THE LIFTING LINE METHOD

AIMS

- To introduce the lifting line method for the modelling of 3D wing flows
- To derive the fundamental lifting line equation of the classical formulation
- To solve the equation for an elliptical circulation distribution
- To solve the equation for a general distribution
- To identify the limitations of the classical formulation and briefly explain how the method can be extended.

1 INTRODUCTION

The lifting line method developed by Prandtl and colleagues represents the flow about a finite wing using a system of vortices. The method produces solutions where the motion of the air surrounding the wing is similar to that produced by a lifting wing. It is thus able to give practical first predictions of the aerodynamic characteristics of finite wings. Prandtl realised that the non-physical results of the horseshoe model i.e. the infinite values of downwash predicted at the wing tips could be overcome by
superimposing horseshoe vortices of varying strength with the bound vortices along a single line, the 'lifting line' commonly located at the quarter chord. This better models the real flow where the 'tip' vortices do not originate just at wing tips as in the horseshoe vortex , but are in fact formed from the roll-up of the wake shed over the span . This means that the model should have a bound circulation Γ which varies with span with this bound circulation progressively shed along the span, from centreline to tips. This model of superimposed horseshoe vortices is illustrated below.

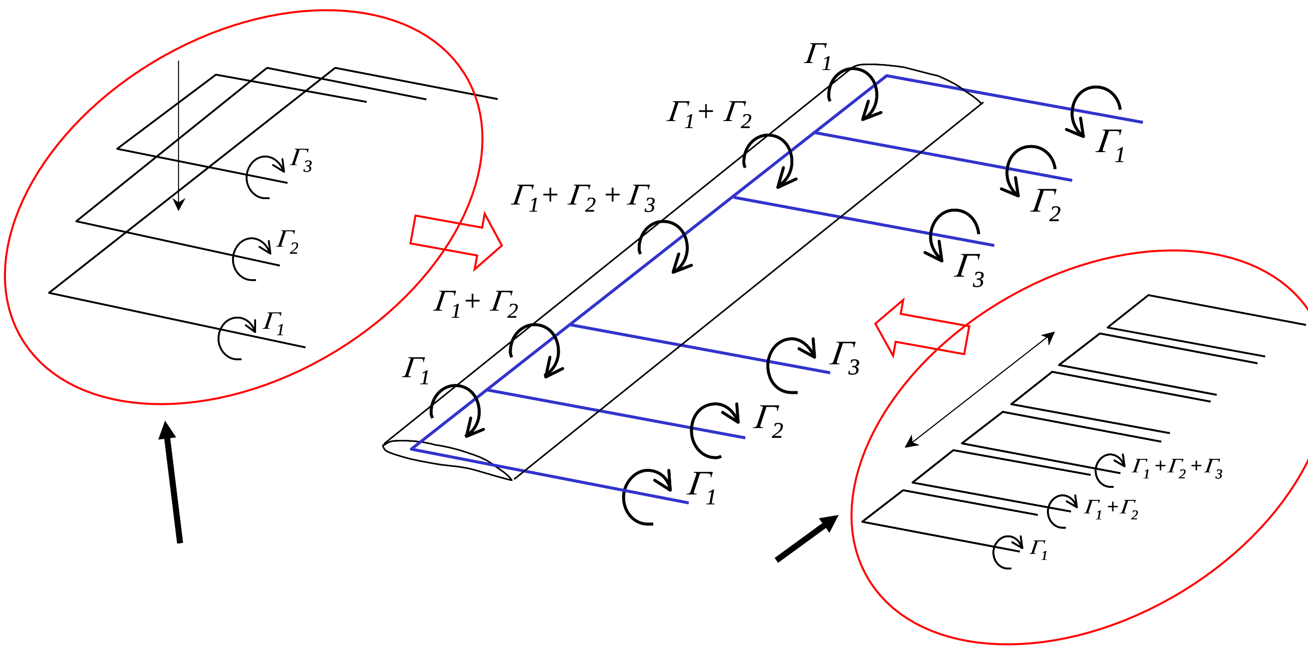


FIGURE 1

The resulting model has a lifting line with bound circulation $\Gamma(y)$ varying with span y, see figure 2. Each time the circulation jumps another vortex is shed into the wake with circulation equal to the jump or change in circulation i.e. there are multiple vortex pairs shed into the wake across the span. The magnitude of the circulation shed into the wake by each half of the wing is equal to the circulation at the mid point of the wing.

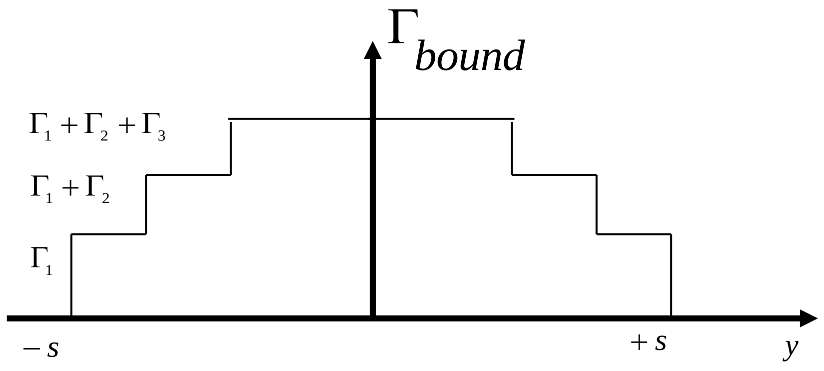


FIGURE 2

If the above model is taken to the limiting case when an infinite number of horseshoe vortices is superimposed each of vanishingly small strength dΓ, then this gives a continuous bound circulation distribution $\Gamma(y)$ on the lifting line. Due to Helmholz theorems this bound circulation is coupled with a continuous vortex sheet shed from the wing trailing edge of strength $\gamma_x(y)$ (the x indicates the orientation of the sheet). Note that Γ_0 , the circulation at the origin, equals the magnitude of the circulation about the trailing vortices of each half of the wing.

