sec6

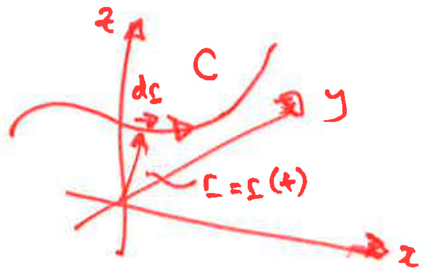
INTRO

So far

- integrate over a line / curve

$$\int_C \varphi \, |d\underline{r}|$$

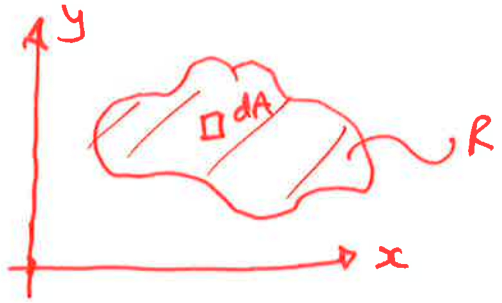
$$\int_C \underline{F} \cdot d\underline{r}$$



change variables $\quad d\underline{r} = \frac{d\underline{r}}{dt} dt$

- integrate over 2D surface (x-y plane)

$$\iint_R \varphi(x,y) \, dA$$

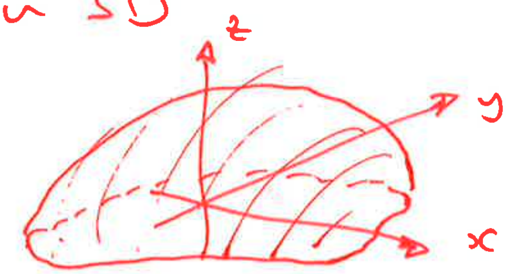


$$dA = dx\,dy = |J| \, du\,dv$$

$$\hookrightarrow |J| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$x = x(u,v)$
$y = y(u,v)$

- Next : integrate over a <u>general</u> 2D surface embedded in 3D
  - scalar fields
  - vector fields.



6. Integration over surfaces

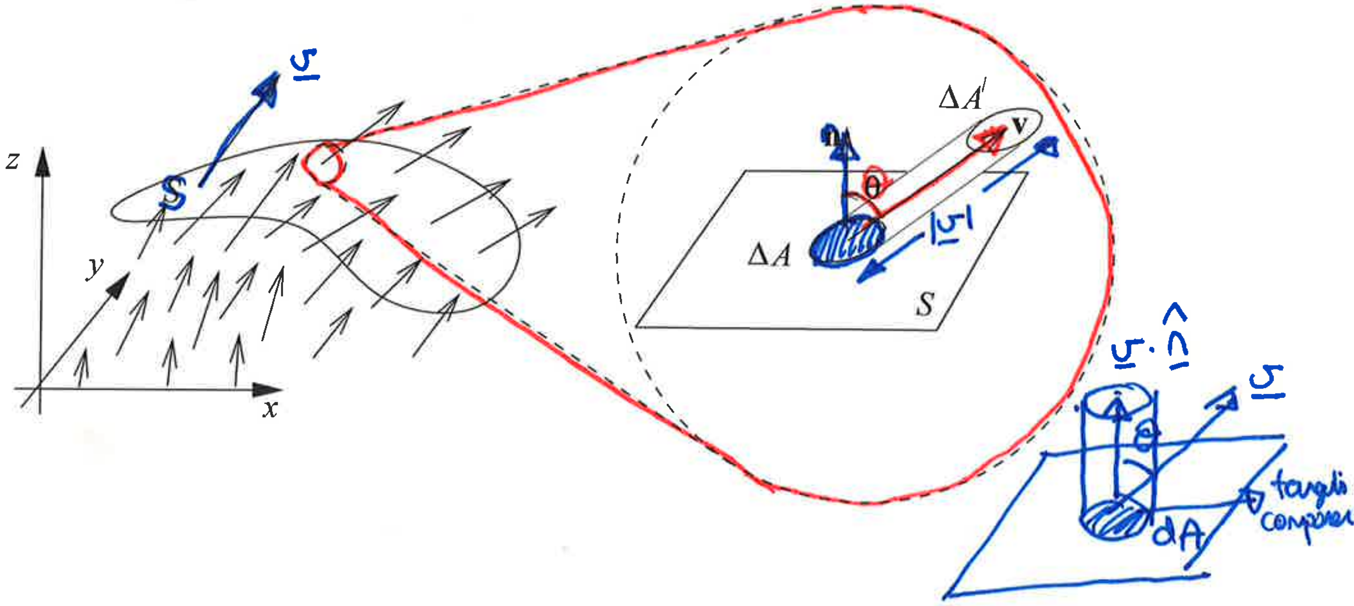
How do we integrate over a surface $S(x, y, z) = 0$? Before we do that we had better find out how to parametrise the surface, using two parameters. What is the notion of area dS? What is a flux integral and what does it represent physically?

