Advanced Bending and Torsion
Shear Centre of Complex Thin-Walled Sections

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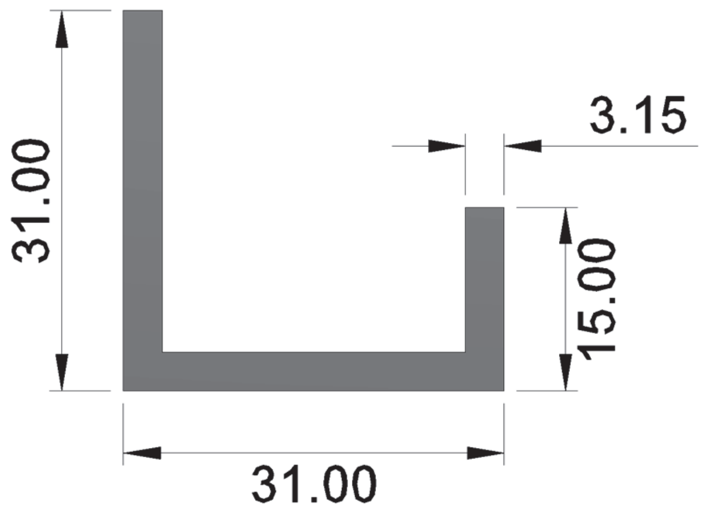
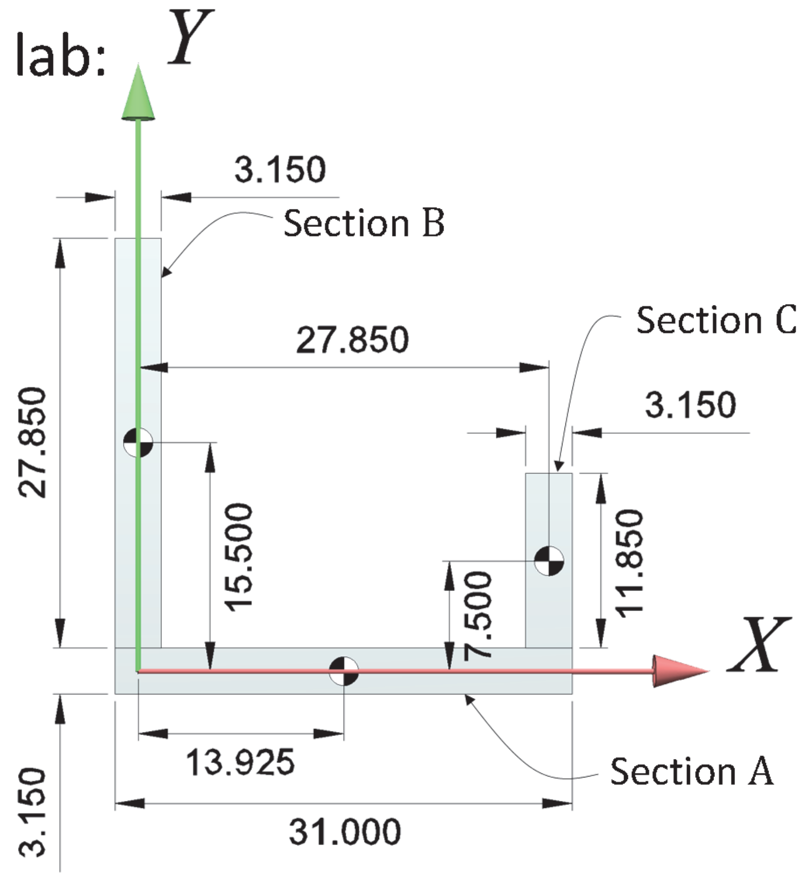
Lipped Section Example

