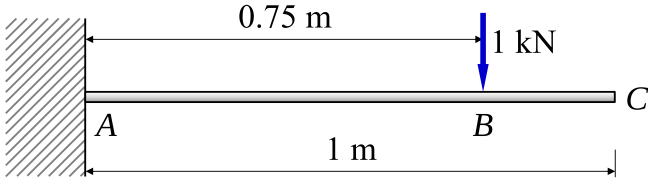
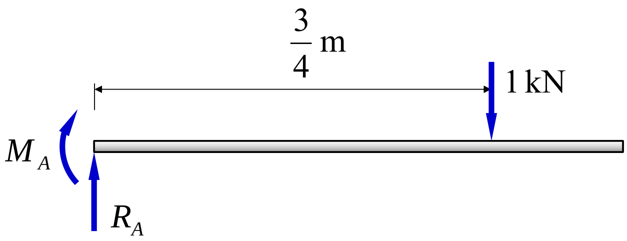
Example 2.3.2(b) – Plot the curvature, slope and deflection for the cantilever beam below. The beam is made of aluminium alloy with $E = 70\,\text{GPa}$ and has a solid square cross-section measuring 40 mm × 40 mm.



We start by finding support reactions.



$$\sum M_{@A}^{CW} = 0,$$

$$M_A + (1\,\text{kN})\left(\frac{3}{4}\,\text{m}\right) = 0 \qquad \therefore \qquad M_A = -\frac{3}{4}\,\text{kN m}.$$

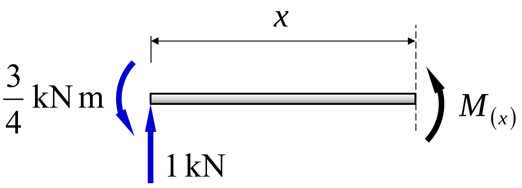
Vertical equilibrium gives,

$$\sum F = 0,$$

$$R_A - (1\,\text{kN}) = 0 \qquad \therefore \qquad R_A = 1\,\text{kN}.$$

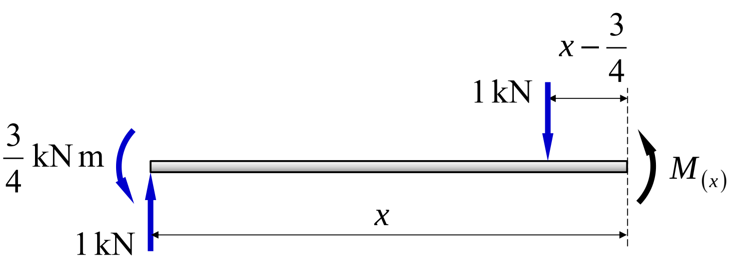
We get two different moment equations, depending on where we section the beam.

Sectioning between points A and B:



$$M_{(x)} + \left(\frac{3}{4}\,\text{kN m}\right) - (1\,\text{kN})(x) = 0 \qquad \therefore \qquad M_{(x)} = x - \frac{3}{4}$$

Sectioning between points B and C:



$$M_{(x)} + \left(\frac{3}{4}\,\text{kN m}\right) - (1\,\text{kN})(x) + (1\,\text{kN})\left(x - \frac{3}{4}\,\text{m}\right) = 0 \qquad \therefore \qquad M_{(x)} = x - \frac{3}{4} - \left(x - \frac{3}{4}\right)$$

In order to **combine both equations in one**, we use the Heaviside step function,

$$M_{(x)} = x - \frac{3}{4} - \left[\left(x - \frac{3}{4}\right) H\left(x - \frac{3}{4}\right)\right].$$

The curvature equation is therefore,

$$M_{(x)} = EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = x - \frac{3}{4} - \left[\left(x-\frac{3}{4}\right)H\left(x-\frac{3}{4}\right)\right]. \qquad (1)$$

Integrating once gives the slope,

$$EI\;\phi_{(x)} = EI\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2}x^2 - \frac{3}{4}x - \left[\frac{1}{2}\left(x-\frac{3}{4}\right)^2 H\left(x-\frac{3}{4}\right)\right] + A. \qquad (2)$$

Integrating again gives the deflection,

$$EI\;v_{(x)} = \frac{1}{6}x^3 - \frac{3}{8}x^2 - \left[\frac{1}{6}\left(x-\frac{3}{4}\right)^3 H\left(x-\frac{3}{4}\right)\right] + Ax + B. \qquad (3)$$

At the built-in end the beam cannot translate nor rotate. The boundary conditions are therefore,

$x = 0,\ \phi = 0 \quad \therefore \quad A = 0$,

and,

$x = 0,\ v = 0 \quad \therefore \quad B = 0$.

In order to draw the graphs we need to define a few points.

- First we compute the flexural modulus.