6.1 Scalar and vector integrals over surfaces

MOTIVATION: Consider a steady fluid motion (e.g. airflow); what is the flux (volume of fluid per unit time) flowing through a given surface?

E.g. what is the air intake rate into the orifice of a jet engine? How much blood flows through a given membrane etc?

Consider fluid flowing across a surface S with velocity $\boldsymbol{v}$. In unit time a small piece of area ΔA moves to $\Delta A'$



The volume of fluid flowing through ΔA in unit time is given by the volume of the column in the above figure

$$
\begin{aligned}
\text{area} \times \text{flow/unit time} &= \Delta A\, |\boldsymbol{v}| \cos\theta \\
&= \Delta A\, (\boldsymbol{v} \cdot \hat{\boldsymbol{n}}),
\end{aligned}
$$

ΔA times component of flow normal to surface.

where $\hat{\boldsymbol{n}}$ is the unit normal to the surface S at this point. We need the component of the vector field $\boldsymbol{v}(\boldsymbol{r})$ that is **normal to the surface**.

Summing up all the small contributions from the ΔA's across the surface S, the the integral we require is:

Definition *The* **Flux integral** *of a vector field* $\boldsymbol{v}$ *through a surface* S *is*

$$\boxed{\int\int_S \boldsymbol{v} \cdot \boldsymbol{dA} := \int\int_S \boldsymbol{v} \cdot \hat{\boldsymbol{n}} dA}$$ $\hat{n}dA \equiv \underline{dA}$

where dA *is an infinitesimal piece of area of the surface and* $\hat{\boldsymbol{n}}$ *is the unit normal (writing* $\boldsymbol{dA} := \hat{\boldsymbol{n}} dA$*).*

Applications of the Flux Integral

In fluid dynamics, the flow through a surface S is given by

$$\Phi = \int\int_S \boldsymbol{v} \cdot \hat{\boldsymbol{n}} dA,$$

where $\boldsymbol{v}$ is the fluid velocity vector field.

In thermodynamics, the heat flow through a surface S is given by

$$\Phi = \int\int_S \boldsymbol{H} \cdot \hat{\boldsymbol{n}} dA,$$

where $\boldsymbol{H}$ is the heat flow vector field.

In electrostatics, the electric and magnetic fluxes through a surface S are given by

$$\Phi_E = \int\int_S \boldsymbol{E} \cdot \hat{\boldsymbol{n}} dA, \qquad \Phi_B = \int\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} dA,$$

where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field.

But, how do we calculate the flux integral? Need to find $\boldsymbol{dA}$:

Worked example 6.1 *Compute the flux of water through the plane*

$$S : y = x, \quad 0 \leq x \leq 2, \quad 0 \leq z \leq 3,$$

where the velocity vector is

$$\boldsymbol{v} = 3z^2\boldsymbol{i} + 6\boldsymbol{j} + 6xz\boldsymbol{k}. \quad = (3z^2, 6, 6xz)$$

How would you calculate the flux through $S : y = x^2$*?*

We shall return to this example later after learning how to parameterise surfaces S.

Another type of surface integral is the scalar surface integral, which is the integral over a surface of a scalar field f.

Definition *The* **surface integral** *of a scalar field f on a surface S is*

$$\boxed{\int\int_S f dA := \int\int_S f|\boldsymbol{dA}|}$$

$d\underline{A} = \hat{\underline{n}}\, dA$
$|d\underline{A}| = dA \quad (|\hat{\underline{n}}| = 1)$

Examples, include mass, surface area and moments of inertia of plates, shells etc. Where these quantities are defined as for 2D flat surfaces, and 3D volumes in the last Chapter:

$$\text{Surface Area} \quad A = \int\int_S dA,$$

$$\text{Surface Mass} \quad M = \int\int_S \rho(x,y,z)dA,$$

$$\text{Centre of gravity} \quad \bar{x} = \frac{1}{M}\int\int_S x\rho(x,y,z)dA, \quad \text{etc.}$$

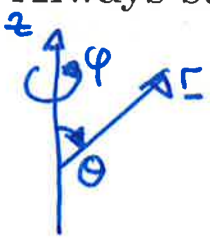
similarly for y & z

$$\text{Moment of intertia} \quad I_z = \int\int_S (x^2+y^2)\rho(x,y,z)dA, \quad \text{etc.}$$

Worked example 6.2 *Find the area and moment of inertia about the z axis of the uniform (constant density ρ_0) spherical shell of mass M and radius a (with co-ordinates $x^2+y^2+z^2 = a^2$).*

In all these examples we need to parameterise the surface S and calculate $\boldsymbol{dA}$ (or $|\boldsymbol{dA}|$) for a general surface S.

