Roughly : Vector calculus = modelling tools for engineering systems.

Linear Systems & Partial Differential Equations

= analysis tools for engineering systems.

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EMAT20200, weeks 10–15

0. Contents

1. Fourier series } frequency analysis
2. Fourier transform
3. Laplace transform } stability analysis
4. Introduction to partial differential equations (PDEs)
5. Solving PDEs (1): separation of variables
6. Solving PDEs (2): d'Alembert method

analysis tools for ODE systems.

analysis tools for PDE systems.

e.g. $\frac{\partial u}{\partial t} = k\nabla^2 u$

Support

- On Blackboard:
  - Example sheets and solutions
  - Annotated lecture notes
  - Past exam papers and solutions and formula sheets
  - Discussion forum
- Drop-in sessions: 1-1 help from PGs
  - Tuesdays and Thursdays 1-2pm (QB 1.68) and 5-6pm (MVB cafe/atrium)
  - I will be at the Thursday 5-6pm session

best places for 1-1 help.

- QMP quizzes → zero-weighted!
- Homework (formative): due 12noon Friday 8 February ~~2018~~ 2019 (week 14)
- Class test (formative)

* Textbook: James et al Modern Engineering Mathematics.

1. Fourier series

↳ Frequency analysis, of periodic signals (or functions)

- What is the difference between a periodic signal and a sinusoidal signal?
- Periodic signals and functions.
- Odd and even functions.
- How sinusoidal is a signal? Expressing periodic functions in terms of sines and cosines. Half-range sine and cosine series.
- Lots of worked examples!
- The Gibbs phenomenon.

[James Advanced MEM (4th Edn) Ch. 7]

$m\ddot{x} + c\dot{x} + kx = f(t)$ forcing

We are used to seeing signals represented as waveforms; e.g.:

? Use a Fourier series.

- the output of an AC circuit,
- the trace of accelerations in an earthquake,
- representation of music or speech (e.g. in an MP3 player),
- or the characteristic vibrations of a structural component,
- or even the number of hours of daylight over the course of a year.

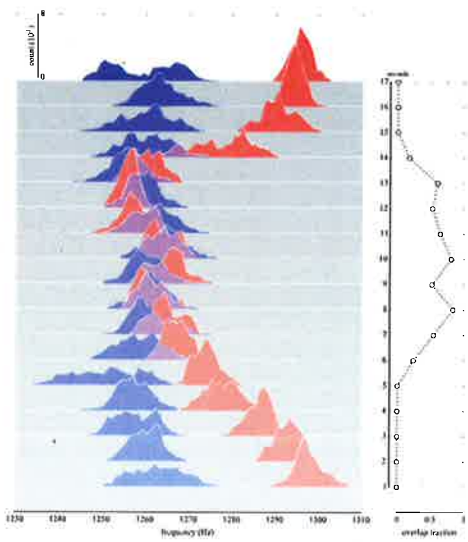
audio, video compression → Fourier series. ↳ processing

Sometimes these signals look sinusoidal, sometimes they do not.

How can we tell 'how sinusoidal' a signal is? In other words, how close is it to a "pure frequency"?

In this chapter we are going to deal with stationary signals, which have a fixed frequency (=2π/period). These are periodic functions.

Engineering application: frequency analysis



- How do mosquitoes communicate with each other?
- Mosquitoes collaborate to form a swarm, compete to find a mate
- Interaction in the frequency domain: they listen and respond to each others' wing beats
- Implications for swarm control, robotics, UAVs, nanoparticles, sensor networks, ...

TIME-SCALING OF PERIODIC FUNCTIONS.

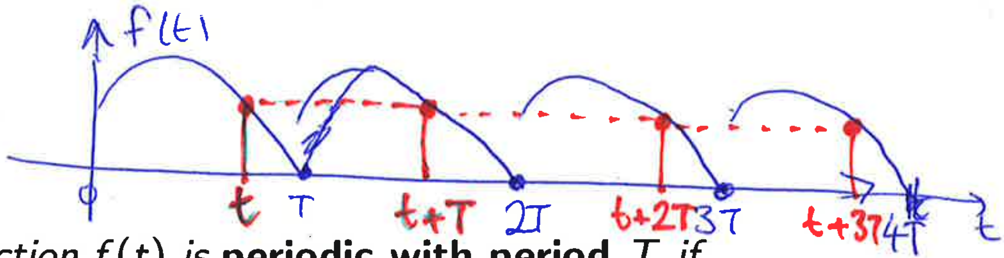
If $f(t)$ has period T then $f(at)$ has period $\frac{T}{a}$
(for any constant a)

Why? Why T/a?

