StM3 – Composite Laminate Analysis

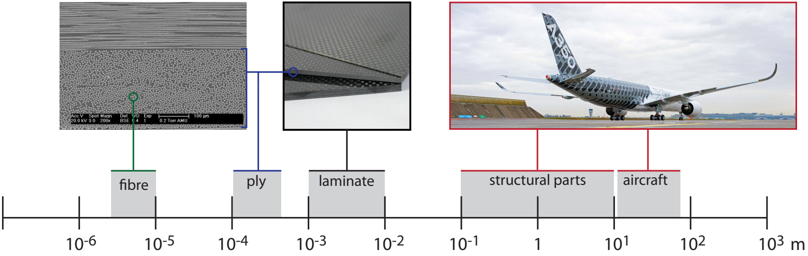
Lecture 4 :
- micromechanics of uni-directional lamina (ctd.)
- Classical Laminate Theory

2019/2020

Mark Schenk – M.Schenk@bristol.ac.uk

Lecture Outline

- finish micromechanics of composite ply: failure
- introduce Classical Laminate Theory (CLT)



Summary

- **micromechanics**: interaction of constituents (fibre/matrix)
- homogenise using Representative Volume Element (RVE)
- key parameter: fibre volume fraction V_f
- calculate macroscopic ply stiffness (E_{11},E_{22},G_{12},ν_{12}) using (inverse) rule of mixtures or Halpin-Tsai
- load transfer fibre/matrix in short-fibre composites

Micro-Mechanics of Strength

micromechanics of **composite strength** more complex

various mechanisms for composite laminate failure:

- fibre fracture,
- matrix micro-cracking,
- debonding (separation of fibres and matrix),
- delamination (separation of laminae), *etc.*

aim: *qualitatively* describe failure modes for different loading

Longitudinal Tensile Strength (X_t) – I

longitudinal tensile (X_t): found using rule of mixtures:

$$X_t = \sigma_{f_{\max}} V_f + (\sigma_m)_{\varepsilon_{f\,\max}} (1 - V_f)$$

$(\sigma_m)_{\varepsilon_{f\,\max}}$ is matrix stress at maximum tensile strain in fibres

assumes that (i) all fibres have same strength, and (ii) matrix has higher strain to failure than fibres

observed tensile failure of composites is sudden, but actual failure mechanism is more intricate (and interesting!)

Longitudinal Tensile Strength (X_t) – II

QUIZ

A single fibre has characteristic fibre strength σ_0.

Will a *bundle of loose fibres* and a *fibre-reinforced polymer composite*, be **stronger** or **weaker** than σ_0?

	stronger	**weaker**
fibre bundle		
composite		

How did you come to this conclusion?

Longitudinal Tensile Strength (X_t) – III

individual fibre: strength of brittle fibre dominated by *defects*

strength σ described by Weibull probability distribution

$$p(\sigma) = 1 - \exp\left[-\left(\frac{L}{L_0}\right)\left(\frac{\sigma}{\sigma_0}\right)^m\right]$$

with fibre length L, characteristic fibre strength σ_0 for length L_0

NB: longer fibres, higher defect probability, lower strength!

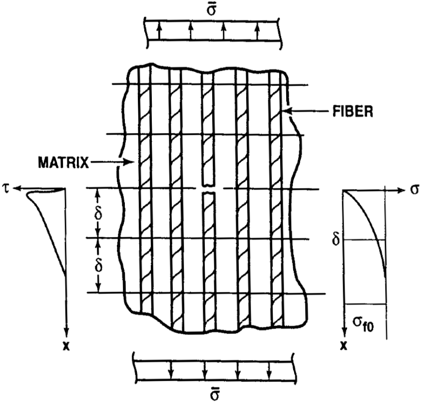
---

fibre bundle: as individual fibres break, load is transferred to remaining fibres; bundle is *weaker* than characteristic fibre strength

Longitudinal Tensile Strength (X_t) – IV

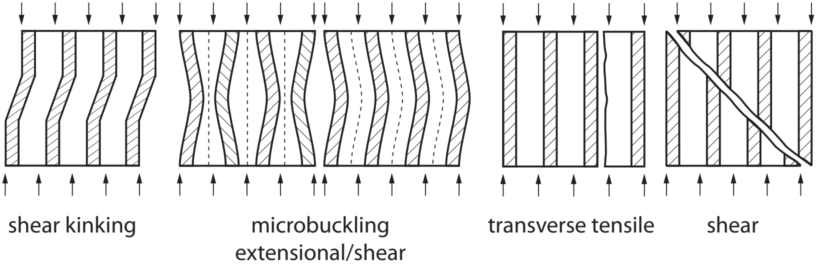
composite material:
matrix redistributes load around fibre fracture, similar to load transfer for short-fibre composites

fractured fibre continues to carry load; composite is *stronger* than characteristic fibre strength!



