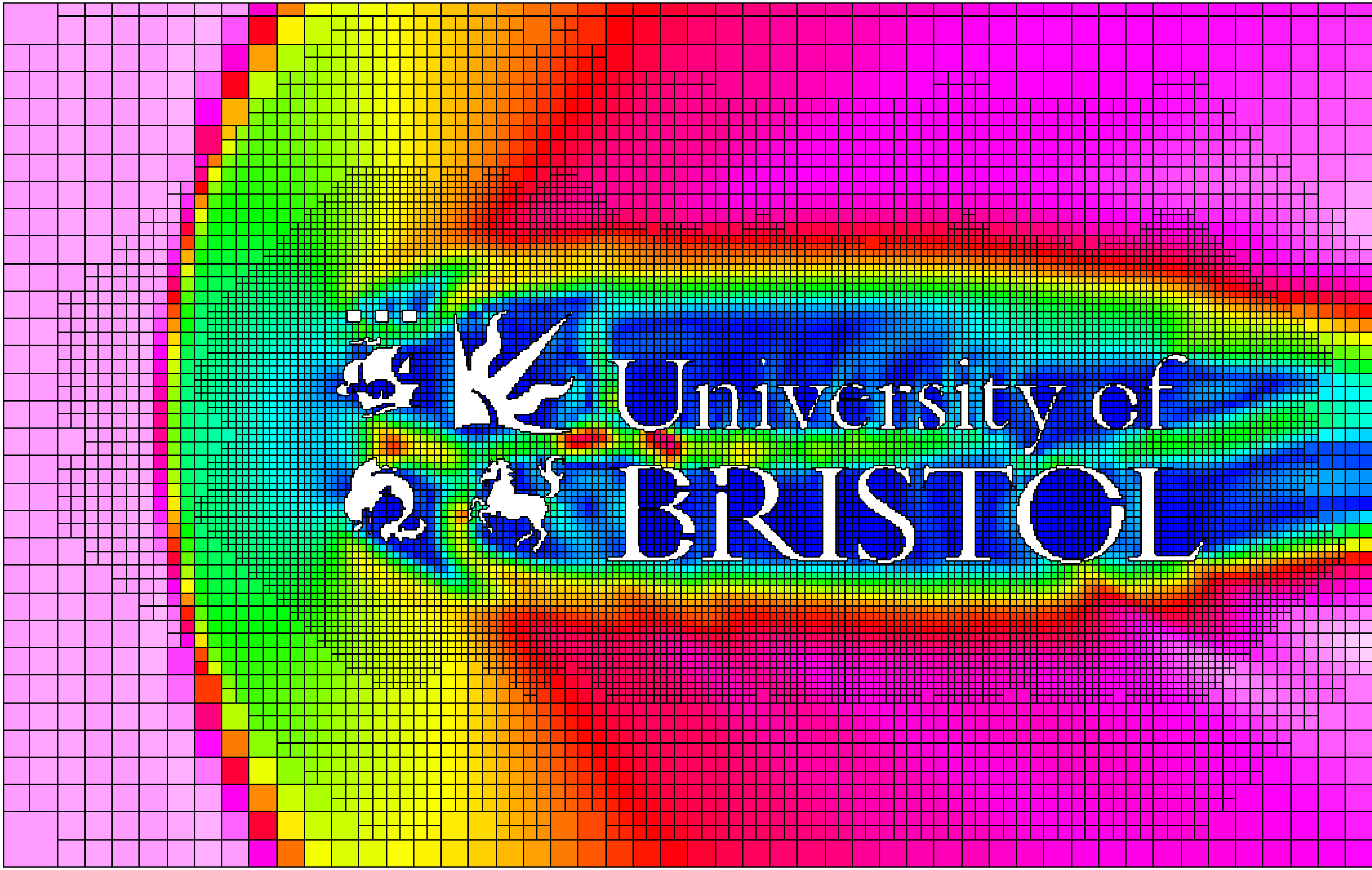




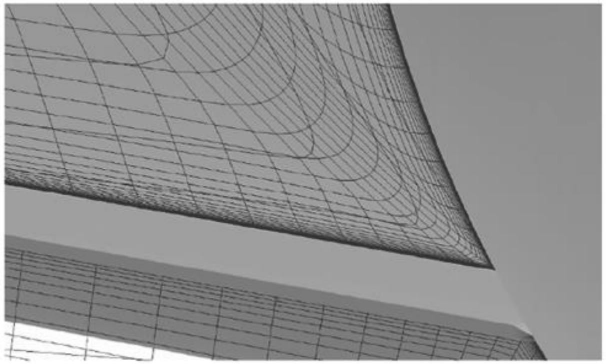


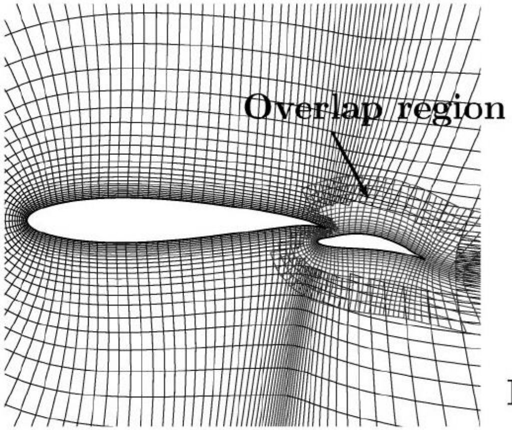


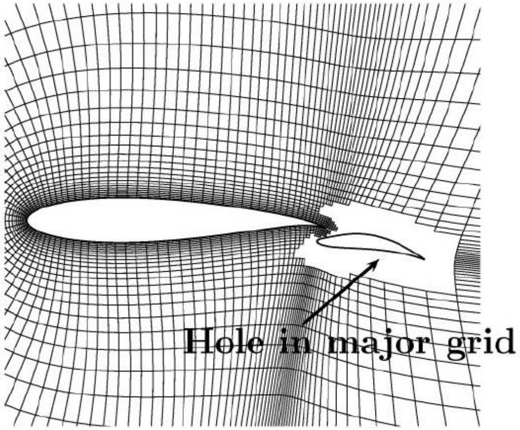


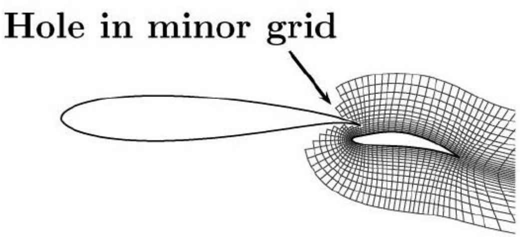

University of
BRISTOL

Some geometries can be tricky…



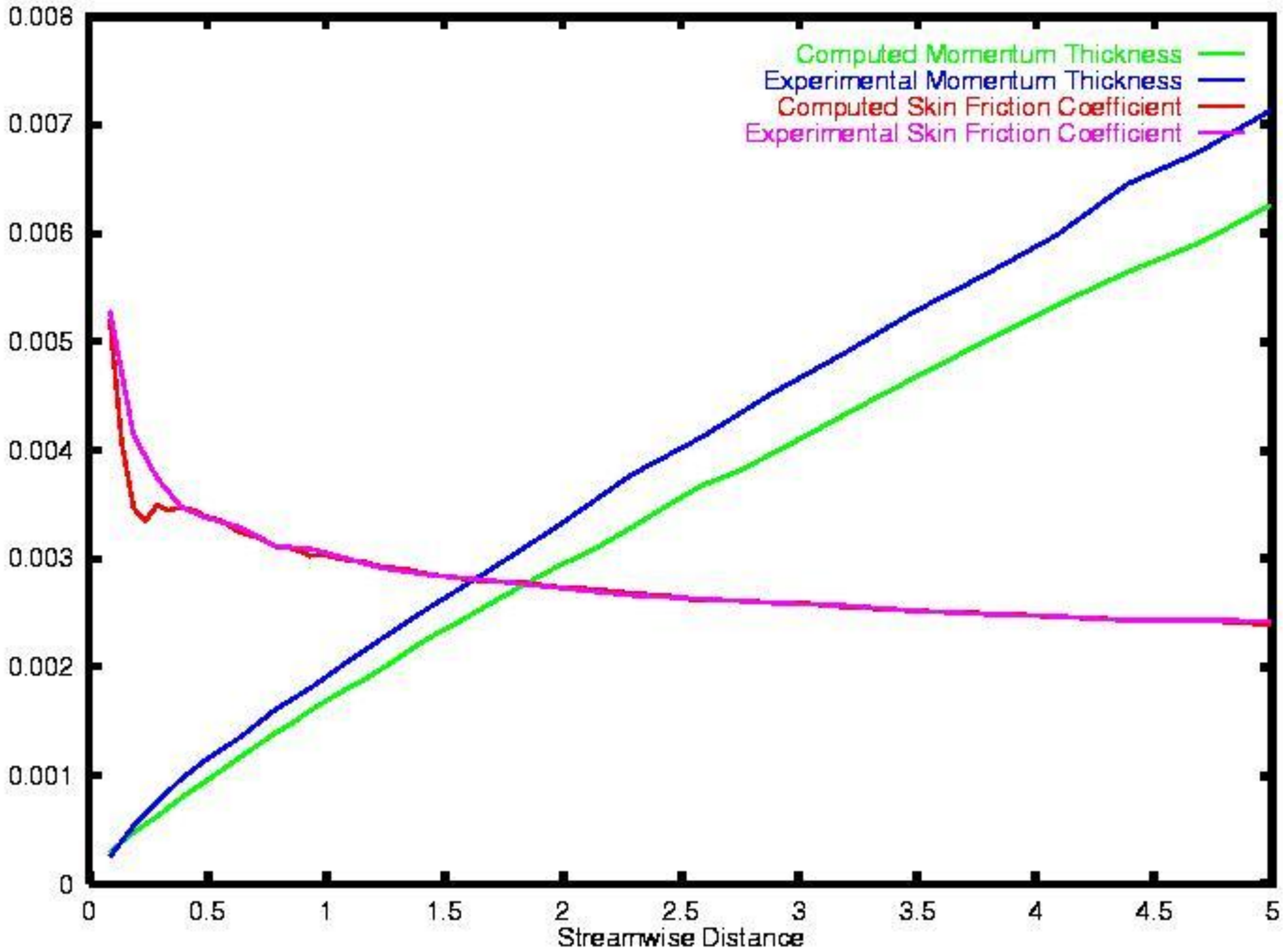






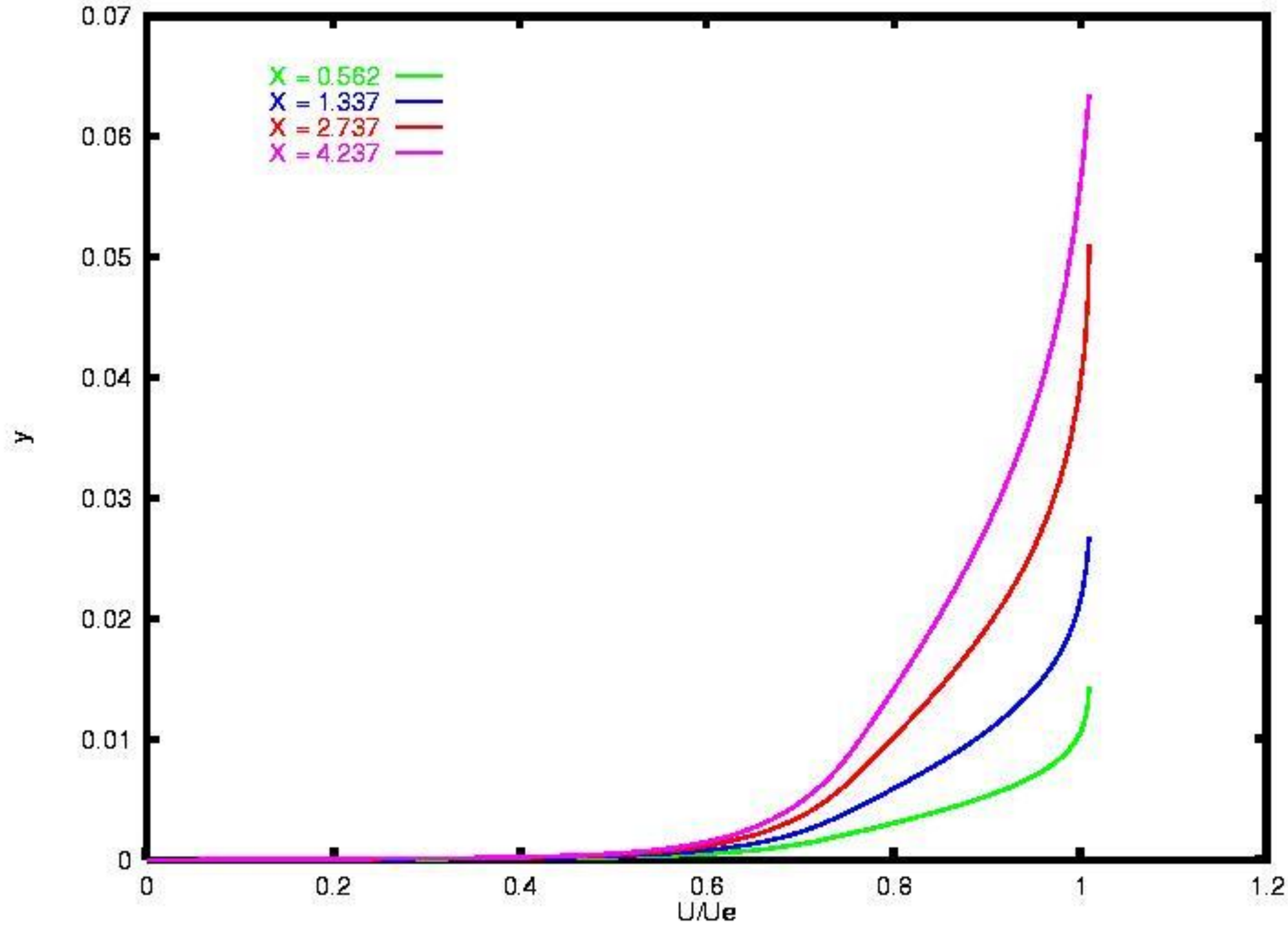
=?