Let's define $h(t) = f(at)$
& check that $h(t + T/a) = h(t)$

$$\rightarrow \quad h(t + T/a) = f(a(t + T/a))$$
$$= f(at + T)$$
$$= f(at) \quad \text{f has period } T$$
$$= h(t)$$

Periodic functions



Definition: *A function* $f(t)$ *is* **periodic with period** T *if* $f(t+T) = f(t)$ *for all* t*, and* $T > 0$ *is the smallest positive constant for which this is true.* $f(t + kT) = f(t)$ for all integer k.

The simplest periodic functions that we use every day are the *trigonometric functions* $\sin(t)$ and $\cos(t)$ which both have period 2π.

SUMS OF PERIODIC FUNCTIONS

Note that sums of periodic functions are also periodic. That is, if $f(t)$ and $g(t)$ are periodic with the same period T, then so is $\underbrace{af(t) + bg(t)}_{\text{period } T}$ for any scalars a and b.

Moreover, if $f(t)$ is periodic with period T and $g(t)$ is periodic with period T/n for any integer $n > 0$, then $\underbrace{a\underbrace{f(t)}_{\text{period } T} + b\underbrace{g(t)}_{\text{period } T/n}}_{\text{period } T}$ is also periodic with period T

BUT if $f(t)$ has period T & $g(t)$ has period $2T/3$ then $af(t)+bg(t)$ does not have period T

if " & $g(t)$ has period $\sqrt{2}\,T$ then $af(t)+bg(t)$ does not have period T.

Periods, frequencies, examples

NOTE: In these lectures 'frequency' ω means circular angular frequency and is measured in radians per second. To convert to Hz we have to divide by 2π, i.e.

If f has period T, then its frequency ω is $\omega = \dfrac{2\pi}{T}$ [rad s^{-1}]

In Hertz, frequency f is $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ [Hz] [s^{-1}]

Worked example 1.1 *Calculate the minimum period* and frequency *of the following functions:*

(a) $\sin 2t$,

(b) $\tan t$,

(c) $b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t$,

(d) $\cos \frac{2\pi t}{L}$

(e) $\sin \frac{2n\pi t}{L}$

(f) $\cos^2(2\pi t)$

Ex 1.1

We'll use the fact that

$\sin t$ is periodic, with period $T = 2\pi$, frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

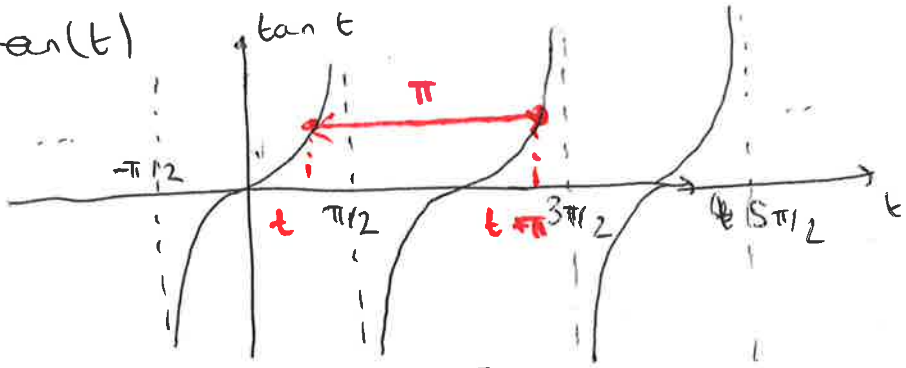
(a) $\sin(\underbrace{2}_{=a}t) = \sin(at)$ & use the time-scaling rule

↳ has period $T = \frac{2\pi}{a} = \frac{2\pi}{2} = \pi$

frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

lesson 1: two functions with the same period (and frequency) need not be the same.