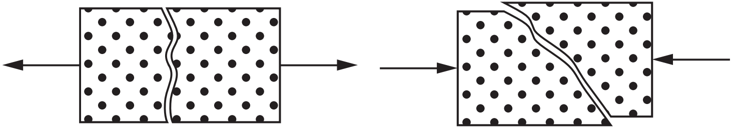
Longitudinal Compressive Strength (X_c)

longitudinal compressive (X_c): different failure modes



Transverse Strength (Y_t, Y_c)

transverse tensile (Y_t): strain concentrations around fibres, leads to matrix micro-cracks, which coalesce and propagate; strain concentrations increase for higher V_f and E_f



transverse compressive (Y_c): matrix shear failure and debonding

Summary

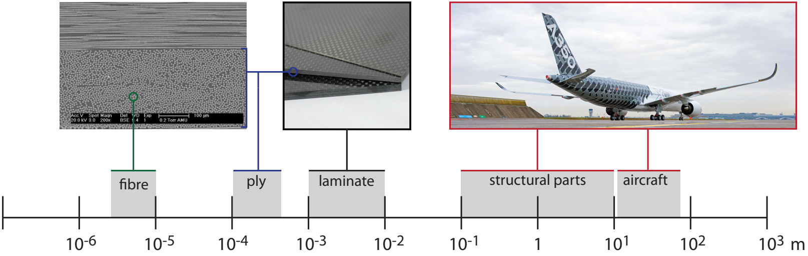
- **micromechanics**: interaction of constituents (fibre/matrix)
- homogenise using Representative Volume Element (RVE)
- key parameter is fibre volume fraction V_f
- calculate ply stiffness (E_{11},E_{22},G_{12},ν_{12})
  use (inverse) rule of mixtures or Halpin-Tsai
- strength is more complex than stiffness

Revision Objectives

Revision Objectives:

- describe different micromechanical composite failure modes;
- calculate longitudinal tensile strength X_t using rule of mixtures;

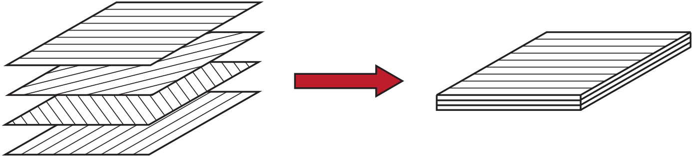
Composite Laminate Analysis – I



mechanics of composite laminate plate,
with multiple plies oriented at different angles

Composite Laminate Analysis – II

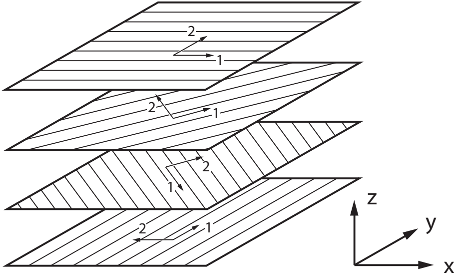
composite laminate:
combine multiple composite layers into single structural element



composite lay-up: **tailored structural properties**

- ply material properties and thickness
- ply orientation with respect to structural axes
- ply stacking sequence

Composite Laminate Analysis – III



lay-up notation:

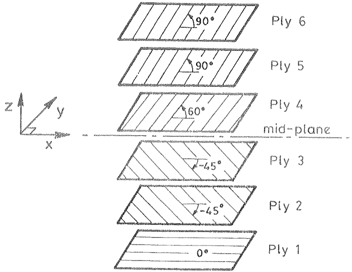
- ply numbering bottom-up in positive z-direction
- angle θ defines orientation of ply material axes (CCW)

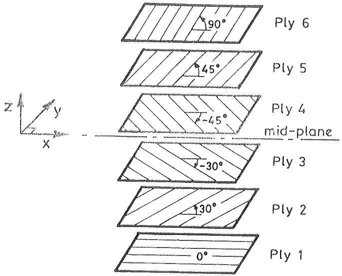
Composite Laminate Analysis – IV

- repeating ply angle θ in n successive layers: θ_n
- symmetry around xz-plane in successive layers: $\pm$ or $\mp$
- symmetry around mid-plane:

  $[0/90/90/0] = [0/90]_S$ and $[0/90/0] = [0/\bar{90}]_S$

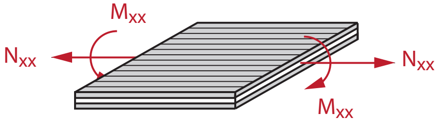
Composite Laminate Analysis – V





Classical Laminate Theory

Classical Laminate Theory:
composite *plate* model : in-plane and out-of-plane deformations

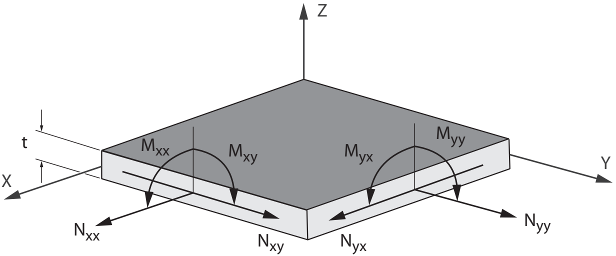


underlying assumptions:

- all plies are macroscopically homogeneous and linear-elastic
- all plies are perfectly bonded: strain continuity
- each ply is assumed to be in plane stress

Plate Strains and Deformations – I

applied loads (per unit length) will result in deformations



in-plane strains (ε_{xx}, ε_{yy}, γ_{xy})
out-of-plane plate curvature and twist (κ_{xx}, κ_{yy}, κ_{xy})

Plate Strains and Deformations – II

Kirchoff-Love plate model: cross-sections remain straight

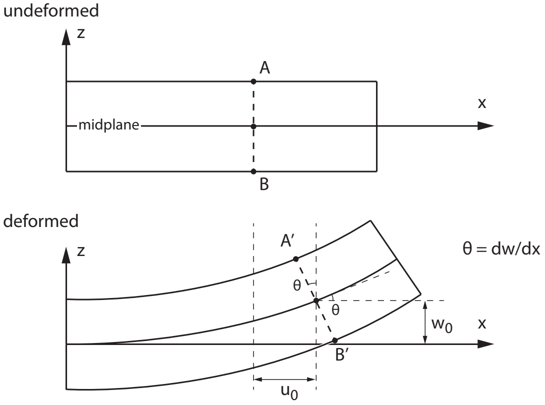


Plate Strains and Deformations – III

displacement u of point on cross-section:

$$u = u_0 - z\theta$$

$$= u_0 - z\frac{\partial w_0}{\partial x}$$

distance z from midplane, and $\theta = \partial w_0/\partial x$

similarly, for displacement v in y-direction:

$$v = v_0 - z\frac{\partial w_0}{\partial y}$$

Plate Strains and Deformations – IV

strains are calculated as follows (see StM2):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} = \varepsilon_{xx}^0 + z\kappa_{xx}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} = \varepsilon_{yy}^0 + z\kappa_{yy}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y} = \gamma_{xy}^0 + z\kappa_{xy}$$

ε_{xx}^0, ε_{yy}^0, γ_{xy}^0 are midplane strains

κ_{xx}, κ_{yy}, κ_{xy} are plate curvatures

Plate Strains and Deformations – V

strains at any point on the cross-section:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

function of midplane strains and curvatures, and distance z

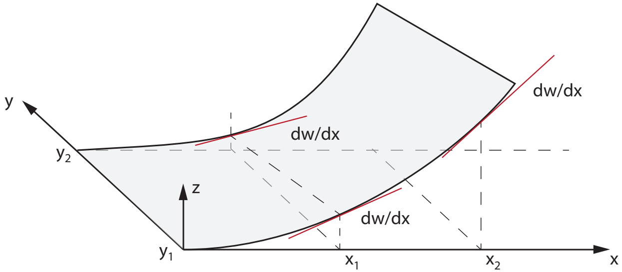
curvatures (κ_{xx}, κ_{yy}) and twist (κ_{xy}):

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} = - \begin{bmatrix} \partial^2 w_0 / \partial x^2 \\ \partial^2 w_0 / \partial y^2 \\ 2\partial^2 w_0 / \partial x \partial y \end{bmatrix}$$

