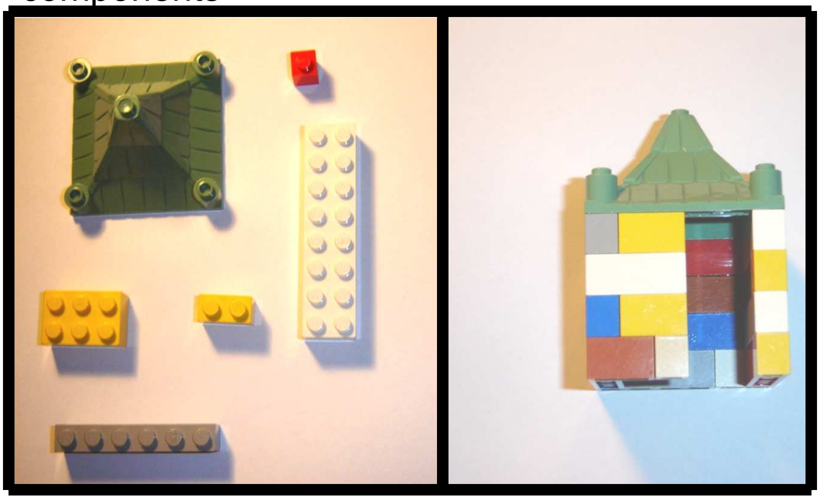
Elementary 2D Potential Solutions

Start with simple components $\xrightarrow{\text{combine}}$ Build up more complex models



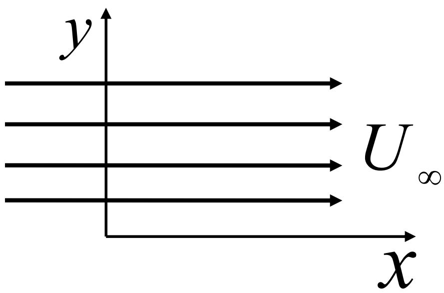
Similarly for solutions of Laplace's equation, because it is LINEAR

1. Uniform flow
2. Point source / sink
3. Doublet
4. 2D Point vortex

$\xrightarrow{\text{combine}}$ More complex solutions, to model real flows

Elementary Flow 1- Uniform Horizontal Flow

Sketch of the streamlines of the flow



Velocities

$$u = U_\infty\ ,\ \ v = 0$$

Potential function

$$\phi = U_\infty x + \text{constant}$$

Stream function

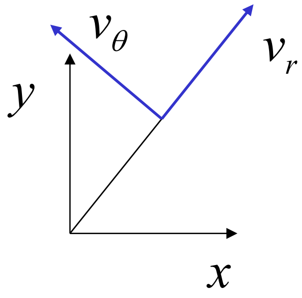
$$\psi = U_\infty y + \text{constant}$$

Before the next lecture, prove to yourself how the potential and stream functions are obtained

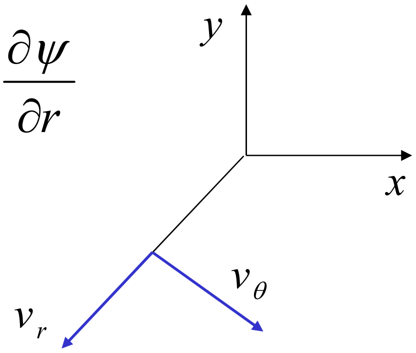
Short Aside: Velocity field in 2D polar coordinates

It is sometimes easier for the other elementary flows to work in (r, θ) coordinates.

$$v_r = \frac{\partial \phi}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$



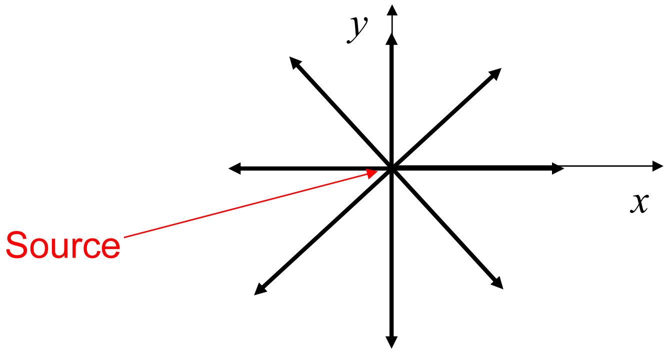
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$



