Signals part 6 – Discrete time / frequency transformations and spectra

More Time - Frequency transforms

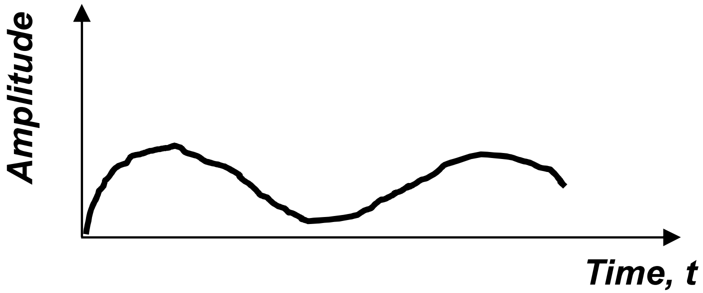
- There are several related integral transforms that convert between the time and frequency domains:

Time	Frequency
Continuous time, t	Continuous frequency, ω,
Discrete time, n	Continuous complex frequency, s
	Continuous frequency normalised to sampling period, Ω
	Discrete frequency, normalised to sequence length, k
	Discrete complex frequency, z

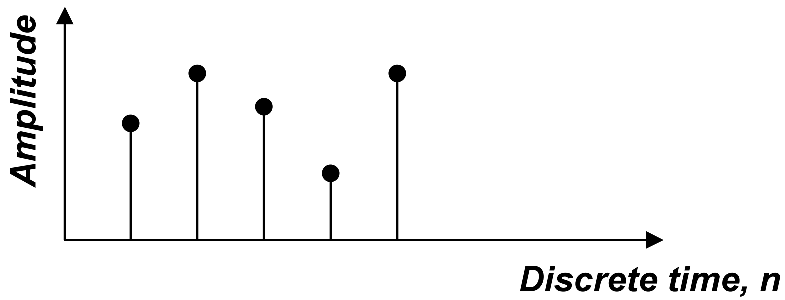
Discrete domains – why do we have them?

- In analogue system a signal can have a value at any number of infinitely small time steps.
- In the frequency domain it could have infinitely many frequency components.
- We can't represent this in digital systems – discretise signals in both time and frequency domains.
    - This is separate from quantisation of magnitude in digital systems

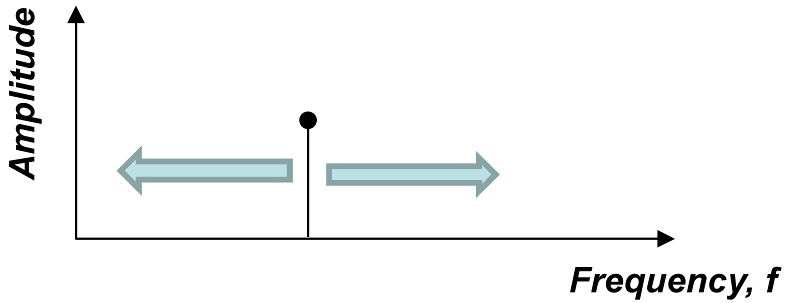
Discrete domains



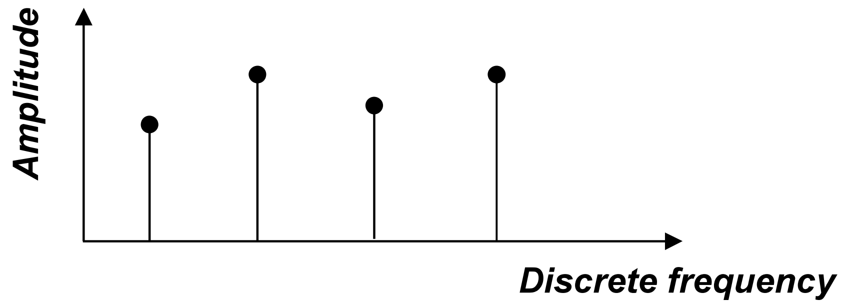
Signal exists for any t



Signal exists only for integers of ‘n’



Components can exist at any f



Components only at any defined frequencies

Discrete time Fourier transform - DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

n = current sample
Ω = normalised frequency, radians per sample.
$0 \leq \Omega < 2\pi$

Discrete time so correlation integral becomes a summation

Discrete time series

Phasor at 'Ω'

$$x[n] = \frac{1}{2\pi} \int X(\Omega) e^{j\Omega n} d\Omega$$

The inverse DTFT is an integral because the frequency domain is continuous

- The Discrete Time Fourier Transform maps a discrete time signal to a continuous frequency domain.
- Ω can take any value hence the spectrum is continuous
- Because it is sampled, the continuous spectrum is still subject to aliasing, i.e. Ω is only unique over the range $0 \leq \Omega < 2\pi$

Discrete Fourier transform - DFT

- In a real system having infinite number of frequencies is not possible – we need to discretise the frequency domain – the DFT.