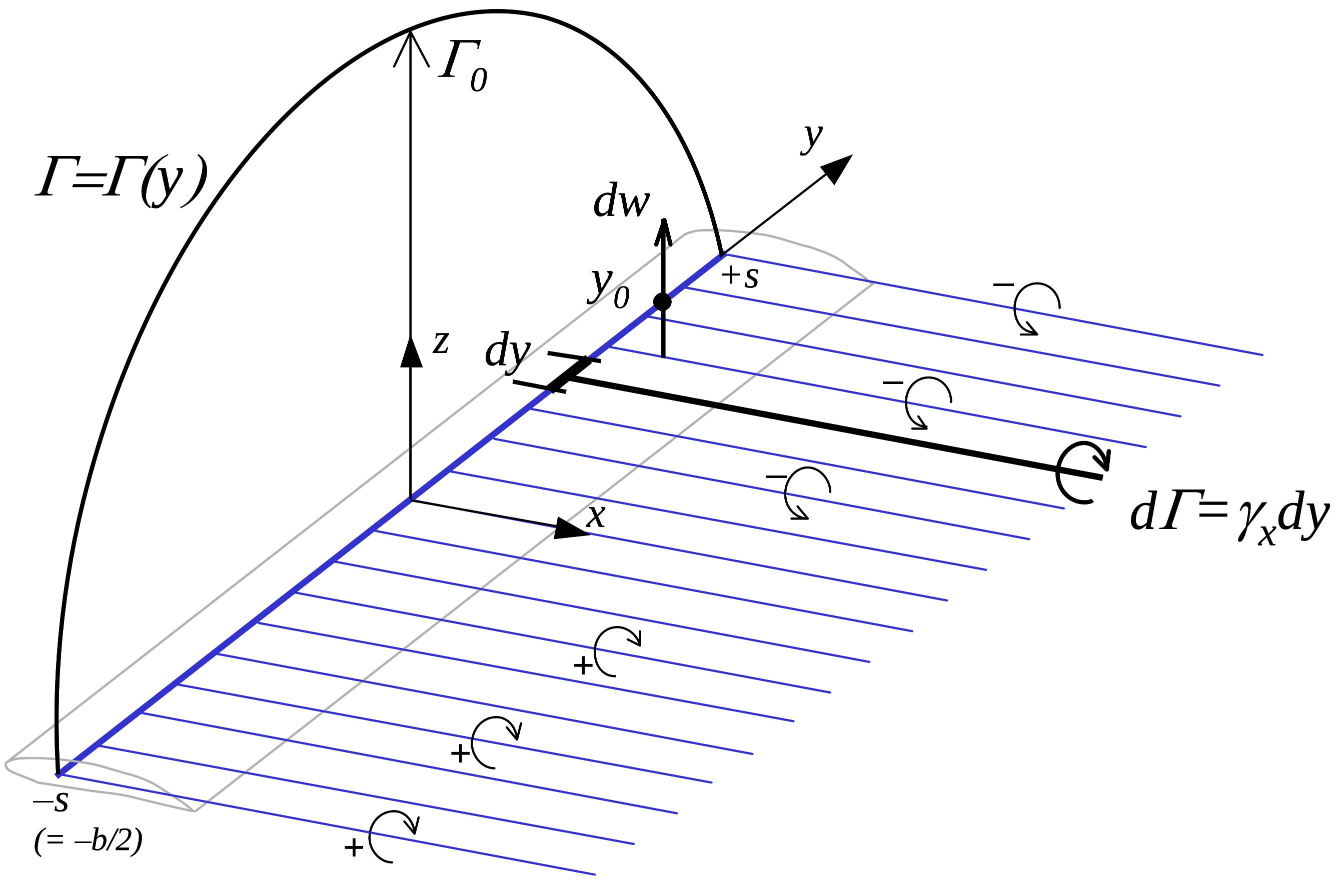


FIGURE 3

In this limiting case when the wake becomes a vortex sheet, consider the circulation at a point y on the lifting line where the circulation is $\Gamma(y)$. The change in circulation over a segment of the bound vortex of length dy is $d\Gamma = (d\Gamma / dy)dy$. Extrapolating from the finite number of vortices case the corresponding trailing vortex strength must equal the change in circulation strength along the lifting line.

$$d\Gamma_{wake} = d\Gamma_{bound} = \left(\frac{d\Gamma_{bound}}{dy}\right)dy = \gamma_x(y)dy$$

To work out the effect of the model, need to find the downwash velocity induced by wake element $d\Gamma$ at an arbitrary point on the bound vortex, at y_0 say.

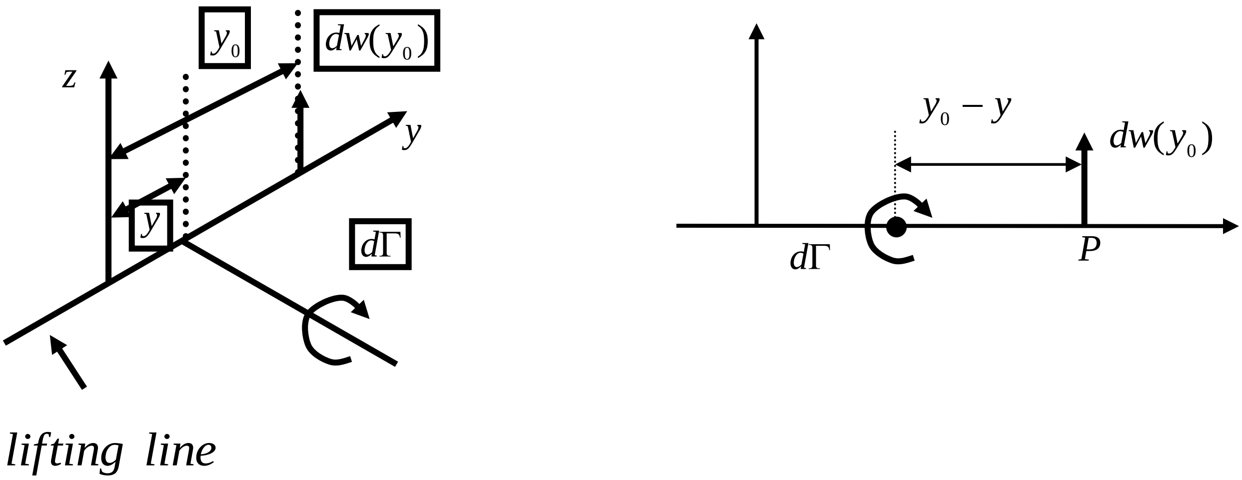


FIGURE 4

The geometry is shown in figure 4, note a vortex element with positive circulation is considered. The Biot-Savart law for a semi-infinite vortex is used i.e. the magnitude of velocity V is given by $V=\Gamma/(4\pi h)$ and this is in the downwards direction so

$$dw(y_0) = \frac{-d\Gamma(y)}{4\pi(y_0 - y)} = \frac{-(d\Gamma/dy)dy}{4\pi(y_0 - y)}$$

