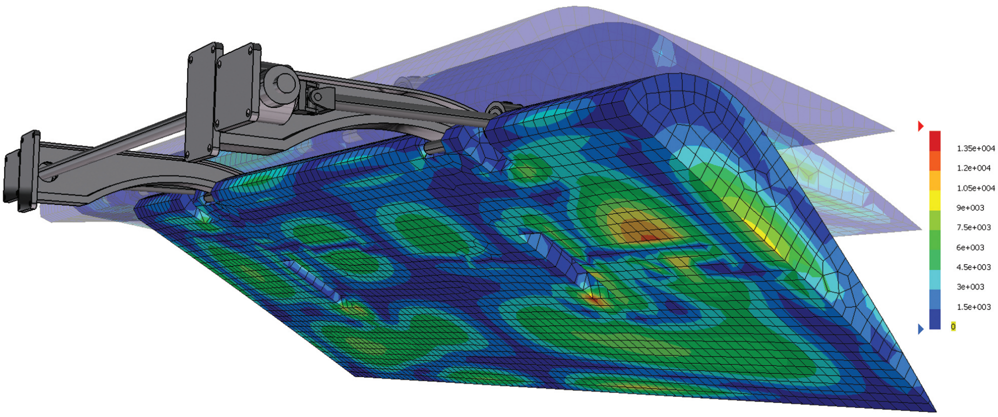
Structures and Materials 2 (AENG21200)

2D Elasticity

2018/2019

Handout 1 – 2D Stress Analysis

In this handout we shall not deal with how the stress varies within an elastic structure under the effect of external forces, *e.g.* bending of beams, stress concentrations around a hole, or aerodynamic loads on a wing flap. Instead, we shall investigate the concept and properties of stress *at a point*.



The objective is to provide a deeper understanding of stress, which in turn will enable you to interpret the results of your structural calculations. The ability to analyse combined stress states will also allow us to formulate theories of failure, which is the subject of Handout 4.

1.1 Stress Definitions

The concept of *stress* was introduced in the theory of elasticity by Cauchy (1789–1857) around 1822[1]. The following definition was later given by Saint-Venant (1797–1886):

> *"The total stress on an infinitesimal element of a plane taken within a deformed elastic body is defined as the resultant of all the actions of the molecules situated on one side of the plane upon molecules on the other side..."*

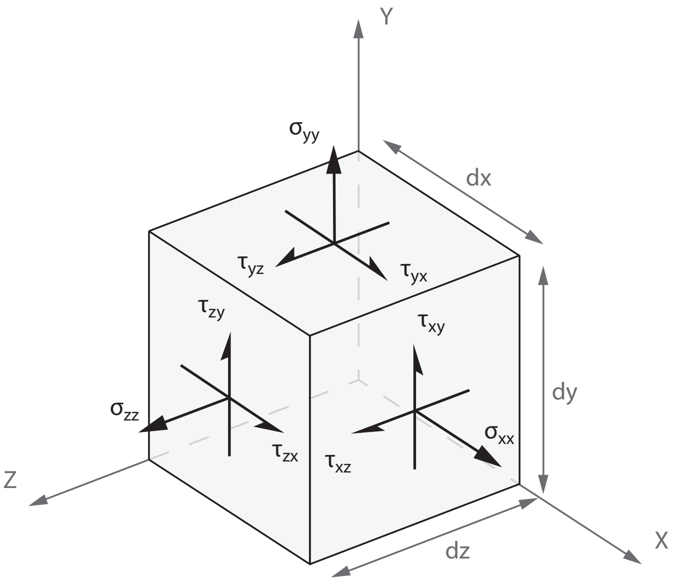
Key words are *infinitesimal element*, which enables us to define stress at a 'point' in an elastic body, and *plane*, which means that there is an orientation/direction associated with stress.

Stress is defined as force per unit area[2] with units of MPa [N/mm^2].

[1] An excellent and fascinating history of the theory of elasticity and structures up to the 1950s is found in "History of Strength of Materials" by Stephen P. Timoshenko (TA405 TIM).

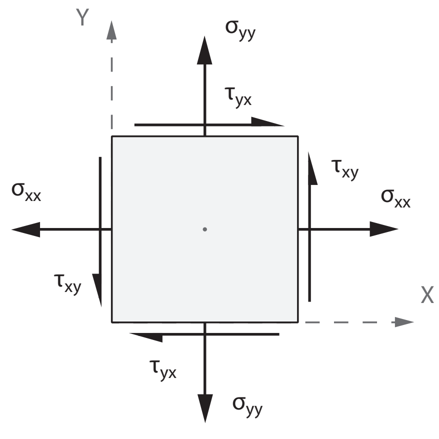
[2] One might rightfully wonder whether stress should be defined as the force per unit area of the material in its *deformed* or *underformed* configuration, since an elastic body will deform under the effect of the applied stress. For instance, in tensile tests of ductile rods the cross-sectional area will change noticeable in the 'necking' region, and the shape of the stress-strain curve will change significantly when substituting the deformed cross-sectional area. However, we shall limit ourselves to linear elasticity, where the strains are small enough for the change in area to be insignificant. Fortunately, the vast majority of engineering problems are accurately described by linear elasticity!

An *infinitesimal* element of material with dimensions $dx \times dy \times dz$ is subjected to direct stresses σ_{ii} and shear stresses τ_{ij}. Direct stresses act normal to a face, whereas shear stresses act parallel.



The first index is the face (defined by the direction of it normal vector) on which the stress acts, and the second the direction of the stress. A shear stress τ_{ij} is taken to be positive if it acts in the positive j direction on the face with positive i; similarly, on a negative face of the element, a shear stress is positive when it acts in the negative direction of an axis.

The shear stresses are not independent, by virtue of *complementary shear*. Consider an infinitesimal 2D element with dimensions $dx \times dy$ (dz is out-of-plane).



Moment equilibrium around the centre of the infinitesimal element:

$$\underbrace{\underbrace{\tau_{xy}\,dy\,dz}_{\text{force}}\,dx}_{\text{CCW couple}} = \tau_{yx}\,dx\,dz\,dy$$

$$\tau_{xy}\,\cancel{dx\,dy\,dz} = \tau_{yx}\,\cancel{dx\,dy\,dz}$$

$$\tau_{xy} = \tau_{yx}$$

Repeating for the remaining axes gives $\tau_{ij} = \tau_{ji}$, reducing the number of independent stress variables.

Thus, a 3D stress state is described by 6 variables, conveniently expressed in a *symmetric* 3×3 matrix:

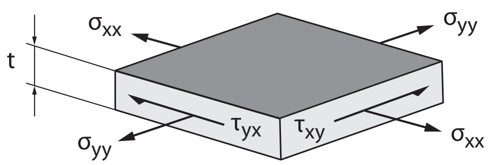
$$\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

The symmetry is a result of the complementary shear, and means that $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^T$. This formulation is known as the **Cauchy stress tensor**, and is used in more mathematical descriptions of the theory of elasticity.

NB: in many structural engineering texts $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ are simply referred to as $(\sigma_x, \sigma_y, \sigma_z)$.

1.2 Plane Stress

Many engineering applications make use of thin-walled structures, for example aircraft fuselages and wing skins, where the wall thickness is much smaller than other dimensions of the structure. This allows us to make a simplifying assumption that the stress state is uniform across the section.



Furthermore, consider a small element cut through the thickness of the surface. Since it is a free surface there can be no out-of-plane stresses, and thus:

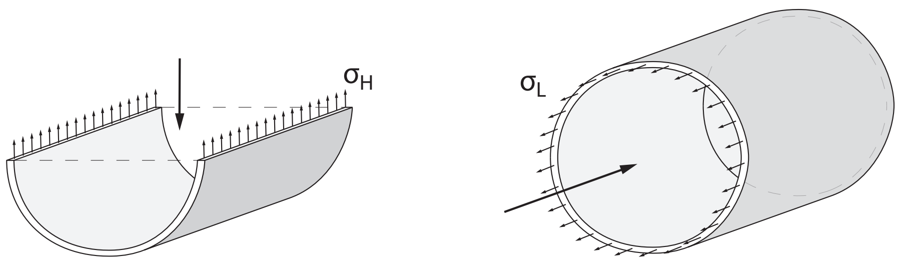
$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

The three-dimensional stress state thus reduces to a two-dimensional stress state known as **plane stress**, with three independent stress variables: σ_{xx}, σ_{yy}, and τ_{xy}. For many engineering applications plane stress is a very useful approximation.

NB: this is different from *plane strain* where out-of-plane deformations are zero $\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$. Plane strain is used to represent conditions deep within a material, *i.e.* inside thick structures, whereas plane stress is used to represent conditions at the surface of a material and in thin-walled structures.

Example 1.1 – Cylindrical Pressure Vessel

In general, determining the 2D stress distribution in a structure can be challenging. For a cylindrical pressure vessel, however, the membrane stresses can straightforwardly be derived from equilibrium considerations. The pressure vessel has radius r, wall thickness t and gauge pressure p.

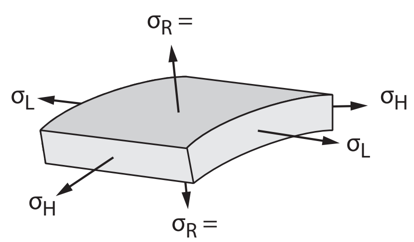


Hoop stress σ_H and longitudinal stress σ_L are given by:

$$\sigma_H =$$

$$\sigma_L =$$

The stress on the inner surface is non-zero ($\sigma_R = -p$).

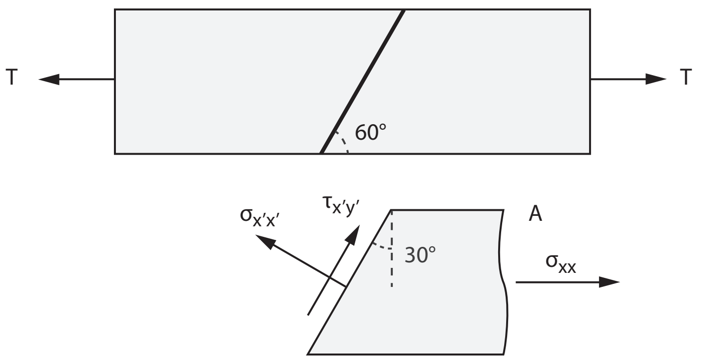


Nonetheless, we can approximate the stress state as plane stress. For thin-walled structures where $r/t \gg 1$ the membrane stresses, σ_H and σ_L are much greater than the internal pressure, and the through-thickness stress is therefore assumed to be negligible.

Example 1.2 – Stresses in a Weld

A thin plate with cross-sectional area A is loaded in tension with T. The plate consists of two welded parts, with the weld inclined at 60° to the direction of loading.

Q: What are the tensile and shear stresses in the weld?



Direct stress $\sigma_{x'x'}$ on the inclined section is at $\theta = 30^\circ$ to σ_{xx}.

Resolving forces perpendicular to the weld:

$$\sigma_{x'x'} \frac{A}{\cos\theta} =$$

$$\sigma_{x'x'} =$$

Resolving forces parallel to the weld:

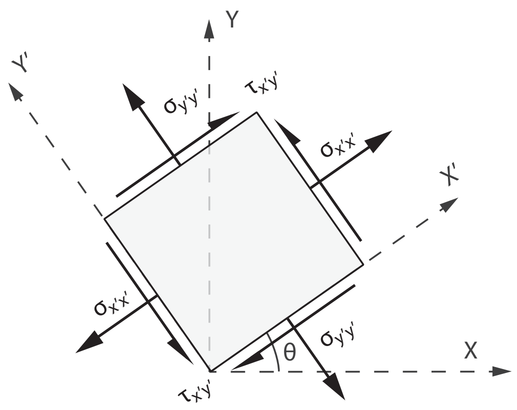
$$\tau_{x'y'} \frac{A}{\cos\theta} +$$

gives:

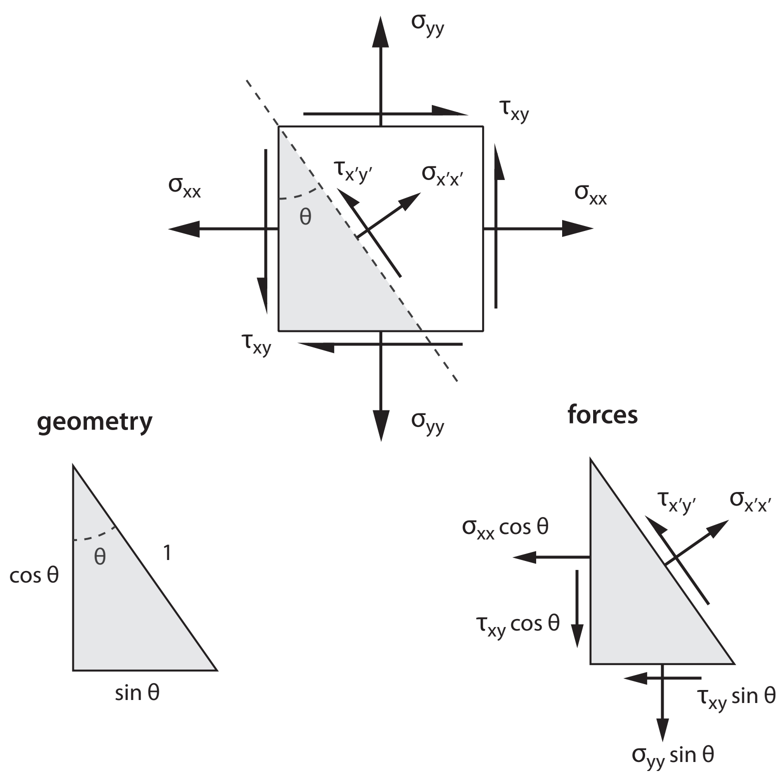
$$\tau_{x'y'} =$$

1.3 Stress Transformations

Up to this point we have defined the stresses in a convenient, but arbitrary, XY coordinate system. What are the stresses in another set of axes, $X'Y'$, at an angle θ to the original coordinate system?



To find the stresses in the new coordinate system, consider the equilibrium of an infinitesimal element (with unit depth) cut at an angle θ (measured CCW from the Y-axis) on which a normal stress $\sigma_{x'x'}$ and shear stress $\tau_{x'y'}$ act. Note that we are looking at *force* equilibrium, so the stresses must be multiplied with the area over which they act.



Resolving forces perpendicular and parallel to the cut plane:

$$\sigma_{x'x'} = \quad (\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta)\cos\theta + (\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)\sin\theta$$

$$\tau_{x'y'} = -(\sigma_{xx}\cos\theta + \tau_{xy}\sin\theta)\sin\theta + (\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)\cos\theta$$

Reshuffle and combine into a transformation matrix $\boldsymbol{T}$:

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{1.1}$$

where $\sigma_{y'y'}$ is found using $\theta_{y'} = \theta_{x'} + \pi/2$.

These **stress transformation** equations will enable you to calculate the stresses in any given direction, and form the core of 2D stress analysis.

The stress transformation equations can be rewritten using the standard trigonometric double-angle relationships to find an alternative formulation:

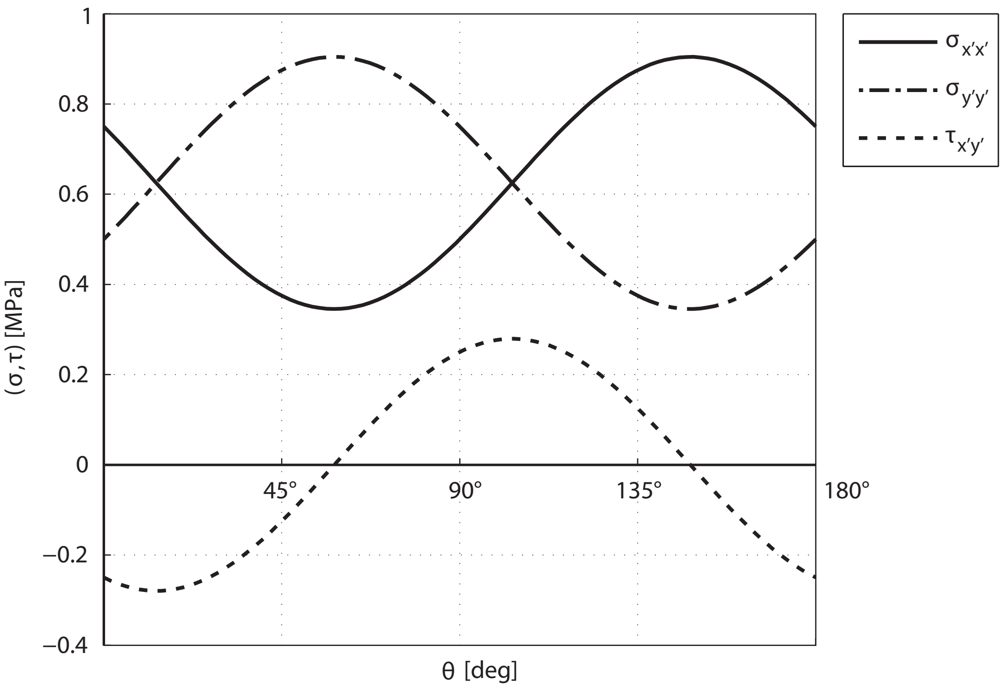
$$\sigma_{x'x'} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \tag{1.2}$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta \tag{1.3}$$

which will turn out to be particularly useful later in this handout.

1.4 Properties of Stress

The stress transformation equations enable us to calculate the stresses in any direction. As we rotate through different angles $\theta \in [0, \pi]$ the direct stresses $\sigma_{x'x'}$ and $\sigma_{y'y'}$ will vary periodically.