Integral Parameters



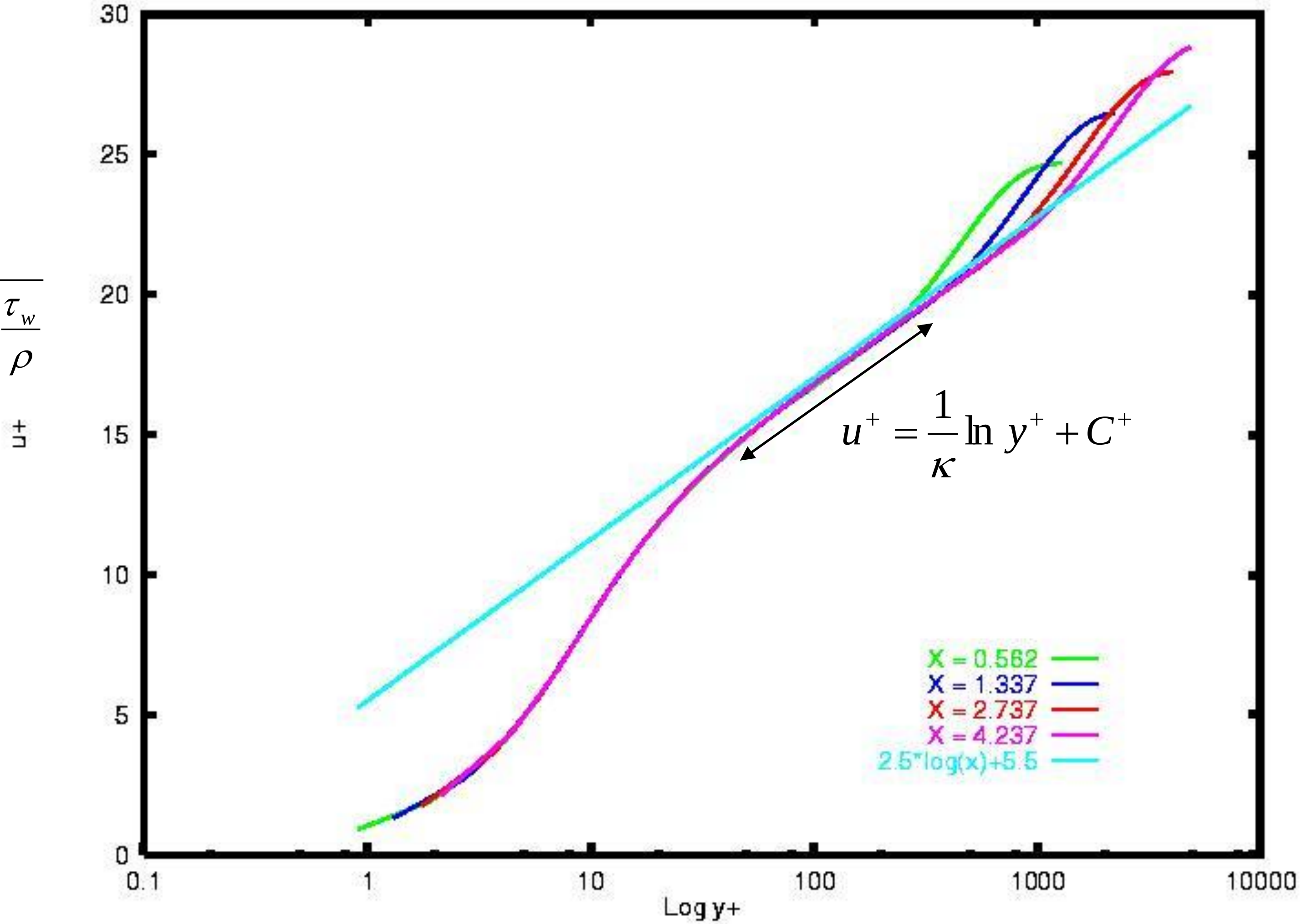
Turbulent flat plate simulation

Velocity Profiles



Non-Dimensional Velocity Profiles

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

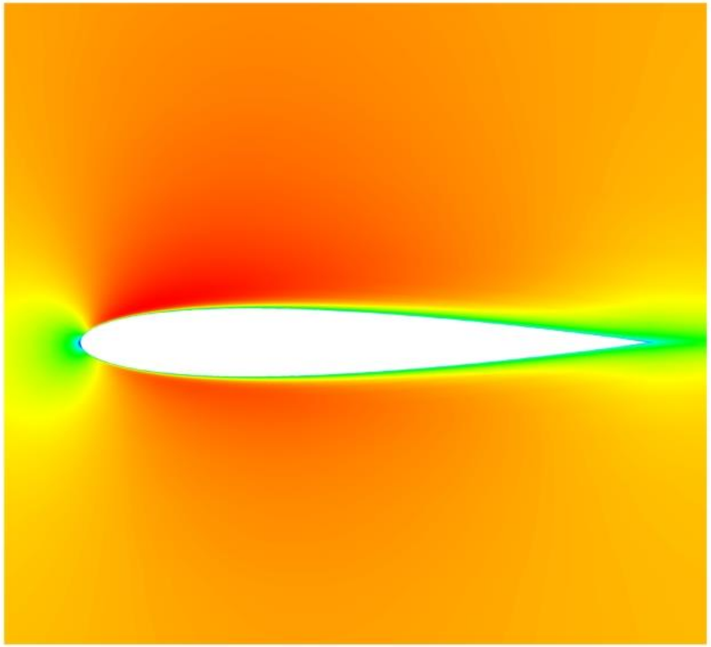


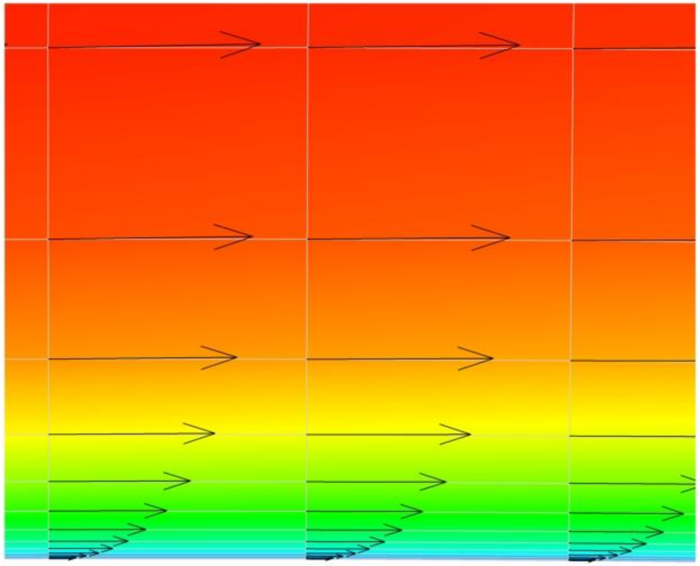
$$u^+ = \frac{1}{\kappa}\ln y^+ + C^+$$

$$u^+ = \frac{u}{u_\tau}$$

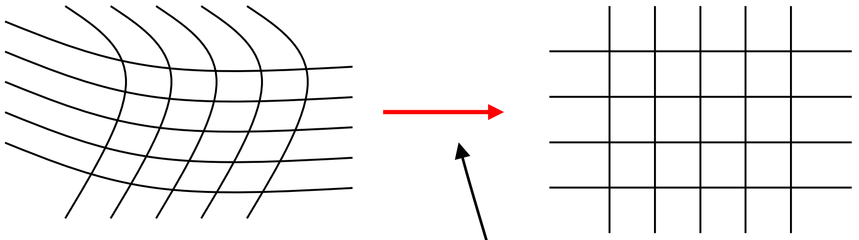
$$y^+ = \frac{y u_\tau}{\nu}$$

0012





Other issues



- Must define outer edge of computed zone
  - I.e. δ.
  - this may require iteration
- May need to transform to intermediate plane
  - i.e. *(x,y)* to (ξ,η) to remove `skew' from cells
- But for laminar flows, once b.l. equations are represented by finite differences/volumes, solution may be achieved.
- Turbulent flows require an additional step…

Turbulence modelling

- For turbulent flows, equations contain *Reynolds Stresses*

$$\rho\left(\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}\right)=-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}-\rho\overline{u'v'}\right)$$

- Remember from first turbulence lecture, we introduced *eddy viscosity* (ν_t), so that

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\left(\nu+\nu_t\right)\frac{\partial^2 u}{\partial y^2}$$

- You may wonder - what happens without ν_t?

- Laminar bl separates very easily and no steady

solutions are found -> ‘wrong’ result

Turbulence modelling

- So, we just need to find some way of representing eddy viscosity
- This however, can be a very complex process
- Involves empirical relationships
- Simplest methods are called *zero* equation methods,
  - don’t require any extra flow equations

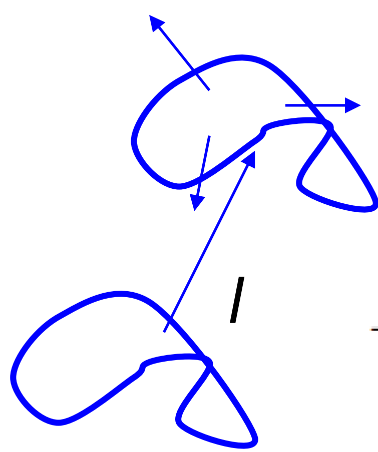
Zero Equation Methods

- Based on Prandtls *mixing length hypothesis*, which draws an analogy with molecular mean free paths.
- i.e. it is assumed that a `packet' of turbulence travels some distance *l* as a cohesive unit, then interacts with the surrounding flow:

$$u_2 \approx u_1 + l_1 \frac{\partial u}{\partial y}$$

$$|u'| \approx |u_2 - u_1| = l_1 \left| \frac{\partial u}{\partial y} \right|$$

$$-\rho \overline{u'v'} \approx const.\rho l_1^2 \left(\frac{\partial u}{\partial y} \right) \left| \frac{\partial u}{\partial y} \right|$$



$$-\rho \overline{u'v'} = \tau_t = \mu_t \frac{\partial u}{\partial y}$$

$$\mu_t = \rho l^2 \left| \frac{\partial u}{\partial y} \right|$$

Zero Equation Methods

- The eddy viscosity is given by

$$\mu_t = \rho L^2 \left| \frac{\partial u}{\partial y} \right| \qquad \text{i.e.,} \qquad \nu_t = L^2 \left| \frac{\partial u}{\partial y} \right|$$

- where L is a new constant, near the wall given by

$$L = Ky,$$ (*K* is the von Karman constant)

- Satisfactory near the wall outside of the viscous sub layer. Corrections are applied for intermittency in outer layer, and laminar viscosity v. near the wall

Turbulence models

- You will encounter zero, one equation and two equation models.
- This number refers to the number of additional differential equations solved to find ν_t
- Remember they are semi-empirical=semi-fictional! They do **not** have the solid Newtonian foundations of the Navier-Stokes equations
- Nevertheless, often they look similar, and their construction is based on sound experimental observation and experience

Turbulence models

- Cebbeci-Smith (0)
- Baldwin-Lomax (0)
- Spalart-Allmaras (1)
- Wilcox $k-\omega$ (2)
- Widely used in industry
- But note we have not covered transition in the context of these model – v. tricky

Spalart-Allmaras (1 equation model)

Creates Spreads Destroys Transition

$$\frac{\partial \tilde{\nu}}{\partial t}+u_j\frac{\partial \tilde{\nu}}{\partial x_j}=C_{b1}[1-f_{t2}]\tilde{S}\tilde{\nu}+\frac{1}{\sigma}\{\nabla\cdot[(\nu+\tilde{\nu})\nabla\tilde{\nu}]+C_{b2}|\nabla\nu|^2\}-\left[C_{w1}f_w-\frac{C_{b1}}{\kappa^2}f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2+f_{t1}\Delta U^2$$

Advection – moves ν around and analogous to $\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}$ in Navier-Stokes. Moderately 'physical'

These 'source' terms create/destroy/spread ν depending the time averaged values. Very empirical

Just an example –

don't memorise!

Summary

- You need a qualitative understanding of how a numerical method functions, meshing and turbulence modelling issues, and the mixing length hypothesis
- Appreciate the idea that it is possible to model the boundary layer independently, or coupled in a full Navier-Stokes method
- Next lecture (final one) – real boundary layer effects

Tom Rendall
thomas.rendall@bristol.ac.uk

Aerodynamics 3

Realistic Boundary Layers

(chapter 19 in notes)

University of BRISTOL

Today

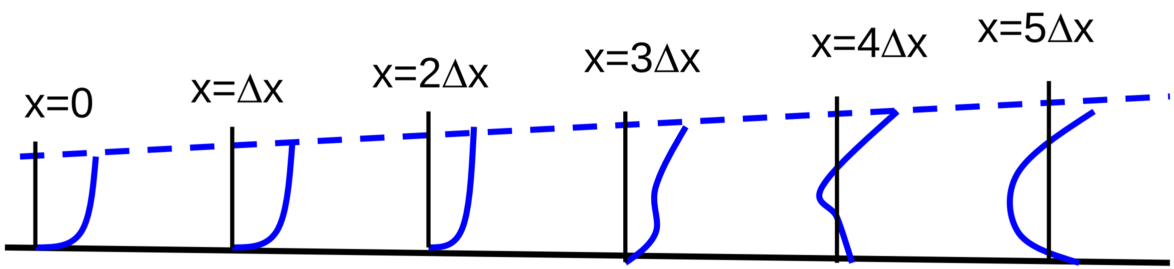
- Separation
- Shock-boundary layer interaction
- Laminar aerofoils and boundary layer suction
- Lift enhancement by blowing

Separation

- So far in the course we have only considered boundary layers which experience relatively mild external gradients, or in the case of a flat plate, no gradients at all.
  - results in thin, attached boundary layers
  - true for most of the flow over streamlined bodies at small incidence
- However, not true all the time

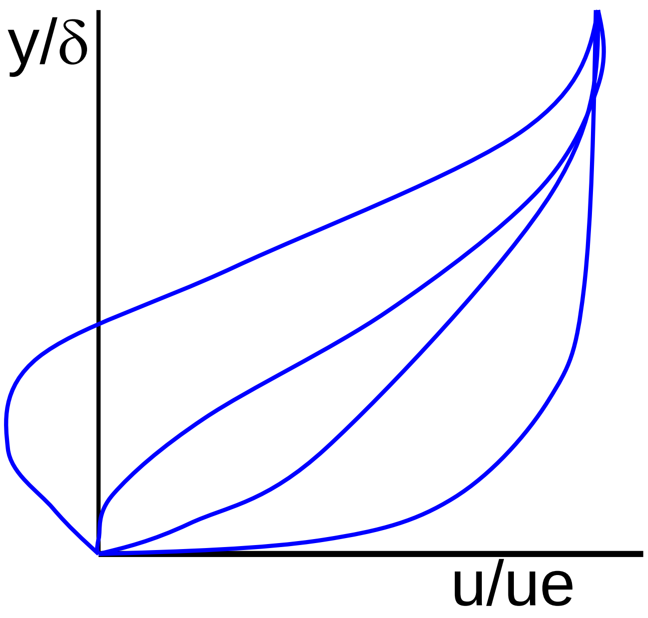
Separation

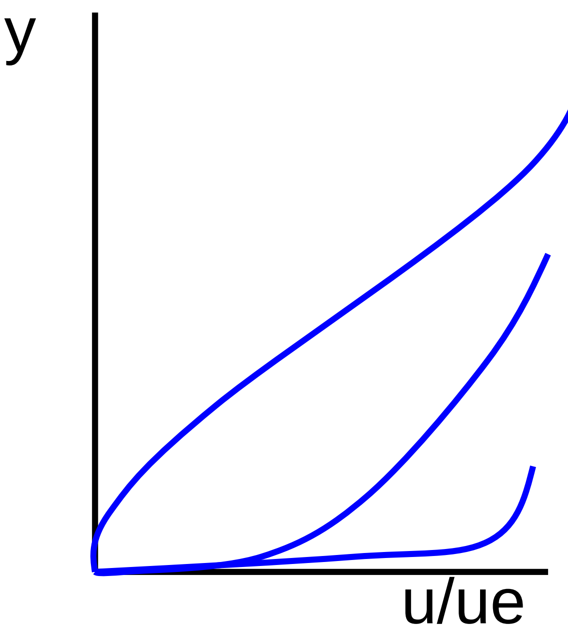
- Adverse pressure gradient (i.e. $\dfrac{dP}{dx} > 0, \rightarrow \dfrac{dU_e}{dx} < 0$)
- Occurs over rear part of most vehicles
  - in generating lift, air is accelerated
  - flow must therefore slow down towards rear stagnation region
- Some of the kinetic energy of the flow must turn to pressure (i.e. potential energy) to get through the adverse gradient, causes velocity profile to change:



Leads to

- Greater growth of boundary layer, hence higher drag
- Change in H due to change in velocity profile



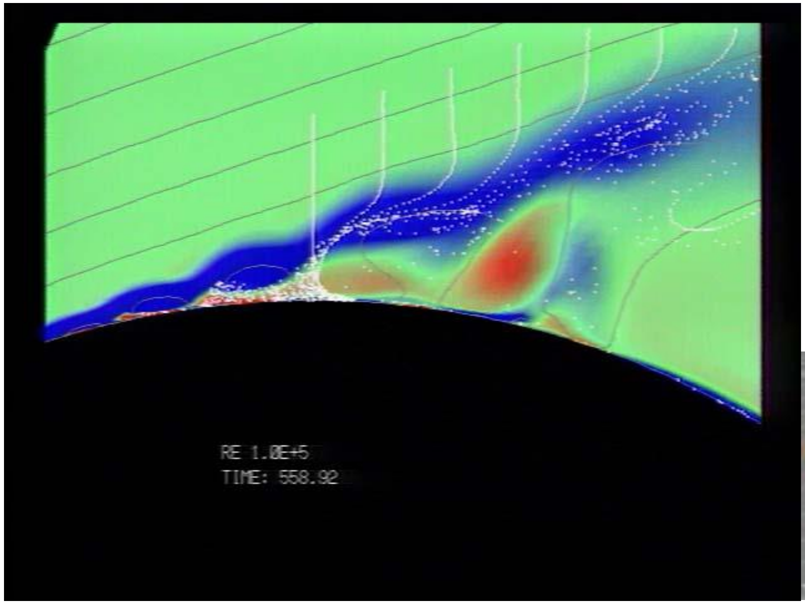


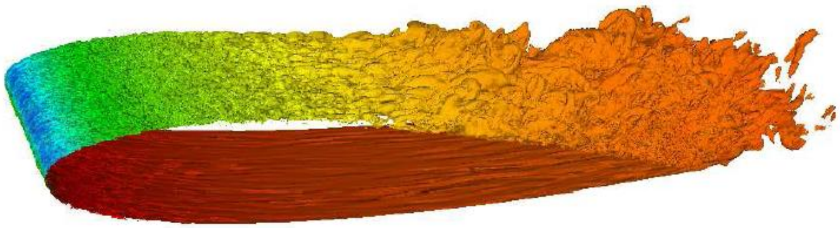
Results in

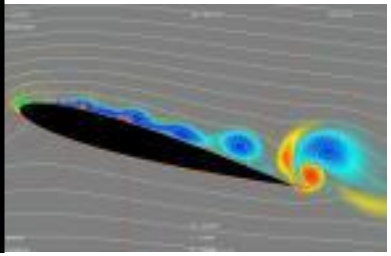
- zero skin friction at separation
- reversed flows, and changes in flow behaviour
  - unsteady
  - three dimensional
  - dimensions significant w.r.t. body
- Therefore violates most of the assumptions we have made in this course – methods described cannot be used for any significant separation (exception is separation bubble, but needs care)