- Lipped section seen in the Structures lab:





$$A_{\mathrm{A}} = (31.00)(3.15)\ \mathrm{mm}^2 \qquad A_{\mathrm{B}} = (3.15)(27.85)\ \mathrm{mm}^2 \qquad A_{\mathrm{C}} = (3.15)(11.85)\ \mathrm{mm}^2$$

$$A_{\mathrm{A}} = 97.65\ \mathrm{mm}^2 \qquad A_{\mathrm{B}} = 87.73\ \mathrm{mm}^2 \qquad A_{\mathrm{C}} = 37.33\ \mathrm{mm}^2$$

$$\bar{X}_{\mathrm{A}} = 13.925\ \mathrm{mm} \qquad \bar{X}_{\mathrm{B}} = 0 \qquad \bar{X}_{\mathrm{C}} = 27.85\ \mathrm{mm}$$

$$\bar{Y}_{\mathrm{A}} = 0 \qquad \bar{Y}_{\mathrm{B}} = 15.50\ \mathrm{mm} \qquad \bar{Y}_{\mathrm{C}} = 7.50\ \mathrm{mm}$$

- Centroid of the compound section:

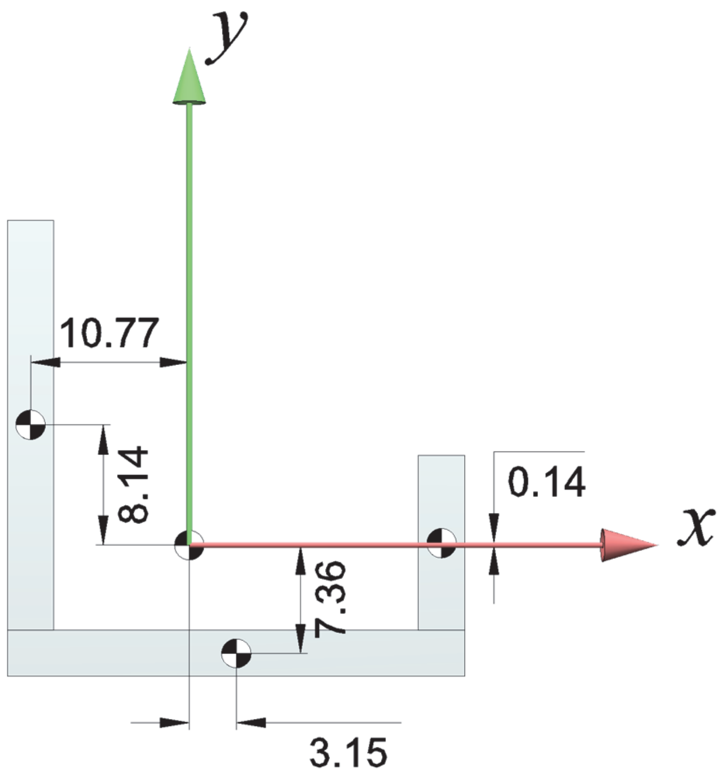
$$\bar{X} = \frac{\sum \bar{X}_i\, A_i}{\sum A_i} = \frac{\bar{X}_{\mathrm{A}}\, A_{\mathrm{A}} + \bar{X}_{\mathrm{B}}\, A_{\mathrm{B}} + \bar{X}_{\mathrm{C}}\, A_{\mathrm{C}}}{A_{\mathrm{A}} + A_{\mathrm{B}} + A_{\mathrm{C}}} = \frac{(13.925)(97.65) + (0)(87.73) + (27.85)(37.33)}{(97.65) + (87.73) + (37.33)}$$