(b) $\tan(t)$



has period $T = \pi$

frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$.

(c) $b_1 \sin t \quad + \quad b_2 \sin 2t \quad + \quad b_3 \sin 3t$

$b_1 \sin t$: period 2π, freq $\frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$b_2 \sin 2t$: period $\frac{2\pi}{2} = \pi$, freq. $\frac{2\pi}{\pi} = 2$

$b_3 \sin 3t$: period $\frac{2\pi}{3}$, freq. $\frac{2\pi}{2\pi/3} = 3$

period 2π (using sums of periodic functions rule)

period $T = 2\pi$

frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

moral: a periodic function f with frequency ω can contain components with frequency $\omega, 2\omega, 3\omega, 4\omega,$

(d) $\cos \underbrace{\frac{2\pi t}{L}}$ $=a$ — we know that $\cos t$ has period $T = 2\pi$, freq. $\omega = \frac{2\pi}{T} = 1$

$\cos(at)$ has period $T/a = \frac{2\pi}{a}$, freq. $\omega = \frac{2\pi}{T/a} = 2\pi / (2\pi/a) = a$

↳ period $\frac{2\pi}{2\pi/L} = L$

freq. $2\pi/L$

(e) $\sin\left(\frac{2n\pi t}{L}\right)$ a

$\hookrightarrow$ period $= \frac{2\pi}{2n\pi/L} = \frac{L}{n}$

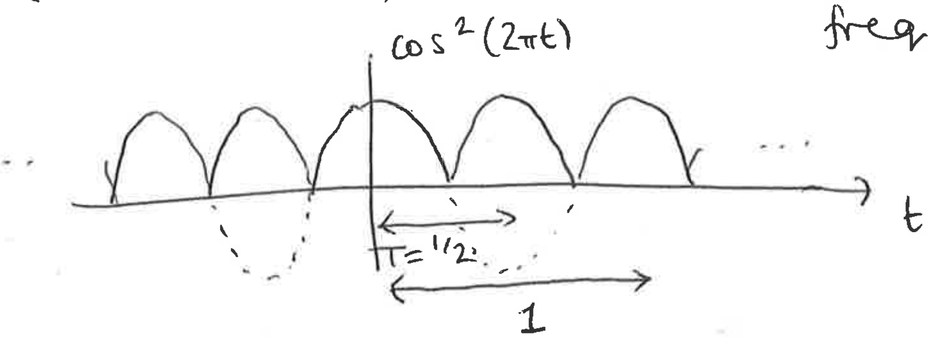
freq. $\frac{2n\pi}{L}$

moral

$\cos at$, $\sin at$ } have freq. a

(f) $\cos^2(2\pi t)$ $\longrightarrow$ period $= 1/2$

freq $= \frac{2\pi}{1/2} = 4\pi$



moral:

nonlinear combinations of periodic functions do complicated things to the period.

Fourier series: the method

We want to find a way of representing periodic functions in terms of sines and cosines.

That is precisely what Fourier series are. We will first present the calculation of Fourier series as a black box method, before coming back to *why* it works.

By computing a function's Fourier series, we figure out a combination of cos and sin functions that approximates our function. Consider a periodic function of period T:

$$f(t+T) = f(t).$$

Q. How close is $f(t)$ to $\cos(\omega t)$ or $\sin(\omega t)$, where $\omega = (2\pi/T)$?

a tool for frequency analysis of periodic functions.

Finding the Fourier series

Answer: *Pretty much any f* can be expanded as a series of sines and cosines!

periodic, with period T, freq. $\omega = \frac{2\pi}{T}$

THE FOURIER SERIES OF f: how much frequency $n\omega$ is in f.

$$f(t) \overset{*}{\sim} \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \underbrace{\cos(n\omega t)}_{} + \sum_{n=1}^{\infty} b_n \underbrace{\sin(n\omega t)}_{} = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

periodic functions frequency $n\omega$

$|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$

where $\omega = \frac{2\pi}{T}$,

DC offset average of f over one period.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

FOURIER COEFFICIENTS. (how to find a_n, b_n).

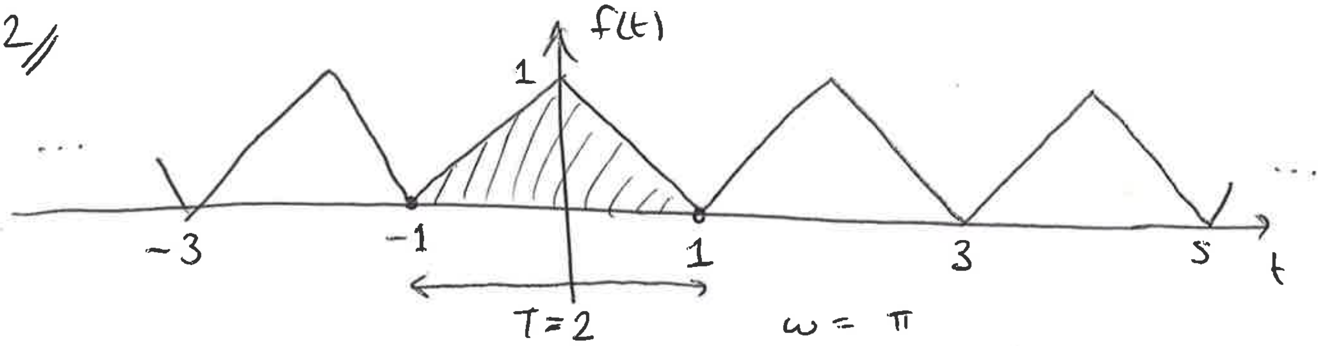
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt \qquad (1)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt$$

integrals over one period of f

* $\sim$ means = unless f is discontinuous. (then the Fourier Series converges to the midpoint of the discontinuity)

Ex 1.2



$\omega = \pi$

Fourier series.

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt = \int_{-1}^{1} f(t)\cos(n\pi t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt = \int_{-1}^{1} f(t)\sin(n\pi t)\,dt.$$

Step 1: calculate the a_n's.

$\int uv' = uv - \int u'v$

$$a_n = \int_{-1}^{0} \underbrace{(1+t)}_{u}\,\underbrace{\cos(n\pi t)}_{v'}\,dt + \int_0^1 \underbrace{(1-t)}_{u}\,\underbrace{\cos(n\pi t)}_{v'}\,dt$$

integrate by parts.

$$= \left[\underbrace{(1+t)}_{u}\cdot\underbrace{\cancel{\frac{1}{n\pi}\sin(n\pi t)}}_{v}\right]_{-1}^{0} - \int_{-1}^{0} \underbrace{1}_{u'}\cdot\underbrace{\frac{1}{n\pi}\sin(n\pi t)}_{v}\,dt$$

$$+ \left[\underbrace{(1-t)}_{u}\cdot\underbrace{\cancel{\frac{1}{n\pi}\sin(n\pi t)}}_{v}\right]_{0}^{1} - \int_{0}^{1} \underbrace{-1}_{u'}\cdot\underbrace{\frac{1}{n\pi}\sin(n\pi t)}_{v}\,dt.$$

$$= -\int_{-1}^{0}\frac{1}{n\pi}\sin(n\pi t)\,dt + \int_0^1 \frac{1}{n\pi}\sin(n\pi t)\,dt$$

$$= \left[\frac{1}{(n\pi)^2}\cos(n\pi t)\right]_{-1}^{0} + \left[-\frac{1}{(n\pi)^2}\cos(n\pi t)\right]_0^1$$

$$= \frac{1}{(n\pi)^2}\Big(1 - \underbrace{\cos(-n\pi)}_{\cos(n\pi)}\Big) - \frac{1}{(n\pi)^2}\big(\cos(n\pi) - 1\big)$$

$$\boxed{a_n = \frac{2}{(n\pi)^2}\big(1-\cos(n\pi)\big) = \frac{2}{(n\pi)^2}\big(1-(-1)^n\big)}$$

for n integer $\cos(n\pi) = (-1)^n$

Step 2: calculate a_0

$$a_0 = \int_{-1}^{1} f(t)\,dt = \text{area under curve of } f \text{ over 1 period.}$$

$$= \tfrac{1}{2}\cdot 2\cdot 1 = 1 \qquad \boxed{a_0 = 1}$$

low freq. components are most important

high freq. $\propto \frac{1}{n^2} \to 0$ quickly.

- we've done signal compression

- note half the Fourier coefficients are zero!

$$a_n = \begin{cases} 0 & n \text{ is even } \geq 2 \\ \frac{4}{(n\pi)^2} & n \text{ is odd} \end{cases}$$

$$= \frac{4}{(2m-1)^2\pi^2} \qquad n = 2m-1 \qquad m = 1, 2, 3, 4, \ldots$$

We can use this information to write a better Fourier series. → sum over the nonzero coefficients only.

$$\boxed{f(t) \sim \frac{1}{2} + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi^2} \cos\left((2m-1)\pi t\right)}$$

Step 3: calculate b_n

$$b_n = \int_{-1}^{0} \underbrace{(1+t)}_{u} \underbrace{\sin(n\pi t)}_{v'}\, dt + \int_0^1 \underbrace{(1-t)}_{u} \underbrace{\sin(n\pi t)}_{v'}\, dt$$

$$= \left[\underbrace{(1+t)}_{u} \cdot \underbrace{-\frac{1}{n\pi}\cos(n\pi t)}_{v} \right]_{-1}^{0} - \int_{-1}^{0} \underbrace{1}_{u'} \cdot \underbrace{-\frac{1}{n\pi}\cos(n\pi t)\, dt}_{v}$$

$$+ \left[\underbrace{(1-t)}_{u} \cdot \underbrace{-\frac{1}{n\pi}\cos(n\pi t)}_{v} \right]_{0}^{1} - \int_0^1 \underbrace{-1}_{u'} \cdot \underbrace{-\frac{1}{n\pi}\cos(n\pi t)}_{v}\, dt$$

$$= \cancel{\left(1 \cdot -\frac{1}{n\pi} - 0\right) + \left(0 \cdot - -\frac{1}{n\pi}\right)}^{\,0}$$

$$+ \int_{-1}^{0} \frac{1}{n\pi}\cos(n\pi t)\, dt - \int_0^1 \frac{1}{n\pi}\cos(n\pi t)\, dt$$

$$= \left[\frac{1}{(n\pi)^2}\sin(n\pi t)\right]_{-1}^{0} - \left[\frac{1}{(n\pi)^2}\sin(n\pi t)\right]_{0}^{1}$$

$$= \frac{1}{(n\pi)^2}\left(0 - \sin(-n\pi)\right) - \frac{1}{(n\pi)^2}\left(\sin(n\pi) - 0\right)$$

$$\boxed{b_n = 0}$$

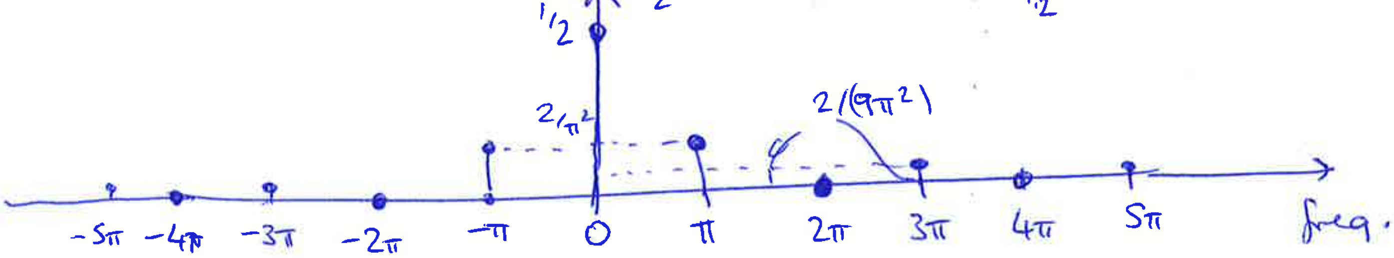
$\sin(n\pi) = 0$ for all integer n.

Step 4: put it all together

$$\boxed{f(t) \sim \underbrace{\frac{1}{2}}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underbrace{\frac{2}{(n\pi)^2}(1-(-1)^n)}_{a_n} \underbrace{\cos(n\pi t)}_{\cos(n\omega t)}}$$

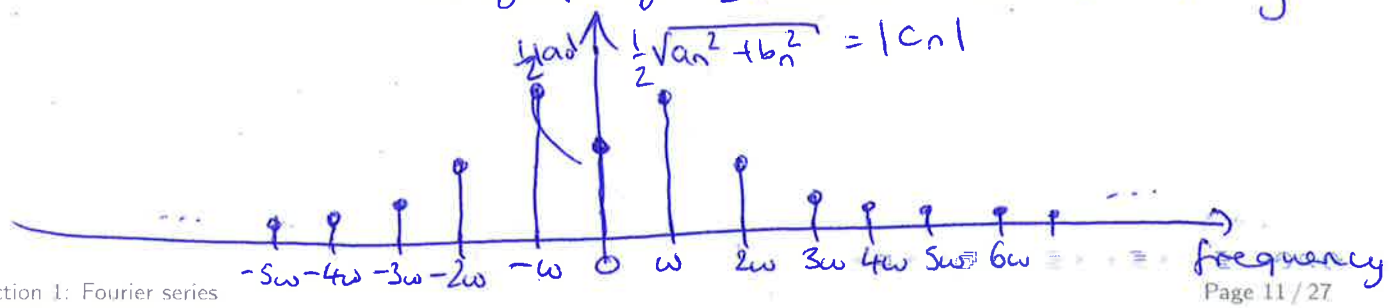
- So what?
  - we've frequency analysed $f(t)$
  - plot the line spectrum to visualise

$$\frac{1}{2}\sqrt{a_n^2 + b_n^2} = \begin{cases} \frac{1}{(n\pi)^2}(1-(-1)^n) & |n| \geq 1 \\ 1/2 & n = 0 \end{cases}$$



- This expansion is called the Fourier series representation of f.
- It can be shown that the Fourier series converges to f as $n \to \infty$ provided f is continuous.
- We say that $\underbrace{a_n \cos(n\omega t) + b_n \sin(n\omega t)}_{\text{the bit of } f \text{ with frequency } n\omega}$ is the *nth harmonic component* of f.
- Average power in the *nth harmonic component* is $\frac{1}{2}(a_n^2 + b_n^2)$
- We say that $a_0/2$ is the DC component of f.
- Visualise the Fourier series in a *line spectrum*

↳ a graph of $\frac{1}{2}\sqrt{a_n^2 + b_n^2}$ vs. frequency



Worked Example 1.2

Calculate the Fourier series approximation to the periodic function that has period $T = 2$ and can be expressed on its fundamental domain as

frequency $\omega = \frac{2\pi}{T} = \pi$

$$|t| = \begin{cases} t & t > 0 \\ -t & t < 0 \end{cases}$$

$$f(t) = 1 - |t|, \quad -1 < t < 1 \,. \quad = \begin{cases} 1 - t & 0 \le t < 1 \\ 1 + t & -1 \le t < 0 \end{cases}$$

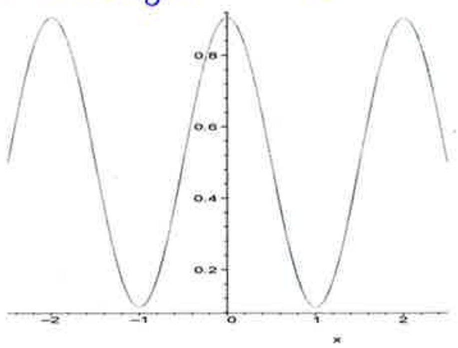
In this case the answer is

$$\begin{aligned} f(t) &\sim \frac{1}{2} + \frac{4}{\pi^2}\cos(\pi t) + \frac{4}{9\pi^2}\cos(3\pi t) + \frac{4}{25\pi^2}\cos(5\pi t) + \ldots \\ &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi^2}\cos[(2m-1)\pi t] \end{aligned}$$

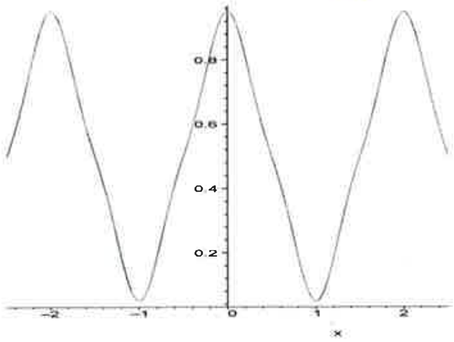
Looks pretty good ...

plot graphs of truncated Fourier series.

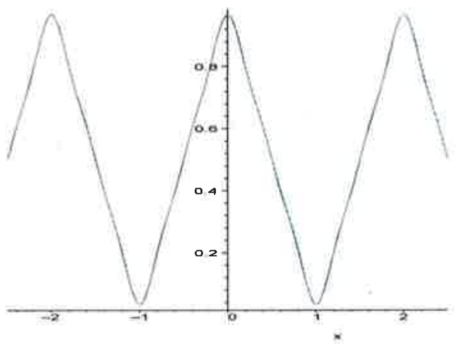
first 2 nonzero terms

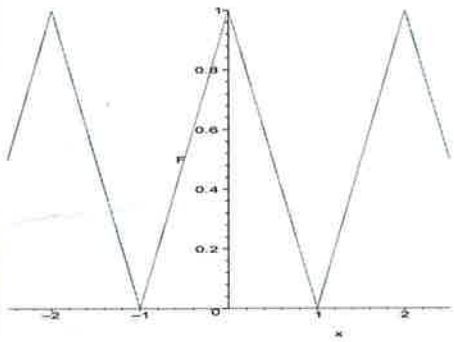


first 3 non-zero terms



first 4



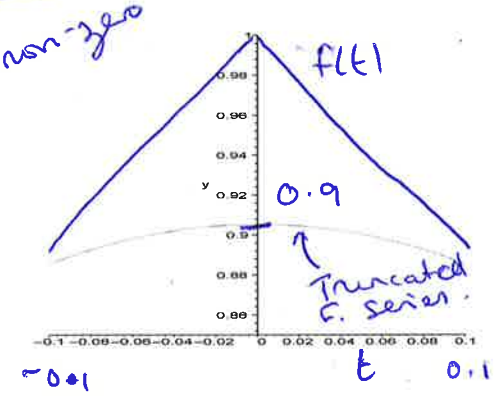


original function

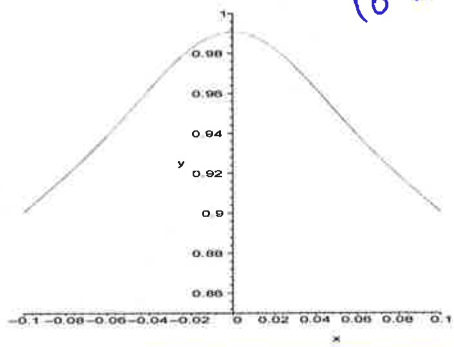
... except near to the corner point

E.g. plotting a zoom of the series truncated after the 2nd, 10th and 100th nonzero term:

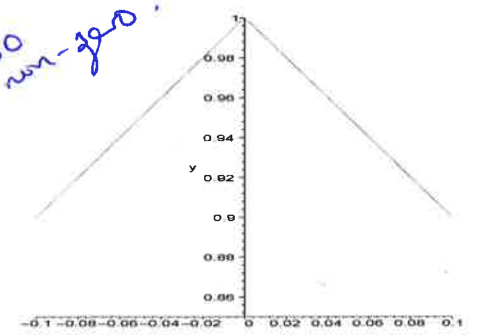
2 non-zero

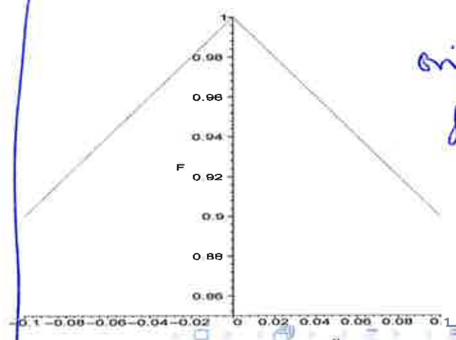


10 non-zero



100 non-zero.