Can we make general observations that will apply to any stress state?

Maximum/Minimum Direct Stress

To find directions in which the direct stresses are *maximum* or *minimum*, differentiate the stress transformation equation for $\sigma_{x'x'}$ (Equation 1.2) with respect to 2θ:

$$\frac{d\sigma_{x'x'}}{d\,(2\theta)} = -\frac{1}{2}\,(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

to find

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{1.4}$$

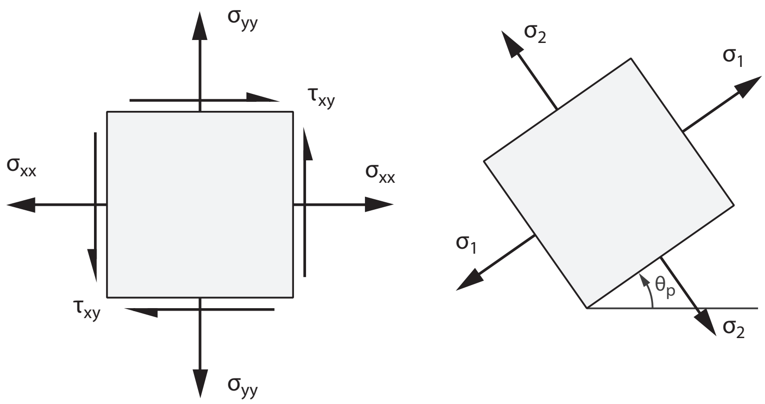
with two solutions: θ_p and $\theta_p + \pi/2$.

These angles describe two *perpendicular* planes where the direct stress is either maximum or minimum; these are known as the **principal directions**. The **principal stresses** are found by substituting θ_p into the stress transformation equation (Equation 1.1).

To find the magnitude of the shear stress in those directions, substitute $\sin 2\theta = \tan 2\theta\,\cos 2\theta$ into Equation 1.3, and use the result for the principal directions:

$$\begin{aligned}
\tau_{x'y'} &= -\frac{1}{2}\,(\sigma_{xx} - \sigma_{yy})\sin 2\theta_p + \tau_{xy}\cos 2\theta_p \\
&= -\frac{1}{2}\,(\sigma_{xx} - \sigma_{yy})\,\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\cos 2\theta_p + \tau_{xy}\cos 2\theta_p \\
&= -\tau_{xy}\cos 2\theta_p + \tau_{xy}\cos 2\theta_p \\
&= 0
\end{aligned}$$

In other words, the planes where the direct stress $\sigma_{x'x'}$ is maximum or minimum carry no shear stress!



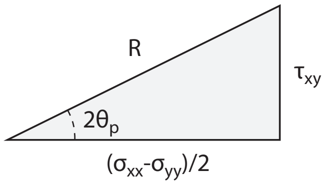
This is a very powerful result, as it holds true for any[3] state of stress!

[3] Fascinatingly, this result also extends to any three-dimensional state of stress: it is always possible to find three orthogonal directions in which the shear stresses vanish — these are the directions of principal stress. In fact, the principal stresses and the principal directions can be found as the eigenvalues and eigenvectors of the Cauchy stress tensor. Recall that a symmetric matrix will have real eigenvalues, and can be diagonalised by pre- and post-multiplying with a transformation matrix $\boldsymbol{U}$:

$$\boldsymbol{U}'\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}\boldsymbol{U} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

where $\boldsymbol{U} = \begin{bmatrix} \hat{\boldsymbol{u}}_1 & \hat{\boldsymbol{u}}_2 & \hat{\boldsymbol{u}}_3 \end{bmatrix}$ with $\hat{\boldsymbol{u}}_i$ the normalized eigenvectors that define the planes of principal stress.

From Equation 1.4 we can derive further relationships.



The hypotenuse is expressed as:

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

and the other geometric relationships are:

$$\frac{\sigma_{xx} - \sigma_{yy}}{2} = R\,\cos 2\theta_p$$

$$\tau_{xy} = R\,\sin 2\theta_p$$

Substituting these into Equation 1.2 yields:

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{1.5}$$

This allows us to calculate both principal stresses directly, but does not let us know which principal stress acts on which principal plane.

Mean Direct Stress

From the stress transformation equations (Equation 1.1)

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

it is seen that the mean direct stress:

$$\frac{\sigma_{x'x'} + \sigma_{y'y'}}{2} = \frac{\sigma_{xx}\left(\cos^2\theta + \sin^2\theta\right) + \sigma_{yy}\left(\cos^2\theta + \sin^2\theta\right)}{2}$$

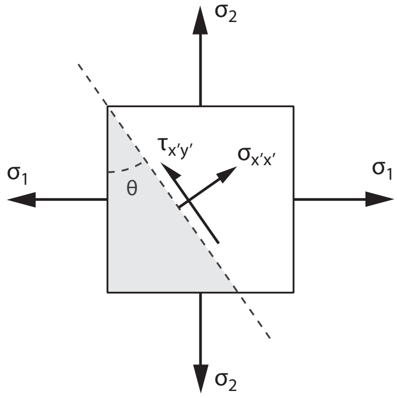
$$= \frac{\sigma_{xx} + \sigma_{yy}}{2} = C$$

is *constant* for any transformed coordinate system.

Maximum/Minimum Shear Stress

Taking the derivative of the transformation equation for $\tau_{x'y'}$ (Equation 1.3) with respect to 2θ, it is found that the directions of maximum and minimum shear stress are also perpendicular.

Next, consider an element under principal stresses σ_1 and σ_2 (recall, $\tau_{12} = 0$).



Equation 1.3 gives the shear stress at an angle θ to the principal axes:

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$$

which has a maximum/minimum

$$\tau_{\text{max,min}} = \pm\frac{1}{2}(\sigma_1 - \sigma_2) \tag{1.6}$$

at $\theta = \pi/4$. The planes of maximum/minimum shear stress are at 45° to the planes of principal stress.

$$\theta_s = \theta_p \pm \frac{\pi}{4}$$

Using Equation 1.5 and Equation 1.6 we can write:

$$\tau_{\text{max,min}} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{1.7}$$

This completes the full toolset to calculate the principal stresses, maximum shear stress, and principal directions for any given state of stress.

Example 1.3 – Stress Calculations

A finite element calculation of a section of an aircraft fuselage has given $\sigma_{xx} = -75$ MPa, $\sigma_{yy} = 210$ MPa and $\tau_{xy} = -200$ MPa. Determine (a) the principal stresses, (b) the maximum shear stresses and associated normal stresses. Sketch a correctly oriented infinitesimal element for both load cases.

(a) The principal stresses are given by

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

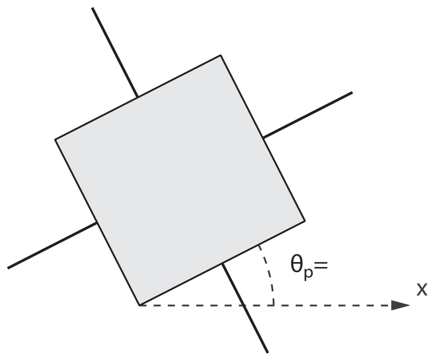
as $\sigma_1 = 313$ MPa and $\sigma_2 = -178$MPa.

The principal directions are found using:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

which gives $\theta_p = 27.3°$ and $\theta_p = 27.3° + 90° = 117.3°$

In order to find which principal stress corresponds to which principal direction, substitute θ_p into Equation 1.1.

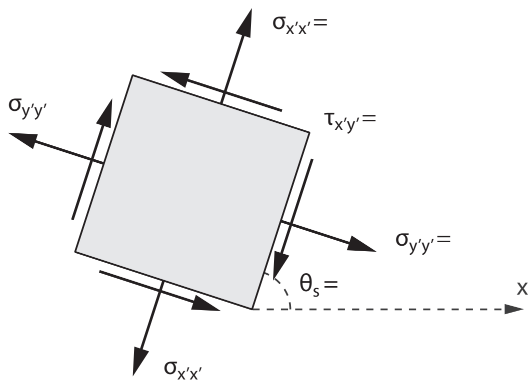


(b) The maximum shear stress is given by:

$$\tau_{\text{max,min}} = \pm\frac{\sigma_1 - \sigma_2}{2} =$$

These planes will be at 45° to the directions of principal stress, so $\theta_s = \theta_p + 45°$ and $\theta_s = \theta_p - 45°$.

To find which direction corresponds to maximum/minimum shear stress, and to find the associated direct stresses, substitute θ_s into Equation 1.1.



The direct stresses on planes with maximum/minimum shear stress are equal to $(\sigma_{xx} + \sigma_{yy})/2$.

Example 1.4 – Pure Shear

Consider a plate subjected to pure shear, where $\sigma_{xx} = \sigma_{yy} = 0$. Using Equation 1.4,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

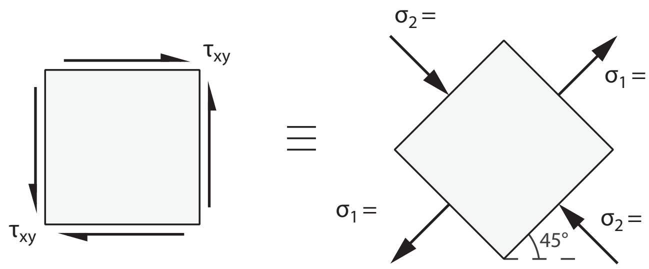
we find the principal directions as $45°$ and $135°$. Substituting into Equation 1.2,

$$\sigma_{x'x'} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta$$

we find the principal stresses as:

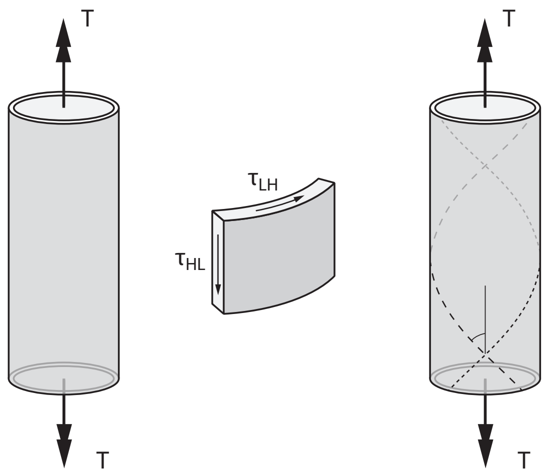
$\sigma_1 =$ at $45°$

$\sigma_2 =$ at $135°$



Thus a case of pure shear gives rise to direct tensile and compressive stresses of equal magnitude to that of the applied shear at $\pm 45°$ to the direction of shear.

A situation where we might encounter pure shear, is a thin-walled cylindrical shaft in torsion. In that case the principal stress directions can be interpreted as forming a helix at 45° to the longitudinal axis.



1.5 Mohr's Circle

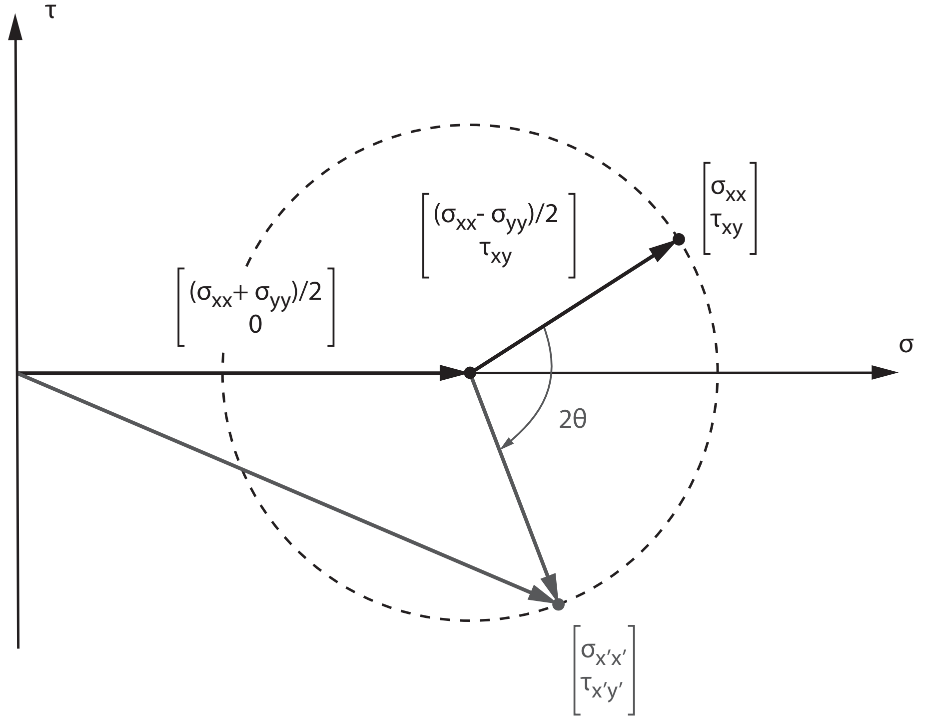
Mohr's circle is a very useful device for visualising stress, and performing quick calculations. The method was introduced by civil engineer Otto Mohr (1835–1918) in 1882 as a graphical representation of the state of stress occurring in a material.

Rewriting transformation Equations 1.2 and 1.3 into a matrix formulation:

$$\begin{bmatrix} \sigma_{x'x'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} (\sigma_{xx} + \sigma_{yy})/2 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} (\sigma_{xx} - \sigma_{yy})/2 \\ \tau_{xy} \end{bmatrix}$$

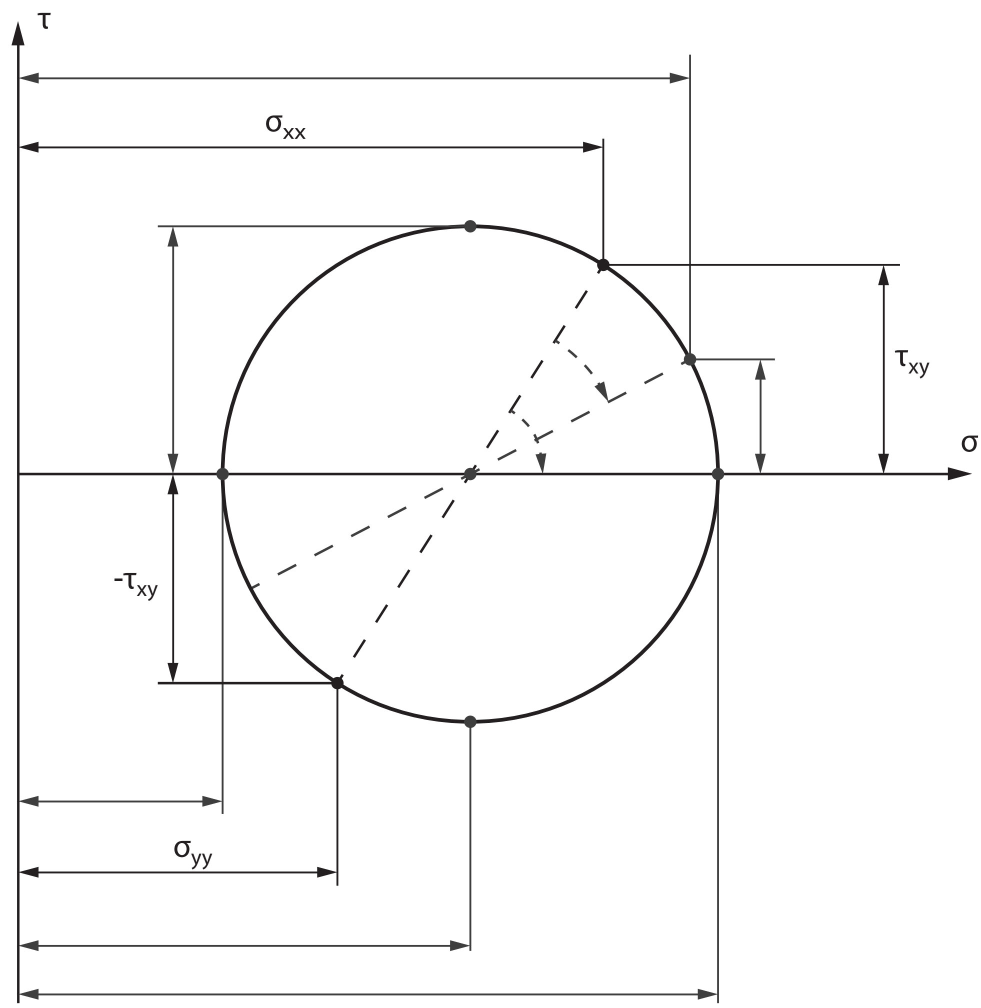
Here you will recognise a rotation matrix for a clockwise rotation by an angle 2θ.

Plotting the transformed stresses for $\theta \in [0, \pi]$ on a set of axes with $\sigma_{x'x'}$ on the abscissa and τ_{xy} on the ordinate, traces out Mohr's circle for stress:



NB: please note the sign convention used for Mohr's circle: a θ **CCW** rotation of the coordinate system is represented by a 2θ **CW** rotation on Mohr's circle! You may find an alternative sign conventions in certain textbooks, where the shear stress is plotted negative upwards to let the rotation of the vector to be counter-clockwise.

The key thing to remember is that Mohr's circle is nothing more than a graphical representation of the stress transformation equations, but one that can provide new insights and improve intuition of the state of stress.



The midpoint of Mohr's circle:

$$C = \frac{\sigma_{xx} + \sigma_{yy}}{2} \tag{1.8}$$

represents the mean value of the normal stresses, and is *invariant* to the choice of coordinate system.