$$\bar{X} = 10.77 \text{ mm}$$

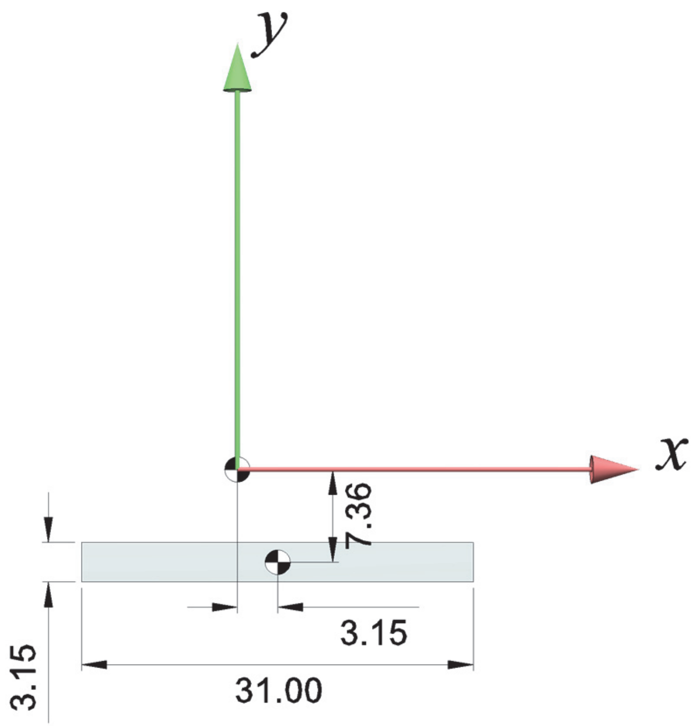
$$\bar{Y} = \frac{\sum \bar{Y}_i\, A_i}{\sum A_i} = \frac{\bar{Y}_{\mathrm{A}}\, A_{\mathrm{A}} + \bar{Y}_{\mathrm{B}}\, A_{\mathrm{B}} + \bar{Y}_{\mathrm{C}}\, A_{\mathrm{C}}}{A_{\mathrm{A}} + A_{\mathrm{B}} + A_{\mathrm{C}}} = \frac{(0)(97.65) + (15.50)(87.73) + (7.50)(37.33)}{(97.65) + (87.73) + (37.33)}$$

$$\bar{Y} = 7.36 \text{ mm}$$

- New coordinates:



- Parallel axis theorem for section A:



$$I_{x_A x_A} = \frac{b\,h^3}{12} = \frac{(31.00)(3.15)^3}{12} = 80.74\ \text{mm}^4$$

$$\bar{y}_A = \bar{Y}_A - \bar{Y} = 0 - 7.36 = -7.36\ \text{mm}$$

$$I_{xx}^{A} = I_{x_A x_A} + A_A(\bar{y}_A)^2$$

$$I_{xx}^{A} = 5{,}374.43\ \text{mm}^4$$

$$I_{y_A y_A} = \frac{b\,h^3}{12} = \frac{(3.15)(31.00)^3}{12} = 7{,}820.14\ \text{mm}^4$$

$$\bar{x}_A = \bar{X}_A - \bar{X} = 13.925 - 10.77 = 3.15\ \text{mm}$$

$$I_{yy}^{A} = I_{y_A y_A} + A_A(\bar{x}_A)^2$$

$$I_{yy}^{A} = 8{,}789.90\ \text{mm}^4$$

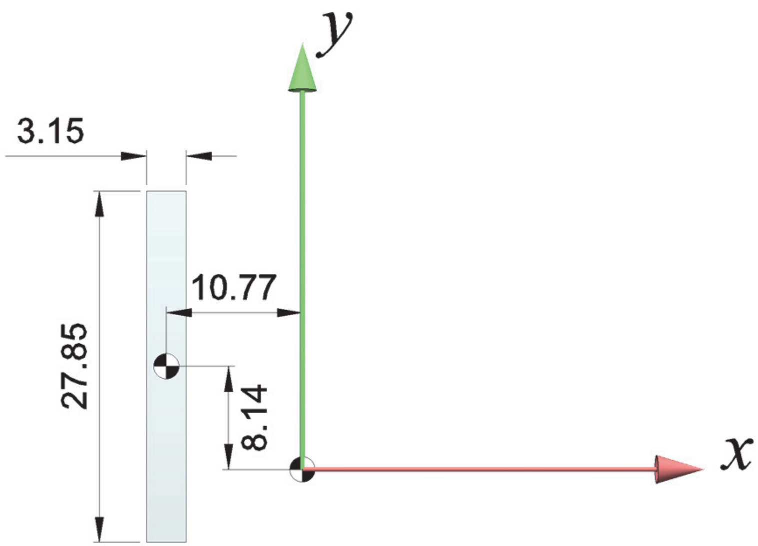
$I_{x_A\,y_A} = 0$ (symmetric cross-section)

$$I_{xy}^{A} = I_{x_A y_A} + A_A(\bar{x}_A \bar{y}_A)$$

$$I_{xy}^{A} = 0 + (97.65)(3.15)(-7.36)$$

$$I_{xy}^{A} = -2{,}265.75\ \text{mm}^4$$

- Parallel axis theorem for section B:



$$I_{x_B x_B} = \frac{b\,h^3}{12} = \frac{(3.15)(27.85)^3}{12} = 5{,}670.29 \text{ mm}^4$$

$$\bar{y}_B = \bar{Y}_B - \bar{Y} = 15.50 - 7.36 = 8.14 \text{ mm}$$

$$I_{xx}^B = I_{x_B x_B} + A_B(\bar{y}_B)^2$$

$$I_{xx}^B = 11{,}479.08 \text{ mm}^4$$

$$I_{y_B y_B} = \frac{b\,h^3}{12} = \frac{(27.85)(3.15)^3}{12} = 72.54 \text{ mm}^4$$

$$\bar{x}_B = \bar{X}_B - \bar{X} = 0 - 10.77 = -10.77 \text{ mm}$$

$$I_{yy}^B = I_{y_B y_B} + A_B(\bar{x}_B)^2$$

$$I_{yy}^B = 10{,}255.22 \text{ mm}^4$$

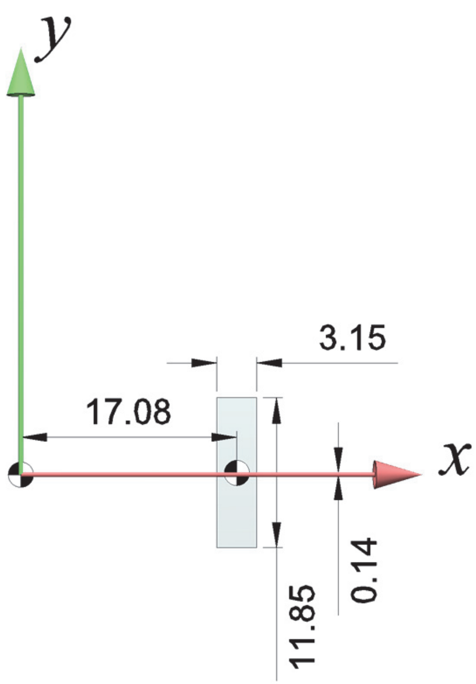
$I_{x_B\,y_B} = 0$ (symmetric cross-section)

$$I_{xy}^B = I_{x_B\,y_B} + A_B(\bar{x}_B \bar{y}_B)$$

$$I_{xy}^B = 0 + (87.73)(-10.77)(8.14)$$

$$I_{xy}^B = -7{,}690.84 \text{ mm}^4$$

- Parallel axis theorem for section C:



$$I_{x_C x_C} = \frac{b\,h^3}{12} = \frac{(3.15)(11.85)^3}{12} = 436.80\ \mathrm{mm}^4$$

$$\bar{y}_C = \bar{Y}_C - \bar{Y} = 7.50 - 7.36 = 0.14\ \mathrm{mm}$$

$$I_{xx}^{C} = I_{x_B x_B} + A_C(\bar{y}_C)^2$$

$$I_{xx}^{C} = 437.50\ \mathrm{mm}^4$$

$$I_{y_C y_C} = \frac{b\,h^3}{12} = \frac{(11.85)(3.15)^3}{12} = 30.87\ \mathrm{mm}^4$$

$$\bar{x}_C = \bar{X}_C - \bar{X} = 27.85 - 10.77 = 17.08\ \mathrm{mm}$$

$$I_{yy}^{C} = I_{y_C y_C} + A_C(\bar{x}_C)^2$$

$$I_{yy}^{C} = 10{,}915.62\ \mathrm{mm}^4$$

$I_{x_C\,y_C} = 0$ (symmetric cross-section)

