EMAT10100 Engineering Maths I
Lecture 6: sinh, cosh and tanh

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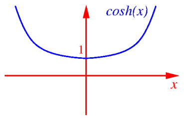
Recap of polar and exponential form

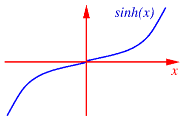
- Recall:
$$z \;=\; x + jy \;=\; r\,(\cos\theta + j\sin\theta) \;=\; re^{j\theta}$$
so
$$\cos\theta = \text{Re}\left(e^{j\theta}\right)$$
$$\sin\theta = \text{Im}\left(e^{j\theta}\right)$$
(the link between $\cos$, $\sin$, and $\exp$)
- This time:
  - hyperbolic functions $\sinh$, $\cosh$ and $\tanh$
  - connections to trig functions $\sin$, $\cos$ and $\tan$

Hyperbolic functions: definitions

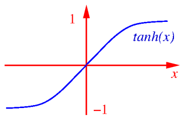
- $$\cosh x := \frac{1}{2}\left(e^{x} + e^{-x}\right)$$



- $$\sinh x := \frac{1}{2}\left(e^{x} + e^{-x}\right)$$



- $$\tanh(x) := \frac{\sinh x}{\cosh x}$$



- Homework: full details in *James* 2.7.4–5

Properties of hyperbolic functions

- Multi-angle formulae obey Osborn's rule

  "every formula for trig is true for hyperbolic funcs. except change the sign every time you have sine times sine":

  - $\cos^2 x + \sin^2 x = 1 \;\Rightarrow \cosh^2 x - \sinh^2 x = 1$
  - $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \;\Rightarrow$
    $\sinh(A+B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$
  - $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \Rightarrow$
    $\cosh(A+B) = \cosh(A)\cosh(B) + \sinh(A)\sinh(B)$
  - etc. . . .
- Proof: see link to trig functions in 2 slides time

Engineering HOT SPOT

Graph $\cosh(x)$ called a catenary

- the shape of a chain
- hanging under gravity
- used in structural design
- need math modelling to understand deformation under load:

Relationship to trig functions

- From exponential forms

$$\mathrm{e}^{+j\theta} = \cos\theta + j\sin\theta$$
$$\mathrm{e}^{-j\theta} = \cos\theta - j\sin\theta$$

- Therefore:

$$\cos\theta = \frac{1}{2}\left(\mathrm{e}^{+j\theta} + \mathrm{e}^{-j\theta}\right),$$
$$\sin\theta = \frac{1}{2j}\left(\mathrm{e}^{+j\theta} - \mathrm{e}^{-j\theta}\right)$$

- Let $\theta = jx$ (x real), then from definitions

$$\cos(jx) = (1/2)(e^{-x} + e^{x}) = \cosh(x)$$
$$\sin(jx) = (1/2j)(e^{-x} - e^{x}) = j\sinh(x)$$
$$\tan(jx) = j\tanh(x) \quad \text{(by dividing the two above)}$$

Application: simplifying trig powers

I can easily integrate $\int \sin(n\theta)d\theta \quad or \int \cos(n\theta)d\theta$

but not so easily $\int \sin^n(\theta)d\theta \quad or \int \cos^n(\theta)d\theta$

- So, how to express $\sin^n(\theta)$ or $\cos^n(\theta)$ in terms of linear combinations of $\sin(k\theta)\cos(k\theta)$ for other k?
- Answer: Write

$$\cos^n\theta = \left[\frac{1}{2}(e^{j\theta} + e^{-j\theta})\right]^n, \sin^n\theta = \left[\frac{1}{2j}(e^{j\theta} - e^{-j\theta})\right]^n$$

- . . . and expand out the bracket using binomial theorem:

$$(a+b)^n = \sum_{i=0}^{n} K_i a^{n-i} b^i$$

where the K_i are the nth row of Pascal's triangle

Example $\sin^3(\theta)$

note: James explains this using $z = e^{j\theta}$, $z^{-1} = e^{-j\theta}$

$$\sin^3\theta = \left[\frac{e^{j\theta} - e^{-j\theta}}{2j}\right]^3$$
$$= \frac{1}{2^3 j^3}\left[(e^{j\theta})^3 + 3(e^{j\theta})^2(-e^{-j\theta}) + 3(e^{j\theta})(-e^{-j\theta})^2 - (-e^{-j\theta})^3\right]$$
$$= -\frac{1}{2^3 j}\left[e^{3j\theta} - 3e^{j\theta} + 3e^{-j\theta} - e^{-3j\theta}\right]$$
$$= -\frac{1}{2^2}\left[\frac{e^{3j\theta} - e^{-3j\theta}}{2j} - 3\left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)\right]$$
$$= -\frac{1}{4}[\sin(3\theta) - 3\sin\theta] = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

- Exercise: Find $\cos^5(\theta)$ in terms of linear functions of $\cos(k\theta)$ for various k.

Complex arguments for sin, cos etc.

- Use addition formula:

$$\begin{aligned} \sin(z) &= \sin(x + jy) \\ &= \sin(x)\cos(jy) + \cos(x)\sin(jy) \end{aligned}$$

- so

$$\sin(z) = \sin(x)\cosh(y) + j\cos(x)\sinh(y)$$

- Similarly

$$\cos(z) = \cos(x)\cosh(y) - j\sin(x)\sinh(y)$$

- Exercise: find values of z such that $\cos z = 2$.

Summary (of complex numbers)

- $j = \sqrt{-1}$
- $z = x + jy \quad re^{j\theta} = r[\cos\theta + j\sin\theta]$
- use polar form to find nth roots $z^n - (a + jb) = 0$
- $\ln(z) = \ln(r) + j(\theta + 2n\pi)$,
- $\cos(jx) = \cosh(x), \quad \sin(jx) = j\sinh(x)$
- use de Moivre's theorem to find $\sin(n\theta)$, $\cos(n\theta)$ in terms of $\sin(\theta)$, $\cos(\theta)$

$$\cos(n\theta) = \text{Re}\,[\cos(n\theta) + j\sin(n\theta)]$$

- use $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ & $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ to simplify $\cos^n(\theta)$, $\sin^n(\theta)$ etc.

Homework

- Read James Secs. 2.7.4, 3.2.9, 3.3.2
- Do James exercise 3.2.11 Qs. 29, 30(a),(d), 31
  & exercises 3.3.3 Qs. 35, 36(a),(b)
- The next Questionmark test is ready qmp.bris.ac.uk
- Next time, new topic: Vectors
- Please try to get up to date on all the complex numbers homeworks before then use the drop in sessions