(b) Similarly to (a), we have $F(s+6)$ where $F(s) = \frac{1}{s^2+1}$. F has inverse Laplace transform $\sin(t)$ so the answer is

$$L^{-1}\left[\frac{1}{(s+6)^2+1}\right] = \sin(t)e^{-6t}$$

(c) We know that the inverse Laplace transform of $\frac{1}{s^2+1}$ is $\sin(t)$. For this one we need to use the second shifting theorem, and we get

$$L^{-1}\left[\frac{e^{-3s}}{s^2+1}\right] = H(t-3)\sin(t-3)$$

Inverting when there are repeated roots

The Shift Theorem can be used in the solution of differential equations in which the quadratic denominator $s^2 + ps + q$ has repeated or complex roots:

Example 1:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$$

with initial conditions $y(0) = 2$ and:

$$\frac{dy}{dt}(0) = 10$$

time-domain system for $y(t)$

in s-space this gives:

$$(s^2 + 6s + 9)Y(s) = 2s + 22$$

1. take the Laplace transform of the system.

$$\mathcal{L}\left[\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y\right] = \mathcal{L}[0]$$

Common mistake! don't forget to L.T. the rhs

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + 6\left(sY(s) - y(0)\right) + 9Y(s) = 0$$

$y(0) = 2$, $y'(0) = 10$

$$s^2Y(s) - 2s - 10 + 6sY(s) - 12 + 9Y(s) = 0$$

$$\boxed{(s^2 + 6s + 9)Y(s) - 2s - 22 = 0}$$

s-domain system for Y(s) (no t terms here!)

2. Solve for Y(s)

$$Y(s) = \frac{2s + 22}{s^2 + 6s + 9}$$

factorize bottom line (two real roots, repeated s=3 twice)

3. turn into known Laplace transforms (to be able to take inverse L.T.)

$$= \frac{2s+22}{(s+3)^2}$$

$$= \frac{2(s+3) + 16}{(s+3)^2}$$

or use partial fractions.

$$= \frac{2}{s+3} + \frac{16}{(s+3)^2}$$

$$\frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$$Y(s) = 2\mathcal{L}[e^{-3t}] + 16[te^{-3t}]$$

use 1st shifting thm.

$\mathcal{L}[1] = 1/s$

$\mathcal{L}[t] = \frac{1}{s^2}$

4. take inverse L.T. to ~~find~~ y(t)

$$\boxed{y(t) = 2e^{-3t} + 16te^{-3t}}$$

Repeated roots continued

The quadratic $s^2 + 6s + 9 = 0$ has two equal roots: $s_1 = s_2 = -3$ hence the partial fraction expansion is:

$$Y(s) = \frac{2s + 22}{(s+3)^2} = \frac{2}{s+3} + \frac{16}{(s+3)^2}$$

Using the Shift theorem:

$$L^{-1}\left[\frac{2}{s+3}\right] = e^{-3t}L^{-1}\left[\frac{2}{s}\right] = 2e^{-3t}$$

and:

$$L^{-1}\left[\frac{16}{(s+3)^2}\right] = e^{-3t}L^{-1}\left[\frac{16}{s^2}\right] = 16te^{-3t}$$

so:

Example 2: (using 1st shifting theorem)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0, \quad y(0) = 4, \quad \frac{dy}{dt}(0) = 4$$

t-domain system for y(t)

In s-space this gives:

$$(s^2 + 4s + 13)Y(s) = 4s + 20$$

The quadratic $s^2 + 4s + 13 = 0$ has complex roots: $s = -2 \pm 3j$ thus we complete the square in this case:

$$s^2 + 4s + 13 = (s+2)^2 + 3^2$$

1. Take the Laplace transform

$$L\left[\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y\right] = L[0]$$

$$\underbrace{s^2Y(s) - sy(0) - y'(0)} + 4\underbrace{\left(sY(s) - y(0)\right)} + 13Y(s) = 0$$

$$y(0) = 4, \quad y'(0) = 4, \quad y(0) = 4$$

$$s^2Y(s) - 4s - 4 + 4sY(s) - 16 + 13Y(s) = 0$$

$$\boxed{\underbrace{(s^2 + 4s + 13)}Y(s) - 4s - 20 = 0}$$

S-domain system for Y(s)

check! do the coefficients & powers of s match the original ODE?