The radius of Mohr's circle:

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{1.9}$$

is equal to the maximum/minimum shear stress:

$$\tau_{\mathsf{max,min}} = \pm R$$

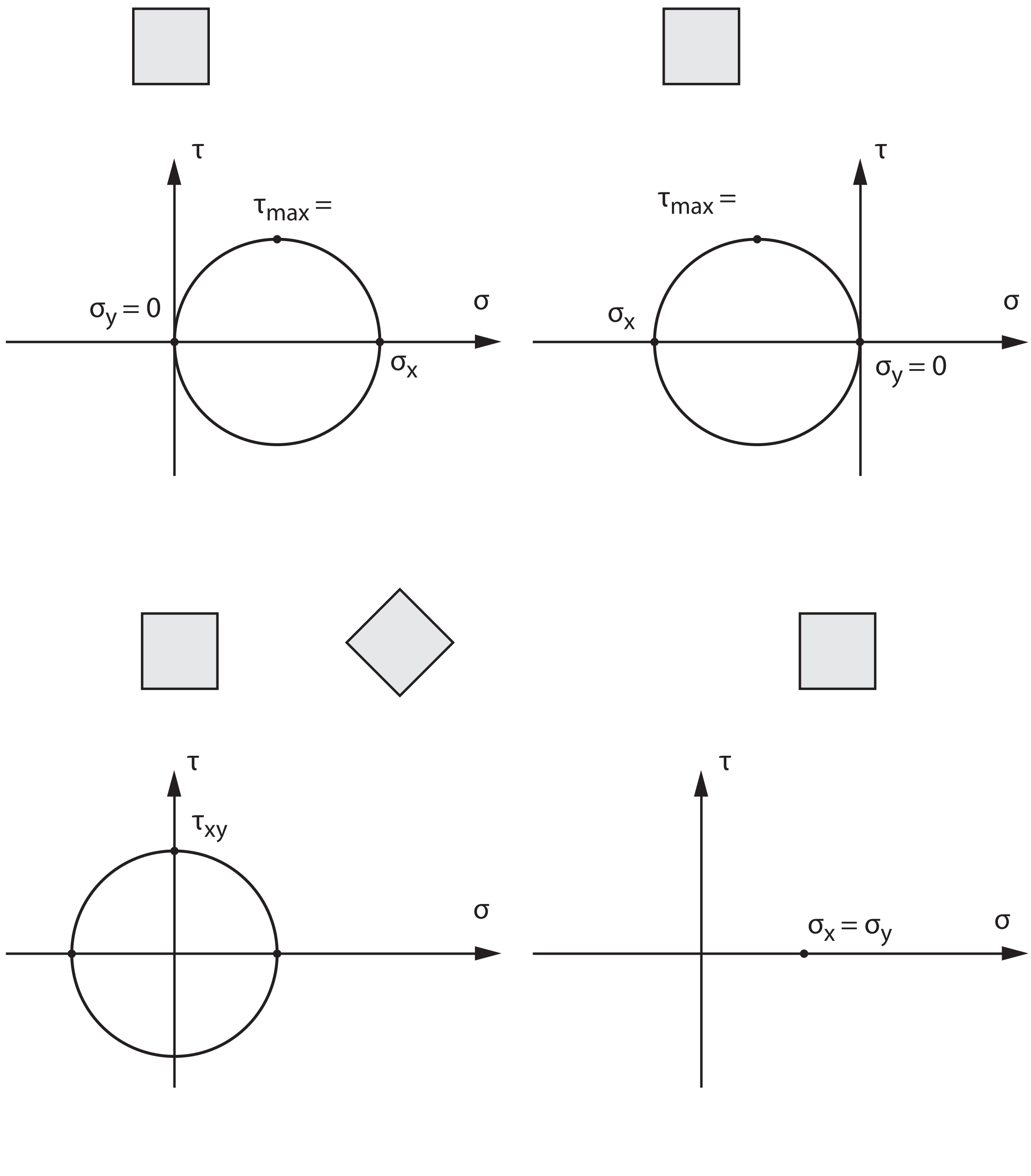
The points where Mohr's circle crosses the horizontal axis represent the principal stresses:

$$\sigma_{1,2} = C \pm R$$

The principal directions can then be found from geometry.

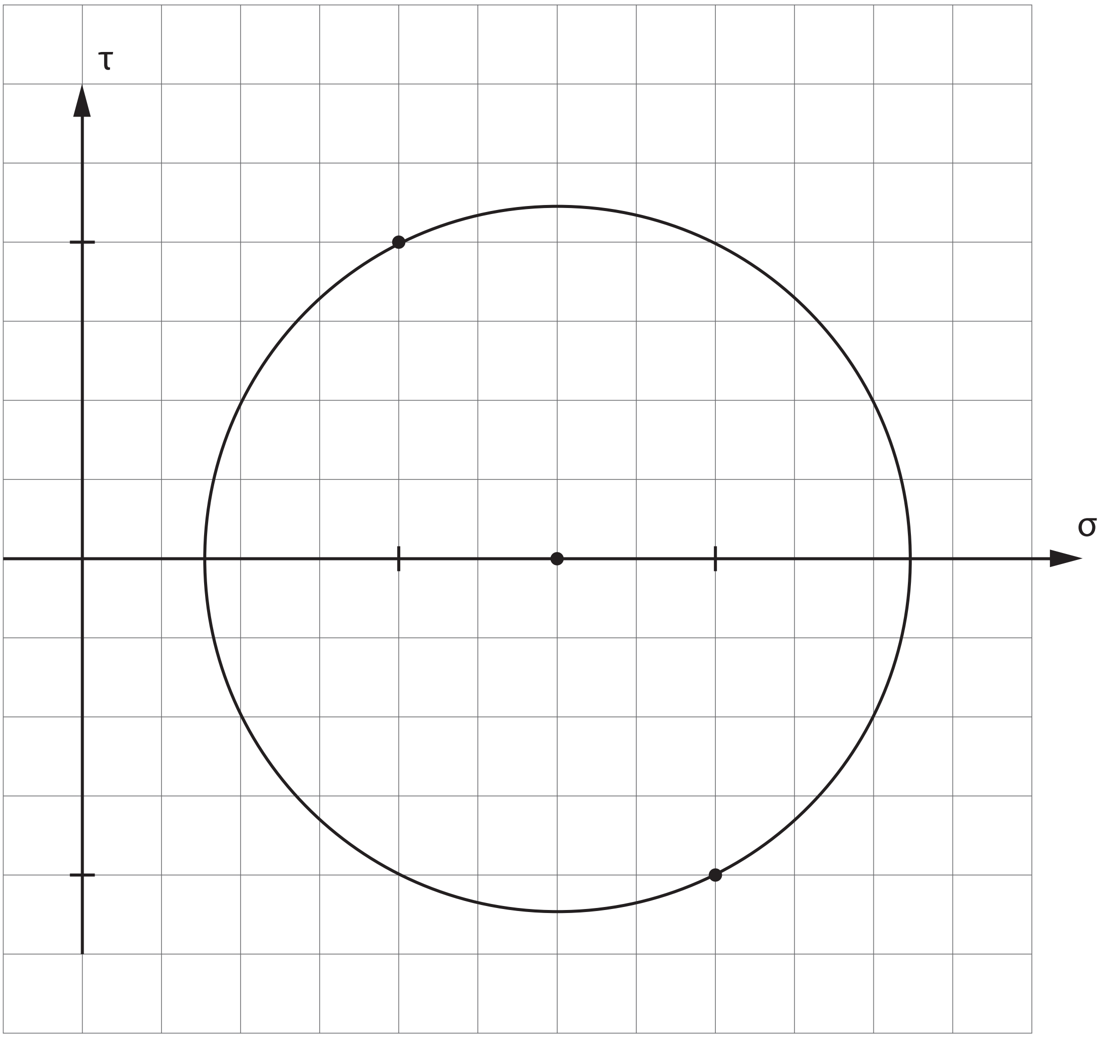
Example 1.5 – Stress States

For the Mohr's circles shown, draw the corresponding stress states.



Example 1.6 – Constructing Mohr's Circle

For a stress state $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = 200$ MPa, and $\tau_{xy} = 100$ MPa, find the principal stress state using Mohr's circle.



Recipe for constructing Mohr's circle:

1. draw a coordinate system with σ on the horizontal and τ on the vertical axis;
2. plot points (σ_{xx}, τ_{xy}) and $(\sigma_{yy}, -\tau_{xy})$;
3. draw a straight line connecting these points to find the centre of Mohr's circle where the line crosses the horizontal-axis: $C = (\sigma_{xx} + \sigma_{yy})/2$
4. draw Mohr's circle through the points, with radius $R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$
5. the points where the circle crosses the horizontal axis are the principal stresses;
6. find the angle θ_p for maximum/minimum principal stress with respect to the x-axis

1.6 Summary

In this handout we have discussed the concept and definitions of stress, and focused on plane stress: σ_{xx}, σ_{yy} and τ_{xy}. This is a useful engineering approximation for the stress state in thin-walled structures, or at the surface of thick structure.

By considering the equilibrium of an infinitesimal element, we derived the stress transformation equations, which enable us to find the magnitudes of the stress in different directions. Mohr's circle is a valuable tool to visualise the stress transformation equations, and help perform calculations.

Studying the stress transformation equations revealed the concept of principal stresses: for any state of 2D stress we can find two perpendicular directions where the direct stresses are at maximum and minimum. What is more, the shear stress is zero on those planes.

Revision Objectives Handout 1:

- be familiar with stress notations, and the concept of 2D plane stress: σ_{xx}, σ_{yy}, τ_{xy}
- derive the stress transformation equations

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

- calculate stresses in a Cartesian coordinate system which has undergone a rotation from the original;
- explain the concept of principal stresses and directions;
- recall and apply the equations for principal stresses, maximum shear stress and directions;

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\text{max,min}} = \pm\frac{\sigma_1 - \sigma_2}{2} = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

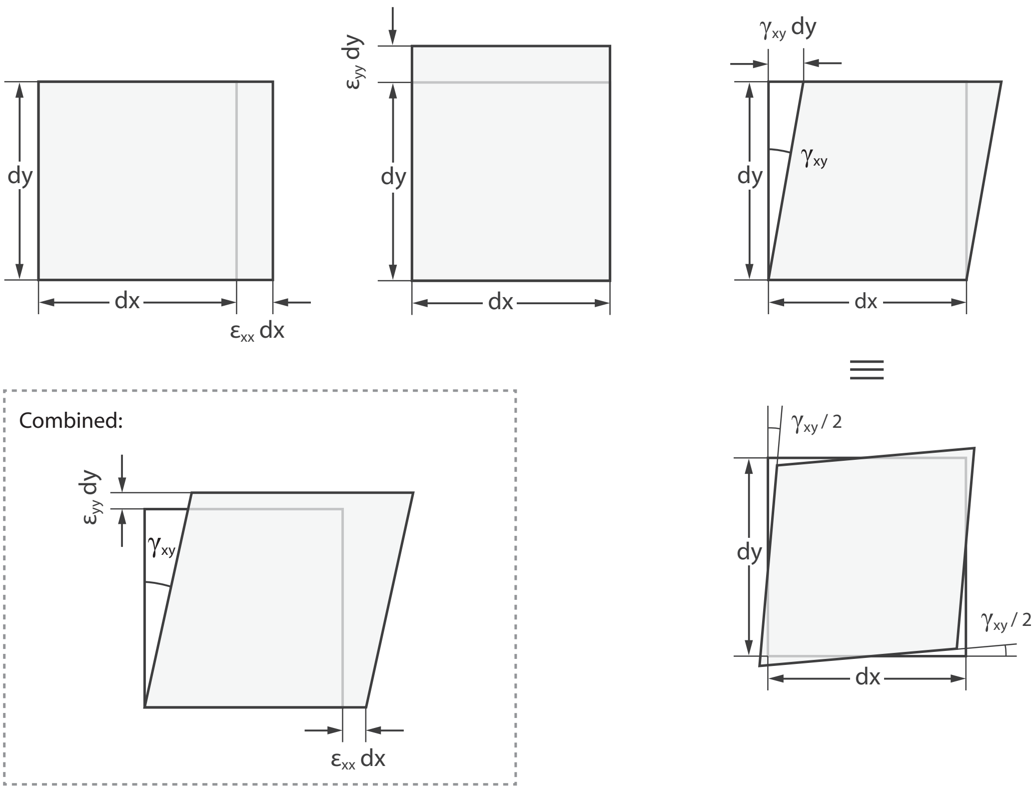
- draw a Mohr's circle for a 2D state of stress, and use it to support calculations using stress transformation quations;

Handout 2 – 2D Strain and Strain Measurements

This handout introduces the properties of two-dimensional strain. Deriving the strain transformation equations leads to a Mohr's circle of strain. Experimental strain measurement techniques will be described, focusing on the use of strain gauges.

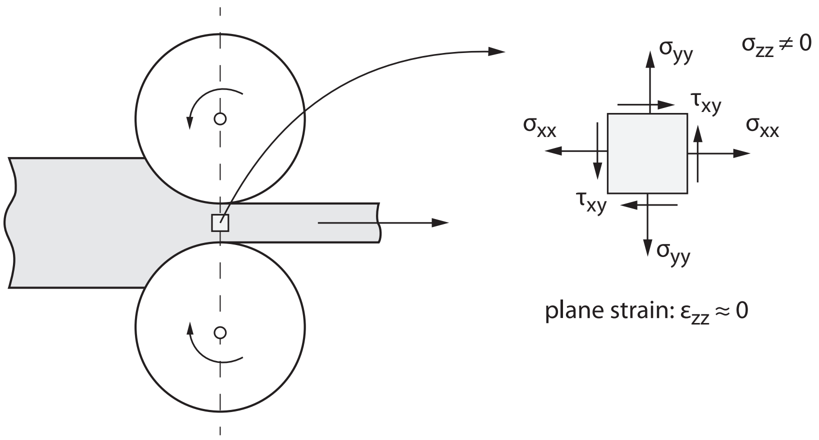
2.1 Plane Strain and Plane Stress

Two-dimensional deformations are composed of direct strains ε_{xx}, ε_{yy} and a shear strain γ_{xy}. The shear strain represents the change in angle between two orthogonal directions; note that a shear deformation is volume preserving. If the only deformations are those in the XY plane, $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$, the element is considered to be in *plane strain*.



The similarity between the definitions of *plane strain* and *plane stress* (where σ_{zz}, τ_{xz} and τ_{yz} are zero) invites the misunderstanding these can occur simultaneously. In fact, for an element in plane stress the through-thickness strain ε_{zz} will in general be non-zero due to Poisson's ratio effects, and for an element in plane strain the out-of-plane stress will generally be non-zero to enforce the plane strain condition.

These two assumptions are used for very different applications: while plane stress is used for thin-walled structures, plane strain is used to describe the stress deep inside an elastic body (where z-dimensions are much greater than x and y). One application of plane strain would be in forming processes, such as rolling, drawing and forging, where flow in a particular direction is constrained by the geometry of the machinery.



Despite the fundamental differences in applications, the stress and strain transformations for plane stress and plane strain are identical, as the out-of-plane direct stresses and strains do not affect the in-plane transformation equations.

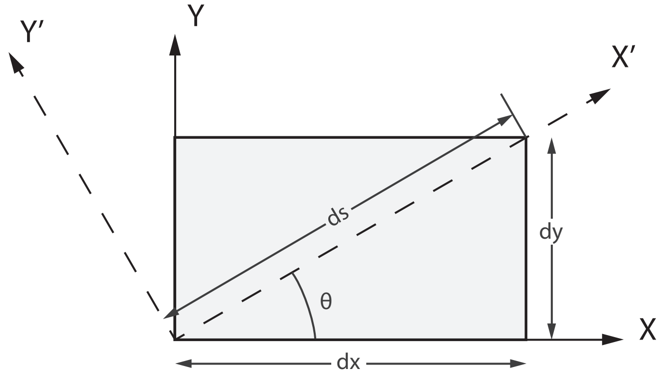
Strain Tensor As with stresses, we can define a strain tensor $\bar{\varepsilon}$:

$$\bar{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

Here we use the *mathematical* shear strain $\varepsilon_{xy} = \gamma_{xy}/2$ rather than the *engineering* shear strain γ_{xy}. This allows us to exploit the parallels with the Cauchy stress tensor.

2.2 Strain Transformation – Engineering Approach

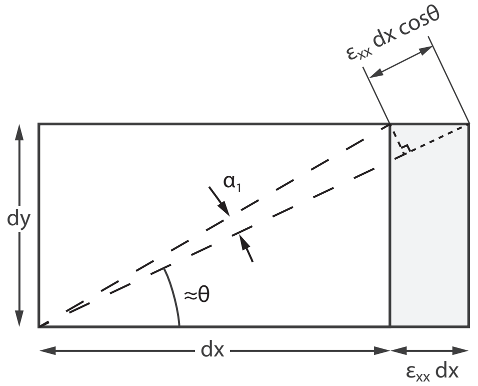
As with the stress calculations, we are interested in strains in directions other than the XY axes. Whereas the stress transformation equations followed from *equilibrium* considerations, for strain we shall derive these from *compatibility* of the deformed configuration. Let us consider an infinitesimal element with dimensions $dx \times dy$, with its diagonal conveniently aligned with the X' axis.