To obtain the downwash induced by the entire vortex sheet this is integrated from tip to tip ($y = -s$ *to* $+s$)

$$\boxed{w(y_0) = \frac{-1}{4\pi}\int_{-s}^{+s}\frac{(d\Gamma/dy)dy}{(y_0 - y)}}$$

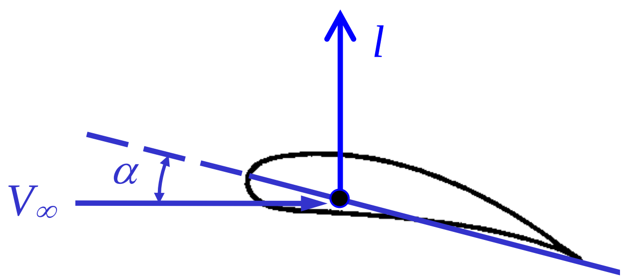
The velocity w is positive upwards in the z direction and so usually has a negative value (i.e. acts downwards) for simple wing geometry where the whole of the wing experiences a downwash

If the circulation distribution Γ*(y)* were known then the corresponding downwash, lift *l(y)* and drag *d(y)* distributions could be worked out.

BUT there is a circular relationship between downwash and circulation – if we change the local downwash, the local effective angle of attack changes, so lift changes, so circulation changes and so downwash changes.

Therefore to determine circulation and hence lift distribution some additional information is needed: A 3D equivalent of the Kutta condition is required to select the correct circulation distribution. Several ways to do this, only the *simplest classical method* developed by Prandtl will be used here. This treats each chordwise wing section as if it were a 2D aerofoil (METHOD 1 in lecture presentation) and is therefore valid only if cross flow velocities are small i.e. high aspect ratio wings, low sweep etc.

Starting from basic 2D aerofoil flow

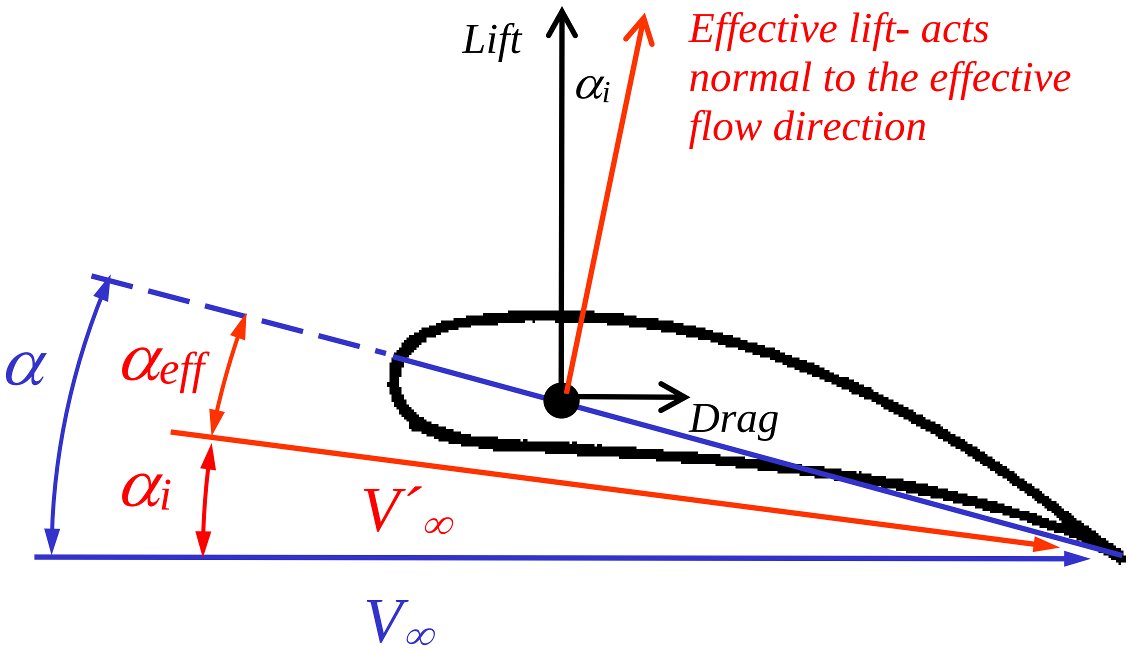


$$l = \rho_\infty V_\infty \Gamma \,, \; d = 0$$
$$c_l = a_0\left(\alpha - \alpha_{l=0}\right)$$

2D lift curve slope $a_0 \approx 2\pi$

$\alpha_{l=0}$ accounts for effects of camber

For a 3D wing the downwash velocity *w(y)* reduces effective incidence by α_i the induced incidence so the lift coefficient is reduced. In addition the lift vector is rotated clockwise by α_i so there is a drag component.