Always start by drawing a picture!



find dA on a shell

in spherical polars $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$dA = a^2 \sin\theta\, d\theta\, d\phi$

$$A = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} dA = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} a^2 \sin\theta\, d\theta\, d\phi = a^2 \underbrace{\left[-\cos\theta\right]_{\theta=0}^{\pi}}_{2} \underbrace{\left[\phi\right]_0^{2\pi}}_{2\pi} = \underline{4\pi a^2}$$

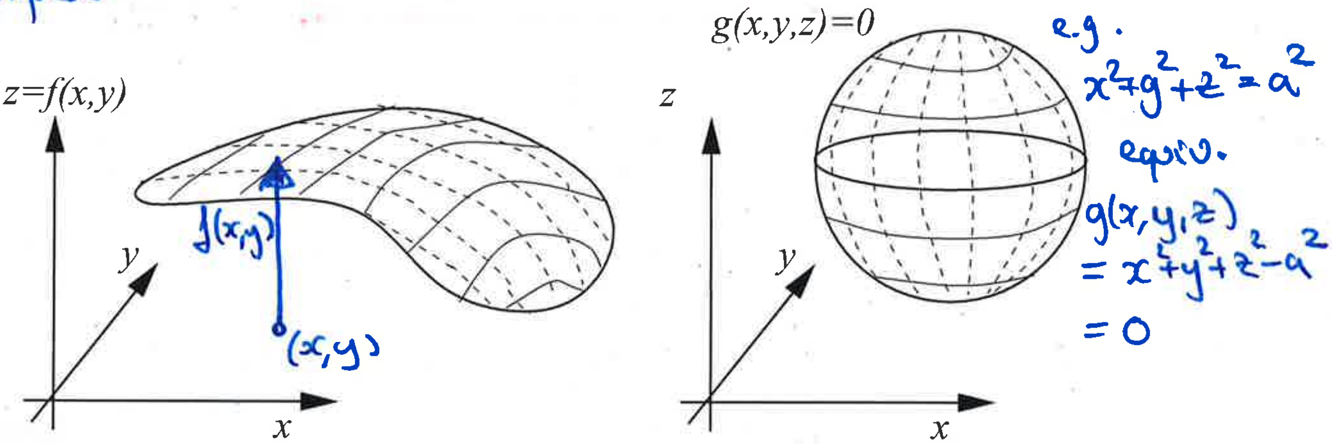
6.2 Parameterisation of surfaces

2D surfaces in our 3D world can be (locally) represented by a single scalar equation

REP^N I – height function REP^N II – constraint eq^n

examples of defining surfaces

$$z = f(x, y) \quad \text{or} \quad g(x, y, z) = 0.$$



But in order to perform vector calculus on surfaces, it is so much easier to have a **parametric representation**. Surfaces are 2-dimensional objects, therefore they need two parameters to describe them

$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}.$$

Compare this situation with that for curves, which are 1-D objects and hence we parameterised them with a single parameter as $\boldsymbol{r}(t)$.

Sometimes the choice of parameters is obvious:

- if the surface can be written as $z = \underbrace{f(x,y)}_{\text{height function}}$ then we can choose $\underline{(u,v) = (x,y)}$ so that $z = f(u,v) = f(x,y)$

$$\boxed{\boldsymbol{r}(u,v) = u\boldsymbol{i} + v\boldsymbol{j} + f(u,v)\boldsymbol{k}.}$$

e.g. a plane $(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = \boldsymbol{0}$ with normal $\boldsymbol{n} = (a,b,c)$

Writing $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ can rearrange as

example for plane

$$f(u,v) = z = z_0 - \frac{a(u - x_0) + b(v - y_0)}{c}$$

height fn rep^n of plane

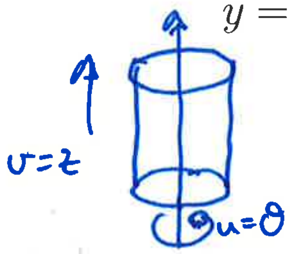
- if one co-ordinate does not appear in $g(x,y,z) = 0$

e.g. $\underbrace{g(x,y) = 0}_{\text{constraining a curve in xy plane}} \Rightarrow$ can write $\underbrace{z = v}_{\text{coord does not appear}}$ and $\underbrace{x = x(u),\ y = y(u)}_{\text{parametric eqn for curve}}$

makes it tractable

For other surfaces, we can use an angular formulation.