2. solve for Y(s)

$$Y(s) = \frac{4s + 20}{s^2 + 4s + 13}$$

factorize the bottom line?

can't! the polynomial has complex roots

Complete the square

3. reduce to known transforms

$$= \frac{4s + 20}{(s+2)^2 + 9}$$

$$= \frac{4s + 20}{(s+2)^2 + 3^2}$$

use 1st shifting thm.

$$= \frac{4(s+2) + 12}{(s+2)^2 + 3^2}$$

$$Y(s) = 4 \cdot \frac{s+2}{(s+2)^2 + 3^2} + \cancel{12}\,4 \cdot \frac{3}{(s+2)^2 + 3^2}$$

4. take the inverse L.T. to find $y(t)$

$$y(t) = 4 . e^{-2t} \cos 3t + 4 . e^{-2t} \sin 3t$$

General toolkit:

1. take the L.T. of the ode system for $y(t)$ to find the S-domain system for $Y(s)$
2. solve for $Y(s)$
3. polynomial on bottom line
   - → real (non repeated) roots → partial fractions
   - → " (repeated) roots → "
   - → complex roots → complete the square.

   & use the 1st shifting theorem.
4. take the inverse L.T. to find $y(t)$.

We can therefore write:

$$Y(s) = \frac{4s+20}{s^2+4s+13} = \frac{4(s+2)+12}{(s+2)^2+3^2}$$

Using the Shift theorem this gives:

$$L^{-1}\left[\frac{(s+2)}{(s+2)^2+3^2}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] = e^{-2t}\cos 3t$$

and

$$L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right] = e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right] = e^{-2t}\sin 3t$$

so:

$$y(t) = e^{-2t}(4\cos 3t + 4\sin 3t)$$

Piecewise continuous functions

Example:

Suppose we have the differential equation

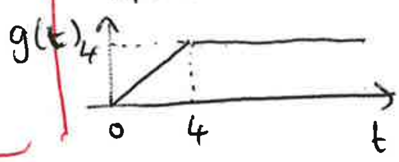
$$y' + 3y = g(t); \quad y(0) = 1$$

with $g(t)$ given by

$$g(t) = \begin{cases} t & 0 < t < 4 \\ 4 & 4 \leq t < \infty \end{cases}.$$

time-domain system for y(t)

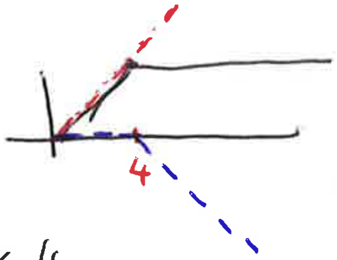
input



Take the Laplace transform of both sides and solve for $Y(s)$. Note: the problem can then be solved by taking inverse transforms!

~~g(t)~~) To solve with Laplace Transforms we need to know $G(s)$

To do so, we note that:



$$g(t) = t - (t-4)\,H(t-4)$$

$$= \begin{cases} t - 0 = t & t < 4 \\ t - (t-4)\cdot 1 = 4 & t > 4 \end{cases}$$

$$G(s) = L[t] - L[(t-4)\,H(t-4)]$$

$$= \frac{1}{s^2} - \frac{e^{-4s}}{s^2}$$

2nd shifting theorem.

1. take the L.T. of the ode system

$$L[y' + 3y] = L[g] = G(s)$$

$$sY(s) - \underbrace{y(0)}_{=1} + 3Y(s) = \frac{1}{s^2} - \frac{e^{-4s}}{s^2}$$

$$sY(s) - 1 + 3Y(s) = \frac{1}{s^2} - \frac{e^{-4s}}{s^2}$$

$$\boxed{(s+3)Y(s) - 1 = \frac{1}{s^2} - \frac{e^{-4s}}{s^2}}$$

s-domain system for $Y(s)$

2. Solve for $Y(s)$

easy | quite easy (partial fractions) | (uses 2nd shifting theorem)

$$Y(s) = \boxed{\frac{1}{s+3}} + \boxed{\frac{1}{s^2(s+3)}} - \boxed{\frac{e^{-4s}}{s^2(s+3)}}$$

not that hard either (time-shifted version of 2nd term)

3. reduce to known transforms.

$$\frac{1}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} = -\frac{1}{3}\cdot\frac{1}{s} + \frac{1}{3}\cdot\frac{1}{s^2} + \frac{1}{9}\,\frac{1}{s+3}$$

$$Y(s) = L\left[e^{-3t}\right] + L\left[-\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot t + \frac{1}{9}\cdot e^{-3t}\right]$$

$$- L\left[\left(-\frac{1}{3}\cdot 1 + \frac{1}{3}(t-4) + \frac{1}{9}e^{-3(t-4)}\right)H(t-4)\right]$$

4. take the inverse L.T. to find $y(t)$

$$\boxed{\begin{aligned} y(t) &= e^{-3t} - \frac{1}{3} + \frac{1}{3}t + \frac{1}{9}e^{-3t} \\ &\quad - \left(-\frac{1}{3} + \frac{1}{3}(t-4) + \frac{1}{9}e^{-3(t-4)}\right)H(t-4) \end{aligned}}$$