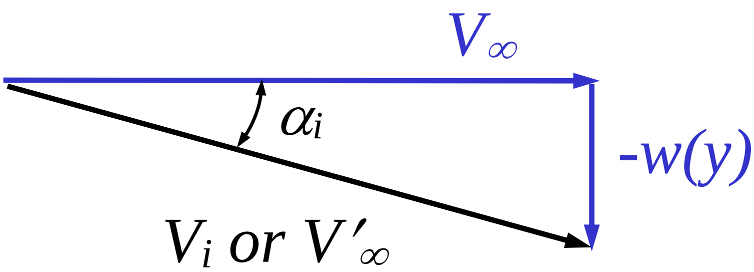


Then resolving gives the lift and drag per unit span

$$l = l_{eff}\cos\alpha_i \approx l_{eff}$$
$$d = l_{eff}\sin\alpha_i \approx l_{eff}\alpha_i \approx l\alpha_i$$

Note, that $l_{eff}, l, d, \alpha_i, \alpha_{eff}, \alpha_{L=0}$ and c are all functions of the spanwise coordinate, y.

Looking in more detail at the induced incidence and making the usual small angle assumptions



$$\alpha_i(y) \approx \frac{-w(y)}{V_\infty}$$
$$V'_\infty \approx V_\infty$$

Using the 2D lift expression, but replacing the geometric incidence with the effective incidence gives the effective lift per unit span which acts normal to the effective flow direction as

$$l_{eff} = \left[\tfrac{1}{2}\rho_\infty V_\infty^2 c\right]a_0\left(\alpha_{eff} - \alpha_{l=0}\right)$$
$$\alpha_{eff} = \alpha - \alpha_i = \alpha + \frac{w}{V_\infty}$$

and hence since $l \approx l_{eff}$ the lift coefficient per unit span is

$$c_l = \frac{l}{1/2\rho_\infty V_\infty^2 c} = a_0(\alpha - \alpha_i - \alpha_{l=0})$$

Another expression is available from an application of the 2D Kutta-Joukowsky theorem to the local section

$$c_l(y) = \frac{l}{1/2\rho_\infty V_\infty^2 c} = \frac{\rho_\infty V_\infty \Gamma}{1/2\rho_\infty V_\infty^2 c} = \frac{2\Gamma}{V_\infty c}$$

Equating these two expressions gives

$$\alpha = \frac{2\Gamma}{a_0\, V_\infty c} + \alpha_{l=0} + \alpha_i$$

Applying this equation at a spanwise location y_0 gives

$$\alpha(y_0) = \frac{2\Gamma(y_0)}{a_0(y_0) V_\infty c(y_0)} + \alpha_{l=0}(y_0) + \alpha_i(y_0)$$

Then substituting for α_i and using the integral expression derived above for the downwash velocity gives the following

$$\alpha(y_0) = \frac{2\Gamma(y_0)}{a_0(y_0) V_\infty c(y_0)} + \alpha_{l=0}(y_0) + \frac{1}{4\pi V_\infty} \int_{-s}^{+s} \frac{(d\Gamma/dy)dy}{(y_0 - y)}$$

This is the FUNDAMENTAL LIFTING LINE EQUATION derived by Prandtl. In this equation the only unknown is the circulation distribution $\Gamma(y_0)$, since $a_{0,}\alpha, c, V_\infty$ and $\alpha_{l=0}$ are known for a given design of wing, at a given angle of attack in a freestream with given velocity. Note that for thin untwisted wings can assume the thin aerofoil result that $a_0 \approx 2\pi$ for all sections.

Once the circulation distribution is obtained from this equation the lift distribution can be found

$$l(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

where the lower case *l* indicates that this is a lift per unit span. Integrating this local lift, dropping the subscript on *y* for simplicity, the total lift is given by

$$L = \int_{-s}^{s} l(y)dy = \rho_\infty V_\infty \int_{-s}^{s} \Gamma(y)dy$$

$$C_L = \frac{2}{V_\infty S} \int_{-s}^{+s} \Gamma(y)dy$$

The drag distribution can be found from

$$d(y_0) \approx l(y_0)\alpha_i(y_0),$$

This can be integrated to obtain the total induced drag as

$$D_i = \rho_\infty V_\infty \int_{-s}^{+s} \Gamma(y)\alpha_i(y)dy$$

$$C_{D_i} = \frac{2}{V_\infty S}\int_{-s}^{+s}\Gamma(y)\alpha_i(y)dy$$

Solving the fundamental lifting line equation is not in general straightforward. However there is one simple special case that turns out to be particularly relevant.

2 CLASSICAL LIFTING LINE METHOD FOR AN ELLIPTIC SPANWISE CIRCULATION DISTRIBUTION

Consider the case for an elliptic spanwise circulation distribution.

$$\Gamma = \Gamma_0\sqrt{1-\left(\frac{y}{s}\right)^2}$$

where Γ_0 is constant and the wing is defined by $-s \le y \le +s$ Note that the sectional lift per unit span is given by

$$l(y_0) = \rho_\infty V_\infty \Gamma(y_0) = \rho_\infty V_\infty \Gamma_0\sqrt{1-\left(\frac{y_0}{s}\right)^2}$$

and therefore also has an elliptical distribution.

2.1 Downwash and Induced Incidence

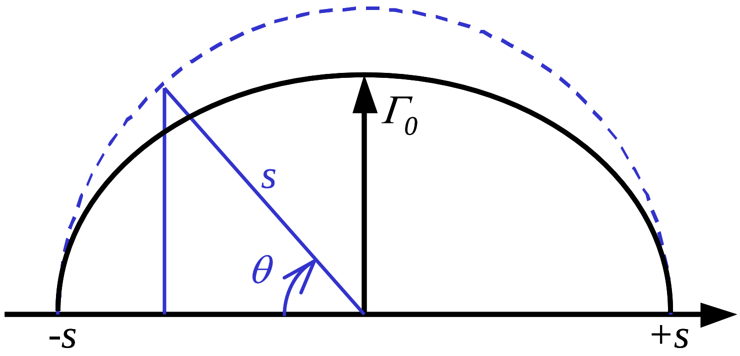
Now the expression for the downwash for an elliptic circulation is given by

$$w(y_0) = -\frac{1}{4\pi}\int_{-s}^{+s}\frac{(d\Gamma/dy)}{y_0-y}dy$$

where the derivative of the circulation is given by

$$\frac{d\Gamma}{dy} = \Gamma_0\frac{1}{2\sqrt{1-\left(\frac{y}{s}\right)^2}}\left(-2\frac{y}{s^2}\right) = \frac{-y\Gamma_0}{s\sqrt{s^2-y^2}}$$

