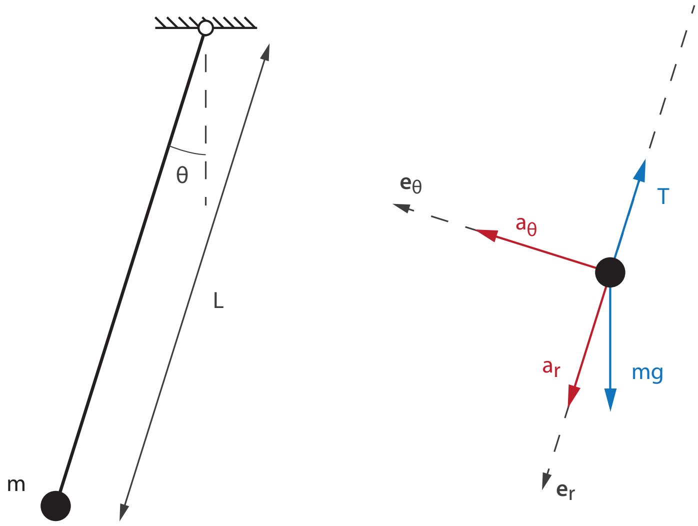
Handout 6 – Pendulums

A classic example to illustrate the effect of moments of inertia is the comparison between a **mathematical** and **compound** pendulum.

6.1 Mathematical Pendulum

A mathematical pendulum consists of a mass m attached to a weightless string of length L.



Its equations of motion are described by the dynamics of a particle. From the FBD we can write:

$$\sum F_r : \qquad mg\cos\theta - T = -m\dot{\theta}^2 L$$

$$\sum F_\theta : \qquad -mg\sin\theta = m\ddot{\theta}L$$

Using small-angle approximation, the equations of motion are obtained as an ordinary differential equation:

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

This ODE has a solution of the form:

$$\theta = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right)$$

and with the initial conditions (at $t = 0$, $\theta = \theta_0$ and $\dot{\theta} = 0$) we find

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{L}}t\right)$$

This describes a simple harmonic motion, and the pendulum therefore has an oscillation frequency[1]

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \tag{6.1}$$

which is a function of the pendulum length L, but is independent of the mass m. A longer pendulum will have a lower frequency and thus longer oscillation period.

String Tension From the solution for $\theta(t)$ we can solve for the tension T in the string:

$$T = mg\cos\theta + m\dot{\theta}^2 L$$

where we can substitute the solution for the ODE, as well its derivative:

$$\dot{\theta} = -\theta_0\sqrt{\frac{g}{L}}\sin\left(\sqrt{\frac{g}{L}}t\right)$$

to find an expression for the tension at any angle θ.

Alternatively, consider the two configurations where the tension is at maximum or minimum. As the pendulum reaches it maximum deflection, $\theta = \pm\theta_0$ and $\dot{\theta} = 0$, the tension T is at a minimum:

$$T_{\text{min}} = mg\cos\theta_0 \approx mg\left(1 - \frac{\theta_0^2}{2}\right)$$

where we use the MacLaurin series to linearise $\cos\theta$ around $\theta = 0$.

At the midpoint of the pendulum swing, $\theta = 0$ and $\dot{\theta} = \dot{\theta}_{\text{max}}$, the tension T is maximum:

$$\begin{aligned} T_{\text{max}} = mg + mL\dot{\theta}_{\text{max}}^2 &= mg + mL\theta_0^2\frac{g}{L} \\ &= mg\left(1 + \theta_0^2\right) \end{aligned}$$

The variation of the tension T is relatively small (of the order of θ_0^2), but for a large mass m it can result in measurable changes in length of the pendulum, which affect accurate measurements of gravity using pendulums.

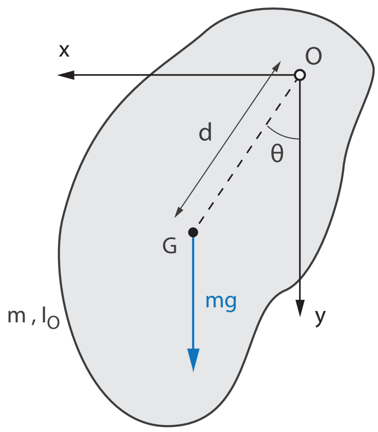
[1] A mostly irrelevant, but fascinating aside: you may have noticed that g is approximately equal to π^2. This is the result of an early definition of a meter. Pendulums were used for accurate time keeping; for a pendulum with an oscillation period of 2 seconds (*i.e.* a single swing takes 1 second), the required length is:

$$L = \frac{g}{\pi^2}$$

If that length is defined to be 1 meter, then g would be equal to π^2. The definition of the meter has moved on a bit since then, which means g is no longer precisely equal to π^2.