For background

Elementary Flow 2- Point Source/Sink Flow

Sketch of the streamlines of the flow



Velocities

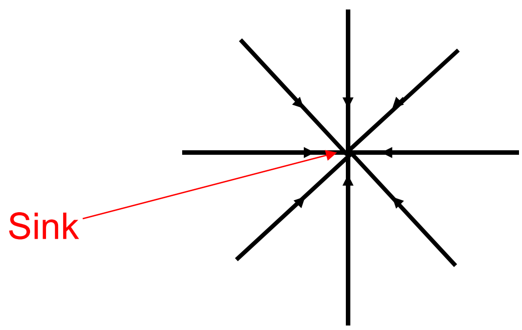
$$v_r = \frac{\Lambda}{2\pi r} \qquad v_\theta = 0$$

Potential function

$$\phi = \frac{\Lambda}{2\pi}\ln r + \text{constant}$$

Stream function

$$\psi = \frac{\Lambda}{2\pi}\theta + \text{constant}$$



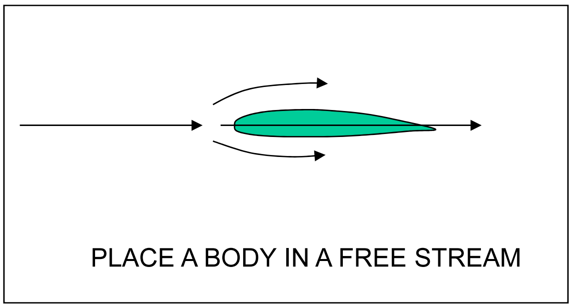
Λ, the source/sink strength is the volume flow rate (per unit depth) out of the source, or into the sink

- The radial velocity is singular at the origin since

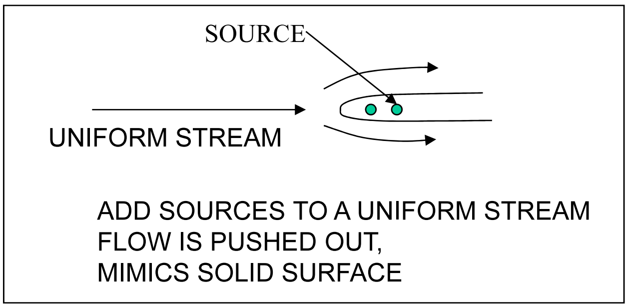
$$v_r \to \infty \qquad \text{as} \qquad r \to 0$$

- and source flow satisfies Laplace's equation and has a potential function except at the *singularity* at the origin.
- Source flow is a mathematical abstraction and singularities can't exist in the flow solution domain since real flow velocities can't tend to infinity.
- However the behaviour of *theoretical model* away from the source mimics the effect of solid objects in a real flow so it is in fact a useful tool.

REAL FLOW

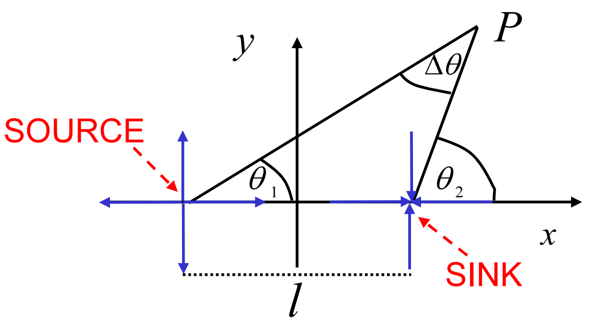


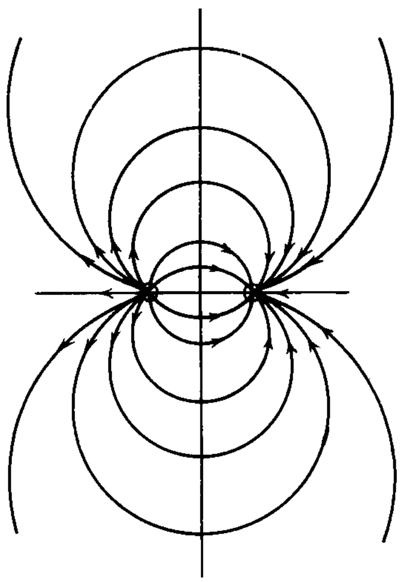
POTENTIAL FLOW MODEL



Superposition of a Source & Sink

- Consider a source of strength Λ and a sink of equal, but opposite strength separated by distance l





