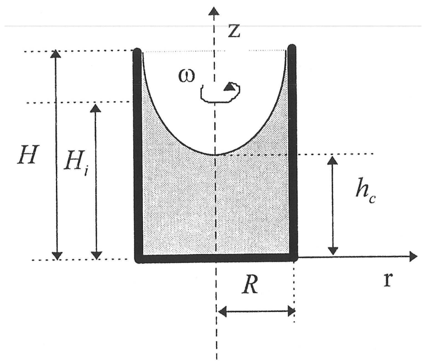
Example: Rigid body rotation of a liquid



The container is initially filled to height H_1. Find angular speed, the shape of the free surface and the pressure at the bottom corner of the container if the liquid surface just touches the lip at height H.

Rotational symmetry, so consider r and z only.
Centripetal acceleration provides a pressure gradient in r

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \qquad \frac{\partial p}{\partial z} = -\rho g$$

Integrating the centripetal terms $\quad p = \tfrac{1}{2}\rho\omega^2 r^2 + f(z)$

Taking the partial differential w.r.t. z (r fixed) gives

$$\frac{\partial p}{\partial z} = \frac{\partial f}{\partial z} = \frac{df}{dz} \quad \rightarrow \quad \frac{df}{dz} = -\rho g \quad \rightarrow \quad f(z) = -\rho g z + C \rightarrow \quad p = \tfrac{1}{2}\rho\omega^2 r^2 - \rho g z + C$$

Complete centripetal equation for p and evaluating the constant at the centre $p(r=0, z=h_c) = p_a$

$$p = \tfrac{1}{2}\rho\omega^2 r^2 + p_a - \rho g(z - h_c)$$

At the lip with maximum angular speed ω=Ω and pressure given by $p(r=R, z=H) = p_a$

$$\tfrac{1}{2}\rho\Omega^2 R^2 = \rho g(H - h_c)$$

The same result for all the surface points shows that the local height above the centreline increases with r^2, forming a parabaloid

$$(z - h_c) = \left(\frac{\Omega^2}{2g}\right) r^2$$

Example: Rigid body rotation of a liquid (2)

The area under the parabaloid is exactly half the base times the height so the original height of the liquid is half way between the minimum and maximum fluid heights

$$h_c = 2H_i - H \qquad \leftarrow \qquad H_i = \frac{H + h_c}{2}$$

Substituting into the equation for angular speed

$$\tfrac{1}{2}\rho\Omega^2 R^2 = \rho g(H - h_c) \quad \rightarrow \quad \Omega^2 = \frac{2g(H - h_c)}{R^2} \quad \rightarrow \quad \Omega^2 = \frac{2g(H - (2H_i - H))}{R^2}$$

$$\Omega = \sqrt{\frac{4g(H - H_i)}{R^2}}$$

The equation for the free surface therefore becomes

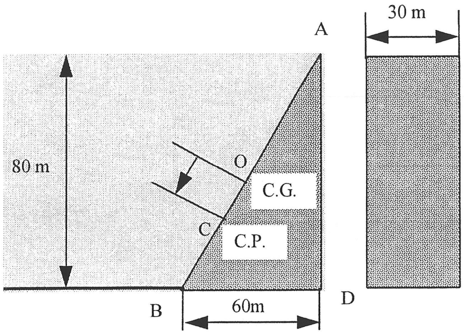
$$\tfrac{1}{2}\rho\Omega^2 r^2 = \rho g(z - h_c) \quad \rightarrow \quad z = \frac{\Omega^2 r^2}{2g} + h_c \quad \rightarrow \quad z = \frac{4g(H - H_i)}{2gR^2} r^2 + (2H_i - H)$$

$$z = \left[2\tfrac{(H - H_i)}{R^2}\right] r^2 + (2H_i - H)$$

And the pressure at the bottom corner

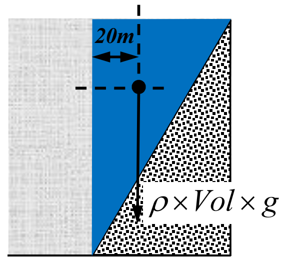
$$p = p_a + 2\rho g(H - H_i) + \rho g(2H_i - H) = p_a + \rho g H$$

Example: Hydrostatic Thrust on Submerged Surface



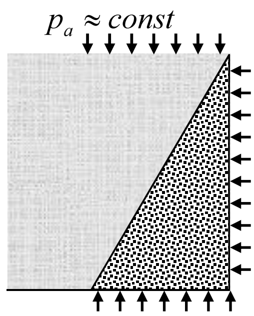
Find the magnitude and point of action of the hydrostatic force exerted on the dam.

Let us consider the vertical and horizontal components of the force (F_v & F_H) separately (we will also look at the problem for a general submerged surface as we go)



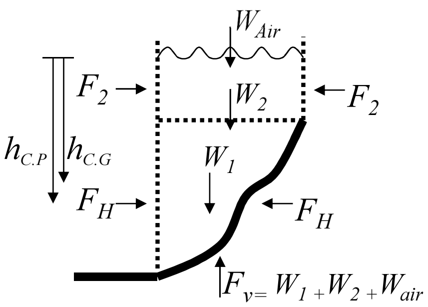
Vertical Force: Can be calculated as the integral of the vertical component of pressure force over the area. Gives a simple relation for planar surfaces but complex otherwise.
As the fluid is static so the forces balance. Hence, F_v equals the **effective weight of fluid above**.