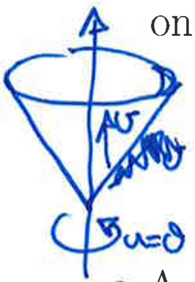
- A cylinder of radius a. Parametrise the circular base as $x = a\cos u$, $y = a\sin u$, and the height z as v. So that



$$\boldsymbol{r}(u,v) = (\underset{\theta}{a\cos u}, \underset{\theta}{a\sin u}, \underset{z}{v})$$

cylindrical polars (θ, z)

- A cone $x^2 + y^2 = z^2$. This is similar, except the radius a depends on the z co-ordinate:



$$\boldsymbol{r}(u,v) = (v\cos \underset{\theta}{u}, v\sin u, \underset{z}{v})$$

cylindrical polars

- A sphere $x^2 + y^2 + z^2 = a^2$. Here we use the *polar* $u = \theta$ and *azimuthal* $v = \phi$ angles (latitude and longitude) with ranges $0 \le \theta < \pi, 0 \le \phi < 2\pi$ so that

spherical polars.

$$\boldsymbol{r}(\underset{\theta}{u}, \underset{\phi}{v}) = (a\sin u\cos v, a\sin u\sin v, a\cos u)$$

Standard expressions in spherical polars.

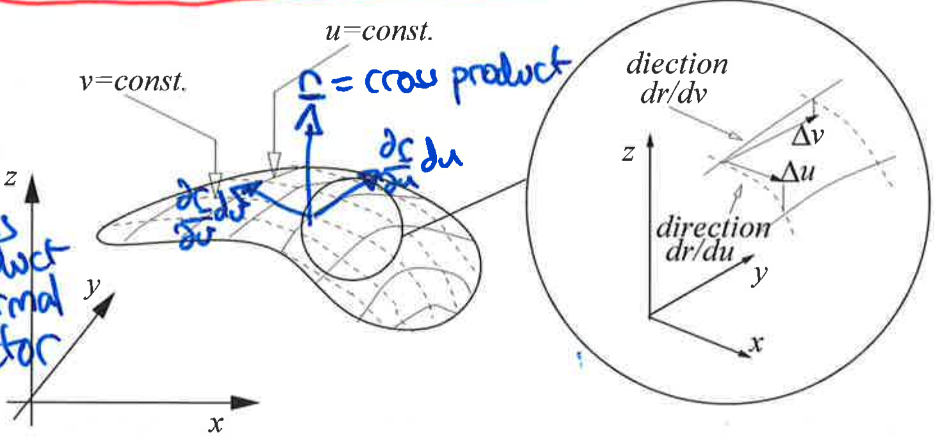
So, how to calculate dA on an arbitrary surface?

Consider a small piece of surface $\boldsymbol{r}(u, v)$, then the infinitesimal area vector is

$$\boxed{\boldsymbol{dA} := \hat{\boldsymbol{n}}\, dA = (\boldsymbol{r}_u \times \boldsymbol{r}_v) dudv = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) dudv.}$$

small changes in u & v give tangent vectors in surface

$$\left.\begin{matrix} \frac{\partial r}{\partial u} du \\ \frac{\partial r}{\partial v} dv \end{matrix}\right\} \text{cross product normal vector}$$



Return to worked example 6.1. *Calculate the infinitesimal area vector* $\boldsymbol{dA}$ *and hence evaluate the flux integral.*
Also calculate the flux integral through the surface $S : y = x^2$.

Worked example 6.3 *What happens if we calculate* $\left(\frac{\partial \boldsymbol{r}}{\partial v} \times \frac{\partial \boldsymbol{r}}{\partial u} dudv\right)$ *instead of* $\left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) dudv$ *in worked example 6.1?*

~~reflects~~ Means that normals can point in or out

This leads us to the notion of the orientation of a surface. We may choose a parameterisation so that the normal vector points inwards or outwards. A surface is said to be **orientable** if a label (e.g. outwards) can be assigned to a normal direction at a point, and that this labeling can be continued in a unique and continuous way throughout the entire surface.

Q.) Can you think of an example of a non-orientable surface?

MORAL: When evaluating flux integrals, it is important to check whether the normal vector we compute points in the correct direction to make physical sense for the problem at hand. Otherwise we get precisely the negative of the true answer.

6.3 **Evaluation of surface integrals**

Considering how we solved worked example 6.1, we have the following method for **evaluating flux integrals**

$$\int\int_S \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{dA} = \int\int_S \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{n} dA$$

1. Draw a sketch
2. Express the surface S as $\boldsymbol{r}(u, v)$ with two parameters u and v.
3. Calculate the limits a, b and c, d on the parameters u and v respectively
4. Calculate the area vector $\boldsymbol{dA} = (\boldsymbol{r}_u \times \boldsymbol{r}_v) dudv = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} dudv$ and check that this normal vector points in the right direction.
5. form the dot product and evaluate the integral as a double integral

$$\int_c^d \int_a^b \boldsymbol{F}(\boldsymbol{r}(u, v)) \cdot \underbrace{\overset{\partial r/\partial u \times \partial r/\partial v}{(\boldsymbol{r}_u \times \boldsymbol{r}_v)}}_{dA} dudv$$

