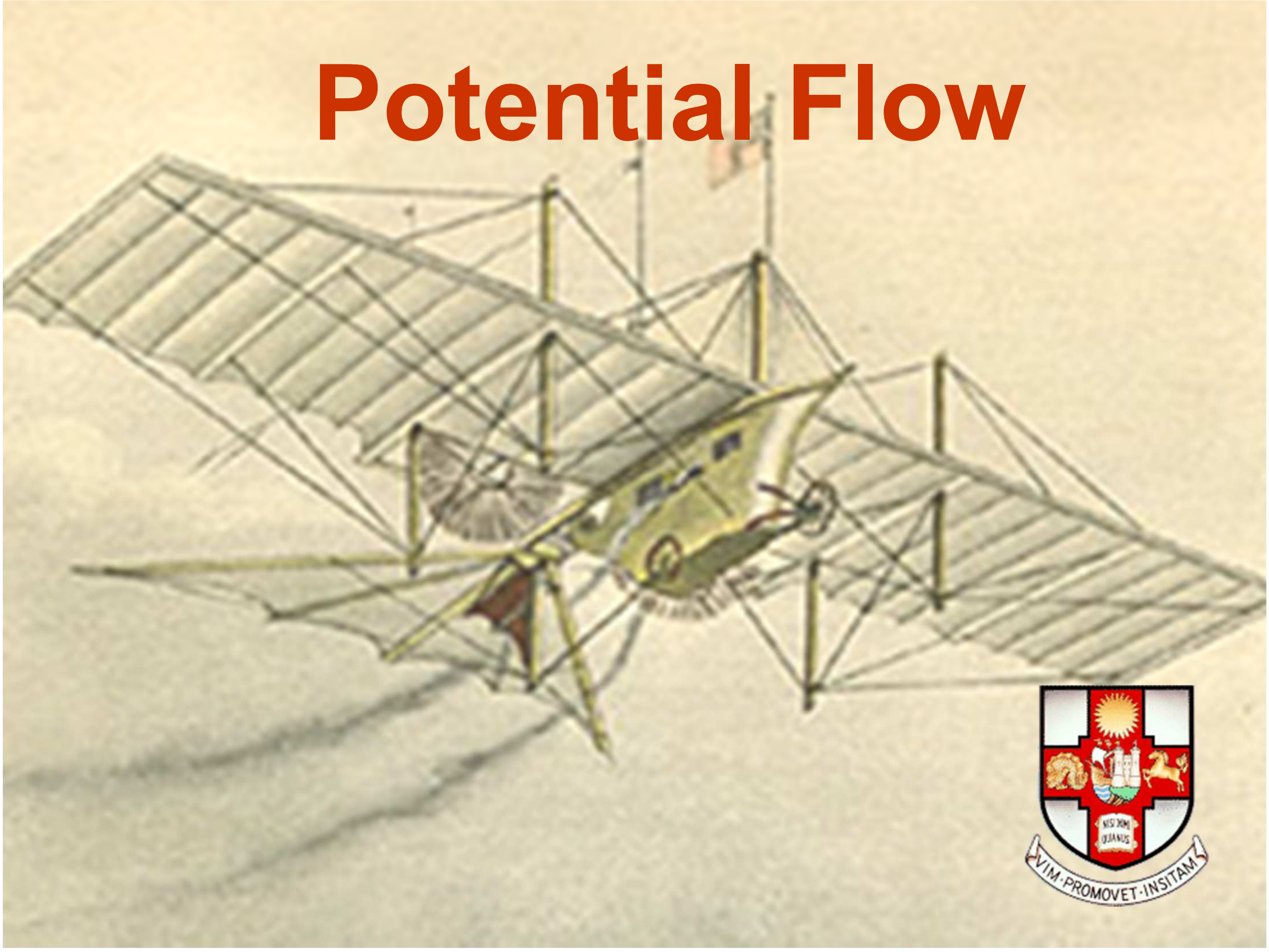
Potential Flow



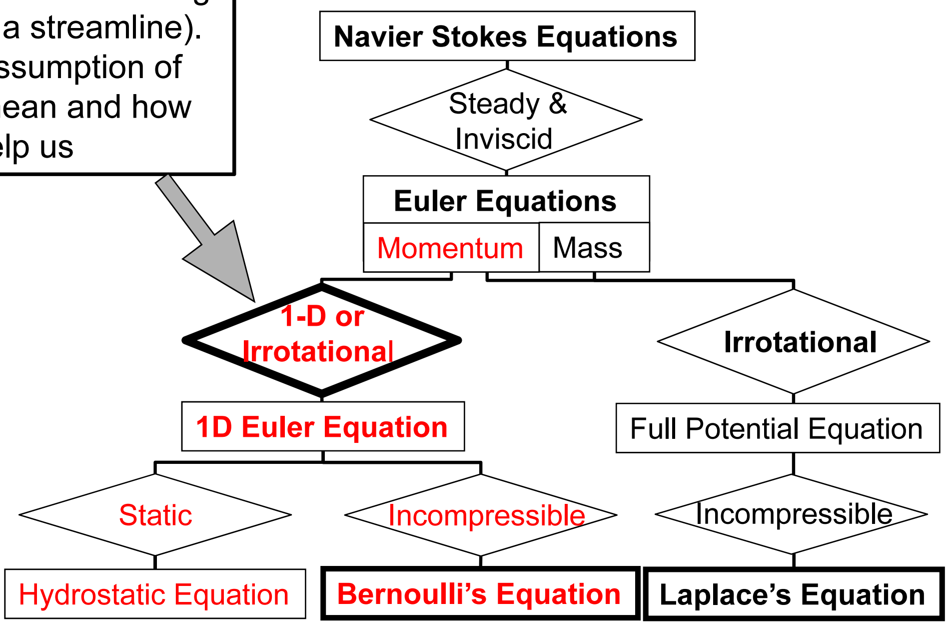
From Navier-Stokes to Laplace's Equation

Aims for this lecture

- To explain the assumptions which need to be made to simplify the 3 momentum equations of the Navier-Stokes equations to a 1D equation
- Assuming an irrotational flow, to show that a velocity potential can be defined.
- To show that if a velocity potential can be defined then the potential satisfies Laplace's equation
- To introduce and give key facts about the stream function.
- Use example questions to illustrate different methods to find stream lines.

From Navier-Stokes to Laplace & Bernoulli's equations

We have looked at the derivation for Bernoulli's equation assuming 1-D flow (or along a streamline). What does the assumption of irrotational flow mean and how does it help us

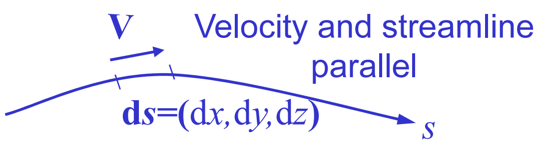


From Navier-Stokes to Bernoulli's equation: The assumptions

1. **Viscous effects are negligible** - Valid for many real flows where viscous effects are confined to a narrow band near the body surface.
2. **Steady Flow**
3. **No Body forces**. - Such as gravity
4. **a) Flow along a streamline**

$$\underline{\mathbf{ds}} \times \underline{\mathbf{V}} = \underline{\mathbf{0}}$$

$$(dx, dy, dz) \times (u, v, w) = (0,0,0)$$



4. **b) Irrotational Flow -** The vorticity (or rotational velocity) is defined as zero everywhere in the flow field for irrotational flow.

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u, v, w) = (0,0,0)$$

5. **Incompressible Flow**
6. **Adiabatic Flow**

This gives Bernoulli's equation without the hydrostatic term (as we have ignored body forces) valid for

i) any two points along a streamline **OR**

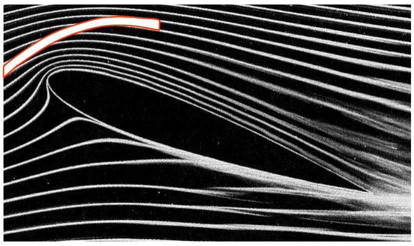
ii) two points anywhere in an irrotational flow

$$\boxed{p + \frac{1}{2}\rho V^2 = \text{constant}}$$

Why isn't Bernoulli's Equation Enough

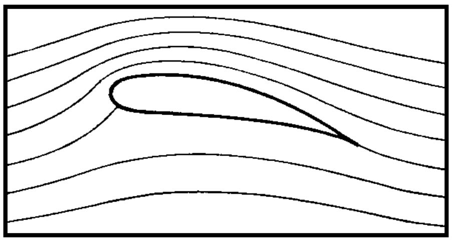
- Consider applying Bernoullis equation along each streamline for steady inviscid incompressible flow
- Even if we know the initial pressure and velocity along each streamline, Bernoulli's equation does not give us the streamline position so that we can integrate along it.

