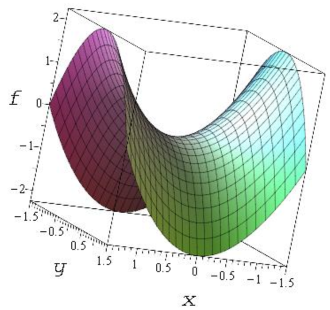
EMAT10100 Engineering Maths I
Lecture 26: Partial Differentiation

John Hogan & Alan Champneys

Functions of several variables

- So far, only dealt with functions of single variable $f(x)$
- More typical in engineering to use functions depending on several variables $z = f(x, y, w, u, ...)$
- for simplicity, we consider functions of 2 or 3 variables
  (like matrices & vectors, the principles apply in more dimensions)



Derivative in two dimensions

- Recall definition for a function of one variable $f(x)$

  $\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
- The partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

  $=$ "derivative in x direction treating y as a constant"
- Similarly, partial derivative of f with respect to (w.r.t.) y:

  $\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$
- Sometimes use abbreviations: $\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial f}{\partial y} = f_y.$

  note 'curly d' ∂ is NOT δ

in 3D

- What if f is a function of many variables?

  e.g. $f(x, y, z) = x^2yz^3 + 3xz - 2y$
- in this case we would have three partial derivatives
- e.g. $\frac{\partial f}{\partial y}$ assumes x and z are constants:

$$\frac{\partial f}{\partial x} = 2xyz^3 + 3z, \quad \frac{\partial f}{\partial y} = x^2z^3 - 2, \quad \frac{\partial f}{\partial z} = 3x^2yz^2 + 3x.$$

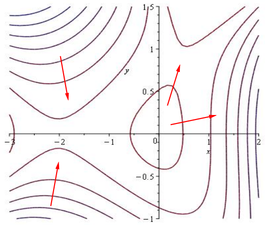
- Exercises
  1. Find partial derivatives w.r.t. x and y of $f(x, y) = x^2e^{-y} + 3y$
  2. Find partial derivatives w.r.t. x, y and z of $f(x, y, z) = 3x^2 \ln y + z^2 \sin(y)(e^x + z)$

Gradient vector

- Note that gradient, or slope, in two dimensions is a vector:

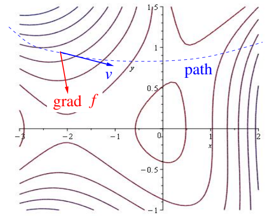
$$\nabla f = \text{grad } f = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)$$

- direction is uphill (in which f is increasing the most)



- But how to define the slope (rate of change) of a function $f(x, y)$ in an arbitrary direction?
  If I walk in an arbitrary direction, what will be the slope I feel?

The directional derivative



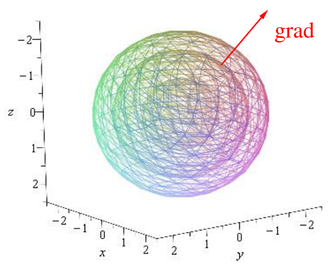
- First compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- Then add a direction by defining a vector $\mathbf{v}$,
- Then we define directional derivative of $f(x, y)$ in direction of $\mathbf{v}$ by taking a dot product:

$$f_{\hat{\mathbf{v}}} := \nabla f \cdot \frac{\mathbf{v}}{|\mathbf{v}|},$$

where $\frac{\mathbf{v}}{|\mathbf{v}|} \equiv \hat{\mathbf{v}}$ is unit vector in the direction of $\mathbf{v}$.

Higher dimensional analogies

- Note that $\nabla f = (f_x, f_y, f_z)$ and directional derivative are well defined in 3D and in higher dimensions



- Exercise Compute the directional derivative of $f(x, y, z) = x^2y^2z^3$ at the point $(3, 2, 1)$ in the direction of the vector $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Chain rule for partial derivatives

- recall chain rule (function of a function) for differentiation in 1D

$$\frac{\mathrm{d}}{\mathrm{d}\,x} f(g(x)) = \frac{\mathrm{d}\,g}{\mathrm{d}\,x} \cdot \frac{\mathrm{d}\,f}{\mathrm{d}\,g}$$

- Consider two-variable function: $f(x, y)$ and suppose $x = x(s, t)$, $y = y(s, t)$
- Q. how to evaluate $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$?
- A. use the Chain Rule for two variables:

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}\end{aligned}$$

Example: change of co-ordinates

- Common requirement, change co-ordinates, e.g.: CARTESIAN $\longrightarrow$ POLAR
- e.g. given $f(x,y) = x^3 - xy + y^3$
- Assume x and y have been expressed in polar coordinates:
  $x = r\cos(\theta) := x(r,\theta), \quad y = r\sin(\theta) := y(r,\theta)$
- Compute partial derivatives of f w.r.t r and θ.
- Solution: to find $\frac{\partial f}{\partial r}$, write $f(x,y) = f(x(r,\theta), y(r,\theta))$ and use chain rule:

$$\Rightarrow \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

- Now, since

$$f(x,y) = x^3 - xy + y^3 \Rightarrow \begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 - y, \\ \frac{\partial f}{\partial y} &= -x + 3y^2. \end{aligned}$$

and $x(r,\theta) = r\cos(\theta) \Rightarrow \frac{\partial x}{\partial r} = \cos(\theta)$,
$y(r,\theta) = r\sin(\theta) \Rightarrow \frac{\partial y}{\partial r} = \sin(\theta)$.

- Combining these we obtain:

$$\begin{aligned} \frac{\partial f}{\partial r} &= (3x^2 - y)\cos(\theta) + (3y^2 - x)\sin(\theta), \\ \Rightarrow \quad \frac{\partial f}{\partial r} &= \left[3r^2\cos^2(\theta) - r\sin(\theta)\right]\cos(\theta) + \left[3r^2\sin^2(\theta) - r\cos(\theta)\right]\sin(\theta). \\ &= 3r^2[\cos^3(\theta) + \sin^3(\theta)] - 2r\cos(\theta)\sin(\theta) \end{aligned}$$

- Exercise compute $\frac{\partial f}{\partial \theta}$ in a similar manner.

Other versions of the chain rule

- If $f(x(t), y(t))$:

$$\frac{\mathrm{d}\,f}{\mathrm{d}\,t} = \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,t}$$

- If $f(x(s,t), y(s,t), z(s,t))$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$$

- etc.

Exercises

1. Elliptic co-ordinates can be defined by

$$x(s,t) = \cosh s \cos t, \qquad y(s,t) = \sinh s \sin t$$

   Use the chain rule to find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ for $f(x,y) = x^2 + y^2$

2. A space rocket has a trajectory in three-dimensional space given by

$$x(t) = at, \quad y(t) = b\sin(\omega t), \quad z(t) = ct^2 - kt^3,$$

   Use the chain rule to calculate the speed

$$\frac{\mathrm{d}\,r}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\sqrt{x(t)^2 + y(t)^2 + z(t)^2}$$

   in terms of the constants a, b, ω, c and k

EMAT10100 Engineering Maths I Lecture 26: More on Partial Differentiation

John Hogan & Alan Champneys

This lecture

There are two topics in this lecture:

- I. Higher-order partial derivatives
- II. Total differentials and error estimation

I. Higher order derivatives

- in 1D we have the notion of 2nd derivatives, 3rd derivatives etc: $\frac{d}{dx}[f(x)] = f'(x)$, $\frac{d}{dx}[f'(x)] = f''(x)$, $\dots$ $\frac{d}{dx}[f^{(n-1)}(x)] = f^{(n)}(x)$.
- Similarly, we can define higher-order partial derivatives:

  $\frac{\partial}{\partial x}[f(x,y)] = f_x(x,y)$, $\frac{\partial}{\partial x}[f_x(x,y)] = f_{xx}(x,y)$, etc.

  $\frac{\partial}{\partial y}[f(x,y)] = f_y(x,y)$, $\frac{\partial}{\partial y}[f_y(x,y)] = f_{yy}(x,y)$, etc.
- There is another 2nd derivative though: what happens though if we differentiate $f_x(x,y)$ w.r.t. y or equivalently $f_y(x,y)$ w.r.t. x?
- $\Rightarrow$ mixed derivatives:

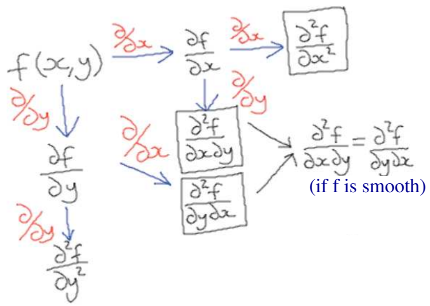
$$\frac{\partial f_x}{\partial y} = f_{xy}, \quad \frac{\partial f_y}{\partial x} = f_{yx}$$

- Note that $f_{xy} = f_{yx}$ provided the functions f_x and f_y are smooth$^{(1)}$

  (1) = always read the small print

Calculating 2nd derivatives

- general principles:



- Exercise: Compute the 2nd partial derivatives of

$$f(x,y) = x^2y^3 + 3y + x$$

and show explicitly that $f_{xy} = f_{yx}$ in this case.