- The stream function at any point *P* subtending an angle $\Delta\theta$ from the pair is

$$\psi = \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2 = \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = -\frac{\Lambda}{2\pi}\Delta\theta$$

- Some trigonometry shows the streamlines correspond to a series of circles, that pass through the source and sink

Elementary Flow 3- Doublet

- Now to obtain a doublet move the source and sink closer together, i.e. let $l \to 0$
- but at the same time increase the source/sink strength, so that

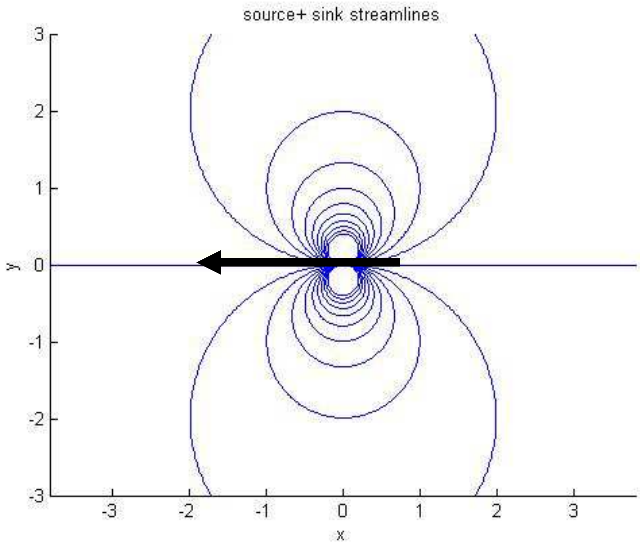
$$\Lambda l = \text{constant} \quad = \kappa$$

- In the limit the source and sink are superimposed in a doublet of finite strength κ with stream function

$$\psi_{doublet} = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$

- The potential for a doublet is

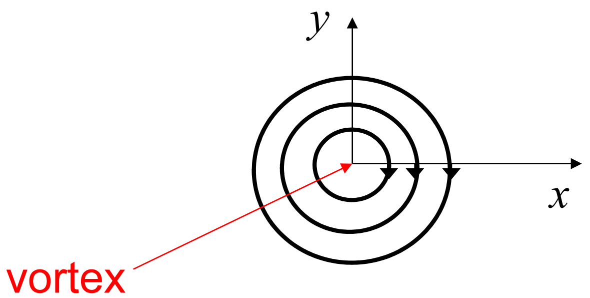
$$\phi_{doublet} = +\frac{\kappa}{2\pi}\frac{\cos\theta}{r}$$



- The doublet has a direction, or axis, associated with it. The positive direction is given by an arrow drawn from the sink to the source (right to left in this case).

Elementary Flow 4- Point Vortex Flow

Sketch of the streamlines of the flow



Velocities $v_r = 0 \quad v_\theta = -\dfrac{\Gamma_V}{2\pi r}$

Potential function

$$\phi = -\frac{\Gamma_V}{2\pi}\theta + \text{constant}$$

Stream function

$$\psi = +\frac{\Gamma_V}{2\pi}\ln r + \text{constant}$$

The tangential velocity is singular at the origin since

$$v_\theta \to \infty \qquad \text{as} \qquad r \to 0$$

Point vortex flow satisfies Laplace's equation and has a potential function except at the *singularity* at the origin.

In a point vortex flow, fluid moves around circular paths and is irrotational except at the singularity or vortex.