It is easiest to evaluate the downwash incidence by making the substitution $y = -s\cos\theta,\ dy = s\sin\theta\, d\theta$



This substitution means that the wing $-s \le y \le +s$ maps to $0 \le \theta \le \pi$ with $\theta = 0$ corresponding to $y = -s$ and $\theta = \pi$ corresponding to $y = +s$. Then

$$\Gamma = \Gamma_0 \sin\theta$$
$$\frac{d\Gamma}{dy} = \frac{s\cos\theta\,\Gamma_0}{s\sqrt{s^2 - s^2\cos^2\theta}} = \frac{\cos\theta\,\Gamma_0}{s\sin\theta}$$

Hence

$$w(y_0) = -\frac{\Gamma_0}{4\pi s}\int_0^\pi \frac{\cos\theta}{(\cos\theta - \cos\theta_0)}\,d\theta$$

So since

$$\int_0^\pi \frac{\cos n\theta}{(\cos\theta - \cos\theta_0)}\,d\theta = \frac{\pi\sin n\theta_0}{\sin\theta_0}$$

finally

$$w(y_0) = -\frac{\Gamma_0}{4s}$$

So the downwash is constant across the wing (no y or θ dependence). This is also true of the induced incidence since

$$\alpha_i(y_0) = -\frac{w(y_0)}{V_\infty} = \frac{\Gamma_0}{4V_\infty s}$$

Also the effective incidence is

$$\alpha_{eff}(y_0) = \alpha(y_0) - \alpha_i(y_0) = \alpha(y_0) - \frac{\Gamma_0}{4V_\infty s}$$

and this constant across the span if $\alpha(y_0)$ is constant i.e. no geometric twist.

2.2 Total Lift

Consider the expression for the lift of a wing with an elliptical circulation distribution (use 3D Kutta Joukowski theorem)

$$
\begin{aligned}
L &= \rho_\infty V_\infty \int_{-s}^{+s} \Gamma(y)\,dy = \rho_\infty V_\infty \Gamma_0 \int_{-s}^{+s} \sqrt{1-(y/s)^2}\,dy \\
&= \rho_\infty V_\infty \int_0^\pi (\Gamma_0 \sin\theta)\, s \sin\theta \, d\theta \\
&= \rho_\infty V_\infty \Gamma_0 \, s \int_0^\pi \left(\sin^2\theta\right) d\theta \\
&= \rho_\infty V_\infty \Gamma_0 \, s \int_0^\pi \frac{1}{2}(1-\cos 2\theta)\, d\theta \\
&= \frac{1}{2}\rho_\infty V_\infty \Gamma_0 \, s \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^\pi \\
&= \frac{1}{2}\rho_\infty V_\infty \Gamma_0 \, s\pi
\end{aligned}
$$

Hence the lift coefficient is

$$
\begin{aligned}
C_L &= \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{\frac{1}{2}\rho_\infty V_\infty \Gamma_0 \, s\pi}{\frac{1}{2}\rho_\infty V_\infty^2 S} \\
&= \frac{\Gamma_0 \, s\pi}{V_\infty S}
\end{aligned}
$$

and thus can show that

$$
\begin{aligned}
C_L &= \frac{s\Gamma_0 \pi}{V_\infty S} = \frac{\Gamma_0 4s^2\pi}{4V_\infty sS} = \alpha_i \pi \frac{(2s)^2}{S} = \alpha_i \pi \frac{b^2}{S} \\
C_L &= \alpha_i \pi \; AR
\end{aligned}
$$

So the lift coefficient only depends on the induced incidence and the wing aspect ratio and so is constant. This can be rearranged to give

$$
\alpha_i = \frac{C_L}{\pi \; AR}
$$

2.3 Drag

Consider now the induced drag coefficient

$$
\begin{aligned}
C_{D_i} &= \frac{D}{1/2\rho_\infty V_\infty^2 S} = \frac{2}{\rho_\infty V_\infty^2 S} \int_{-s}^{+s} \rho_\infty (-w(y))\Gamma(y)dy = \frac{2}{\rho_\infty V_\infty^2 S} \int_{-s}^{+s} \rho_\infty V_\infty (-w(y)/V_\infty)\Gamma(y)dy \\
&= \frac{2}{V_\infty S} \int_{-s}^{+s} \Gamma(y)\alpha_i(y)dy
\end{aligned}
$$

since induced incidence is constant. So

$$
C_{D_i} = \frac{2\alpha_i}{V_\infty S} \int_{-s}^{+s} \Gamma(y)\, dy
$$

but

$$C_L = \frac{2}{V_\infty S}\int_{-s}^{s}\Gamma dy$$

so that

$$C_{D_i} = \alpha_i \, C_L$$

and since the induced incidence is given by

$$\alpha_i = \frac{C_L}{\pi \; AR}$$

finally the induced drag is given by

$$C_{D_i} = \frac{C_L^2}{\pi \; AR}$$

The induced drag increases with C_L^2. Thus lift generation comes at the price of induced drag and the drag coefficient increases rapidly as lift increases to become a substantial part of the overall drag on an aircraft at high lift e.g. when an aircraft is flying slowly such as at take off and landing. Typically at cruise conditions induced drag would be around 25% of the total drag. Another thing to note is that the induced drag is inversely proportional to aspect ratio - this is why gliders have high aspect ratio wings. However there is a trade off against weight, structural strength and friction drag.

2.4 Sectional Lift

Now in the classical method each chordwise section of the wing is treated as a 2D aerofoil, thus at a particular spanwise location

$$c_l(y_0) \;=\; a_0(y_0)\left(\alpha_{eff}(y_0) - \alpha_{l=0}(y_0)\right)$$

For the special case of an **untwisted wing** the local sectional lift coefficient predicted for an elliptical circulation distribution is

$$c_l \;=\; a_0\left(\alpha - \frac{\Gamma_0}{4V_\infty s} - \alpha_{l=0}\right)$$

and this is also <u>constant</u> since $\alpha_{l=0}$ and a_0 are constant for zero aerodynamic twist, α is constant for a wing with zero geometric twist and the induced incidence is constant for an elliptic circulation distribution.