Partial differential equations

- equations that involve partial derivatives of physical quantities that evolve in space and time
- e.g. wave equation for displacement of $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$c =$ wave speed

- Exercise show that the function

$$u(x,t) = \cos(kx + \phi)\cos(kct + \psi)$$

solves wave eqn. for any constants k, ϕ and ψ.

- to find the constants, we need to use boundary conditions (see 2nd year Maths)

II Total differentials & error analysis

- Suppose we have a physical quantity $u = f(x,y)$
- If we can estimate error in x and y, how do we estimate error in u?
- More generally, given small changes in x and y, what is corresponding small change in u?
- Let Δx and Δy be changes x and y, then

$$\begin{aligned}
\Delta u &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] + [f(x,y+\Delta y) - f(x,y)] \\
&= \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x + \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}\Delta y \\
&\approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y
\end{aligned}$$

The total differential

- This gives good approximation of the total increment to a function, given small increment in each of its variables.
- That is

$$\Delta u \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

- By taking limits as small quantities $\to 0$

$$\Delta x \to dx, \quad \Delta y \to dy \quad \Delta u \to du$$

- Then, formally, we define the total differential of u as:

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Example

- Compute the total differential of

$$z = x^2y^3 := f(x,y).$$

- Solution: We need to compute

$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$

$$\frac{\partial f}{\partial x} = 2xy^3, \qquad \frac{\partial f}{\partial y} = 3x^2y^2,$$
$$\Rightarrow \quad dz = 2xy^3dx + 3x^2y^2dy$$

- Exercise compute the total differential of

$$f(r,\theta,\phi) = r^2\sin^2(\theta)\cos(\phi)$$

Application to error analysis

- Example The volume of a cylinder of radius r cm and height h cm is given by $V(r.h), = \pi r^2 h := V(r, h)$
  and that measurements are taken such that:
  $r = 3 \pm 0.01$ cm $:= r_0 \pm \Delta r$, $\quad h = 5 \pm 0.005$ cm $:= h_0 \pm \Delta h$, find the maximum possible error in the calculation of V.
- Solution: We want to compute ΔV, given Δr and Δh.

$$\begin{aligned} |\Delta V| &\approx |\frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h|, \\ &\leq |(2\pi r_0 h_0)||\Delta r| + |(\pi r_0^2)||\Delta h| \end{aligned}$$

- Substituting in the values of r_0, Δr, h_0 and Δh we obtain

$$|\Delta V| \approx 0.3\pi + 0.045\pi$$

Therefore we can state that $V = \pi(45 \pm 0.345)$

Relative and percentage error

- relative error in $f = \frac{\Delta f}{f}$, $\qquad$ % error in $f = \frac{\Delta f}{f} \times 100.$
- From previous expressions we get:
  $\frac{\Delta f}{f} \approx \frac{\partial f}{\partial x}\frac{\Delta x}{f} + \frac{\partial f}{\partial y}\frac{\Delta y}{f}$
- Example find relative error in $f(x, y) = (x^2 + y^2 + xy)$ in terms of relative errors in x and y.
- Solution : from above formula

$$\begin{aligned} \frac{\Delta f}{f} &= (2x + y)\frac{\Delta x}{x^2 + y^2 + xy} + (2y + x)\frac{\Delta y}{x^2 + y^2 + xy} \\ &= \frac{2x^2 + xy}{x^2 + y^2 + xy}\left(\frac{\Delta x}{x}\right) + \frac{2y^2 + xy}{x^2 + y^2 + xy}\left(\frac{\Delta y}{y}\right) \end{aligned}$$

- Exercise If $v = \sqrt{(3x/y)}$ find maximum percentage error in v due to errors of 1% in x and 3% in y.

Homework

- Please read *James*
  - Section 9.6.1–9.6.9
- and do exercises 4th edition
  - Sec. 9.6.4, Qns. 39–41, 44,
  - Sec. 9.6.6, Qns. 47, 48, 54, 55
  - Sec. 9.6.8 Qns. 56–59
  - Sec. 9.6.10 Qns. 65,67,69,72
- and do exercises 5th edition
  - Sec. 9.6.4, Qns. 39–41, 44,
  - Sec. 9.6.6, Qns. 47, 48, 54, 55
  - Sec. 9.6.8 Qns. 57–60
  - Sec. 9.6.10 Qns. 69,70,75