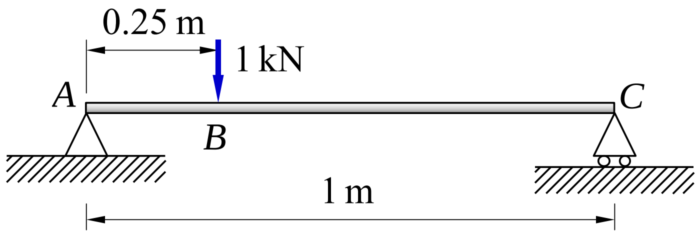
Example 2.3.2(a) – Plot the curvature, slope and deflection for the simply-supported beam below. The beam is made of aluminium alloy with $E = 70\ \mathrm{GPa}$ and has a solid square cross-section measuring 40 mm × 40 mm.



We start by finding support reactions,

$$\sum M_{@A}^{CW} = 0,$$

$$M_A + (1\ \mathrm{kN})\left(\frac{1}{4}\ \mathrm{m}\right) - (R_C)(1\ \mathrm{m}) = 0.$$

The extremity A is pinned, therefore $M_A = 0$ and,

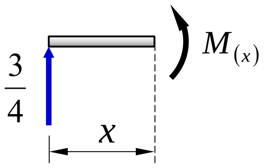
$$R_C = (1\ \mathrm{kN})\left(\frac{1}{4}\ \mathrm{m}\right)\left(\frac{1}{1\ \mathrm{m}}\right) \qquad \therefore \qquad R_C = \frac{1}{4}\ \mathrm{kN}.$$

Vertical equilibrium gives,

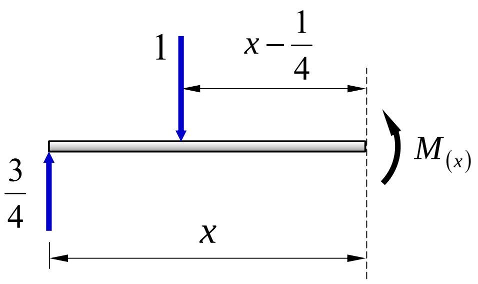
$$\sum F = 0,$$

$$R_A + R_C - (1\ \mathrm{kN}) = 0 \qquad \therefore \qquad R_A = -\frac{3}{4}\ \mathrm{kN}.$$

We find two different moment equations, depending on where we section the beam,



$$M_{(x)} - \left(\frac{3}{4}\right)(x) = 0 \qquad \therefore \qquad M_{(x)} = \frac{3}{4}x$$



$$M_{(x)} - \left(\frac{3}{4}\right)(x) + (1)\left(x - \frac{1}{4}\right) = 0 \qquad \therefore \qquad M_{(x)} = \frac{3}{4}x - \left(x - \frac{1}{4}\right)$$

In order to **combine both equations into one**, we use the Heaviside step function,

$$M_{(x)} = \frac{3}{4}x - \left[\left(x - \frac{1}{4}\right)H\left(x - \frac{1}{4}\right)\right].$$

The curvature equation is therefore,

$$M_{(x)} = EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{3}{4}x - \left[\left(x-\frac{1}{4}\right)H\left(x-\frac{1}{4}\right)\right]. \qquad (1)$$

Integrating once gives the slope,

$$EI\,\phi_{(x)} = EI\,\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3}{8}x^2 - \left[\frac{1}{2}\left(x-\frac{1}{4}\right)^2 H\left(x-\frac{1}{4}\right)\right] + A. \qquad (2)$$

Integrating again gives the deflection,

$$EI\,v_{(x)} = \frac{1}{8}x^3 - \left[\frac{1}{6}\left(x-\frac{1}{4}\right)^3 H\left(x-\frac{1}{4}\right)\right] + Ax + B. \qquad (3)$$

The first boundary condition is,

$$x = 0,\; v = 0 \qquad \therefore \qquad B = 0.$$

And the second boundary condition is:

$$x = 1,\; v = 0 \qquad \therefore \qquad \frac{1}{8}(1)^3 - \frac{1}{6}\left(1-\frac{1}{4}\right)^3 + A(1) = 0 \qquad \therefore \qquad A = -\frac{7}{128}\ \mathrm{kN\ m^2}.$$

In order to draw the three graphs we need to define a few points.

- First we compute the <u>flexural modulus</u>.

$$EI = \left(70\cdot 10^9\,\frac{\mathrm{N}}{\mathrm{m}^2}\right)\left[\frac{(0.04\ \mathrm{m})^4}{12}\right] \qquad \therefore \qquad EI = 14.9\bar{3}\ \mathrm{kN\ m^2}.$$

- <u>Point of maximum downward deflection</u>. The span *AC* will 'sag' under the applied load (*i.e.* it will have a 'smiley face' type deformation), and the maximum downward deflection will not be at *B* nor at the mid-span, but somewhere else along the span *BC*. This minimum in $v_{(x)}$ is characterised by a <u>zero first derivative</u> (*i.e.* $\phi(x) = 0$), therefore,

$$EI\,\phi_{x_{(v_{\min})}} = 0 \qquad \therefore \qquad \frac{3}{8}x^2_{(v_{\min})} - \frac{1}{2}\left[x_{(v_{\min})} - \frac{1}{4}\right]^2 - \frac{7}{128} = 0 \qquad \therefore \qquad -\frac{1}{8}x^2_{(v_{\min})} + \frac{1}{4}x_{(v_{\min})} - \frac{11}{128} = 0.$$

The roots of a 2nd order polynomial are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which gives $\begin{cases} x_+ = 0.44\ \mathrm{m} \\ x_- = 1.56\ \mathrm{m} \end{cases}$.

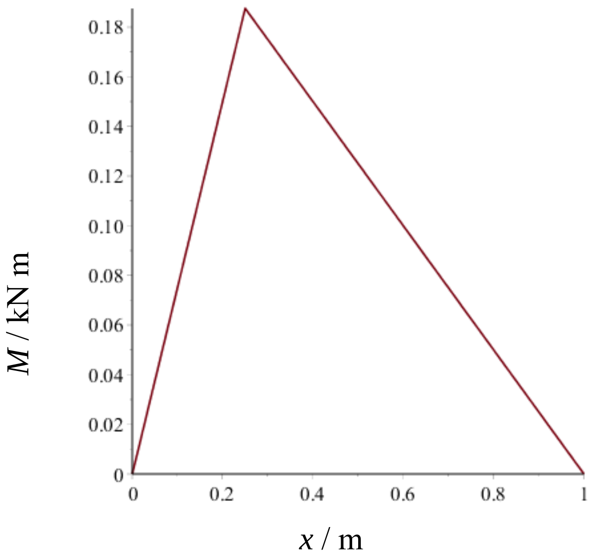
The beam is 1 m long so only the first root is valid, and we have $x_{(v_{\min})} = 0.44\ \mathrm{m}$.

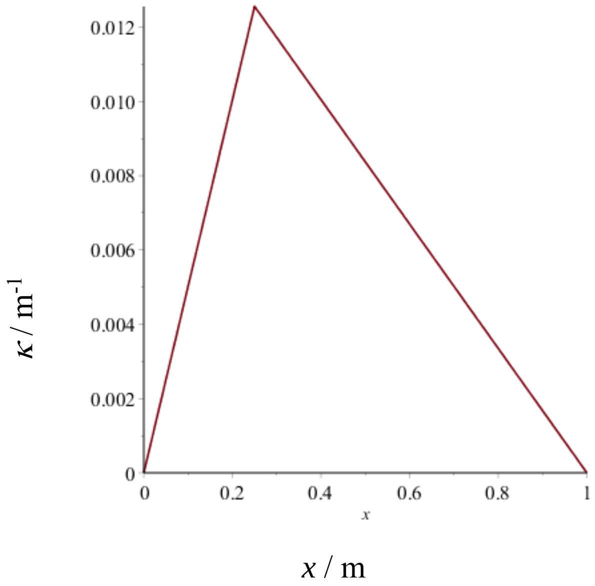
- <u>Maximum downward deflection</u>. Replacing $x = 0.44$ m in equation (3) gives,

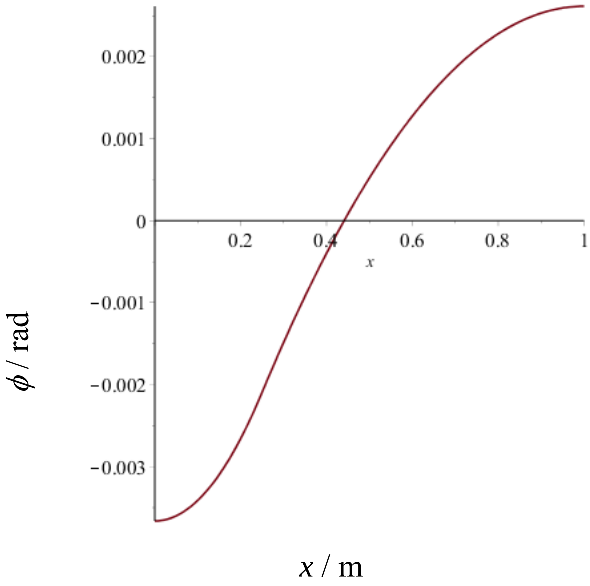
$$EI\,v_{\min} = \frac{1}{8}(0.44)^3 - \frac{1}{6}(0.44 - 0.25)^3 - \frac{7}{128}(0.44) \qquad \therefore \qquad v_{\min} \cong 0.975\ \mathrm{mm}$$

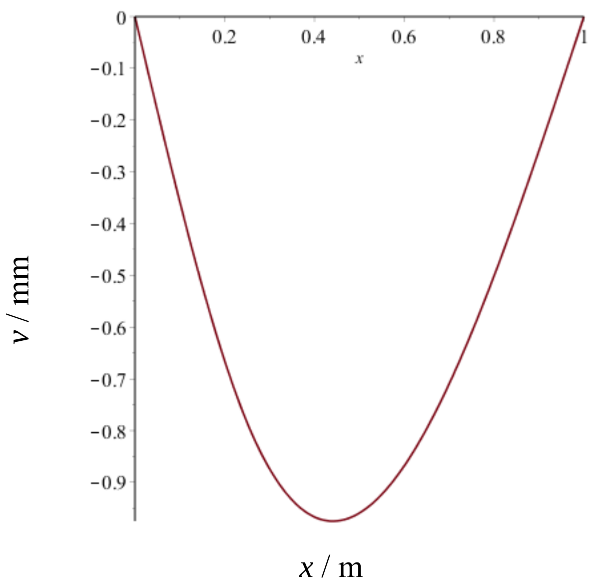
- <u>Local curvature</u>. Remember that $M = \kappa \cdot EI$ $\qquad \therefore \qquad \kappa = \dfrac{M}{EI}$ $\qquad (4)$

Further points may be found by substituting *x* values in equations (1), (2), (3) and (4). The final graphs are shown below.


0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
0
0.2
0.4
0.6
0.8
1
M / kN m
x / m


0.012
0.010
0.008
0.006
0.004
0.002
0
0
0.2
0.4
0.6
0.8
1
x
κ / m-1
x / m


0.002
0.001
0
−0.001
−0.002
−0.003
0.2
0.4
0.6
0.8
1
x
ϕ / rad
x / m


0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
−0.9
0.2
0.4
0.6
0.8
1
x
v / mm
x / m