Advanced Bending and Torsion
Shear Stresses in Solid Section Beams

Dr Luiz Kawashita

Luiz.Kawashita@bristol.ac.uk

30 October 2018

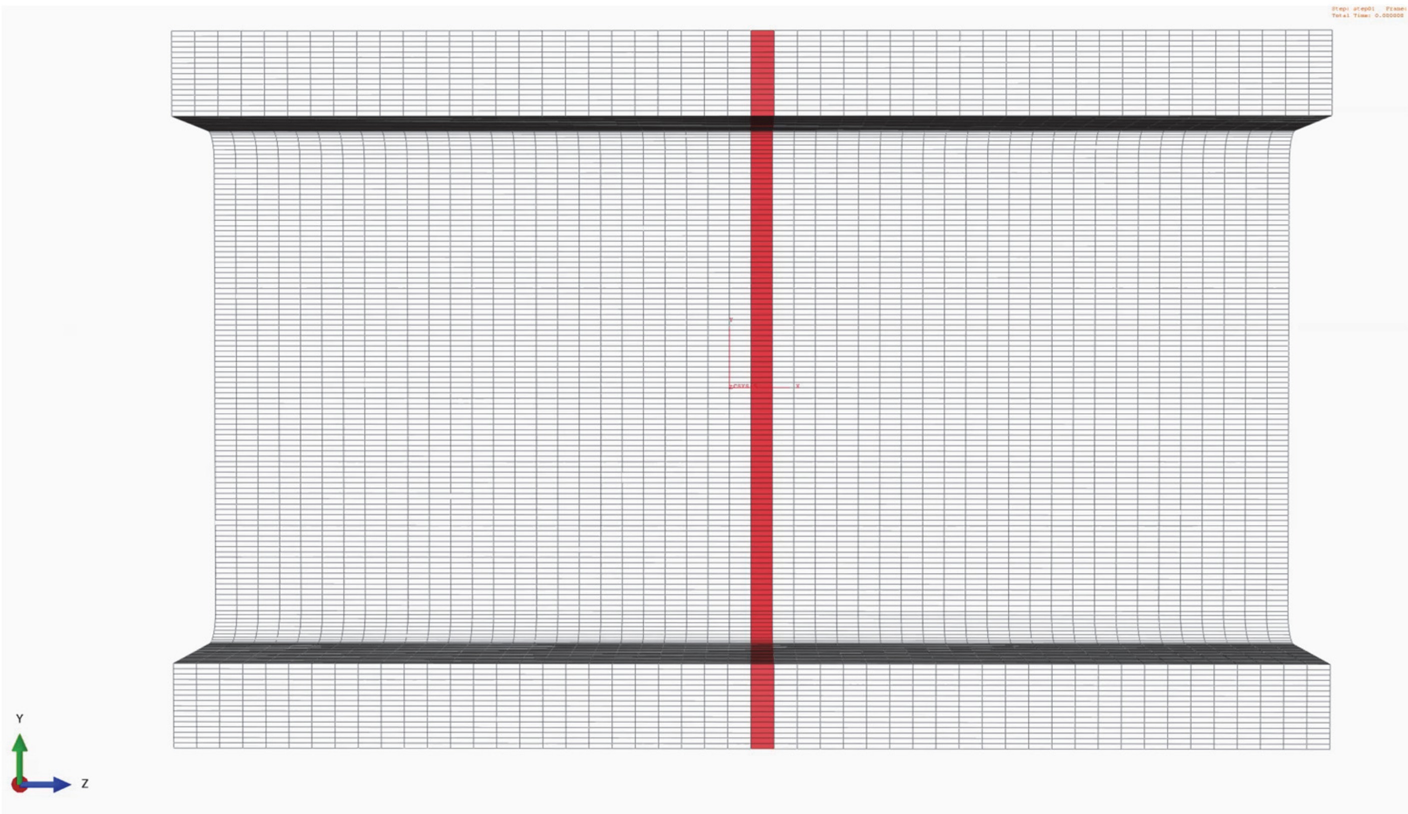
University of BRISTOL

Topics

1. Direct stresses in beams
    1. Off-axis loading of symmetric cross-sections
    2. Transformation of bending axes
    3. Unsymmetric cross-sections
    4. Composite (multi-material) beams
2. Shear stresses in beams
    1. Solid cross-sections
    2. Thin-walled open cross-sections
    3. Thin-walled closed cross-sections

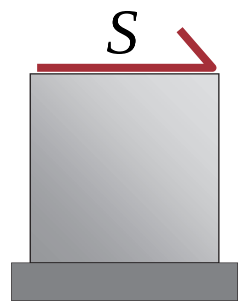
Shear Strains in Beams

- All bending analyses so far assumed that:
  - Plane sections remain plane
  - Shear deformation is negligible
- In real life cross-sections **warp** due to shear stresses
  - Assuming linear elasticity, let us look first at **shear strains**:

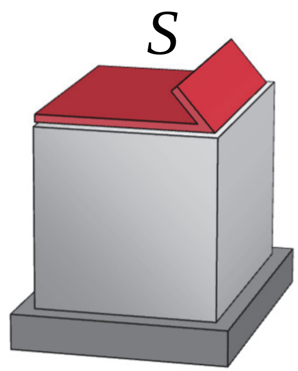


StM1 Recap: Shear Stresses and Strains

- Consider a cube element of material subject to a 'sliding force' (*i.e.* a force tangential to the surface) of intensity S



- The **shear stress** τ is a measure of 'force per unit area' where the force is tangential to the surface



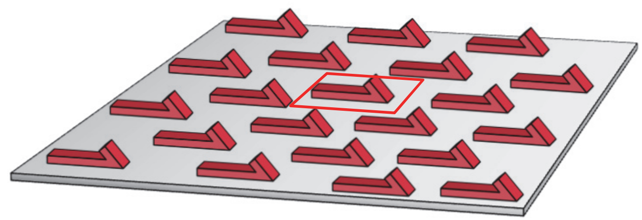
- It is a field property like the 'direct stress' σ; it can vary continuously within a body and can be considered at a point:

$$\tau = \lim_{\delta A \to 0} \frac{\delta S}{\delta A}$$

τ = 'tau'