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

N = total number of samples in sequence

k = cycles per sequence length

Ω = 2πk/N

k only takes values 0≤k<N-1

Hence 0≤Ω<2π(N-1/N)

1. First we will define our DTFT in terms frequency, so we add a 2π – this is just convention
2. Next we introduce a fixed number of samples, N, of our sequence. Since there are now a finite number of frequencies that we are looking for, the integral becomes a summation.
3. k is defined as cycles over the whole sequence length, i.e. 0-N.
4. Then we discretise frequency – we know the maximum value must be less than 1 cycle per sample to avoid aliasing, and the minimum is ‘0’ (i.e. DC) but what about each step in-between these two limits?
5. If we have an ‘N’ sample long discrete sequence then we can resolve frequency into ‘N’ steps. We do this by setting 0≤k<N-1, and dividing by ‘N’

Discrete Fourier transform - DFT

$$X[k]=\sum_{n=0}^{N-1}x[n]\underbrace{e^{-j2\pi kn/N}}_{\textit{So what is this?}}$$

n = current sample
k = cycles per sequence (Ω = 2πk/N)
N = Total number of samples
k takes integer values from 0<k<N-1

- Remember $e^{-j\omega t}$ is a continuous-time phasor (vector rotating on complex plane) representing a single frequency component at 'ω'.
- In the DFT time is discretised so 't' becomes 'n' .
- We are using frequency rather than radians, so we introduce '2π'.
- Our frequency per sample is given by 'k/N':
- To avoid aliasing the maximum frequency per sample we allow is <1. The lowest value will be '0' i.e. DC.
- We want a limited number of frequency values to exist – we want the same number of frequencies as we have samples - so we discretise by letting 'k' range 0≤k≤N-1, resulting in 'N' distinct frequencies.

Example of calculating maximum and difference between discrete frequencies:

$\Omega = 2\pi k/N$

From definition of normalised frequency:

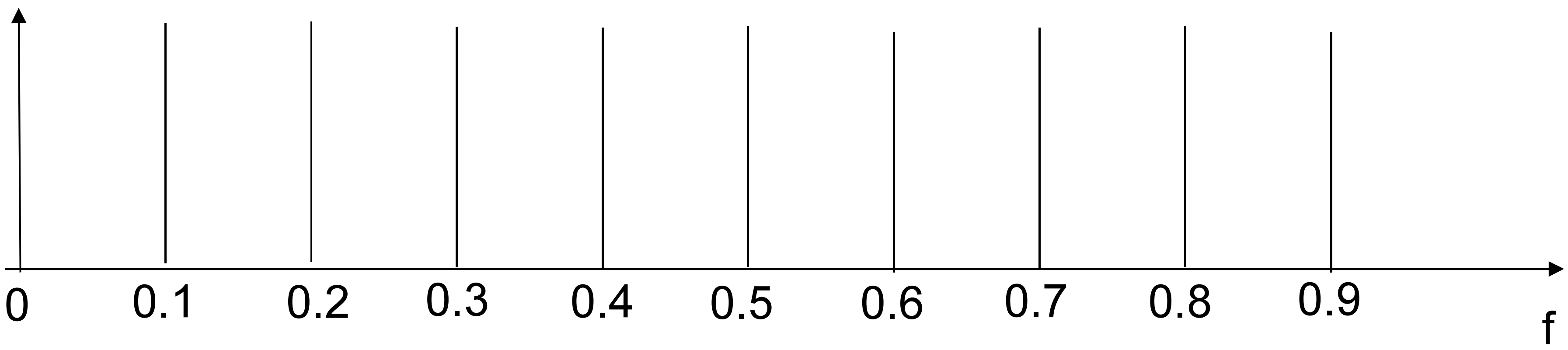
$\Omega = \omega T = 2\pi fT, \; k/NT = f$