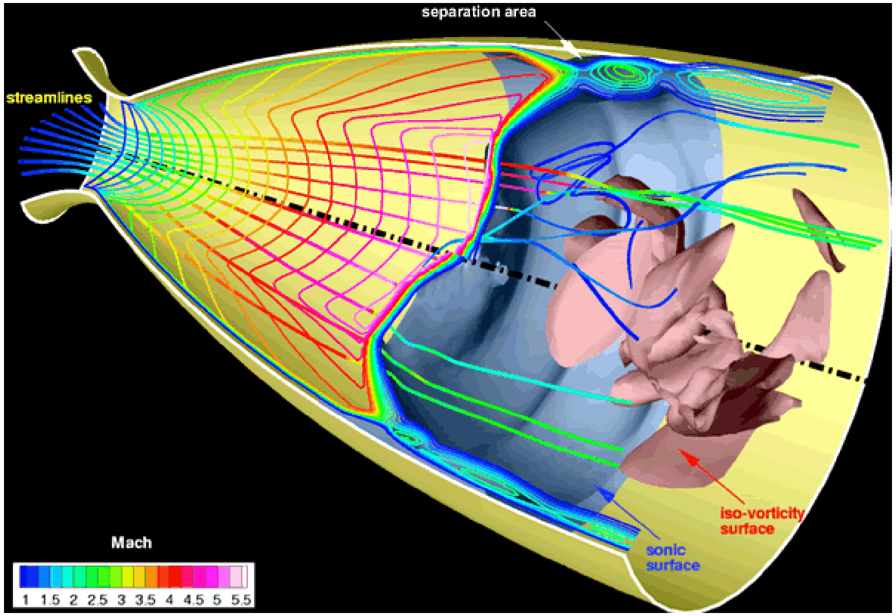
Location of separation

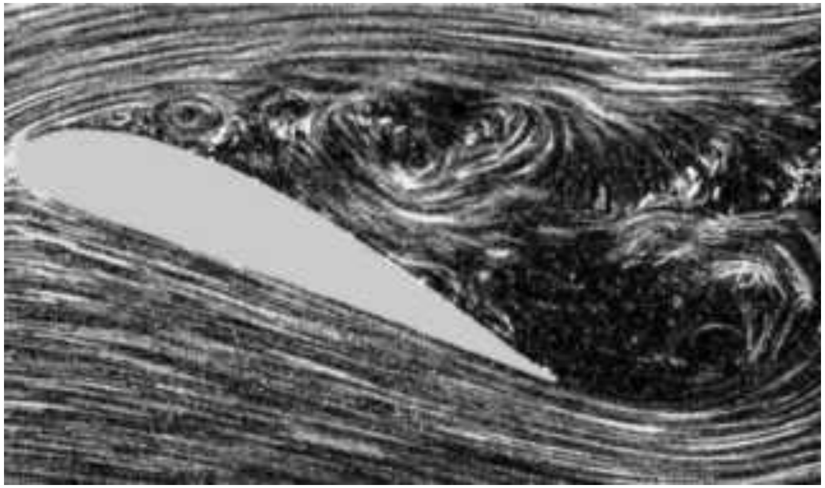
- Location can be very difficult to predict
- depends on pressure gradient, plus
  - geometry
  - laminar/turbulent
  - shock wave interaction
  - turbulence in main flow
  - 3D effects (i.e. gradients in other directions)
- Can try to predict using NS solutions + turb model, but results should be viewed with caution


RE 1.0E+5
TIME: 558.92





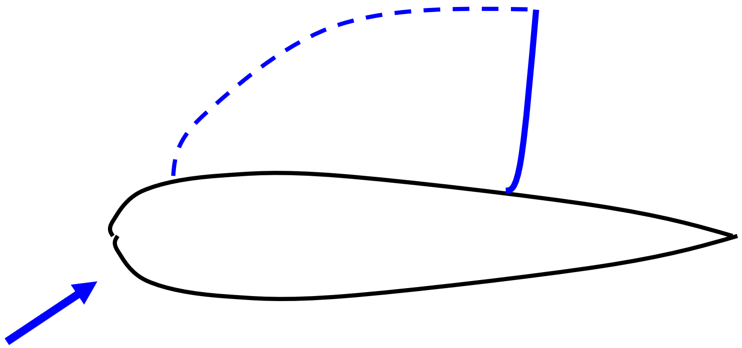

separation area
streamlines
iso-vorticity surface
sonic surface
Mach
1 1.5 2 2.5 3 3.5 4 4.5 5 5.5

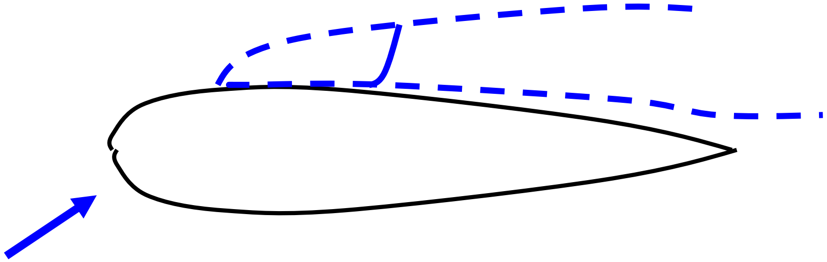


Shock interactions

- Obviously only occurs in supersonic flows, or flows with regions of supersonic flow, e.g.
  - flow over wings of aircraft in transonic flows
  - supersonic vehicles
  - rocket and jet exhausts
- Boundary layer allows communication of presence of the shock wave upstream at the wall.
- Adverse gradient causes thickening of boundary layer, and influences main flow:

On a transonic aerofoil

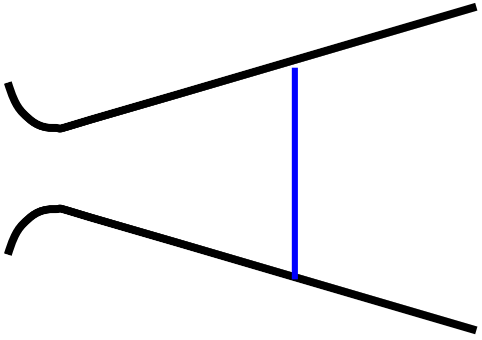


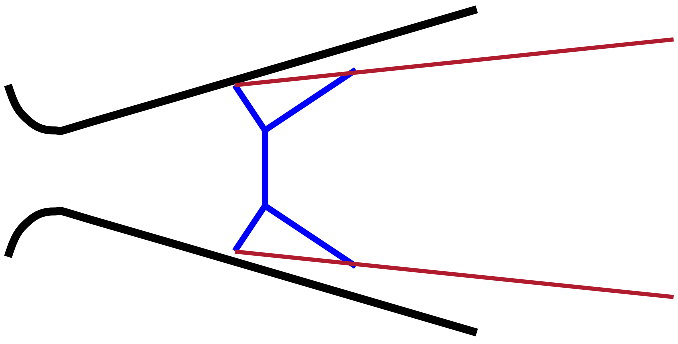


Inviscid

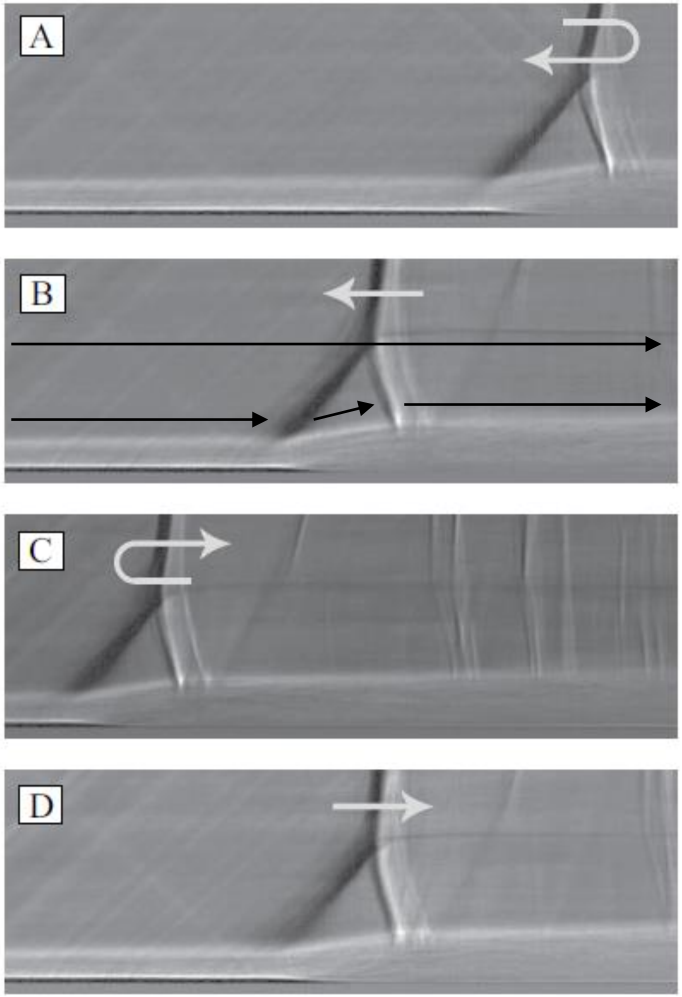
Viscous

In a nozzle

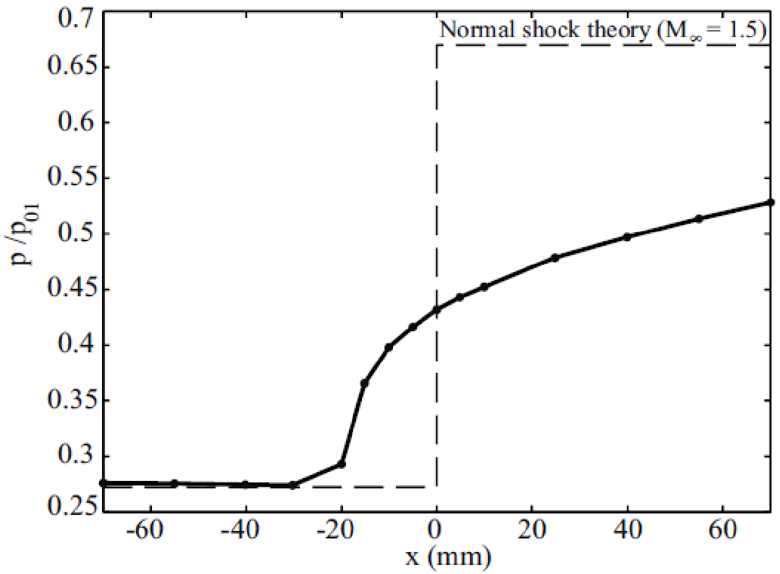


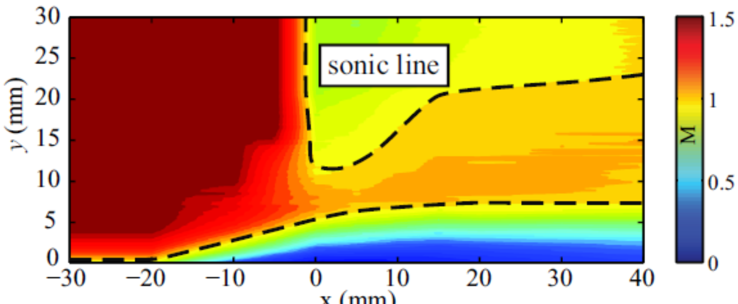


Normal shock-boundary layer interaction



(measured at wall)

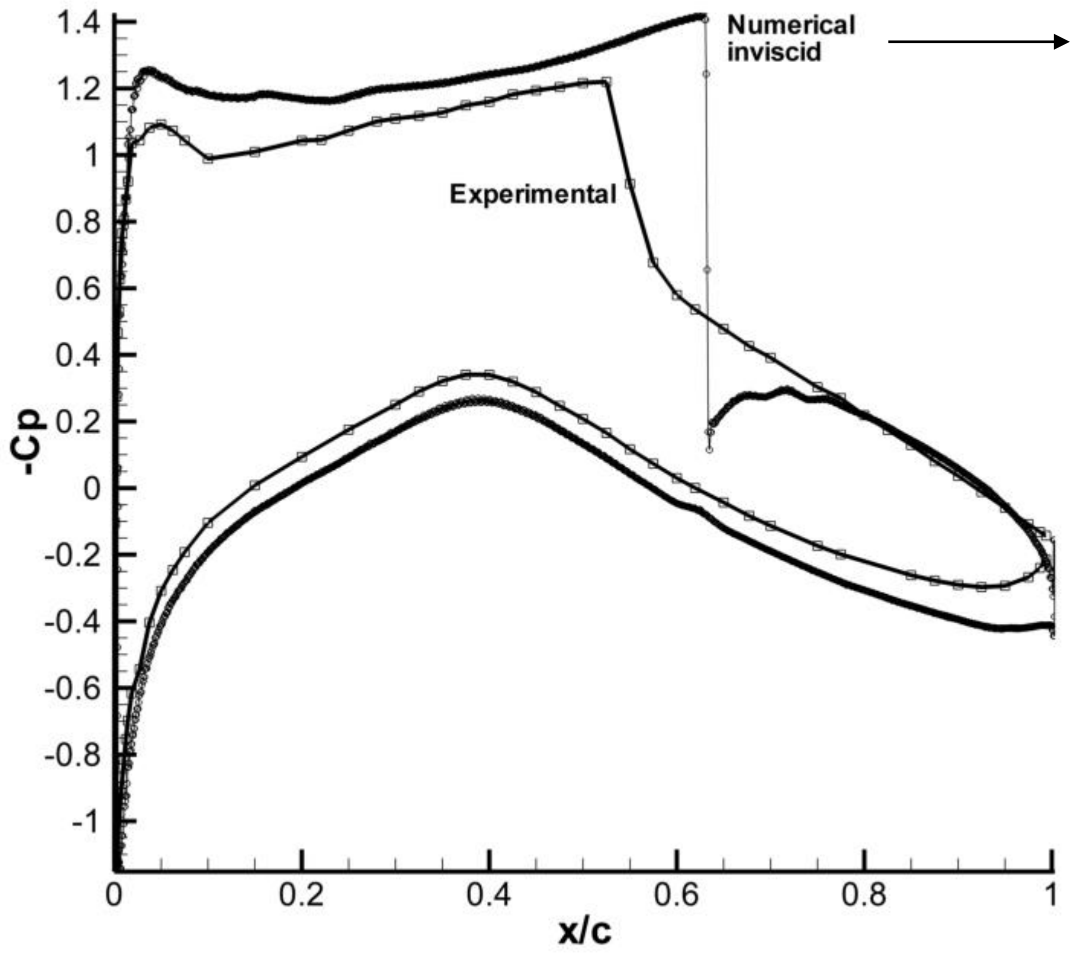


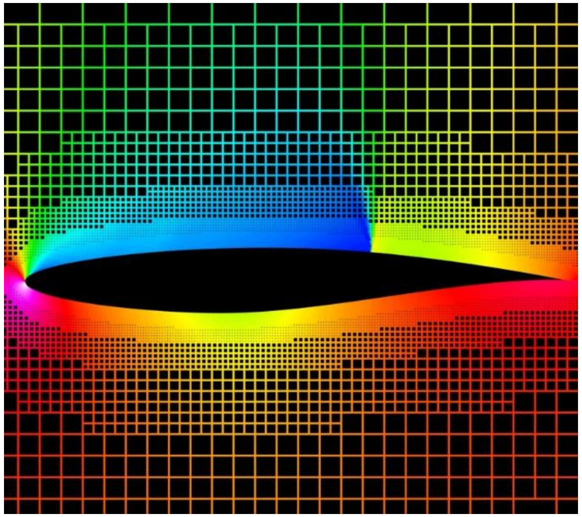


(Laser Doppler experimental picture of

Mach number)

