FLUIDS I
Example sheet 5: Potential Flow (short revision)

1. What are the fundamental principles used to derive the Navier-Stokes?

Conservation of mass

Conservation of momentum or Newton's second law or force =rate of change of momentum
Conservation of energy

2. The usual form of Bernoulli's equation is found by integrating the 1D Euler equation. What assumptions are needed to first derive the 1D Euler equation $dp = \rho \mathbf{V} d\mathbf{V}$? What further assumption is necessary to integrate this equation to get the familiar form of Bernoulli's equation $p + 1/2\rho\mathbf{V}^2 =$ constant ?

Steady
Inviscid (no viscous forces)
No body forces
Along streamline or anywhere in an irrotational flow.
Incompressible flow

3. For what type of flow can a velocity potential be defined?

Irrotational flow

4. The velocity potential if it exists satisfies Laplace's equation. Give expressions for u, v and w in terms of the potential? Why is it advantageous to be able to solve Laplace's equation?

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}$$

The advantages are (a) only one equation rather than 3 momentum equations to solve before velocity field known (b) superposition of solutions

5. What is the name given to a line whose tangent an any point is in the direction of the local fluid velocity?

Streamline

6. What is the equation for a streamline in 2D flow (assume that we are working in the x-y plane)?

$$\frac{dy}{dx} = \frac{v}{u}$$

7. Using the above equation obtain an equation for the streamlines of a flow which has u=$w\ x$ and v=-$w\ y$, where w is a constant. Also sketch the streamlines.

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{\omega y}{\omega x} = -\frac{y}{x}$$

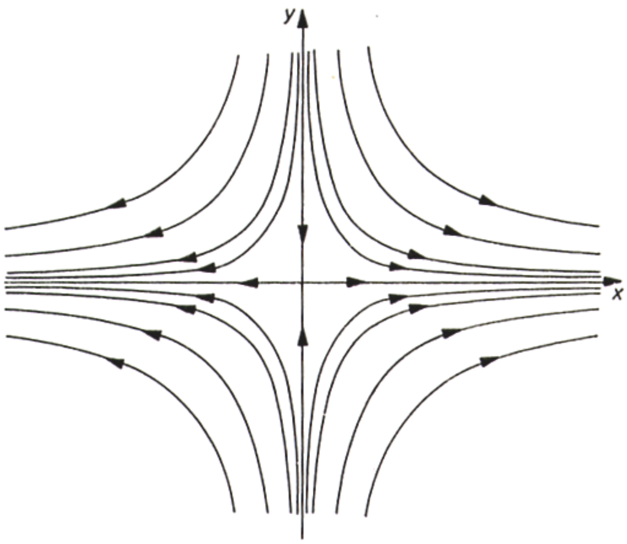
$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\ln(|y|) + \ln(|x|) = c$$

$$\ln(|x||y|) = c$$

$$|x||y| = e^{c} = c^{*}$$

$$|y| = \frac{c^{*}}{|x|}$$



8. For the flow in question 7 find the expressions for the potential and streamfunction.

$$u = \frac{\partial \phi}{\partial x} = \omega x \quad \Rightarrow \phi = \frac{\omega}{2}x^2 + f(y) \quad \Rightarrow \frac{\partial \phi}{\partial y} = f'(y)$$

$$v = \frac{\partial \phi}{\partial y} = -\omega y = f'(y) \quad \Rightarrow f(y) = -\frac{\omega}{2}y^2 + c$$

SO $\phi = \frac{\omega}{2}(x^2 - y^2) + c$

$$u = \frac{\partial \psi}{\partial y} = \omega x \quad \Rightarrow \psi = \omega xy + f(x) \quad \Rightarrow \frac{\partial \psi}{\partial x} = \omega y + f'(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\omega y = -\omega y - f'(x) \quad \Rightarrow f'(x) = 0 \Rightarrow f(x) = c$$

SO $\psi = \omega xy + c$

9. Give expressions for the velocities u and v in terms of (a) the stream function ψ, (b) the potential ϕ.

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \qquad\qquad u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