If N = 100 and T = 1 seconds

$f_{max} = 99/100T = 0.99$ Hz

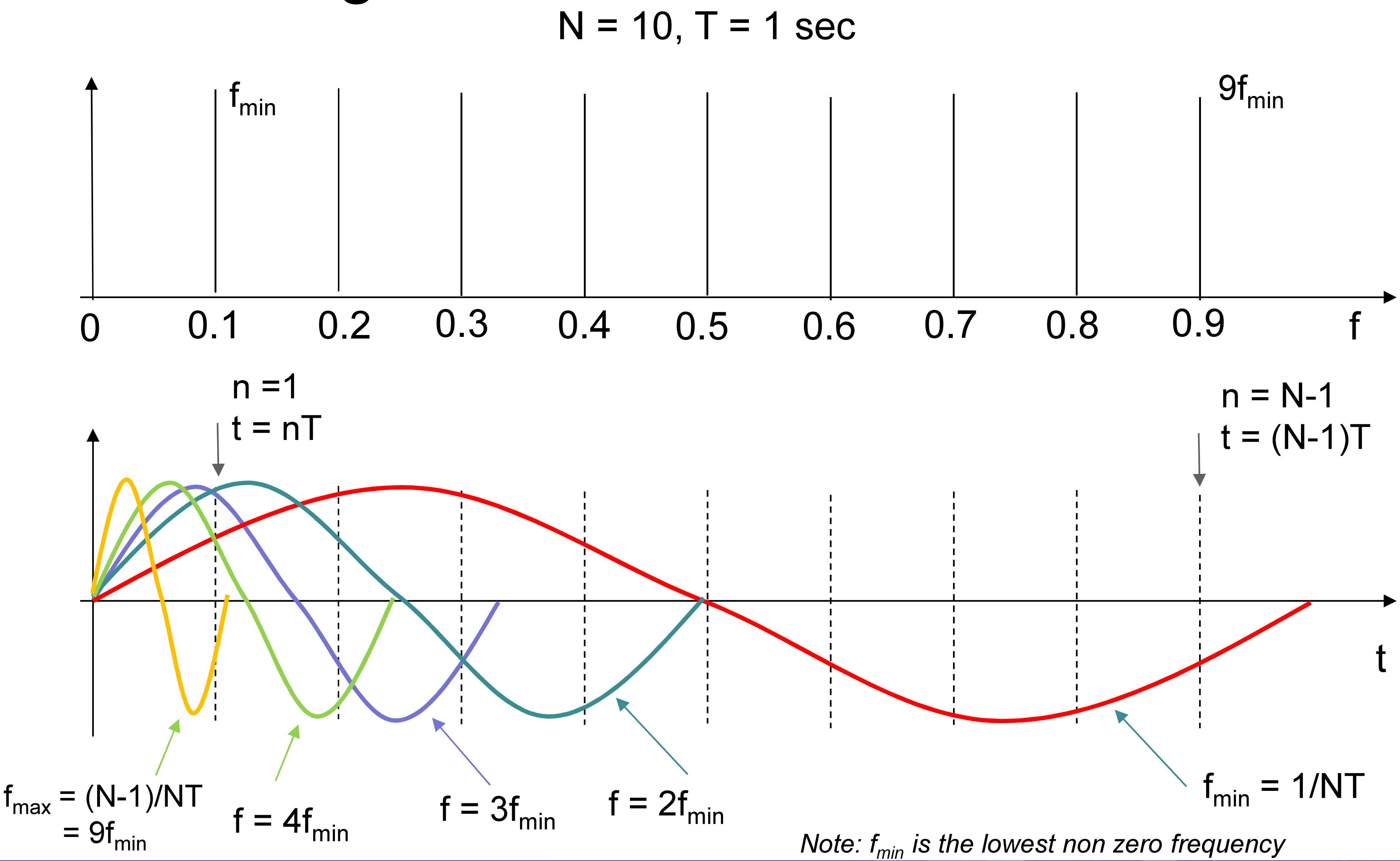
$\Delta f = 1/100T = 0.01$ Hz

Visualising the DFT



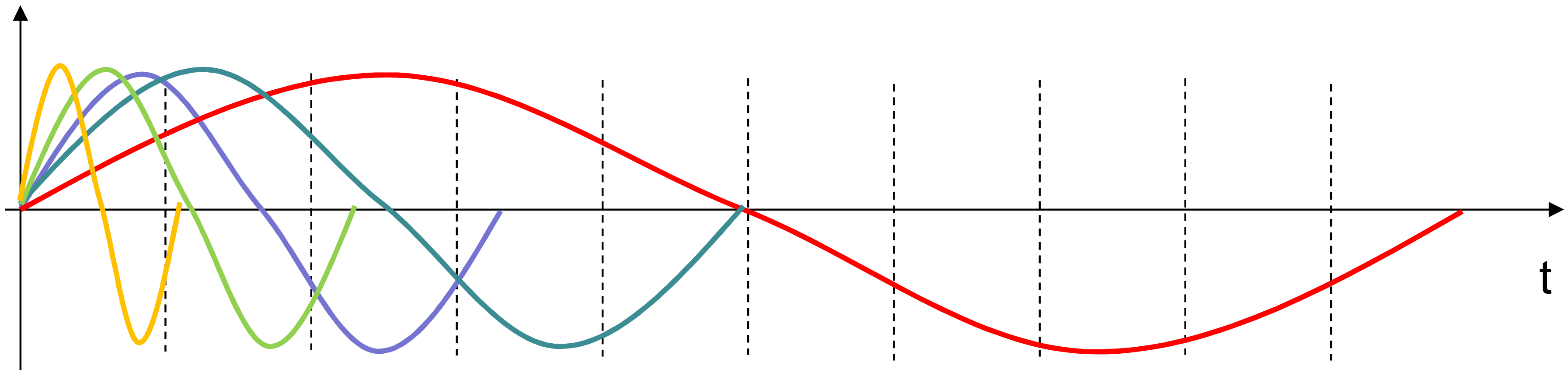
- Consider a sequence 10 samples long, sampled at T = 1 sec.
- There are 10 samples so we want 10 equally spaced frequencies.
- Since $0 \le k \le N-1$, these will be at 0,1,2,3,4,5,6,7,8,9 cycles per sequence, or 0,1/10, 2/10, 3/10, - , 9/10 cycles per sample.