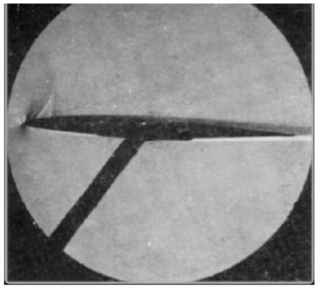
Cp comparison RAE 2822, M=0.729, 2.31deg



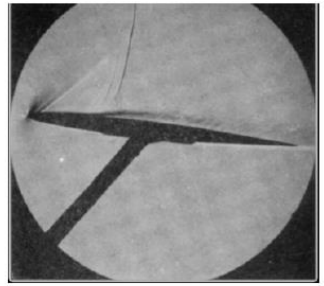


Shock induced separation

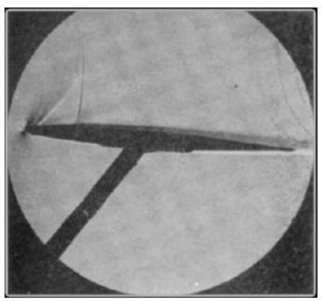
M=0.75, 6% thick



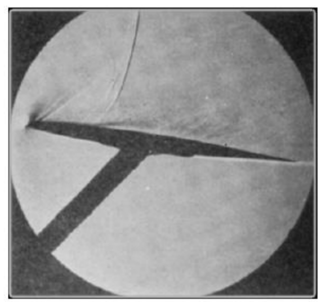
a. α = 2.7 deg



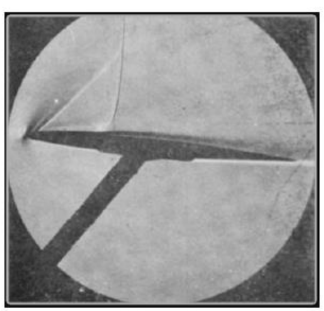
d. α = 5.7 deg



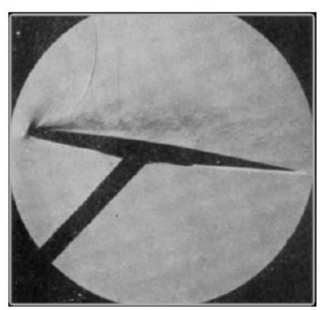
b. α = 3.7 deg



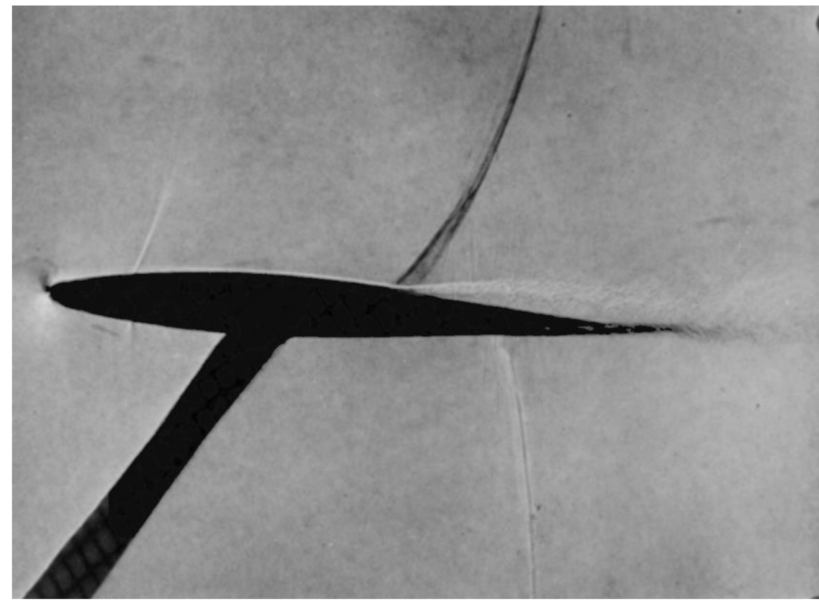
e. α = 6.7 deg



e. α = 4.7 deg



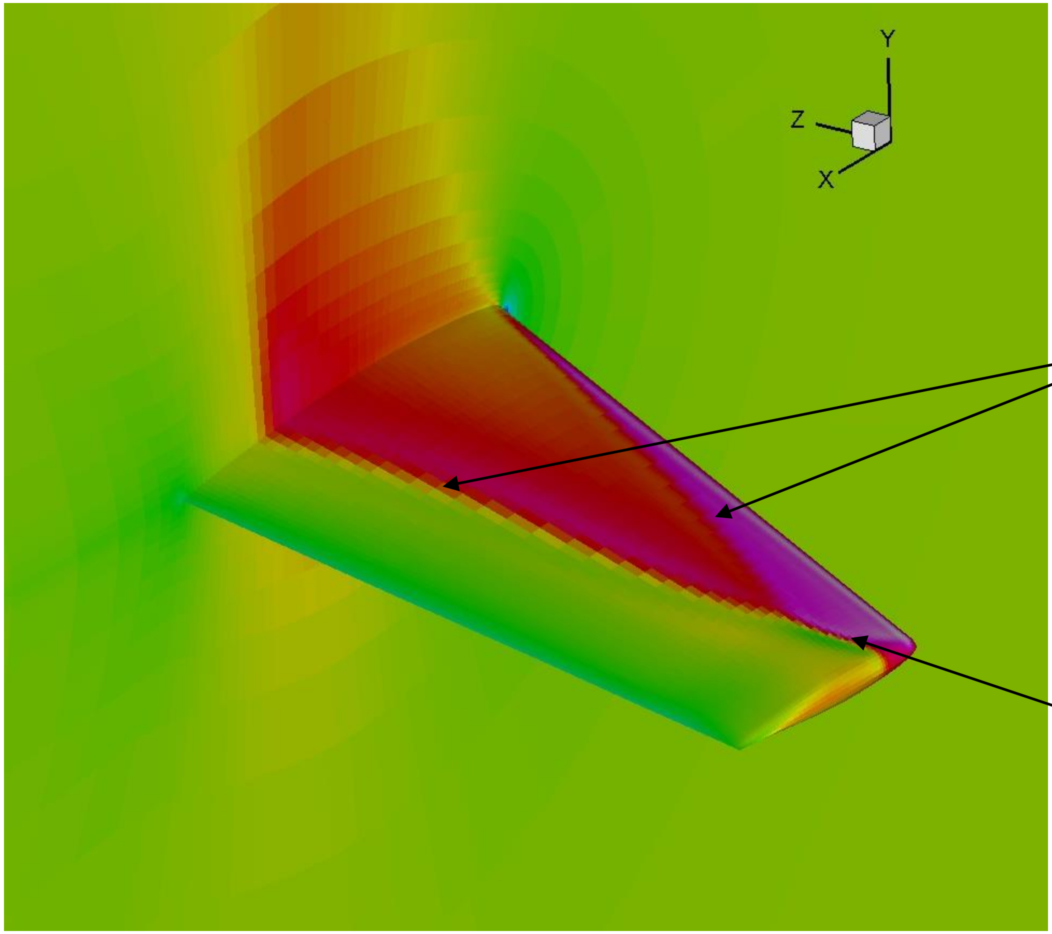
f. α = 7.7 deg



M=0.88, AoA=2deg, 10% thick

3D inviscid

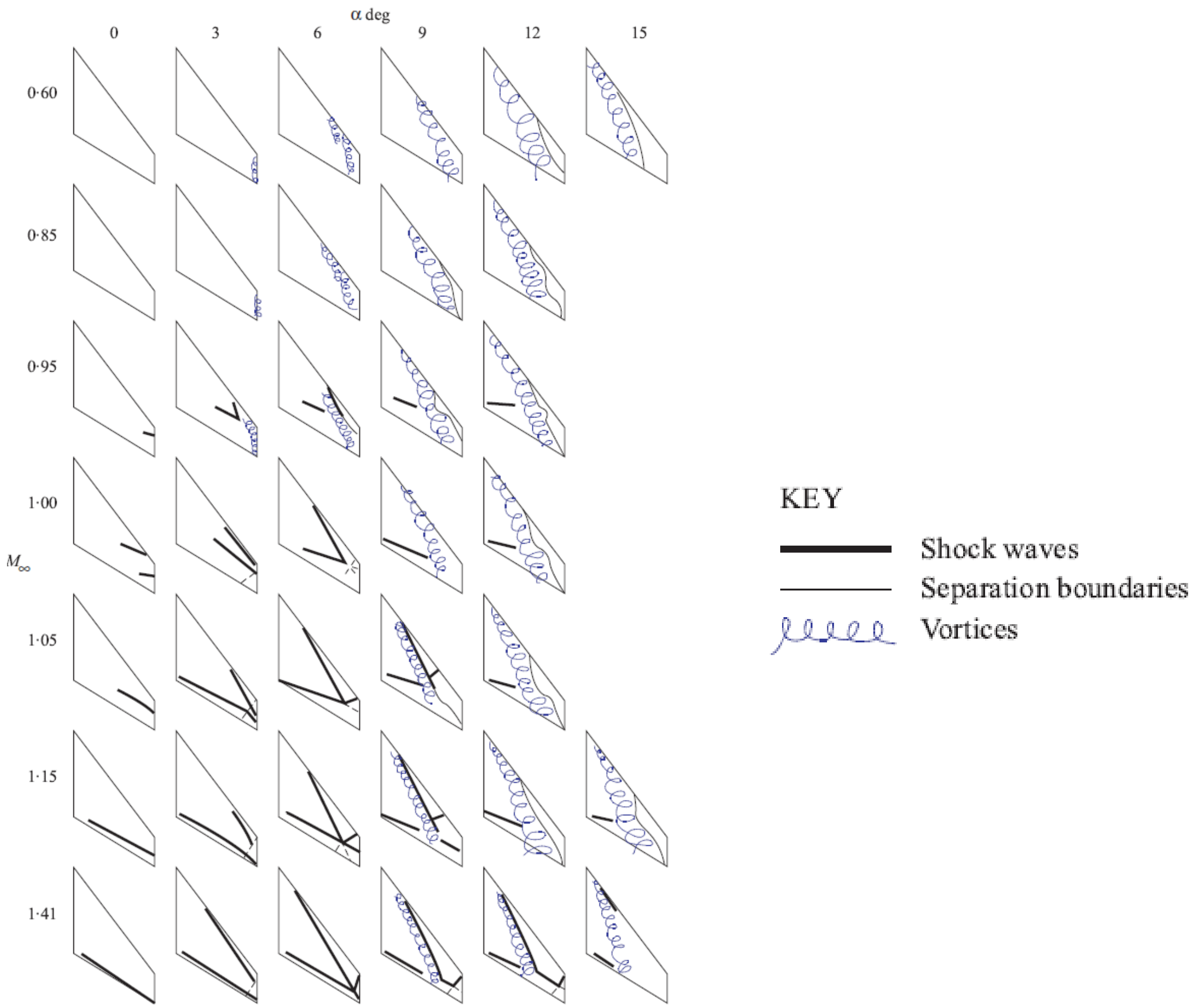
SBLI even more complicated in 3D



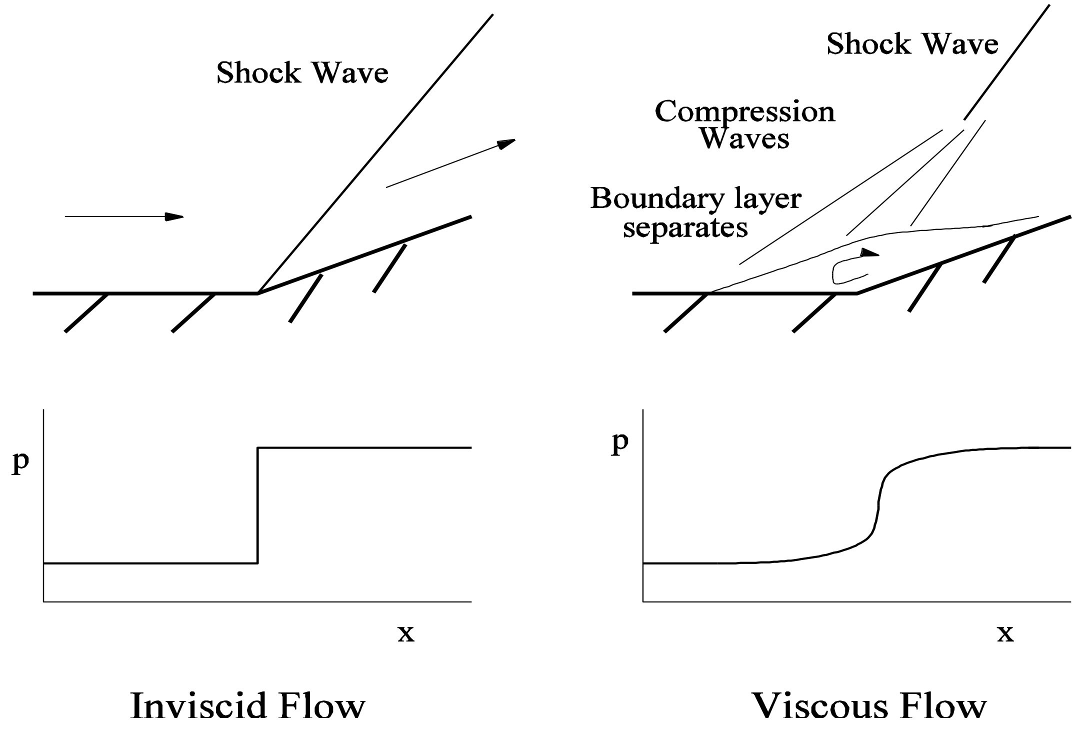
Two shocks

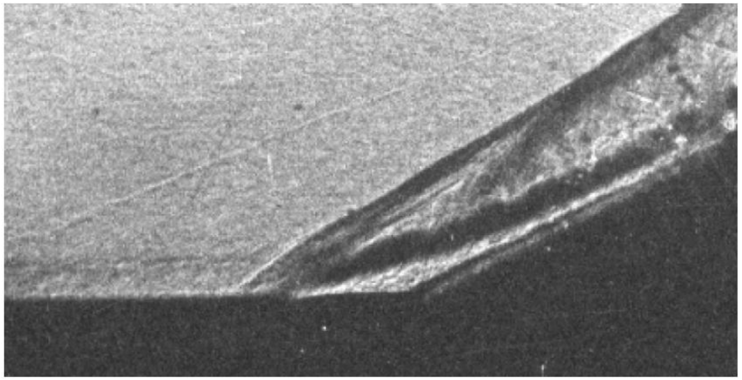
Merge at tip

3D shock+viscous effects



Oblique shock-boundary layer interaction



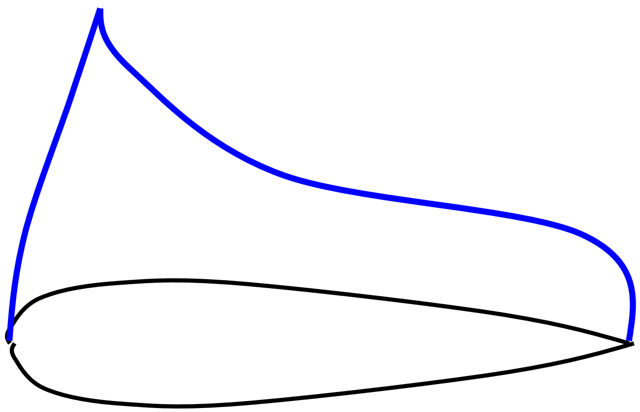


Drag Reduction

- We have already seen that the boundary layer properties effect the overall flow in terms of drag, separation, etc.
- If we can control the boundary layer, can control drag, at least to some extent
- Two examples: laminar flow by design, and by suction

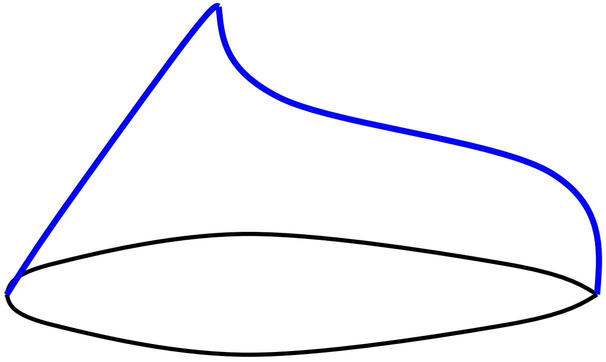
Laminar flow aerofoils

- Concept is simple enough
  - laminar flows have less drag (provided no separation)
  - More laminar boundary layer, less drag!
- Dates from about 1940
- Transition on an aerofoil is dominated by pressure gradient
- Can control the pressure gradient geometrically