- Recall mathematical definition of irrotationality

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(u,v,w)=(0,0,0)$$

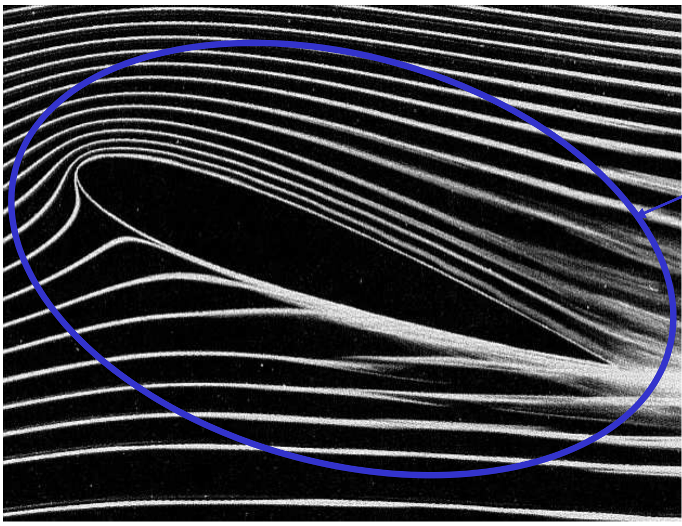
- Vorticity at a point is equivalent to twice the angular velocity of a fluid element there. In 2D this means

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\times(u,v)=2(0,0,\omega)$$

- So in an irrotational vortex flow fluid, elements can move in a circular path, but do not rotate
- To investigate the significance of the point vortex strength need to introduce the concept of flow **circulation**.

Circulation – Common Misconception

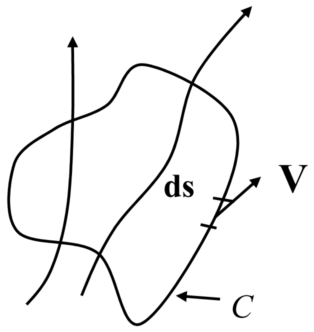
- Everyday usage -circulation means to move in a circle or circuit.
- In aerodynamics if there is non-zero circulation then the fluid elements are *not* necessarily moving along a circular path.
- Example flow past an aerofoil-fluid elements don't move in circles, but circulation calculated around contour enclosing aerofoil is non-zero



Circulation around closed path non zero

What is Circulation?

- Circulation is defined in terms of the integral of the velocity component along the closed curve c



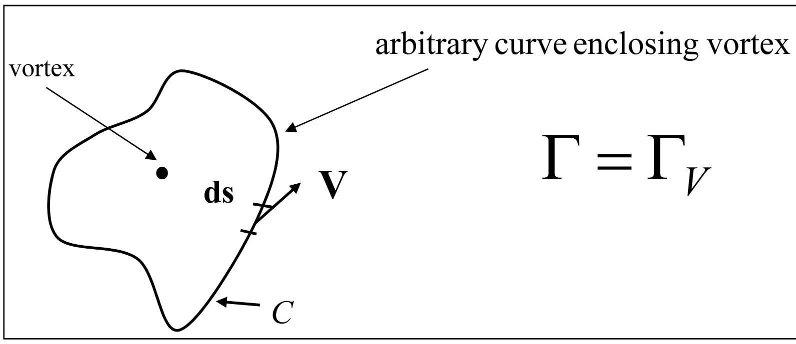
$$\Gamma = -\oint_c \mathbf{V}\cdot\mathbf{ds} = \underbrace{-\oiint_S \left(\nabla\times\mathbf{V}\right)\cdot\mathbf{dS}}_{\text{from Stokes' theorem}}$$

- Positive sense of line integrals - anti-clockwise
- Aeronautical convention - circulation Γ is positive in the clockwise direction, used in this course

For background

- If the vorticity is zero everywhere (irrotational flow) within the closed contour C, then circulation Γ =0
- Note same nomenclature for vorticity & circulation, we will see why shortly

Point Vortex Strength & Circulation



$$\Gamma = \Gamma_V$$

Q: Why is there circulation in an irrotational potential flow?