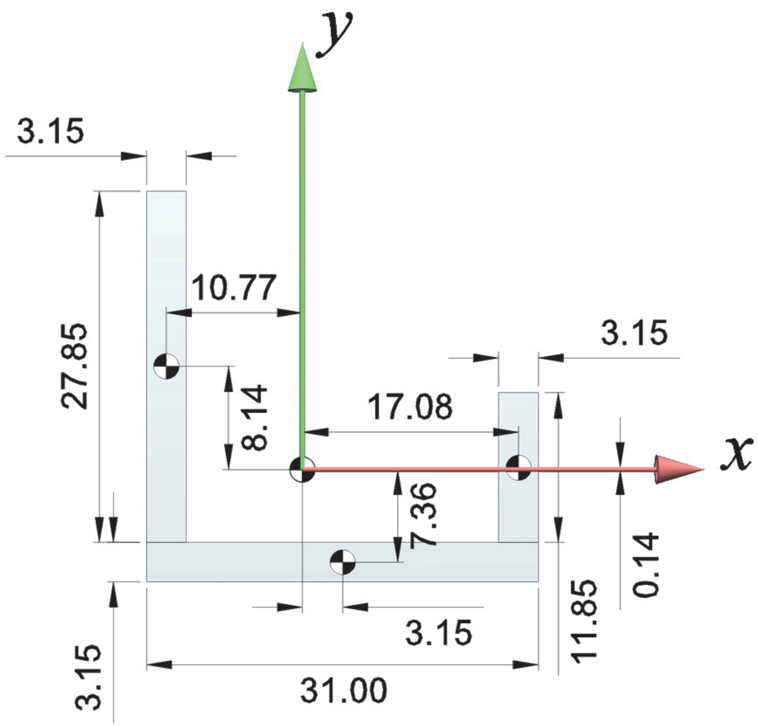
$$I_{xy}^{C} = I_{x_C\,y_C} + A_C(\bar{x}_C \bar{y}_C)$$

$$I_{xy}^{C} = 0 + (37.33)(17.08)(0.14)$$

$$I_{xy}^{C} = 87.45\ \mathrm{mm}^4$$

Lipped Section Example

- Finally, for the compound section:



$$I_{xx} = I_{xx}^{\mathrm{A}} + I_{xx}^{\mathrm{B}} + I_{xx}^{\mathrm{C}}$$

$$I_{xx} = 17{,}291.01\ \mathrm{mm}^4$$

$$I_{yy} = I_{yy}^{\mathrm{A}} + I_{yy}^{\mathrm{B}} + I_{yy}^{\mathrm{C}}$$

$$I_{yy} = 29{,}960.73\ \mathrm{mm}^4$$

$$I_{xy} = I_{xy}^{\mathrm{A}} + I_{xy}^{\mathrm{B}} + I_{xy}^{\mathrm{C}}$$

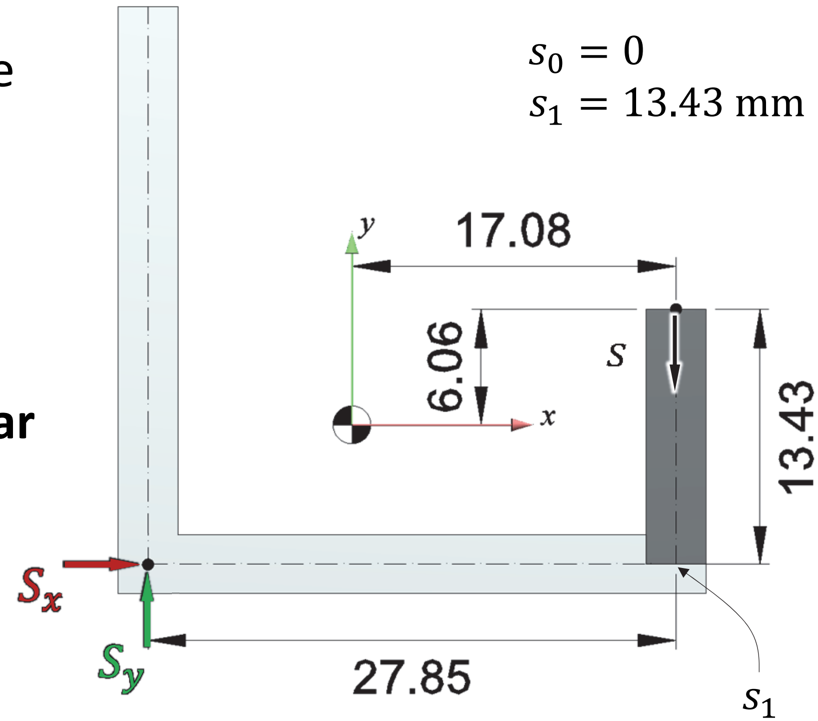
$$I_{xy} = -9{,}869.13\ \mathrm{mm}^4$$

Lipped Section Example

- **Shear centre**:
  - Most convenient reference point is the bottom left corner (origin of X, Y)
  - Only stresses along section C will generate moments about this reference point (!)
  - We integrate shear flow from $s = 0$ to $s = s_1 = 13.425\text{ mm}$
  - **Important**: shear stresses and **shear flow** are defined in terms of $\boldsymbol{x}, \boldsymbol{y}$ while the **shear centre** is defined in terms of $\boldsymbol{X}, \boldsymbol{Y}$