BUT

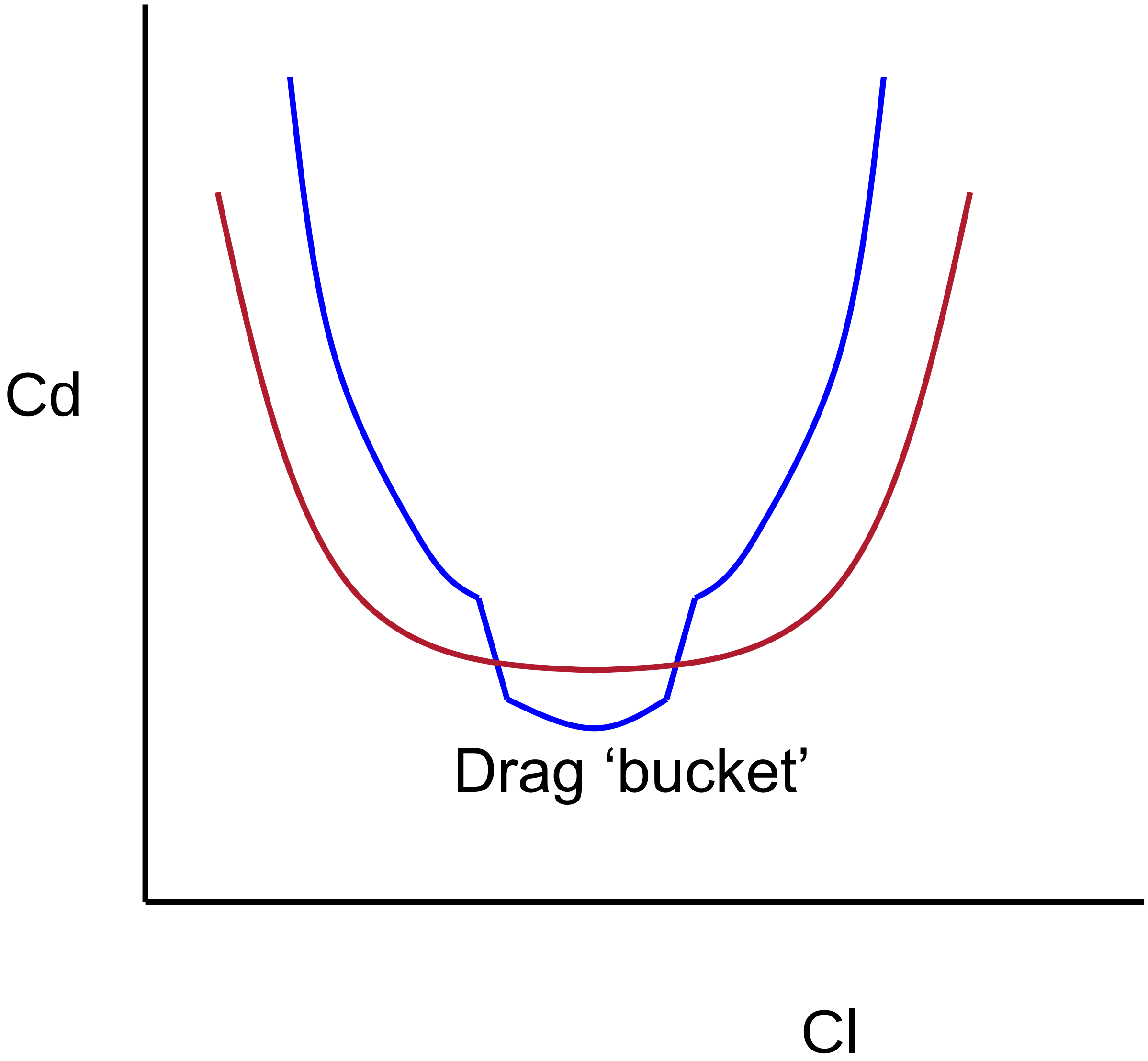
Susceptible to contamination – insects, rain, paint, dirt



By shifting thickness aft, reduce adverse pressure gradient = more laminar flow, less drag

However

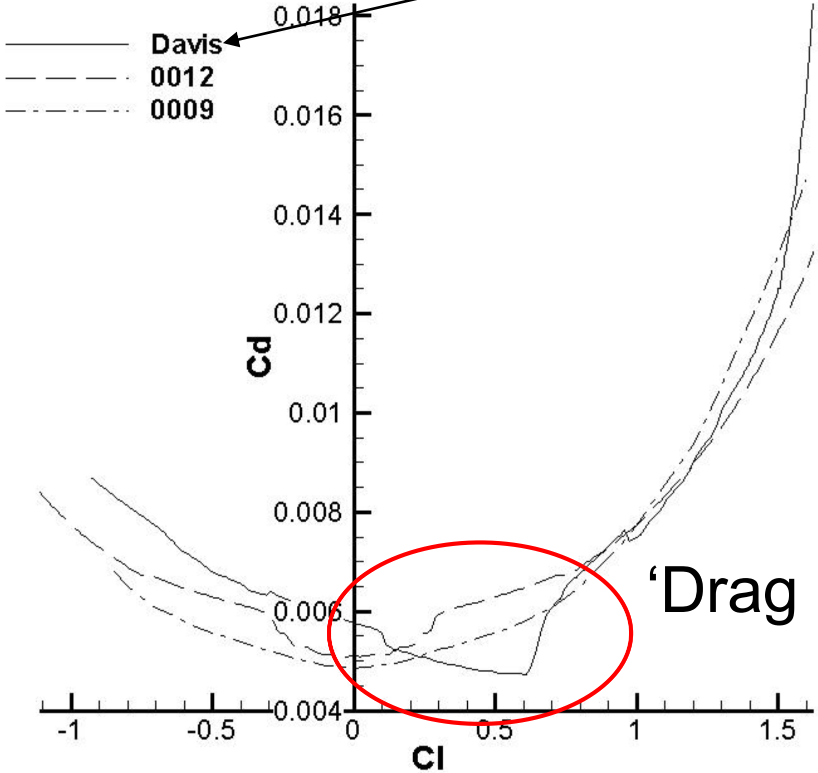
- Changing incidence moves Cp minima forward (as it does on all aerofoils
- In laminar case, moves more and faster
- means that get good performance only for a narrow range of alpha (drag bucket)
- Not good for transonic flight – laminar bl more likely to undergo shock induced separation


Cd
Drag ‘bucket’
Cl

B-17/B-24

The Davis Wing and the Problem of Airfoil Design: Uncertainty and Growth in Engineering Knowledge
Author(s): Walter G. Vincenti
Source: *Technology and Culture*, Vol. 27, No. 4, Special Issue: Engineering in the Twentieth Century (Oct., 1986), pp. 717-758
http://www.jstor.org/stable/3105326

Davis aerofoil showed larger region of laminar flow



'Drag bucket'

B-24

Davis

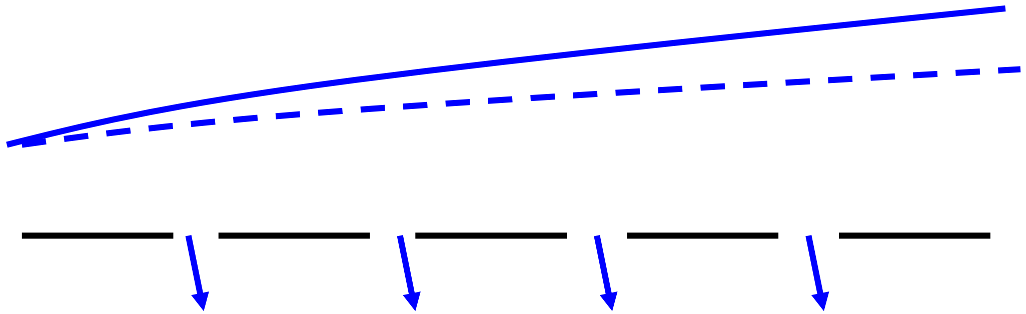
B-17

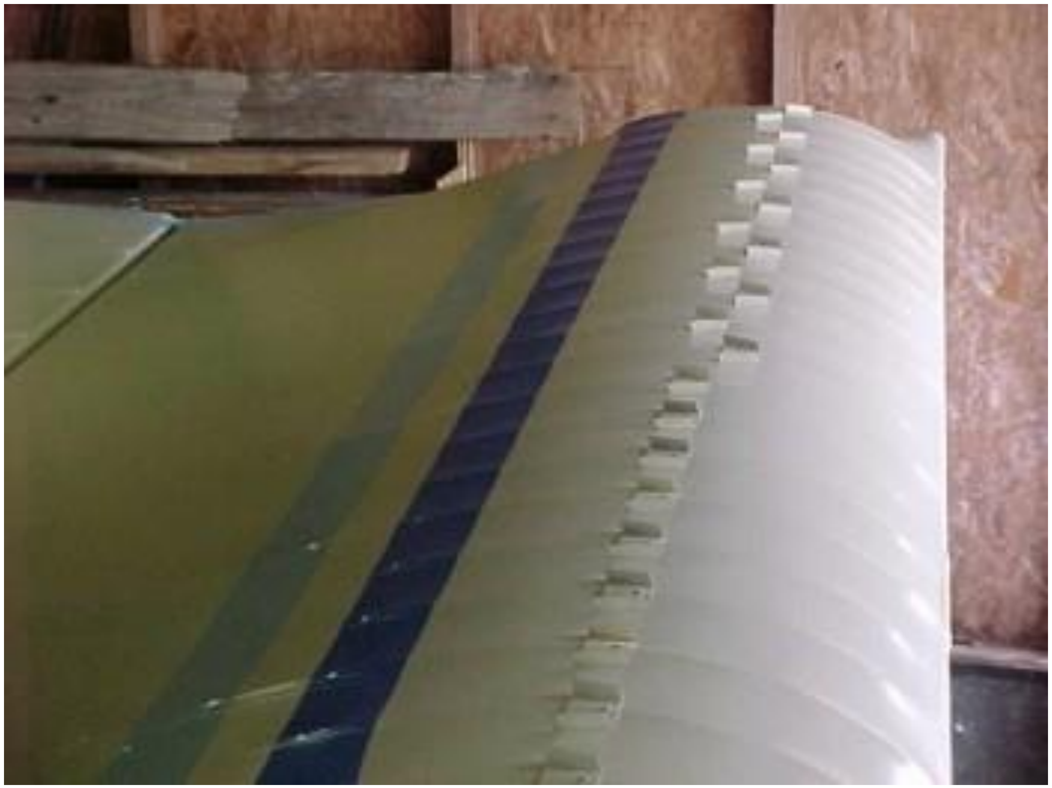
NACA

Any differences most likely due to AR difference (>11 compared to 7.5)

Boundary Layer suction

- If you take mass out of the boundary layer it behaves as if it had a lower Re
- Can do this by having a porous wall and sucking
  - is therefore an `active' measure, unlike laminar flow aerofoils which just rely on geometry
- extends amount of laminar flow
- requires power
  - Also problems of dirt, blockage of very small holes, etc!
  - Possibly 15% drag reduction – but system also requires power to run. Complicated to implement + maintenance costs





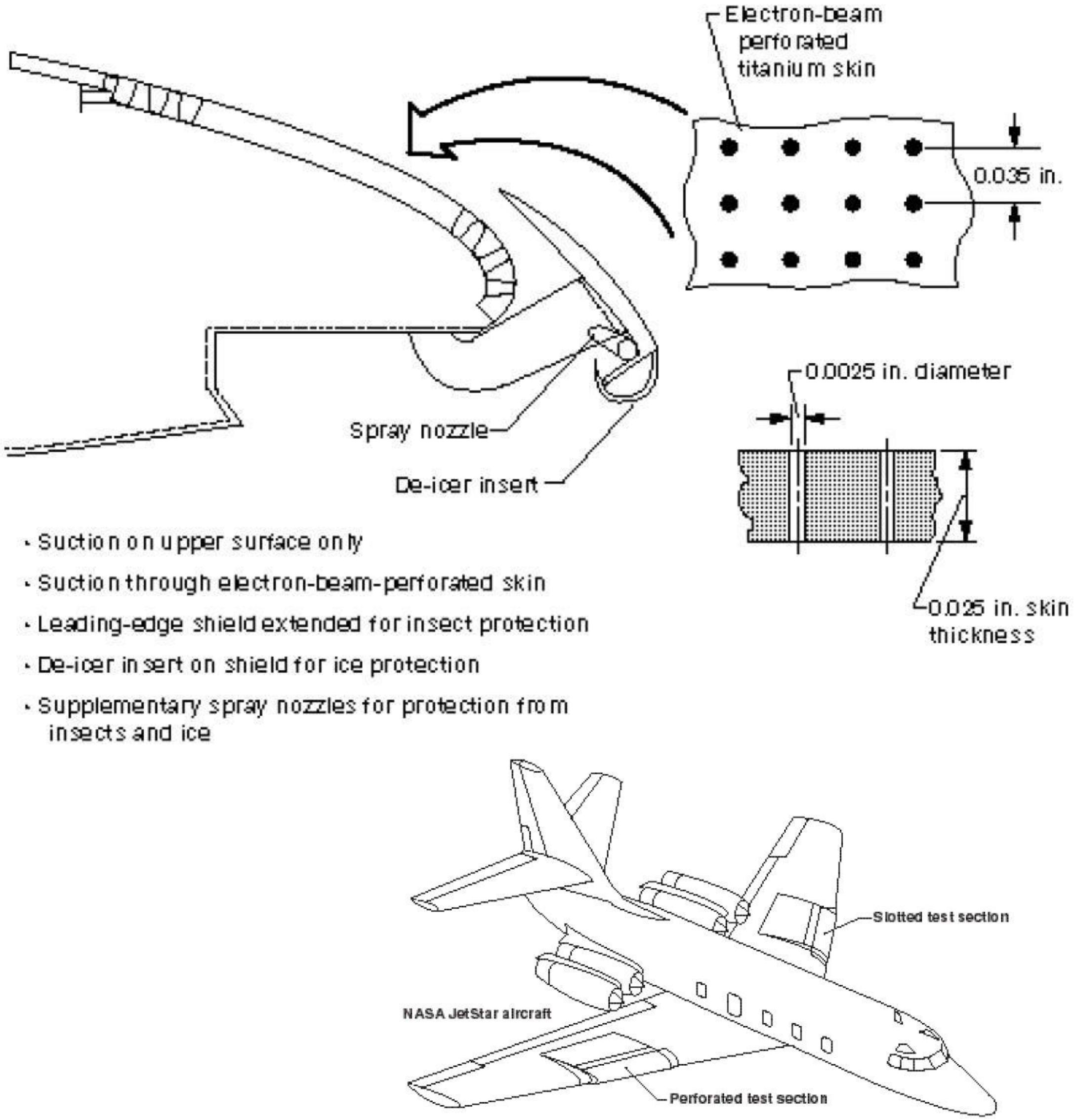


Figure 12. Leading-Edge Flight-Test program perforated test article.

Lift enhancement

- Can increase Cl by injecting high speed air across a flap, or over the leading edge of a wing
- Contributes to lift directly, and also prevents separation of the boundary layer, raising Clmax

Boundary layer blowing

Stall speed<30mph!

Hunting H.126

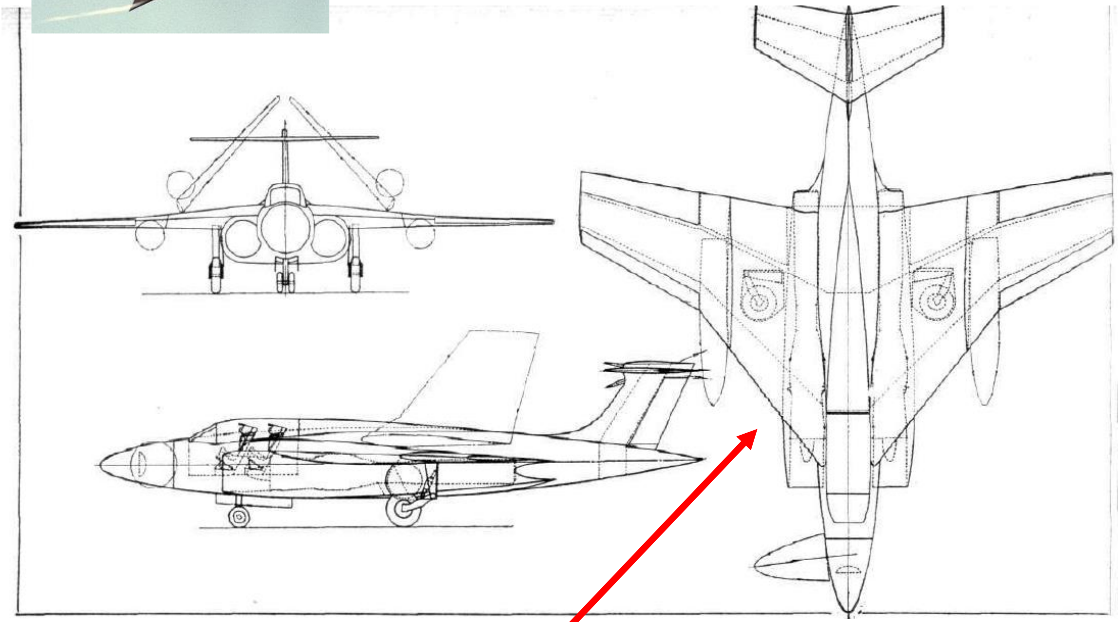
Also works for de-icing!

