StM3 – Composite Laminate Analysis

Revision : Classical Laminate Theory

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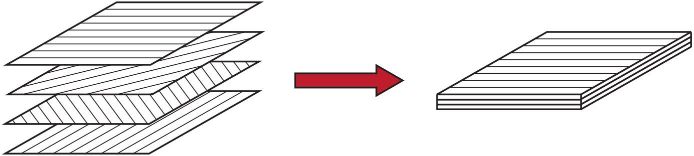
Composite Laminate

Composite Laminate Analysis

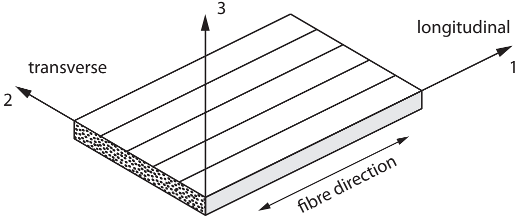


structural hierarchy

- fibres and matrix combine into uni-directional ply
- multiply plies form a composite laminate plate

Specially Orthotropic Material Model – I

composite ply is *not* isotropic: mechanical properties differ along the direction of the fibres and perpendicular to the fibres



sign convention: 123 refers to the natural axes of the material, and *xyz* refers to structural axes

Specially Orthotropic Material Model – II

compliance matrix $\boldsymbol{S}$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} \tag{1.5}$$

with

$$\begin{aligned} S_{11} &= \frac{1}{E_{11}} & S_{22} &= \frac{1}{E_{22}} \\ S_{12} &= -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} & S_{66} &= \frac{1}{G_{12}} \end{aligned} \tag{1.6}$$

Specially Orthotropic Material Model – III

reduced stiffness matrix $\boldsymbol{Q}$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \tag{1.7}$$

with

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{1-\nu_{12}\nu_{21}} & Q_{22} &= \frac{E_{22}}{1-\nu_{12}\nu_{21}} \\ Q_{12} &= \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} & Q_{66} &= G_{12} \end{aligned} \tag{1.8}$$

Specially Orthotropic Material Model – IV

Voigt Notation: $i, j \in 1, 2, 6$

$$\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}$$

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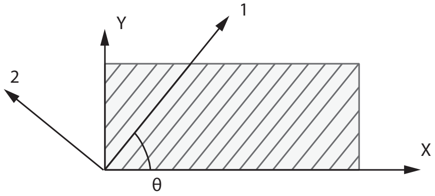
homogenised elastic components: E_{11}, E_{22}, G_{12} and ν_{12}

reciprocal theorem: $\nu_{21}/E_{22} = \nu_{12}/E_{11}$

Generally Orthotropic Material – I

material axes (123) not aligned with the structural axes (xyz)



UD inclined to structural axes: *generally orthotropic material*

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

$$\bar{\boldsymbol{Q}} = \boldsymbol{T}^{-1}\boldsymbol{QRTR}^{-1}$$

Generally Orthotropic Material – II

with components $\bar{Q}_{ij}$

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

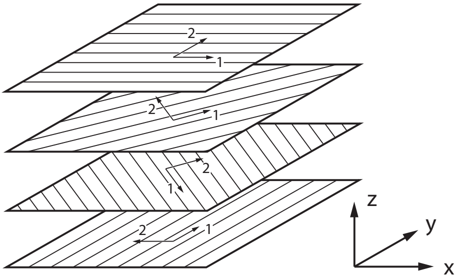
$$\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\bar{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta$$

$$\bar{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta$$

Composite Laminate

composite laminate:
combine multiple composite layers into single structural element

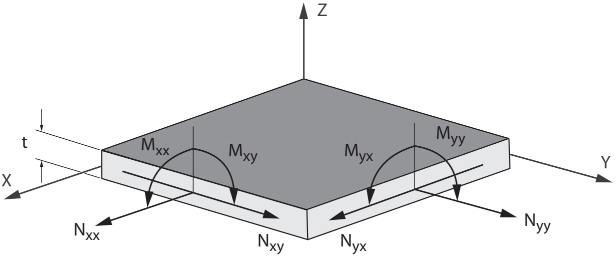


lay-up notation:

- ply numbering bottom-up in positive z-direction
- angle θ defines orientation of ply material axes (CCW)

Composite Plate Model – I

plate model: in-plane and out-of-plane loads and deformations



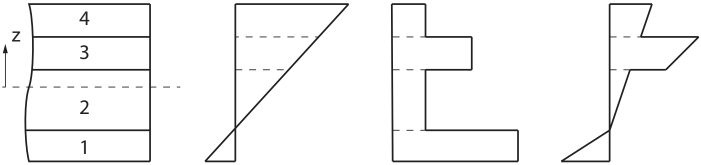
in-plane (N_{xx}, N_{yy}, N_{xy}) and out-of-plane (M_{xx}, M_{yy}, M_{xy}) loads