original function

even function : $f(t) = f(-t)$ for all t

odd function : $f(t) = -f(-t)$ for all t

- The convergence in Worked Example 1.2 is rapid. What does this mean? It means that although formally the whole infinite sum converges to f, to get a good approximation we only need the first few terms!
- There are no sine terms in Worked Example 1.2. Why not?
  – because $f(t)$ ~~w~~ is an even function

Theorem (Fourier series of odd and even functions):

If $f(t)$ is an even function then

- $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$
- $b_n = 0$ for all $n \geq 1$
- $a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega t)\,dt$

Fourier series has only cos terms, no sin

If $f(t)$ is an odd function then

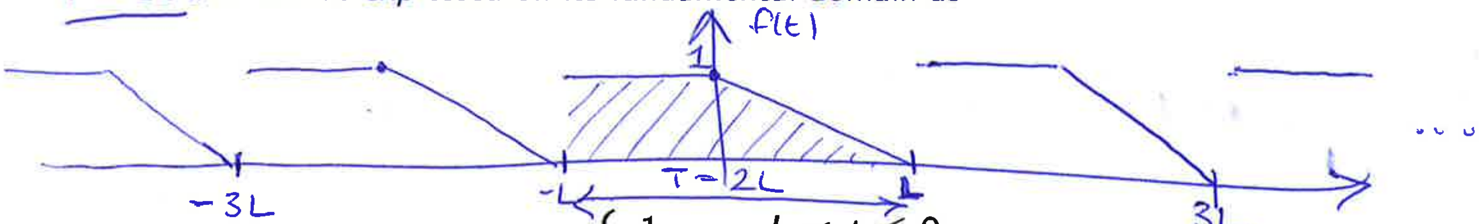
- $f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$
- $a_n = 0$ for all $n \geq 0$
- $b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega t)\,dt$

Fourier series has only sin terms, no cos.

Worked Example 1.3

& frequency $\omega = \frac{2\pi}{T} = \frac{\pi}{L}$

Calculate the Fourier series approximation to the periodic function that has period $T = 2L$ and can be expressed on its fundamental domain as



$$f(t) = \begin{cases} 1\,, & -L < t \leq 0\,, \\ 1 - \frac{t}{L} & 0 < t \leq L\,. \end{cases}$$

The answer is

$$f(t) \sim \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi t}{L}\right) + \frac{(-1)^n}{n\pi}\sin\left(\frac{n\pi t}{L}\right)\right)$$

Ex 1.3

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt$$

Step 1 : calculate a_n $(n \geq 1)$

$$a_n = \frac{1}{L}\left\{\int_{-L}^{0} 1.\cos\left(\frac{n\pi t}{L}\right) dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)\cos\left(\frac{n\pi t}{L}\right) dt\right\}$$

homework!

$$= \frac{1-(-1)^n}{n^2\pi^2}$$

Step 2 : calculate a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{1}{L}\left(L\times 1 + \frac{1}{2}\times L\times 1\right) = \frac{1}{L}\cdot\frac{3}{2}L = \frac{3}{2}$$

Step 3 : calculate b_n $(n \geq 1)$

$$b_n = \frac{1}{L}\left\{\int_{-L}^{0} 1.\sin\left(\frac{n\pi t}{L}\right) dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)\sin\left(\frac{n\pi t}{L}\right) dt\right\}.$$

homework!

$$= \frac{(-1)^n}{n\pi}$$

Step 4: put it all together

$$\boxed{f(t) \sim \frac{3}{4} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi t}{L}\right) + \frac{(-1)^n}{n\pi}\sin\left(\frac{n\pi t}{L}\right)}$$

plot the line spectrum

line heights decay slower than before. $\propto \frac{1}{n}$ (slow)

high frequencies more important?