- Since the sampling period is 1 sec, in this case the frequency will be the same as the cycles per sample

Visualising the DFT



Note: f_{min} is the lowest non zero frequency

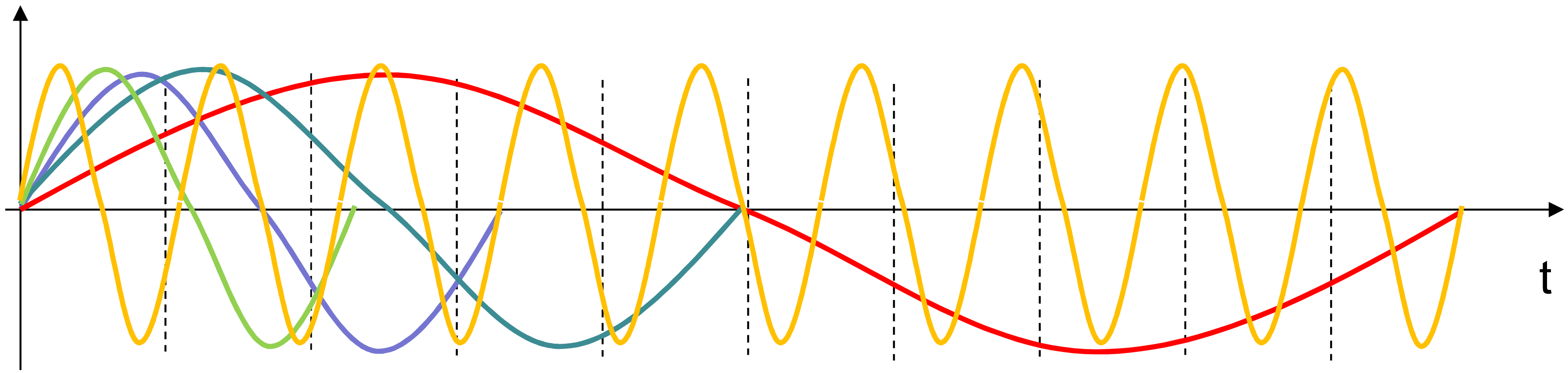
The FFT



- Consider the low frequency component

- There are many more samples than we need to identify this harmonic, so we could reduce the number of samples