Composite Plate Model – II

resulting midplane strains $\boldsymbol{\varepsilon}^0$ and out-of-plane curvatures $\boldsymbol{\kappa}$

strain linear along cross-section, but stress discontinuous:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$



strain distribution

lamina stiffness

stress distribution

ABD-matrix – I

mechanics of composite laminate is described by ABD-matrix

$$\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]$$

$$\left[\begin{array}{c} \boldsymbol{N} \\ \boldsymbol{M} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{array}\right] \tag{3.2}$$

ABD-matrix – II

A is the **extensional stiffness matrix**

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right) \tag{3.3}$$

B is the **coupling stiffness matrix**

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \tag{3.4}$$

D is the **bending stiffness matrix**

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \tag{3.5}$$

ABD-matrix – III

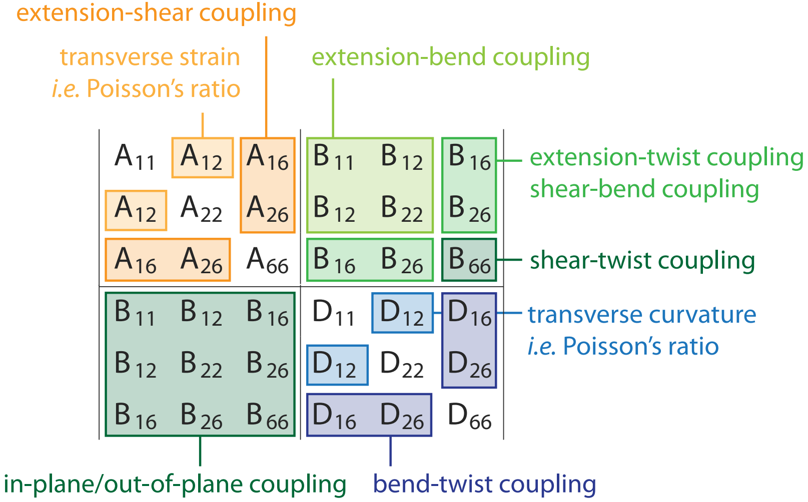
laminate with n plies and total thickness h

ply numbering $k = 1 \ldots n$ in positive z-direction

location h_k of top of each ply measured from geometric mid-plane;

ply thickness: $t_k = h_k - h_{k-1}$

ABD-matrix – IV



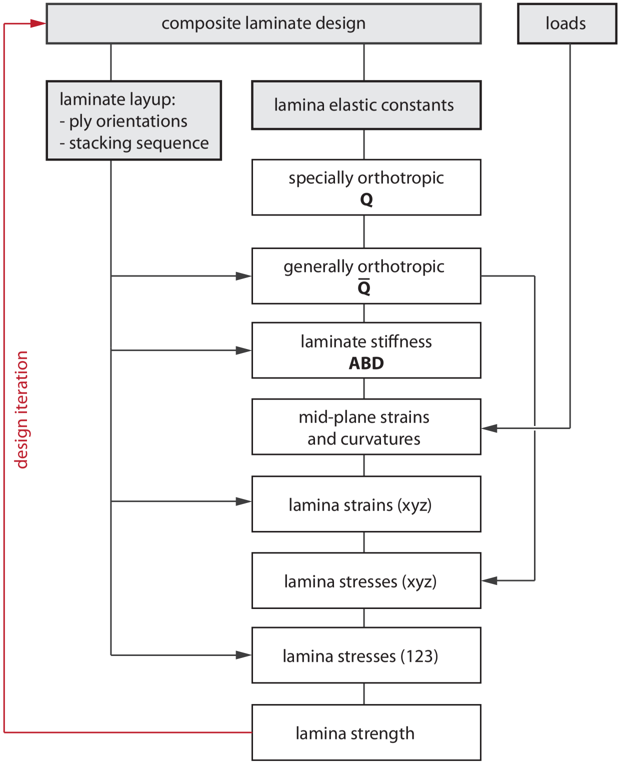
Special Laminate Types

balanced laminate: $A_{16} = A_{26} = 0$

symmetric laminate: $B_{ij} = 0$

quasi-isotropic laminate: same in-plane properties in all directions

anti-symmetric laminate: $D_{16} = D_{26} = 0$



composite laminate design
loads
laminate layup:
- ply orientations
- stacking sequence
lamina elastic constants
specially orthotropic
Q
generally orthotropic
Q̄
laminate stiffness
ABD
mid-plane strains
and curvatures
lamina strains (xyz)
lamina stresses (xyz)
lamina stresses (123)
lamina strength
design iteration

Thermal Effects

thermal effects on composite reponse (*e.g.* warping after cure)

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \begin{bmatrix} \varepsilon^0 \\ \boldsymbol{\kappa} \end{bmatrix} - \begin{bmatrix} \boldsymbol{N}^T \\ \boldsymbol{M}^T \end{bmatrix}$$

thermal loads and moments:

$$\left[\boldsymbol{N}^T\right] = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (\alpha)_k (h_k - h_{k-1}) \Delta T$$

$$\left[\boldsymbol{M}^T\right] = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (\alpha)_k \left(h_k^2 - h_{k-1}^2\right) \Delta T$$

with CTEs: α_{xx}, α_{yy}, α_{xy} and change in temperature ΔT