A: because of the presence of vortices (or a doublet gradient but we not explore this idea this year)

For a “2D point vortex flow” at the origin, vorticity is infinite (but zero everywhere else!). This infinite vorticity over an infinitely small point gives a finite circulation. So:

The circulation is finite as long as the contour of integration encloses point vortices and is equal to the sum of vorticity
Enclosed sources and sinks do not create circulation (and therefore lift)

Learning Outcomes:
"What you should have learnt so far"

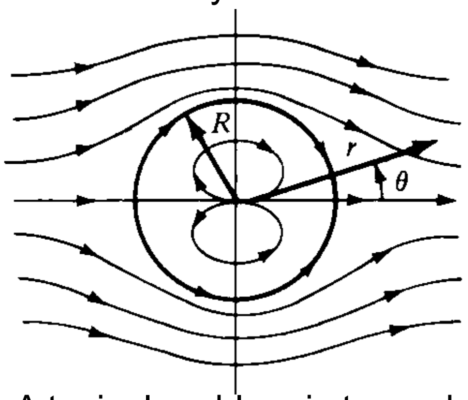
- Describe the 4 basic elementary flows
- Sketch the stream lines and potential lines for each elementary flow
- Explain how a doublet is obtained from a source/sink pair
- Explain the basic concept of circulation and state that circulation around a curve is zero if the flow is irrotational everywhere within the curve.
- Explain that there can be circulation in an irrotational flow if there are singularities present.

Aims for this lecture

- To explain how more complex 2D potential solutions can be found
- Singularities are defined as if they were placed at the origin. Introduce the principles of applying singularities away from the origin
- Begin first example question

GENERAL COMMENTS ON POTENTIAL MODELS

- No flow can cross a stream line in the potential model. Mimics the property of a solid boundary, so any stream line can be regarded as a solid boundary.

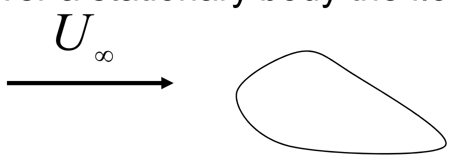


- No singularities in a real flow, so any singularities in potential model must be located outside the domain modelling the flow (Sources Sinks Doublets and Vortices are placed inside solid bodies or outside solid walls for internal flows)

- A typical problem is to model flow about a body travelling at a uniform speed into still air. This is equivalent to the flow about a body in a uniform stream (using the Galilean transformation)

BOUNDARY CONDITIONS

For the external flow over a stationary body the flow boundary conditions are



(1) Far away from the body (towards infinity) the flow velocity approaches uniform stream conditions

(2) There must be no flow perpendicular to the solid surface i.e. surface of the body must be a stream line of the flow. So the same equation solved over different boundary shapes leads to different solutions.

Note: (1) is satisfied if sources/sinks and point vortices are added to a uniform flow, as the velocity of these basic elements tends to zero far away.

In satisfying (2) there are a number possibilities but these all use the same idea. Adding each source/doublet/vortex to a uniform free stream adds parameters of **position** and **strength** that can be adjusted to produce the required streamline that is taken as the body shape.

Web Site Link

- There are a number of programs available on the web that allow you to look at combining the elementary flows, which plot out streamlines.
- The following example was checked Oct 2009, but others could be found by searching for potential flow programs

www.aoe.vt.edu/~devenpor/aoe5104/ifm/ifm.html

a) Try an onset flow of 1, 4 sources of strength 1, 4 sources of strength -0.5 and 8 of strength -0.25

b) Try an onset flow of 1 and a doublet of strength -8

c) Add a vortex of strength -4 directly on top of the doublet

Change of Origin For Elementary Potential Flow Solutions

Consider an elementary potential flow solution (source, doublet, vortex) where the centre of the element is at the Cartesian coordinate origin (0,0). The horizontal and vertical components of velocity at any point (x,y) can be written in the general form (see hand out)

$$u = \Theta\, f(x,y), \qquad v = \Theta\, g(x,y)$$

Where Θ is the element strength (i.e. Λ, Γ or κ) and f, g are functions of the position (x,y). As an example consider a source of strength Λ

$$u = \frac{+\Lambda}{2\pi}\frac{x}{\left(x^2+y^2\right)}, \quad v = \frac{+\Lambda}{2\pi}\frac{y}{\left(x^2+y^2\right)} \quad \rightarrow \left\{ \begin{array}{ll} \Theta = \Lambda, & f = \dfrac{x}{2\pi\left(x^2+y^2\right)} \\ & g = \dfrac{y}{2\pi\left(x^2+y^2\right)} \end{array} \right.$$

