Numerical methods

Lecture 6: Sequences and series: sequences

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Sequences

Many problems in mathematics and Engineering involve *sequences* and *series*.
A *sequence* is an ordered set of numbers (called *terms*) which we denote as

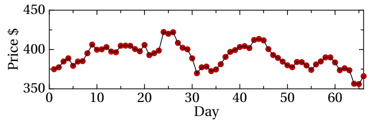
$$\{a_n\}_{n=1}^{\infty} = a_1,\ a_2,\ a_3,\ \ldots, a_n, \ldots$$

Example: Fibonacci sequence:

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 0, a_2 = 1$$

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \ldots$$

Stock market prices (Apple - USD):



Sequences and series

In this part of the unit we look at two different but related things. Don't get them mixed up!

A *sequence* is a an *ordered set* of numbers that goes on forever:

$$\{a_n\}_{n=1}^{\infty} = a_1,\ a_2,\ a_3,\ \ldots, a_n, \ldots$$

A *series* is the *sum* of the terms of a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

Right now we will look at sequences but we will come back to series later. . .

Converging sequences

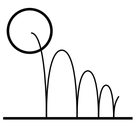
Consider the sequence

$$\begin{aligned} a_1 &= 0.9 \\ a_2 &= 0.99 \\ a_3 &= 0.999 \\ a_4 &= 0.9999 \\ &\ldots \\ a_n &= 1 - 10^{-n} \end{aligned}$$

The terms of the sequence are converging towards 1. They will never quite reach 1 but can become *arbitrarily close* for large enough n.
We say that the sequence has limit 1, or that it converges to 1, or that $a_n \to 1$ as $n \to \infty$.

Height of a bouncing ball

Consider a bouncing ball



The heights of the bounces form a sequence

$$h_0,\ h_1,\ h_2,\ \ldots$$

The basic equation connecting the height h of a bounce with the speed v at the time the ball lands is $v^2 = 2gh$.

If the ball starts from rest at height h_0 then it will have speed $v_0 = \sqrt{2gh_0}$ when it lands.

Height of a bouncing ball

The ball lands with speed $v_0 = \sqrt{2gh_0}$.

If it bounces with coefficient of restitution e (where $0 < e < 1$) then the upwards speed after the bounce is given by $v_1 = ev_0 = e\sqrt{2gh_0}$.

This means it will now bounce to a height $h_1 = \frac{v_1^2}{2g} = e^2 h_0$.

We have then the sequences of bounce heights and bounce velocities

$$\{h_n\} = h_0,\ e^2h_0,\ e^4h_0,\ e^6h_0,\ \ldots$$
$$\{v_n\} = v_0,\ ev_0,\ e^2v_0,\ e^3v_0,\ \ldots$$

which both converge to zero since $0 < e < 1$

Abstract definition of convergence

Definition (Convergence of a sequence)

A sequence $\{a_n\}$ converges to a limit L if

for all $\epsilon > 0$ there exists an N such that $|a_n - L| < \epsilon$ for all $n > N$.

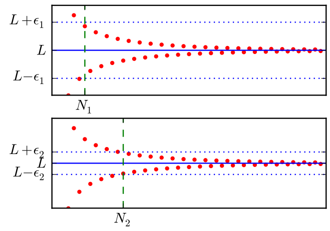
In other words we can get as close as we like to L (a distance ϵ) and *stay close* if we wait for a long enough time N.

If the sequence converges to some L we say that the limit *exists* and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty$$

Convergence in pictures

We need $L - \epsilon < a_n < L + \epsilon$ for all $n > N$. When ϵ is smaller we need a bigger N.



The sequence converges if we can find an N for any ϵ.

Convergence using the definition

Consider the sequence $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots,\ \frac{1}{n},\ \dots$ Show that this sequence converges i.e. that $\lim_{n\to\infty} \frac{1}{n}$ exists.

Pick any $\epsilon > 0$. Then we want to find N so that, for all $n > N$

$$|a_n - L| < \epsilon$$

We have $a_n = \frac{1}{n}$ and $L = 0$, so we want to find N so that, $\left|\frac{1}{n}\right| < \epsilon$ for all $n > N$.
Since $\epsilon \neq 0$ we can choose N to be any integer bigger than $\frac{1}{\epsilon}$ and we find that

$$\left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} < \epsilon$$

So $\lim_{n\to\infty} \frac{1}{n}$ exists and is equal to 0.

Properties of limits

We usually find limits of sequences in terms of other sequences we already know.

Theorem

If $\{a_n\}$ is a converging sequence with limit L (i.e. $\lim_{n\to\infty} a_n = L$) and f is a function that is continuous at L then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

Examples: If $\{a_n\}$ is a converging sequence and α is a constant

- $\lim_{n\to\infty} (\alpha + a_n) = \alpha + \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} (\alpha a_n) = \alpha \lim_{n\to\infty} a_n$

Properties of limits

Suppose we have two converging sequences $\{a_n\}$ and $\{b_n\}$ then

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ providing that $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$.