10. For a uniform flow parallel to the x axis, derive expressions for the stream function and the potential

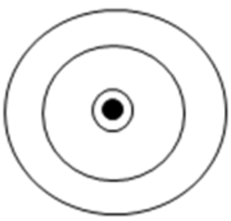
$$u = \frac{\partial \phi}{\partial x} = U_\infty \quad v = \frac{\partial \phi}{\partial y} = 0 \quad \Rightarrow \phi = U_\infty x + c$$

$$u = \frac{\partial \psi}{\partial y} = U_\infty \quad v = -\frac{\partial \psi}{\partial x} = 0 \quad \Rightarrow \psi = U_\infty y + c$$

11. Give expressions for the radial and tangential velocity components v_r, v_θ in terms of the stream function and the potential

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad v_\theta = -\frac{\partial \psi}{\partial r} \qquad\qquad v_r = \frac{\partial \phi}{\partial r} \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

12. For a source $\phi_{source} = \frac{\Lambda}{2\pi}\ln r$ and $\psi_{source} = \frac{\Lambda}{2\pi}\theta$ Sketch the equipotentials and streamlines



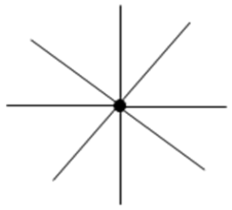
Equipotentials

Streamlines

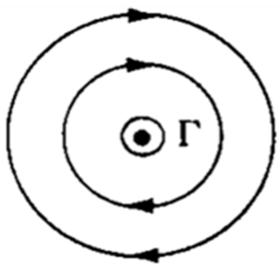
13. What is the line integral equation for circulation?

$$\Gamma = -\oint_c \mathbf{V} \cdot \mathbf{ds}$$

14. For a vortex $\phi_{vortex} = -\frac{\Gamma}{2\pi}\theta$ and $\psi_{vortex} = \frac{\Gamma}{2\pi}\ln r$ Sketch the equipotentials and Streamlines.



Equipotentials



Streamlines

15. Explain briefly in words how a doublet flow is obtained from a combination of elementary flows.

A source and sink of equal strength Λ are placed a distance l apart. Then the distance between the source and sink is reduced keeping the term Λl constant. In the limit as the distance tends to zero the flow is a doublet flow.

16. What elementary flows can be combined to produce a 2D potential model for a non-lifting flow over a circular cylinder

A doublet and a free steam

17. Derive an expression for the pressure coefficient Cp in terms of velocity from Bernoulli's equation.

$$p + \frac{1}{2}\rho V^2 = p_\infty + \frac{1}{2}\rho_\infty V_\infty^2 \quad \text{but } \rho = \rho_\infty \quad \text{incompressible flow}$$

Rearranging

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 1 - \frac{V^2}{V_\infty^2}$$

$$C_p = 1 - \frac{V^2}{V_\infty^2}$$

18. Briefly describe the physical significance of the stream function ψ in incompressible flow. What are the limits on its application, compared with the potential function ϕ?

Streamlines are lines of constant stream function. The change in ψ between streamlines corresponds to the volume flow (per unit depth) between these lines or the mass flow between the lines scaled by the density.

ψ is only defined for 2D flow whereas ϕ exists for 3D flows. ψ can be used in rotational flows, whereas ϕ implies irrotationality.

19. The non-lifting flow over a cylinder has a stream function given by $\psi = U_\infty y - \frac{\kappa}{2\pi}\frac{\sin\theta}{r}$

(a) This flow is modelled as a combination of which two flows?

Uniform stream and doublet

(b) Put the stream function into consistent (r, θ) coordinates assuming that $R^2 = \frac{\kappa}{2\pi U_\infty}$

$$\psi = U_\infty r \sin\theta\left(1 - \frac{\kappa}{2\pi r^2 U_\infty}\right) = U_\infty r \sin\theta\left(1 - \frac{R^2}{r^2}\right)$$

(c) Find expressions for the radial and tangential velocity components v_r , v_θ

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right) \qquad v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta\left(1 + \frac{R^2}{r^2}\right)$$

(d) What is the pressure coefficient distribution for the cylinder which has radius R?

$$v_r = 0, \qquad v_\theta = -2U_\infty \sin\theta$$

$$c_p = 1 - \left(\frac{V}{U_\infty}\right)^2 = 1 - 4\sin^2\theta$$

20. State D'Alembert's paradox.

The drag of <u>any</u> 2D body placed in a stream of inviscid incompressible fluid with or without circulation is always zero. (Note real flows have non-zero drag so whilst an inviscid flow model can give a reasonable estimate of lift it cannot model real drag, need viscosity for that)