However if the position of the elementary solution is moved to $(\hat{x}, \hat{y})$ then the velocities at (x,y) can now be written as

$$u = \Theta\, f\left((x-\hat{x}),(y-\hat{y})\right), \qquad v = \Theta\, g\left((x-\hat{x}),(y-\hat{y})\right)$$

Or for the source previously defined

$$u = \frac{+\Lambda}{2\pi}\frac{(x-\hat{x})}{\left((x-\hat{x})^2+(y-\hat{y})^2\right)}, \quad v = \frac{+\Lambda}{2\pi}\frac{(y-\hat{y})}{\left((x-\hat{x})^2+(y-\hat{y})^2\right)}$$

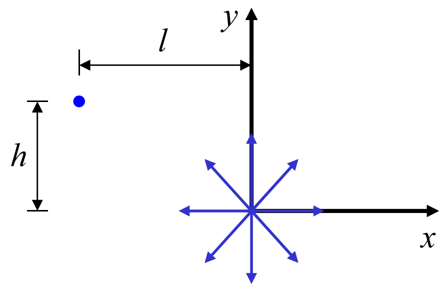
Change of Origin (2)

The choice of the origin location has no effect on the solution. Consider the evaluation of the horizontal component of velocity at a point a fixed relative distance from a source. Evaluating the velocity with three different origins

i) Origin at the source

$$x = -l,\ \ y = h,\ \ \hat{x} = 0,\ \ \hat{y} = 0$$

$$u = \frac{-\Lambda}{2\pi}\frac{l}{\left(l^2 + h^2\right)}$$



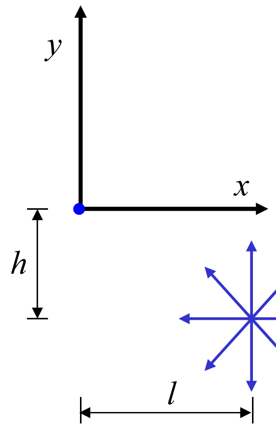
ii) Origin at the evaluation point

$$x = 0,\ \ y = 0,\ \ \hat{x} = l,\ \ \hat{y} = -h$$

$$u = \frac{+\Lambda}{2\pi}\frac{(x-\hat{x})}{\left((x-\hat{x})^2 + (y-\hat{y})^2\right)}$$

$$u = \frac{+\Lambda}{2\pi}\frac{(0-l)}{\left((0-l)^2 + (0+h)^2\right)}$$

$$u = \frac{-\Lambda}{2\pi}\frac{l}{\left(l^2 + h^2\right)}$$



Change of Origin (3)

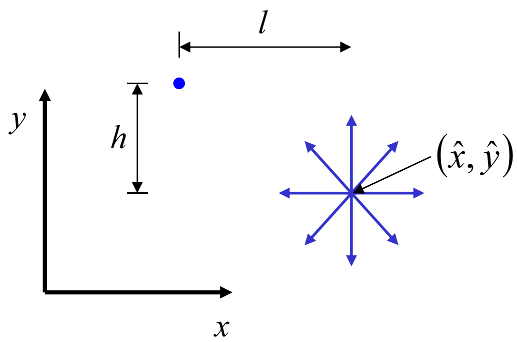
iii) Origin at a general point

$$x = \hat{x} - l,\ \ y = \hat{y} + h,$$

$$u = \frac{+\Lambda}{2\pi}\frac{(x-\hat{x})}{\left((x-\hat{x})^2 + (y-\hat{y})^2\right)}$$

$$u = \frac{+\Lambda}{2\pi}\frac{(\hat{x}-l-\hat{x})}{\left((\hat{x}-l-\hat{x})^2 + (\hat{y}+h-\hat{y})^2\right)}$$

$$u = \frac{-\Lambda}{2\pi}\frac{l}{\left(l^2 + h^2\right)}$$



So we see that the position of the origin makes no difference to the solution but it does make the evaluations more or less algebraically complex.

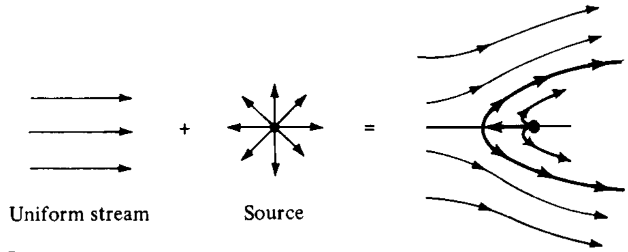
We can apply the same principle in cylindrical-polar coordinates but this is generally not required in the questions you will be asked.