Unknown streamline path so cannot apply Bernoulli's.Eqn



- If we make the further assumption of irrotational flow then we still need the pressure throughout the domain to give us the velocity field or the velocity field to give us the pressure distribution.

e.g. Pressure at any point can be calculated only if the velocity is known at that point and pressure & velocity are known at an arbitrary point



Laplace's Equation

- Kelvin's Theorem: "in the absence of viscous forces and discontinuities the flow will remain irrotational"
- Irrotational flow defined by

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(u,v,w)=\left(\left[\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right],\left[\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right],\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]\right)=(0,0,0)$$

- Discontinuities come from free surfaces (multi-phase flows) or shock waves (only present in compressible flows). Inviscid incompressible flows are irrotational.
- Irrotational flow is a good approximation other than at the surface and in the wake for flows with lift
- For incompressible flow the mass conservation equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Density cancels, so this is equivalent to conservation of fluid volume

- This is combined with the definition of irrotational flow to give us our flow equation

Laplace's Equation 2 :(Velocity Potential)

- **irrotationality** guarantees the existence of a scalar 'velocity potential' function ϕ.
- ie $u = \frac{\partial \phi}{\partial x}, \ v = \frac{\partial \phi}{\partial y}, \ w = \frac{\partial \phi}{\partial z}$ if the flow is irrotational.
- We have changed the problem, now need to find ϕ instead of u, v & w
- name 'potential' is significant
  - direct analogue to 'potential' in electrodynamics
  - flow only occurs where there is a difference (gradient) in potential
- only defined for irrotational flows
  - therefore such flows usually referred to as 'potential flows'
  - exists in 3D We will only consider 2D for simplicity
  - exists in unsteady and compressible flows

    Laplace's equation replaced by more complex equations though still used in engineering analysis

Laplace's Equation (3)

- for **incompressible** and irrotational flow consider continuity applied to a velocity field defined by a velocity potential

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Laplace equation (1789)

- **linear** 2nd order partial differential equation
  - well-understood equation – over 2 centuries of study!
  - 1 differential equation to solve for 3 velocity components
- solutions can be superimposed
  - flow models built up from 'elementary' flow solutions

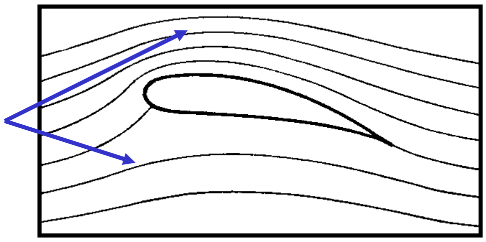
If $\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0$ and $\frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0$ then $\frac{\partial^2 (\phi_1 + \phi_2)}{\partial x^2} + \frac{\partial^2 (\phi_1 + \phi_2)}{\partial y^2} + \frac{\partial^2 (\phi_1 + \phi_2)}{\partial z^2} = 0$

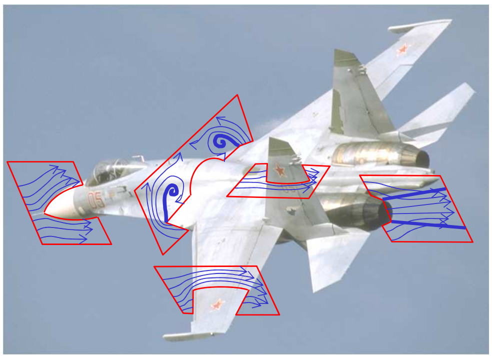
- boundary conditions required

Usually solid boundaries define as streamlines & far field undisturbed

How to Solve Potential Flows

- Solve Laplace's equation to find the potential in the domain
- Differentiate to find the velocity at all points
- Use Bernoulli to find the pressure at all points given the pressure and velocity at a single point (far field conditions)
- Same process in 2D & 3D, just 2D this year





Why Start in 2D

- Clearer demonstration of fundamentals
- Applicable to many practical 3D flows
- Mechanism for lift generation
- Stream function available in 2D only

Introduction of Stream Function

- In a 2D incompressible flow a function $\psi(x,y)$ exists such that on streamlines:

$$\psi(x,y) = c$$

where c is an arbitrary constant. This function is called the stream function (Note this equation is an alternative integrated form of the 2D equation for a streamline).

- The stream function is related to flow velocities via

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

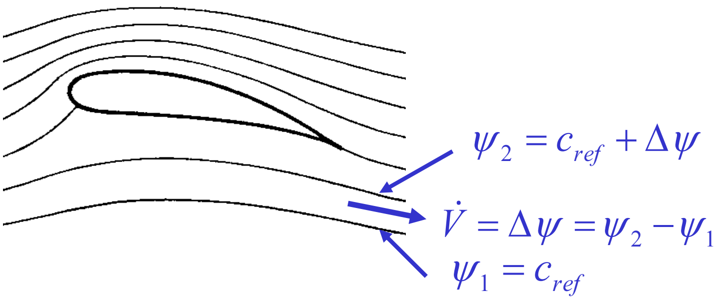
Two equations for a streamline are: $dy/dx = v/u$ and $\psi(x,y) = c$.

- Physical interpretation of the stream function is that the increment in the stream function between two streamlines in the flow corresponds to the volume flow rate between the lines

$$\boxed{\Delta\psi = \text{Volume flow rate}}$$

or since density is constant

$$\boxed{\rho\,\Delta\psi = \text{Mass flow rate}}$$



Example, find stream function from velocity

If the fluid velocity components are given by $u = -\omega\, y, \quad v = \omega\, x$

then find an expression for the stream function and hence an equation for the stream lines.

$$u = \partial\psi/\partial y = -\omega y \qquad \text{Integrating w.r.t. } y \Rightarrow \psi = -\frac{1}{2}\omega y^2 + f(x)$$

$$\text{Differentiate w.r.t. } x \Rightarrow \partial\psi/\partial x = df(x)/dx = -\omega x$$

$$\text{Integrating w.r.t. } x \Rightarrow \quad f(x) = -\frac{1}{2}\omega x^2 + const \;\; \text{(can ignore const)}$$

$$\psi = -\frac{1}{2}\omega(y^2 + x^2)$$

$$\text{Streamlines}: \;\; \psi = const = -\frac{1}{2}\omega(y^2 + x^2) \;\Rightarrow\; y^2 + x^2 = const$$

$$\text{Alternatively Streamlines } \frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y}$$

$$ydy + xdx = 0$$

$$\text{Integrating} \Rightarrow y^2 + x^2 = const$$

BUT this doesn't give us the stream function as requested.

Relationship of Stream function and Potential

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}$$

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}$$

Potential and stream function are clearly related. In fact they are ORTHOGONAL functions i.e. are perpendicular to each other where they cross (except at stagnation points where $\mathbf{V} = \mathbf{0}$).

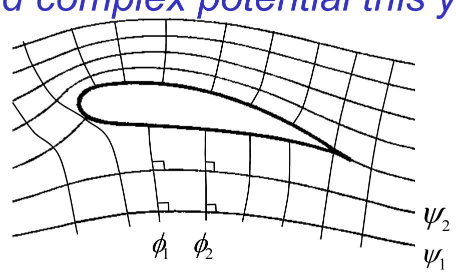
Both satisfy Laplace's equation so can have superposition of solutions

$$\nabla^2\psi = \nabla^2\phi = 0$$

Can be combined as a COMPLEX POTENTIAL $\quad W(Z) = \phi + i\psi$

Do not need complex potential this year

Lines cross perpendicularly. Note it is often easier to sketch the stream lines of a flow rather than equi-potentials because of the tangential relationship to the velocity vector.



Learning Outcomes: "What you should have learnt so far"

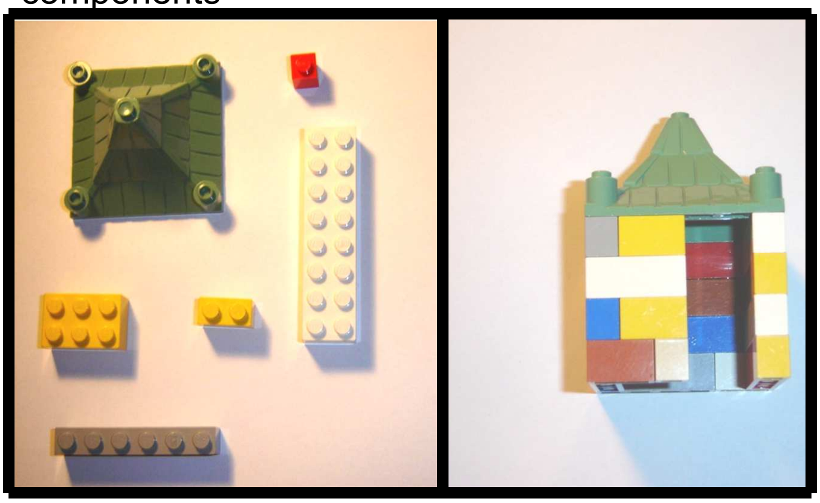
- State the assumptions needed to reduce the 3 momentum equations of the Euler equations to a 1D equation (note two cases).
- Understand that a velocity potential may only be defined for an irrotational flow which satisfies Laplace's equation. This means only 1 equation needs to be solved for the three components of velocity.
- Be aware that because Laplace's equation is a linear equation so solutions can be found by superposition of other solutions.
- Give a physical interpretation of the stream function, know that it is constant on a stream line
- Find the stream function and stream lines from a velocity distribution.