Buccaneer

Requirements – **carrier based** with **high top speed**

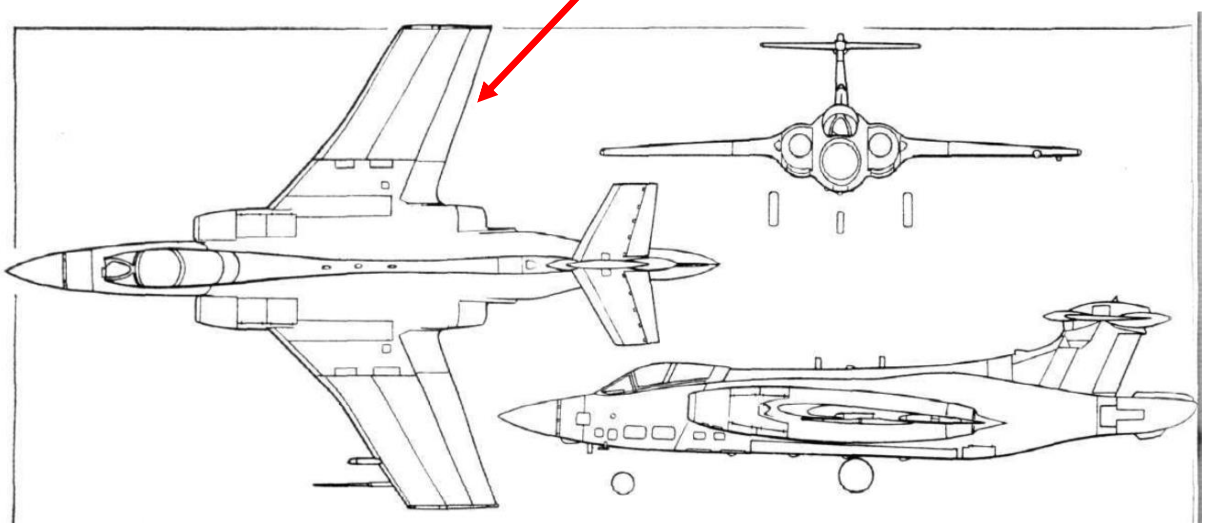
No bl blowing

=big wing, low top speed

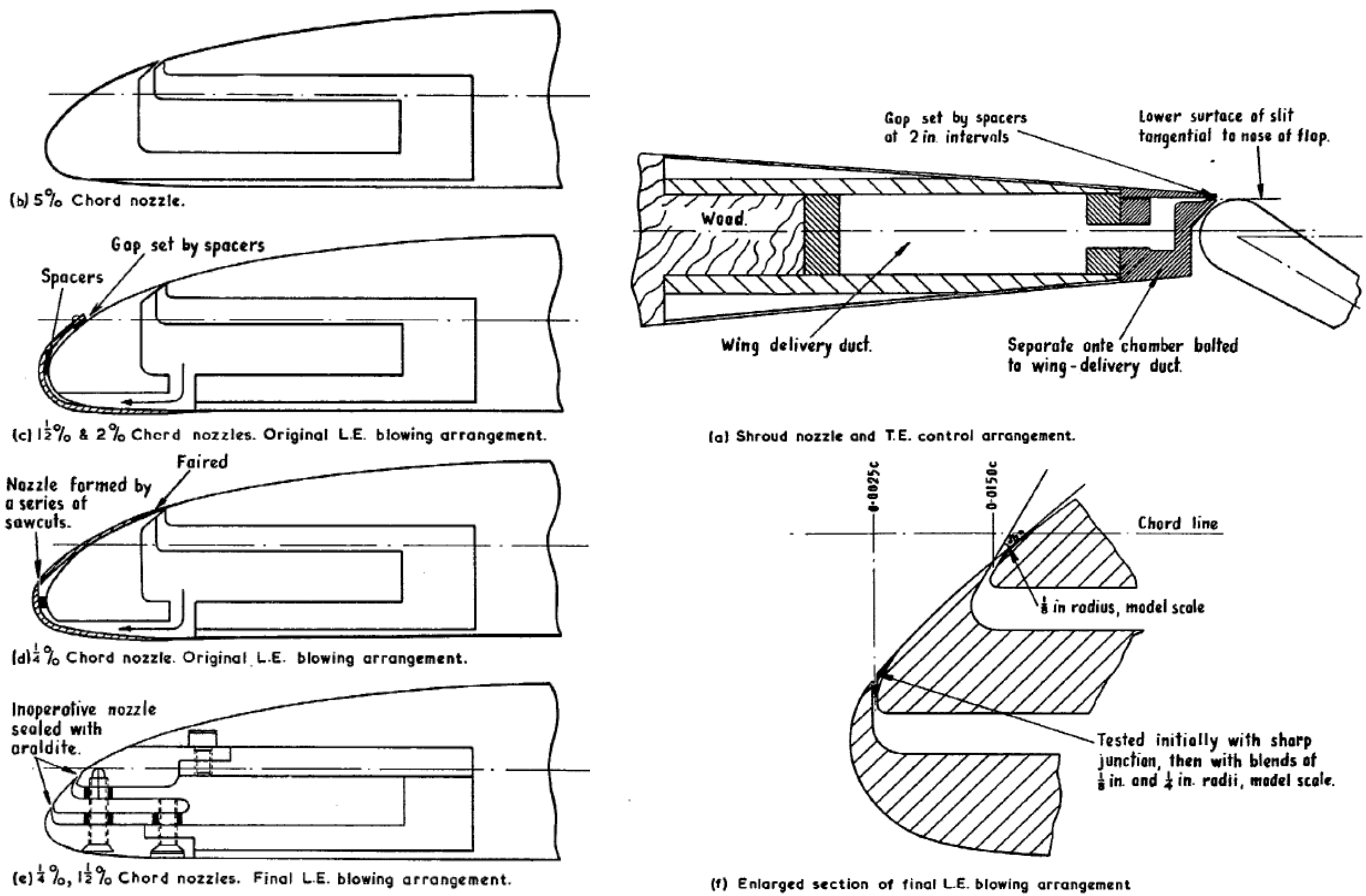


With bl blowing

=small wing, high top speed



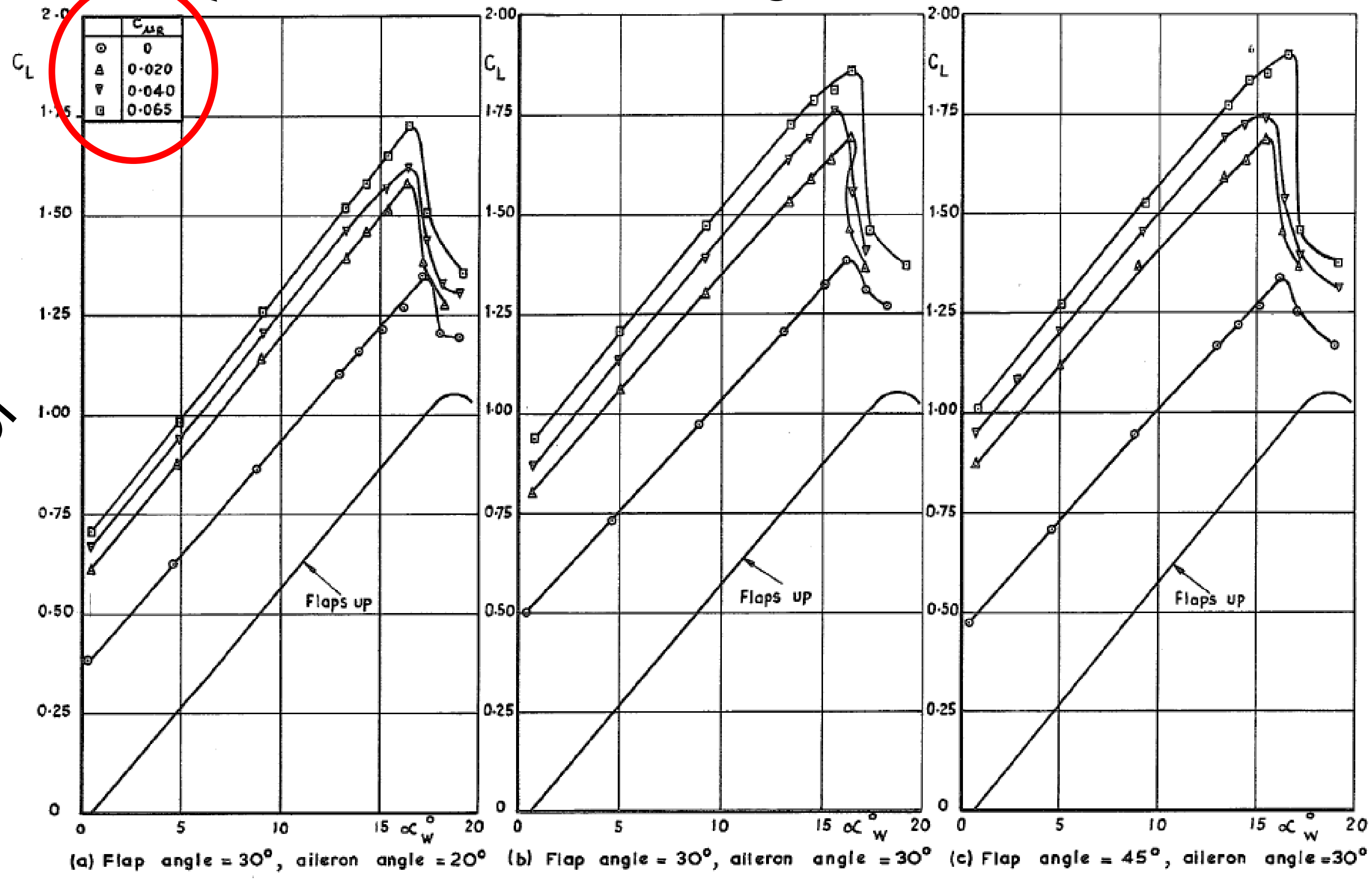
Buccaneer wind tunnel model arrangement



Effectiveness of flap bl blowing - Buccaneer

Amount of blowing

Wing Cl



(aileron drooped to behave as a flap)

Tom Rendall

thomas.rendall@bristol.ac.uk

Aerodynamics 3

Solution methods for aerodynamics

(chapter 20 in notes)

University of BRISTOL

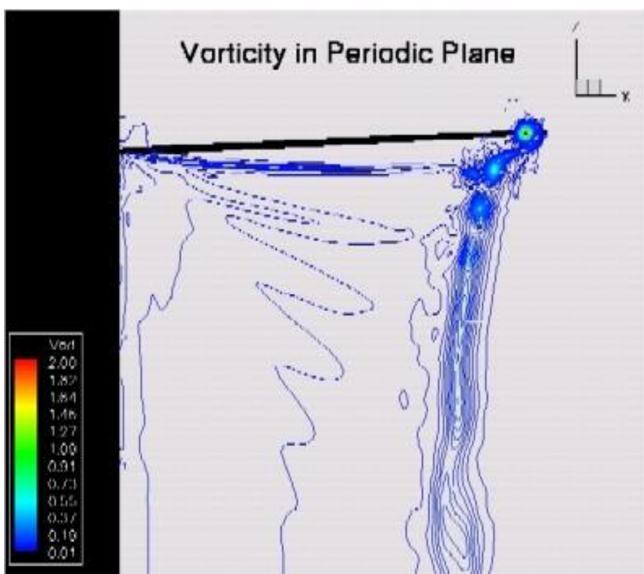
Introduction

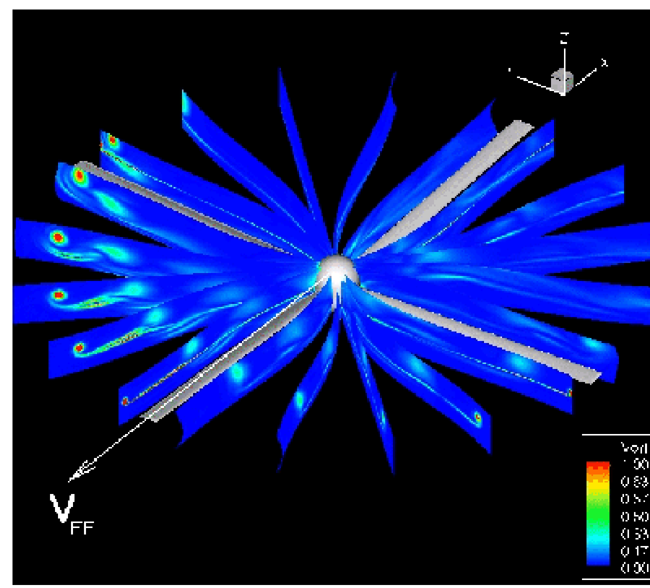
- There are a wide range of aero methods, which differ by many 100s of times in terms of accuracy, speed and range of applicability
- Even with the most elaborate methods some problems are still very hard to do – eg stall
- The key is applying the right method for what it is that you want to know. Navier-Stokes modelling is not always the way forward! Your job is evidently to get a reasonable answer in a reasonable timeframe
- Quick look at developing areas

University of BRISTOL

Introduction

- 2D/3D? Viscous/inviscid? Turbulence model? Attached/separated? Steady/unsteady? Rotating frame of reference? – eg. rotors, turbines
- Will consider influence of 2D/3D, compressibility, viscosity and time dependence today





Introduction

Joe Sutter
“Father of the 747”

- Aerodynamics and aeroplanes existed long before computers. They managed to fly, and even flew efficiently and safely
- Preliminary design used analytical and semi-emiprical tools. These are still used today and are available in NASA and ESDU reports, all of which you can access. They are the **first stop when making any estimate**
- Detailed design used wind tunnel results
- Nowadays we have a third alternative – a computational method

University of BRISTOL

Examples

You wish to determine the position of the neutral point on an aircraft

Attached flow

yes

incompressible

yes

VLM or data sheets

You wish to determine Clmax for an aerofoil

Standard design

yes

Look up data in a table/ semi-empirical

no

NS CFD

Error ~10%

University of BRISTOL

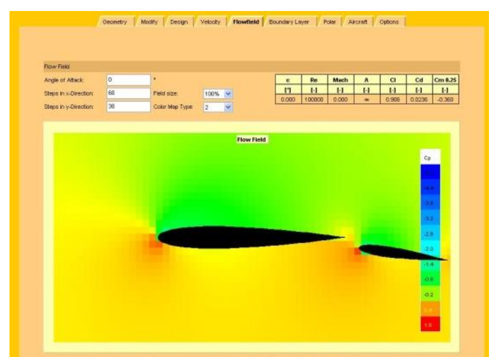
Options

- **3D** only a small complication, but can increase CFD computation time ++. Negligible cost for panel methods
- **Compressibility** no complication for a CFD approach. Not feasible for panel methods
- **Viscosity** a complication for panel methods. Requires much bigger higher quality meshes for CFD, with stretched layers near the walls
- **Unsteady** a big complication for panel methods (exception is the VLM/DLM) – especially if wake important – eg rotors

2D incompressible inviscid/viscous

- Inviscid - panel methods/lumped vortex method (2D vortex lattice). Good for multi-element problems (attached only), or wall/ground effects. Easy to write your own (about 100-500 lines) – also freely available. Try Javafoil.
- Viscous - panel method + IBL (integral boundary layer) model - eg. Xfoil. 10000s of lines - freely available. Attached only.
- Could use a NS solver if separation important, but for attached flows panel method + IBL faster and just as accurate, if not more so!

