Maths for Aerodynamics 2

Aerodynamics 2
AENG21100

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Mathematics for Aerodynamics 2

- Maths is used extensively in Aerodynamics 2
- Key things you will need to use:
  - Differentiation and integration of simple functions
  - Curve sketching
- You will see, but won't need to use:
  - Vector calculus
- This hand out contains a revision test on maths you will use as well as a brief description of vector calculus notation you might encounter. Answers will be given on the BB site.

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Revision Self Assessment Test

- Differentiation
- Integration
- Curve sketching
- Other useful maths

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Differentiate the following functions with respect to x

x	$\sin x$
x^2	$\cos x$
x^n	$\tan^{-1} x$
$\frac{1}{x}$	$\frac{1}{x^2}$
$\ln x$	$x \sin x$
$x^2 \ln x$	$\frac{1}{x} \ln x$

Note: $\frac{d}{dx}\big(u(x)v(x)\big) = u\frac{dv}{dx} + v\frac{du}{dx}$

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Function of a function differentiation-differentiate with respect to x

$\ln(x^2)$ $\quad\quad$ $\sin x^2$

$(1+x^2)^2$ $\quad\quad$ $\cos(1/x)$

$\sin(x^n)$ $\quad\quad$ $\tan^{-1}(x^3)$

$\dfrac{1}{(1+\ln x)}$ $\quad\quad$ $\dfrac{1}{(1+\sin x)^2}$

Note: $\dfrac{d}{dx}\left(f(u(x))\right)=\dfrac{df}{du}\dfrac{du}{dx}$

Example: $\dfrac{d}{dx}\left(\sin(\ln x)\right)=\cos(\ln x)\left(\dfrac{1}{x}\right)$

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Partial Differentiation-Differentiate each function with respect to x and y

$x+y$

x^2y+xy^3

$\sin(x+y)$

$\cos(xy)$

$\tan^{-1}\left(\dfrac{y}{x}\right)$

$\dfrac{1}{xy}$

$\dfrac{1}{y}\cos(x)$

$y^2\sin(x^2)$

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Integrate the following functions with respect to x

x	$\sin x$
x^2	$\cos x$
x^n	c (constant)
$\dfrac{1}{x}$	$\dfrac{1}{x^2}$
$\dfrac{1}{1+x^2}$	

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Example of solving for a function given its partial derivatives

$$\phi_x = x^2 y + y\cos x \qquad \phi_y = \frac{1}{3}x^3 + \sin x + y$$

Integrate ϕ_x with respect to $x \Rightarrow \phi = \frac{1}{3}x^3 y + y\sin x + f(y)$

Differentiate ϕ with respect to $y \Rightarrow \phi_y = \frac{1}{3}x^3 + \sin x + f'(y)$

Comparing to ϕ_y $\;f'(y) = y \Rightarrow f(y) = \frac{1}{2}\mathrm{y}^2$

So $\phi = \frac{1}{3}x^3 y + y\sin x + \frac{1}{2}\mathrm{y}^2$

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Solve for ϕ given its partial derivatives

1) $\phi_x = xy + \sin y \cos x \qquad \phi_y = \frac{1}{2}x^2 + \cos y \sin x + \frac{1}{2}y^2$

2) $\phi_x = \frac{\ln y}{x} + 1 \qquad \phi_y = \frac{\ln x}{y}$

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Sketch the following curves

$y = mx + c \quad (c, m$ *constants)*

$x^2 + y^2 = c \quad (c, m$ *constant)*

$y = \frac{1}{x}$

Note: you should recognise these curves, if you don't check the notes on the next page.

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More on curve sketching

When sketching a curve in the *x y* plane which is not immediately recognisable you should:

(1) Find out what happens as $y \to 0$ & $x \to 0$ for example are there any places where the curve crosses the axes

(2) Consider the behaviour as $x \to \pm\infty$

(3) Look for any points where *y* tends to infinity and then consider the behaviour if this point is approached from either side

(4) Sometimes you might what to find max/mins via differentiation

(5) Choose what to plot on the axes to simplify the sketch and to make evaluation of sample points easier.

(6) Evaluate a few sample points if necessary.

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Example 1

1) $y = \dfrac{x-3}{(x-1)^2}$

$(i) \quad y = 0 \Rightarrow x = 3, \;\; x = 0 \Rightarrow y = -3$

$(ii) \;\; x \to \pm\infty \;\; |x-3| << (x-1)^2 \quad so \quad y \to \pm 0$

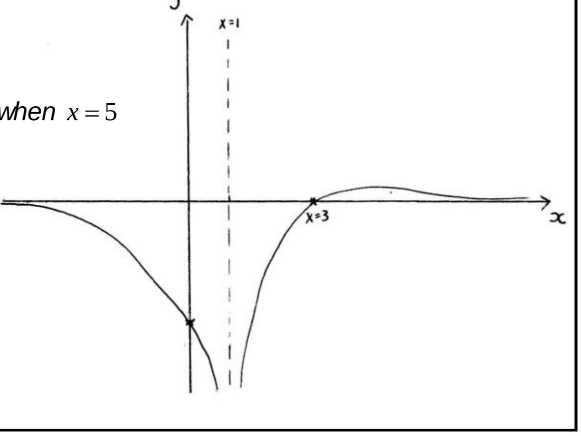
$(iii) \; y \to \infty$ *when* $x = 1. \;\; For \;\; x = 1 \pm \delta$

$(x-1)^2 > 0$

$(x-3) < 0$

$y < 0$

$(iv) \;\; \dfrac{dy}{dx} = \dfrac{5-x}{(x-1)^3} \;\; \Rightarrow \dfrac{dy}{dx} = 0$ *when* $x = 5$



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Example 2

2) $y=-\frac{1}{2}\left(1+\frac{\sqrt{1+(x/2)^2}}{(x/2)}\right)$ *or* $y=-\frac{1}{2}\left(1+\frac{\sqrt{1+(x')^2}}{(x')}\right)$ *where* $x'=x/2$

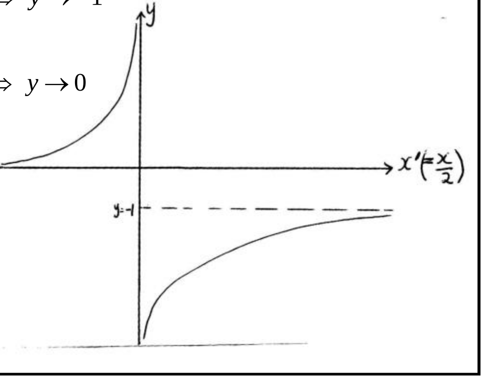
(i) $x'\to 0 \Rightarrow \frac{\sqrt{1+(x')^2}}{(x')}\to\frac{1}{0}\to\infty\Rightarrow y\to\infty$

$x'\to+\delta \quad y<0$ *and* $x'\to-\delta \quad y>0$

No solution for $y=0$

(ii) $x\to+\infty \quad \frac{\sqrt{1+(x')^2}}{(x')}\to\frac{\sqrt{(x')^2}}{(x')}\to+1\Rightarrow y\to-1$

$x\to-\infty \quad \frac{\sqrt{1+(x')^2}}{(x')}\to\frac{\sqrt{(x')^2}}{(x')}\to-1\Rightarrow y\to 0$



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Sketch the following curve

1) $y=\frac{8-x}{(x-2)^2}$

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Miscellaneous

(1) Fill in the brackets

$$\ln a + \ln b = \ln\left(\qquad \right)$$

$$\ln a - \ln b = \ln\left(\qquad \right)$$

(2) If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$ where A and B are either both zero or both infinite then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

is called indeterminate $0/0$ or ∞/∞ . However the limit can be evaluated using L'Hospital's Rule. Make sure you know how to do this.

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Vector Calculus

- The equations of fluid dynamics are most simply written using vector calculus. You will be covering this topic in Eng. Maths this year.
- Main definitions included here so that the meaning of terms arising in the introductory material are clear.

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- The gradient or grad or del or nabla operator

$$\text{grad} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \nabla$$

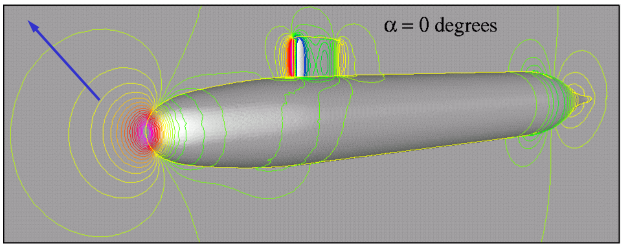
- Applying gradient to a scalar, e.g. for pressure

$$\text{grad } p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k} = \nabla p$$

partial derivative $\quad \frac{\partial p}{\partial x} = \left.\frac{p(t, x+\delta, y, z) - p(t, x, y, z)}{\delta}\right|_{\delta \to 0} = \left.\frac{dp}{dx}\right|_{\text{t,y,z=const}}$

grad p is

normal to contours (or surfaces of constant p)



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- What is the dot product of two vectors?

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = (b_1, b_2, b_3)$$
$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

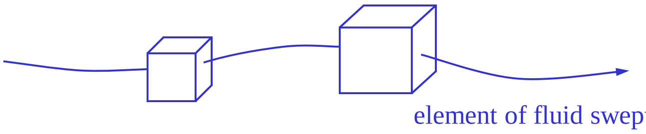
$\mathbf{a}\cdot\hat{\mathbf{b}}$ is how much of **a** is in the direction of **b**

- So what if we take dot product of grad and a vector

If $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ then

$$\nabla\cdot\mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div}\mathbf{V}$$

The rate of change of the volume of a moving fluid element, per unit volume



element of fluid swept along a streamline

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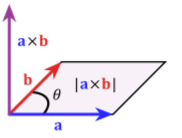
- What is the cross product of two vectors?

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = (b_1, b_2, b_3)$$
$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

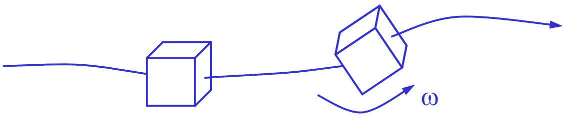
Area of a parallelogram= magnitude of a x b



- What if we take a cross product of grad and a vector

If $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ then

$$\nabla\times\mathbf{V} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \mathrm{curl}\,\mathbf{V}$$



element of fluid moving along a streamline and spinning at a rate $\omega = \frac{1}{2}\nabla\times\mathbf{V}$

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- A useful definition is the ‘substantial’ or ‘material’ derivative, e.g. for density

substantial derivative

$$\frac{D\rho}{Dt} = \underbrace{\frac{\partial\rho}{\partial t}}_{\text{local derivative}} + \underbrace{u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}}_{\text{convective derivative}} = \frac{\partial\rho}{\partial t} + (\mathbf{V}\cdot\nabla)\rho$$

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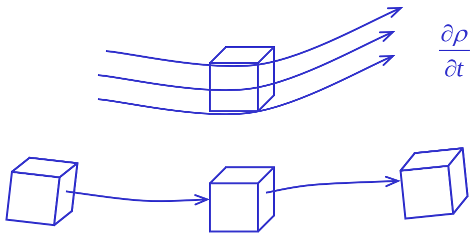
What is the difference between a substantive derivative and a partial derivative with respect to time?



$\frac{\partial \rho}{\partial t}$ Time rate of change at a fixed point

$\frac{D\rho}{Dt}$

$(\mathbf{V}.\nabla)$ convective derivative: the time rate of change due to the movement of the fluid element through a flow field with changing properties

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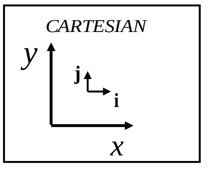
- If considering 2D flows, $w=0$ and derivatives with respect to z are zero then



GRADIENT $$\nabla p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j}$$

DIVERGENCE $$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

CURL $$\nabla \times \mathbf{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$ in z direction only

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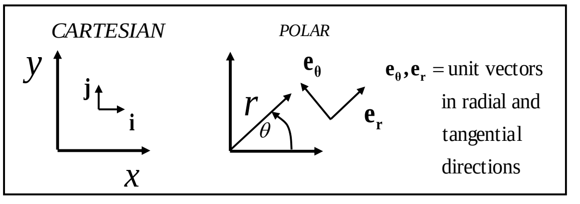
- For some 2D cases it is easier to work in polar coordinates



GRADIENT

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta$$

DIVERGENCE

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

CURL

$$\nabla \times \mathbf{V} = \left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{k}$$

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