The FFT



- Consider the high frequency component
- There are many more cycles than we need to identify this harmonic.
- We could reduce the period over which we calculate

Why do these things? Because the DFT is very computational expensive

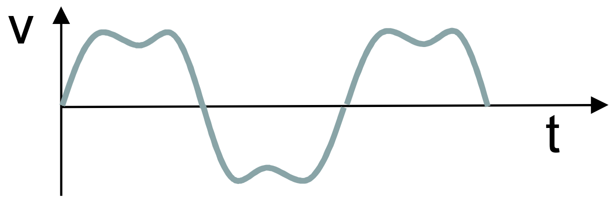
Fast Fourier Transform - FFT

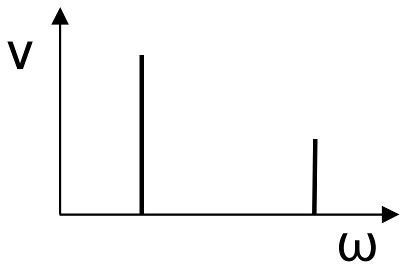
- The ‘FFT’ describes a range of algorithms that increase the speed of the calculating a DFT.
  - For long sequences a DFT is very computationally expensive as the number of computations follows ‘N^2’

- By recognising that in any ‘N’ long sequence, there are many more cycles than needed to find the high frequencies, and many more samples than required to calculate the low frequencies, the computational complexity can be reduced to ‘*NlogN*’
  - For large N this can be many orders of magnitude.

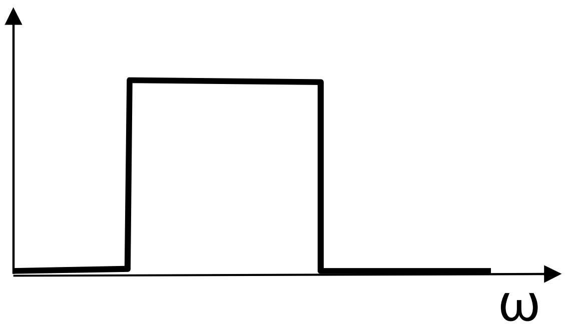
Frequency spectrums

- So far we have considered signals made up from particular frequency components
  - Even our continuous time examples map to individual frequency components