(note that the limits a, b may depend on v).

vector field as function of u, v

substitute $\underline{F}(u,v) = (x(u,v), y(u,v), z(u,v))$

Worked example 6.4 *Find the ~~(volume)~~ flux of the vector field*

$$\boldsymbol{v} = \frac{1}{16}(x^2 - y^2)\boldsymbol{i} + \frac{xy}{8}\boldsymbol{j} + x\boldsymbol{k} = \left(\frac{1}{16}(x^2-y^2), \frac{xy}{8}, x\right)$$

outwards through, (i) the curved surface S_1*, (ii) the top surface* S_2 *of a cylinder of radius* 4*. Cylinder axis along z-axis, limits* $0 \leq z \leq 1$.

To **evaluate scalar surface integrals**,

$$\int\int_S f(\boldsymbol{r})dA = \int\int_S f(\boldsymbol{r})|\boldsymbol{dA}|$$

we need to compute a similar parameterisation of the surface

1. Draw a sketch
2. Express the surface S as $\boldsymbol{r}(u, v)$ with two parameters u and v.
3. Calculate the limits a, b and c, d on the parameters u and v respectively
4. Calculate the magnitude of the area element vector
$dA = |\boldsymbol{dA}| = |\boldsymbol{r}_u \times \boldsymbol{r}_v| dudv = \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| dudv$
5. Evaluate the integral as a double integral

$$\int_c^d \int_a^b f(\boldsymbol{r}(u, v))\overbrace{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}^{\partial r/\partial u \times \partial r/\partial v} dudv$$

(note that the limits a, b may depend on v).

Scalar field as function of u, v

substitute $r(u,v) = (x(u,v), y(u,v), z(u,v))$

Return to worked example 6.2 *Find the surface area and moment of inertia about the z-axis of the uniform spherical shell of mass M and radius a (with co-ordinates* $x^2 + y^2 + z^2 = a^2$*).*

There are plenty more examples of both kinds of surface integrals in Example sheet 7.

REMARK Some textbooks suggest other ways of evaluating surface integrals, for example by projecting onto the (x, y)-plane. The approach we adopt here is the most general, and is described in Kreyszig Section 9.5.

Summary

- Two kinds of surface integral:

$$\int\int_S \boldsymbol{F}(r) \cdot \hat{\boldsymbol{n}} \quad dA, \quad \text{and} \quad \int\int_S f(r) dA.$$

- Evaluate by parametrising the surface using two parameters u, v.

Extra: Proof of area element formula (non-examinable)

Result: For a surface $\boldsymbol{r}(u,v)$, the infinitesimal area vector is

$$\boldsymbol{dA}(u,v) := \hat{\boldsymbol{n}}(u,v)\,dA = (\boldsymbol{r}_u \times \boldsymbol{r}_v)dudv = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) dudv.$$

Proof: Two vectors in the first-order approximation (tangent space) to the surface S at a point parametrised by (u_0, v_0) are

$$\left.\frac{\partial \boldsymbol{r}}{\partial u}\right|_{u_0,v_0} \Delta u \quad \text{and} \quad \left.\frac{\partial \boldsymbol{r}}{\partial v}\right|_{u_0,v_0} \Delta v\,,$$

where Δu and Δv are small displacements in the u and v directions.

To see this recall from Taylor's theorem that to first order:

$$\begin{aligned}\boldsymbol{r}(u_0+\Delta u, v_0) &= \boldsymbol{r}(u_0,v_0) + \left(\frac{\partial u}{\partial u}, \frac{\partial u}{\partial u}, \frac{\partial u}{\partial u}\right)\Delta u + O\left(\Delta u^2\right),\\ &= \boldsymbol{r}(u_0,v_0) + \frac{\partial \boldsymbol{r}}{\partial u}\Delta u + O\left(\Delta u^2\right),\end{aligned}$$

and similarly for a small change in the Δv in the v-direction.

Now, the area ΔA on the tangent space approximation to S is the area of the parallelogram spanned by the two vectors $\frac{\partial \boldsymbol{r}}{\partial u}\Delta u$ and $\frac{\partial \boldsymbol{r}}{\partial v}\Delta v$.

Also the unit normal vector to S is perpendicular to both of them. Recall the physical definition of the cross-product; $\boldsymbol{a}\times\boldsymbol{b}$. Its magnitude is precisely the area of the parallelogram defined by the two vectors. Also, its direction is the direction $\hat{\boldsymbol{n}}$ normal to the two vectors.