6.2 Compound Pendulum

A compound pendulum consists of a body with mass m hinged around a point O, at a distance d from the centre of mass G. This is a more realistic representation of physical pendulums. In the *Moments of Inertia* lab next term you will be asked to experimentally investigate the properties of a squash racket, by modelling it as a compound pendulum.



The dynamics of the pendulum are now described by its moment of inertia I_O.

$$\sum M_O : \qquad -mgd\sin\theta = I_O\ddot{\theta}$$

Using small-angle approximations, this results in the following equation of motion:

$$\ddot{\theta} + \frac{mgd}{I_O}\theta = 0$$

Noting the similarity with the equations of the mathematical pendulum, we write:

$$\ddot{\theta} + \frac{g}{L_{\text{eq}}}\theta = 0$$

and define an equivalent length:

$$L_{\text{eq}} = \frac{I_O}{md}$$

The oscillation frequency of the compound pendulum is therefore:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L_{\text{eq}}}} = \frac{1}{2\pi}\sqrt{\frac{mgd}{I_O}}$$

Now let us consider varying the distance d and explore its effect on the frequency of the pendulum. Crucially, I_O will also vary with distance d; using the parallel axis theorem:

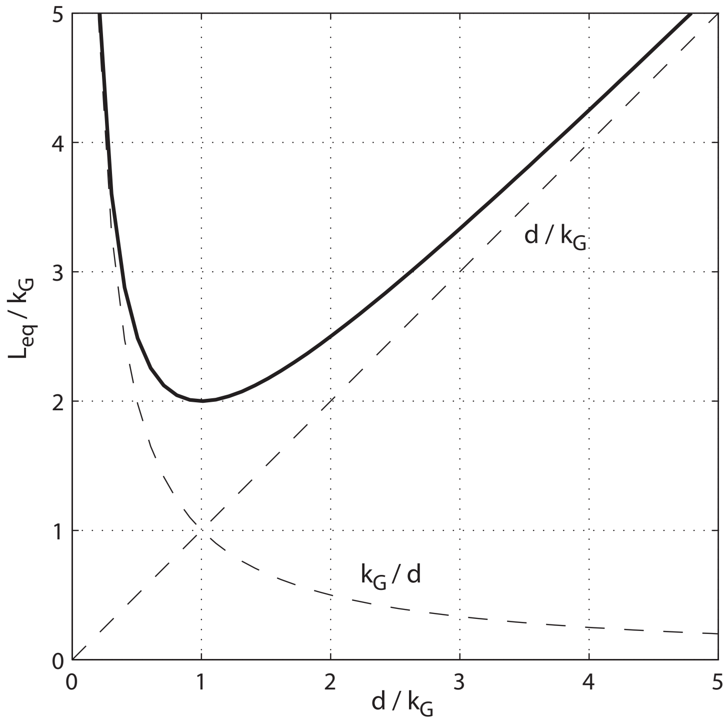
$$I_O = I_G + md^2 = m\left(k_G^2 + d^2\right)$$

where k_G is the radius of gyration of the body. Remember that the moment of inertia of a body is equivalent to placing a point mass at a radius k_G from the centre of mass. This gives:

$$L_{\text{eq}} = \frac{I_O}{md} = \frac{k_G^2}{d} + d$$

which can be non-dimensionalised as follows:

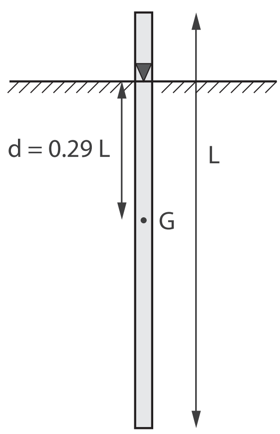
$$\frac{L_{\text{eq}}}{k_G} = \frac{k_G}{d} + \frac{d}{k_G}$$



Note that a compound pendulum is always slower than a mathematical pendulum of equal mass m and distance d between the pivot point and centre of mass; in the above graph[2] the mathematical pendulum is represented by the line $L_{eq} = d$. This slowdown is a result of the moment of inertia of the compound pendulum.

The minimum $L_{\mathsf{eq}} = 2k_G$ at $d = k_G$ corresponds to the *fastest* pendulum. For larger values of d the moment of inertia increases and the oscillation period becomes longer; for smaller values of d the pendulum also oscillates more slowly. Why is that? And what does $d/k_G = 0$ represent?

As an example, consider a slender, uniform bar of length L. The moment of inertia $I_G = \frac{mL^2}{12}$, and thus $k_G = \frac{L}{\sqrt{12}} \approx 0.29L$. Placing the pivot at $d = k_G$ produces the fastest pendulum. More interestingly, however, the frequency is also insensitive to changes in length d. This enables the construction of accurate pendulums which are insensitive to the wear of the knife edge pivot; these were used for time keeping.