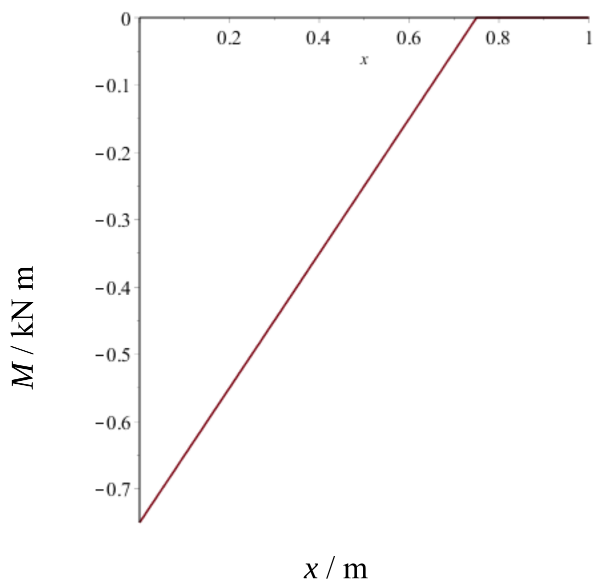
$$EI = \left(70\cdot 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}\right)\left[\frac{(0.04\ \mathrm{m})^4}{12}\right] \quad \therefore \quad EI = 14.9\overline{3}\ \mathrm{kN\ m}^2.$$

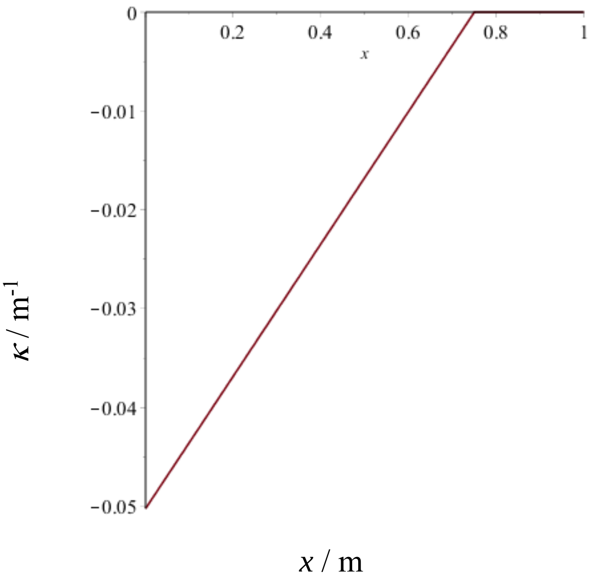
- Maximum downward deflection. This will obviously be at x = 1 m, so equation (3) gives,

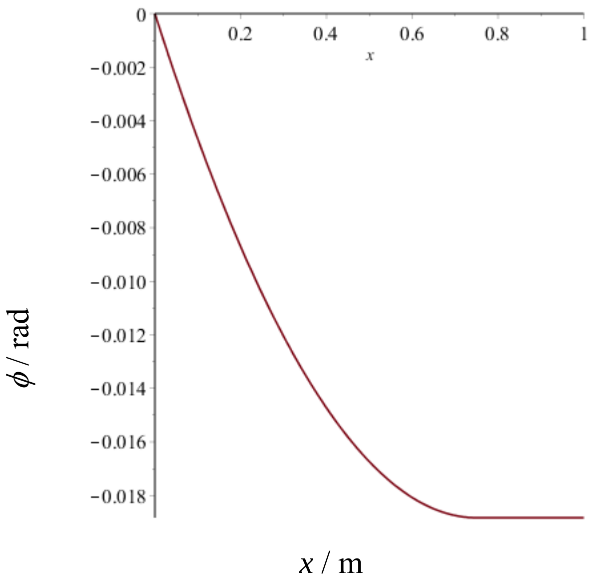
$$EI\;v_{\min} = \frac{1}{6}(1)^3 - \frac{3}{8}(1)^2 - \frac{1}{6}\left(1-\frac{3}{4}\right)^3 \quad \therefore \quad v_{\min} \cong -14.125\ \mathrm{mm}$$

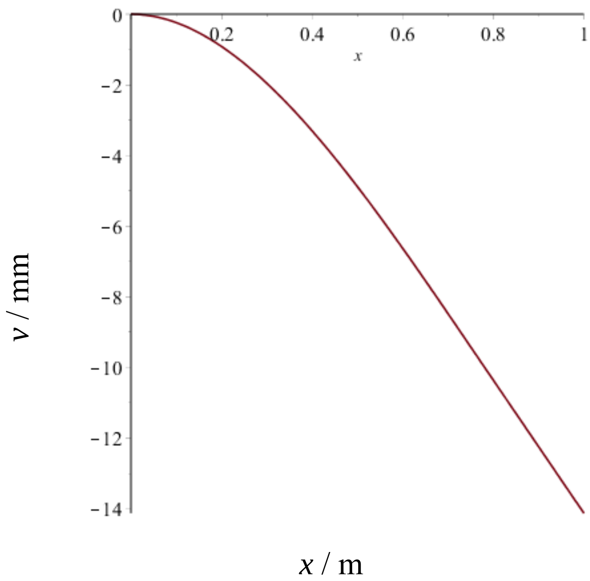
- Local curvature. Remember that $M = \kappa \cdot EI \quad \therefore \quad \kappa = \dfrac{M}{EI}$ (4)

Further points may be found by substituting x values in equations (1), (2), (3) and (4). The final graphs are shown below.




0
0.2
0.4
0.6
0.8
1
x
−0.01
−0.02
−0.03
−0.04
−0.05
κ / m^-1
x / m


0
0.2
0.4
0.6
0.8
1
x
−0.002
−0.004
−0.006
−0.008
−0.010
−0.012
−0.014
−0.016
−0.018
ϕ / rad
x / m


0
0.2
0.4
0.6
0.8
1
x
−2
−4
−6
−8
−10
−12
−14
v / mm
x / m