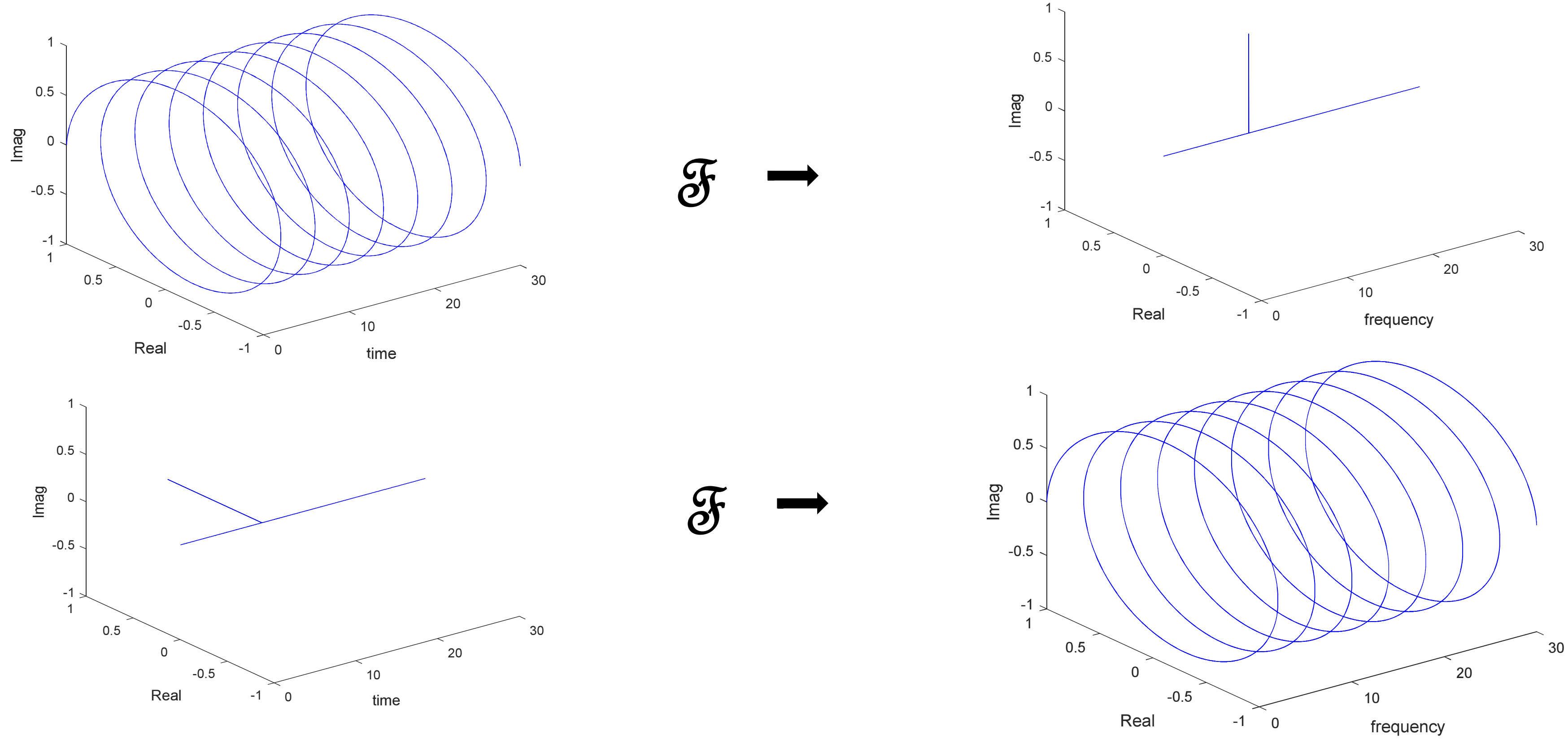


- What is the significance of a continuous spectrum in the frequency domain?



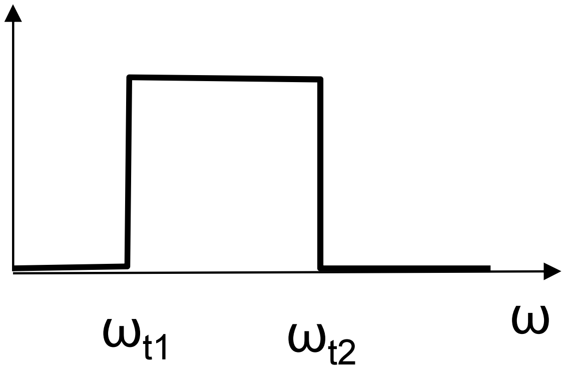
If the y axis units are magnitude, then this doesn't make sense – it implies we have an infinite number of frequency components all with non-zero magnitude

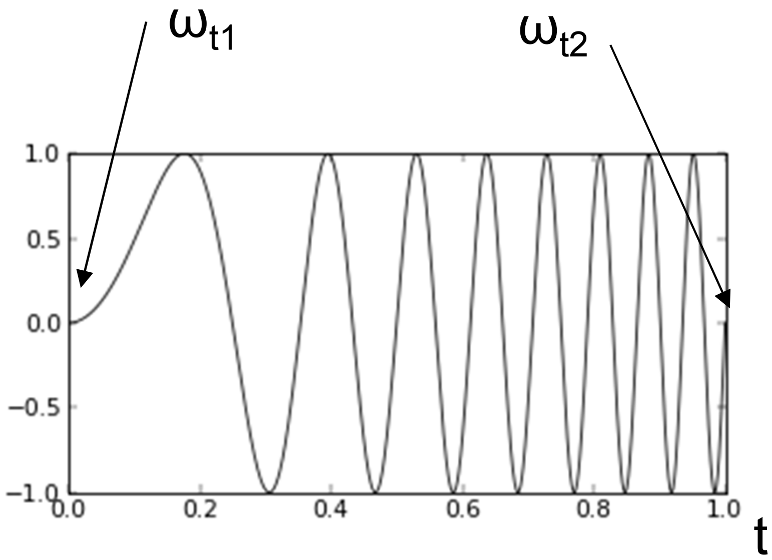
Transform and inverse of Impulses



- One example of a continuous spectrum can be seen by considering the Fourier transform of a impulse function in the time domain. Note the amplitude of impulses and time/frequency domains are not drawn infinite!
- Because the signals are infinite, this example only partially answers our questions, but it does show the link between d/dt and higher frequencies (and vice-versa).