Transfer functions

We can use Laplace transforms to analyse input-output systems if they involve derivatives.

u(t) y(t)

Consider a linear input-output system whose output $\cancel{u}(t)$ can be expressed in terms of the input $\cancel{y(t)}$ via the differential equation

input output

u(t) $\longrightarrow \boxed{\dfrac{dy}{dt}+2y=u} \longrightarrow$ y(t)

t-domain system

$\mathcal{L}$ Laplace transforms: $\mathcal{L}^{-1}$

$U(s) \longrightarrow \boxed{-y_0+sY+2Y=U} \longrightarrow Y(s)$

s-domain system

If we take $y_0=0$ then we have

solve in (s-domain)

y(0)

output $\boxed{Y(s)=\dfrac{1}{s+2}U(s)}$ input

the effect of the system

TRANSFER FUNCTION

G(s)

The transfer function: definition

- The differential relation between $y(t)$ and $u(t)$ is replaced by an algebraic relation between $Y(s)$ and $U(s)$.
- The factor in the example $1/(s+2)$ is a property of the system itself (called the **plant** in control engineering, and not of either $u(t)$ or $y(t)$.
- It is called the transfer function, $G(s)$, of the system

Definition: For an autonomous (time invariant) linear system the *transfer function* $G(s)$ is the ratio $Y(s)/U(s)$ of the Laplace transform of the output to the Laplace transform of the input: $Y(s)=G(s)U(s)$.

In general the transfer function of a system is of the form:

$$G(s) = \frac{Q(s)}{P(s)}$$

roots of P(s) tell you about the stability of the system.

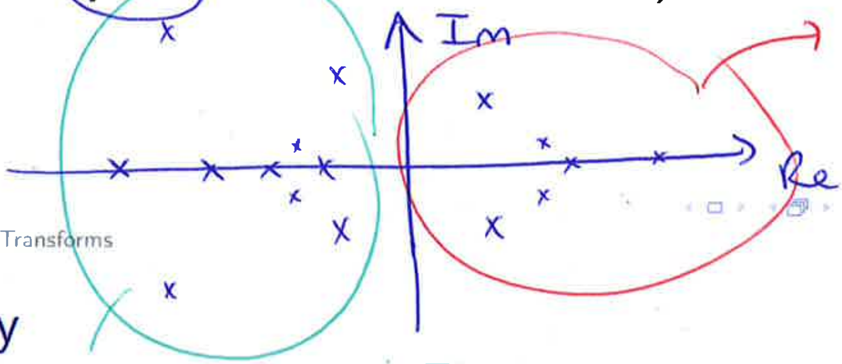
where $P(s)$ and $Q(s)$ are polynomial functions of s.

The degree of the polynomial $P(s)$ is called the *order* of the system.

for real systems, roots of P are real nos, or complex conjugate pairs

A system is said to be **asymptotically stable** if the zeros of $P(s)$ (called the **poles of the transfer function**) are in the **left-half plane**.

roots of P.



roots with +ve real part ⇒ instability

decaying perturbations

Summary

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt$$

- No simple formula for inverse transform
- Use tables to evaluate transforms and inverse transforms.
- Use derivative function to reduce

$$L\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

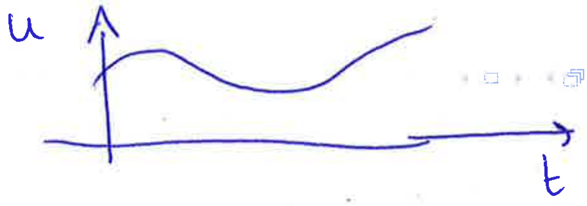
- Hence solve ODEs.
- Transfer function $G(s)$ links output $y(t)$ to input $u(t)$ via $Y(s) = G(s)U(s)$

4. Introduction to partial differential equations (PDEs)

In first year we learnt about **ordinary differential equations (ODEs)**, and how to solve them; e.g. the unknown or unknowns are functions of one coordinate

- Linear first-order equations: $\frac{du}{dt} + f(t)u = g(t)$ subject to an initial condition $u(0) = u_0$
- Separable equations $\frac{du}{dt} = \frac{f(t)}{g(u)}$ subject to an initial condition $u(0) = u_0$
- Linear constant coefficient 2nd-order equations $a\frac{d^2u}{dt^2} + b\frac{du}{dt} + cu = f(t)$ subject to a pair of initial conditions, e.g. $u(0) = u_0$, $\frac{du}{dt}(0) = v_0$, or separated boundary conditions $u(0) = u_0$, $u(L) = u_L$.