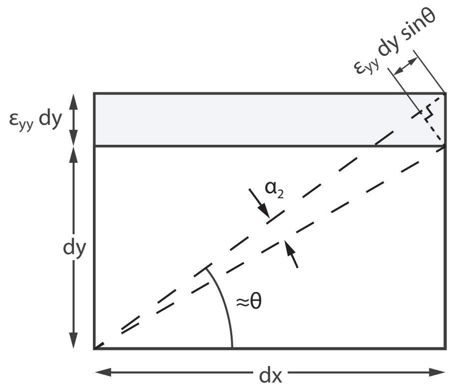


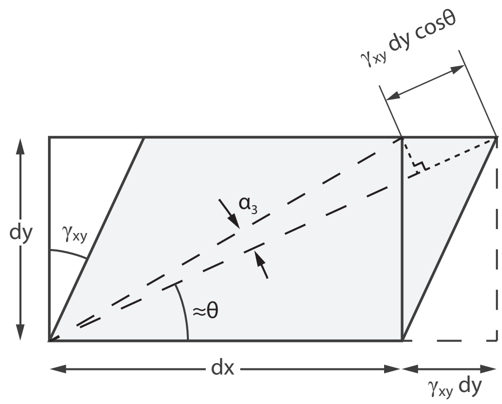
By superimposing three plane strain deformations (ε_{xx}, ε_{yy} and γ_{xy}) we can formulate expressions for the direct strain $\varepsilon_{x'x'}$ (*i.e.* change in length of the diagonal) and shear strain $\gamma_{x'y'}$ (*i.e.* change in orientation of diagonal).

2.2.1 Strain Transformation: Direct Strain

First we consider the direct strain $\varepsilon_{x'x'}$ at an angle θ to the XY axes; this is found by calculating the change in length of the diagonal of our infinitesimal element. Throughout the analysis we assume that the change in orientation of the diagonal, α_i, is very small and therefore $\theta + \alpha_i \approx \theta$.







The total increase in length of the diagonal is given as:

$$\Delta d = \varepsilon_{xx} dx \cos\theta + \varepsilon_{yy} dy \sin\theta + \gamma_{xy} dy \cos\theta$$

and the direct strain of the diagonal is therefore:

$$\begin{aligned}\varepsilon_{x'x'} = \frac{\Delta d}{ds} &= \varepsilon_{xx}\frac{dx}{ds}\cos\theta + \varepsilon_{yy}\frac{dy}{ds}\sin\theta + \gamma_{xy}\frac{dy}{ds}\cos\theta \\ &= \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \end{aligned} \tag{2.1}$$

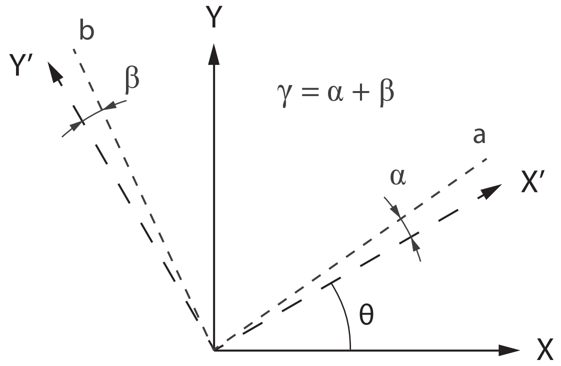
where $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$ (see undeformed geometry).

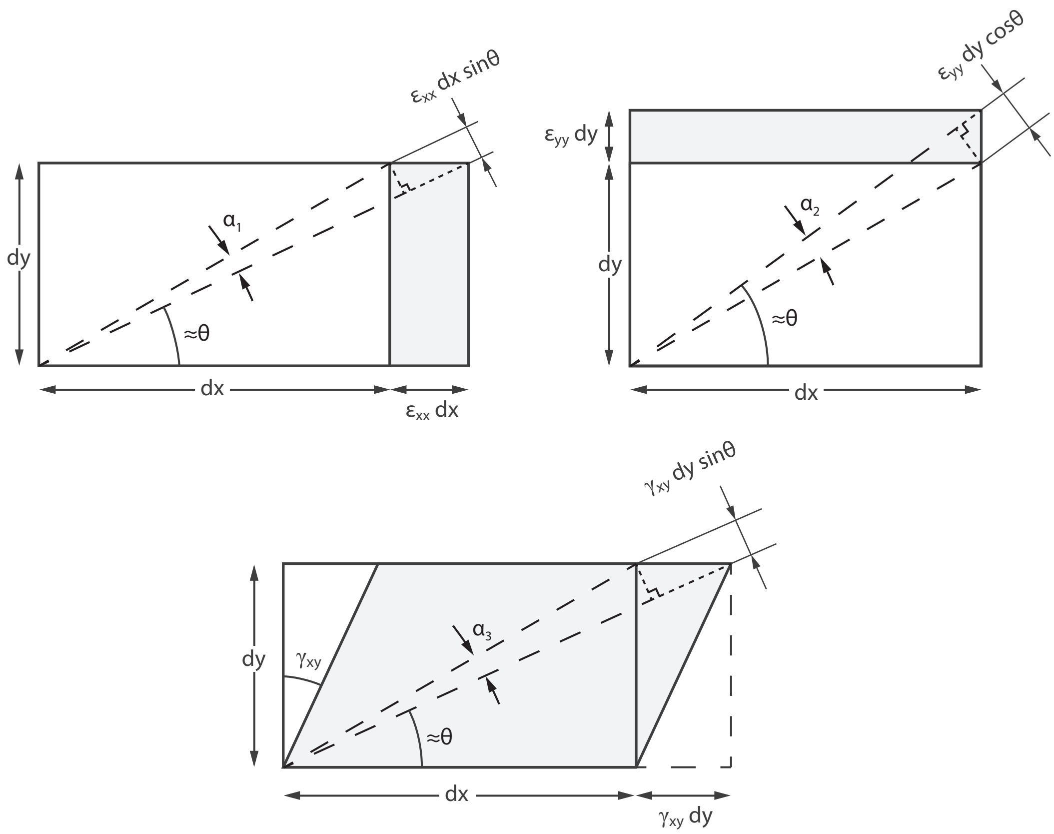
For the strain in Y' direction, we subsitute $\theta_{Y'} = \theta + \pi/2$ to find:

$$\varepsilon_{y'y'} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta \tag{2.2}$$

2.2.2 Strain Transformation: Shear Strain

Next, we consider the change in orientation of the diagonal of the infinitesimal element. The shear strain $\gamma_{x'y'}$ is equal to the change in angle between lines that were initially along the X' and Y' axes.





In our analysis we assume that the changes in angle α_i are small, and therefore $\tan\alpha_i \simeq \alpha_i$. This gives the total change in angle α as:

$$
\begin{aligned}
\alpha &= -\alpha_1 + \alpha_2 - \alpha_3 \\
&= -\frac{\varepsilon_{xx} dx \sin\theta}{ds} + \frac{\varepsilon_{yy} dy \cos\theta}{ds} - \frac{\gamma_{xy} dy \sin\theta}{ds} \\
&= -\varepsilon_{xx}\cos\theta\sin\theta + \varepsilon_{yy}\sin\theta\cos\theta - \gamma_{xy}\sin^2\theta
\end{aligned}
$$

again using $dx/ds = \cos\theta$ and $dy/ds = \sin\theta$ from the undeformed geometry. The rotation β of the Y' axis is then found by substituting $\theta + \pi/2$, and correcting for the counter-clockwise rotation (β is taken to be positive in clockwise direction) to give:

$$\beta = -\varepsilon_{xx}\cos\theta\sin\theta + \varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}\cos^2\theta$$

and with $\gamma_{x'y'} = \alpha + \beta$ we find:

$$\gamma_{x'y'} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \tag{2.3}$$

2.2.3 Combined Strain Transformation

We can now combine our strain transformation equations into a single transformation matrix $\boldsymbol{T}$:

$$\begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \gamma_{x'y'}/2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy}/2 \end{bmatrix} \tag{2.4}$$

where we have used $\gamma_{xy}/2$. Note that this is the same[4] transformation matrix as derived for plane stress even though they have been derived using completely different methods, *i.e.* geometry and equilibrium. It is also worth emphasising that these transformation equations are independent of the material properties!

2.2.4 Mohr's Circle of Strain

The analogy between the transformation matrices for stress and strain naturally leads to corresponding results, and we can find principal strains, maximum shear strains, and formulate a Mohr's circle for strain.

Principal directions:

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{2.5}$$

Principal strains:

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{2.6}$$

Maximum shear strains:

$$\gamma_{\text{max,min}} = \pm\sqrt{\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2 + \gamma_{xy}^2} = \varepsilon_1 - \varepsilon_2 \tag{2.7}$$

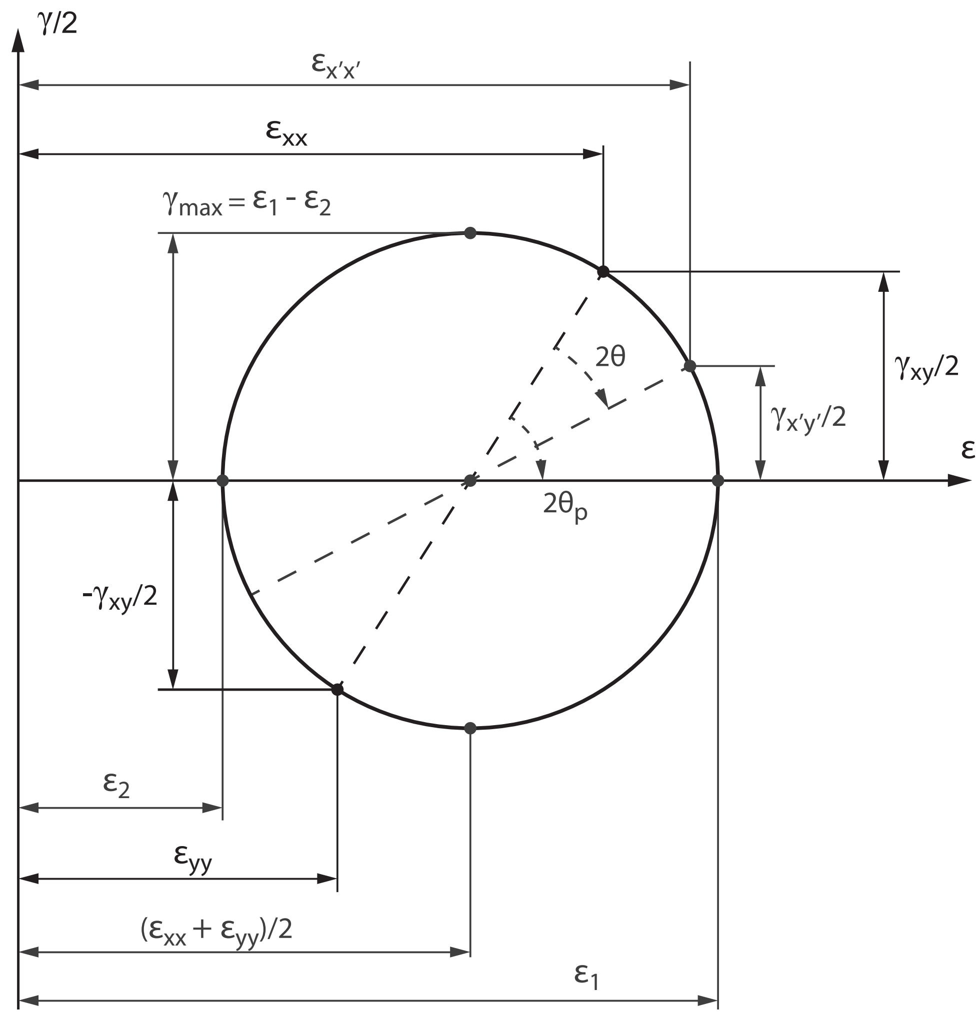
The above equations were derived by substituting ε_{xx}, ε_{yy} and $\gamma_{xy}/2$ for σ_{xx}, σ_{yy} and τ_{xy}, respectively, in Equations 1.4, 1.5 and 1.7.

[4] Looking back to your notes on beam deflections, you will find that this is the same coordinate transformation equation as for the second moments of area I_{xx}, I_{yy} and I_{xy}:

$$\begin{bmatrix} I_{x'x'} \\ I_{y'y'} \\ -I_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} I_{xx} \\ I_{yy} \\ -I_{xy} \end{bmatrix}$$

It turns out that stress, strain and second moment of area are all examples of second-rank tensors, and therefore share the same properties such as principal directions. Subtleties in sign conventions exist, so take care when revising from other sources, such as textbooks.

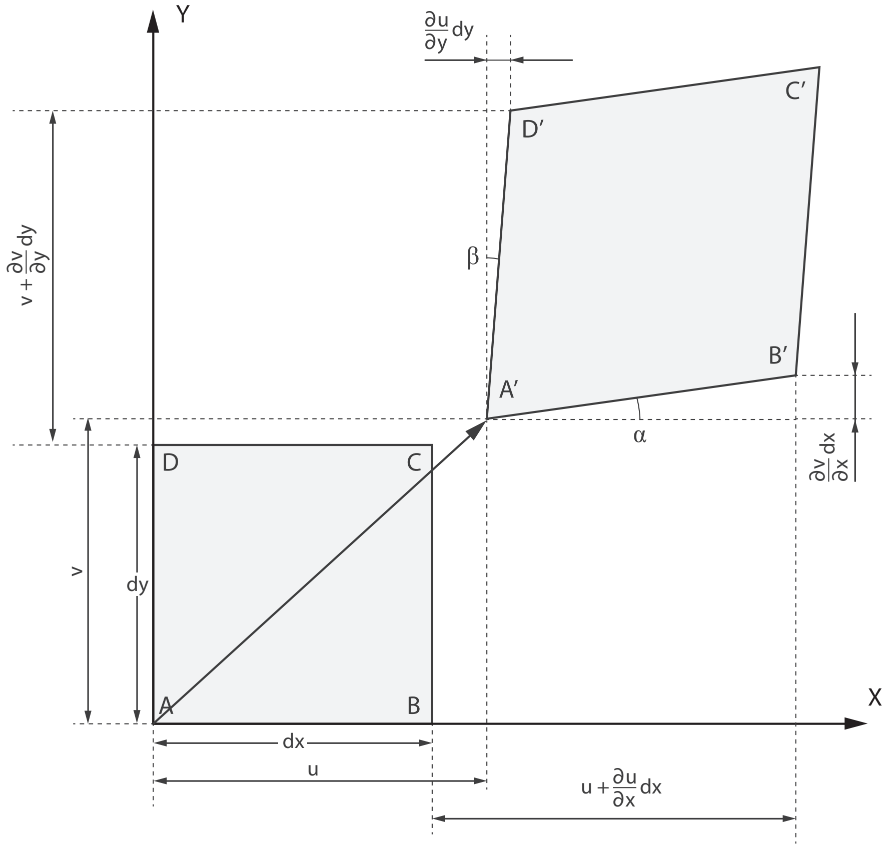
Mohr's circle for strain:



Please note we have maintained the sign convention we used for Mohr's circle for stress, *i.e.* a CCW rotation of the coordinate system is represented by a CW rotation on Mohr's circle. Some textbooks use a convention where $\gamma_{xy}/2$ is plotted positive downward, to ensure all rotations are CCW.

2.3 Strain Transformation – Mathematical Approach

An alternative method to deriving the strain transformation equations, is to approach strain from a mathematical point of view. Let $u(x, y)$ and $v(x, y)$ describe the displacement field of points (x, y) in the deformed body. The deformation of an infinitesimal element ABCD can then be described in terms of u and v.



The displaced position of corner A is given by (u, v), and the position of the other points can then be described using a Taylor expansion. For example, for point B:

$$u_B = u + \frac{\partial u}{\partial x}dx + \frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(dx)^2 + \dots$$

Taking only the linear terms, the direct strains are then found as:

$$\varepsilon_{xx} = \frac{\left(u + \frac{\partial u}{\partial x}dx\right) - u}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\left(v + \frac{\partial v}{\partial y}dy\right) - v}{dy} = \frac{\partial v}{\partial y}$$

The shear strain $\gamma_{xy} = \alpha + \beta$, which can be found as:

$$\alpha = \frac{\partial v}{\partial x} \qquad \beta = \frac{\partial u}{\partial y} \qquad \therefore \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

where use is made of small-strain approximations ($\tan\alpha \approx \alpha$, $\partial u/\partial x \ll 1$, and $\partial v/\partial y \ll 1$).

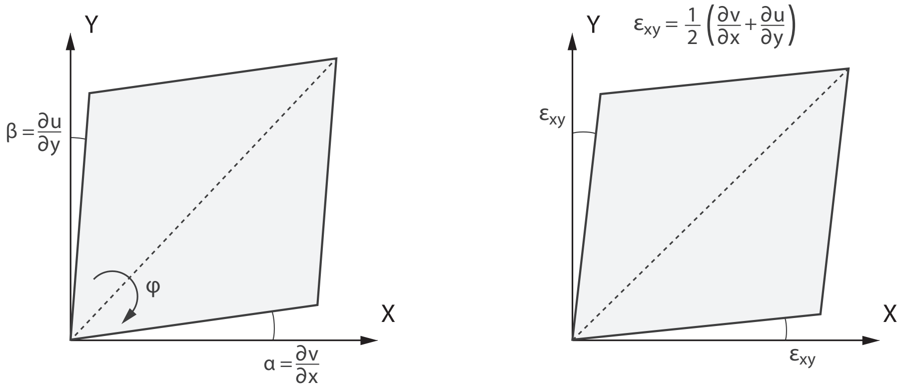
The three strain components are now given as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{2.8}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{2.9}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2.10}$$

Note that the infinitesimal element undergoes a small rigid-body rotation, φ, in addition to a (pure) shear deformation, ε_{xy}.