Some simple examples

- Find
  - $\lim_{n\to\infty} \dfrac{5}{n^2}$

$$\lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty} \left(\frac{1}{n}\cdot\frac{1}{n}\right) = 5\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right) = 5\cdot 0\cdot 0 = 0$$

  - $\lim_{n\to\infty} \dfrac{n+1}{2n-1}$

$$\lim_{n\to\infty} \frac{n+1}{2n-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(2-\frac{1}{n}\right)}$$
$$= \frac{1+\lim_{n\to\infty}\frac{1}{n}}{2-\lim_{n\to\infty}\frac{1}{n}} = \frac{1+0}{2-0} = \frac{1}{2}$$

  - $\lim_{n\to\infty} \dfrac{n+1}{2n^2-5} = \lim_{n\to\infty} \dfrac{n+1}{2n^2-5} = \lim_{n\to\infty} \dfrac{\frac{1}{n^2}+\frac{1}{n}}{2-\frac{5}{n^2}} = \frac{0+0}{2-0} = 0$

Exercises

Which of the following sequences $\{a_n\}$ converge as $n \to \infty$?
For those that do, find $\lim_{n\to\infty} a_n$

1. $$a_n = \frac{2n + n^3}{3n^3 + 3n^2 - 2}$$
2. $$a_n = \cos(n\pi)$$
3. $$a_n = \frac{3n + (-1)^n}{n^3 + 2}$$

Recursive sequences

In a recursive sequence each term is defined in terms of the previous term i.e. we have

$$a_{n+1} = g(a_n)$$

If the sequence converges then for large n we must have that $a_{n+1} \approx a_n$ or

$$a_n \approx g(a_n)$$

If $a_n \to L$ this must become exact so that

$$L = g(L).$$

We say that L is a *fixed point* of the iteration.

Recursive sequences

Consider the sequence given by

$$a_{n+1} = \frac{a_n + 2}{a_n}$$

with $a_0 = 1.5$. If it converges what limit could it have?

If it converges to L then $L = g(L)$ so that

$$L = \frac{L+2}{L} \implies L^2 = L + 2 \implies (L+1)(L-2) = 0$$

so $L = -1$ or $L = 2$.

Which root does it converge to? What happens instead if $a_n = 1$?

Geometric sequences

We saw earlier the example of the bouncing ball that leads to a geometric sequence for the bounce speeds (and heights):

$$\{v_n\} = v_0,\ ev_0,\ e^2v_0,\ e^3v_0,\ \ldots,\ e^nv_0,\ \ldots$$

This is known as a geometric sequence. We can define a geometric sequence recursively with

$$a_{n+1} = ra_n$$

for some number r which will be the *ratio* of successive terms. If it converges to L then we see that

$$L = rL \quad \implies \quad L(1-r) = 0$$

so for $r \neq 1$ it must converge to zero *if it converges*.

Geometric sequences

Suppose we have a geometric sequence defined by

$$a_{n+1} = ra_n$$

for some $r \neq 1$ (if $r = 1$ the sequence isn't very interesting).

The sequence has terms that look like

$$\{a_n\} = a_0,\ ra_0,\ r^2a_0,\ \ldots,\ r^na_0,\ \ldots$$

The sequence will converge if $|r| < 1$ and in this case it will always converge to zero.

Theory of fixed point iteration

Previously (last lecture) we saw that we can use fixed point iteration

$$x_{n+1} = g(x_n)$$

to find a solution to the equation $x = g(x)$. We can easily see now that if fixed point iteration converges to a limit L then L must be a root of the equation i.e. $L = g(L)$.

The question is when will fixed point iteration converge?

Suppose $\hat{x}$ is a solution of $x = g(x)$ so that $\hat{x} = g(\hat{x})$. We want $x_n \to \hat{x}$. If we denote the difference between x_n and $\hat{x}$ as E_n then we have

$$x_n = \hat{x} + E_n$$

If $x_n \to \hat{x}$ as $n \to \infty$ then we need $E_n \to 0$. Now $x_{n+1} = g(x_n)$ becomes

$$\hat{x} + E_{n+1} = g(\hat{x} + E_n)$$

We now use a Taylor series for g to find that

$$\hat{x} + E_{n+1} = g(\hat{x}) + g'(\hat{x})E_n + \frac{1}{2}g''(\hat{x})E_n^2 + \cdots$$

Since $g(\hat{x}) = \hat{x}$ and E_n^2 is small this gives

$$E_{n+1} \approx g'(\hat{x})E_n$$

Hence $\{E_n\}$ is approximately a geometric sequence with $r = g'(\hat{x})$.

Fixed point iteration again

Remember this sequence

$$a_{n+1} = \frac{a_n + 2}{a_n}$$

Which is of the form $x_{n+1} = g(x_n)$ with $g(x) = \frac{x+2}{x}$ and has two possible limiting values $L = -1$ or $L = 2$ (the roots of $x = g(x)$).