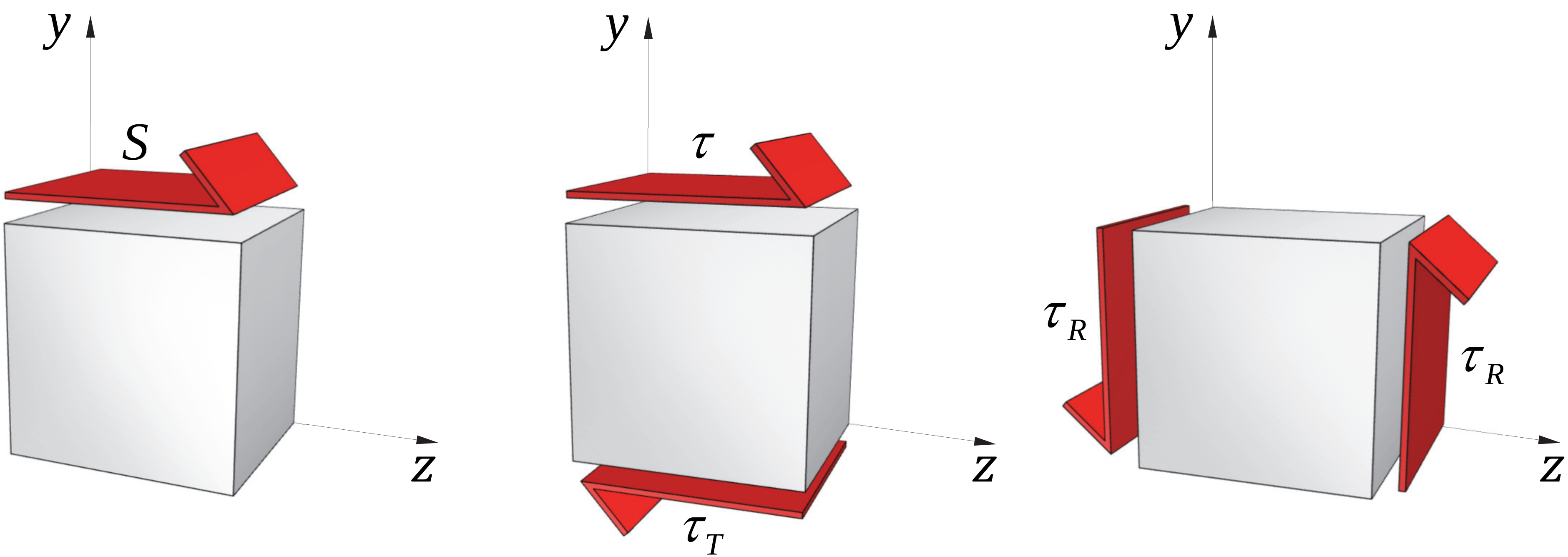
- Same units as for other stresses, *i.e.*

$$1\,\mathrm{Pa} = 1\,\frac{\mathrm{N}}{\mathrm{m}^2} = 10^{-6}\,\mathrm{MPa} = 10^{-6}\,\frac{\mathrm{N}}{\mathrm{mm}^2}$$



StM1 Recap: Complementary Shear Stresses

- For equilibrium, **complementary shear stresses** must exist to balance translational and rotational tendencies



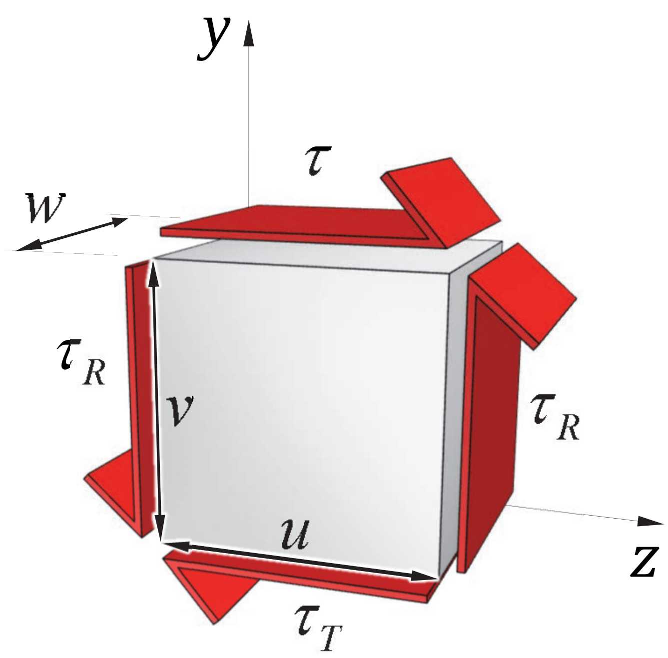
Applied shear force

Shear stresses for translational balance

Shear stresses for rotational balance

StM1 Recap: Complementary Shear Stresses

- In order to balance translational and rotational tendencies the magnitudes of the shear stress components are related:



$$\sum F_z = 0$$

Translational equilibrium along axis z

$$\tau\,(u\,w) - \tau_T\,(u\,w) = 0$$

$$\boxed{\tau = \tau_T}$$

$$\sum M_x = 0$$

Rotational equilibrium about axis z

$$\tau\,(u\,w)(v) - \tau_R\,(v\,w)(u) = 0$$

$$\boxed{\tau = \tau_R}$$

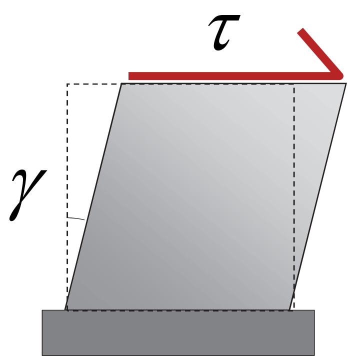
- *i.e.* **all complementary stresses are equal**

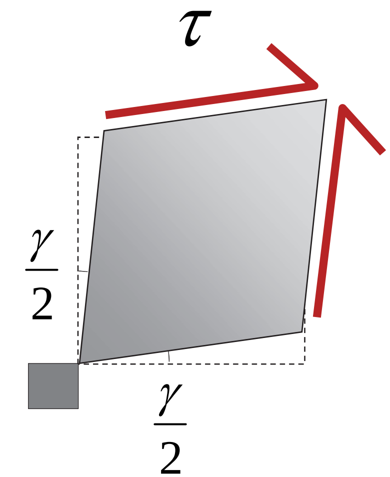
StM1 Recap: Shear Strains

- **Shear strain** γ: angular rotation in radians (non-dimensional)

Simple Shear
'Element fixed along an edge'

Pure Shear
'Element fixed at a corner'

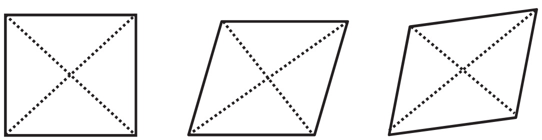




γ = 'gamma'

- Element 'edges' (or 'shear planes') do not change length but simply translate or rotate

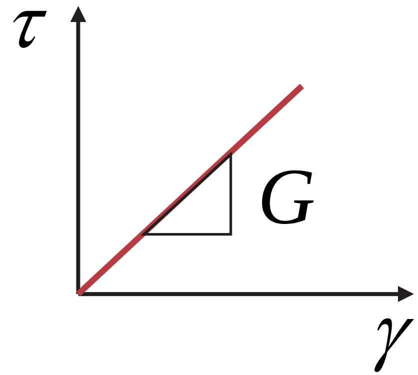
- Element 'diagonals' **<u>do</u>** change length:
  - *i.e.* shear = diagonal 'tension' and 'compression'