$$(X_0, Y_0) = (27.85\text{ mm}, 13.425\text{ mm})$$
$$(x_0, y_0) = (17.08\text{ mm}, 6.06\text{ mm})$$

$$s_0 = 0$$
$$s_1 = 13.43\text{ mm}$$



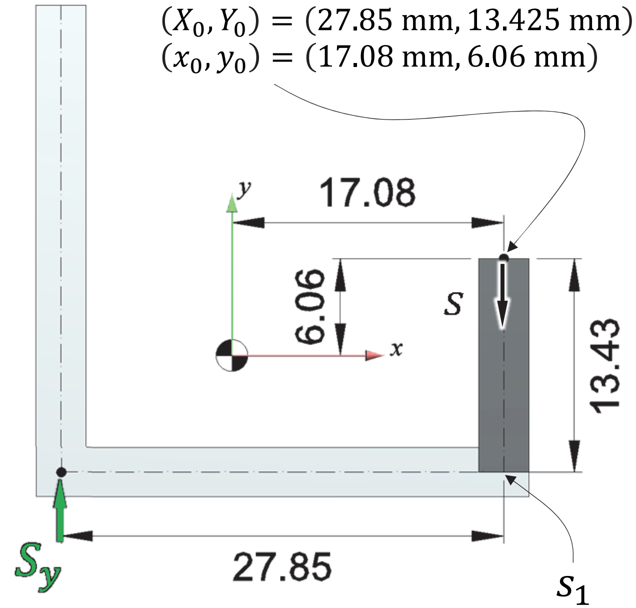
Equations:

Shear centre: $S_y\, e_x = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{ds}$ $S_x\, e_y = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{ds}$

Shear flow: $-q_s = \left(\dfrac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \displaystyle\int_0^s x\, t\, \mathrm{ds} + \left(\dfrac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{ds}$

Lipped Section Example – Shear Centre e_x

$$-q_s = \left(\frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{d}s$$



To find e_x we apply S_y, make $S_x = 0$ and therefore:

$$-q_s = \left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{d}s$$

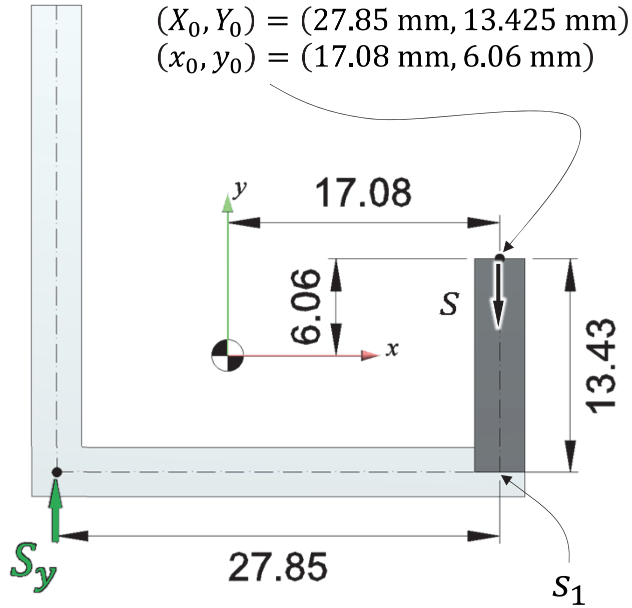
Note that here $x_{(s)} = x_0 = 17.08\ \mathrm{mm}$, while $y_{(s)} = y_0 - s$ and therefore:

$$-q_s = \left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s (y_0 - s)\, t\, \mathrm{d}s$$

$$-q_s = \left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, t\, s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \left(y_0\, s - \frac{s^2}{2}\right) t$$