Frequency spectrums

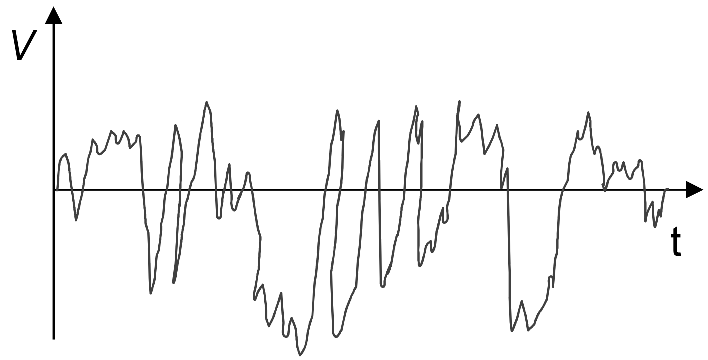




- A second way to interpret our spectrum is through the chirp signal – a sinusoid linearly changing frequency with time.
- Chirps are very useful for testing the frequency response of systems

Frequency spectrums - Noise signals

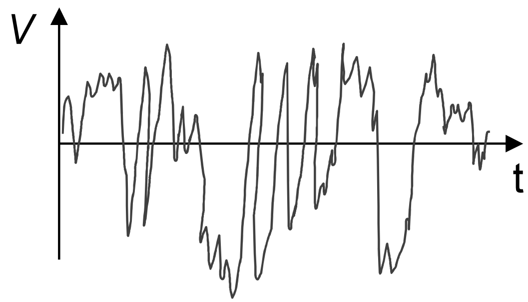
- To look at this another way lets consider ‘noise’, the most commonly encountered random signal.
  - ‘Noise’ describes unwanted frequency components in the ‘background’ of a signal. Many processes that create noise are unknown (if they are known it is common to refer to them as ‘interference’) and they are typically described using stochastic tools: probability theory etc.



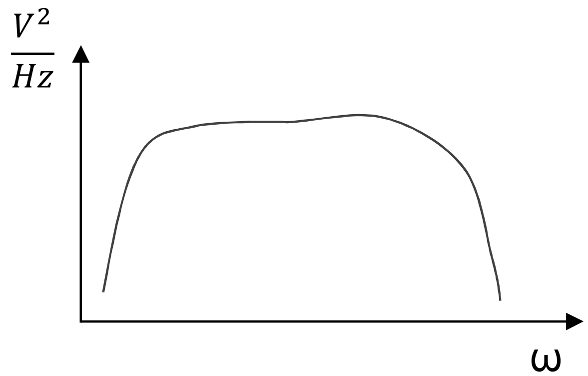
- In the time domain it doesn’t make sense to describe the instantaneous value of such a signal at a point in time – we use variance to tell us the range of values the signal will take over a period of time (noting that variance is the value squared, so a little like our power signal)

Power spectral density

- By describing the variance we have captured the average 'power' of a signal.
- Whether a signal is expressed in the time or frequency domain, we expect the energy (and power) to be the same. Hence if we take the Fourier transform of the time domain signal squared (known as the Power Spectral Density) and calculate the energy in the frequency domain the results will match.