It was shown above that the sectional lift coefficient is <u>also</u> given by

$$c_l(y) = \frac{2\Gamma(y)}{V_\infty c(y)}$$

and for an untwisted wing with an elliptical circulation distribution this has to be constant and therefore the chord length distribution must also be elliptical.

Thus an elliptical planform without twist will produce an elliptical lift distribution, such a planform theoretically gives the minimum induced drag (see later in these notes).

2.5 3D Lift Slope for an Untwisted Wing

The sectional lift is assumed to be

$$c_l = a_0(\alpha_{eff} - \alpha_{l=0})$$

and for the special case of an **untwisted wing with elliptical circulation distribution** this is constant and so

$$c_l = C_L$$

Then concentrating on the expression derived from the 2D lift slope

$$\begin{aligned} C_L &= a_0(\alpha - \alpha_i - \alpha_{l=0}) \\ &= a_0(\alpha - \frac{C_L}{\pi\ AR} - \alpha_{l=0}) \end{aligned}$$

so on rearranging

$$C_L = \left(\frac{a_0}{1 + \dfrac{a_0}{\pi\ AR}}\right)(\alpha - \alpha_{l=0}) = a(\alpha - \alpha_{l=0})$$

The lift curve slope in 3D is then obtained

$$a = \frac{dC_L}{d\alpha} = \frac{a_0}{\left(1 + a_0 \dfrac{1}{\pi AR}\right)}$$

This lift curve slope in 3D is less than the 2D lift slope and is a function of aspect ratio.

3 RESULTS FOR CLASSICAL LIFTING LINE METHOD FOR A GENERAL SPANWISE CIRCULATION DISTRIBUTION

Results for a general spanwise circulation distribution are given in this section. A general symmetric circulation distribution can be represented by a Fourier series

$$\Gamma(\theta) = 4sV_\infty \sum_{n=1}^{N} A_n \sin n\theta$$

where $y = -s\cos\theta$. Note that for an elliptic distribution

$$\Gamma(\theta) = \Gamma_0 \sin\theta$$

i.e. N=1 and $A_1 = \Gamma_0 /(4sV_\infty) = \alpha_i$. For more detail of how results in this section are derived see reference document RefI5_LiftingLine_Extra.pdf available from the Reference section of the Aero2 Blackboard site.

3.1 Induced Incidence

Now the expression for the induced incidence for a general circulation is given by

$$\alpha_i = \sum_{n=1}^{N} nA_n \frac{\sin n\theta_0}{\sin\theta_0}$$

3.2 Total Lift

The following expression is obtained for the lift coefficient of the wing

$$C_L = A_1 \pi \, AR$$

So the lift coefficient only depends on the first term of the Fourier series and the wing aspect ratio. The other terms A_n provide the characteristic variations in lift across the wing, but do not influence the overall wing lift. For an elliptic wing

$$C_L = \alpha_i \pi \, AR$$

so A_1 can be thought of as effective "mean or average" induced incidence for the wing.

3.3 Drag

The induced drag coefficient is given

$$C_{D_i} = \frac{C_L^2}{\pi \, AR}\left(1+\delta\right)$$

where $\delta \geq 0$ is a function of the Fourier coefficients. For an elliptical circulation distribution $\delta = 0$ so

$$C_{D_i} > (C_{D_i})_{elliptical}$$

and the elliptical distribution gives minimum induced drag for a given lift.

3.4 3D Lift Slope for an Untwisted Wing

It can be shown that for an untwisted wing

$$C_L = \frac{a_0(\alpha - \alpha_{l=0})}{\left(1 + a_0 \dfrac{1}{\pi AR}(1+\tau)\right)}$$

where τ is a function of the Fourier coefficients. The lift curve slope in 3D is then given by

$$a = \frac{dC_L}{d\alpha} = \frac{a_0}{\left(1 + a_0 \dfrac{1}{\pi AR}(1+\tau)\right)}$$

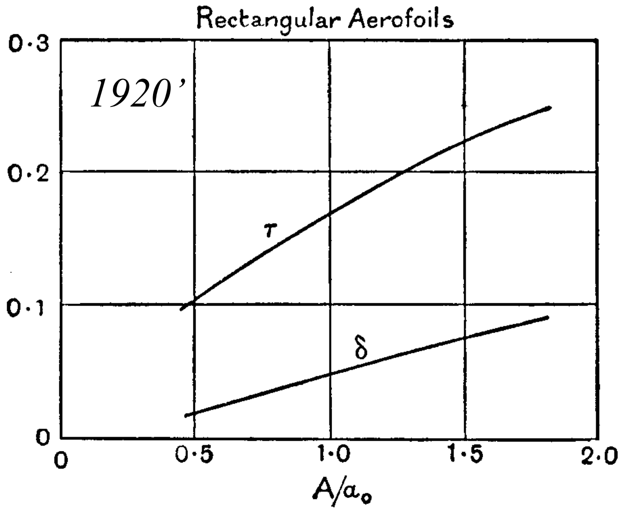
For an elliptic circulation distribution τ =0, generally typical values of τ are in the range 0.05 to 0.25.

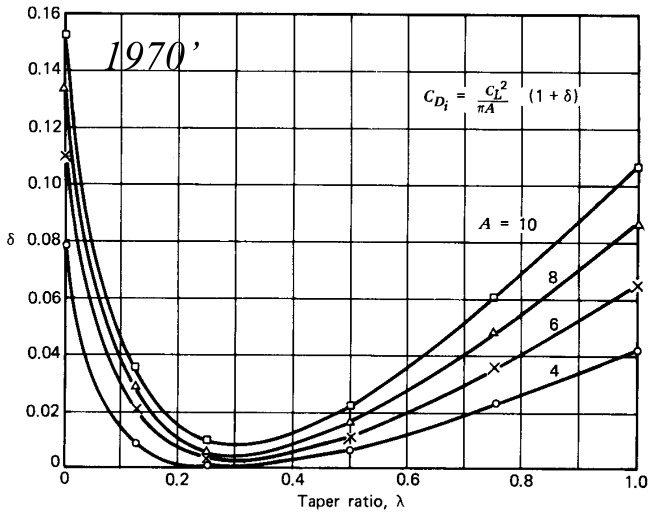
3.5 Values for Correction Factors

The induced drag and 3D lift slope depend on planform-dependent correction factors τ and δ.

