

# EMAT10100 Engineering Maths I

## Lecture 19: Max, Min and Graph Sketching

John Hogan & Alan Champneys

## Looking back, looking forward

### last time:

- Functions

$$f : X \rightarrow Y \quad f(x) = y$$

- functions as graphs  $f(x)$
- continuity  $\Rightarrow$  **continuous** functions
- definition of derivative  $\frac{df}{dx}$ ,  $f'(x)$ ,  $\dot{x}(t)$
- derivatives of common functions
- differentiability  $\Rightarrow$  **smooth** functions
- 2nd and higher derivatives

### This time

- Maxima, minima and saddle points
- Tips for graph sketching.

## Interpretation of 2nd derivative

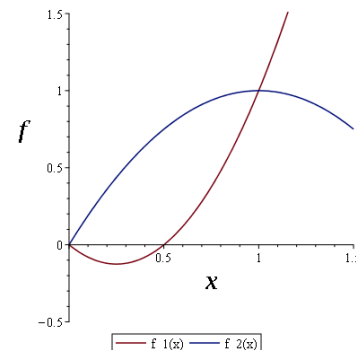
- Example:** compute the second derivatives of the following two functions

$$f_1(x) = 2x^2 - x$$

$$f_2(x) = 2x - x^2$$

nb. for both:

$$f(0) = 0, f(1) = 1$$



- But  $f_1''(x) = 4 > 0$ ,  $f_2''(x) = -2 < 0$

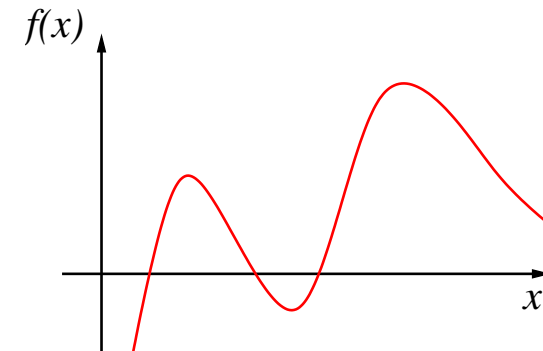
- What's the difference when we plot the functions?

- $f_1''(x) \Rightarrow$  positive curvature  $\Rightarrow$  "steepening"  $\Rightarrow$  **convex**

- $f_2''(x) \Rightarrow$  negative curvature  $\Rightarrow$  "flattening"  $\Rightarrow$  **concave**

## Exercise

- Consider the following graph  $f(x)$

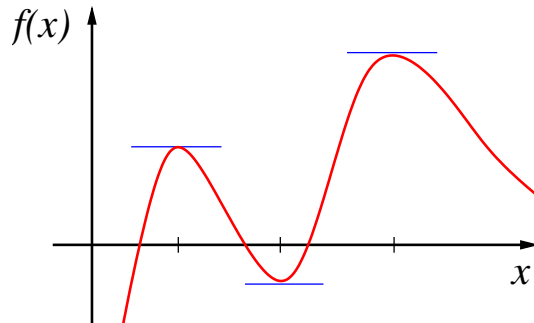


Identify the portions where  $f(x) > 0$ , where  $f'(x) > 0$ , and where  $f''(x) > 0$ ?

- Points where  $f''(x) = 0$  are called **inflection points**

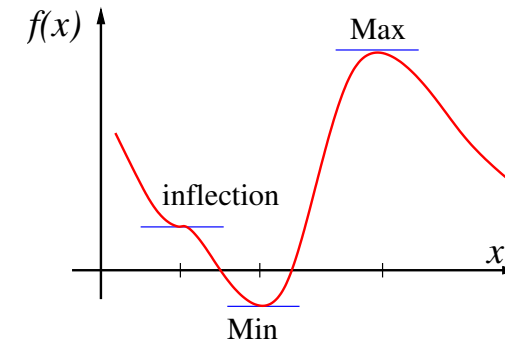
## Stationary points

- Given a smooth function  $f(x)$ , a point  $x$  at which  $f'(x) = 0$  is called a **stationary point**, or **extremum**.
- At such points, the graph is locally flat



## Max, min and inflection points I

- There are three kinds of stationary point: *maximum*, *minimum* & *inflection points*



- Q. How do we decide which is which, without drawing the graph?
- A. Through the 2nd derivative

## Max, min and inflection points II

- At a **maximum** the slope  $f'(x)$  goes from positive to negative, hence

$$\text{Maximum: } f'(x) = 0, \quad f''(x) < 0$$

- At a **minimum** the slope  $f'(x)$  goes from negative to positive, hence

$$\text{Minimum: } f'(x) = 0, \quad f''(x) > 0$$

- At an **inflection (extremum) point**

$$\text{Inflection: } f'(x) = 0, \quad f''(x) = 0$$

## Example

- Exercise:** Consider the function

$$f(x) = -3x^5 + 5x^3$$

- find all the stationary points
- classify each as being a maximum, minimum or inflection point
- hence sketch a graph of the function  $f(x)$  for  $-2 < x < 2$