The mathematical shear strain is

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

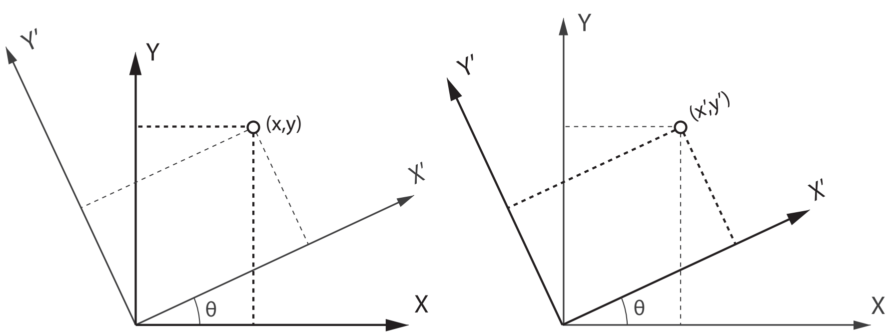
and the rigid-body rotation φ is therefore:

$$\varphi = \alpha - \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

2.3.1 Coordinate Transformation *(not examinable)*

Now consider the coordinate transformation from XY to $X'Y'$ in the form of rotation matrices, to find

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$



Substituting into mathematical formulations for strain in transformed coordinates $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$ and $\gamma_{x'y'}$ gives:

$$\begin{aligned}
\varepsilon_{x'x'} = \frac{\partial u'}{\partial x'} &= \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} \\
&= \left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial v}{\partial x}\right)\cos\theta + \left(\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial v}{\partial y}\right)\sin\theta \\
&= \cos^2\theta\frac{\partial u}{\partial x} + \sin^2\theta\frac{\partial v}{\partial y} + 2\sin\theta\cos\theta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + 2\gamma_{xy}\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
\varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} &= \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} \\
&= \left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial v}{\partial x}\right)\sin\theta + \left(-\sin\theta\frac{\partial u}{\partial y} + \cos\theta\frac{\partial v}{\partial y}\right)\cos\theta \\
&= \sin^2\theta\frac{\partial u}{\partial x} + \cos^2\theta\frac{\partial v}{\partial y} - 2\sin\theta\cos\theta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - 2\gamma_{xy}\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
\gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} &= \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} \\
&= \left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial v}{\partial x}\right)\cos\theta + \left(-\sin\theta\frac{\partial u}{\partial y} + \cos\theta\frac{\partial v}{\partial y}\right)\sin\theta \ldots \\
&\qquad \ldots - \left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial v}{\partial x}\right)\sin\theta + \left(\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial v}{\partial y}\right)\cos\theta \\
&= -2\sin\theta\cos\theta\frac{\partial u}{\partial x} + 2\sin\theta\cos\theta\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\cos^2\theta - \sin^2\theta\right) \\
&= -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right)
\end{aligned}$$

which is identical to the strain transformation equations derived previously. While the engineering approach provides a more intuitive understanding for plane strain, the mathematical approach can straightforwardly be extended to three-dimensional states of strain.

NB: this derivation is *not* examinable!

2.4 Stress/Strain Measurements

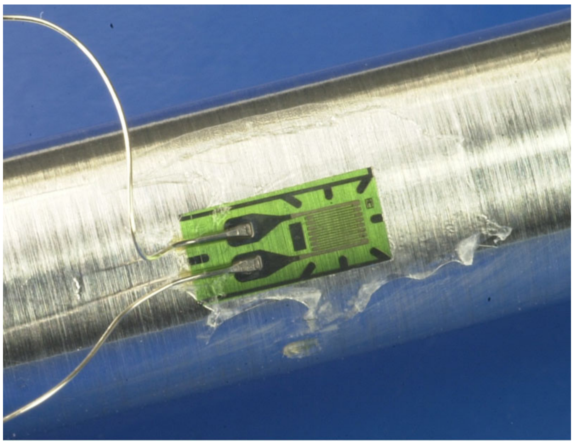
Many classic problems in structural mechanics have been solved analytically, and the validity of those solutions has been painstakingly verified using experiments. Modern structural analysis often makes use of Finite Element Analysis (FEA) to calculate the stress and strain, which may give engineers a false sense of certainty regarding the correctness of the results. In cases where material properties are not well established, or where the loading conditions are not precisely known *a priori* (*e.g.* fluid structure interaction) methods are still needed to measure the stress and strain experimentally.

A state of plane stress is determined by measuring σ_{xx}, σ_{yy} and τ_{xy} at a point. Measuring those stresses directly, however, turns out to be rather challenging. Instead, often the strains on the surface of the body are measured, allowing us to then reconstruct the state of stress. Keep in mind that this relies on knowing the material model that relates stress and strain — the topic of the next handout!

In *full field* measurements, the strains are measured across the surface of the structure simultaneously. This provides valuable insight into the global stress distributions, before focusing in on highly stressed areas. Such techniques include **photoelasticity** and **digital image correlation (DIC)**. More commonly, however, strains are measured at discrete points on a surface, using **strain gauges**. These provide accurate results and do not require intensive post-processing (such as is the case for DIC).

2.4.1 Strain Gauges

A strain gauge consists of a long thin wire, looping back and forth, which is attached to the surface of the structure being measured.

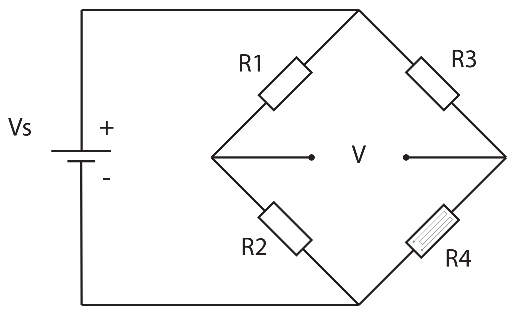


The electrical resistance R of the wire varies with change of length, *i.e.* strain; the change in resistance is further amplified by the change in cross-section due to its Poisson's ratio. As the strain gauge is bonded to the surface, it will experience the same strain as the structure. Note the strain gauge can only measure strains in the direction it is oriented, as transverse and shear strains will not change the length of the wire.

The change in resistance ΔR of the strain gauge is amalgamated into a Gauge Factor κ, which depends on the wire material used.

$$\frac{\Delta R}{R} = \kappa\,\varepsilon$$

Using strain gauges, the direct stress in a specific direction can be determined by measuring its change in resistance using a Wheatstone bridge.



The measured voltage V is given as:

$$V/V_s = \frac{R_4}{R_3 + R_4} - \frac{R_2}{R_1 + R_2}$$

When $R_1/R_2 = R_3/R_4$ the Wheatstone bridge is balanced, and $V = 0$. For the Quarter-Bridge configuration shown, where $R_4 = R + \Delta R$ is the strain gauge, and assuming $R_1 = R_2$ and $R_3 = R$, the measured voltage as a function of the strain is:

$$\frac{V}{V_s} = \frac{\Delta R}{4R + 2\Delta R} \approx \frac{\kappa}{4}\varepsilon$$

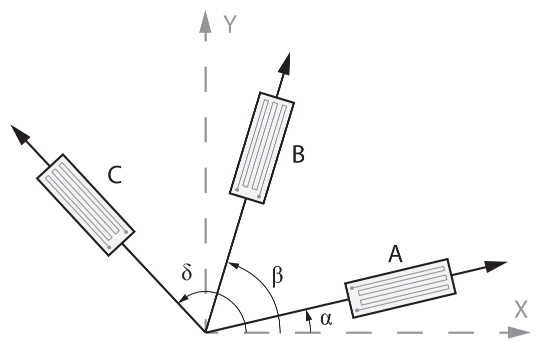
The measured strains will be in the order of micro-strains ($\mu\varepsilon = 1 \cdot 10^{-6}$), and the signal will therefore need to be further amplified. There is an art and skill to using strain gauges, for example, to compensate for temperature variations — this is beyond the scope of this course, but will be covered again in 3rd year Signals, Sensors and Controls.

NB: the details of the Wheatstone bridge are non-examinable.

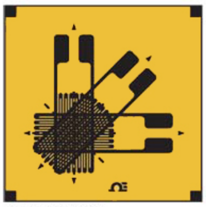
Strain Gauge Rosettes To fully reconstruct a state of stress, three independent strain measurements in three different directions are required. Calculating the strains in a desired coordinate system can then be achieved using the strain transformation equations:

$$\begin{aligned} \varepsilon_A &= \varepsilon_{xx}\cos^2\alpha + \varepsilon_{yy}\sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha \\ \varepsilon_B &= \varepsilon_{xx}\cos^2\beta + \varepsilon_{yy}\sin^2\beta + \gamma_{xy}\sin\beta\cos\beta \\ \varepsilon_C &= \varepsilon_{xx}\cos^2\delta + \varepsilon_{yy}\sin^2\delta + \gamma_{xy}\sin\delta\cos\delta \end{aligned} \tag{2.11}$$

where α, β and δ are the angles of the three strain gauges with respect to the axes of interest. Solving the three equations simultaneously will give the strains ε_{xx}, ε_{yy} and γ_{xy}, which then enable the stresses and principal stresses and strains to be found.



A variety of standard strain gauge configurations is available, with strain gauges oriented at different angles, and are often referred to as *rosettes*.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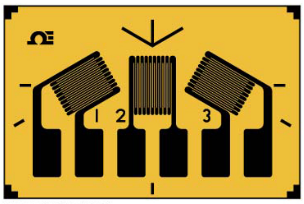

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Tee : 0/90° rosette Only applicable if the principal strain directions are known *a priori* through other considerations, such as cylindrical pressure vessels or shafts in torsion. See for example Example 3.3.

Rectangular: 45° rosette A common strain gauge configuration is the 45°rosette, where $\alpha = 0$, $\beta = 45°$ and $\delta = 90°$ and therefore:

$$\varepsilon_0 = \varepsilon_{xx}$$

$$\varepsilon_{90} = \varepsilon_{yy}$$

$$\varepsilon_{45} = \frac{\varepsilon_0 + \varepsilon_{90} + \gamma_{xy}}{2}$$

and thus

$$\gamma_{xy} = 2\varepsilon_{45} - \varepsilon_0 - \varepsilon_{90}$$

After some re-arranging, it may be shown that the principal stresses are given by:

$$\sigma_{1,2} = \frac{E}{2}\left[\frac{\varepsilon_0 + \varepsilon_{90}}{1-\nu} \pm \frac{1}{1+\nu}\sqrt{2\left(\varepsilon_0 - \varepsilon_{45}\right)^2 + 2\left(\varepsilon_{45} - \varepsilon_{90}\right)^2}\right]$$

Delta : 60° Another common configuration is the Delta rosette, where the strain gauges are set at 60°angles, and the measured strains are therefore ε_0, ε_{60} and ε_{120}. If we align the x-axis with the 0°direction, we obtain:

$$\varepsilon_0 = \varepsilon_{xx}$$

$$\varepsilon_{60} = \frac{\varepsilon_{xx} + 3\varepsilon_{yy} + \sqrt{3}\gamma_{xy}}{4}$$

$$\varepsilon_{120} = \frac{\varepsilon_{xx} + 3\varepsilon_{yy} - \sqrt{3}\gamma_{xy}}{4}$$

Adding and subtracting ε_{60} and ε_{120} respectively gives the following relationships:

$$\varepsilon_{yy} = \frac{-\varepsilon_0 + 2\varepsilon_{60} + 2\varepsilon_{120}}{3}$$

$$\gamma_{xy} = \frac{2\varepsilon_{60} - 2\varepsilon_{120}}{\sqrt{3}}$$

Finding an expression for the principal stresses of the Delta rosette is then a matter of some mildly tedious algebra, and ultimately yields:

$$\sigma_{1,2} = \frac{1}{3}\left[\frac{E}{1-\nu}\left(\varepsilon_0 + \varepsilon_{60} + \varepsilon_{120}\right) \pm \frac{E}{1+\nu}\sqrt{2\left(\varepsilon_0 - \varepsilon_{60}\right)^2 + 2\left(\varepsilon_{60} - \varepsilon_{120}\right)^2 + 2\left(\varepsilon_{120} - \varepsilon_0\right)^2}\right]$$

NB: do not memorise equations for specific rosettes, but use Equation 2.11

Example 2.1 – Strain Gauge Measurements

Q: A 45° rosette is used to measure the strains on the surface of a thin-walled aircraft fuselage. Derive the shear strain γ_{xy} from the strain gauge measurements.

A: As strain gauges A and C are aligned with the XY axes, the direct strains are known directly:

$$\varepsilon_{xx} = \varepsilon_A$$

$$\varepsilon_{yy} = \varepsilon_C$$

For the shear strain, use the strain transformation equation:

$$\varepsilon_B = \frac{1}{2}\left[(\varepsilon_{xx} + \varepsilon_{yy}) + (\varepsilon_{xx} - \varepsilon_{yy})\cos 2\beta + \gamma_{xy}\sin 2\beta\right]$$

where for $\beta = \pi/4$ we find:

$$\varepsilon_B = \frac{1}{2}\left[(\varepsilon_A + \varepsilon_C) + \gamma_{xy}\right]$$

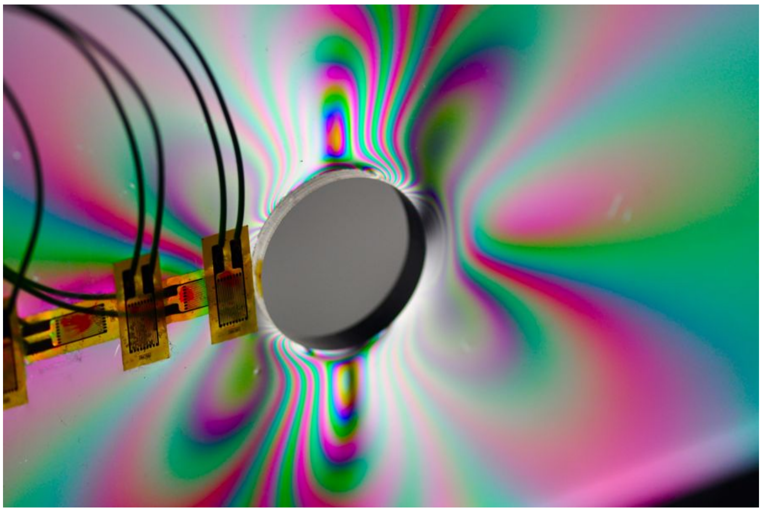
which provides:

$$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C$$

Once ε_{xx}, ε_{yy} and γ_{xy} are derived, the standard equations for principal strains and directions can be employed to further analyse the strain state.

2.4.2 Photoelasticity

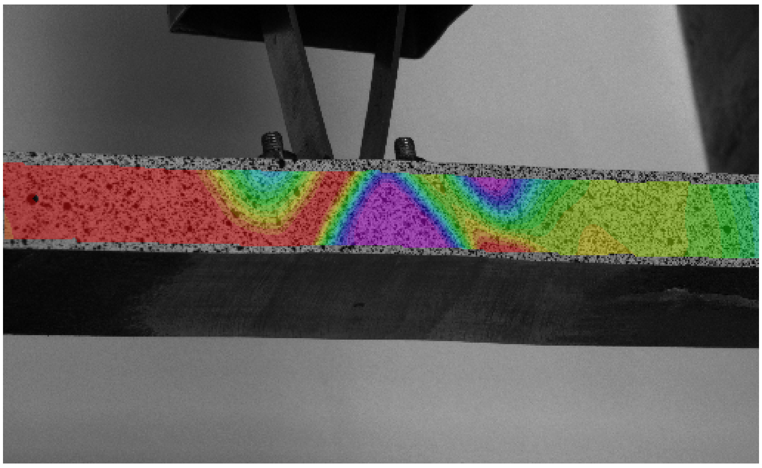
Photoelasticity is a classic full-field stress measurement technique. It relies on a photoelastic effect, where the material refractive index changes with stress. A photoelastic coating is applied to the structure, and the magnitude and direction of the stresses at any point can then be determined by examining the fringe patterns produced by a polarised light source.



This allows imaging of iso-static (where the *difference in principal stresses* is equal) and iso-clinic (where either of the principal directions coincides with the axis of polarisation) fringes. Although photo-elasticity gives a good insight into the stress distribution, it only visualises the *difference* in principal stresses, not the magnitude of the stresses.

2.4.3 Digital Image Correlation (DIC)

Another full-field measurement technique is Digital Image Correlation (DIC). A fine speckle pattern is applied to the structure, and the relative displacement of the speckles is tracked with a camera during the loading of the structure, enabling the strains to be calculated; see Section 2.3. Using stochastic cross-correlation methods, subpixel accuracy can be achieved.



The computationally intensive post-processing of high-resolution images precluded the practical use of DIC until relatively recently. What is more, the measurements need to be done under controlled circumstances (*e.g.* constant lighting conditions) so will not replace strain gauges in all applications. Nonetheless, its ability to perform contactless, full-field strain measurements is powerful!

2.5 Summary

In this handout we introduced the concept of plane strain and its relationship to plane stress. The strain transformation equations were derived from *compatibility* considerations, and were found to be identical to the stress transformation equations. This analogy allowed us to formulate concepts such as principal strains and directions, and establish a Mohr's circle for strain.

The strain transformation equations enabled us to reconstruct a state of stress from experimental strain measurements using strain gauges. Two alternative, full-field stress/strain measurement techniques, photo-elasticity and Digital Image Correlation, were also briefly introduced.

Revision Objectives Handout 2:

- explain the differences between plane stress and plane strain;
- derive strain transformation equations from first principles (geometrically only);
- appreciate and make use of parallels between strain and stress in terms of principal axes, transformation equations and Mohr's circle;
- calculate principal strains and directions, maximum/minimum shear strains;
- express strains in terms of a displacement field u, v;
- transform strains from experimental strain gauge readings (given transformation equations);
- describe different experimental stress/strain measurement techniques;

Handout 3 – Constitutive Model

After studying the properties of stress and strain, we turn our attention to the relationship *between* stress and strain, to formulate the *constitutive equations*, or *material model*. Many engineering materials are considered:

homogeneous:	properties are the same in all locations;
isotropic:	properties are the same in all directions;
linear-elastic:	there is a linear relationship between strains and stresses.