Interested students should take a look at

http://www.grc.nasa.gov/WWW/K-12/airplane/foil3.html

Elementary 2D Potential Solutions

Start with simple components $\xrightarrow{\text{combine}}$ Build up more complex models



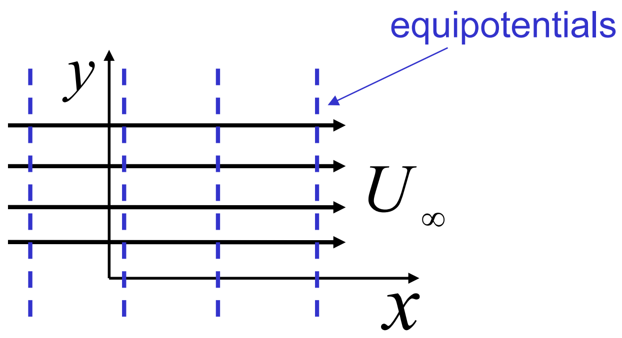
Similarly for solutions of Laplace's equation, because it is LINEAR

1. Uniform flow
2. Point source / sink
3. Doublet
4. 2D Point vortex

$\xrightarrow{\text{combine}}$ More complex solutions, to model real flows

Elementary Flow 1- Uniform Horizontal Flow

Sketch of the streamlines of the flow



Velocities

$$u = U_\infty\,, \quad v = 0$$

Potential function

$$\phi = U_\infty x + \text{constant}$$

Stream function

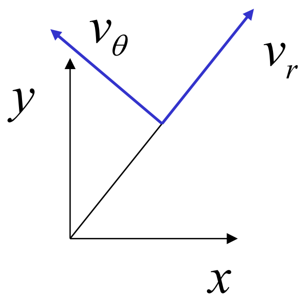
$$\psi = U_\infty y + \text{constant}$$

Before the next lecture, prove to yourself how the potential and stream functions are obtained

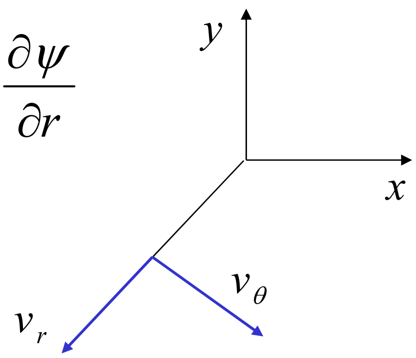
Short Aside: Velocity field in 2D polar coordinates

It is sometimes easier for the other elementary flows to work in (r, θ) coordinates.

$$v_r = \frac{\partial \phi}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$



$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$



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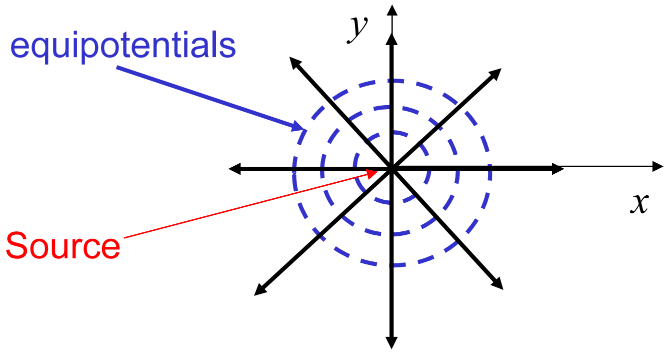
Elementary Flow 2- Point Source/Sink Flow

Sketch of the streamlines of the flow



Velocities

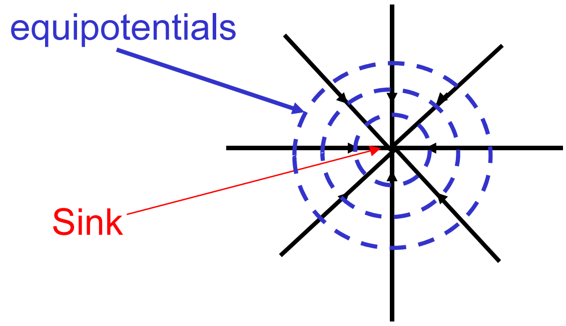
$$v_r = \frac{\Lambda}{2\pi r} \qquad v_\theta = 0$$

Potential function

$$\phi = \frac{\Lambda}{2\pi} \ln r + \text{constant}$$

Stream function

$$\psi = \frac{\Lambda}{2\pi} \theta + \text{constant}$$



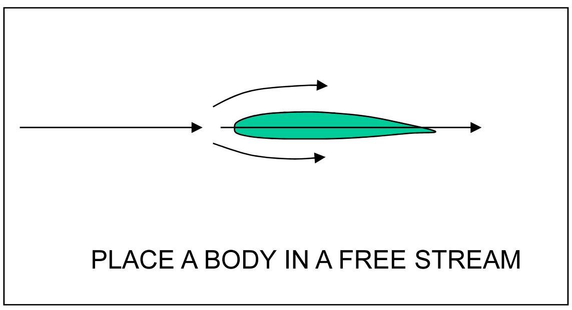
Λ, the source/sink strength is the volume flow rate (per unit depth) out of the source, or into the sink

- The radial velocity is singular at the origin since

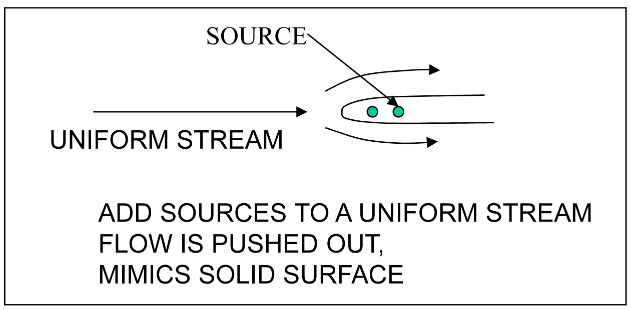
$$v_r \to \infty \qquad \text{as} \qquad r \to 0$$

- and source flow satisfies Laplace's equation and has a potential function except at the *singularity* at the origin.
- Source flow is a mathematical abstraction and singularities can't exist in the flow solution domain since real flow velocities can't tend to infinity.
- However the behaviour of *theoretical model* away from the source mimics the effect of solid objects in a real flow so it is in fact a useful tool.

REAL FLOW



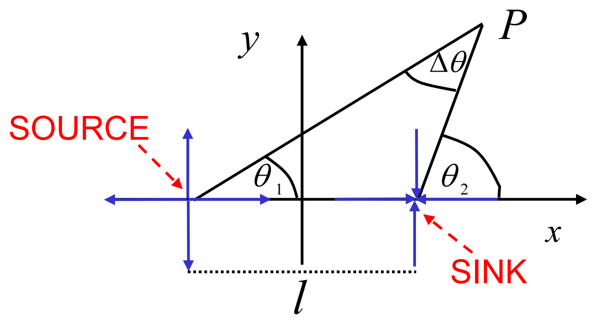
POTENTIAL FLOW MODEL

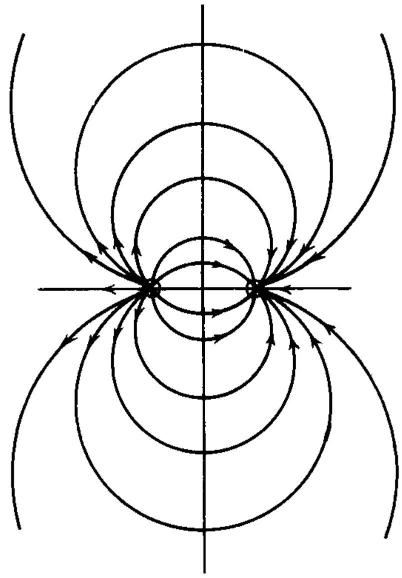


We can use sources and sinks to model just about any inviscid flow, as long as there is no lift.
In this case we need to use other singularities (vorticies or doublets) to provide circulation

Superposition of a Source & Sink

- Consider a source of strength Λ and a sink of equal, but opposite strength separated by distance l





- The stream function at any point P subtending an angle $\Delta\theta$ from the pair is

$$\psi = \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2 = \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{\Lambda}{2\pi}\Delta\theta$$

- Some trigonometry shows the streamlines correspond to a series of circles, that pass through the source and sink

Elementary Flow 3- Doublet

- Now to obtain a doublet move the source and sink closer together, i.e. let $l \to 0$
- but at the same time increase the source/sink strength, so that

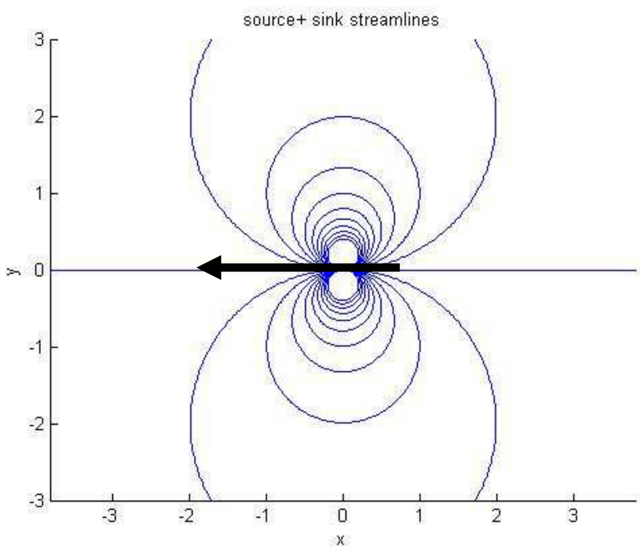
$$\Lambda l = \text{constant} = \kappa$$

- In the limit the source and sink are superimposed in a <u>doublet</u> of finite strength κ with stream function

$$\psi_{doublet} = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

- The potential for a doublet is

$$\phi_{doublet} = +\frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$

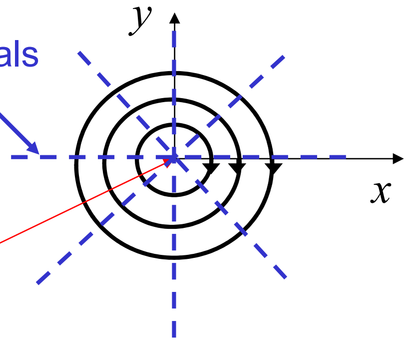


- The doublet has a direction, or axis, associated with it. The positive direction is given by an arrow drawn from the sink to the source

(right to left in this case).

Elementary Flow 4- Point Vortex Flow

Sketch of the streamlines of the flow



Velocities $v_r = 0 \quad v_\theta = -\dfrac{\Gamma_V}{2\pi r}$

Potential function

$$\phi = -\frac{\Gamma_V}{2\pi}\theta + \text{constant}$$

Stream function

$$\psi = +\frac{\Gamma_V}{2\pi}\ln r + \text{constant}$$

The tangential velocity is singular at the origin since

$$v_\theta \to \infty \qquad \text{as} \qquad r \to 0$$

Point vortex flow satisfies Laplace's equation and has a potential function except at the *singularity* at the origin.