- Confusion avoidance:** Note the double use of the term **inflection point**

- Inflection point: at which  $f''(x) = 0$
- Inflection extremum point: at which  $f''(x) = 0$  AND  $f'(x) = 0$

- e.g. consider previous example:

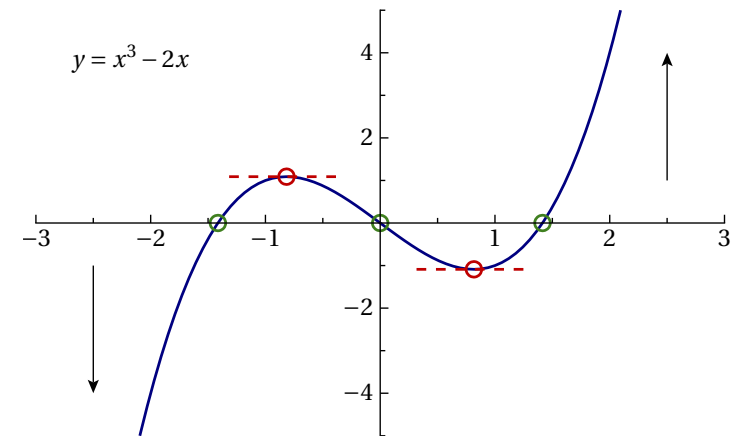
$$f''(x) = -60x^3 + 30x = 0 \Rightarrow x = 0, \quad x = \pm \frac{1}{\sqrt{2}}$$

but only  $x = 0$  has  $f'(x) = 0$  also.

## Graph sketching - some tips

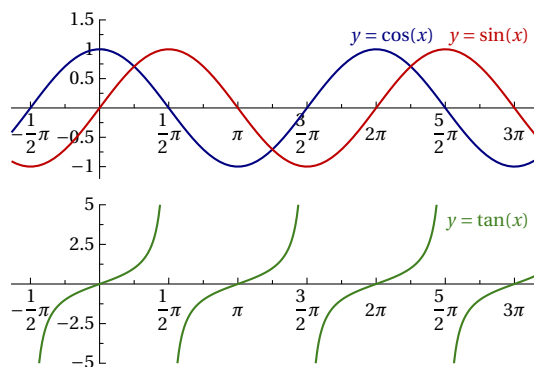
- ✳ graph sketching is **not** the same as accurate plotting
- ✳ its about plotting approximate shape of functions
- ✳ piecing together evidence bit by bit
- ✳ know how to sketch simple functions:  
(see [James](#) Sec.2.3–2.7) also see formula sheet
  - ▶ straight lines  $y = ax + b$
  - ▶ polynomials  $y = ax^2 + bx + c$ ,  $y = ax^3 + bx^2 + cx + d$
  - ▶ rational functions  $y = \frac{ax + b}{cx + d}$
  - ▶ trig functions  $y = \sin x$ ,  $\cos x$ ,  $\tan x$ ,  $\operatorname{cosec} x$ ,  $\sec x$ ,  $\cot x$
  - ▶ exponentials & hyperbolic functions  $y = e^{ax}$ ,  $y = \sinh x$ ,  $\cosh x$ ,  $\tanh x$

## tips for sketching polynomials



- ✳ Find the zeros of the polynomial
- ✳ Find the minima/maxima of the polynomial ( $\frac{dy}{dx} = 0$ )
- ✳ Find what happens as  $x \rightarrow \pm\infty$

## tips for sketching trig functions



✳ **Exercise** plot graphs of the following

1.  $\sin(2x)$ , 2.  $\sin(x^2)$ , 3.  $\sin(1/x)$  (**hard!**)

## Homework

A lot of material this week. To some this will be revision, but if it's unfamiliar, you'll need to catch up in the **reading week** next week (week 8), as well as catching up with the trickier matrix stuff we did last week.

As a **minimum** I suggest all attempt:

1. **2nd derivatives, stationary points etc.**
  - ▶ read [James](#) sec. 8.5
  - ▶ do exercises:
    - ▶ 4th edition 8.5.2 Qns. 72-75
    - ▶ 5th edition 8.5.2 Qns 79 (a),(b); 80 (a)-(c); 81-82
2. **graph sketching**
  - ▶ Skim read [James](#) secs. 2.3–2.7 & practice sketching (crucial skill)
3. Don't forget the **assessed homework** (given out on Monday, also on Blackboard). Due in **noon Tues 22th Nov**. Remember, this doesn't count to final unit mark, but will be marked for you to get some **formative feedback**.