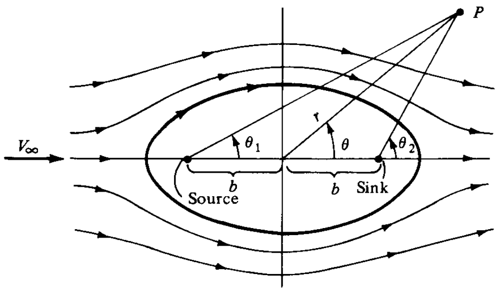
Simple Non-lifting Flows

There are three main examples of combining elementary flows to give simple non-lifting flows covered in most text books

(1) uniform flow +point source

=semi-infinite body

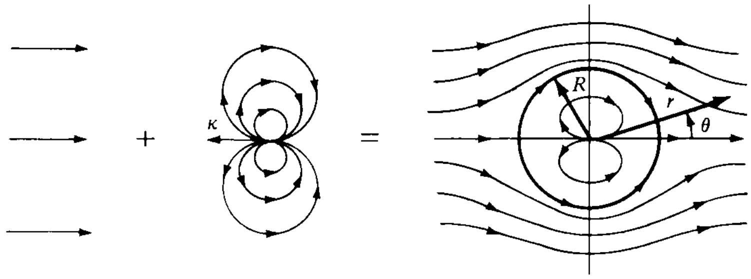




(2) uniform flow + point source and sink = axisymmetric body (eg Rankine oval)

(3) uniform flow + doublet

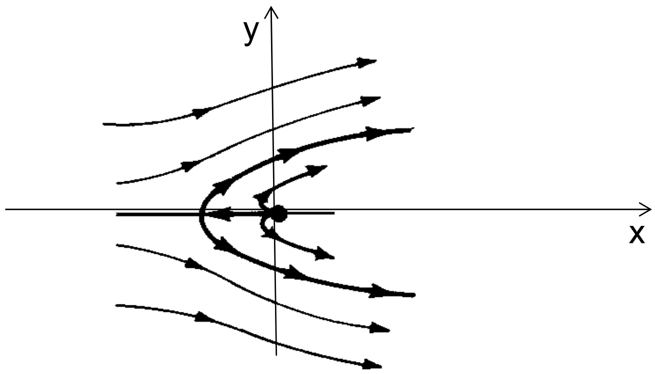
= flow around a cylinder



STEP BY STEP FLOW INVESTIGATION (Source+Freestream)

- All questions involving a source and free stream should be started with the *same initial* steps even if the actual quantities requested differ.

- STEP 1 Make an initial sketch of the situation.



- STEP 2 Write down the stream function of the combined flow (may not be required)

$$\psi = U_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta \qquad \psi = U_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

(polar) (cartesian)

STEP BY STEP FLOW INVESTIGATION (2)

- STEP 3 Obtain the velocity field by differentiation of the stream function, but easier to use given equations.

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta + \frac{\Lambda}{2\pi r} \qquad u = \frac{\partial \psi}{\partial y} = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2}$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta \qquad v = -\frac{\partial \psi}{\partial x} = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2}$$

$$(\text{polar}) \qquad (\text{cartesian})$$

- STEP 4 Find the stagnation points (x_{stag},y_{stag}) i.e. where the velocity is zero. Using a Cartesian coordinate system

$$v = \frac{\Lambda}{2\pi}\frac{y_{stag}}{x_{stag}^2 + y_{stag}^2} = 0$$

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{x_{stag}}{x_{stag}^2 + y_{stag}^2} = 0$$

Could have used polar velocity components, the location of the stagnation point would then be defined by

$$\theta_{stag} = \pi$$

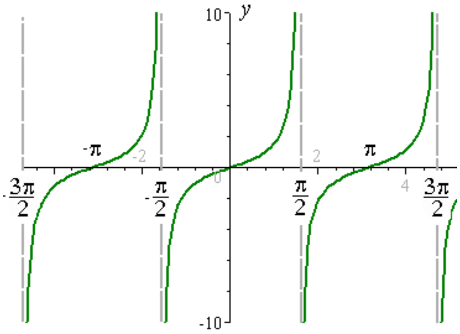
$$r_{stag} = \Lambda/(2\pi U_\infty)$$

STEP BY STEP FLOW INVESTIGATION (3)

- STEP 5 Link the **source strength** to the **ultimate height** of the body.