In a point vortex flow, fluid moves around circular paths and is irrotational except at the singularity or vortex.

So how is it possible that a flow which is apparently rotating is irrotational?

- Recall mathematical definition of irrotationality

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(u,v,w) = (0,0,0)$$

- Vorticity at a point is equivalent to twice the angular velocity of a fluid element there. In 2D this means

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\times(u,v) = 2(0,0,\omega)$$

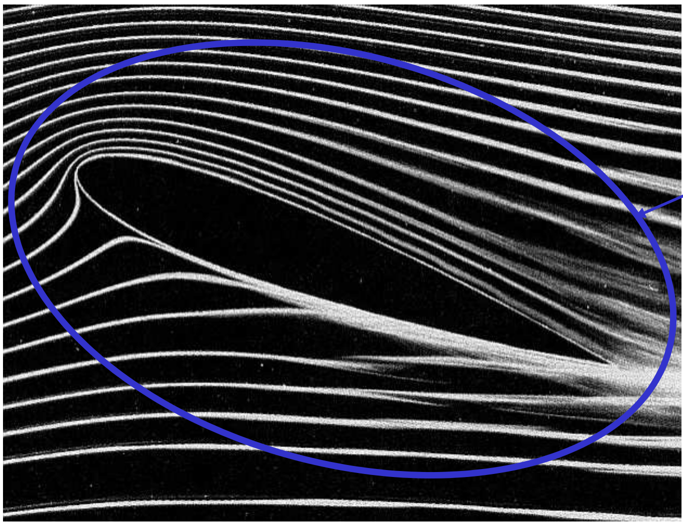
- So in an irrotational vortex flow fluid, elements can move in a circular path, but do not rotate

- To investigate the significance of the point vortex strength need to introduce the concept of flow **circulation**.

We will not need circulation in calculations this year. However it is an important concept in understanding lift, needed in later years.

Circulation – Common Misconception

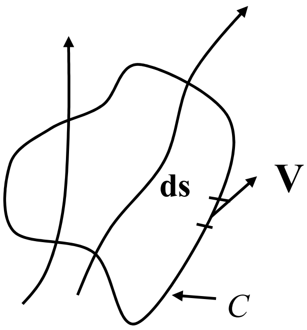
- Everyday usage -circulation means to move in a circle or circuit.
- In aerodynamics if there is non-zero circulation then the fluid elements are *not* necessarily moving along a circular path.
- Example flow past an aerofoil-fluid elements don't move in circles, but circulation calculated around contour enclosing aerofoil is non-zero



Circulation around closed path non zero

What is Circulation?

- Circulation is defined in terms of the integral of the velocity component <u>along</u> the closed curve c



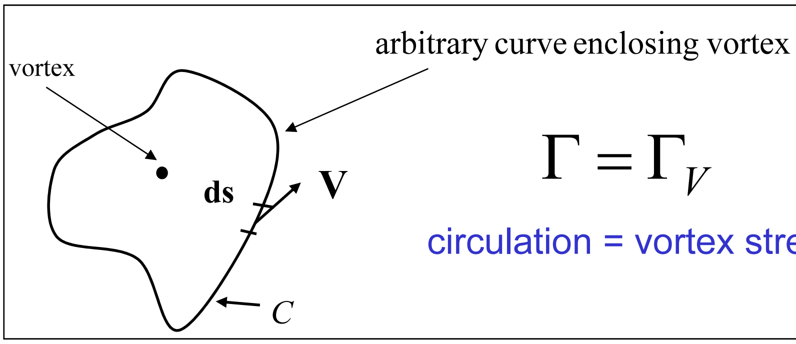
$$\Gamma = -\oint_c \mathbf{V}\cdot\mathbf{ds} = \underbrace{-\oiint_S (\nabla\times\mathbf{V})\cdot\mathbf{dS}}_{\text{from Stokes' theorem}}$$

- Positive sense of line integrals - <u>anti-clockwise</u>
- Aeronautical convention - circulation Γ is positive in the <u>clockwise</u> direction, used in this course

For background

- If the vorticity is zero everywhere (irrotational flow) within the closed contour C, then circulation Γ =0
- Note same nomenclature for vorticity & circulation, we will see why shortly

Point Vortex Strength & Circulation



$$\Gamma = \Gamma_V$$

circulation = vortex strength.

Q: Why is there circulation in an irrotational potential flow?

A: because of the presence of vortices (or a doublet gradient but we not explore this idea this year)

For a “2D point vortex flow” at the origin, vorticity is infinite (but zero everywhere else!). This infinite vorticity over an infinitely small point gives a finite circulation. So:

The circulation is finite as long as the contour of integration encloses point vortices and is equal to the sum of vorticity
Enclosed sources and sinks do not create circulation (and therefore lift)

Learning Outcomes: “What you should have learnt so far”

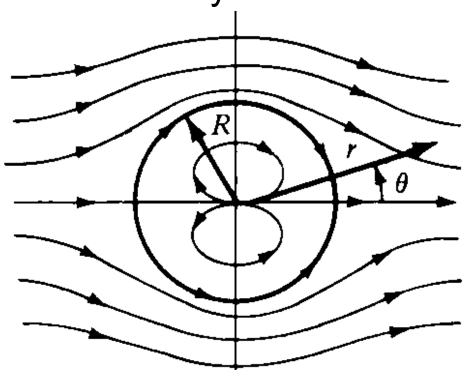
- Describe the 4 basic elementary flows
- Explain the basic concept of circulation and state that circulation around a curve is zero if the flow is irrotational everywhere within the curve.
- Explain that there can be circulation in an irrotational flow if there are singularities present.

Aims for this lecture

- To explain how more complex 2D potential solutions can be found
- Singularities are defined as if they were placed at the origin. Introduce the principles of applying singularities away from the origin
- Begin first example question

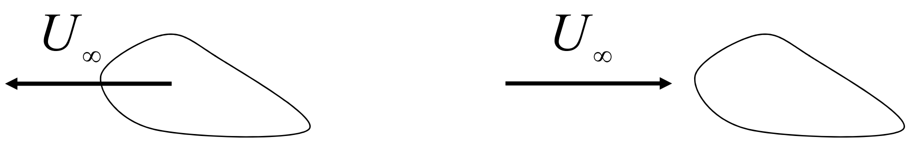
GENERAL COMMENTS ON POTENTIAL MODELS

- No flow can cross a stream line in the potential model. Mimics the property of a solid boundary, so any stream line can be regarded as a solid boundary.



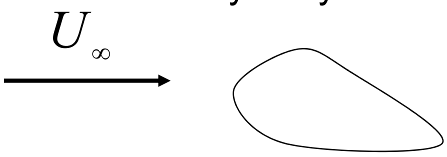
- No singularities in a real flow, so any singularities in potential model must be located outside the domain modelling the flow (Sources Sinks Doublets and Vortices are placed inside solid bodies or outside solid walls for internal flows)

- A typical problem is to model flow about a body travelling at a uniform speed into still air. This is equivalent to the flow about a body in a uniform stream (using the Galilean transformation)



BOUNDARY CONDITIONS

For the external flow over a stationary body the flow boundary conditions are



(1) Far away from the body (towards infinity) the flow velocity approaches uniform stream conditions

(2) There must be no flow perpendicular to the solid surface i.e. surface of the body must be a stream line of the flow. So the same equation solved over different boundary shapes leads to different solutions.

Note: (1) is satisfied if sources/sinks and point vortices are added to a uniform flow, as the velocity of these basic elements tends to zero far away.

In satisfying (2) there are a number possibilities but these all use the same idea. Adding each source/doublet/vortex to a uniform free stream adds parameters of **position** and **strength** that can be adjusted to produce the required streamline that is taken as the body shape.
Numerical applications distribute the singularities in a pre-described way and adjust the strengths to produce the required streamlines.

Web Site Link

- There are a number of programs available on the web that allow you to look at combining the elementary flows, which plot out streamlines.
- The following example was checked Oct 2009, but others could be found by searching for potential flow programs

www.aoe.vt.edu/~devenpor/aoe5104/ifm/ifm.html

a) Try an onset flow of 1, 4 sources of strength 1, 4 sources of strength -0.5 and 8 of strength -0.25

b) Try an onset flow of 1 and a doublet of strength -8

c) Add a vortex of strength -4 directly on top of the doublet

Change of Origin For Elementary Potential Flow Solutions

Consider an elementary potential flow solution (source, doublet, vortex) where the centre of the element is at the Cartesian coordinate origin (0,0). The horizontal and vertical components of velocity at any point (x,y) can be written in the general form (see hand out)

$$u = \Theta\, f(x,y), \qquad v = \Theta\, g(x,y)$$

Where Θ is the element strength (i.e. Λ, Γ or κ) and f, g are functions of the position (x,y). As an example consider a source of strength Λ

$$u = \frac{+\Lambda}{2\pi}\frac{x}{\left(x^2+y^2\right)}, \quad v = \frac{+\Lambda}{2\pi}\frac{y}{\left(x^2+y^2\right)} \quad \rightarrow \begin{cases} \Theta = \Lambda, & f = \dfrac{x}{2\pi\left(x^2+y^2\right)} \\ & g = \dfrac{y}{2\pi\left(x^2+y^2\right)} \end{cases}$$

However if the position of the elementary solution is moved to $(\hat{x},\hat{y})$ then the velocities at (x,y) can now be written as

$$u = \Theta\, f\left((x-\hat{x}),(y-\hat{y})\right), \qquad v = \Theta\, g\left((x-\hat{x}),(y-\hat{y})\right)$$

Or for the source previously defined

$$u = \frac{+\Lambda}{2\pi}\frac{(x-\hat{x})}{\left((x-\hat{x})^2+(y-\hat{y})^2\right)}, \quad v = \frac{+\Lambda}{2\pi}\frac{(y-\hat{y})}{\left((x-\hat{x})^2+(y-\hat{y})^2\right)}$$