$$a = \frac{a_0}{1 + \frac{a_0}{\pi AR}(1+\tau)} \qquad C_{D_i} = \frac{(1+\delta)C_L^2}{\pi AR} = \frac{C_L^2}{e\pi AR}$$

Figures show calculations of the variation of τ and δ with aspect ratio and δ with taper ratio



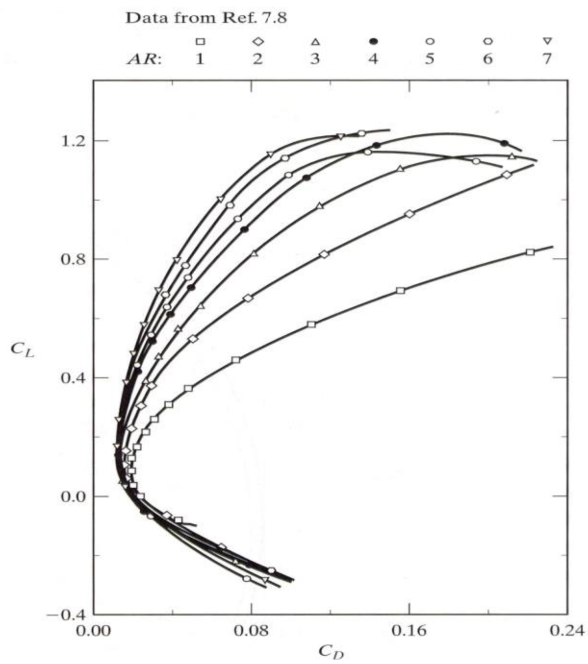


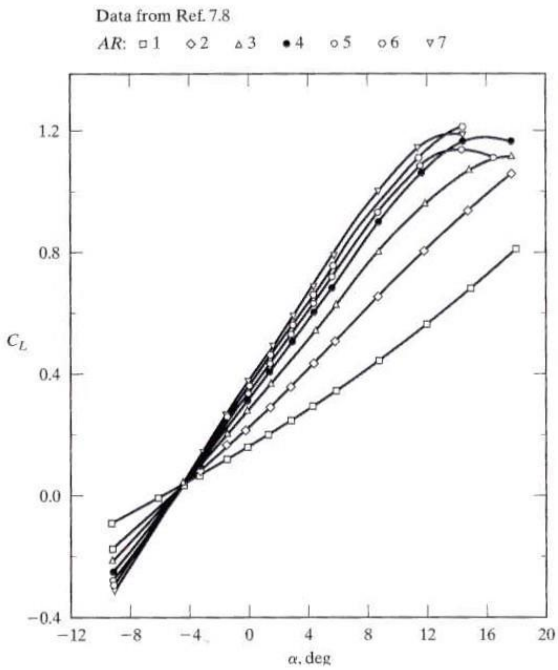
For an elliptic wing $\delta = \tau = 0$ this is the minimum drag case.

4 COMPARING THEORY WITH EXPERIMENT

4.1 Experimental Data for Wings of Different Aspect Ratio

The experimental data shown in this section has been obtained for untwisted rectangular wings with the same aerofoil section, but different aspect ratios. This figure shows lift and drag versus incidence for the wings

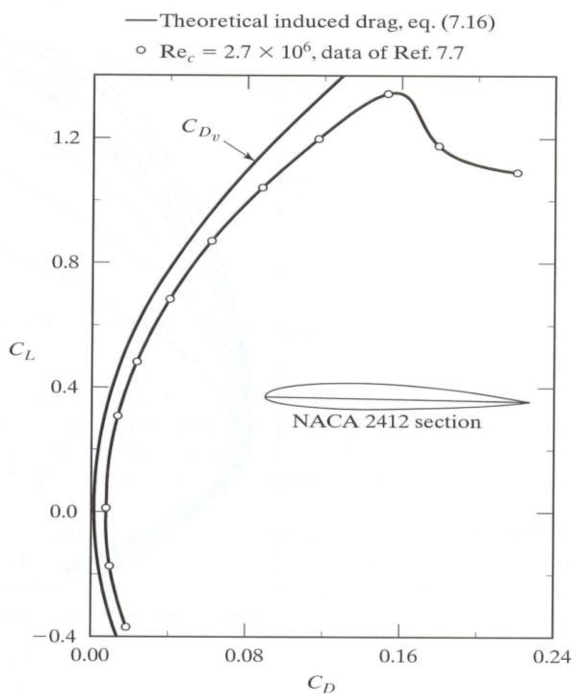




In the following subsections comparisons are made between theory and this experimental data

4.2 Drag Polars

The question is does the theoretical result match experimental observations?



The shape is correct, but there is an almost constant difference between the theoretical drag and the experimental drag

WHY? Because for incompressible flow

$$C_D = C_{D0} + kC_L^2$$

where C_{D0} is the drag at zero lift or the parasitic drag. Note in this formula compressibility effects are not included, in that case there would be an additional wave drag term.

There is reasonable correlation up to 20deg angle of attack. At highest angles of attack near stall viscous drag increases and also the assumptions made in deriving lifting line (that there is a linear lift curve) break down.

