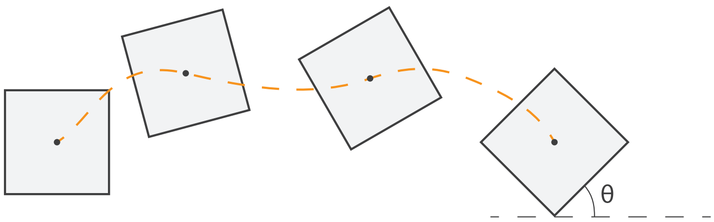
Handout 4 – Kinematics of Rigid Bodies

Meriam & Kraige, Dynamics: 5/1–5/6

In mechanics we generally consider the dynamics of rigid bodies, which are assumed not to deform under the applied forces. This means that the distances between points within a rigid body remain constant. Even in cases where this is not strictly true, for example in aircraft where the wings deflect due to aerodynamic loads, in mechanics we shall consider the body to remain rigid.

In this unit we restrict ourselves to planar kinematics and dynamics of rigid bodies. General plane motion consists of **translation** and **rotation** of a body. Translation is described by the same equations as in kinematics of particles, whereas rotations form the topic of this handout.



4.1 Planar Kinematics : Rotation

The angular position θ with respect to a reference axis describes the rotation of the rigid body. The time derivatives give angular velocity ω and angular acceleration α:

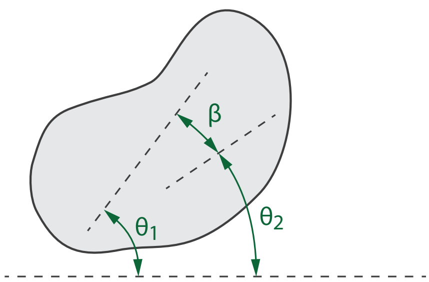
$$\omega = \frac{d\theta}{dt} = \dot{\theta} \tag{4.1}$$

$$\alpha = \frac{d\omega}{dt} = \ddot{\theta} \tag{4.2}$$

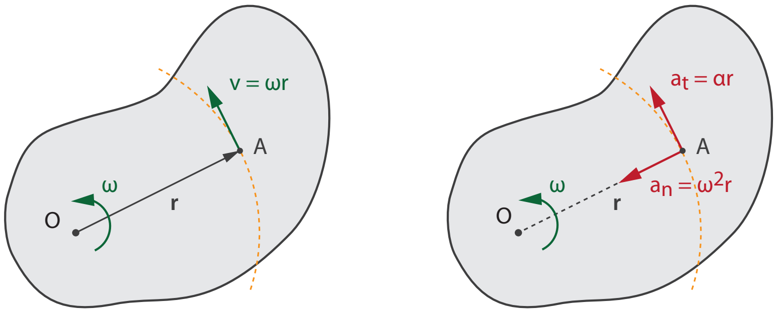
Analogous to the (curvi)linear kinematics of particles:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\,\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega \qquad \rightarrow \qquad \alpha\, d\theta = \omega\, d\omega \tag{4.3}$$

Note that the choice of the reference line only adds a constant offset β, and any line on a rigid body will therefore have the same angular velocity and acceleration.



Velocities and Accelerations on Rigid Body Consider a rigid body with a fixed point of rotation, O, with angular velocity ω and acceleration α.



The velocity v and accelerations (a_n and a_t) of a point A at a distance r from the rotation axis are:

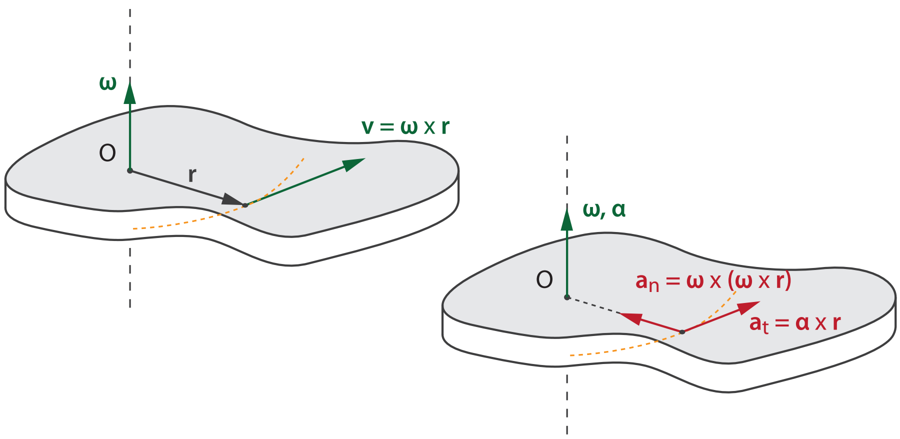
$$v = \omega\, r$$

$$a_n = \omega^2\, r = \frac{v^2}{r}$$

$$a_t = \alpha\, r$$

Note that these are equations for polar coordinates with a constant r (as the body is rigid).

Vector Formulation These equations may also be expressed in vector form, which allow for extension to 3D kinematics and dynamics. Consider a planar body, rotating around a normal vector through a point O.



The velocity vector $\boldsymbol{v}$ is:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{4.4}$$

For planar kinematics, $\boldsymbol{\omega} = \omega \boldsymbol{k}$. Take care to maintain the order of multiplication, as $\boldsymbol{r} \times \boldsymbol{\omega} = -\boldsymbol{v}$. The acceleration of the point is found by differentiating the vector expression for velocity:

$$\begin{aligned}\boldsymbol{a} = \dot{\boldsymbol{v}} &= \boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} \\ &= \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})}_{a_n} + \underbrace{\boldsymbol{\alpha} \times \boldsymbol{r}}_{a_t}\end{aligned}$$

where use was made of $\dot{\boldsymbol{r}} = \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. Therefore

$$\boldsymbol{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{4.5}$$

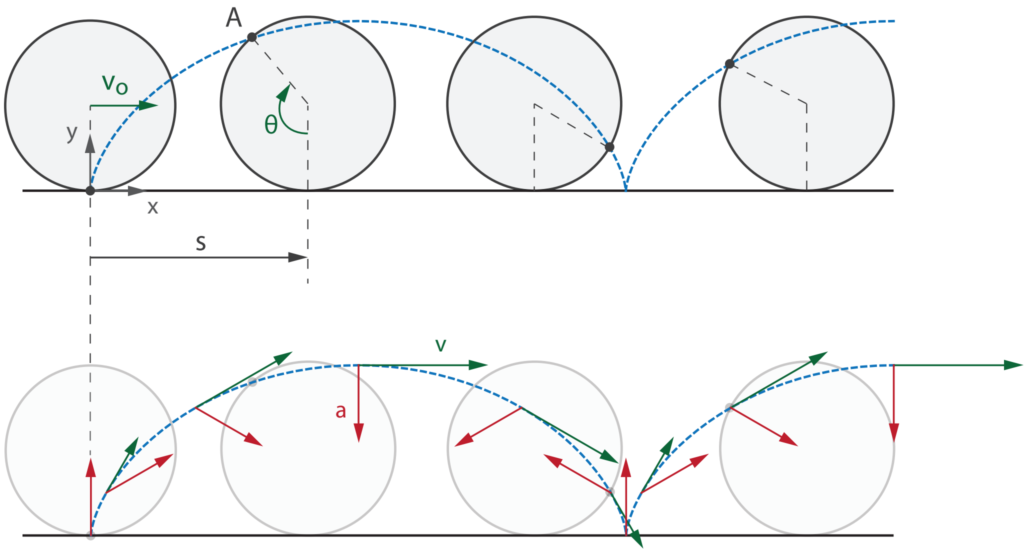
$$\boldsymbol{a}_t = \boldsymbol{\alpha} \times \boldsymbol{r} \tag{4.6}$$

4.2 Planar Kinematics : General Motion : Absolute

The general motion (translation and rotation) of a rigid body, or system of rigid bodies, is often described in **absolute** terms with respect to the chosen coordinate system. The general approach is to express the configuration of the rigid body in terms of geometric relations, before differentiating with respect to time to find velocities and accelerations. This approach is limited to relatively simple problems, where a convenient closed-form solution can be formulated.

Example 4.1 – Rolling Wheel - Pt. I

Let us consider the motion of a wheel: a fairly ubiquitous and somewhat critical mechanical component.