StM1 Recap: Shear Modulus

- For linear elastic behaviour shear stress is proportional to shear strain

$$\boxed{\tau = G\,\gamma}$$

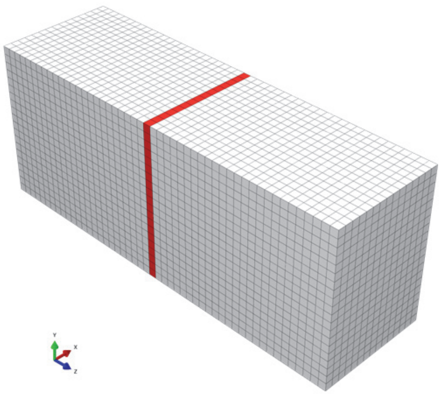


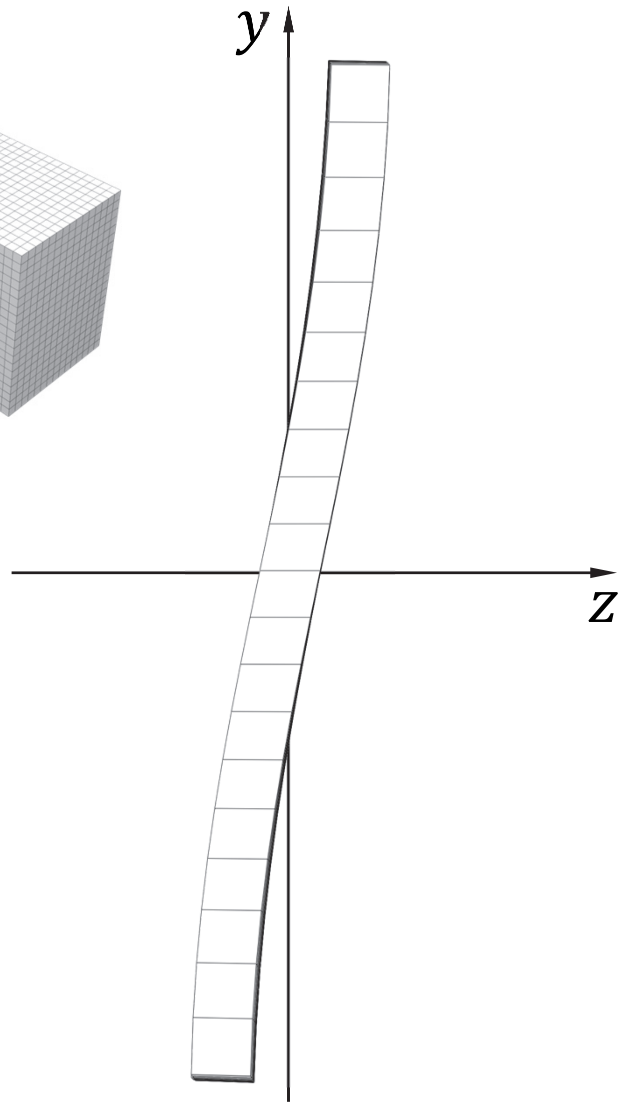
- Where the proportional constant G is the **Shear Modulus**
  - This is a **material property** like Young's modulus E or Poisson's ratio ν
  - In fact, for **isotropic materials** these three properties obey a very simple relationship:

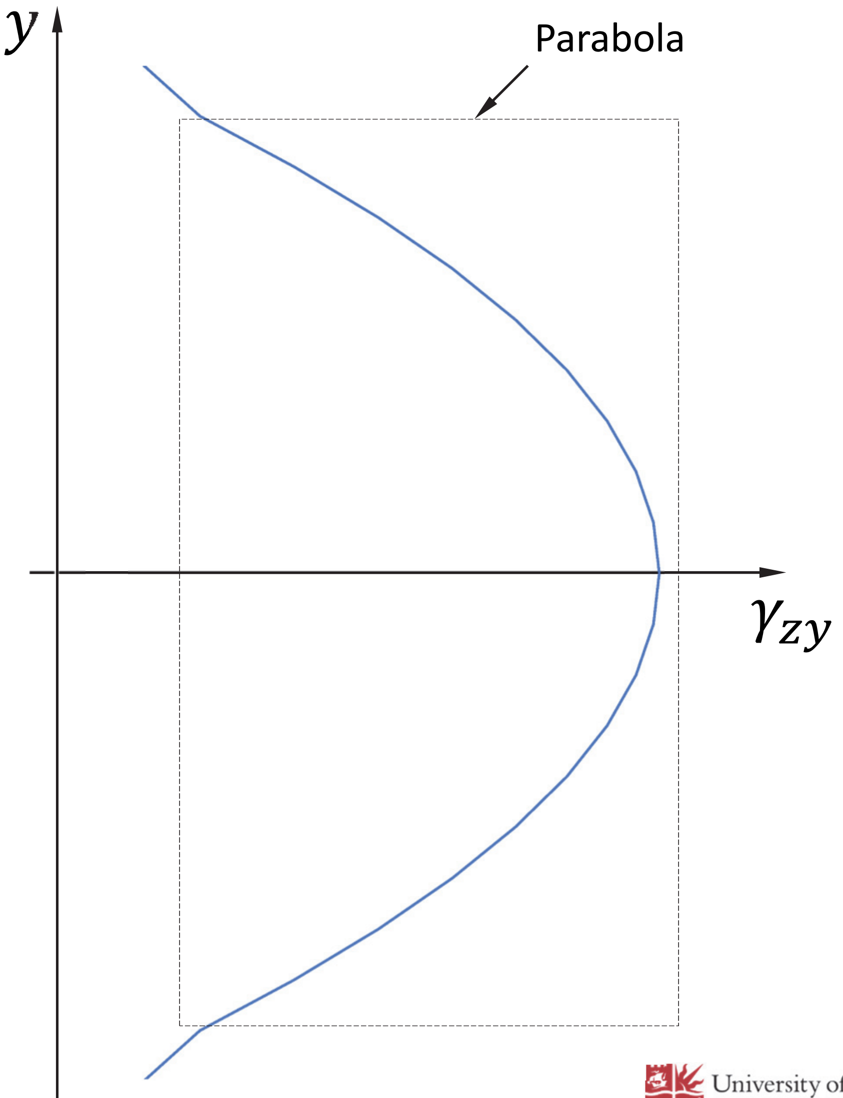
$$\boxed{G = \frac{E}{2(1+\nu)}}$$

Shear Strains in Solid Cross Sections

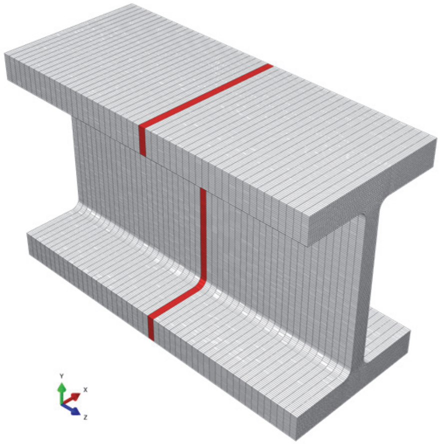
- Shear strains are **angles** which can easily been visualised in FE models, *e.g.* for a solid rectangular cross-section:

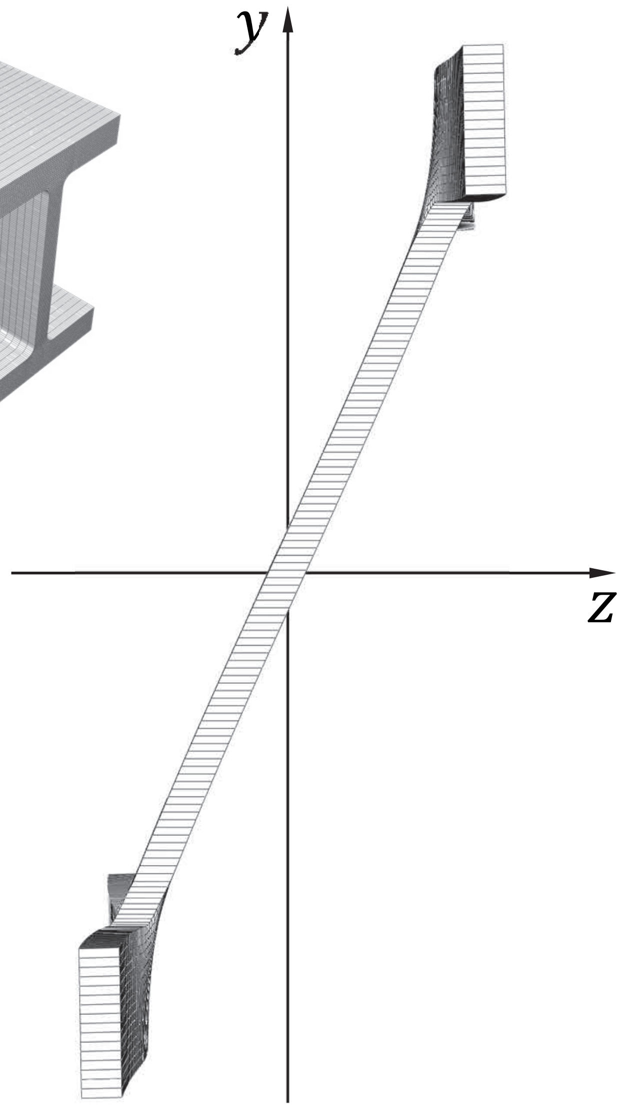


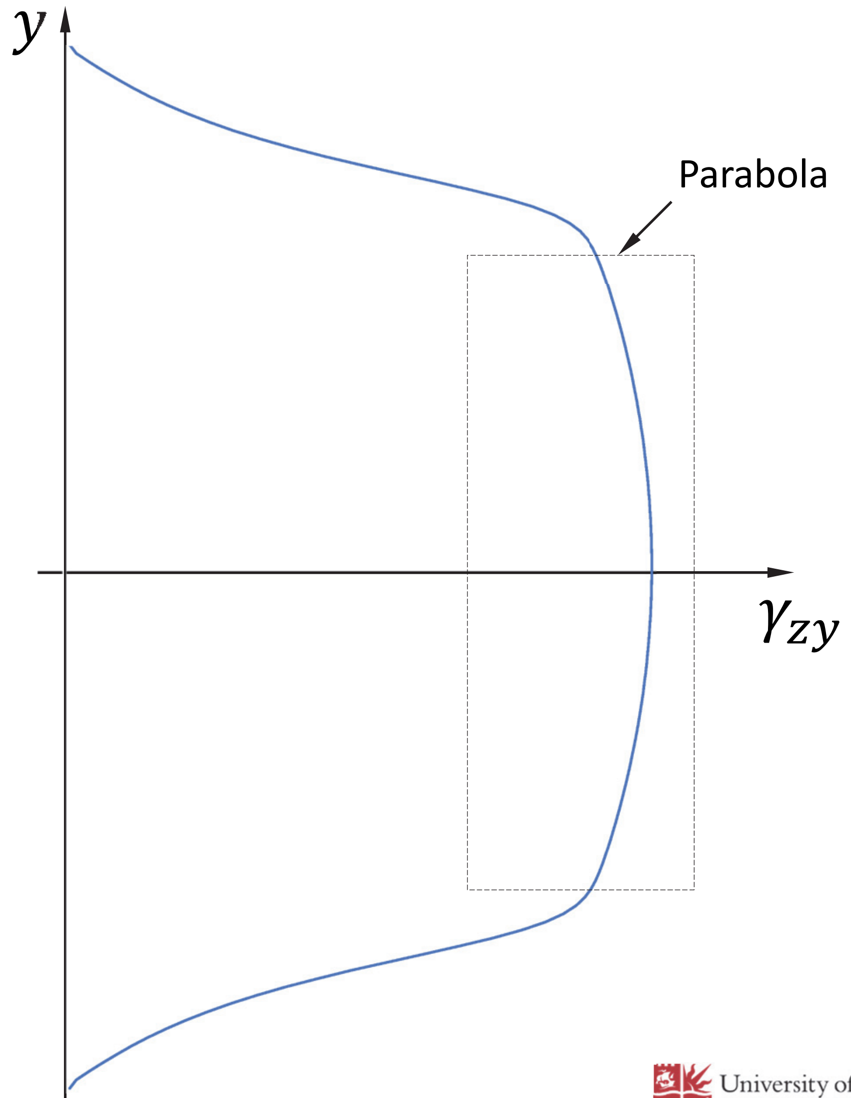




- And the I-section beam seen earlier:







StM1 Recap: Shear Forces *vs.* Bending Moments

- Shear force is the first derivative of the bending moment:

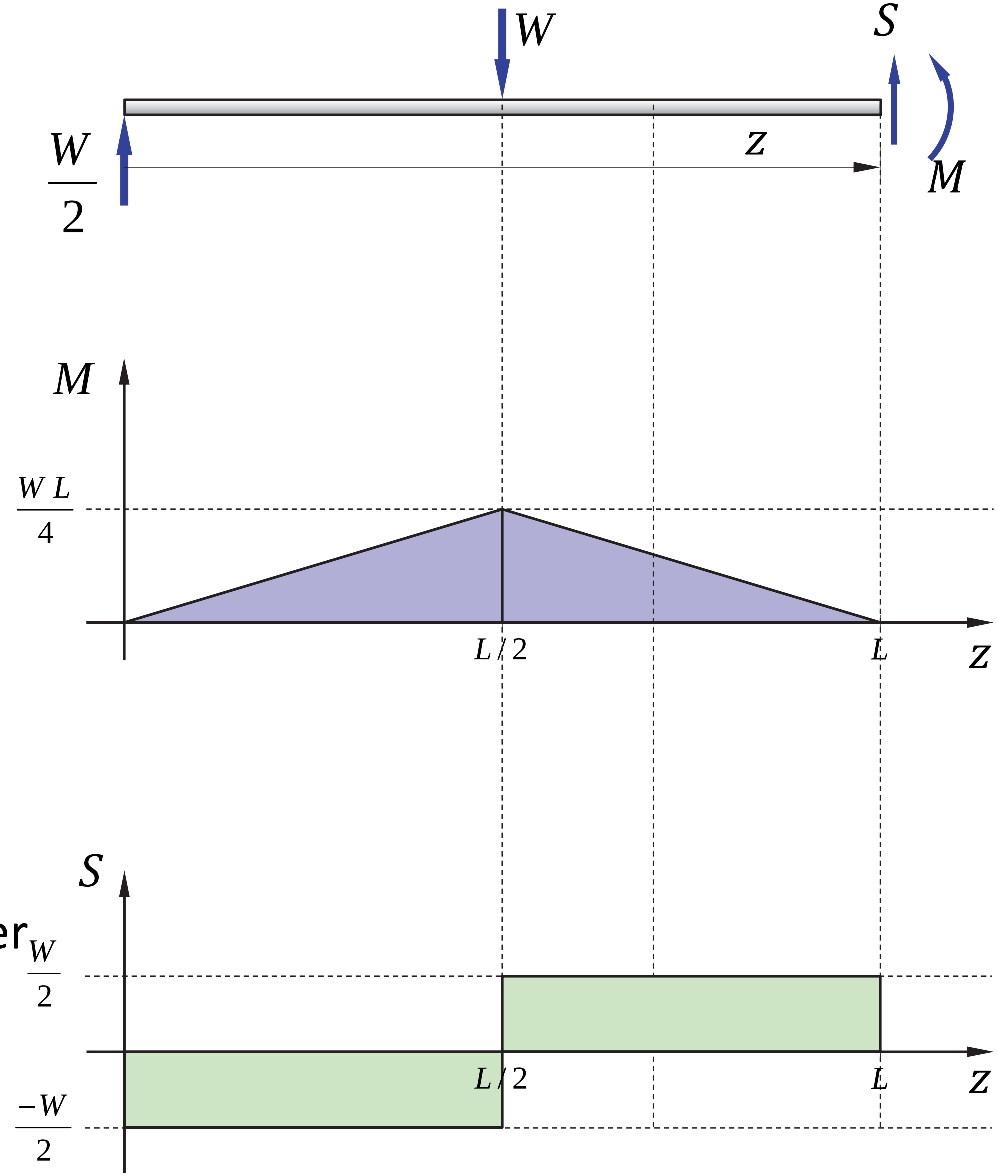
$$\boxed{S = -\frac{\mathrm{d}M}{\mathrm{d}z}}$$