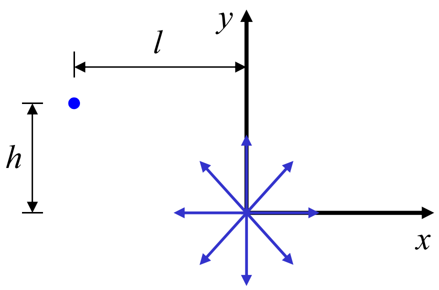
Change of Origin (2)

The choice of the origin location has no effect on the solution. Consider the evaluation of the horizontal component of velocity at a point a fixed relative distance from a source. Evaluating the velocity with three different origins

i) Origin at the source

$$x = -l, \; y = h, \; \hat{x} = 0, \; \hat{y} = 0$$

$$u = \frac{-\Lambda}{2\pi}\frac{l}{\left(l^2+h^2\right)}$$



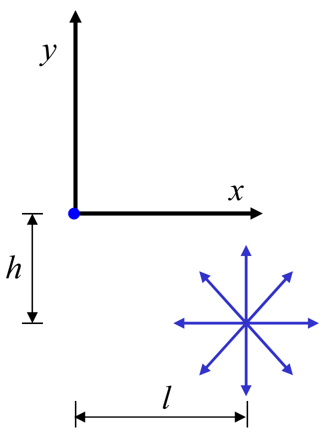
ii) Origin at the evaluation point

$$x = 0, \; y = 0, \; \hat{x} = l, \; \hat{y} = -h$$

$$u = \frac{+\Lambda}{2\pi}\frac{(x-\hat{x})}{\left((x-\hat{x})^2+(y-\hat{y})^2\right)}$$

$$u = \frac{+\Lambda}{2\pi}\frac{(0-l)}{\left((0-l)^2+(0+h)^2\right)}$$

$$u = \frac{-\Lambda}{2\pi}\frac{l}{\left(l^2+h^2\right)}$$



Change of Origin (3)

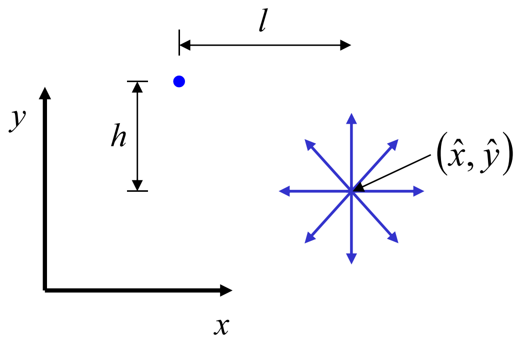
iii) Origin at a general point

$$x = \hat{x} - l, \quad y = \hat{y} + h,$$

$$u = \frac{+\Lambda}{2\pi}\frac{(x-\hat{x})}{\left((x-\hat{x})^2 + (y-\hat{y})^2\right)}$$

$$u = \frac{+\Lambda}{2\pi}\frac{(\hat{x}-l-\hat{x})}{\left((\hat{x}-l-\hat{x})^2 + (\hat{y}+h-\hat{y})^2\right)}$$

$$u = \frac{-\Lambda}{2\pi}\frac{l}{\left(l^2 + h^2\right)}$$



So we see that the position of the origin makes no difference to the solution but it does make the evaluations more or less algebraically complex.

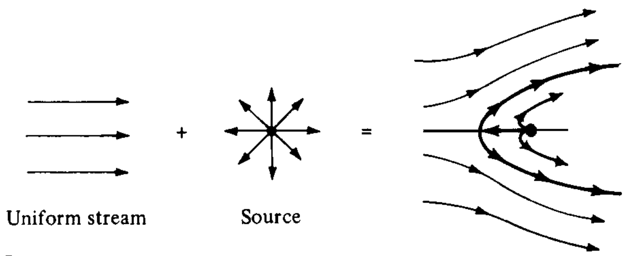
We can apply the same principle in cylindrical-polar coordinates but this is generally not required in the questions you will be asked.

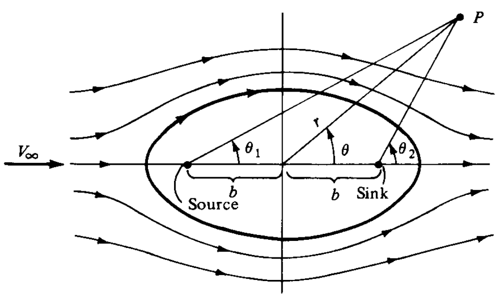
Simple Non-lifting Flows

There are three main examples of combining elementary flows to give simple non-lifting flows covered in most text books

(1) uniform flow +point source

=semi-infinite body

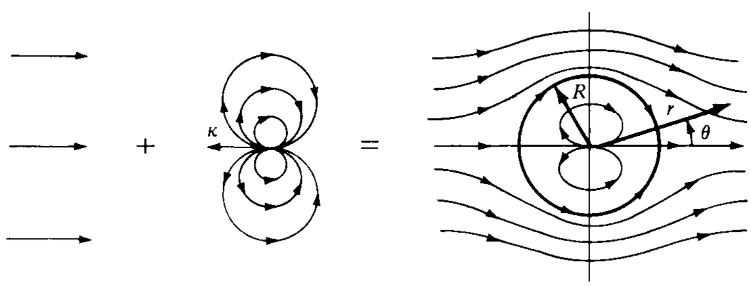




(2) uniform flow + point source and sink = axisymmetric body (eg Rankine oval)

(3) uniform flow + doublet

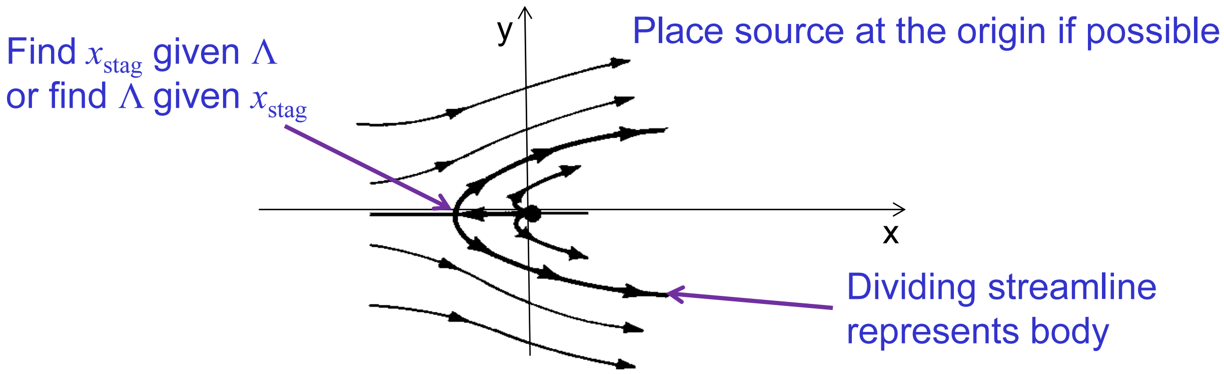
= flow around a cylinder



STEP BY STEP FLOW INVESTIGATION (Source+Freestream)

- All questions involving a source and free stream should be started with the *same initial* steps even if the actual quantities requested differ.

- STEP 1 Make an initial sketch of the situation.



- STEP 2 Write down the stream function of the combined flow (may not be required)

$$\psi = U_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta \quad \text{(polar)} \qquad \psi = U_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \quad \text{(cartesian)}$$

STEP BY STEP FLOW INVESTIGATION (2)

- STEP 3 Obtain the velocity field by differentiation of the stream function, but easier to use given equations.

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty \cos\theta + \frac{\Lambda}{2\pi r} \qquad u = \frac{\partial\psi}{\partial y} = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty \sin\theta \quad \text{(polar)} \qquad v = -\frac{\partial\psi}{\partial x} = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2} \quad \text{(cartesian)}$$

- STEP 4 Find the stagnation points (x_{stag},y_{stag}) i.e. where the velocity is zero. Using a Cartesian coordinate system

$$v = \frac{\Lambda}{2\pi}\frac{y_{stag}}{x_{stag}^2 + y_{stag}^2} = 0 \qquad \Rightarrow y_{stag} = 0$$

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{x_{stag}}{x_{stag}^2 + y_{stag}^2} = 0 \qquad \Rightarrow U_\infty + \frac{\Lambda}{2\pi}\frac{1}{x_{stag}} = 0 \Rightarrow x_{stag} = -\frac{\Lambda}{2\pi U_\infty}$$

Could have used polar velocity components, the location of the stagnation point would then be defined by

$$\theta_{stag} = \pi$$

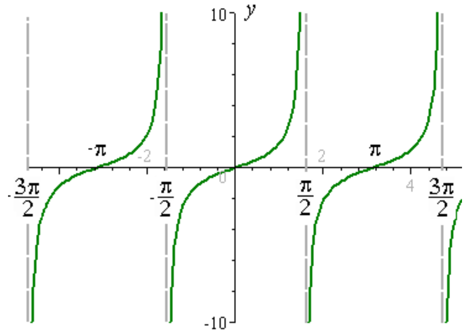
$$r_{stag} = \Lambda/(2\pi U_\infty)$$

STEP BY STEP FLOW INVESTIGATION (3)

- STEP 5 Link the **source strength** to the **ultimate height** of the body.