(a) The stagnation point is on the **dividing stream line**, so use it to find the constant defining this stream line.

$$\psi_{DS_{stag}} = U_\infty 0 + \frac{\Lambda}{2\pi}\underbrace{\tan^{-1}\left(\frac{y_{stag}}{x_{stag}}\right)}_{=\pi,\neq 0} = \frac{\Lambda}{2} \qquad \psi_{DS_{stag}} = U_\infty r \sin\theta_{stag} + \frac{\Lambda}{2\pi}\theta_{stag} = \frac{\Lambda}{2}$$



Equation for dividing streamline therefore.

$$U_\infty y_{DS} + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y_{DS}}{x_{DS}}\right) = \frac{\Lambda}{2}$$

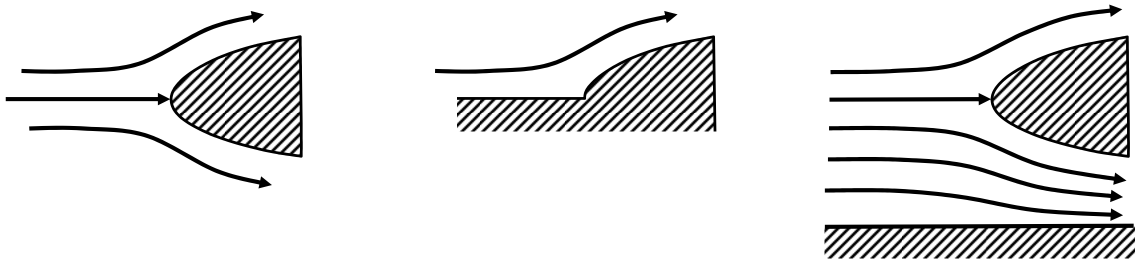
(b) Now consider what happens to y_{DS} as $x_{DS} \rightarrow \infty$

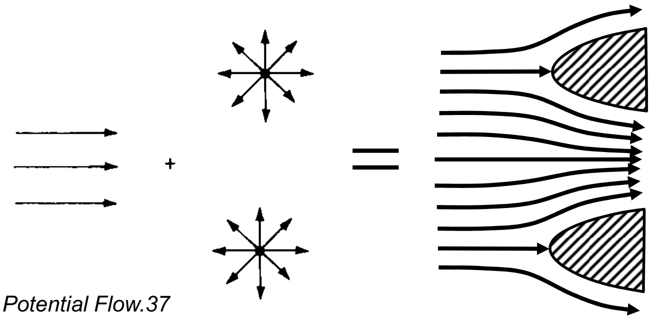
$$\tan^{-1}\left(\frac{y_{DS}}{x_{DS}}\right)\Bigg|_{x_{DS}\rightarrow\infty} \rightarrow 0 \qquad U_\infty y_{DS}\Big|_{x_{DS}\rightarrow\infty} \rightarrow \frac{\Lambda}{2} \qquad y_{DS_{\max}} = \frac{\Lambda}{2U_\infty}$$

$$\text{max - thickness} = \frac{\Lambda}{U_\infty}$$

Related Problems

- This completes the steps common to all problems using a source and a sink. Questions will then have differing extra parts.
- Solid bounding surfaces treated using symmetry. Consider using images.
- For each simple body (bullet, circle, oval) there are 3 related problems:
- Full body Half body Near Surface





Learning Outcomes: "What you should have learnt so far"

- State the boundary conditions on the potential for external aerodynamic flows
- Sketch the streamlines for flows that are modelled as a combination of
  - a uniform stream and a source
  - a uniform stream, a source and a sink
  - a uniform flow and a doublet
- Explain how images can be used to model straight boundaries
- Solve problems for a source+uniform stream
- Solve Problems where the singularity is not at the origin