– *i.e.* $-S$ is the '**slope**' of M

- Conversely, bending moments are the integral of the shear forces:

$$\boxed{M = -\int_0^z S\,\mathrm{d}z}$$

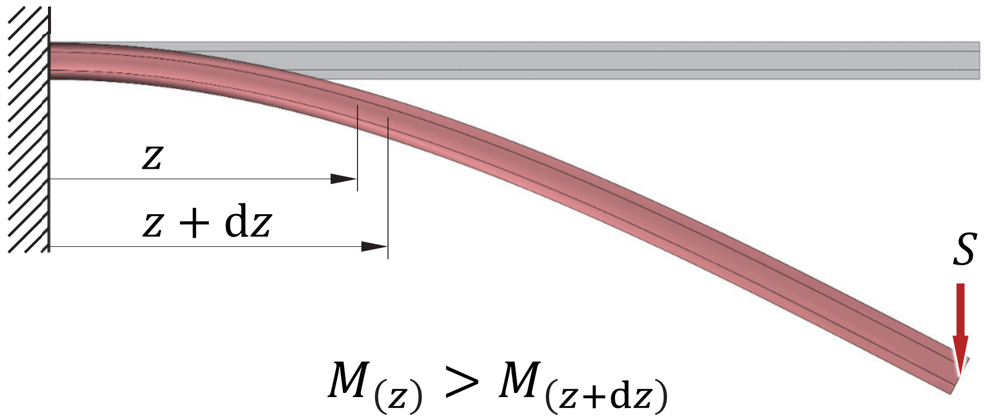
– *i.e.* $-M$ is the '**cumulative area**' under the graph of S



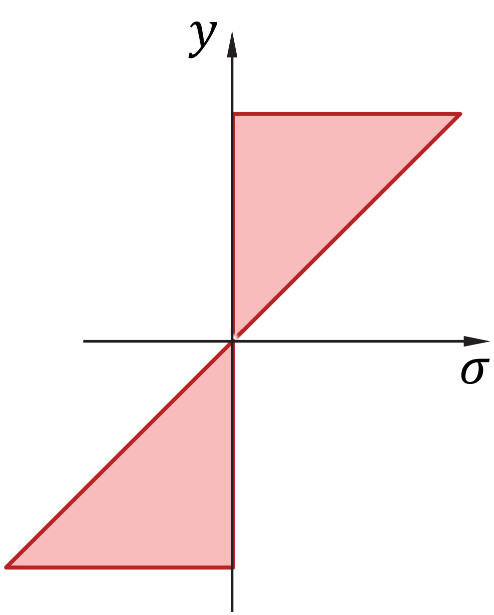
Note: the 'minus' signs in the equations above 'appear/disappear' depending on the adopted sign convention

StM1 Recap: Shear Forces *vs.* Bending Moments

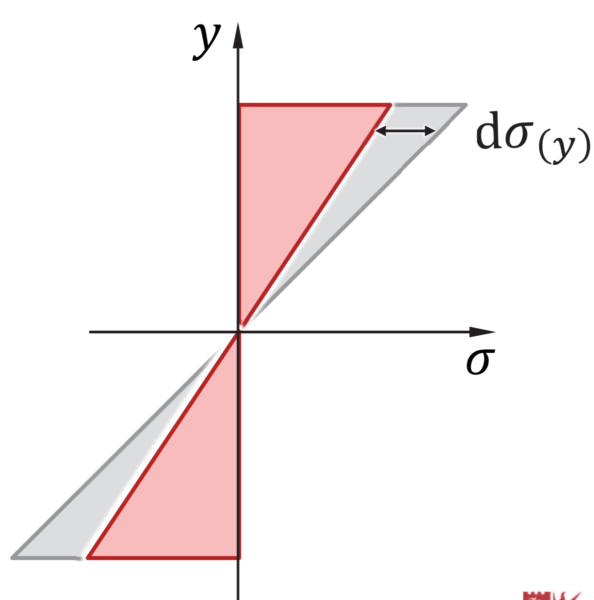
- Remember that stresses σ vary along z in the presence of shear forces:



Stress distribution at z

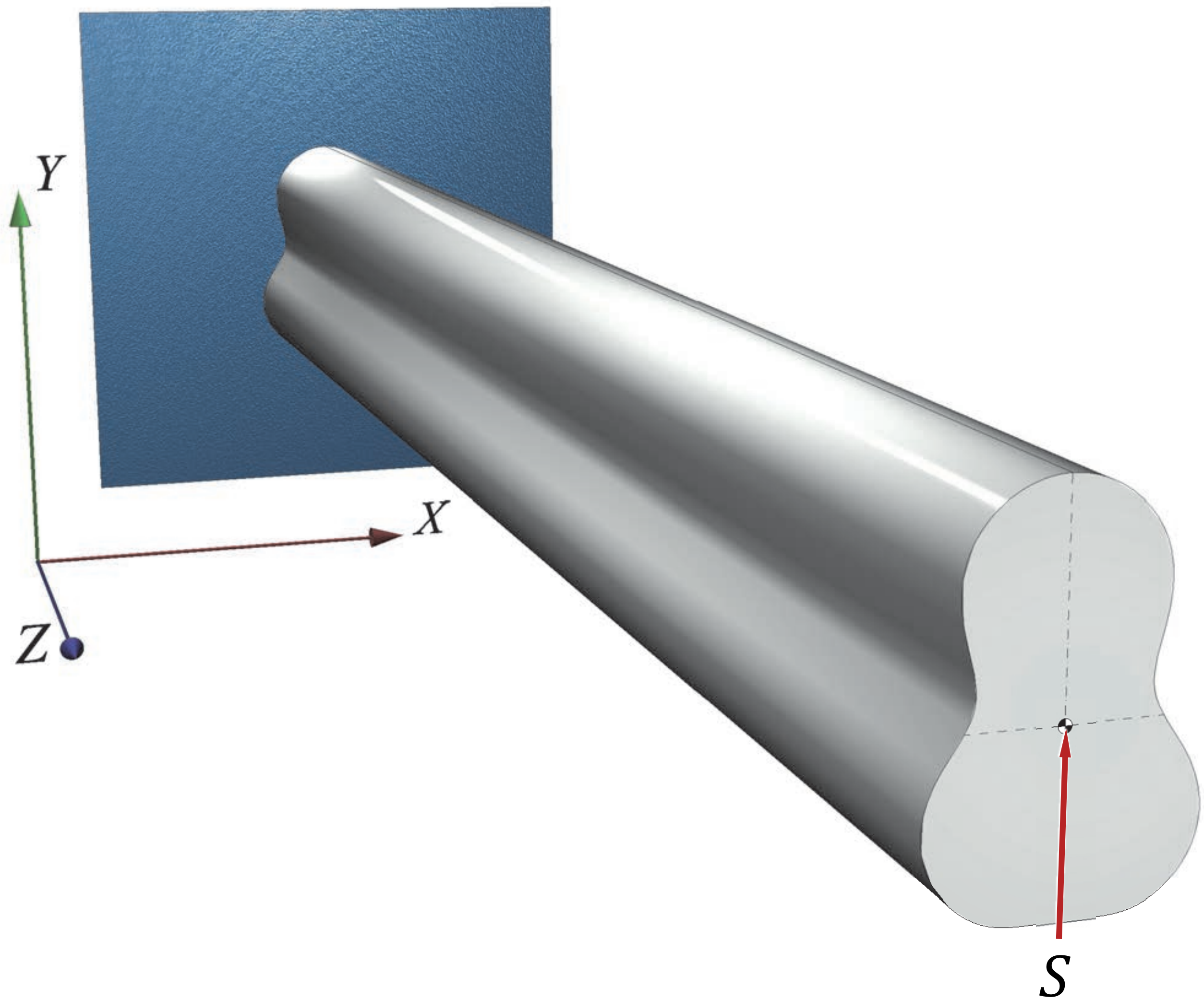


Stress distribution at $z + \mathrm{d}z$



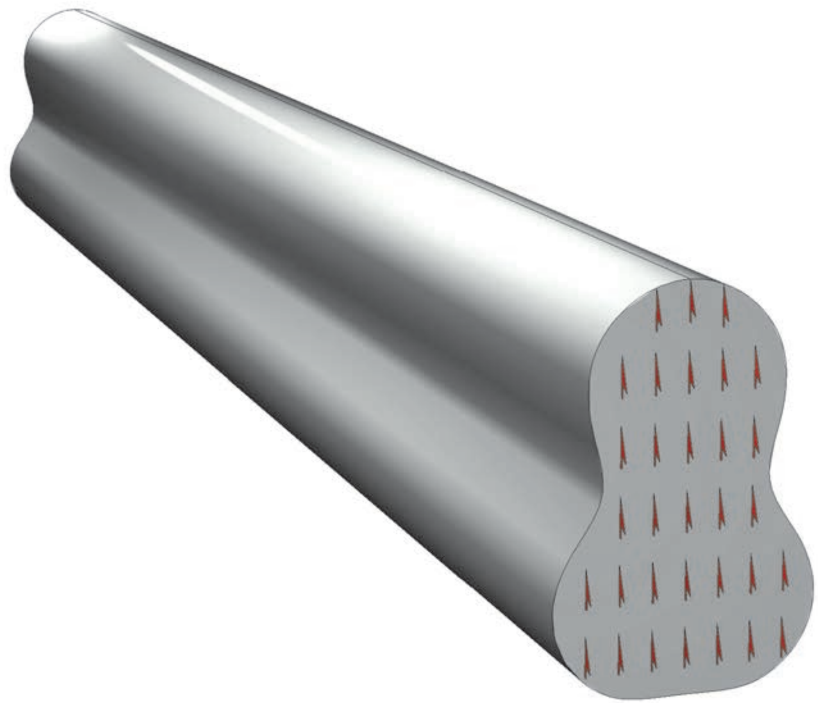
Shear Stresses in Beams – Derivation

- Consider an arbitrary solid cross-section subjected to a **shear force *S* along a principal axis**:



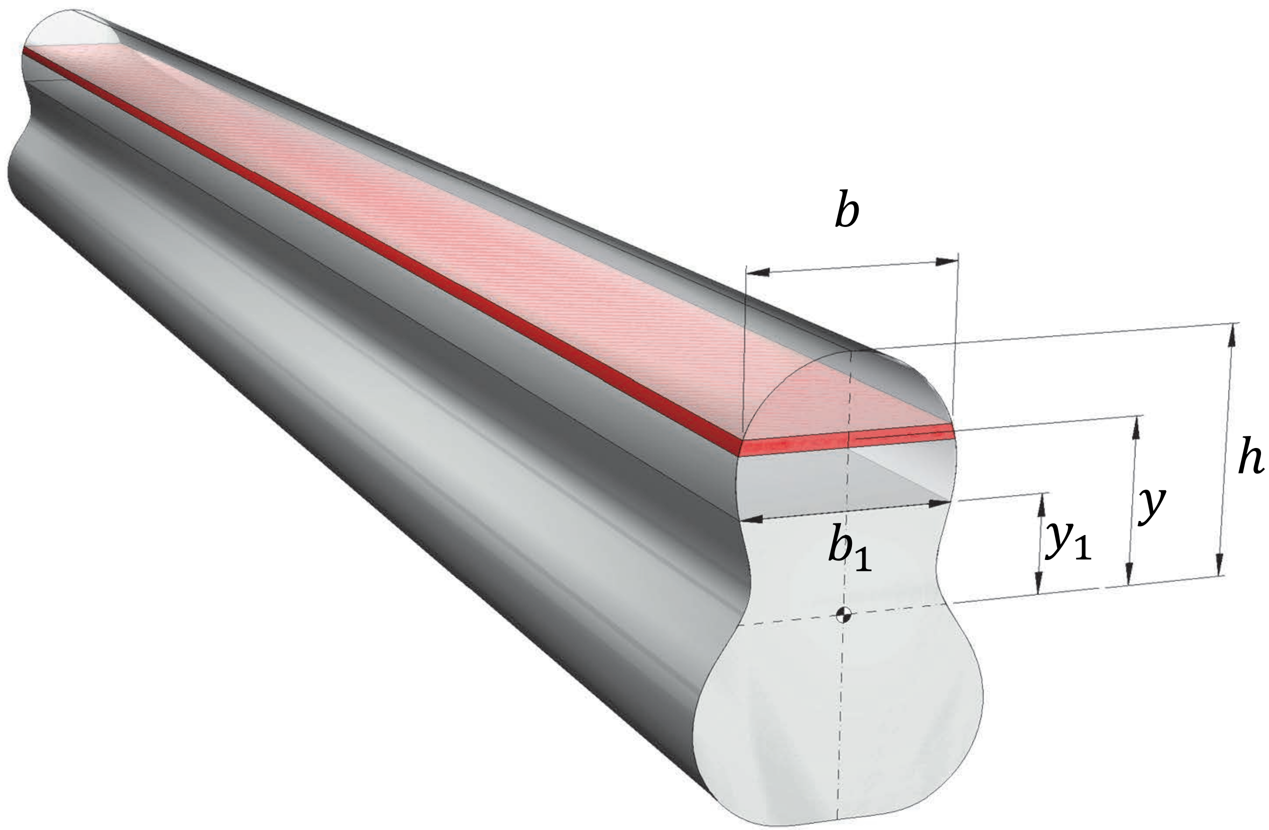
Shear Stresses in Beams – Derivation

- This shear force is 'transmitted' along the beam in the form of shear stresses



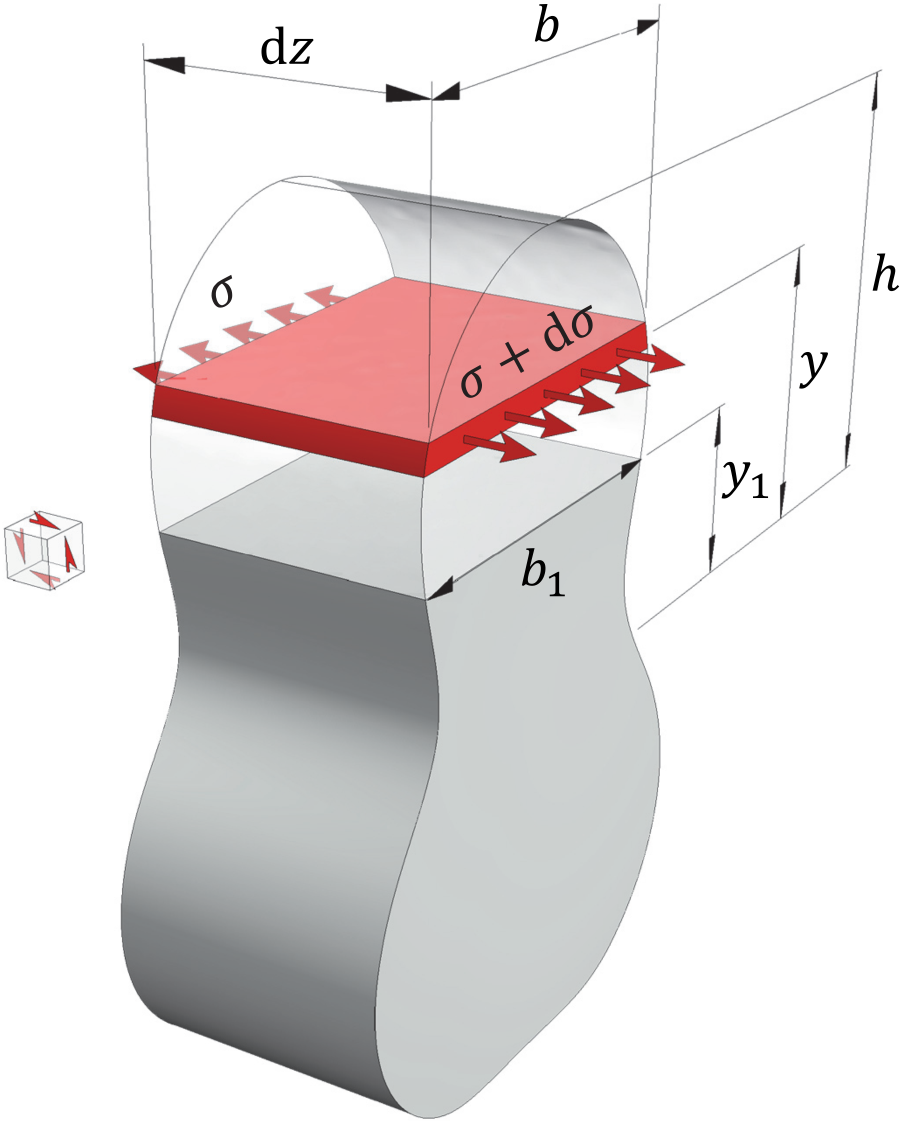
Shear Stresses in Beams – Derivation

- We ‘slice’ the beam transversely to the loading direction and apply equilibrium of internal forces:



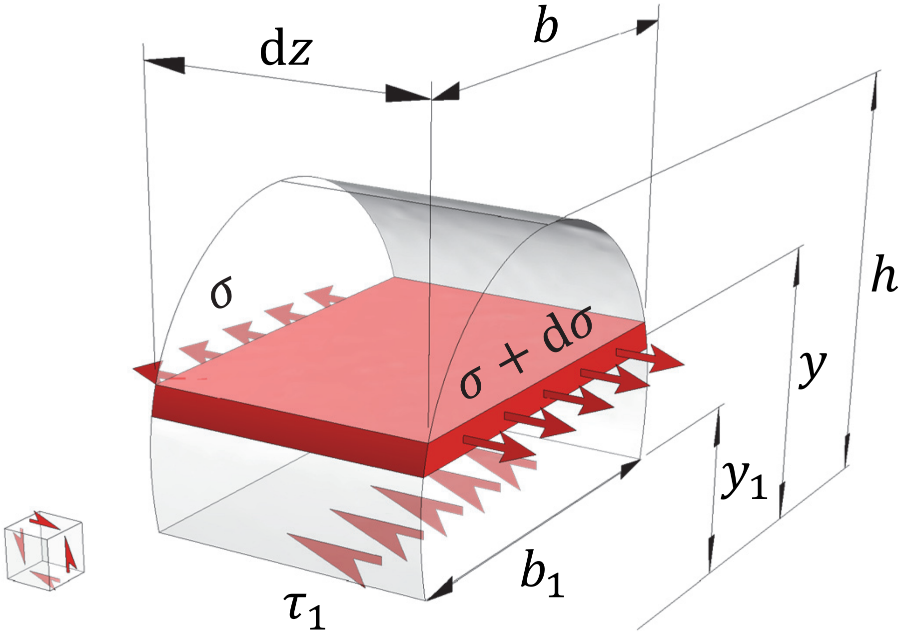
Shear Stresses in Beams – Derivation

- Equilibrium of internal forces for the red slice of thickness $\mathrm{d}y$:



Shear Stresses in Beams – Derivation

- Equilibrium of internal forces for the red slice of thickness $\mathrm{d}y$:



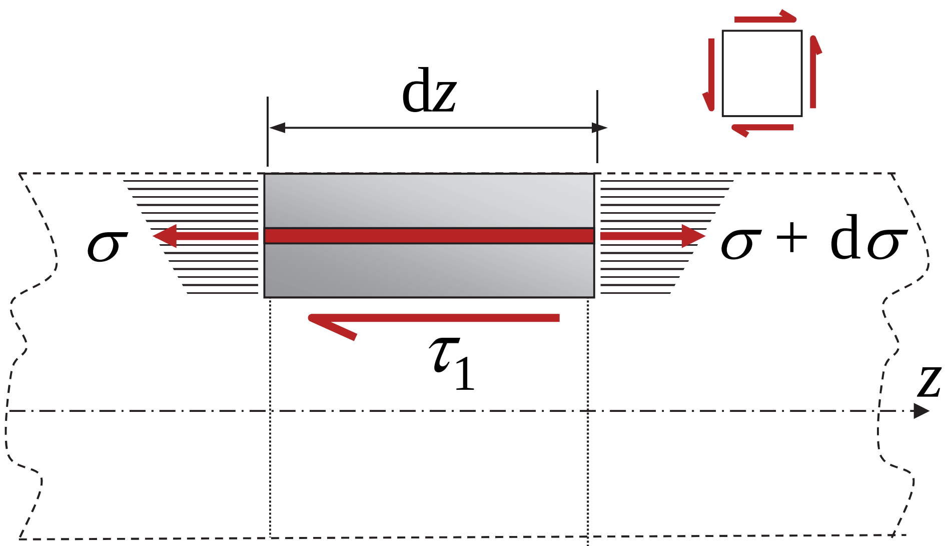
shear force longitudinal force

$$\tau_1\, b_1\, \mathrm{d}z = \int_{y_1}^{h} \mathrm{d}\sigma\, b\, \mathrm{d}y$$

bending formula: $\mathrm{d}\sigma = \dfrac{\mathrm{d}M}{I}\, y$

$$\tau_1 = \frac{\mathrm{d}M}{\mathrm{d}z}\frac{1}{I\, b_1}\int_{y_1}^{h} y\, \mathrm{d}A$$