For the dam in our case $F_v = 1000 \times \frac{60}{2} \times 80 \times 30 \times 9.81 = 7.0632 \times 10^8\,N$

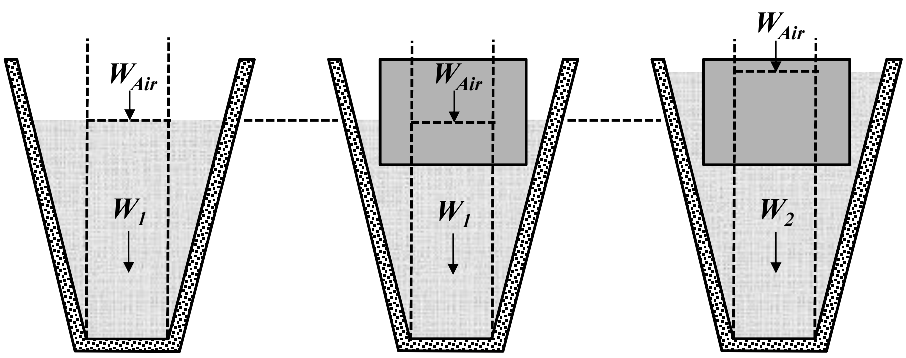


Consider atmospheric pressure to act all around the outer boundary and that it is approximately constant over the height.

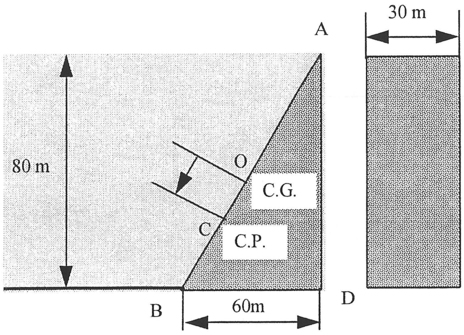
Example: Hydrostatic Thrust on Submerged Surface (2)
Aside:Vertical force on general submerged surfaces



Note the definition “effective weight of fluid above”.
Is there any difference in vertical force at the bottom of a water tank for the cases below?



Example: Hydrostatic Thrust on Submerged Surface (3)



The horizontal force on the submerged surface F_H is the same as the force on a plane formed by projecting the surface horizontally. Atmospheric pressure is ignored as it acts on either side of the dam.

$$F_H = \iint_A p dA = \rho g \iint_A h dA = \rho g h_{C.G} A$$

The horizontal force on a submerged surface is therefore equal to the **pressure at the centre of gravity of the projected shape** times the **area of that projected shape**. Hence for the Dam, the projected shape is a rectangle of 80m by 30m and so

$$F_H = 1000 \times 9.81 \times 40 \times (80 \times 30) = 9.4176 \times 10^8 N$$

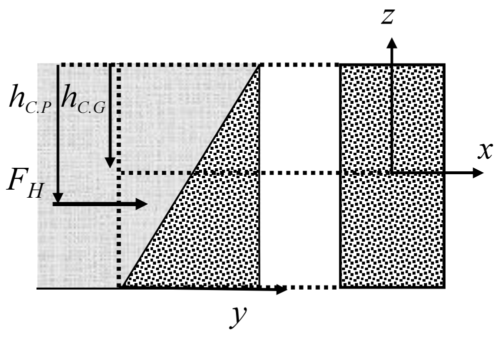
The total force F is then given by

$$F = \sqrt{(F_H)^2 + (F_v)^2} = 1.1772 \times 10^9 N$$

Example: Hydrostatic Thrust on Submerged Surface (4)

However the horizontal force acts through the centre of pressure (C.P.) and not the centre of gravity. The centre of pressure is found by considering moments about the centre of gravity of the projected area:

$$F_H (h_{C.G} - h_{C.P}) = \iint_A z p dA = -\rho g I_{xx}$$ Where I_{xx} is the 2nd moment of area



using our previous formulae for F_H and rearranging

$$h_{C.P} - h_{C.G} = \frac{I_{xx}}{h_{C.G} A}$$

For a rectangle of height H and base b, $I_{xx} = bH^3/12$

Hence the line of action for the horizontal force is $h_{C.P} - h_{C.G} = \dfrac{30 \times 80^3}{12 \times 40 \times 30 \times 80} = 13.333m$

So the horizontal force acts a depth of 53.333m (as does the total force)

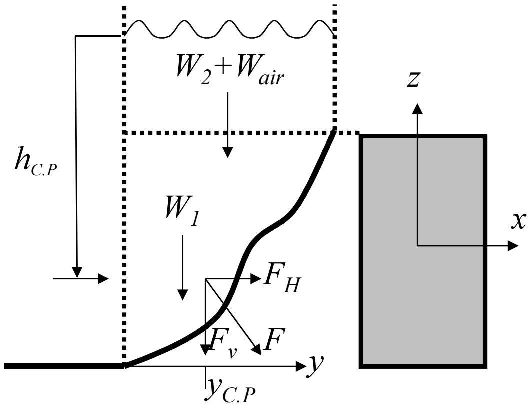
For the current dam case, or any plane surface, this is all we need to know. However for the general case we need a little more work..

Example: Hydrostatic Thrust on Submerged Surface (5)
Aside:Horizontal force on general submerged surfaces

The line of action of the horizontal force (y_{CP}) is also found as the centre of mass of the column of fluid above the surface.

The true point of action of the total thrust F is found as the intersection of the vector F (with origin defined by h_{CP} and y_{CP}) and the surface.

Note that in the planar case (*ie* the dam problem just completed) the origin defined by h_{CP} and y_{CP} lies on the surface.



Learning Outcomes:
"What you should have learnt so far"

- The hydrostatic equation and the constant density hydrostatic equation
- How these relations apply to simple pressure measurement devices and the definition of gauge pressure.
- The definition of standard pressure units
- Buoyancy problems
- How to apply the constant density hydrostatic equation to solve relative steady problems with constant accelerations.
- How to solve thrust problems on simple submerged surfaces