[2]This elegant analysis of the dynamics of the compound pendulum can be found in: J.P. den Hartog (1955), "Mechanics", Dover Publications, reprint of 1948 Edition.

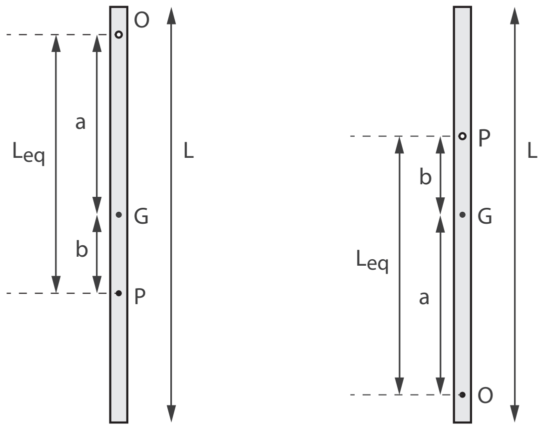
Kater Pendulum Consider a compound pendulum suspended around point O. It will have a centre of percussion P where an applied impulse does not induce a reaction force at the support. It was previously shown that:

$$a\,b = k_G^2$$

where a and b are the distance from points O and P to the centre of mass G (which lie along a straight line). The lengths a and b are interchangeable, and therefore point O is the centre of percussion for point P. The equivalent length of the pendulum is:

$$L_{\text{eq}} = \frac{k_G^2}{a} + a = \frac{a\,b}{a} + a = a + b$$

which is equal to the distance to the centre of percussion.



This is used to great effect in the Kater Pendulum, which was used to accurately measure gravity. It consists of a long bar with two fixed knife edges at an accurately known distance r, and with a movable mass that can be positioned along the bar. The position of the mass (and thus location of centre of mass G) was adjusted until the period T was the same for both pivots. This ensured that each pivot is the centre of percussion for the other, and therefore the equivalent length $L_{\text{eq}} = a + b = r$. From this could be calculated the gravity:

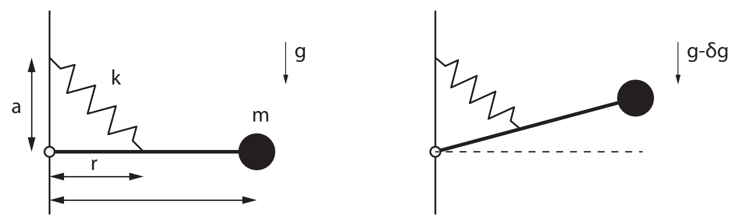
$$g = r\left(\frac{2\pi}{T}\right)^2$$

This pendulum was a standard gravity measuring method until the 1950s, before being replaced by free-fall gravimeters. [3]

[3] Interestingly we have already seen a *relative* gravity measuring device, known as the LaCoste-Romberg gravimeter. It is based on the gravity balancer, which was designed to counterbalance a fixed mass with a zero-length spring:

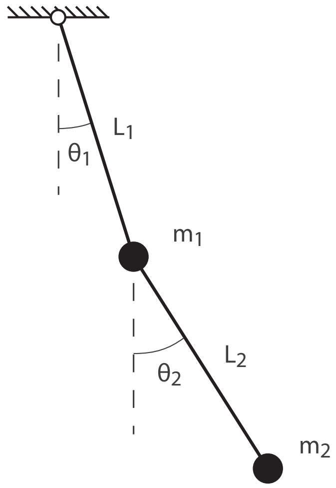
$$k = \frac{mgL}{ar}$$

As a result of the balancing, a small change in gravity δg will cause a large vertical movement, which can be accurately measured.



6.3 Double Pendulum

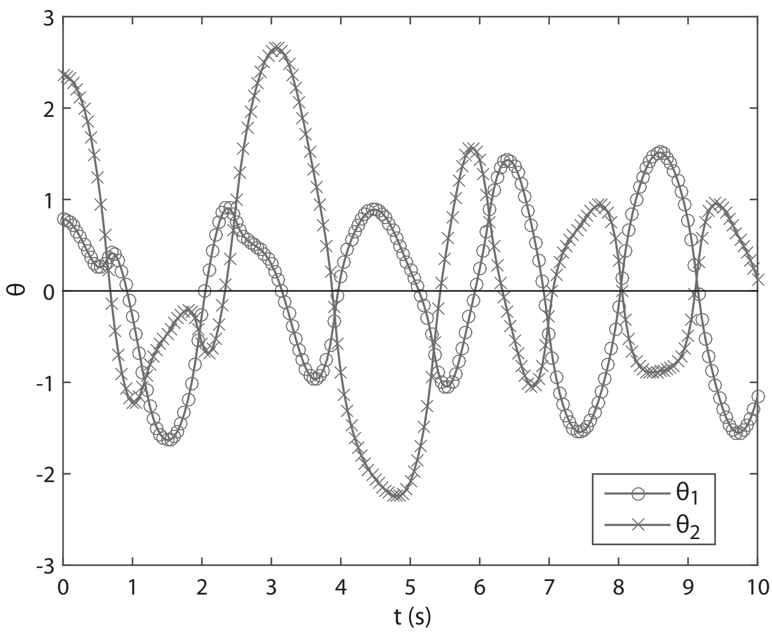
A very interesting situation arises when two pendulums are connected in series to form a **double pendulum**.