$b\, \mathrm{d}y = \mathrm{d}A$

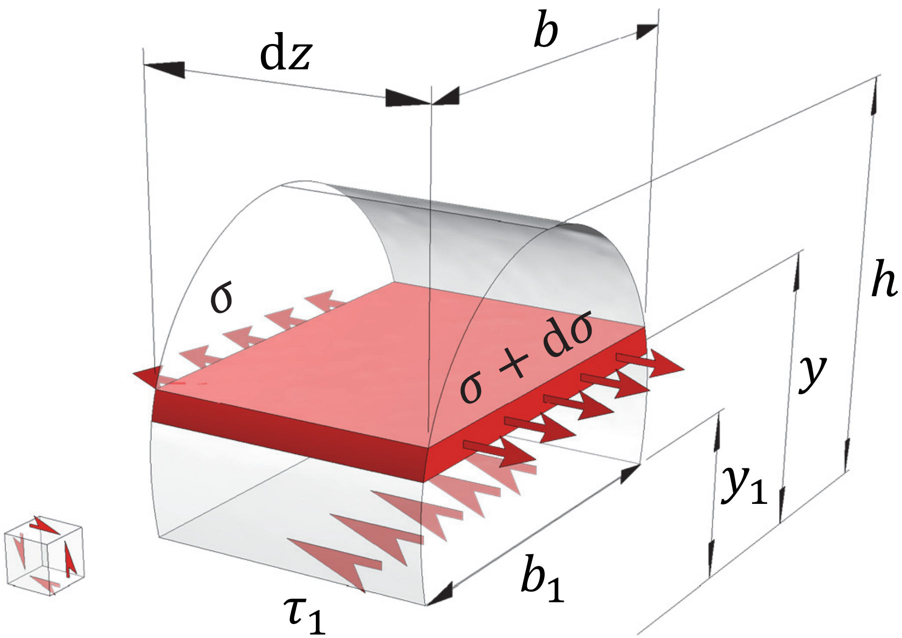


differential beam equations: $\dfrac{\mathrm{d}M}{\mathrm{d}z} = S$

$$\boxed{\tau_1 = \frac{S}{I\, b_1}\int_{y_1}^{h} y\, \mathrm{d}A} = \frac{S\, Q_{xx,1}}{I\, b_1}$$

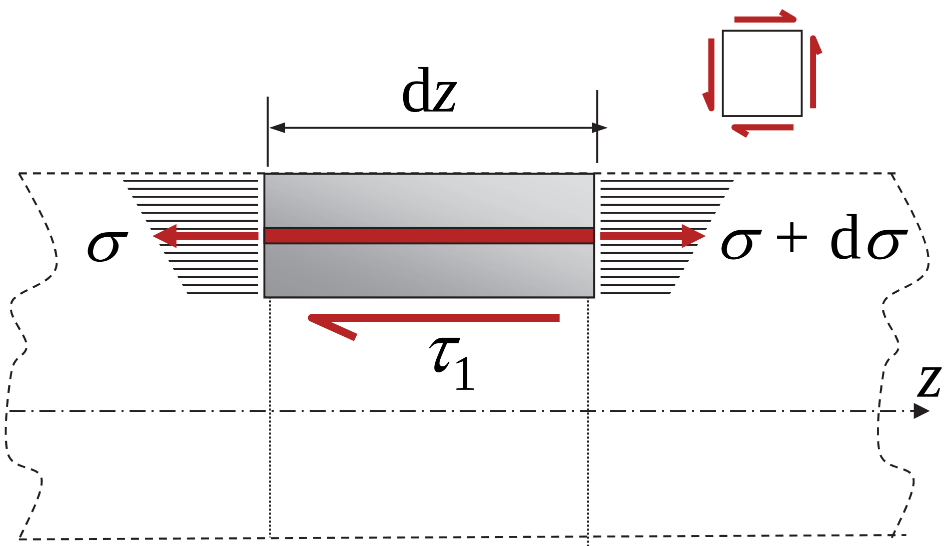
Shear Stresses in Beams – Derivation

- Beam shear formula:



$$\tau_1 = \frac{S}{I\, b_1} \int_{y_1}^{h} y \, \mathrm{d}A = \frac{S\, Q_{xx,1}}{I\, b_1}$$

Note: the first moment of area $Q_{xx,1}$ is maximum when $y_1 = 0$, *i.e.* at the neutral axis

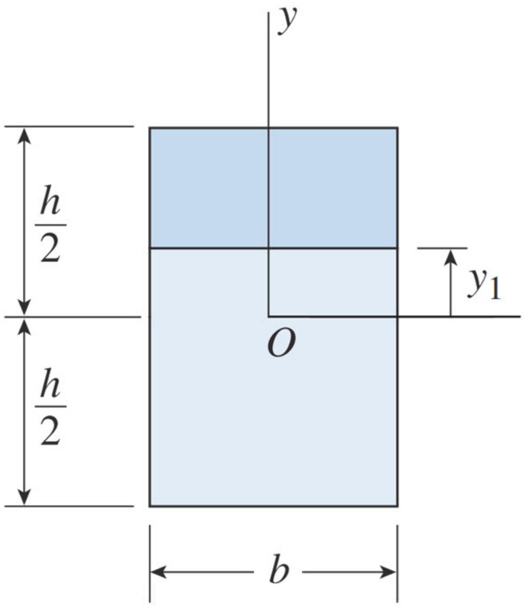


For compound (discrete) sections we have:

$$\tau_1 = \frac{S}{I\, b_1} \sum A_i\, \bar{y}_i$$

Shear Stresses in Beams – Rectangular Cross-Section

- For a solid rectangular cross-section we have:



$$Q_{xx,1} = b\left({}^{h}/_{2} - y_1\right)\left(y_1 \frac{{}^{h}/_{2} - y_1}{2}\right)$$

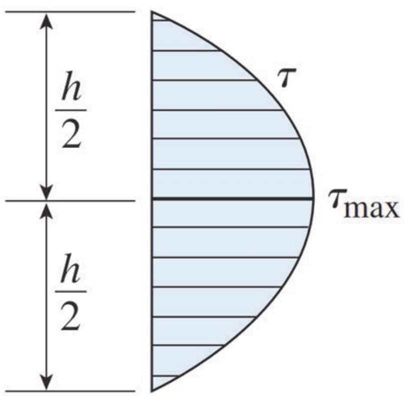
$$Q_{xx,1} = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

Alternatively:

$$Q_{xx,1} = \int_{y_1}^{h} y \, \mathrm{d}A = \int_{y_1}^{h} y \, b \, \mathrm{d}y$$

$$Q_{xx,1} = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

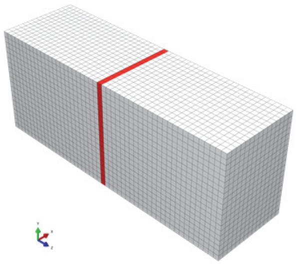
Substituting:



$$\boxed{\tau_1 = \frac{S}{2\,I}\left(\frac{h^2}{4} - y_1^2\right)}$$

Shear Stresses in Beams – Rectangular Cross-Section

- The shear formula above is a good approximation to real shear stress fields:



- However it neglects 3D

effects and Poisson's effect: