3. Laplace Transforms

↳ A tool to solve engineering systems.
(sets of differential equations)

The Laplace transform is another way to transform a function to a different "domain".

- Extracting the decay rate from signals.
- Basic properties.
- Transforms of common functions and the inverse transform.
- Using Laplace transforms to solve linear ODEs. The transfer function.

input ~ sets of ODES output

[James Advanced MEM (4th Edn) Ch. 5 is very comprehensive]

$u(t) \longrightarrow$ SYSTEM $\longrightarrow y(t)$ time domain

$\cancel{\mathcal{F}}$ $\mathcal{L}$ ↓ ↑ $\mathcal{L}^{-1}$ $\cancel{\mathcal{F}^{-1}}$

$U(\cancel{\omega}\,s) \longrightarrow$ SYSTEM $\longrightarrow Y(\cancel{\omega}\,s)$ Laplace / s - ~~frequency~~ domain

multiplication by powers of $\cancel{\omega}$ s

Why?

Laplace transforms have many uses.

- systematic method for solving linear ODEs (e.g. linear AC circuit theory, spring-mass-damper systems etc.)
- they provide a general way to formulate a *transfer function* of an input-output system.

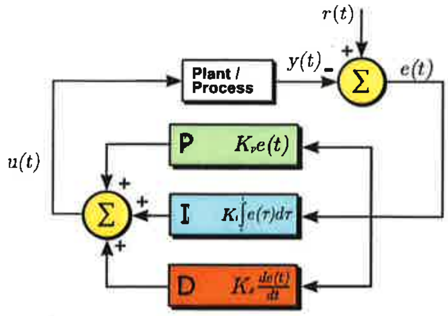
They therefore provide the *basic language of control engineering*.

Transfer function is:

- real-valued (unlike frequency transfer function from Fourier method)
- in the "s-domain", or the "Laplace space"
- useful: given input, use transfer function to get output

The Laplace transform measures characteristic decay rates.

Engineering application: PID control



transfer function

$$G(s) = \frac{K_d s^2 + K_p s + K_i}{s^2}$$

- How can you control a system's output to be what you want? e.g. engine control unit, force-microscope position, streaming video server utilisation, etc.
- Minimise the error between a measured process variable y and a reference signal r
- Using the Laplace domain makes it easy to create rules for how to choose the gains K_p, K_i, K_d

Definition

Recall Fourier transform $\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)\, dt$.

Definition. *The* **Laplace transform** *of a function $f(t)$ is written $L[f(t)]$ (or sometimes $\mathcal{L}[f(t)]$) and is defined by:*

"take the Laplace transform of"

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

- $L[f(t)]$ is not a function of t but it of a (typically positive) *complex* variable s. Thus L is an operator which maps the function $f(t)$ into another function of the variable s. By convention we write $L[f(t)]$ as $F(s)$, so:

$$L[f(t)] = F(s) \qquad L^{-1}[F(s)] = f(t)$$

- There is no simple expression for the inverse transform

use tables to take L^{-1}

time domain $f(t)$ $\xrightarrow{L}$ s-domain $F(s)$ $\xrightarrow{L^{-1}}$ $f(t)$

Linearity

because integration is linear.

- The Laplace transform is a linear operator since:

$$\begin{aligned} L[af(t)+bg(t)] &= \int_0^\infty e^{-st}[af(t)+bg(t)]dt \\ &= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt \\ &= aL[f(t)] + bL[g(t)] \end{aligned}$$

$= aF(s) + bG(s)$

the Laplace transform of a sum = the sum of the Laplace transforms.

Transforms of some common functions

Some examples of Laplace transforms:

$$L[1] = \frac{1}{s} \qquad (s>0) \qquad (1)$$

$$L[t^n] = \frac{n!}{s^{n+1}} \qquad (s>0) \qquad (2)$$

Hence

$$L[t] = \frac{1}{s}L[1] = \frac{1}{s^2}$$

$$L[t^2] = \frac{2}{s}L[t] = \frac{2}{s^3}$$

$$L[t^3] = \frac{3}{s}L[t^2] = \frac{6}{s^4}$$

$$L[1] = \int_0^\infty 1 . e^{-st}\, dt$$

$$= \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=\infty} \qquad Re(s) > 0$$

$e^{-st} \to 0$ as $t \to \infty$

$$= -\frac{1}{s}(0 - 1)$$

$$= \frac{1}{s}$$

$$\int uv' = uv - \int u'v$$

$$L[t^n] = \int_0^\infty \underbrace{t^n}_{u} . \underbrace{e^{-st}}_{v'}\, dt$$

$$= \cancel{\left[\underbrace{t^n}_{u} . \underbrace{-\frac{1}{s} e^{-st}}_{v} \right]_{t=0}^{t=\infty}} - \int_0^\infty \underbrace{n t^{n-1}}_{u'} . \underbrace{\frac{-1}{s} e^{-st}}_{v}\, dt$$

for $s > 0$ $t^n e^{-st} \to 0$ as $t \to \infty$

$$= \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\, dt$$

$$= \frac{n}{s} L[t^{n-1}]$$

$$= \frac{n}{s} . \frac{n-1}{s} L[t^{n-2}]$$

$\vdots$ n times (n integer)

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \cdots . \frac{1}{s} L[1] \quad = \frac{1}{s}$$

$$L[t^n] = \frac{n!}{s^n} . \frac{1}{s} = \frac{n!}{s^{n+1}} \qquad \text{(for integers } n\text{)}$$

<u>and</u> $$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{1}{n!} t^n$$

Bonus! We can take the Laplace transform of <u>any</u> polynomial

$$L\left[t^3 - t + 3\right] = L[t^3] - L[t] + 3L[1]$$

$$= \frac{3!}{s^4} - \frac{1}{s^2} + 3 \cdot \frac{1}{s}$$

& $$L^{-1}\left[\frac{3!}{s^4} - \frac{1}{s^2} + \frac{3}{s}\right] = t^3 - t + 3$$

$\Rightarrow$ We can take the Laplace transform of any time-domain function

(with a Taylor series)

Examples continued

(a constant)

Also,

$$L[e^{-at}] = \frac{1}{s+a} \qquad (s+a>0) \tag{3}$$

and

$$L[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad L[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2} \tag{4}$$

Derivative of a Laplace Transform.

↳ closest we get to the L.T. of a product.

$$L[tf(t)] = -\frac{dF}{ds} \tag{5}$$

where $F(s) = L[f(t)]$.

$\times t$ in time domain $\equiv$ differentiate w.r.t. s & $\times -1$ in s-domain

Consequently,

$$L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n} \tag{6}$$

& hence we can take the L.T. of the product of any polynomial (in t) with any function $f(t)$

$$L[e^{-at}] = \int_0^\infty e^{-at} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-(a+s)t}\, dt$$

$$= \left[-\frac{1}{a+s} e^{-(a+s)t} \right]_0^\infty$$

if $Re(a+s) > 0$ the $e^{-(a+s)t} \to 0$ as $t \to \infty$

$$= -\frac{1}{a+s}(0-1)$$

$$L[e^{-at}] = \frac{1}{a+s}.$$

$$L[\cos(\omega t)] = \int_0^\infty \cos(\omega t)\, e^{-st}\, dt$$

$$L[j\sin(\omega t)] = \int_0^\infty j\sin(\omega t)\, e^{-st}\, dt$$

$-$ $L[\cos(\omega t)] - jL[\sin \omega t]$

$$= \int_0^\infty \underbrace{(\cos \omega t - j \sin \omega t)}_{e^{-j\omega t}}\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-j\omega t} e^{-st}\, dt$$

$$= L[e^{-j\omega t}]$$

using $L[e^{-at}] = \frac{1}{s+a}$

$$= \frac{1}{s+j\omega}$$

$$L[\cos(\omega t)] = Re\left(\frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

$$L[\sin(\omega t)] = -Im\left(\frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

$$\frac{1}{s+j\omega} = \frac{1}{s+j\omega} \cdot \frac{s-j\omega}{s-j\omega} = \frac{s-j\omega}{s^2+\omega^2}$$

$$L[t f(t)] = -\frac{dF}{ds} \quad ?$$

$$\frac{d^n}{ds^n} F(s) = \frac{d^n}{ds^n}\int_0^\infty f(t) e^{-st}\, dt \quad \text{(by defn)}$$

$$= \int_0^\infty \frac{\partial^n}{\partial s^n}\left(f(t) e^{-st}\right) dt$$

(deriv. of an integral is the integral of the derivative if the limits are constant)

$$= \int_0^\infty f(t) (-t)^n e^{-st}\, dt$$

$$= \cancel{(-s)^n} \int_0^\infty (-t)^n f(t)\, e^{-st}\, dt$$

$$\boxed{\cancel{\frac{d^n F}{ds} = (-s)^n L[f(t)] = (-s)^n F(s)}}$$

$$\frac{d^n F}{ds^n} = \int_0^\infty (-1)^n t^n f(t) e^{-st}\, dt$$

$$= L[t^n f(t)] \cdot (-1)^n$$

$$\frac{d^nF}{ds^n} = \frac{d^n}{ds^n}\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{\partial^n}{\partial s^n}\left[f(t)\, e^{-st}\right] dt$$

$$= \int_0^\infty f(t) \cdot \frac{\partial^n}{\partial s^n}\left[e^{-st}\right] dt$$

$$= \int_0^\infty f(t) \cdot (-t)^n \cdot e^{-st}\, dt$$

$$= (-1)^n \int_0^\infty t^n \cdot f(t) \cdot e^{-st}\, dt$$

$$\frac{d^nF}{ds^n} = (-1)^n L\left[t^n f(t) \right]$$

Worked example 3.1

$L[t^2 f(t)] = (-1)^2 F(s) = (-1)^2 L[f(t)]$

Prove (1)–(5). *Use the combined results to find the Laplace transform of* $t^2 \sin(\omega t)$.

$$L[t^2 \sin(\omega t)] = (-1)^2 \frac{d^2}{ds^2} L[\sin \omega t]$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= \cancel{(-1)^2} \frac{d}{ds}\left(-\frac{\omega}{(s^2+\omega^2)^2}\cdot 2s\right)$$

$$= -2\omega \frac{d}{ds}\left(\frac{s}{(s^2+\omega^2)^2}\right)$$

$$= -2\omega \frac{1}{(s^2+\omega^2)^2} + 2\omega \frac{s}{(s^2+\omega^2)^3}\cdot -2\cdot 2s$$

$$= -\frac{2\omega}{(s^2+\omega^2)^2} - \frac{8s^2\omega}{(s^2+\omega^2)^3}$$

Solving simple differential equations

An important property of Laplace transforms is what happens when we take the transform of a derivative:

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0) \tag{7}$$

We transformed the operation of *differentiation* in the time domain to *multiplication* by s, in the s domain. This helps to solve differential equations.

t- domain $\equiv$ s- domain

diff. w.r.t. t $\equiv$ mult by s − a constant

Worked example 3.2 *Prove* (7)

Ex 3.2 //

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \underbrace{\frac{df}{dt}}_{v'} \underbrace{e^{-st}}_{u} \, dt \qquad \left(1 = \frac{d}{dt}\right)$$

parts

$$= \Big[\underbrace{f(t)}_{v}\,\underbrace{e^{-st}}_{u}\Big]_{t=0}^{t=\infty} - \int_0^\infty \underbrace{f(t)}_{v} . \underbrace{-s e^{-st}}_{u'} \, dt$$

provided f grows no faster than exponential & $Re(s) > 0$ $\quad f(t)\, e^{-st} \to 0$ as $t \to \infty$

$$= -f(0) + s \underbrace{\int_0^\infty f(t)\, e^{-st} \, dt}_{F(s) = \mathcal{L}[f(t)]}$$

$$\boxed{\mathcal{L}\left[\frac{df}{dt}\right] = -f(0) + s\, F(s)}$$

Higher derivatives?

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = \mathcal{L}\left[\frac{d}{dt}\left(\frac{df}{dt}\right)\right]$$

$$= -\frac{df}{dt}(0) + s\, \mathcal{L}\left[\frac{df}{dt}\right] \qquad = -f(0) + sF(s)$$

$$= -\frac{df}{dt}(0) - s f(0) + s^2 F(s)$$

Extending this we obtain:

$$
\begin{aligned}
L\left[\frac{d^2 f}{dt^2}\right] &= sL\left[\frac{df}{dt}\right] - \frac{df}{dt}(0) \\
&= s[sL[f(t)] - f(0)] - \frac{df}{dt}(0) \\
&= s^2 F(s) - sf(0) - f'(0)
\end{aligned}
$$

t-domain $\qquad$ s- domain

$\frac{d^n}{dt^n} \equiv \times s^n$ + a polynomial in s. (degree n-1)

and in general:

coeffs are f & its derivatives at time zero. (numbers!)

$$
L\left[\frac{d^n f}{dt^n}\right] = s^n F(s) \underbrace{- s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)}_{\text{poly in } s}
$$

Worked example 3.3

Use the method of Laplace transforms to show that the solution to the differential equation

system tm in time domain (for x(t))

$$
\boxed{\frac{dx}{dt} + 7x = e^{-4t} \qquad \text{given} \quad x(0) = 2}
$$

can be written in the s-domain as

$$
X(s) = \frac{2s + 9}{(s + 7)(s + 4)}.
$$

Ex 3.3

$$\mathcal{L}\left[\frac{dx}{dt} + 7x\right] = \mathcal{L}\left[e^{-4t}\right]$$

take L.T.

$x(0) = 2$

$$\mathcal{L}\left[\frac{dx}{dt}\right] + 7\mathcal{L}[x] = \mathcal{L}\left[e^{-4t}\right]$$

$X(s) = \mathcal{L}[x(t)]$

$$\overbrace{sX(s) - \underset{\overset{\parallel}{2}}{x(0)}} + 7X(s) = \frac{1}{s+4}$$

$$\boxed{sX(s) - 2 + 7X(s) = \frac{1}{s+4}.}$$

system in s-domain for X(s)

this is easy to solve for X(s)

$$(s+7)X(s) = 2 + \frac{1}{s+4}$$

$$\boxed{X(s) = \frac{2}{s+7} + \frac{1}{(s+4)(s+7)}}$$

Solution in s-domain

$2\mathcal{L}\left[e^{-7t}\right]$? $\mathcal{L}[\ ]$

use partial fractions

$$\frac{1}{(s+4)(s+7)} = \frac{A}{s+4} + \frac{B}{s+7} = \frac{1}{3}\frac{1}{s+4} - \frac{1}{3}\frac{1}{s+7}$$

$$1 = A(s+7) + B(s+4)$$

$s=-4$ $\quad 1 = 3A \longrightarrow A = 1/3$

$s=-7$ $\quad 1 = -3B \longrightarrow B = -1/3$

$$\Rightarrow X(s) = \frac{5}{3}\cdot\frac{1}{s+7} + \frac{1}{3}\frac{1}{s+4} = \frac{5}{3}\mathcal{L}\left[e^{-7t}\right] + \frac{1}{3}\mathcal{L}\left[e^{-4t}\right]$$

$\mathcal{L}^{-1}$

$$\boxed{x(t) = \frac{5}{3}e^{-7t} + \frac{1}{3}e^{-4t}}$$

Solution in t-domain.