These assumptions only a cover a subset of potential material properties. For instance, fibre-reinforced composites are not homogeneous (distinct fibres and matrix) or isotropic (material properties can be tailored in different directions) — these aspects will be covered in StM3 Composite Laminate Analysis.

After formulating a generalised Hooke's Law for a linear-elastic material, we formulate the constitutive equations for isotropic materials under plane stress, and derive expressions for a material shear and bulk modulus.

3.1 Generalized Hooke's Law

First, consider a general three-dimensional state of stress and strain, defined by the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$, and corresponding strain tensor $\bar{\varepsilon}$ (with *mathematical* shear strain $\varepsilon_{xy} = \gamma_{xy}/2$):

$$\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \qquad \bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

Assuming a linear material behaviour (*i.e.* the strains are a linear combination of the stress components) we write a 6×6 compliance matrix $\boldsymbol{S}$:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

with 36 material constants S_{ij}, $i, j = 1 - 6$; this is known as the Voigt notation.

The *Maxwell-Betti reciprocal theorem* shows that the compliance matrix must be symmetric, *i.e.* $S_{ij} = S_{ji}$, which reduces the number of elastic constants to 21. This result follows from a thought experiment.

An applied stress σ_{xx} results in elastic strain energy:

$$U_a = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2}S_{11}\sigma_{xx}^2$$

Applying a second stress σ_{yy}, while *maintaining* the first stress σ_{xx} gives:

$$U_b = \frac{1}{2}S_{22}\sigma_{yy}^2 + \sigma_{xx}S_{12}\sigma_{yy}$$

The total elastic strain energy of both stresses becomes:

$$U_a + U_b = \frac{1}{2}S_{11}\sigma_{xx}^2 + \frac{1}{2}S_{22}\sigma_{yy}^2 + \sigma_{xx}S_{12}\sigma_{yy}$$

Repeating the process, but first applying σ_{yy} followed by σ_{xx}, results in:

$$U = \frac{1}{2}S_{22}\sigma_{yy}^2 + \frac{1}{2}S_{11}\sigma_{xx}^2 + \sigma_{yy}S_{21}\sigma_{xx}$$

The total strain energy must be independent of the order in which the loads are applied; therefore $S_{12} = S_{21}$, and in general $S_{ij} = S_{ji}$. Similarly, the stiffness matrix in Finite Element Analysis must be symmetric.

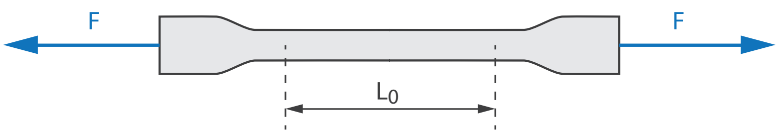
The next step, which is outside the scope of this course, relies on symmetry considerations: for an isotropic material the material properties must be the same in any direction, and therefore any plane is a symmetry plane. Applying those symmetry operations results in the following compliance matrix:

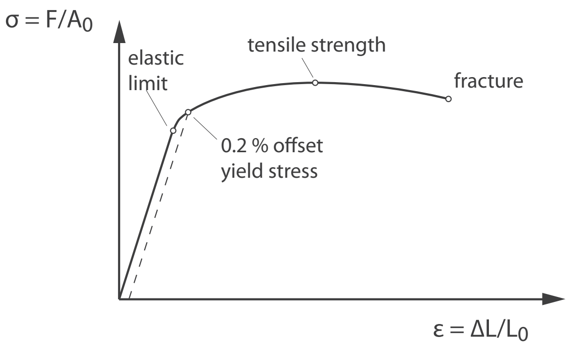
$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

with only two independent compliances S_{11} and S_{12}. For other material properties, *e.g* orthotropic, the symmetry conditions will give a different format for the compliance matrix.

3.2 Isotropic, Linear-Elastic Materials under Plane Stress

We derive the constitutive equations for an isotropic, linear-elastic material under *plane stress* using an engineering approach, based on experimental observations rather than mathematical derivations.





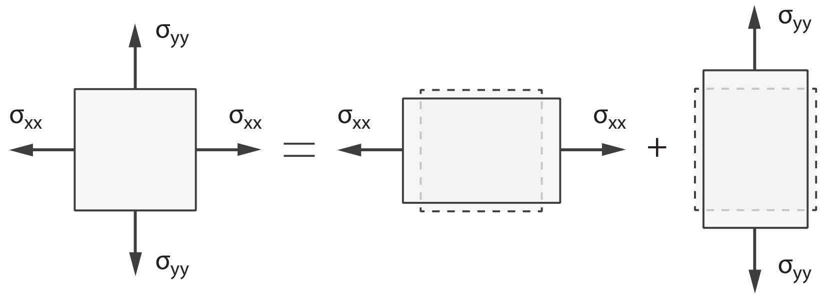
A uni-axial applied direct stress σ_{xx} (with $\sigma_{yy} = \sigma_{zz} = 0$) results in the following strains:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{yy} = -\nu\varepsilon_{xx} = -\nu\frac{\sigma_{xx}}{E}$$

characterised by two material parameters: Young's modulus E and Poisson's ratio ν. The Poisson's ratio is the negative ratio of the transverse and direct strain, for a uniaxial applied stress. For many engineering materials $\nu \approx 0.3$, but we shall derive theoretical bounds later in this handout.

A general bi-axial state of stress can be regarded as a *linear superposition* of two uni-axial stress states.

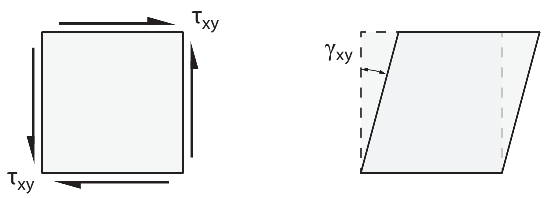


The resulting strains are superimposed:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \\ \varepsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) \end{aligned} \tag{3.1}$$

and can be inverted to express stress as a function of strain:

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \end{aligned} \tag{3.2}$$



The shear deformations are determined by the shear modulus G:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{3.3}$$

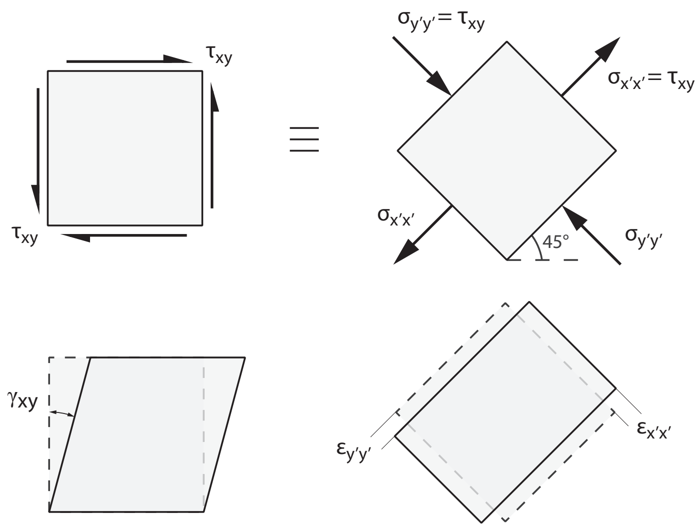
Combining Equations 3.1 and 3.3 into a matrix formulation, gives the compliance matrix:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

It is important to observe that for *isotropic* materials the direct stress σ_{xx} and σ_{yy} do not produce shear strains γ_{xy} with respect to the XY axes. Conversely, a shear stress τ_{xy} will not produce direct strains. This is reflected in the zero coupling terms in the compliance matrix.

3.2.1 Elastic Modulus: Shear Modulus

For isotropic materials the shear modulus G and Young's modulus E are not independent. The relationship between the two elastic moduli is found by observing that pure shear is equivalent to bi-axial tension and compression at 45° to the direction of shear (see Example 1.4). The strains must therefore also be equivalent.



For the case of pure shear, the resulting shear strain:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

can be transformed to the 45° $X'Y'$ coordinate system using the strain transformation equations:

$$\begin{aligned} \varepsilon_{x'x'} &= \sin\theta\cos\theta\,\gamma_{xy} \\ &= \frac{\tau_{xy}}{2G} \end{aligned}$$

The equivalent direct stresses, $\sigma_{x'x'} = \tau_{xy}$ and $\sigma_{y'y'} = -\tau_{xy}$, result in the following strain:

$$\begin{aligned} \varepsilon_{x'x'} &= \frac{1}{E}\left(\sigma_{x'x'} - \nu\sigma_{y'y'}\right) \\ &= \frac{\tau_{xy}}{E}\left(1+\nu\right) \end{aligned}$$

Equating both strains expresses the shear modulus G in terms of E and ν:

$$G = \frac{E}{2\left(1+\nu\right)} \tag{3.4}$$

3.2.2 Plane Stress Stiffness and Compliance Matrices

The material law, *i.e.* the relationship between stress and strain, for a linear-elastic, homogeneous and isotropic material under plane stress can now be described using the following **compliance**

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{3.5}$$

and **stiffness** matrices

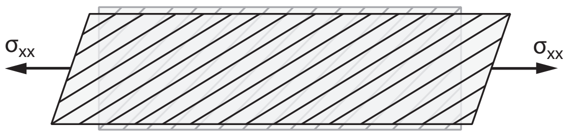
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{3.6}$$

with two material parameters, E and ν.

Other formulations may be found in literature, using the shear modulus G and bulk modulus K as independent parameters (physicists), or the Lamé parameters μ and λ (mathematicans). Most engineering texts, however, will use E and ν as independent parameters.

Note that for isotropic, elastic materials, the principal directions for stress and strain coincide. This is a result of the decoupling between shear and direct strains and is therefore not necessarily the case for anisotropic materials.

Anisotropic Materials For *anisotropic* materials such as fibre reinforced composites, there may exist a coupling between direct and shear effects. For example, consider a composite plate with fibres set an angle to the direction of uni-axial loading.



In this case the direct stresses will result in both direct and shear strains. This means that the compliance matrix will be fully populated:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

where the coupling is described using the $\bar{S}_{16}$ and $\bar{S}_{26}$ matrix terms (the $\bar{S}_{ij}$ notation is introduced in Composite Laminate Analysis in StM3). Such properties are more complicated to characterise, but also provides opportunities for interesting structural behaviour!

Example 3.1 – Plane Stress vs Plane Strain

Using the constitutive equations, we return to a comparison of plane stress and plane strain, and look at the through-thickness strains and stresses.

plane stress: in the case of plane stress $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$. However, the through-thickness strain is non-zero and is given by:

$$\varepsilon_{zz} =$$

$$=$$

In other words, the thickness of the thin-walled structure will change under the applied loads.

plane strain: for plane strain $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$. However, the out-of-plane stresses will be non-zero to satisfy those plane strain conditions:

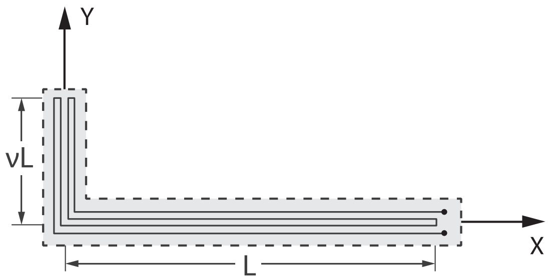
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = 0$$

and therefore:

$$\sigma_{zz} =$$

Example 3.2 – Stress Gauge

Returning to stress and strain measurement, to explore the idea of a *stress* gauge. Consider the following L-shaped strain gauge, where the length of the short leg is νL.



The gauge reading ΔR will be proportional to the change in length, $L\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$. Using Hooke's law:

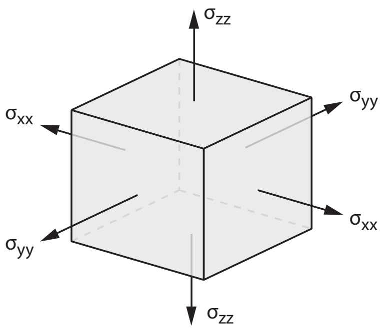
$$\Delta R \propto \frac{L}{E}\left[\sigma_{xx} - \nu\sigma_{yy} + \nu\left(\sigma_{yy} - \nu\sigma_{xx}\right)\right] = \frac{L\left(1-\nu^2\right)}{E}\sigma_{xx}$$

and thus the reading of the stress gauge is proportional to σ_{xx}.

The obvious downside of a stress gauge is that the Poisson's ratio of the material you are measuring must be known accurately *a priori*, making these not much more than an interesting exercise.

3.2.3 Elastic Modulus: Bulk Modulus

The material *bulk modulus* describes how much a material will compress under an external pressure. Consider an infinitesimal element (dimensions $dx \times dy \times dz$) with direct stresses σ_{xx}, σ_{yy} and σ_{zz}.



The element will deform under the applied loads, and its deformed volume V is:

$$
\begin{aligned}
V &= (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\; dx\,dy\,dz \\
&= (1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}+\text{higher order terms})\; dx\,dy\,dz \\
&= (1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})\; dx\,dy\,dz
\end{aligned}
$$

Note that shear stresses (and thus shear strains) will not result in a change of volume. The volumetric, or dilational, strain describes the change in volume of the infinitesimal element:

$$
\begin{aligned}
\frac{\Delta V}{V_0} = \frac{V-V_0}{V_0} &= \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} \\
&= \frac{(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})(1-2\nu)}{E}
\end{aligned}
$$

where we substituted Hooke's law:

$$
\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx}-\nu\left(\sigma_{yy}+\sigma_{zz}\right)\right]; \qquad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy}-\nu\left(\sigma_{xx}+\sigma_{zz}\right)\right]; \qquad \varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{xx}+\sigma_{yy}\right)\right]
$$

For a state of *spherical stress* ($\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$) the volumetric strain is:

$$
\frac{\Delta V}{V_0} = \frac{3\sigma(1-2\nu)}{E}
$$

The bulk modulus K relates the spherical stress to the volumetric strain:

$$
\sigma = K\frac{\Delta V}{V_0}
$$

which gives the following expression

$$
K = \frac{E}{3(1-2\nu)} \tag{3.7}
$$

for the bulk modulus, in terms of E and ν.

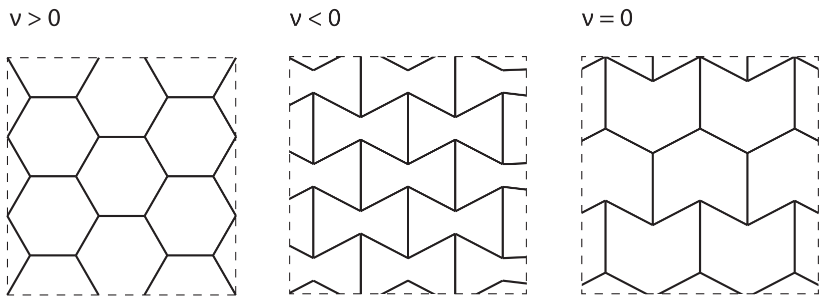
3.2.4 Poisson's ratio

The expressions for the shear modulus (Equation 3.4) and bulk modulus (Equation 3.7) of an isotropic material provide insight into the allowable range of values for Poisson's ratio. The shear and bulk modulus must both be positive and finite, which respectively provide a lower and upper bound:

$$\nu \in \langle -1, 0.5 \rangle$$

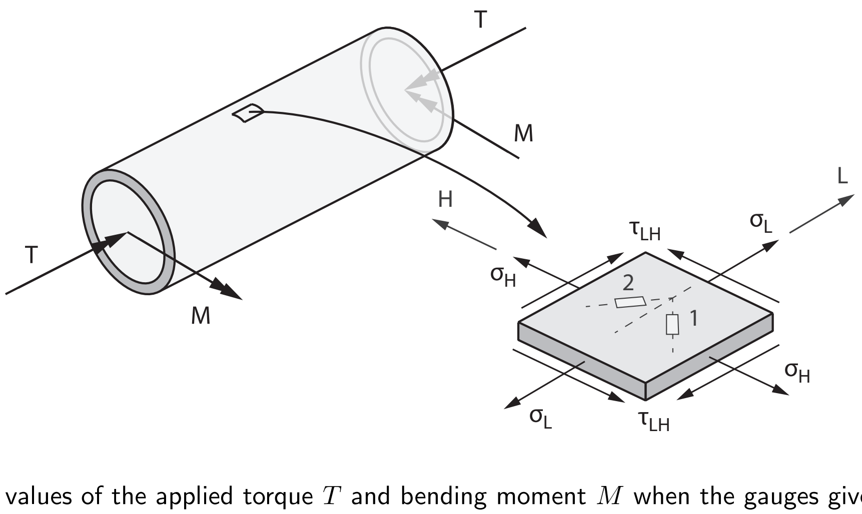
In fact, for most engineering materials $\nu \in [0.2, 0.5\rangle$, and for most metallics $\nu \approx 0.3$. For many rubbery materials $\nu \approx 0.5$, and these are considered effectively incompressible due to their high bulk modulus.

Materials known as *auxetics* have a negative Poisson's ratio (and thus expand transversely under uni-axial tension) as a result of their micro-structure.



Example 3.3 – Cylindrical shaft under combined loading

A uniform, cylindrical, thin-walled tube is subjected to a torque T and a pure bending moment M. The strain is measured at angles of $\pm 45°$ to the longitudinal axis, and the strain gauges lie in the plane of the applied bending moments. The tube has a mean radius $R = 125$ mm and a wall thickness $t = 1.63$ mm. The tube material is aluminium, with $E = 70$ GPa and $\nu = 0.3$.