Assuming there is no slip between the wheel and the ground, the position s, velocity v_O, and acceleration a_O of the wheel axle can be expressed in terms of the rotation:

$$s = \theta R$$

$$v_O = \omega R$$

$$a_O = \alpha R$$

where R is the wheel radius. The position of a point A on the wheel rim over time, known as the *cycloid*, is then found as follows (with initial position $\theta = 0$ chosen as contact point between wheel and the ground).

$$x = s - R\sin\theta = R\,(\theta - \sin\theta)$$

$$y = R - R\cos\theta = R\,(1 - \cos\theta)$$

This can be differentiated (note the use of the chain rule, as θ is a function of t) to find the velocity:

$$\dot{x} = R\dot{\theta}\,(1 - \cos\theta)$$

$$\dot{y} = R\dot{\theta}\sin\theta$$

and again to find the acceleration:

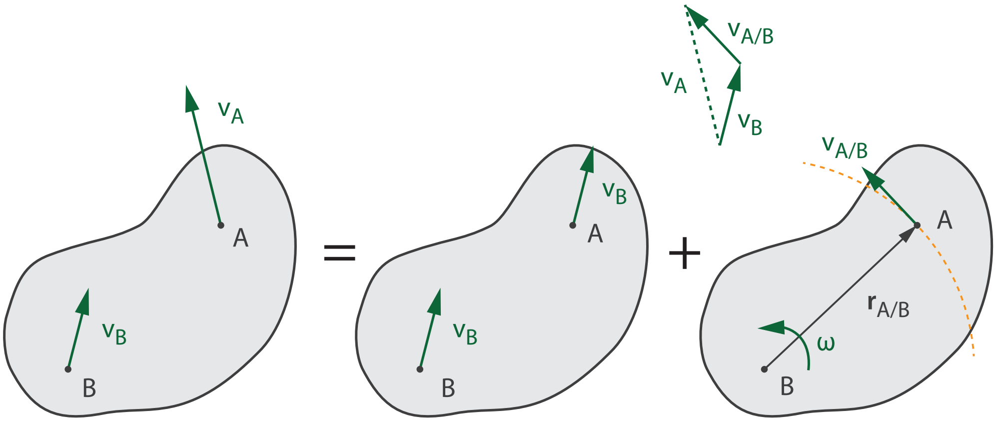
$$\ddot{x} = R\ddot{\theta}\,(1 - \cos\theta) + R\dot{\theta}^2\sin\theta$$

$$\ddot{y} = R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta$$

For the case of constant ω, the velocities and acceleration vectors are plotted along the path.

4.3 Planar Kinematics : General Motion : Relative

The motion of a point on a rigid body can be expressed **relative** to another moving point. Consider two points, A and B, on a rigid body in combined translational and rotational motion.



Relative to point B, the motion of A describes a circular arc as the radial distance between the two points remains unchanged. This means that the relative velocity $\boldsymbol{v}_{A/B}$ is a rotation:

$$\boldsymbol{v}_{A/B} = \boldsymbol{\omega} \times \boldsymbol{r}_{A/B}$$

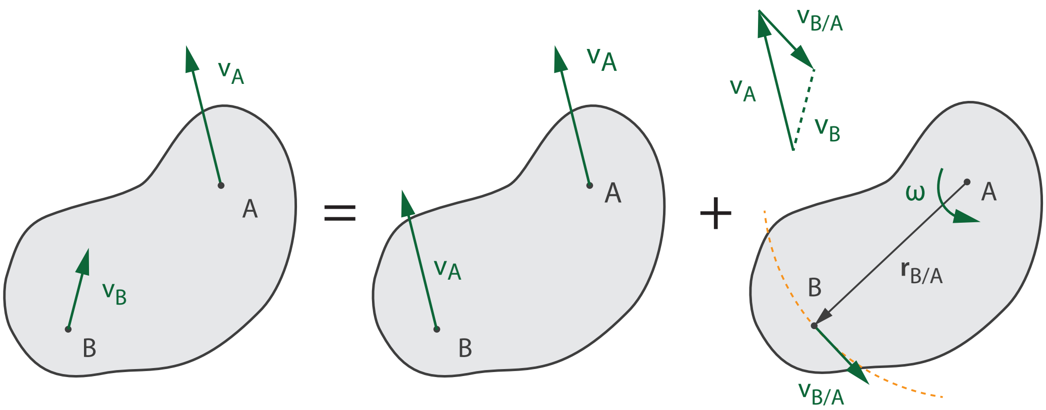
The total velocity $\boldsymbol{v}_A$ is then the vector sum of the velocity $\boldsymbol{v}_B$ of point B, and the relative velocity $\boldsymbol{v}_{A/B}$:

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B} \tag{4.7}$$

The velocities must be identical when taking point A as reference point:

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

Note that $\boldsymbol{v}_{B/A} = -\boldsymbol{v}_{A/B}$, as a result of the reversed direction of the vector $\boldsymbol{r}$. The direction of rotation ω remains unchanged.



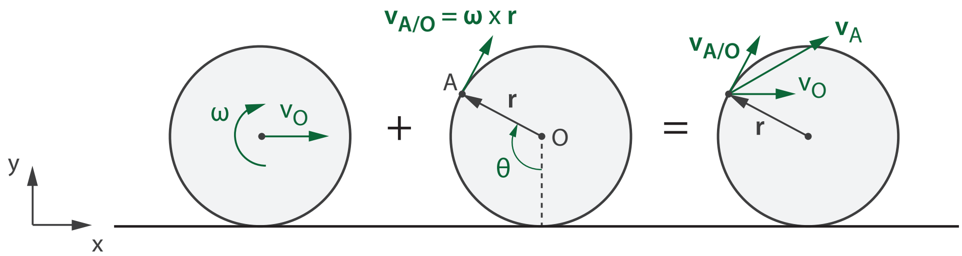
The description of general motion using relatives is powerful for more complicated systems and mechanisms, where closed-form geometric relationships cannot (straightforwardly) be formulated.

Example 4.2 – Rolling Wheel - Pt. II

Consider again the rolling wheel, where the velocity of the axle $v_O = \omega R$. The velocity of a point A on the wheel is then given by:

$$\begin{aligned}\boldsymbol{v}_A &= \boldsymbol{v}_O + \boldsymbol{v}_{A/O} \\ &= \omega R\boldsymbol{i} + \boldsymbol{\omega} \times \boldsymbol{r}\end{aligned}$$

where $\boldsymbol{r}$ is the vector from O to A.



The vector $\boldsymbol{r}$ is described by:

$$\boldsymbol{r} = -R\sin\theta\boldsymbol{i} - R\cos\theta\boldsymbol{j}$$

and with $\boldsymbol{\omega} = -\omega\boldsymbol{k}$ (note the sign convention), we find $\boldsymbol{v}_{A/O}$ as:

$$\begin{aligned}\boldsymbol{v}_{A/O} = \boldsymbol{\omega} \times \boldsymbol{r} &= (-\omega\boldsymbol{k}) \times (-R\sin\theta\boldsymbol{i} - R\cos\theta\boldsymbol{j}) \\ &= -R\omega\cos\theta\boldsymbol{i} + R\omega\sin\theta\boldsymbol{j}\end{aligned}$$

which gives:

$$\begin{aligned}\boldsymbol{v}_A &= \boldsymbol{v}_O + \boldsymbol{v}_{A/O} \\ &= \omega R\boldsymbol{i} + (-\omega R\cos\theta\boldsymbol{i} + \omega R\sin\theta\boldsymbol{j}) \\ &= \omega R\,(1 - \cos\theta)\,\boldsymbol{i} + \omega R\sin\theta\boldsymbol{j}\end{aligned}$$