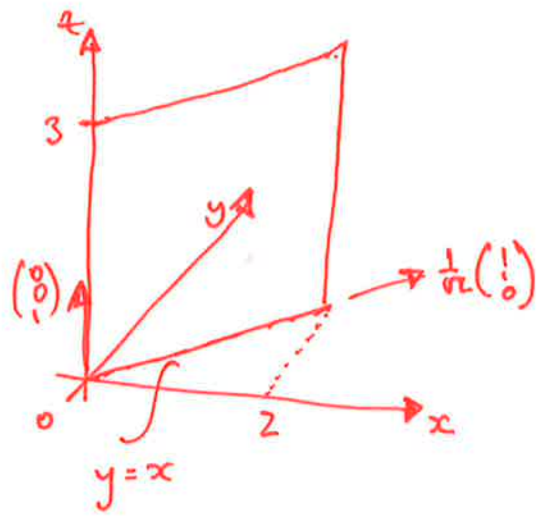
In other words

$$\left|\left(\frac{\partial \boldsymbol{r}}{\partial u}\Delta u\right) \times \left(\frac{\partial \boldsymbol{r}}{\partial v}\Delta v\right)\right| := |\boldsymbol{r}_u \times \boldsymbol{r}_v|\Delta u \Delta v$$

gives the magnitude of ΔA, is by definition normal to both $\boldsymbol{r}_u\Delta u$ and $\boldsymbol{r}_v\Delta v$ and hence normal to S to leading order.

Letting $\Delta u \to du$, $\Delta v \to dv$ and $\Delta A \to dA$ gives result. $\square$

Example 6.1



$$\underline{v} = (3z^2, 6, 6xz)$$

$$\text{Flux} = \iint_S \underline{v}\cdot\hat{\underline{n}}\; dA$$

find surface normal:

2 vectors in surface: $\frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\therefore\ \hat{\underline{n}} = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k}\\ 1 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

parameterize surface: $\begin{cases} x = y = u/\sqrt2 \\ z = v \end{cases}$

will assume $dA = du\,dv$ [will justify this later in section]

region A: $0 < u < 2\sqrt2$
$0 < v < 3$

Then

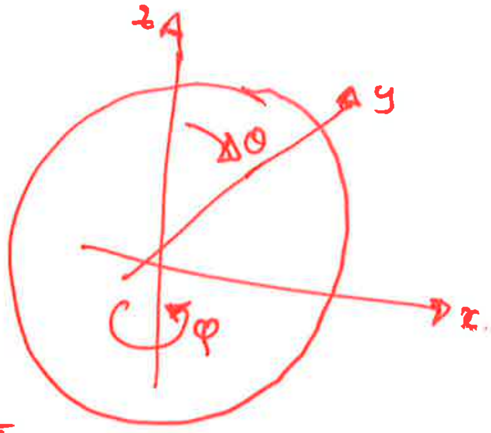
$$\text{Flux} = \int_{v=0}^{3}\int_{u=0}^{2\sqrt2} \left(3v^2, 6, 6uv/\sqrt2\right)\cdot\left(\tfrac{1}{\sqrt2}, \tfrac{-1}{\sqrt2}, 0\right) du\,dv$$

$$= \int_{v=0}^{3}\int_{u=0}^{2\sqrt2}\left(\frac{3v^2}{\sqrt2} - \frac{6}{\sqrt2}\right) du\,dv$$

$$= \Big[u\Big]_0^{2\sqrt2}\left[\frac{v^3}{\sqrt2} - \frac{6v}{\sqrt2}\right]_0^3$$

$$= 2\cancel{\sqrt2}\times\left(\frac{27}{\cancel{\sqrt2}} - \frac{18}{\cancel{\sqrt2}}\right) = \underline{\underline{18}}$$

Example 6.2



know $dA = a^2 \sin\theta \, d\theta \, d\varphi$

[section 5 of notes]

surface : $0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$

$$A = \iint_S dA = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} a^2 \sin\theta \, d\theta \, d\varphi$$

$$= a^2 \left[-\cos\theta\right]_0^{\pi} \left[\varphi\right]_0^{2\pi} = \underline{\underline{4\pi a^2}}$$

$x = a\sin\theta\cos\varphi$
$y = a\sin\theta\sin\varphi$
$z = a\cos\theta$

$$I_z = \rho_0 \iint_S (x^2+y^2)\, dA$$

$$= \rho_0 \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(a^2\sin^2\theta\cos^2\varphi + a^2\sin^2\theta\sin^2\varphi\right) a^2 \sin\theta \, d\theta \, d\varphi$$

$\sin^2\varphi + \cos^2\varphi = 1$

$$= \rho_0 \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} a^4 \sin^3\theta \, d\theta \, d\varphi$$

$\hookrightarrow \sin\theta - \sin\theta\cos^2\theta$

$$= \rho_0 a^4 \left[\varphi\right]_0^{2\pi} \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \underline{\underline{\frac{8}{3}\pi\rho_0 a^4}}$$