Theoretically it will converge to a limit L only if $|g'(L)| < 1$. Since

$$g'(x) = -\frac{2}{x^2}$$

We have $g'(-1) = -2$ and $g'(2) = -\frac{1}{2}$.

Does it converge near -1 (try starting at $x_0 = -1.01$)?

Rate of convergence

For a *geometric sequence*

$$x_{n+1} = rx_n$$

if $|r| < 1$ then the sequence converges to zero. We say that the rate of convergence is $|r|$.

Example: Suppose $x_n = \frac{1}{3}x_n$ and $x_0 = 1$. Then

$$\{x_n\} = 1,\ \frac{1}{3},\ \frac{1}{9},\ \cdots,\ \left(\frac{1}{3}\right)^n,\ \cdots,$$

which converges with rate $|r| = \frac{1}{3}$.

Rate of convergence for fixed point iteration

Fixed point iteration tries to find solutions of $x = g(x)$ using the sequence

$$x_{n+1} = g(x_n)$$

The error $E_n = x_n - \hat{x}$ is approximately a geometric sequence with

$$E_{n+1} = g'(\hat{x})E_n$$

so the rate of convergence of the method is $|g'(\hat{x})|$.

For our example of $x = \frac{x+2}{x}$ fixed point iteration will converge to the root $x = 2$ with rate $|g'(2)| = \frac{1}{2}$.

Rate of convergence for Newton-Raphson

The Newton-Raphson method (last lecture) finds a solution of $f(x) = 0$ with the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We can think of this as fixed-point iteration to solve $x = g(x)$ but with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Now it's easy to see that if this converges to $\hat{x}$ then $\hat{x} = g(\hat{x})$ and also $f(\hat{x}) = 0$ so $\hat{x}$ will be a root of f.

The clever part comes when we check the *rate* of convergence...

Rate of convergence for Newton-Raphson

For fixed-point iteration the error sequence $\{E_n\}$ satisfies

$$E_{n+1} = g'(\hat{x})E_n + \frac{1}{2}g''(\hat{x})E_n^2 + \cdots$$

but if $g(x) = x - \frac{f(x)}{f'(x)}$ we have

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

Now since $f(\hat{x}) = 0$ (since $\hat{x}$ is a root) provided $f'(\hat{x}) \neq 0$ the rate of convergence $|g'(\hat{x})| = 0$ so the error sequence satisfies

$$E_{n+1} = \frac{1}{2}g''(\hat{x})E_n^2 + \cdots$$

In other words provided $f'(\hat{x}) \neq 0$ we have $E_{n+1} \propto E_n^2$.

Example of rates of convergence

Fixed-point iteration converging with rate $g'(\hat{x}) = r = 0.1$ has an error sequence that looks like

$$\{E_n\} = 0.1,\ 0.01,\ 0.001,\ 0.0001,\ \cdots,\ 0.1^{(n+1)},\ ,\cdots$$

Newton-Raphson with quadratic convergence looks like

$$\{E_n\} = 0.1,\ 0.01,\ 0.0001,\ 0.00000001,\ \cdots,\ 0.1^{2^n}\cdots,$$

In this example each iteration of fixed-point gives us 1 extra correct digit. For Newton-Raphson each iteration usually *doubles* the number of correct digits.

Summary of rates of convergence

- Fixed-point iteration solves $x = g(x)$ with $x_{n+1} = g(x_n)$
  - Converges to a root $\hat{x}$ provided initial guess x_0 is close and $|g'(\hat{x})| < 1$.
  - Converges linearly with rate $r = |g'(\hat{x})|$.
  - $|E_{n+1}| \approx r|E_n|$.
- Newton-Raphson iteration solves $f(x) = 0$ with $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
  - Usually converges to a root $\hat{x}$ provided initial guess x_0 is close.
  - Converges quadratically if $f'(\hat{x}) \neq 0$.
  - $|E_{n+1}| \propto |E_n^2|$.
- Intermediate value theorem $f(a)f(b) < 0$ with $a_{n+1} = \frac{1}{2}(a_n + b_n)$ etc.
  - Converges *always*.
  - Converges linearly with rate $\frac{1}{2}$.
  - $|E_{n+1}| \approx \frac{1}{2}|E_n|$.

Summary - sequences

- abstract definition of convergence:
  for all, $\epsilon > 0$ there exists a N such that $|a_n - L| < \epsilon$ for all $n > N$
- use of properties of limits, e.g.
  $\lim_{n\to\infty} a_n = L, \quad \lim_{n\to\infty} b_n = M \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$
- iterative sequences $x_{n+1} = g(x_n)$
- use to find a fixed point $L = g(L)$
- rate of convergence $r = g'(L)$
- need $|r| < 1$ for convergence; rearrange if necessary
- Homework (lectures 1 & 2)
  - *Read* sections 7.1, 7.2.1 & 7.5
  - *Do* exercises 7.5.4: 36–38
  - *Read* section 9.3 & Skim read section 7.4
  - *Do* exercises 9.3.3. 2,4 & 7