$\mathfrak{F}(v^2) \longrightarrow$ PSD

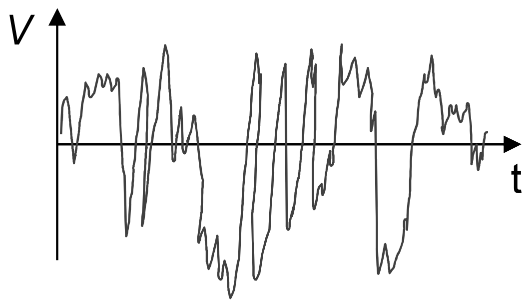


- This is know as Parseval's theorem;

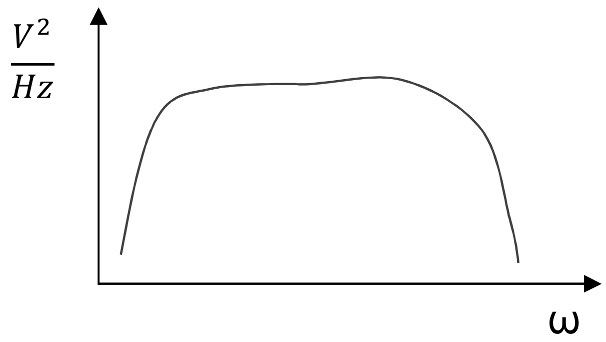
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Power spectral density

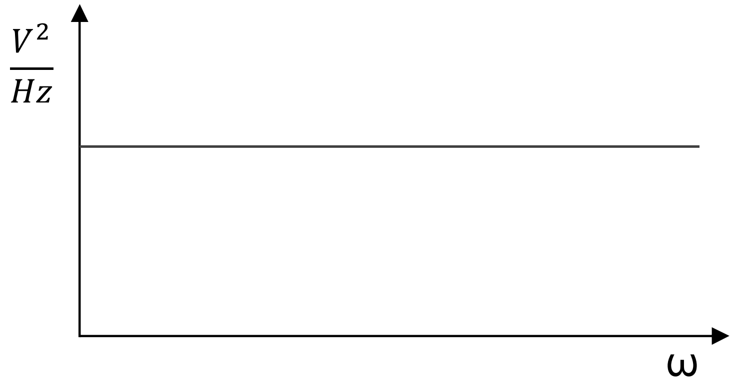
- Although Paseval's theorem provides useful 'constraints' when trying to consider frequency spectrums, it does not in itself imply a continuous spectrum.
- A continuous spectrum only appears from a random signal if the signal continues for all time – although you can begin to approximate this for long signals.
- Instead it tells us where the frequency range over which a signals energy can lie.
- For a short random signal, the frequency spectrum will be random too!

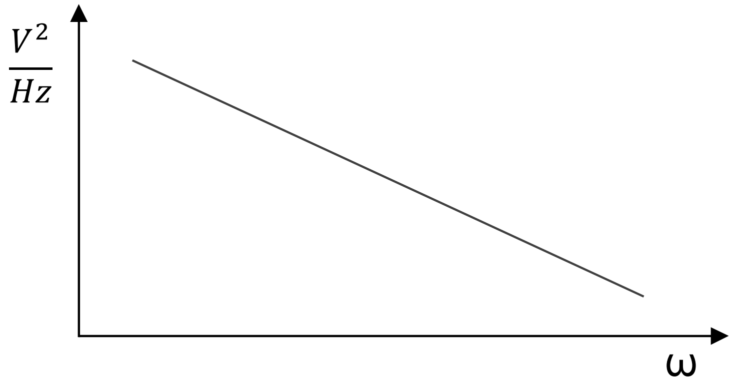


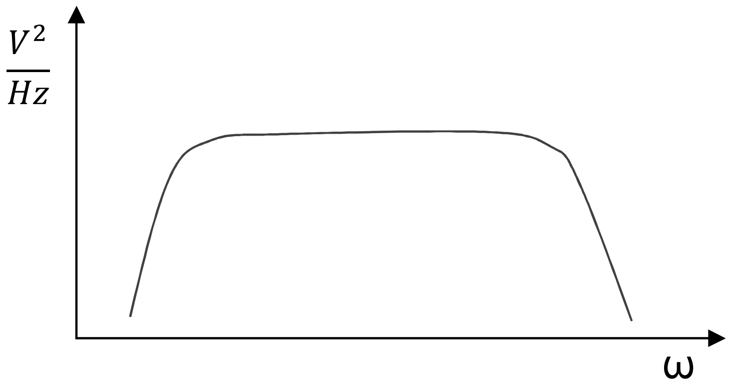
$\mathscr{F}(V^2) \longrightarrow$ PSD



Noise signal spectrums







‘White noise’ is a signal with equal power at all frequencies of interest

‘Pink noise’ has power proportional to 1/ω

‘Band limited noise’ has constant power between an upper and lower frequency