Plate Strains and Deformations – VI

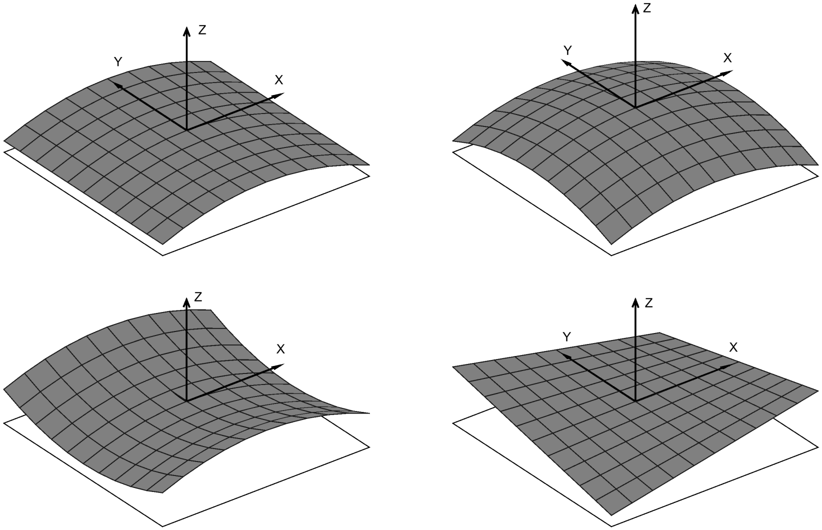
κ_{xx} rate of change in slope $\partial w/\partial x$ with respect to x

κ_{xy} rate of change of slope $\partial w/\partial x$ with respect to y



sign convention!

Plate Strains and Deformations – VII



Laminate Structural Properties – I

expression for strain across thickness of laminate plate:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx}^{0} \\ \varepsilon_{yy}^{0} \\ \gamma_{xy}^{0} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

next steps:

- calculate ply stresses using $\bar{\boldsymbol{Q}}_k$ for each ply k
- calculate stress resultants: N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}
- formulate a structural model: ABD-matrix

Laminate Structural Properties – II

$$\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]$$

Laminate Structural Properties – III

A is the **extensional stiffness matrix**

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1})$$

B is the **coupling stiffness matrix**

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right)$$

D is the **bending stiffness matrix**

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right)$$

Summary

- Classical Laminate Theory:
  describes mechanics of composite laminate plates
- layup notation: $[0/\pm 90/45]_S$
- out-of-plane deformations: curvatures κ_{xx}, κ_{yy}, κ_{xy}
- strain across cross-section

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

- stress resultants N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}

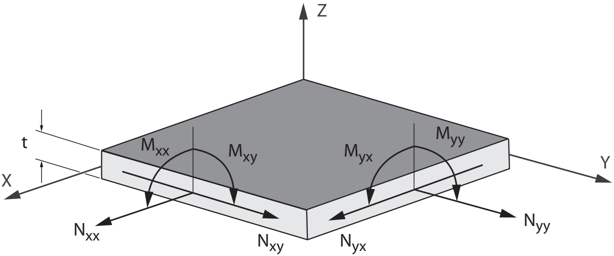
Revision Objectives

Revision Objectives:

- describe assumptions underpinning Classical Laminate Theory;
- use lamination notation for composite laminates (including shorthand);
- interpret geometry of bending/twisting curvatures;
- recall sign convention for applied loads/moments;

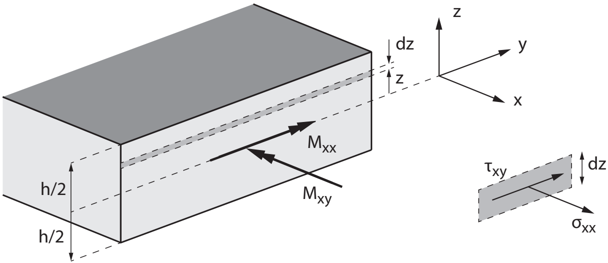
Stress & Stress Resultants – I

stress resultants N and M are per unit width of plate



result of stresses across the laminate cross-section (in z-direction)

Stress & Stress Resultants – II

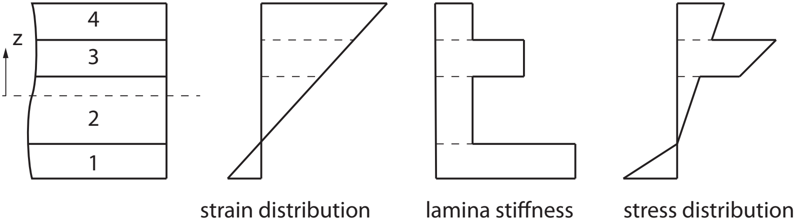


$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} dz \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} dz \quad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} z \, dz \quad M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy} z \, dz \quad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z \, dz$$

Stress & Stress Resultants – III

laminate **stress distribution is discontinuous** across cross-section



note: difference between midplane and neutral axis

Stress & Stress Resultants – IV

laminate stress resultant:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} dz \;\; = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k dz \right)$$