Q: What are the values of the applied torque T and bending moment M when the gauges give tensile strain readings of 823 $\mu\varepsilon$ and 242 $\mu\varepsilon$ respectively?

A: Two strain gauges are not sufficient to uniquely determine the state of stress at a point, unless extra information can be derived from the loading conditions. The thin-walled element where the strain gauges are attached is subjected to a longitudinal stress σ_L due to M, and a shear stress τ_{LH} as a result of T, but no hoop stresses are applied.

The strain transformation equation can therefore be written as:

$$\begin{aligned}\varepsilon_\theta &= \varepsilon_L \cos^2\theta + \varepsilon_H \sin^2\theta + \gamma_{LH} \sin\theta\cos\theta \\ &= \varepsilon_L \left(\cos^2\theta - \nu \sin^2\theta\right) + \gamma_{LH}\sin\theta\cos\theta\end{aligned}$$

where it was observed that:

$$\varepsilon_H =$$

Substitute the strain gauge measurements ($\theta_1 = 45°$, $\varepsilon_1 = 823 \cdot 10^{-6}$; $\theta_2 = -45°$ $\varepsilon_2 = 242 \cdot 10^{-6}$):

$$\begin{aligned}\varepsilon_1 &= \varepsilon_L\,(0.5 - 0.5\,\nu) + 0.5\,\gamma_{LH} = 823\mu\varepsilon \\ &= 0.35\,\varepsilon_L + 0.5\,\gamma_{LH} \\ \varepsilon_2 &= \varepsilon_L\,(0.5 - 0.5\,\nu) - 0.5\,\gamma_{LH} = 242\mu\varepsilon \\ &= 0.35\,\varepsilon_L - 0.5\,\gamma_{LH}\end{aligned}$$

to obtain two linear equations, which are solved to find:

$$\begin{aligned}\varepsilon_L &= \\ \varepsilon_H &= \\ \gamma_{LH} &= \end{aligned}$$

Using Hooke's law:

$$\begin{bmatrix}\sigma_L \\ \sigma_H \\ \tau_{LH}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_L \\ \varepsilon_H \\ \gamma_{LH}\end{bmatrix}$$

provides the stresses:

$$\begin{aligned}\sigma_L &= \\ \sigma_H &= 0 \text{ MPa} \\ \tau_{LH} &= \end{aligned}$$

The applied loads can now be back-calculated from the stresses (using $I = \frac{J}{2} \approx \pi R^3 t$) to give:

$$\begin{aligned}M &= \frac{\sigma_L\, I}{R} = \\ T &= \frac{\tau_{LH}\, J}{R} = \end{aligned}$$

3.3 Summary

In this handout the Generalized Hooke's Law was derived for a homogeneous, isotropic, linear-elastic material under a state of plane stress. The shear modulus G was derived in terms of two independent elastic properties E and ν, by equating the elastic strain energy for a state of pure shear, and that for the equivalent bi-axial tensile/compressive state. The bulk modulus K, which relates the change in material volume to the spherical stress, was derived. The equations for the shear and bulk modulus provide bounds on possible values for Poisson's ratio ν in isotropic materials.

This handout completes our description of plane stress: stress transformation equations (Handout 1), strain transformation equations (Handout 2), and the constitutive equations that relate stress and strain for homogeneous, isotropic, linear-elastic materials (Handout 3). The next handout will focus on using this information to consider material failure criteria.

Revision Objectives Handout 3:

- derive and recall the constitutive equations for isotropic, linear-elastic materials under plane stress

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

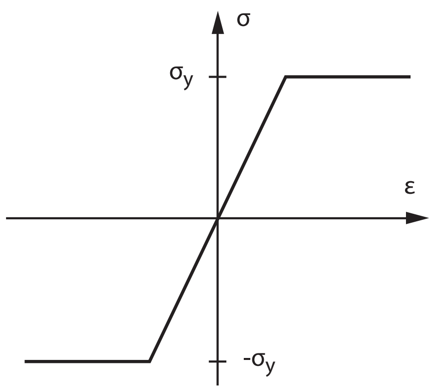
- explain why direct strains and shear strains are decoupled for isotropic materials;
- calculate strains due to applied stresses, and calculate stresses from measured strains;
- derive and recall expressions for shear modulus G and bulk modulus K in terms of E and ν

$$G = \frac{E}{2\,(1+\nu)} \qquad K = \frac{E}{3\,(1-2\nu)}$$

Handout 4 – Failure Criteria

This handout introduces failure criteria, to predict material failure under applied loads. In particular, we are interested in how to extrapolate the results of a uni-axial tensile test to predict the failure under a combined stress state. We shall approach the failure criteria from a macroscopic (elasticity), rather than a micro-mechanical (materials), point of view.

A common simplification is that materials have a distinct yield stress, σ_Y, after which the material fails abruptly (brittle failure) or yields plastically (ductile failure). In practise, this yield point is not sharply defined for ductile materials; nonetheless, we shall here assume an elastic/perfectly-plastic material model.

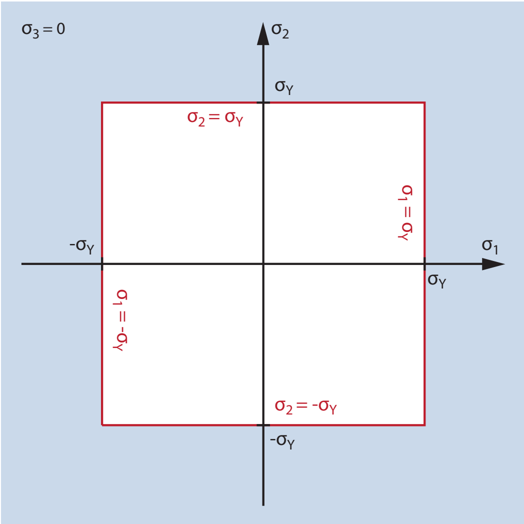


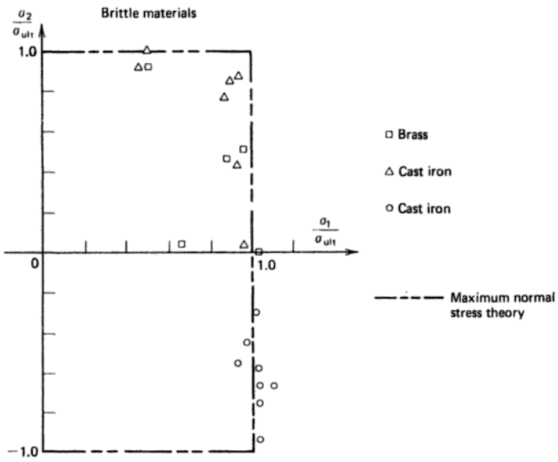
It is assumed that the material behaves linearly up to the yield point, and can therefore be described using the methods developed in this unit. To disassociate the failure criteria from a specific set of axes, these failure criteria are defined in terms of the principal stresses σ_1, σ_2, and σ_3. We shall focus on plane stress, where $\sigma_3 = 0$, but shall see that it is important to explicitly consider the out-of-plane stress in the material failure.

In the principal stress space we aim to find a **yield locus** which separates elastic from plastic deformation; any stress state within its boundaries will therefore be elastic, and failure occurs at the boundary. We are not concerned with the details of the plastic deformations after yield, and shall limit ourselves to the point of onset of failure.

4.1 Maximum Principal Stress

The simplest failure criterion, proposed by **Rankine** (1857), states that failure occurs when the **maximum principal stress** reaches a critical value. The direction of failure is then taken to be the plane of maximum principal stress.





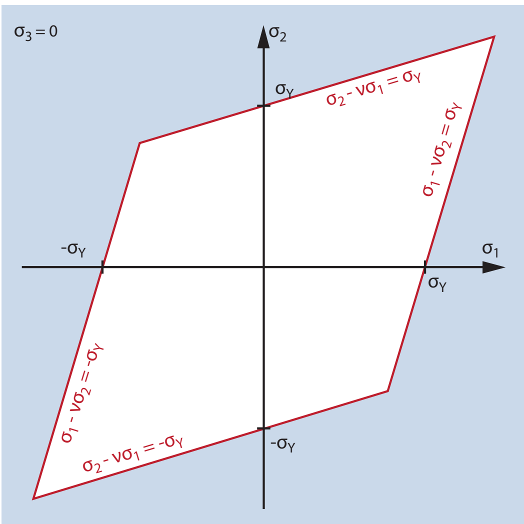
While this is not a very realistic condition for ductile failure, it is quite well suited for predicting brittle failure. This includes ceramic materials, as well as fast fracture (in tension) of metals such as cast iron. The failure in torsion of a piece of blackboard chalk also illustrates this failure mode, as the failure surface will form a 45° helix along a plane with maximum principal stress.

4.2 Maximum Principal Strain

Another theory, due to **Saint-Venant**, predicts failure at a **maximum principal strain**. Assuming that tensile and compressive strength are equal, this failure criterion can be expressed as:

$$\sigma_1 - \nu\,(\sigma_2 + \sigma_3) = \pm\sigma_Y$$
$$\sigma_2 - \nu\,(\sigma_1 + \sigma_3) = \pm\sigma_Y$$

which defines the boundaries for the yield surface, where $\sigma_3 = 0$ for plane stress. This failure criterion is mostly of historical importance, and was favoured by engineers in the 19th century.



4.3 Tresca Yield Criterion

The underlying principle of the **Tresca** yield criterion (1878) is that yielding occurs when the **maximum shear stress** reaches a critical value. This was evidenced by experiments on annealed metals that showed that spherical stress does not cause yield, and it is the *difference* in principal stresses that drives failure.

The critical shear stress can be found from a uni-axial tensile test:

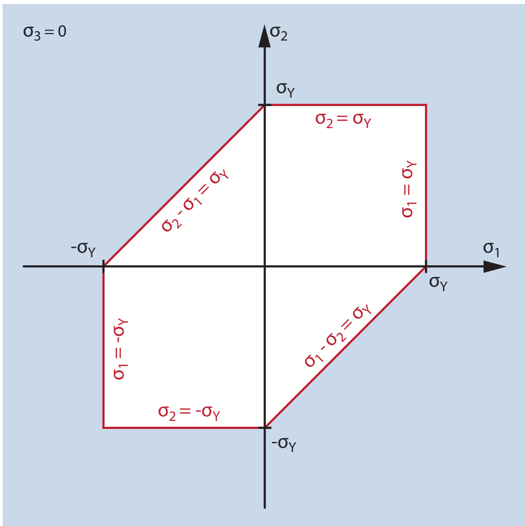
$$\tau_{\text{crit}} = \frac{|\sigma_1 - \sigma_2|}{2} = \frac{\sigma_Y}{2}$$

where $\sigma_2 = \sigma_3 = 0$, and at point of failure $\sigma_1 = \sigma_Y$.

In general, the maximum shear stress is determined by the maximum difference between the three principal stresses, and the Tresca failure criterion becomes:

$$\max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right) = \sigma_Y \tag{4.1}$$

For plane stress $(\sigma_3 = 0)$ the yield locus is drawn as:



with the following bounds:

$$\begin{aligned}
\sigma_1 \geq \sigma_2 \geq \sigma_3(=0) &\rightarrow \sigma_1 - \sigma_3 = \sigma_Y \;\therefore\; \sigma_1 = \sigma_Y \\
\sigma_2 \geq \sigma_1 \geq \sigma_3(=0) &\rightarrow \sigma_2 - \sigma_3 = \sigma_Y \;\therefore\; \sigma_2 = \sigma_Y \\
\sigma_3(=0) \geq \sigma_2 \geq \sigma_1 &\rightarrow \sigma_3 - \sigma_1 = \sigma_Y \;\therefore\; \sigma_1 = -\sigma_Y \\
\sigma_3(=0) \geq \sigma_1 \geq \sigma_2 &\rightarrow \sigma_3 - \sigma_2 = \sigma_Y \;\therefore\; \sigma_2 = -\sigma_Y
\end{aligned}$$

$$\begin{aligned}
\sigma_1 \geq \sigma_3(=0) \geq \sigma_2 &\rightarrow \sigma_1 - \sigma_2 = \sigma_Y \\
\sigma_2 \geq \sigma_3(=0) \geq \sigma_1 &\rightarrow \sigma_2 - \sigma_1 = \sigma_Y
\end{aligned}$$

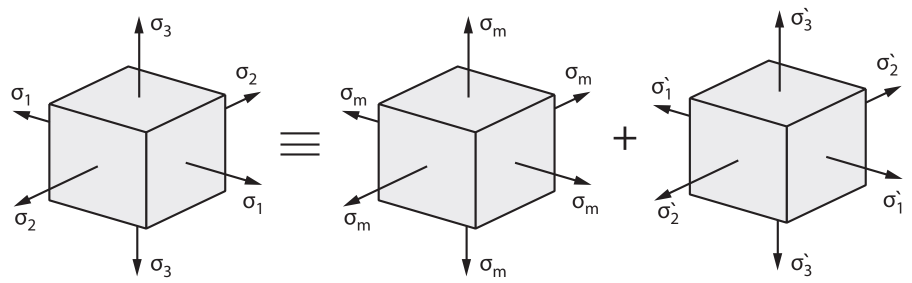
4.4 Von Mises Yield Criterion

Accurate experiments on annealed metals showed that in many cases the ratio of maximum shear stress to uni-axial yield stress was slightly greater than 0.5, and closer to 0.57. This led to the confirmation of a failure criterion based on the **maximum distortion energy**, better known as the **Von Mises** criterion (1913). It is sometimes also referred to as the *Hencky-Huber* or *Maxwell* criterion.

Consider an infinitesimal element in an elastic body subject to principal stresses, σ_1, σ_2 and σ_3. It is possible to replace this state of stress with a statically equivalent set of stresses consisting of a hydrostatic stress σ_m,

$$\sigma_m = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \tag{4.2}$$

plus deviatoric components $\sigma_i' = \sigma_i - \sigma_m$. It is these deviatoric components that induce shear, and therefore initiate yielding in ductile materials. Von Mises proposed that failure occurs when the elastic strain energy due to the deviatoric stresses reaches a critical value.



Let us consider the total energy per unit volume $\hat{U}_{\text{total}}$ of the elastic stresses:

$$\hat{U}_{\text{total}} = \sum_{i=1}^{3} \frac{1}{2}\sigma_i \varepsilon_i$$

where $\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right]$, *etc.* This gives the total elastic energy[5] as:

$$\begin{aligned}\hat{U}_{\text{total}} &= \frac{1}{2E}\sigma_1\left[\sigma_1 - \nu\left(\sigma_2+\sigma_3\right)\right] + \frac{1}{2E}\sigma_2\left[\sigma_2 - \nu\left(\sigma_1+\sigma_3\right)\right] + \frac{1}{2E}\sigma_3\left[\sigma_3 - \nu\left(\sigma_1+\sigma_2\right)\right] \\ &= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right]\end{aligned}$$

If we substitute σ_m (Equation 4.2) for σ_i we obtain the energy associated with the hydrostatic stress state:

$$\begin{aligned}\hat{U}_{\text{hydrostatic}} &= \frac{3\left(1-2\nu\right)}{2E}\sigma_m^2 \\ &= \frac{1-2\nu}{6E}\left(\sigma_1+\sigma_2+\sigma_3\right)^2\end{aligned}$$

Subtracting the hydrostatic component from the total energy, we find:

$$\begin{aligned}\hat{U}_{\text{deviatoric}} &= \frac{1+\nu}{3E}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3\right) \\ &= \frac{1}{12G}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_1-\sigma_3\right)^2\right]\end{aligned}$$

[5] The maximum total strain energy can, in fact, be considered a failure criterion in its own right, and is referred to as the *Beltrami-Haigh* failure criterion. Like the maximum principal strain criterion it has largely been superseded by other failure criteria.

From the uni-axial test with $\sigma_1 = \sigma_Y$ and $\sigma_2 = \sigma_3 = 0$ we find the critical deviatoric strain energy as:

$$\hat{U}_{\text{critical}} = \frac{1}{12G}\left[(\sigma_Y - 0)^2 + (0-0)^2 + (\sigma_Y - 0)^2\right] = \frac{\sigma_Y^2}{6G}$$

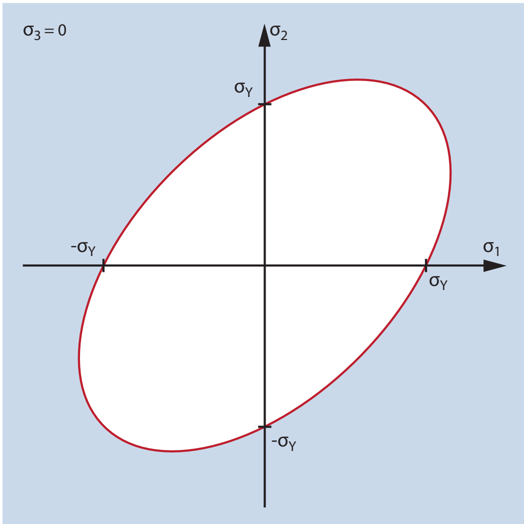
Substituting this critical value into the equation for deviatoric strain energy gives the Von Mises failure criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2\sigma_Y^2 \tag{4.3}$$

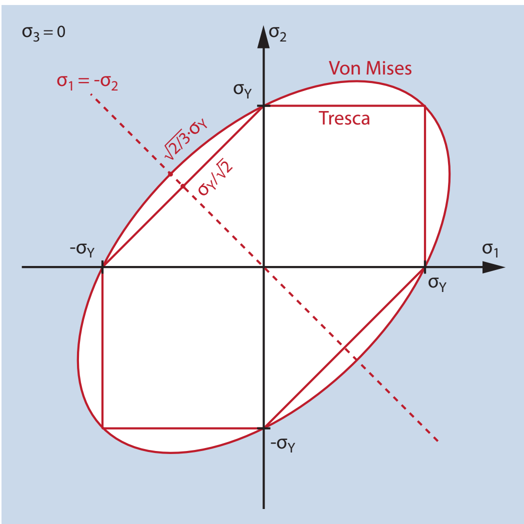
For plane stress ($\sigma_3 = 0$) this condition reduces to:

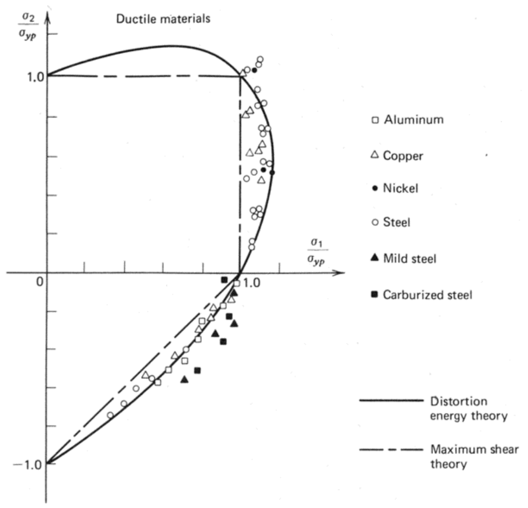
$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2 \tag{4.4}$$

which describes a yield locus in the form of an ellipse at an angle $\pi/4$ to the principal stress axes.