$M = 4\pi a^2 \rho_0 \qquad \therefore \quad I_z = \frac{\overset{2}{\cancel{8}}\cancel{\pi}\rho_0 a^{\overset{2}{\cancel{4}}}}{3} \times \frac{M}{\cancel{4}\cancel{\pi}\cancel{a^2}} = \underline{\underline{\frac{2a^2M}{3}}}$

Example 6.1 (RETURN)

remember parameterisation of plane

$$x = y = u/\sqrt{2}, \quad z = v$$

ie $\underline{r}(u,v) = (u/\sqrt{2}, u/\sqrt{2}, v)$

$$d\underline{A} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\, du\, dv$$

$$= (1/\sqrt{2}, 1/\sqrt{2}, 0) \times (0, 0, 1)\, du\, dv$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} du\, dv$$

$$= (1/\sqrt{2}, -1/\sqrt{2}, 0)\, du\, dv.$$

but this is exactly what we used before!

ie $dA = du\, dv$

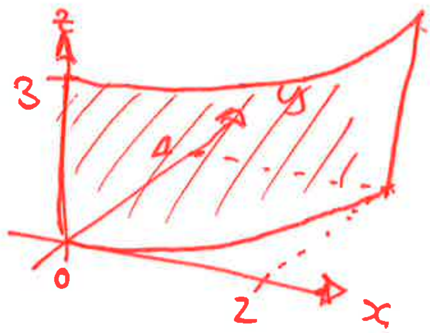
$$\underline{\hat{n}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Same answer!

EXAMPLE 6.1 (RETURN – PART II)

try for parabolic cylinder

$S: y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$



(1) parameterize surface

try $x = u$, $y = x^2 = u^2$, $z = v$

$\therefore \underline{r}(u,v) = (u, u^2, v)$

(2) limits of integration

$0 \le x \le 2 \Rightarrow 0 \le u \le 2$
$0 \le z \le 3 \Rightarrow 0 \le v \le 3$

(3) flux integral $\iint_S \underline{v} \cdot d\underline{A} = \int_{v=0}^{3} \int_{u=0}^{2} \underline{v}(u,v) \cdot d\underline{A}$

$$d\underline{A} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\, du\, dv$$

$$= (1, 2u, 0) \times (0,0,1)\, du\, dv$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} du\, dv = (2u, -1, 0)\, du\, dv$$

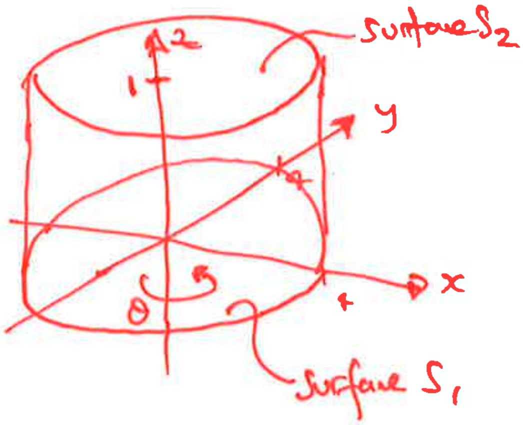
$$\underline{v}(u,v) = (3z^2, 6, 6xz) = (3v^2, 6, 6uv)$$

$$\therefore \iint_S \underline{v} \cdot d\underline{A} = \int_{v=0}^{3} \int_{u=0}^{2} (6uv^2 - 6)\, du\, dv$$

$$= \int_{v=0}^{3} \left[3u^2v^2 - 6u\right]_{u=0}^{2} dv = \int_{v=0}^{3} (12v^2 - 12)\, dv = \left[4v^3 - 12v\right]_0^3$$

$$= 72$$

Example 6.4



(1) <u>parameterize</u> S_1 :

cylindrical polars $\quad x = 4\cos\theta$
$\quad\quad\quad\quad\quad\quad\quad\; y = 4\sin\theta$

so $\underline{r}(\theta, z) = (4\cos\theta, 4\sin\theta, z)$

<u>limits</u> : $0 \le \theta \le 2\pi, \quad 0 \le z \le 1$

<u>integral</u> : $\iint_{S_1} \underline{v} \cdot d\underline{A} = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \underline{v}(\theta, z) \cdot d\underline{A}$

$$\underline{v}(\theta, z) = \left(\frac{1}{16}(x^2 - y^2), \frac{xy}{8}, x \right)$$
$$= (\cos^2\theta - \sin^2\theta, 2\sin\theta\cos\theta, 4\cos\theta)$$