Xfoil example



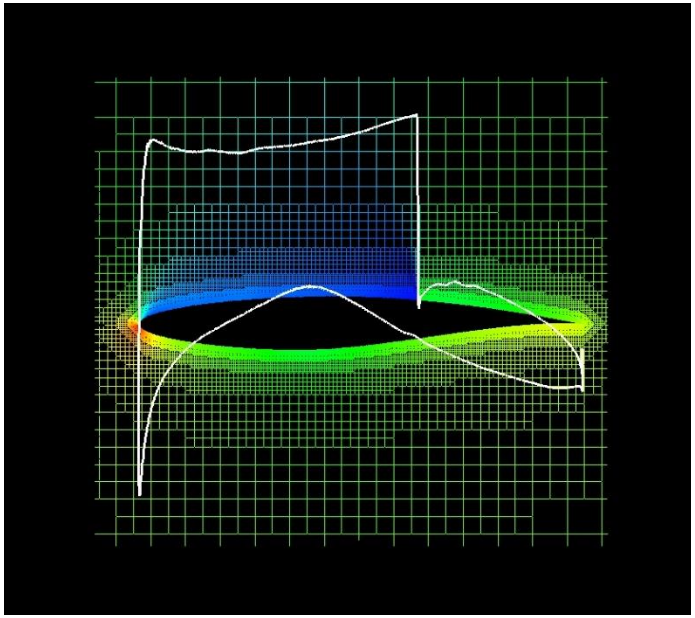
Javafoil

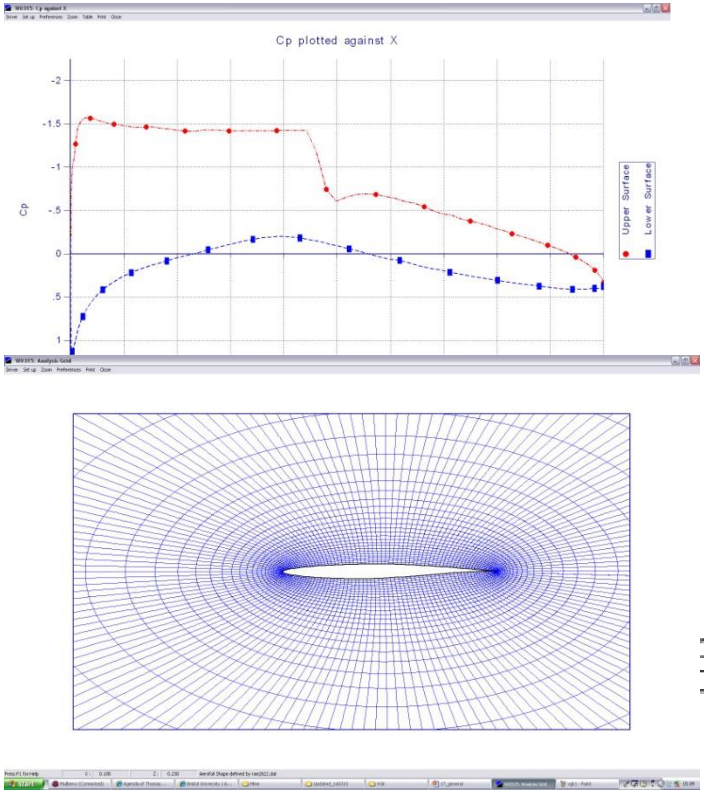
University of BRISTOL

2D compressible inviscid/viscous

- Nonlinearity of compressibility (shocks) requires a CFD approach with a finite volume mesh (if no shocks, Prandtl-Glauert scaling works very well)
- Options are small perturbation potential (rarely used now), full potential (sometimes used), Euler (common) or NS (most detailed option).
- Try VGK (free, single elements only) or MSES (licenced, multi-elements). Both have IBL options. 10000s of lines - freely available. VGK may be accessed via ESDU

VGK example

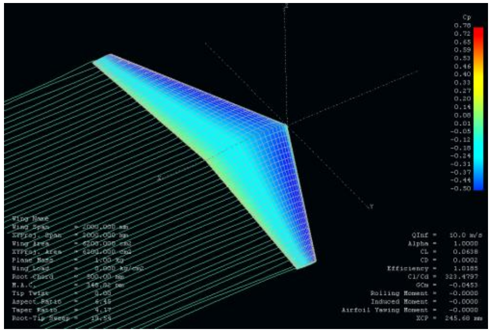




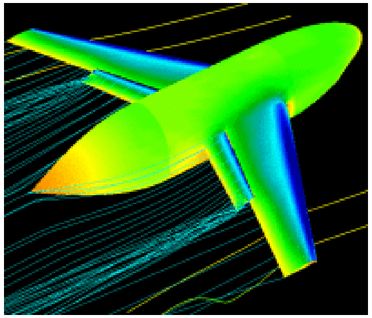
University of BRISTOL

3D incompressible inviscid

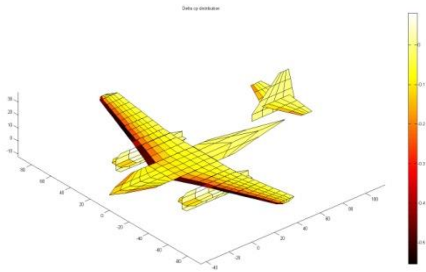
- Vortex lattice - basic method very simple. Basic method 500-1000 lines. Very easy to use - few complications. Try AVL or Tornado - freely available. Wake treatment is straightforward.
- To include thickness with lifting effects is trickier in 3D as the wake becomes more elaborate. Care needed with the Kutta condition. Many complications introduced. 10000s lines. NEWPAN, PANAIR (Boeing) - proprietary. XFLR5 is a free option for wings only.



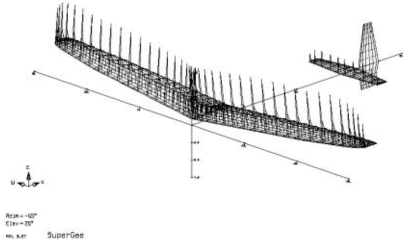
XFLR5



NEWPAN



TORNADO



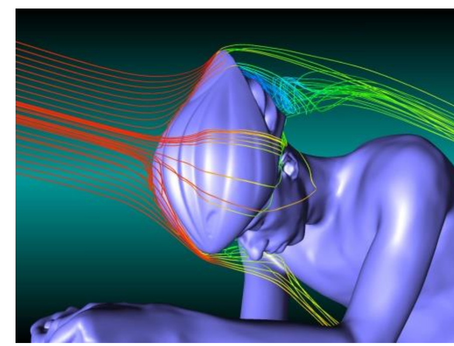
AVL

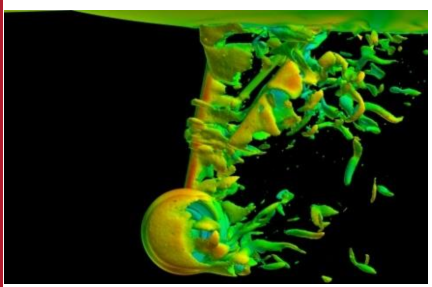
AVL example

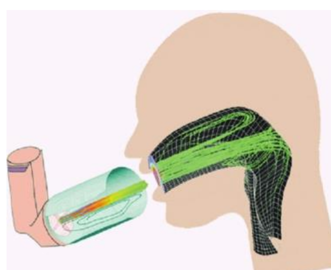
University of BRISTOL

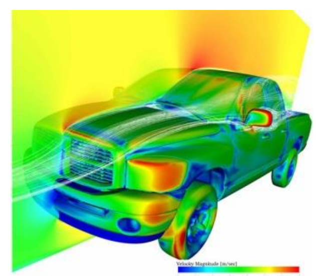
3D incompressible viscous

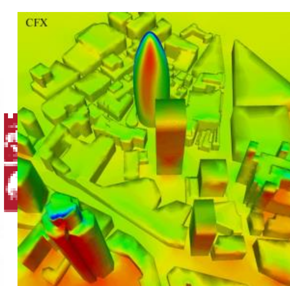
- Tends to be a division between incompressible and compressible NS solvers. Some do both, but this is not the norm. Not much point in incompressible inviscid CFD since this can normally be done faster by panel methods. Incompressible viscous calculations are v. common – cars, asthma inhalers, buildings, weather etc.





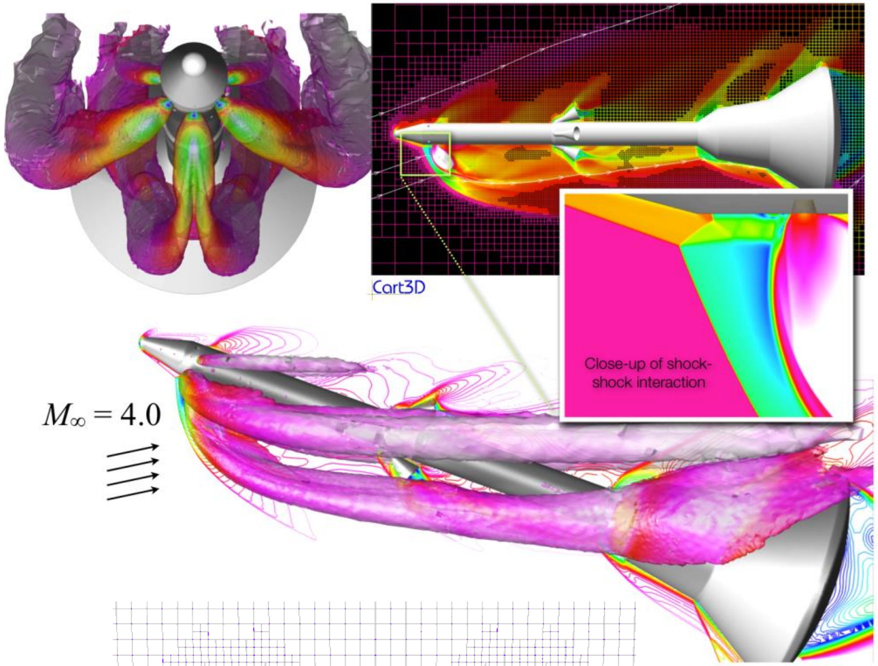


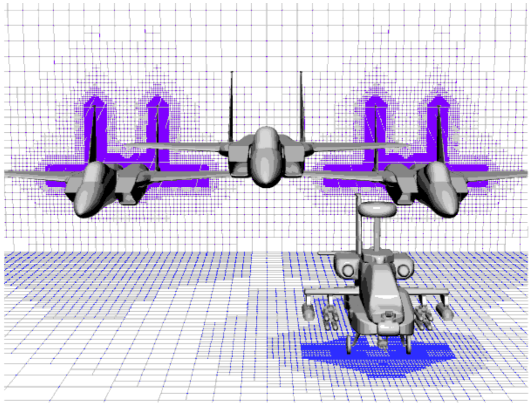


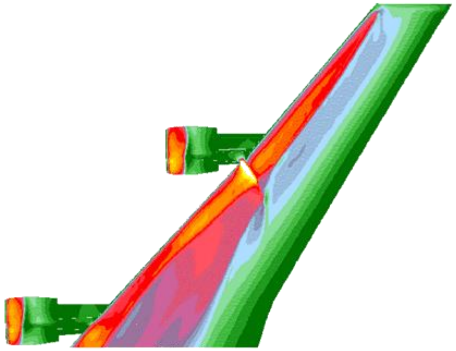


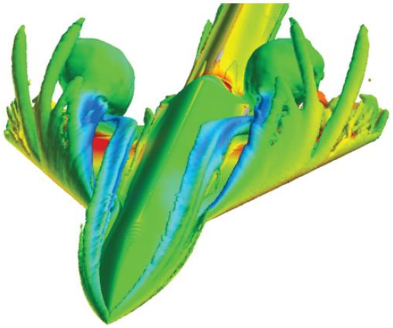
3D compressible viscous/inviscid

- TRANAIR (Boeing, full potential with IBL), Flite3D (Euler with IBL), Cart3D (Euler), Fluent, Star-CD, Start-CCM+ and many others - mostly proprietary. Free exceptions are: OpenFOAM, FreeCFD, *Code_Saturne* – fans of Linux should try CAELinux
- Any Navier-Stokes solver will (should!) also have an inviscid option
- Aerospace tends to use in-house or government developed methods for better accuracy and code transparency
- Generally harder to find a good mesh generator than a good flow solver