(a) The stagnation point is on the **dividing stream line**, so use it to find the constant defining this stream line.

$$\psi_{DS_{stag}} = U_\infty 0 + \frac{\Lambda}{2\pi}\underbrace{\tan^{-1}\left(\frac{y_{stag}}{x_{stag}}\right)}_{=\pi,\neq 0} = \frac{\Lambda}{2} \qquad \psi_{DS_{stag}} = U_\infty r\sin\theta_{stag} + \frac{\Lambda}{2\pi}\theta_{stag} = \frac{\Lambda}{2}$$



Equation for dividing streamline therefore.

$$U_\infty y_{DS} + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y_{DS}}{x_{DS}}\right) = \frac{\Lambda}{2}$$

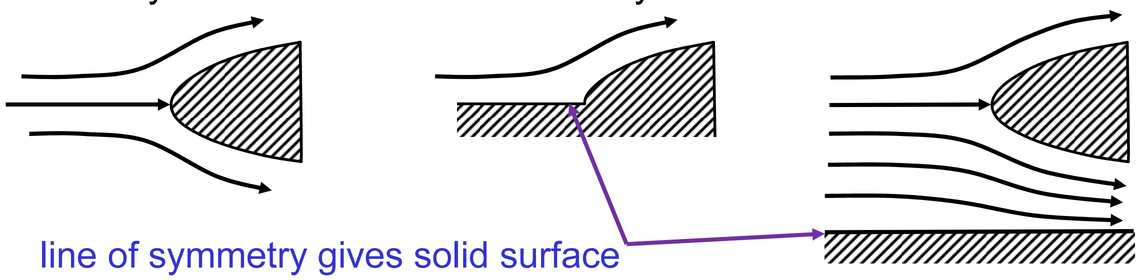
(b) Now consider what happens to y_{DS} as $x_{DS} \to \infty$

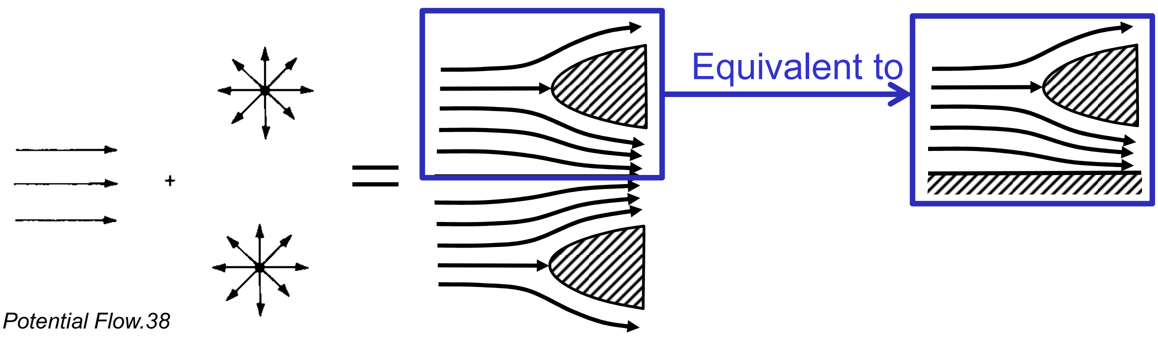
$$\tan^{-1}\left(\frac{y_{DS}}{x_{DS}}\right)\Bigg|_{x_{DS}\to\infty} \to 0 \qquad U_\infty y_{DS}\Big|_{x_{DS}\to\infty} \to \frac{\Lambda}{2} \qquad y_{DS_{max}} = \frac{\Lambda}{2U_\infty}$$

$$\text{max - thickness} = \frac{\Lambda}{U_\infty}$$

Related Problems

- This completes the steps common to all problems using a source and a sink. Questions will then have differing extra parts.
- Solid bounding surfaces treated using symmetry. Consider using images.
- For each simple body (bullet, circle, oval) there are 3 related problems:
- Full body Half body Near Surface





Learning Outcomes:
“What you should have learnt so far”

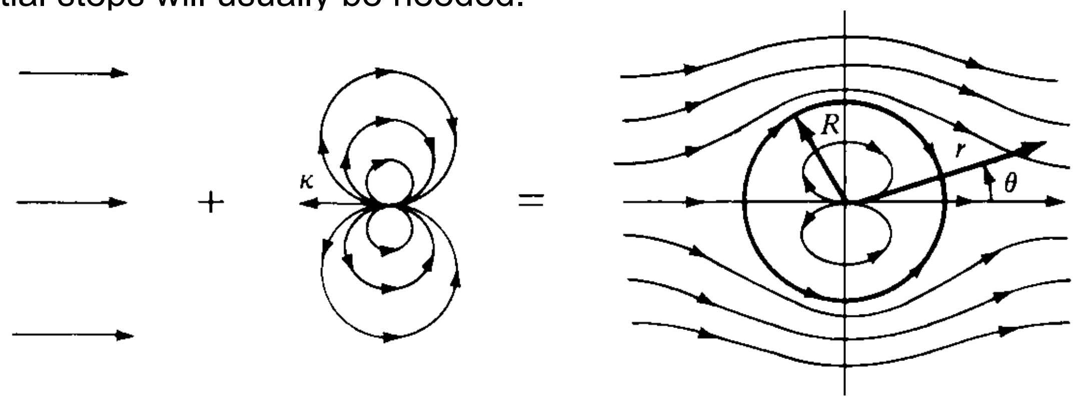
- State the boundary conditions on the potential for external aerodynamic flows
- Sketch the streamlines for flows that are modelled as a combination of
  - a uniform stream and a source
  - a uniform stream, a source and a sink
  - a uniform flow and a doublet
- Explain how images can be used to model straight boundaries
- Solve problems for a source+uniform stream
- Solve Problems where the singularity is not at the origin

Aims for this lecture

- Look at the detailed analysis of the potential flow around a stationary cylinder
- To consider the impact of introducing a point vortex to the non-lifting cylinder flow.
- To gain some insight into mechanism for the generation of lift in a flow

STEP BY STEP FLOW INVESTIGATION
(Flow over Cylinder)

The non-lifting flow over a cylinder is modelled as a free stream+ a doublet. To investigate the model and answer any question the same initial steps will usually be needed.



- STEP 1 Write out the stream function of the combined flow, in consistent coordinates for future calculation of velocities - polar is most suitable.

$$\psi = U_\infty r \sin\theta - \frac{\kappa}{2\pi}\frac{\sin\theta}{r} = U_\infty r\sin\theta\left(1 - \frac{\kappa}{2\pi U_\infty r^2}\right)$$

STEP BY STEP FLOW INVESTIGATION (2)

- STEP 2 Obtain the velocity field by differentiation, or look up components.

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty\cos\theta - \frac{\kappa\cos\theta}{2\pi r^2} = U_\infty\cos\theta\left(1 - \frac{\kappa}{2U_\infty\pi r^2}\right) \quad v_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty\sin\theta\left(1 + \frac{\kappa}{2\pi U_\infty r^2}\right)$$

Notice that when $r = \sqrt{\dfrac{\kappa}{2\pi U_\infty}}$ the radial velocity is zero and

Then introduce the constant R (radius of cylinder) so that $R^2 = \dfrac{\kappa}{2\pi U_\infty}$

So when $r = R$:v_r is zero & the circle is shown to be a flow stream line.

the stream function becomes $\psi = U_\infty r\sin\theta\left(1 - \dfrac{R^2}{r^2}\right)$

Then $\psi = 0$ when $r = R$, i.e. a constant since the circle is a stream line.

Then the velocities at any point become

$$v_r = U_\infty\cos\theta\left(1 - \frac{R^2}{r^2}\right) \qquad v_\theta = -U_\infty\sin\theta\left(1 + \frac{R^2}{r^2}\right)$$

STEP BY STEP FLOW INVESTIGATION (3)

STEP 3 Find the pressure coefficients on the surface:

Apply Bernoulli's equation between the freestream & any surface point

$$p + \frac{1}{2}\rho V^2 = p_\infty + \frac{1}{2}\rho U_\infty^2 \qquad \rightarrow \qquad p - p_\infty = \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho V^2$$

Then using definition of pressure coefficient

$$\boxed{c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2}}$$

the velocities on the cylinder are

$$v_r = 0 \qquad\qquad v_\theta = -2U_\infty \sin\theta$$

Note: stagnation points located where sin θ =0. Two solutions at θ=0 & θ=π

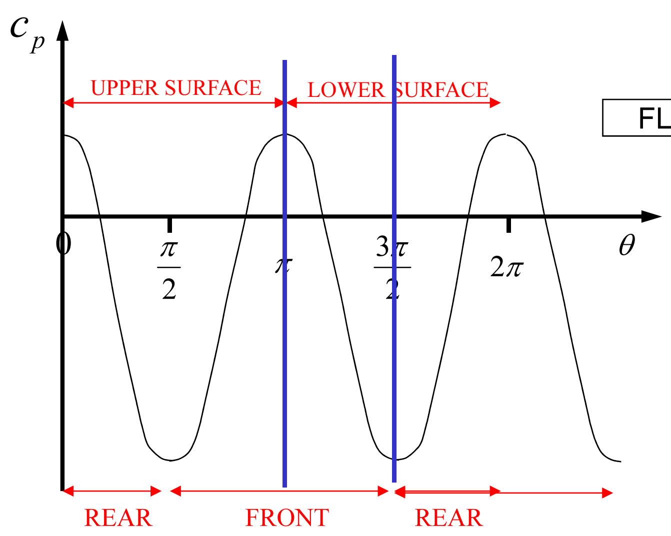
So in this case, on the cylinder $V^2 = v_r^2 + v_\theta^2 = 4U_\infty^2 \sin^2\theta$