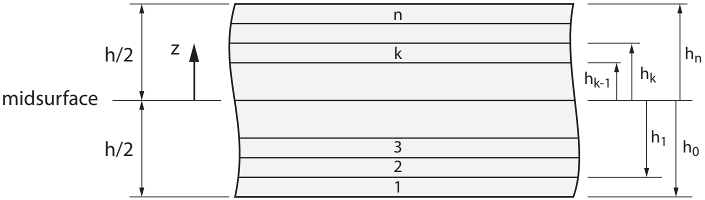
$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} z\, dz = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k z\, dz \right)$$

integrate over each ply k, and sum over n laminate plies

Stress & Stress Resultants – V

laminate with n plies and total thickness h

ply numbering $k = 1 \ldots n$ in positive z-direction



location h_k of top of each ply measured from *geometric* mid-plane

ply thickness: $t_k = h_k - h_{k-1}$

Stress & Stress Resultants – VI

each ply k is assumed to be in plane stress: $\bar{\boldsymbol{Q}}_k$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

where:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

Stress & Stress Resultants – VII

substitute strains to find the in-plane stress resultants:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix} dz \right)$$

$$\ldots + \sum_{k=1}^{n} \left(\int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix} z\, dz \right)$$

Stress & Stress Resultants – VIII

$$\begin{aligned} \text{midplane strains and curvatures}: &\quad \text{constant across all plies } n \\ \text{stiffness matrix } \bar{\boldsymbol{Q}}_k: &\quad \text{constant across ply } k \end{aligned}$$

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} dz \right) \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix}$$

$$\ldots + \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z\, dz \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

Stress & Stress Resultants – IX

in-plane stress resultants are expressed as:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

extensional stiffness matrix $\boldsymbol{A}$ and coupling stiffness matrix $\boldsymbol{B}$

Stress & Stress Resultants – X

similarly, for out-of-plane stress resultants

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z dz \right) \begin{bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{bmatrix}$$

$$\ldots + \left(\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{h_{k-1}}^{h_k} z^2 \, dz \right) \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

Stress & Stress Resultants – XI

$$\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{bmatrix}$$

coupling stiffness matrix $\boldsymbol{B}$ and bending stiffness matrix $\boldsymbol{D}$

ABD-matrix – I

mechanics of composite laminate is described by ABD-matrix

$$\left[\begin{array}{c} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \left[\begin{array}{c} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \gamma^0_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{array}\right]$$

$$\left[\begin{array}{c} \boldsymbol{N} \\ \boldsymbol{M} \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{array}\right] \tag{3.2}$$

ABD-matrix – II

$\boldsymbol{A}$ is the **extensional stiffness matrix**

$$A_{ij} = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k (h_k - h_{k-1}) \tag{3.3}$$

$\boldsymbol{B}$ is the **coupling stiffness matrix**

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \tag{3.4}$$

$\boldsymbol{D}$ is the **bending stiffness matrix**

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^3 - h_{k-1}^3\right) \tag{3.5}$$

ABD-matrix – III

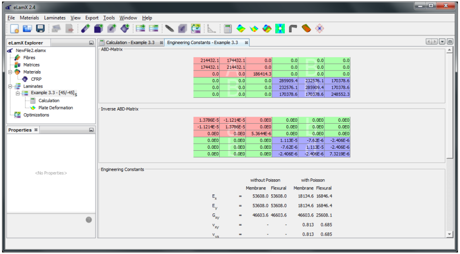
	ε^0_{xx}	ε^0_{yy}	γ^0_{xy}	κ_{xx}	κ_{yy}	κ_{xy}
N_{xx}	A_{11}	A_{12}	A_{16}	B_{11}	B_{12}	B_{16}
N_{yy}	A_{12}	A_{22}	A_{26}	B_{12}	B_{22}	B_{26}
N_{xy}	A_{16}	A_{26}	A_{66}	B_{16}	B_{26}	B_{66}
M_{xx}	B_{11}	B_{12}	B_{16}	D_{11}	D_{12}	D_{16}
M_{yy}	B_{12}	B_{22}	B_{26}	D_{12}	D_{22}	D_{26}
M_{xy}	B_{16}	B_{26}	B_{66}	D_{16}	D_{26}	D_{66}

Composite Laminate Software

repetitive calculations: software!

- eLamX^2 software program

  `https://tu-dresden.de/ing/maschinenwesen/ilr/lft/elamx2/elamx`



- write your own in MATLAB! (or look on Blackboard)