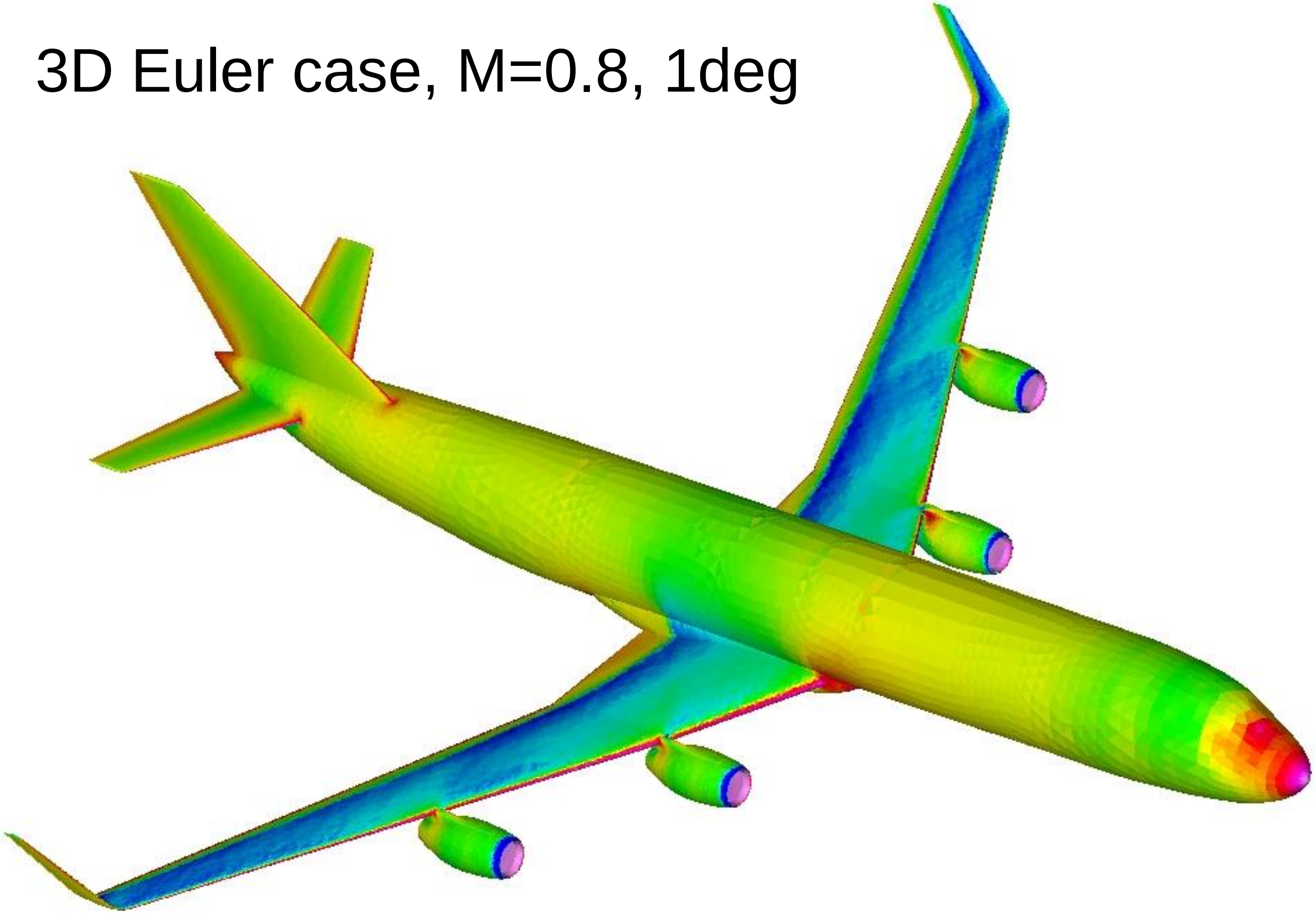
3D compressible viscous/inviscid

FP is a 3D full potential code freely available from ESDU

University of BRISTOL

3D compressible viscous/inviscid

3D Euler case, M=0.8, 1deg

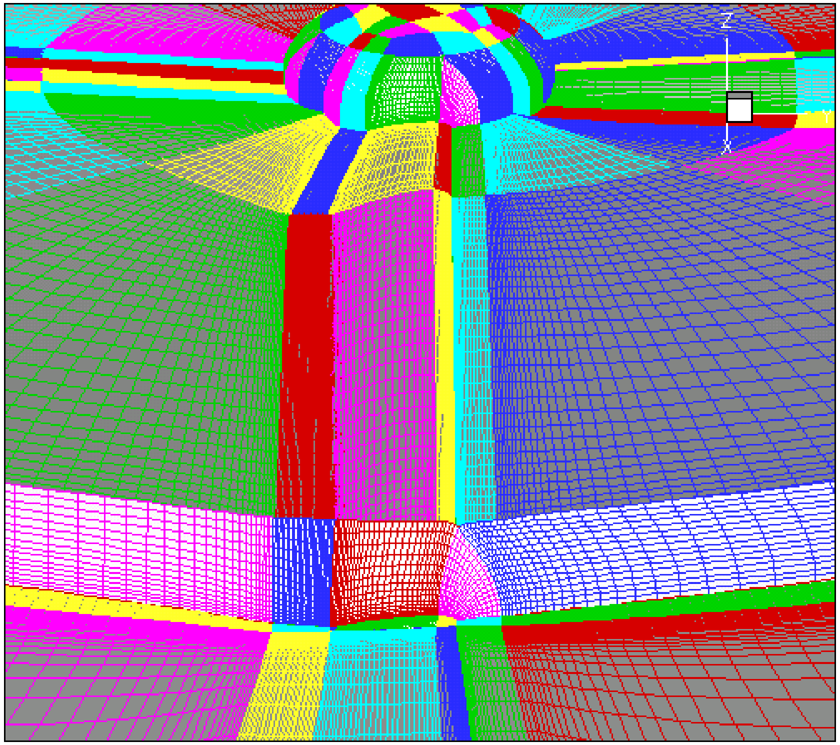


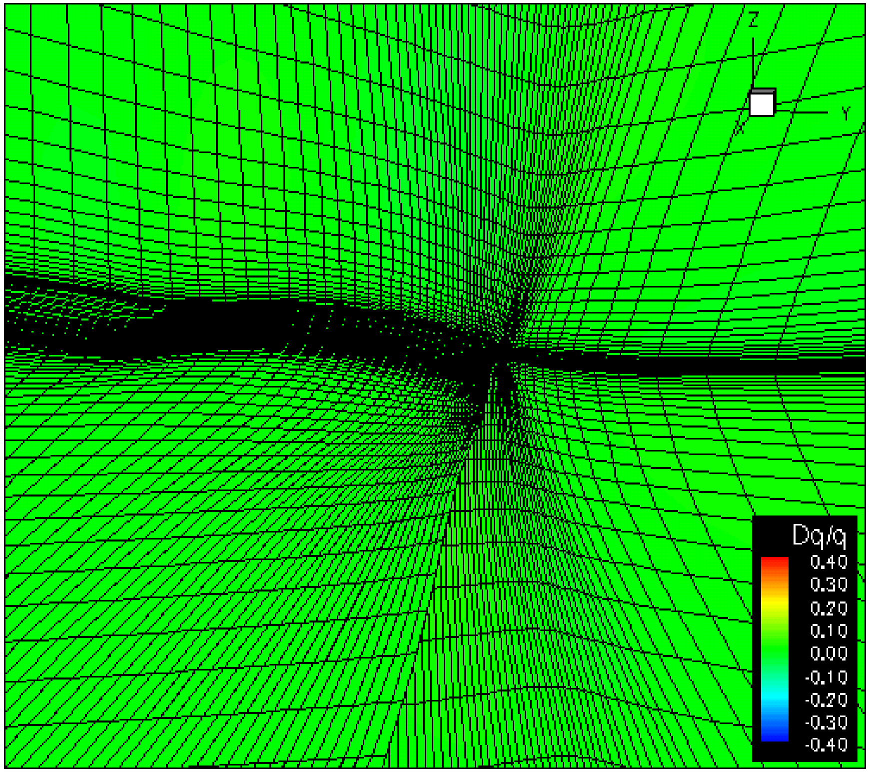
Unsteady

- Panel methods hard because the time-dependent wake is part of the solution. Lots of effort required to make panel methods unsteady - the exception is the Doublet Lattice Method (DLM), which is a (fairly!) easy unsteady vortex lattice method in frequency domain - ideal for flutter. Implemented in NASTRAN/ZAERO.
- CFD codes are simple to make unsteady, viscous or inviscid, compressible or incompressible
- The real problem is the mesh!

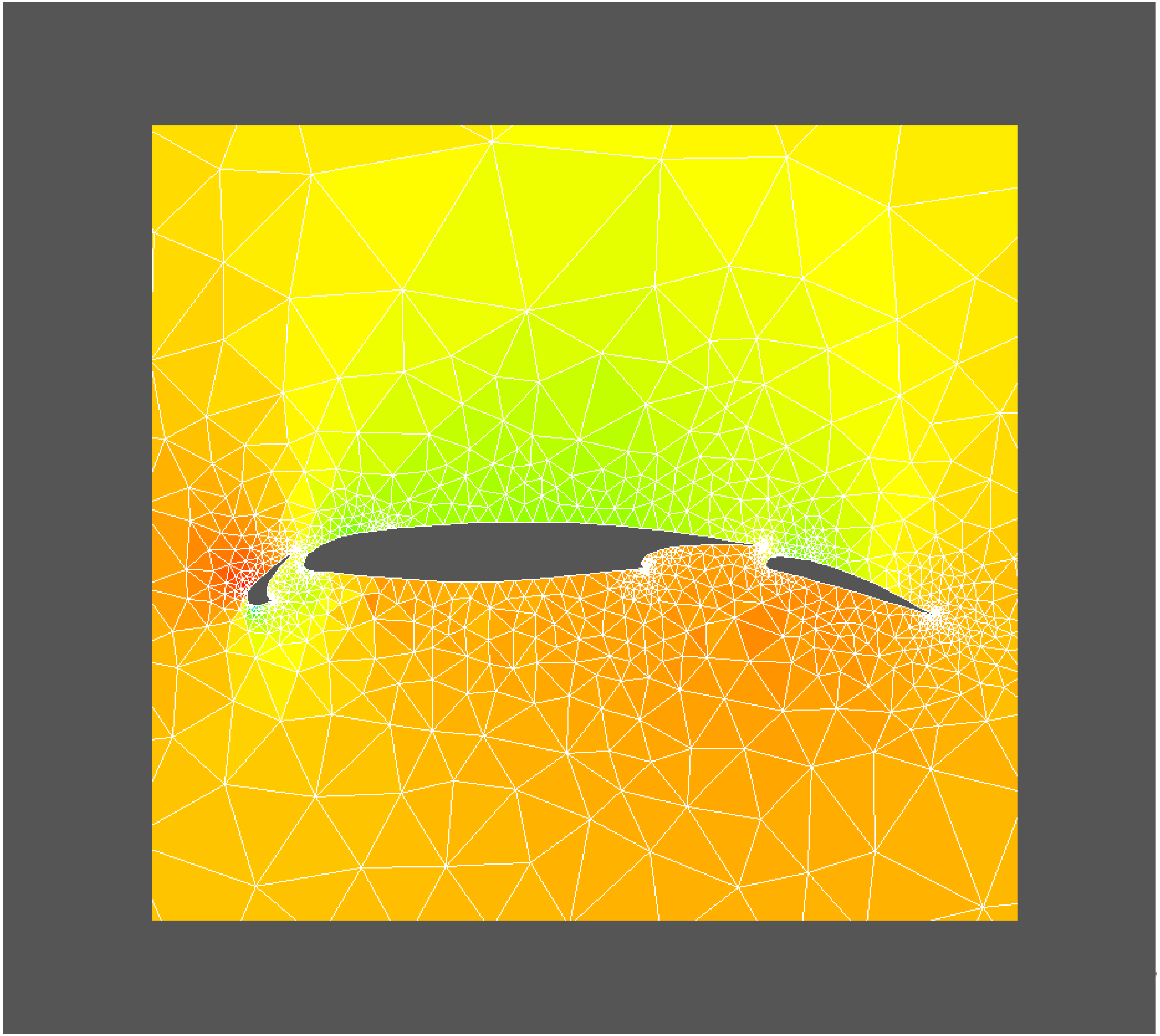
Unsteady using CFD - Mesh motion

- Surprisingly, quite an active area – esp in our group



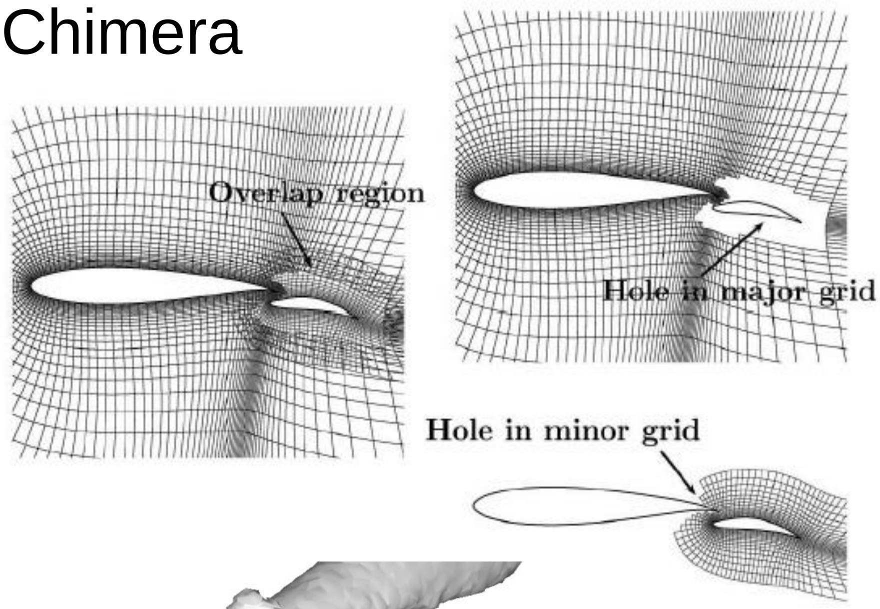


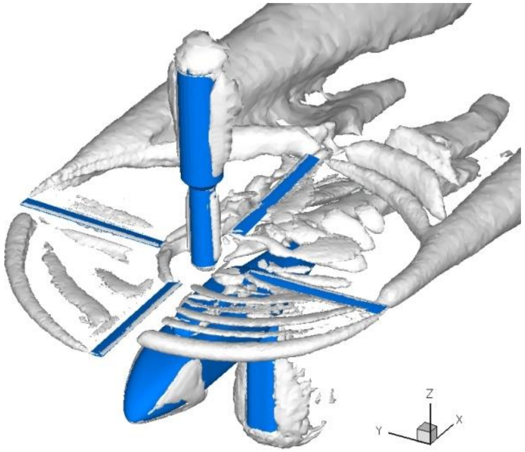
University of BRISTOL



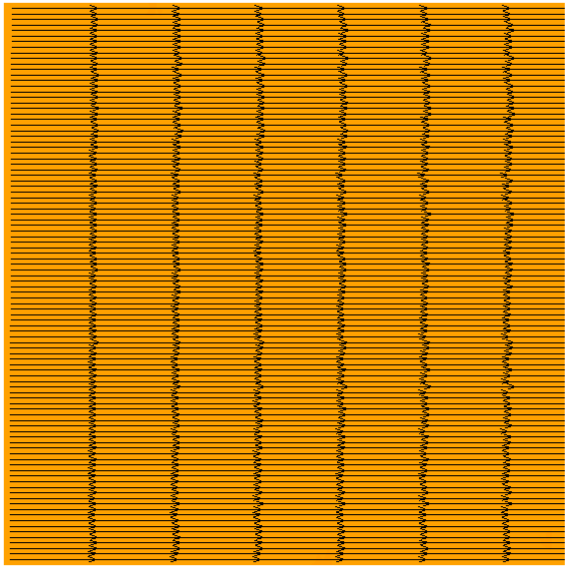
Alternatives to Mesh motion

Chimera

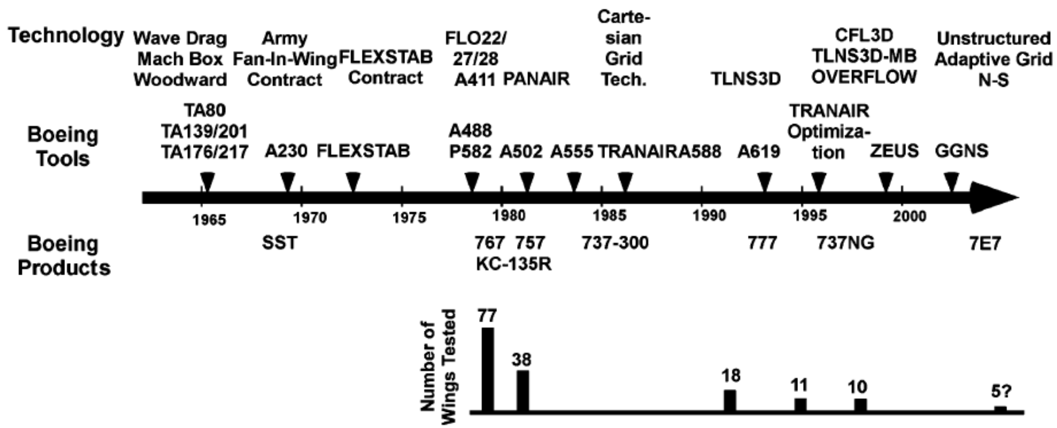


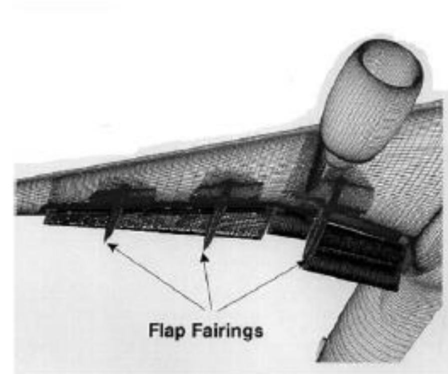


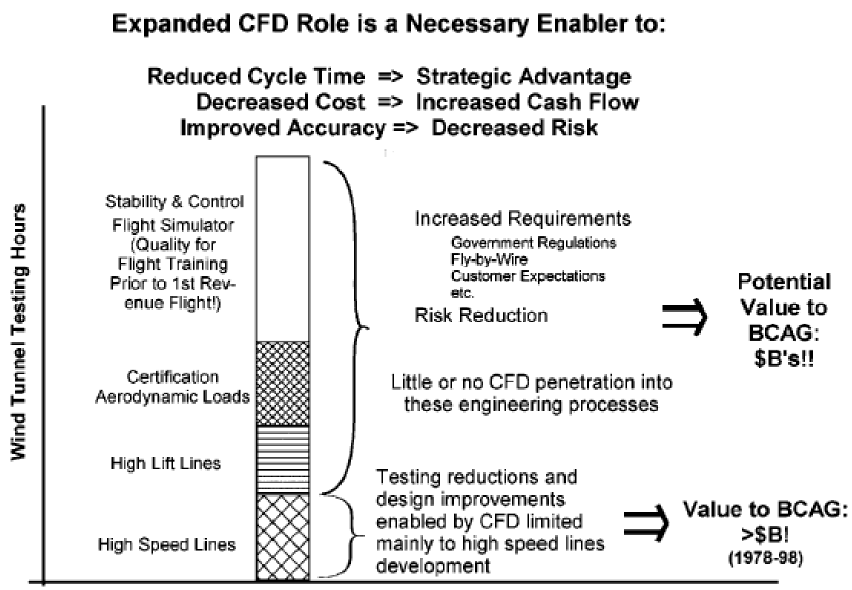
Space-time

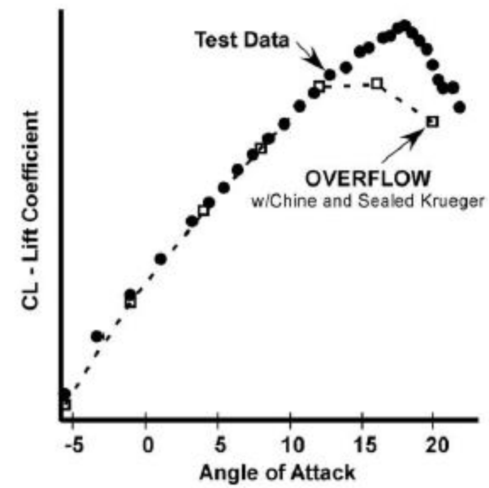


Growing role of CFD









University of BRISTOL

“Thirty years of development and application of CFD at Boeing Commercial Airplanes, Seattle”

Forrester T. Johnson, Edward N. Tinoco, N. Jong Yu

Optimisation

Developing area with significant complexity, and many methods

Probably more common in the future!