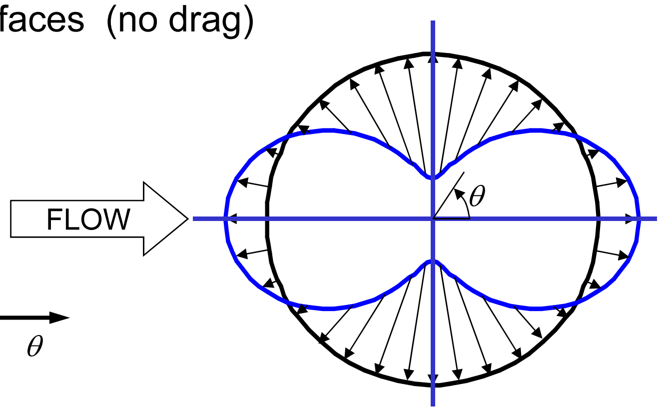
Then the pressure coefficient is given by

$$c_p = 1 - \frac{V^2}{U_\infty^2} = 1 - 4\sin^2\theta$$

Pressure Distribution Around a Cylinder

- Pressure is given by $c_p = 1 - 4\sin^2\theta$
- Symmetry on upper & lower surfaces (no lift)
- Symmetry on front & rear surfaces (no drag)

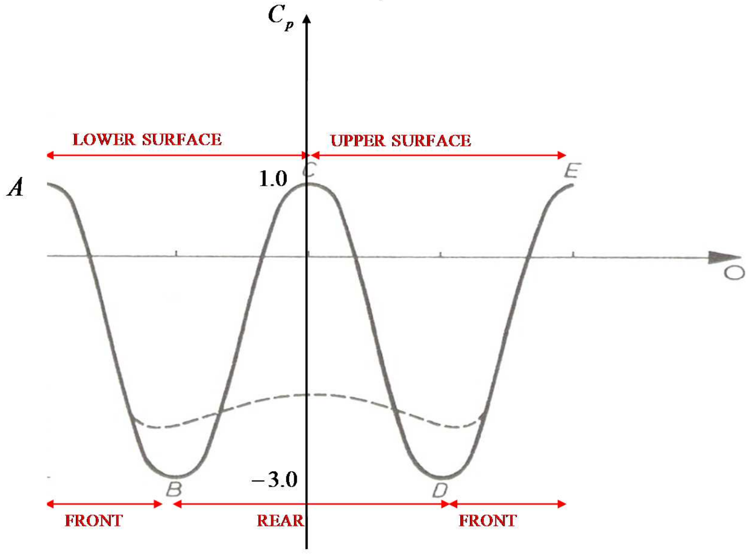




c_p distribution around a cylinder

Forces on Body and D'Alembert's paradox

- pressure distribution is symmetrical top to bottom
  - 'lift coefficient' $C_L = 0$ (particular to this flow)
- pressure distribution is symmetrical left to right
  - 'drag coefficient' $C_D = 0$ (general result)
- Zero drag is in fact, a general result for **2D** potential (or irrotational) flow around closed bodies and is called D'Alembert's paradox



INTEGRATING PRESSURES FOR LIFT COEFFICIENTS

To get lift on cylinder mathematically, consider the forces acting on a small element of its surface. Any other shape follows the same process

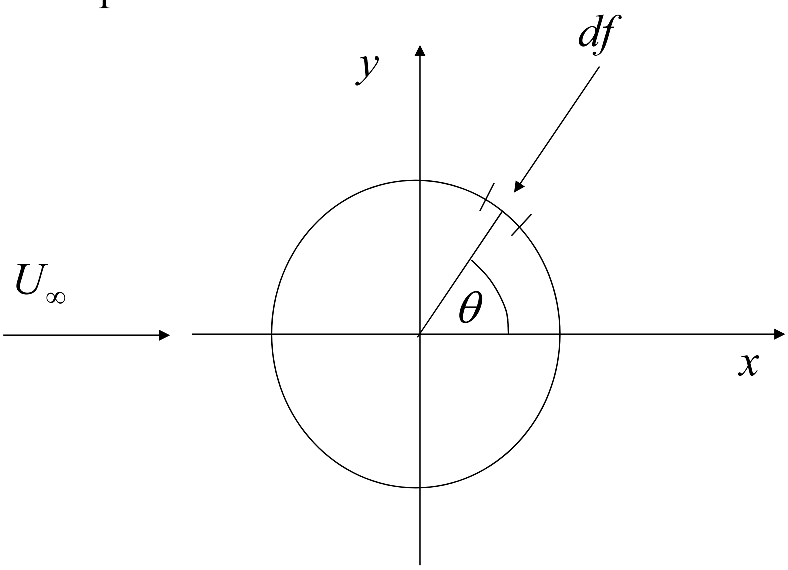
$$df = pds$$

resolving into vertical component

$$dL = -df \sin\theta$$

but as $\quad ds = Rd\theta$

$$dL = -p \sin\theta \, Rd\theta$$



INTEGRATING PRESSURES FOR LIFT COEFFICIENTS(2)

Then the total lift on cylinder is obtained by integration

$$L = -\int_0^{2\pi} p \sin\theta \, R d\theta \quad = -\int_0^{2\pi} (p - p_\infty) \sin\theta \, R d\theta - \int_0^{2\pi} p_\infty \sin\theta \, R d\theta$$

$$L = -\frac{1}{2} \rho_\infty U_\infty^2 R \int_0^{2\pi} c_p \sin\theta \, d\theta$$

Second integral easily shown to be zero

Then substituting in for the pressure coefficient the total lift on a cylinder can be shown to be

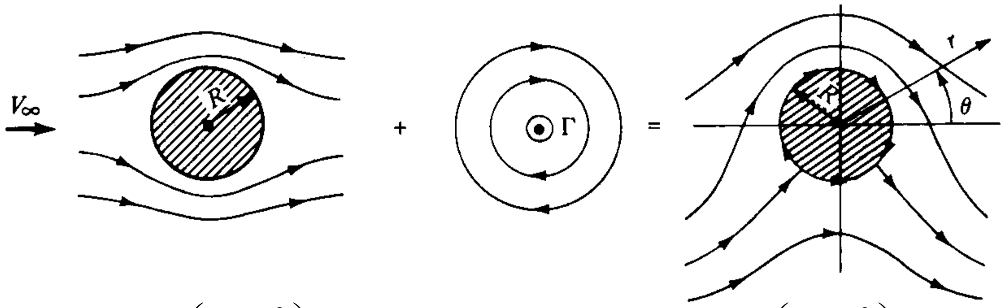
$$L = -\frac{1}{2} \rho_\infty U_\infty^2 R \int_0^{2\pi} \left(1 - 4\sin^2\theta\right) \sin\theta \, d\theta = -\frac{1}{2} \rho_\infty U_\infty^2 R \left(\underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{=0} - 4 \underbrace{\int_0^{2\pi} \sin^3\theta d\theta}_{=0} \right)$$

$$L = 0$$

Remember the definition of Lift Coefficient $\quad C_L = \dfrac{L}{Area \times \frac{1}{2} \rho_\infty U_\infty^2}$

Lifting Cylinder Flow

- Only mechanisms for transmitting forces (& hence lift) to a body is via pressure and shear stress on the body surface
- The potential flow over a rotating cylinder will give us some insight
- A point vortex is added to the previous model of flow over a non-lifting cylinder i.e. the model consists of a uniform stream + doublet + point vortex
- Streamline of vortex is a circle so boundary unchanged



$$\psi \;=\; \underbrace{U_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)}_{\text{Freestream \& doublet}} + \underbrace{\frac{\Gamma}{2\pi} \ln r}_{\text{vortex}} - \underbrace{\frac{\Gamma}{2\pi} \ln R}_{\text{arbitrary constant}} = U_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln\frac{r}{R}$$

- Term added to match streamfunction value at surface to non-lifting case

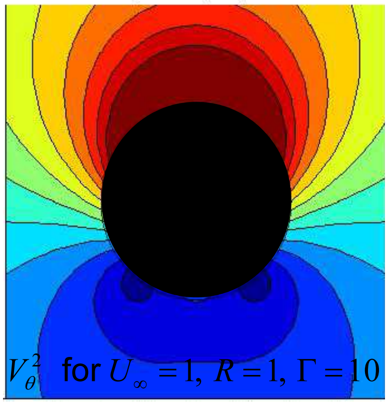
Velocities & Pressure

- Differentiation gives the velocity terms

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \left(1 - \frac{R^2}{r^2}\right)U_\infty \cos\theta, \qquad V_\theta = -\frac{\partial \psi}{\partial r} = -\left(1 + \frac{R^2}{r^2}\right)U_\infty \sin\theta - \frac{\Gamma}{2\pi r}$$

- On the 'surface' of the cylinder (i.e. $r = R$)

$$V_r = 0, \qquad V_\theta = -\underbrace{2U_\infty \sin\theta}_{\text{freestream \& doublet}} - \underbrace{\frac{\Gamma}{2\pi R}}_{\text{vortex}}$$



- Surface pressure found using

$$c_p = 1 - \frac{V^2}{U_\infty^2} = 1 - \frac{V_\theta^2}{U_\infty^2}$$

$$C_p = \underbrace{1 - 4\sin^2\theta}_{\text{basic cylinder flow}} - \underbrace{\frac{2\Gamma \sin\theta}{\pi R U_\infty}}_{\text{asymmetric swirl component}} - \underbrace{\left(\frac{\Gamma}{2\pi R U_\infty}\right)^2}_{\text{constant swirl component}}$$

Terms of Pressure Coefficient

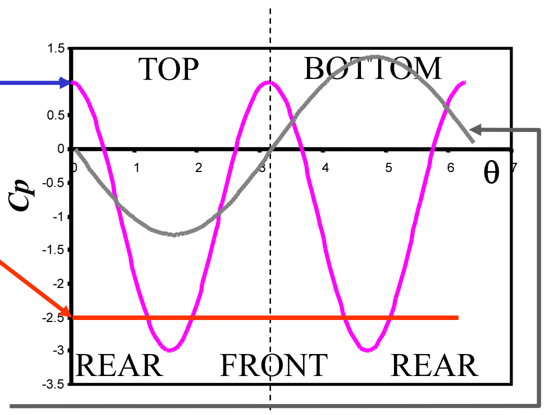
(1) Basic cylinder flow $1 - 4\sin^2\theta$