$$d\underline{A} = \frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial z} \, d\theta \, dz$$
$$= (-4\sin\theta, 4\cos\theta, 0) \times (0, 0, 1) \, d\theta \, dz$$
$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4\sin\theta & 4\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} d\theta \, dz$$
$$= (4\cos\theta, 4\sin\theta, 0) \, d\theta \, dz$$

$$\therefore \iint_{S_1} \underline{v} \cdot d\underline{A} = \int_{z=0}^{1} \int_{\theta=0}^{2\pi} \left\{ 4\cos\theta(\cos^2\theta - \sin^2\theta) + 8\sin^2\theta\cos\theta \right\} d\theta \, dz$$
$$= \left[z \right]_0^1 \int_{\theta=0}^{2\pi} 4\cos\theta \, \underbrace{(\sin^2\theta + \cos^2\theta)}_{1} \, d\theta$$
$$= \underline{\underline{0}}$$

(Disappointingly after hard work
note if $\theta \in [0, \pi/2]$ then non-zero)

EXAMPLES 6.4

(ii) parameterize S_2 : $x = r\cos\theta$, $y = r\sin\theta$, $z = 1$

so $\underline{r} = (r\cos\theta, r\sin\theta, 1)$

limits : $0 \le r \le 4$, $0 \le \theta \le 2\pi$

integral : $\iint_{S_2} \underline{v} \cdot d\underline{A} = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} \underline{v}(r,\theta) \cdot d\underline{A}$

$$\underline{v}(r,\theta) = \left(\tfrac{1}{16}(x^2-y^2), \tfrac{xy}{8}, x\right)$$

$$= \left(\tfrac{r^2}{16}(\cos^2\theta - \sin^2\theta), \tfrac{r^2}{8}\cos\theta\sin\theta, r\cos\theta\right)$$

$$d\underline{A} = \frac{\partial \underline{r}}{\partial r} \times \frac{\partial \underline{r}}{\partial \theta}\, dr\, d\theta$$

$$= (\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0)$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\, d\theta$$

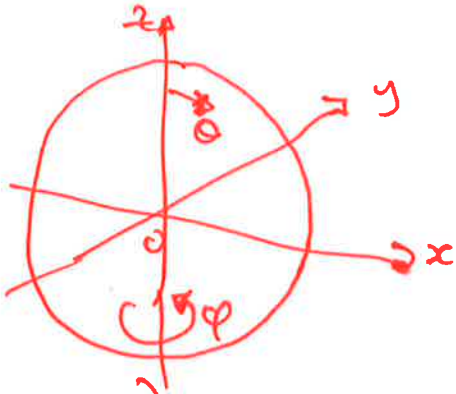
$$= (0, 0, r)\, dr\, d\theta$$

$$\therefore \iint_{S_2} \underline{v} \cdot d\underline{A} = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} r^2 \cos\theta\, dr\, d\theta$$

$$= \left[\frac{r^3}{3}\right]_0^4 \left[\sin\theta\right]_0^{2\pi} = 0$$

[again disappointingly – but if had been $[0, \pi/2]$ then nonzero]

Example 6.2 (Return)



parameterise S : $x = a\sin\theta\cos\varphi$

$y = a\sin\theta\sin\varphi$

$z = a\cos\theta$

$$\underline{r}(\theta,\varphi) = (a\sin\theta\cos\varphi,\ a\sin\theta\sin\varphi,\ a\cos\theta)$$

limits : $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$

integral :

$$d\underline{A} = \frac{\partial \underline{r}}{\partial\theta} \times \frac{\partial \underline{r}}{\partial\varphi}\, d\theta\, d\varphi$$

$$= (a\cos\theta\cos\varphi,\ a\cos\theta\sin\varphi,\ -a\sin\theta) \times (-a\sin\theta\sin\varphi,\ a\sin\theta\cos\varphi,\ 0)\, d\theta d\varphi$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a\cos\theta\cos\varphi & a\cos\theta\sin\varphi & -a\sin\theta \\ -a\sin\theta\sin\varphi & a\sin\theta\cos\varphi & 0 \end{vmatrix} d\theta d\varphi$$

$$= (a^2\sin^2\theta\cos\varphi,\ a^2\sin^2\theta\sin\varphi,\ a\sin\theta\cos\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1})\, d\theta d\varphi$$

$$\therefore |d\underline{A}| = a^2\sqrt{(\sin^4\theta\cos^2\varphi + \sin^4\theta\sin^2\varphi + \sin^2\theta\cos^2\theta)}\, d\theta d\varphi$$

$$= a^2\sqrt{\sin^4\theta + \sin^2\theta\cos^2\theta}\, d\theta d\varphi$$

$$= a^2\sin\theta\, d\theta d\varphi \qquad \left(\text{using } \sin^4\theta + \sin^2\theta\cos^2\theta = \sin^2\theta\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}\right)$$

But this is the differential of area we used before $\Rightarrow$ same answer....