Lipped Section Example – Shear Centre e_x

$$S_y\, e_x = \int (Y\, q_{s,x} - X\, q_{s,y})\, \mathrm{d}s$$



Note that here we only consider the vertical shear flow ($q_{s,y}$) while the 'moment arm' is constant and equal to X_0, therefore:

$$S_y\, e_x = \int_0^{s_1} (-X_0\, q_s)\, \mathrm{d}s$$

$$S_y\, e_x = X_0\, t \int_0^{s_1} \left[\left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, s - \frac{s^2}{2}\right)\right] \mathrm{d}s$$

$$e_x = X_0\, t \left[\left(\frac{I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s^2}{2} + \left(\frac{I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, \frac{s^2}{2} - \frac{s^3}{6}\right)\right]_0^{s_1}$$

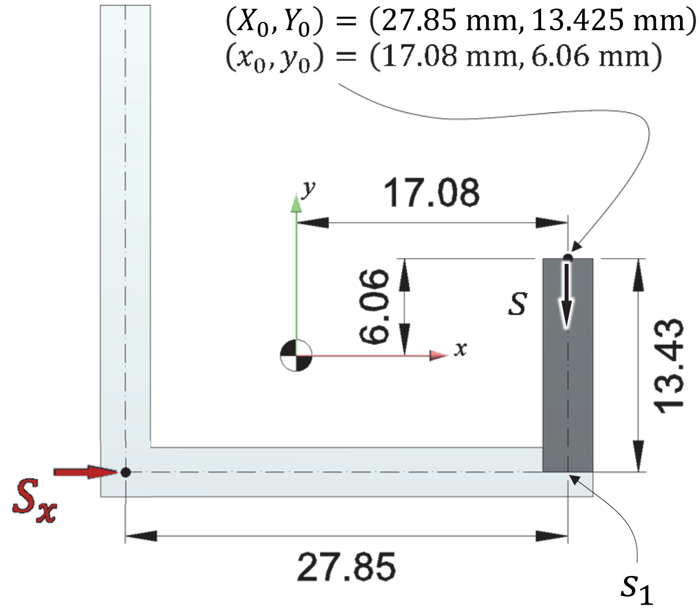
$$e_x = X_0\, t \left[\left(\frac{I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s_1^2}{2} + \left(\frac{I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, \frac{s_1^2}{2} - \frac{s_1^3}{6}\right)\right]$$

$$e_x = (27.85)(3.15)\left\{\left(\frac{-9{,}869.13}{-420{,}651{,}624}\right)(17.08)\frac{(13.425)^2}{2} + \left(\frac{29{,}960.73}{420{,}651{,}624}\right)\left[(6.06)\frac{(13.425)^2}{2} - \frac{(13.425)^3}{6}\right]\right\}$$

$$e_x = 4.06\text{ mm}$$

Lipped Section Example – Shear Centre e_y

$$-q_s = \left(\frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{d}s$$



To find e_y we apply S_x, make $S_y = 0$ and therefore:

$$-q_s = \left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{d}s$$

Note that here $x_{(s)} = x_0 = 17.08$ mm, while $y_{(s)} = y_0 - s$ and therefore:

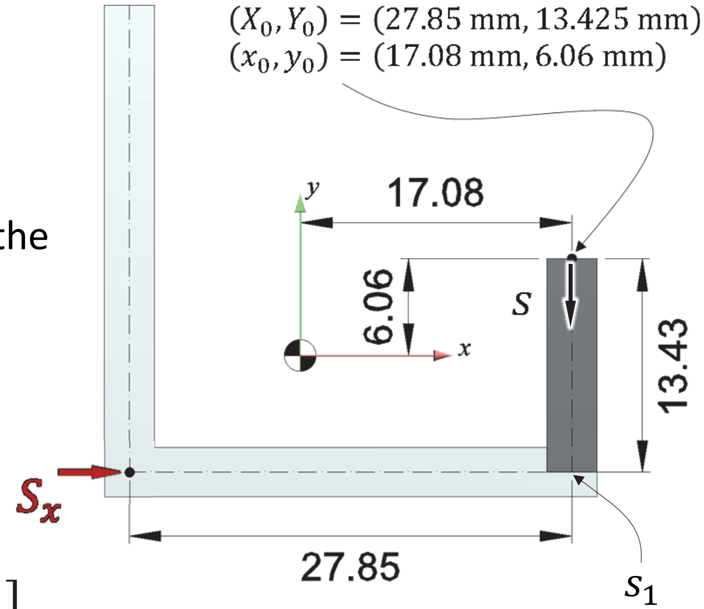
$$-q_s = \left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s (y_0 - s)\, t\, \mathrm{d}s$$

$$-q_s = \left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, t\, s + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \left(y_0\, s - \frac{s^2}{2}\right) t$$