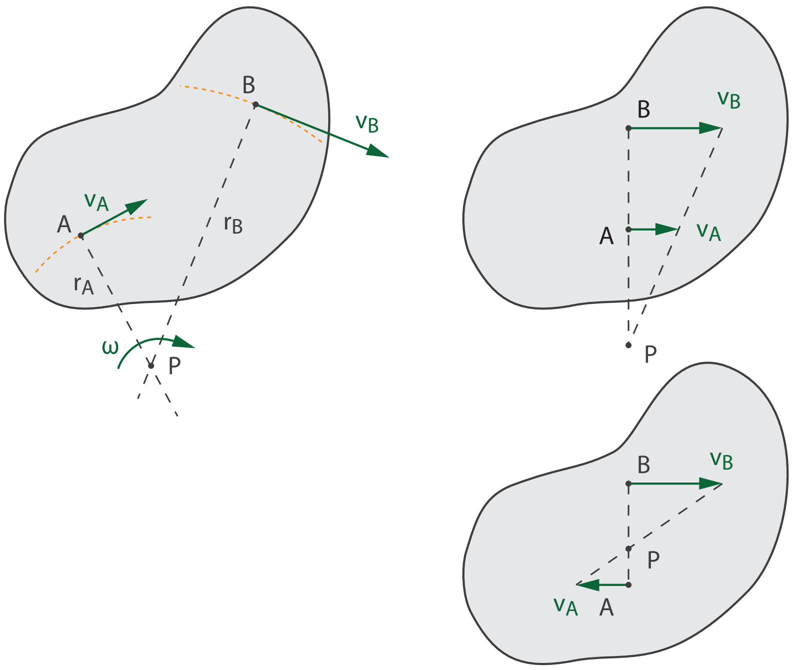
This expression is (obviously) identical to the one found previously describing the motion in absolute terms.

The acceleration of point A can be found in similar fashion:

$$\begin{aligned}\boldsymbol{a}_A &= \boldsymbol{a}_O + \boldsymbol{a}_{A/O} \\ &= \alpha R\boldsymbol{i} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \boldsymbol{\alpha} \times \boldsymbol{r} \\ &= (-\omega\boldsymbol{k}) \times (-\omega\boldsymbol{k} \times (-R\sin\theta\boldsymbol{i} - R\cos\theta\boldsymbol{j})) + (-\alpha\boldsymbol{k}) \times (-R\sin\theta\boldsymbol{i} - R\cos\theta\boldsymbol{j}) \\ &= \alpha R\boldsymbol{i} + \omega^2 R\sin\theta\boldsymbol{i} + \omega^2 R\cos\theta\boldsymbol{j} + \alpha R\sin\theta\boldsymbol{j} - \alpha R\cos\theta\boldsymbol{i} \\ &= \left(\alpha R - \alpha R\cos\theta + \omega^2 R\sin\theta\right)\boldsymbol{i} + \left(\omega^2 R\cos\theta + \alpha R\sin\theta\right)\boldsymbol{j}\end{aligned}$$

4.3.1 Planar Kinematics : Instantaneous Point of Zero Velocity

A special case of relative motion, is to consider a point P where the absolute velocity is zero; this is known as the *instantaneous point of zero velocity*. All points on the rigid body appear to rotate around P, and it therefore also referred to as the *instantaneous centre of rotation*. This point may lie either on or outside the body, and is not fixed in space or relative to the body.

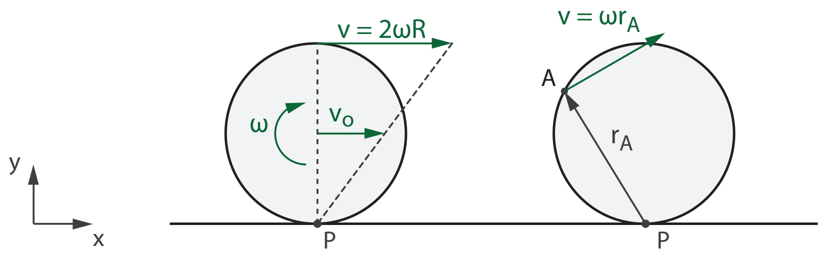


The instantaneous centre of rotation may be located by considering the absolute velocity of two points on a body. The velocities are both tangent to local circular motion, and by taking the normal to both velocity vectors, the instantaneous centre of zero velocity can be found at the intersection of the normals. The angular velocity of the body is therefore $\omega = v_A/r_A$. This is then used to find the absolute velocity of any point on the body, *i.e.* $v_B = \omega r_B$.

Note that at point P the velocity is zero, but the acceleration is not and the point will therefore move in space. The locus of the instantaneous centres in space is known as the *space centrode* wheras the locus of the instantaneous centres relative to the body is the *body centrode*.