- lift and drag contribution are zero

(2) Constant swirl term $-\left(\Gamma/(2\pi R U_\infty)\right)^2$

- lift and drag contribution are zero (note a constant suction contribution)

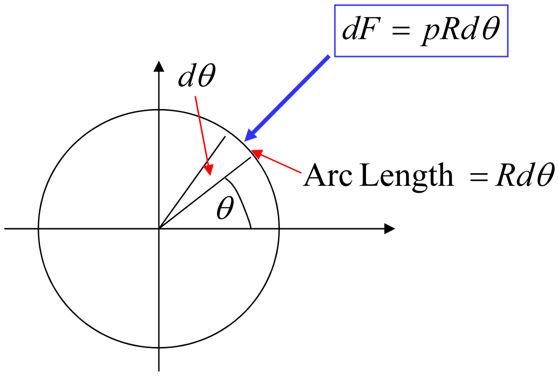
(3) Asymmetric swirl term $-\left(\frac{2\Gamma}{\pi R U_\infty}\right)\sin\theta$



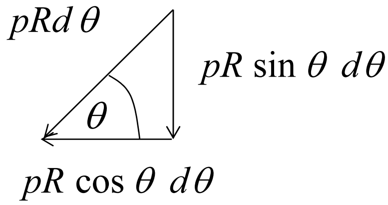
- lift contribution non-zero due to the **combination** of vortex and freestream
- Positive lift from a suction on the upper surface and a positive pressure on the lower surface
- Drag force zero because the pressure distributions on the front and rear halves of cylinder have equal areas of +ve and –ve pressure, therefore everything cancels out

Lift Generation

To evaluate the lift it is easiest to work it out from first principles



Resolve this force to get components



So the expression for lift force in the upwards direction is $l = -\int_0^{2\pi} p(\theta) R \sin\theta \, d\theta$

See handout for derivation

$$\boxed{l = \rho_\infty U_\infty \Gamma}$$

Kutta-Joukowski Theorem (1902/06)

general result for 2D body with 'bound' circulation Γ

Fundamental relation of Fluid Mechanics

In non-dimensional terms (if diameter per unit depth 2Rx1 is reference area)

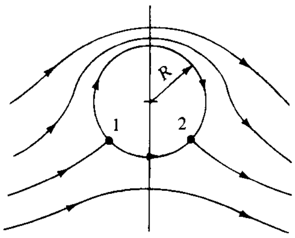
$$C_l = \frac{l}{1/2\rho_\infty U_\infty^2 2R} = \frac{\Gamma}{RU_\infty}$$

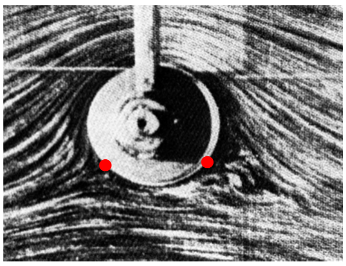
Stagnation Points

$$v_r = 0 \quad \Rightarrow \quad r = R \quad or \quad \cos\theta = 0$$

$$v_\theta = 0 \quad \Rightarrow \quad \sin\theta = \left(\frac{-\Gamma}{2\pi r U_\infty (1 + R^2/r^2)} \right) \qquad \Gamma > 0 \;\rightarrow\; \sin\theta < 0 \;\rightarrow\; y_{\text{stagnation}} < 0$$

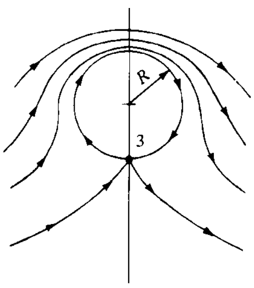
- Case(1): The stagnation points are on the cylinder surface, $r=R$ then
- If $\dfrac{\Gamma}{(4\pi U_\infty R)} < 1$ then there are two values of θ, symmetric about the y-axis



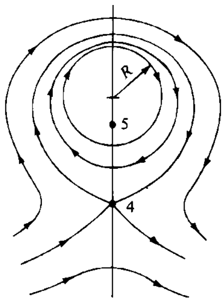


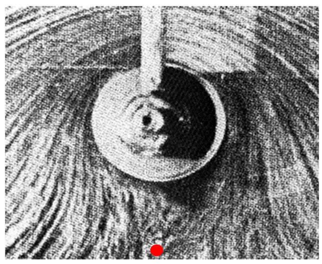
$$\sin\theta = \left(\frac{-\Gamma}{4\pi R U_\infty} \right)$$

- Case(2): $\dfrac{\Gamma}{(4\pi U_\infty R)} = 1$ $\quad \sin\theta = -1$



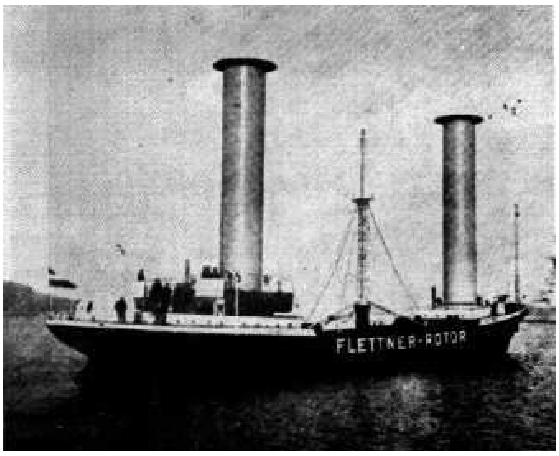
- Case(3): $\dfrac{\Gamma}{(4\pi U_\infty R)} > 1$





Example of real life use of “Lifting Cylinders”

- applied to ship propulsion by Flettner (1920)
  - twin vertical cylinders
  - highly manoeuvrable
  - mechanically over-complex (cost, reliability)



Where Does the Lift Come From .. ?

- Model shows that circulation on its own does NOT lead to lift
- Model indicates that changes in the pressure distribution which lead to lift occur if there is a combination of

  a. circulation (ie flow ***rotation***) and

  b. freestream (ie ***cross-flow***) velocity
  - increased velocity *magnitude* on upper surface
  - reduced velocity magnitude on lower

- Be cautious when reading text books or looking at websites as many give seemingly convincing, but wrong explanations of lift !!
- Arguments and controversy over lift generation have arisen because people misapply Bernoulli’s or Newton’s equations. Both sets of equations do apply, but not in the way that is frequently described in many discussions of lift!!

Learning Outcomes:
"What you should have learnt so far"

- Use suitable potential models to investigate velocities, pressures and forces for a uniform flow and a doublet.
- Explain D'Alembert's paradox
- Derive the velocity and pressure distributions from the stream function for a lifting cylinder
- Derive the Kutta-Joukowski theorem relating lift and circulation
- Distinguish between the elements of the cylinder velocity distribution and hence explain qualitatively how lift is generated

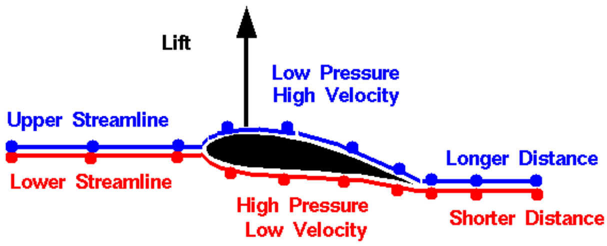
COMMON EXPLANATIONS OF LIFT No 1 - 'Bernoulli'

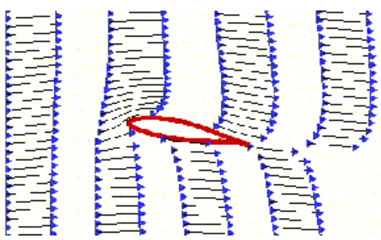
Most common 'popular' explanation

Asserts that air passing over the top of an aerofoil at incidence has further to go than air passing underneath - therefore must go faster to 'catch up' to reach trailing edge at the same time- from Bernoulli's Equation there is a pressure differential, hence lift …

Not entirely correct because

- There is no physical justification for 'equal transit times'
- What about flat plates? Difference in upper and lower path lengths are too small to generate observed levels of lift
- Smoke flow visualisation (or similar) shows upper surface flow moving ahead of the lower surface flow
- Why can planes fly upside down?





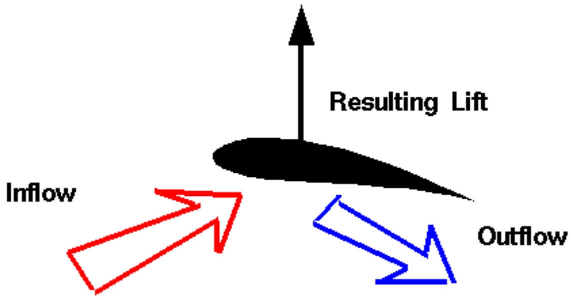
Bernoulli's equation not wrong, equal transit time assumption is in general wrong

COMMON EXPLANATIONS OF LIFT No 2 - ‘Newton’s 2nd law’

Original explanation by Newton assumes that air molecules behave like individual particles. Each particle hits the bottom surface of the wing, bounces and is deflected downward. As the particles strike the bottom surface of the wing, they impart some of their momentum to the wing, thus incrementally nudging the wing upward with every molecular impact.

Not entirely correct because

- The contribution of the top surface is ignored and has a significant contribution.
- Euler later noticed that fluid moving towards an object will deflect before it actually hits the object.



Use of Newton’s equation not wrong, it is just frequently misapplied both in this simple version, but also in more sophisticated attempts to use it.

What Can potential flow methods solve?