Comparing the Von Mises and Tresca condition, it can be seen that they agree quite closely. The largest discrepancy is for pure shear ($\sigma_1 = -\sigma_2$) where the difference is approximately 15%. The Von Mises criterion gives closer correlation with experiments, whereas the Tresca condition gives a useful conservative estimate. Often the decision which failure criterion to use will depend on which one is *easiest* to use for a calculation.





From a more theoretical perspective, it can be observed that the Von Mises criterion is related to the RMS of the principal stress differences, whereas the Tresca accounts for the maximum absolute difference.

Von Mises stress When using Finite Element software to analyse structural components, it is common practise to plot the Von Mises stress to identify the most highly stressed points. The Von Mises stress is calculated as:

$$\sigma_{\text{vm}} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}} \tag{4.5}$$

and it has to remain below the material yield stress (with a suitable margin of safety) to avoid failure.

Example 4.1 – Experimental Yield Envelope

A novel Ceramic Matrix Composite material (CMC) is under development as a possible material for turbine blades in jet engines. Its yield and failure criteria need to be characterised. It is found to yield in uni-axial compression (albeit for small strains) at a stress of 1500 MPa, but is found to fail by a cleavage (brittle) mechanism in uni-axial tension at a stress of 700 MPa.

Q: Sketch the yield and cleavage surfaces in principal stress space and determine combinations of σ_1 and σ_2 where there is a transition from cleavage failure to yielding.

A: In tension a brittle failure is observed, and the Rankine criterion, with $\sigma_Y = 700$ MPa, is used.

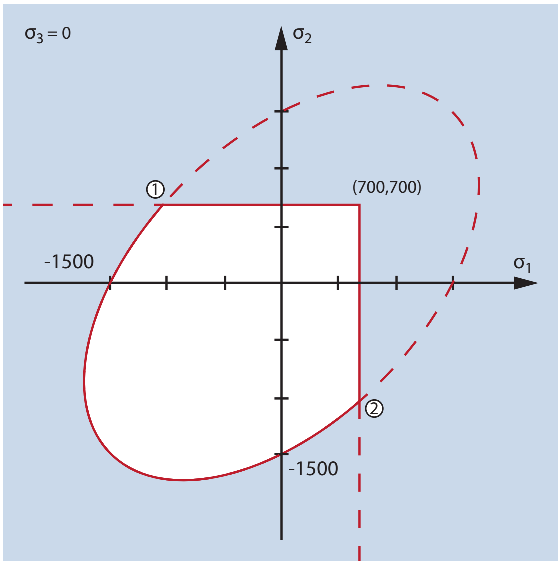
To capture the ductile compressive failure, the Von Mises criterion is assumed, with $\sigma_Y = 1500$ MPa.

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_Y^2$$

which for plane stress ($\sigma_3 = 0$) results in:

$$\sigma_Y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

The combined yield locus is therefore:



At the two points where the lines intersect, the damage mechanism changes. To find point 1, insert $\sigma_2 = 700$ in the equation for the Von Mises failure envelope and solve for σ_1 (similarly for point 2) to find (-1022,700) and (700,-1022).

Example 4.2 – Strength Calculation

Q: A finite element calculation of a thin-walled wing section under plane stress provided the following stresses: $\sigma_{xx} = 50$ MPa, $\sigma_{yy} = 100$ MPa and $\tau_{xy} = -125$ MPa. The material used is Aluminium 6061-T6, with a yield strength $\sigma_Y = 240$ MPa. Using a suitable failure criterion, verify the strength of the structure.

A: Aluminium is a ductile material, and either *Tresca* or *Von Mises* would therefore be a suitable failure criterion. First determine the principal stresses:

$$\sigma_1 = \qquad\qquad\qquad\qquad \sigma_2 =$$

Using the Von Mises failure criterion for plane stress:

$$\sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} =$$

we find that the Aluminium: fails / does not fail

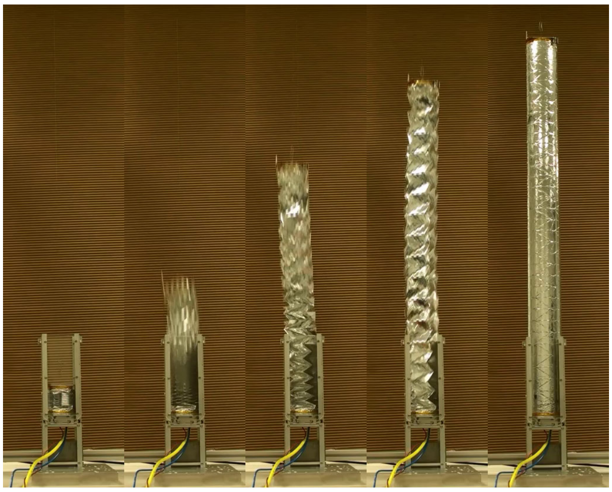
The Tresca criterion considers the maximum difference in principal stresses (in plane stress, $\sigma_3 = 0$)

$$|\sigma_1 - \quad| =$$

and predicts that the Aluminium: fails / does not fail.

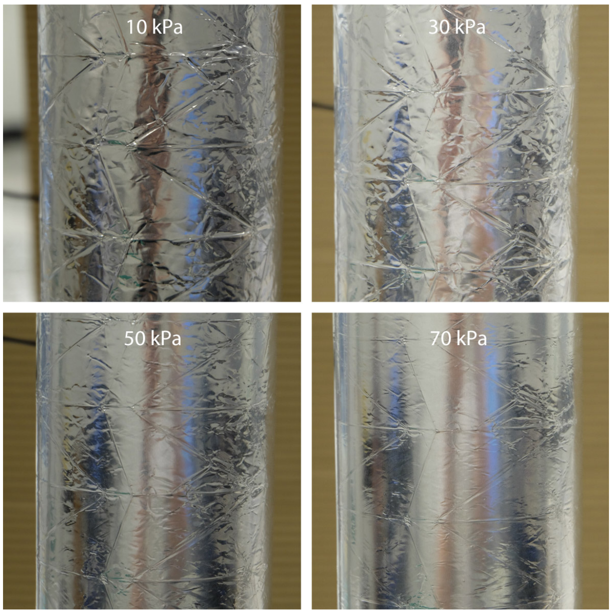
Example 4.3 – Inflatable-Rigidisable Cylinder

For a CubeSat technology demonstration mission, called InflateSail, an inflatable deployable mast was developed. In its deployed configuration the mast has a length $L \approx 1$ m, radius $r = 45$ mm and it is folded down for launch to approximately 65 mm using an origami pattern.



The cylindrical mast is made of a thin laminate, constructed of two layers of Aluminium foil sandwiching a Mylar film (14.5 μm and 16 μm respectively). The aluminium layers provide stiffness and strength, and the polymer layer offers toughness against crack propagation.

After inflation the skin of the cylinder will be wrinkled and creased as a result of the packaging process. To remove these residual creases — and thereby recover the stiffness and strength of the deployed boom — the pressure is increased until the skin material yields and smoothens out. This proces is referred to as strain-rigidisation, and its purpose is to provide stiffness to the boom, even after the inflation gas has been vented.



Q: What is the minimum required inflation pressure to achieve strain-rigidisation? Uni-axial tests suggested a yield stress of approximately 50 MPa for the laminate material.

A: From Example 1.1 recall the hoop stress σ_H, longitudinal stress σ_L and radial stress σ_R equations:

$$\sigma_H = \frac{pr}{t}$$
$$\sigma_L = \frac{pr}{2t} = \frac{1}{2}\sigma_H$$
$$\sigma_R \approx 0$$

with radius r, wall thickness t and gauge pressure p. These are also the principal stresses, as the stress state does not produce shear stresses in the chosen coordinate system.

Assuming plane stress Von Mises conditions, we can use:

$$\sigma_Y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

and substituting $\sigma_1 = \sigma_H$ and $\sigma_2 = \sigma_L = \frac{\sigma_H}{2}$, we find:

$$\sigma_Y^2 = \sigma_H^2 - \frac{1}{2}\sigma_H\sigma_H + \frac{1}{4}\sigma_H^2$$
$$= \frac{3}{4}\sigma_H^2$$

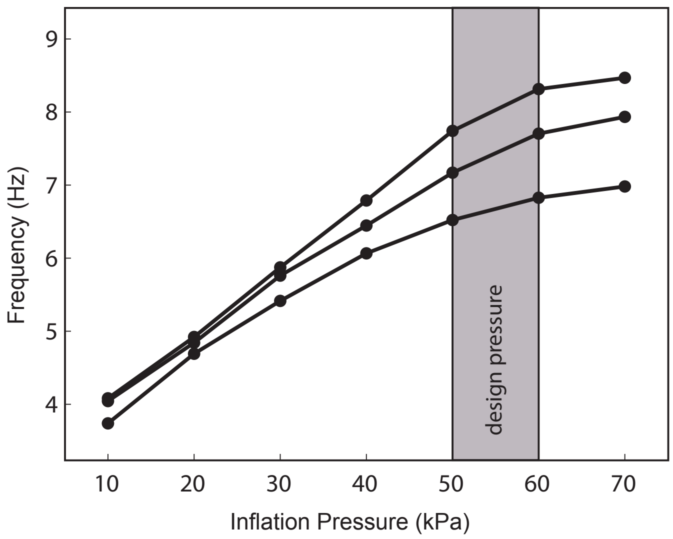
and therefore:

$$p = \sqrt{\frac{4}{3}}\frac{\sigma_Y t}{r}$$

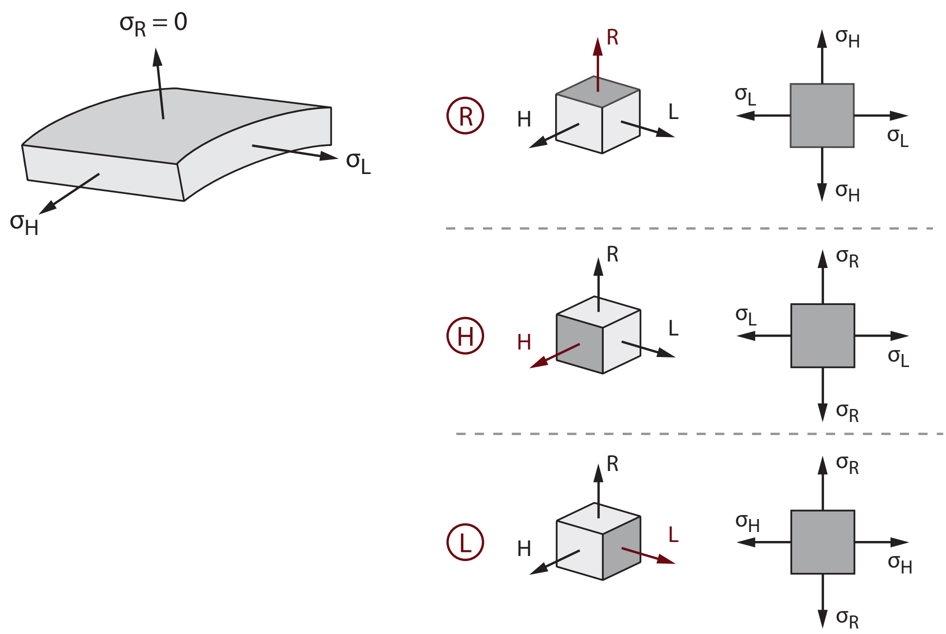
Substituting the values for our specific cylinder gives:

$$p = \sqrt{\frac{4}{3}} \cdot \frac{55 \cdot 10^6 \cdot 45 \cdot 10^{-6}}{45 \cdot 10^{-3}} = 57 \text{ kPa}$$

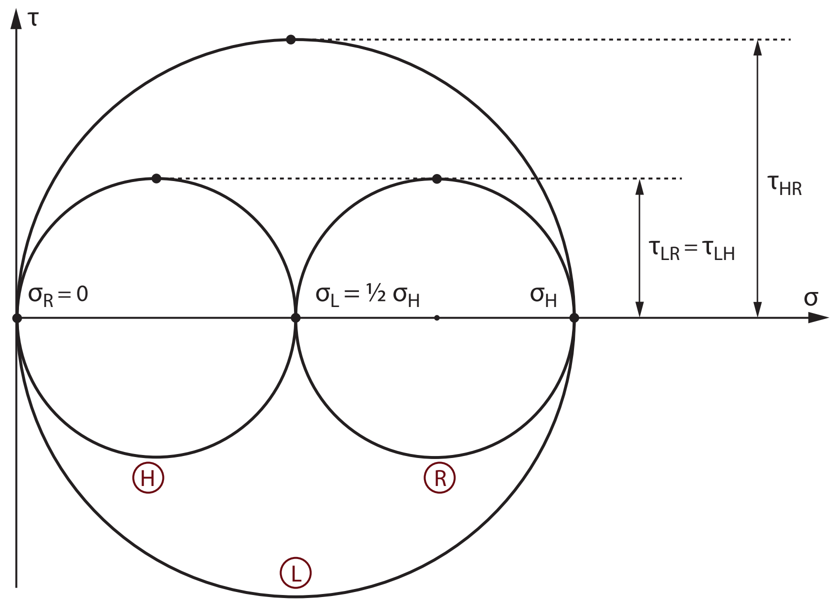
This simple analysis formed the basis of experiments done to verify the recovery of the boom stiffness, for different inflation pressures. This was done non-destructively by measuring the natural frequency of the boom, using small accelerometers attached to the boom tip. It was found that near the predicted rigidisation pressure the boom stiffness no longer increased for greater inflation pressures, indicating that the creases were effectively removed.



Q: A next question would be how the inflatable cylinder *deforms* under the applied pressure, which is where the Tresca yield condition provides more insight. It is important to consider all three principal stresses, to identify the plane in which the maximum shear stress occurs.



Rotating the stress state around the hoop (H), longitudinal (L) and radial (R) vector will provide three Mohr's circles of stress, which can be plotted in a single diagram:

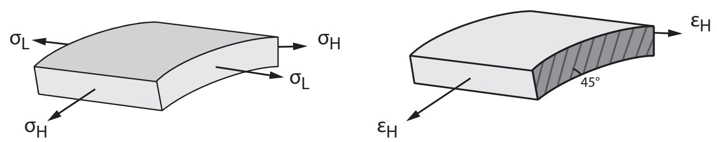


A similar diagram can be constructed for any state of (plane) stress. NB: it is very important to only rotate around vectors of principal directions, as otherwise you lose information about the shear stress on that face!

Looking at the three Mohr's circles of stress, we can see that the greatest shear stress is found in hoop/radial plane (*i.e.* rotating around longitudinal direction):

$$\tau_{\text{max}} = \frac{\sigma_H - \sigma_R}{2} = \frac{\sigma_H}{2}$$

Thus plastic deformation initiates as shear in the hoop direction.



In other words, the diameter increases through plastic deformation, but the cylinder does not permanently change length axially. This has a strong effect on the rigidisation process, as it means that the longitudinal creases will be flattened out most as a result of the strain rigidisation.

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4.5 Summary

A number of failure criteria have been introduced and compared. The criteria are defined in terms of principal stresses, to disassociate the results from a specific set of axes.

The *Rankine* criterion is appropriate for brittle failure, whereas *Tresca* and *Von Mises* apply to ductile materials such as metals. The Tresca criterion assumes yield to occur at the plane of maximum shear stress, while the Von Mises condition uses a maximum distortion energy. In practise both will give usable results, and the selection of the yield criterion depends on which is most convenient for the problem at hand. An example problem highlighted the importance of thinking in 3D when considering failure criteria, even if we are working with plane stress.

It should be noted that in addition to the failure criteria described in this handout, there exist various other theories, to capture the failure modes of specific materials, such as fibre-reinforced composites, concrete or soils.

Revision Objectives Handout 4:

- describe different failure criteria (Rankine, St Venant, Tresca, Von Mises), and draw their failure envelopes in principal stress space;
- select appropriate failure criteria for brittle/ductile materials;
- recall expressions for failure criteria and evaluate the failure load of a structure;