4.3 Effect of Aspect Ratio on Drag

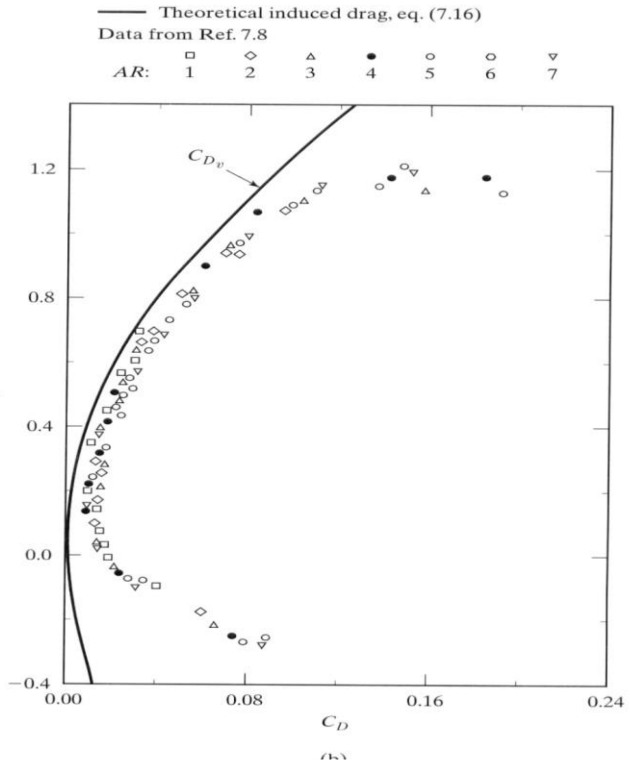
Confirmation of the derived relationship for drag for different aspect ratio wings can be obtained by considering the relationship between drag for two different aspect ratio wings at the same C_L

$$C_{D1} = C_{D01} + \frac{1}{\pi\, AR_1} C_L^2 (1+\delta) \qquad\qquad C_{D2} = C_{D02} + \frac{1}{\pi\, AR_2} C_L^2 (1+\delta)$$

If the aerofoil section is the same then C_{D01} is almost identical to C_{D02} and therefore these can be eliminated to give

$$C_{D1} = C_{D2} + \frac{1}{\pi}\left(\frac{1}{AR_1} - \frac{1}{AR_2}\right) C_L^2 (1+\delta)$$

So experimental data for different aspect ratio wings having the same aerofoil section can be converted to all be for one aspect ratio. If the formula for drag is correct then all the data should overlay on one curve



Good agreement between the different sets of data is shown validating the formula even at low AR.

4.4 Effect of Aspect Ratio on Lift for an Elliptic Circulation Distribution

Confirmation of the derived relationship for lift on untwisted wings with elliptic circulation distribution for different aspect ratios can be obtained by considering the relationship between the geometric angle of attack required to generate the same lift coefficient C_L for two different aspect ratio wings.

$$C_L = a_0\left(\alpha_1 - \alpha_{i1} - \alpha_{l=0}\right) \qquad C_L = a_0\left(\alpha_2 - \alpha_{i2} - \alpha_{l=0}\right)$$

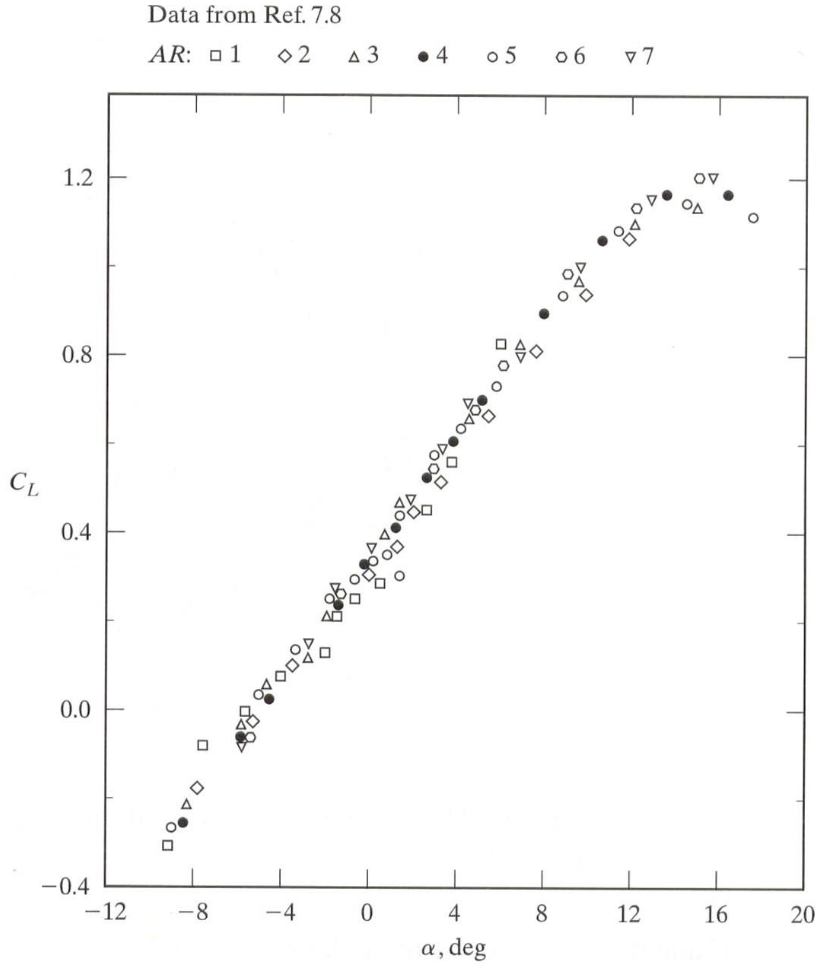
so eliminating $\alpha_{l=0}$, C_L and a_0.

$$\alpha_1 = \alpha_2 + (\alpha_{i1} - \alpha_{i2})$$

Finally it can be shown that

$$\alpha_1 = \alpha_2 + \frac{C_L}{\pi}\left(\frac{1}{AR_1} - \frac{1}{AR_2}\right)$$

So the experimental data for different aspect ratios can be converted to all be for one aspect ratio. If our formula for lift is correct then all the data should overlay on one curve



Good agreement between the different sets of data is shown validating the formula even at low AR.

REVISION OBJECTIVES

You should be able to:

- describe the basic features of lifting line theory and give an outline of the derivation of the "fundamental lifting line" equation.
- derive lift, drag, downwash velocity and induced incidence from a specified circulation distribution
- describe important feature of an elliptical circulation distribution e.g. constant downwash/minimum drag
- state results especially for induced drag and lift curve slope for non-elliptical lift distributions and carry out numerical calculations to evaluate wing characteristics