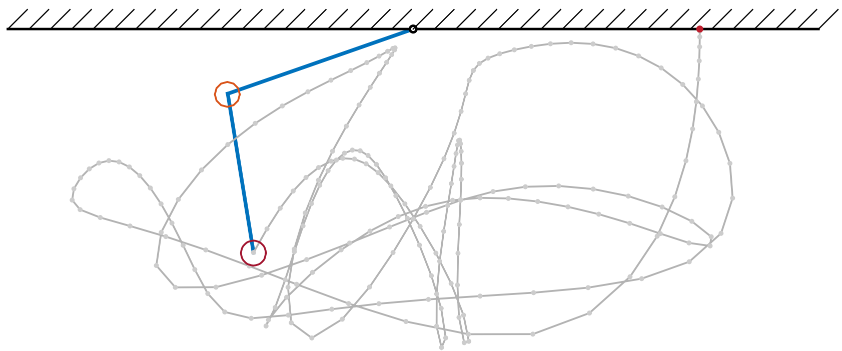
Modelling such multi-body dynamic systems is beyond the scope of this unit, and makes use of the Lagrangian dynamics formulation[4]. The equations of motion[5] are given as:

$$\begin{bmatrix} (m_1+m_2)\,L_1 & m_2 L_2 \cos(\theta_1-\theta_2) \\ m_2 L_1 \cos(\theta_1-\theta_2) & m_2 L_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -m_2 L_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2)\,g \sin\theta_1 \\ +m_2 L_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) - m_2 g \sin\theta_2 \end{bmatrix}$$

These can be integrated using numerical methods (here shown for $m_1 = m_2 = 1$ kg, $L_1 = L_2 = 1$ m, with initial conditions $\theta_1 = \pi/4$, $\theta_2 = 3\pi/4$, and $\dot{\theta}_1 = \dot{\theta}_2 = 0$).

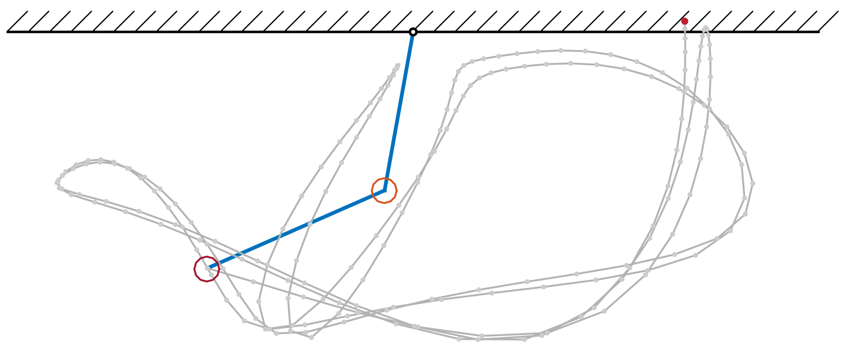


[4]Lagrangian dynamics is an alternative formulation of Newtonian mechanics, and it will be used in your Vibrations 2 unit.
[5]`http://scienceworld.wolfram.com/physics/DoublePendulum.html`



What makes a double pendulum so interesting, is that it is one of the simplest dynamic systems that exhibits *chaos*[6]. In its mathematical definition this means that the system is very sensitive to initial conditions: minor changes in initial condition can result in very large changes in output over time – the butterfly effect. Note that the system is *deterministic*, but *chaotic*!

By means of illustration, consider the same double pendulum, but now with initial condition $\theta_2 = \frac{3}{4}\pi + 0.1$. Note the marked difference in dynamic response!



Revision Objectives Handout 6:

Dynamics of Pendulums

- derive the equations of motion of a mathematical and a compound pendulum for small angles
- recall the equation for the frequency of a mathematical pendulum ($f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$)
- appreciate the effect the moment of inertia has on the period of a compound pendulum

Note: solving the ODE for the pendulums, the Kater pendulum, the double pendulum, and the various footnotes are **not** examinable.

[6]The concept of chaos was discovered in 1963 by Edward Lorenz (1917–2008), who formulated a simple model of atmospheric convection which displayed chaos. In essence, modelling of weather is chaotic!