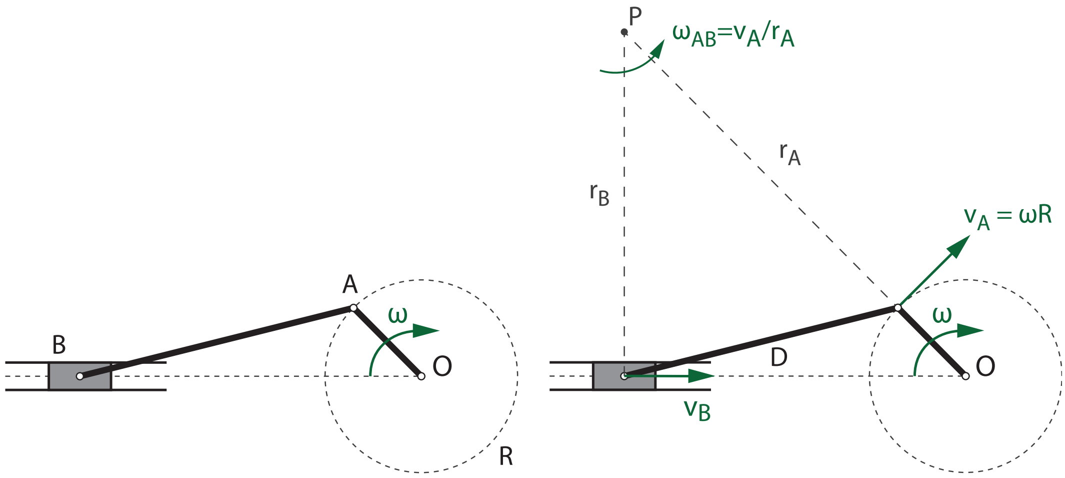
Example 4.3 – Rolling Wheel - Pt. III

Consider again the rolling wheel, where the velocity pole can intuitively be seen to be the contact point between the wheel and the ground. This can also be shown by considering the velocities at the wheel axle (ωR) and top of the wheel ($2\omega R$). The velocity of any point A on the wheel is then written as: $v_A = \omega r_A$.



Example 4.4 – Slider-Crank Mechanism

A very common mechanism is a slider-crank system, where a continuously rotating crank OA drives the linear reciprocating motion of a slider B. The kinematics of the system could be analysed using either absolute or relative motion, but we here graphically derive the velocity of point D midway along the connecting member.



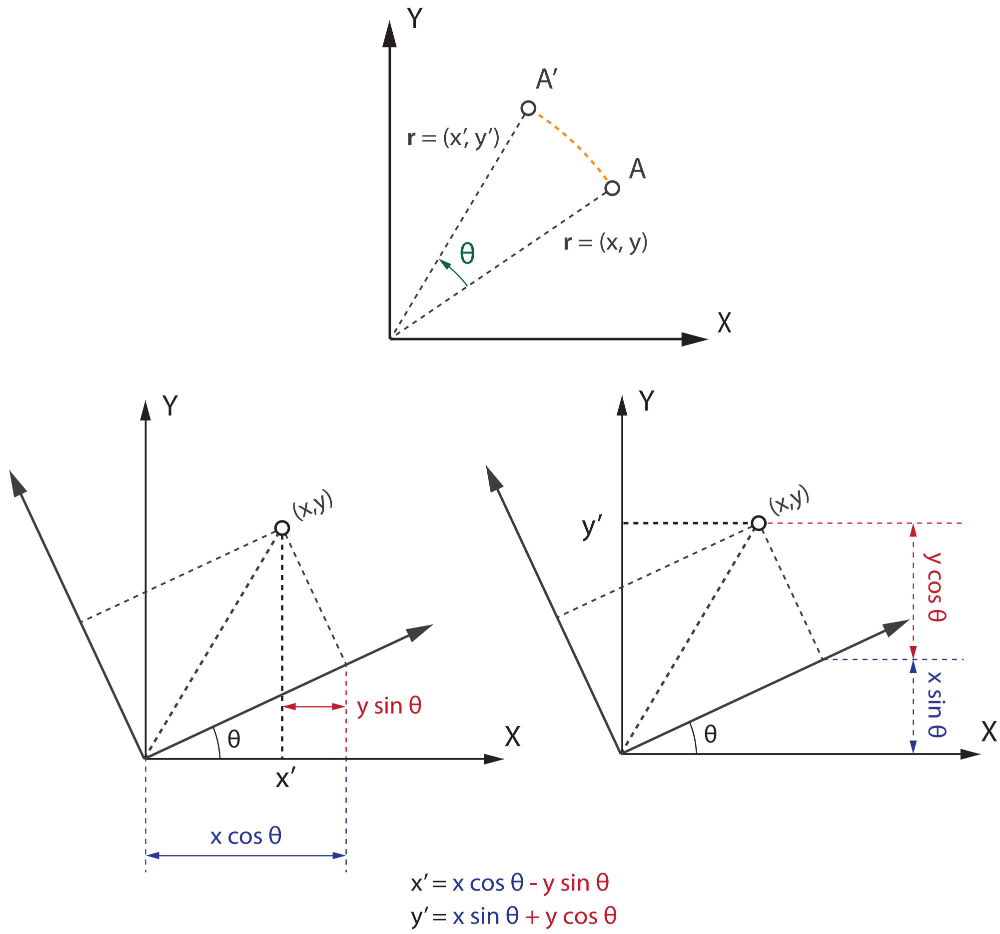
The location of the point of zero velocity P is found at the intersection of the normal to v_A and v_B. The angular velocity of the connecting member AB is then found as:

$$\omega_{AB} = \frac{v_A}{r_A} = \frac{\omega R}{r_A}$$

This can then be used to find the absolute velocity of the slider: $v_B = \omega_{AB} r_B$

4.4 Planar Kinematics: Rotation Matrix

Until now, only *infinitesimal* rotations of rigid bodies have been described, and integrated to find finite rotations. Finite rotations can be expressed using rotation matrices. Consider a point A, which is rotated around the origin over a counter-clockwise angle θ.



The new coordinates are given by:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{R}_z \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation matrices will not be used in this course, but are used extensively in aircraft flight dynamics in later years to describe 3D rotations (*e.g.* pitch, roll, yaw). The rotation matrices around the three axes are:

$$\boldsymbol{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\boldsymbol{R}_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrices and Velocity The expression for the velocity of a point rotating around an axis,

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

can also be found from the rotation matrix. Let a point on a rigid body by described by a location vector $\boldsymbol{r}$. After rotation of the rigid body, its position will be defined by:

$$\boldsymbol{r}' = \boldsymbol{R}\boldsymbol{r}$$

and its velocity by its time derivative:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}'}{dt} = \dot{\boldsymbol{R}}\boldsymbol{r} + \boldsymbol{R}\dot{\boldsymbol{r}}$$

Here $\dot{\boldsymbol{r}} = 0$, as the points lie on a rigid body, and thus:

$$\boldsymbol{v} = \frac{d\boldsymbol{r}'}{dt} = \dot{\boldsymbol{R}}\boldsymbol{r} = \frac{d\boldsymbol{R}}{d\theta}\frac{d\theta}{dt}\boldsymbol{r} = \dot{\theta}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{bmatrix} r_x \\ r_y \end{bmatrix}$$

Evaluating at $\theta = 0$ to find the tangential velocity this reduces to

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} r_x \\ r_y \\ 0 \end{bmatrix} = \omega \begin{bmatrix} -r_y \\ r_x \\ 0 \end{bmatrix}$$

which is the same as

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ 0 \end{bmatrix} = \omega \begin{bmatrix} -r_y \\ r_x \\ 0 \end{bmatrix}$$

It is important to keep in mind that the equations derived throughout this handout have been for instantaneous velocities, and not average velocities over large rotations.

Revision Objectives Handout 4:

Kinematics of Rigid Bodies

- express the relationships between angular displacement θ, velocity ω, and acceleration α
- calculate velocities for rotation around fixed axis ($v = \omega r$, $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$)
- calculate accelerations for rotations around fixed axis ($a_n = \omega^2 r$, $a_t = \ddot{\theta} r$, $\boldsymbol{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \boldsymbol{\alpha} \times \boldsymbol{r}$)
- express velocities of points on a rigid body in general plane motion (combined translation and rotation) in absolute and relative terms ($\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B} = \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{r}_{A/B}$)
- locate the instantaneous centre of zero velocity and use this to calculate velocities on a rigid body

Note: the use of a rotation matrix to describe finite rotations is **not** examinable.