Lipped Section Example – Shear Centre e_y

$$S_x\, e_y = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{d}s$$

Note that here we only consider the vertical shear flow ($q_{s,y}$) while the 'moment arm' is constant and equal to X_0, therefore:



$$S_x\, e_y = \int_0^{s_1} \left(-X_0\, q_s\right) \mathrm{d}s$$

$$S_x\, e_y = X_0\, t \int_0^{s_1} \left[\left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, s + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, s - \frac{s^2}{2}\right)\right] \mathrm{d}s$$

$$e_y = X_0\, t \left[\left(\frac{I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s^2}{2} + \left(\frac{I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, \frac{s^2}{2} - \frac{s^3}{6}\right)\right]_0^{s_1}$$

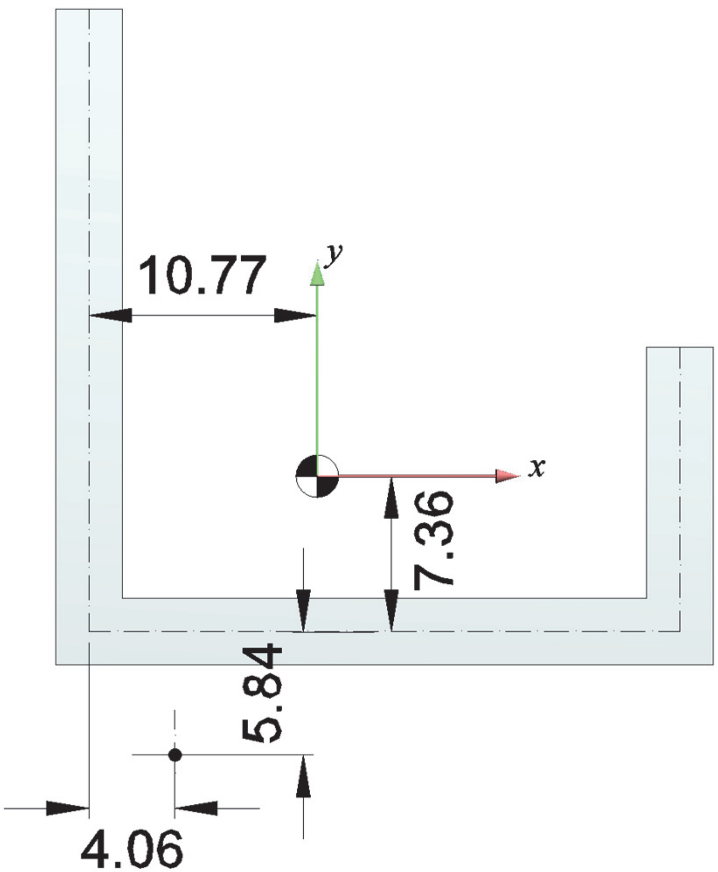
$$e_y = X_0\, t \left[\left(\frac{I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s_1^2}{2} + \left(\frac{I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, \frac{s_1^2}{2} - \frac{s_1^3}{6}\right)\right]$$

$$e_y = (27.85)(3.15)\left\{\left(\frac{17{,}291.01}{-420{,}651{,}624}\right)(17.08)\frac{(13.425)^2}{2} + \left(\frac{-9{,}869.13}{420{,}651{,}624}\right)\left[(6.06)\frac{(13.425)^2}{2} - \frac{(13.425)^3}{6}\right]\right\}$$

$$e_y = -5.84\ \mathrm{mm}$$

Lipped Section Example – Shear Centre

Thin wall (analytical) solution



Full 2D (numerical) solution