The solution $u(t)$ can be thought of as a graph, in $\mathbb{R}^2$, of u against t. In some cases this graph is expressible as a closed form function.



Curve
$u(t)$

What are PDEs?

Partial differential equations (PDEs) have scalar or vector functions that depend on two or more *independent variables*: for example space and time x and t, two spatial co-ordinates x and y.

The simplest kind of PDE is an equation for a single scalar *dependent variable* $u(x,t)$ or $u(x,y)$.

Systems contain partial derivatives

As with ODE's, the *solution* to a PDE is the unknown *function*, ie $u(x,t)$ or $u(x,y)$.

$\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ ...

Again we can think of the solution as a graph, now in $\mathbb{R}^n$ with $n > 2$. E.g., a surface u against x and t.

There are **three great equations** that crop up again and again in physical science and engineering (see the appendix on Blackboard for detailed derivations).

surface
$u(x,t)$

Three great equations

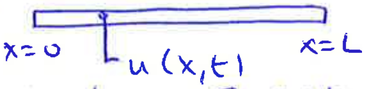
conductivity coefficient

PARABOLIC

in 3D

$\frac{\partial u}{\partial t} = \alpha^2 \nabla^2 u$

- **The heat (or diffusion) equation**: the diffusion of a scalar quantity (heat) through a one-dimensional medium (like a bar)



u = temperature at position x, time t

$$\boxed{\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}}$$

plus
chemical diffusion
disease spread
finance
image processing

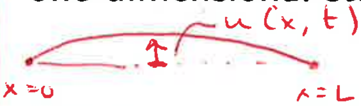
where α^2 is a heat condition (or diffusion) constant

- **The wave equation**: the vibration amplitude of $u(x,t)$ of a one-dimensional string with wave speed c

in 3D

$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

HYPERBOLIC



u displacement at position x, time t

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

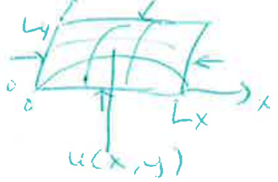
plus
water waves
acoustic / pressure waves
traffic waves
$c = \sqrt{\frac{T}{\rho}}$ — tension, — mass density in string

- **The Laplace equation**: equilibrium configurations $u(x,y)$ of deformed surfaces (e.g. a drum skin)

in 3D

$\nabla^2 u = 0$

ELLIPTIC



$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

plus
electrostatics
gravitation
potential flow

u = steady state displacement of a metal plate at pos'n (x,y)

Other Engineering examples of PDEs: beam equation

Euler-Bernoulli beam equation: derived from force balance, describes the vertical vibration $u(x,t)$ of a one-dimensional beam (or inner ear!)

$$m\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} + EI\frac{\partial^4 u}{\partial x^4} = q(x,t)$$

... or in more dimensions: the displacement $u(\boldsymbol{r},t)$ of a plate or body with bending stiffness

$$m\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} + EI\nabla^2(\nabla^2 u) = q(\boldsymbol{r},t).$$

Other examples: cable equation, reaction diffusion

Cable equation governing the voltage (or current) $u(x,t)$ in a transmission line

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + (RC+LG)\frac{\partial u}{\partial t} + RGu.$$

Reaction diffusion equation: governs the chemical concentration $u(\boldsymbol{r},t)$ of a reactant that both diffuses and reacts with other chemicals

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} + f(u)$$

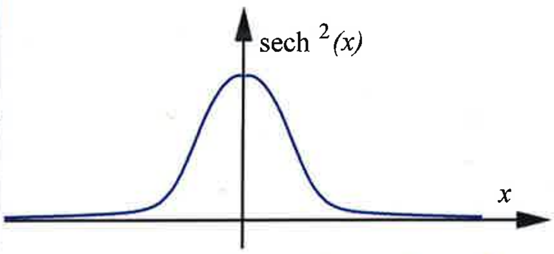
- $f(u)$ nonlinear; summarize interactions other chemicals.
- Systems of reaction diffusion equations can create complex patterns, like complex forms in biology: markings on animal coats or skins, time-dependent spiral waves of chemical concentration.

Other examples: Korteweg-de Vries equation

Korteweg-de Vries (KdV) equation: the governing amplitude $u(x,t)$ of *dispersive* waves on the surface of water.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{3}{2}u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

This is one of the simplest and most important *nonlinear* PDEs. In addition to describing the way that trains of small ripples eventually break up (disperse) it famously has 'soliton' solutions, first observed by John Scott Russell in 1834.



Other examples: Navier Stokes, Schrödinger

Navier Stokes equation: the fundamental force balance equation for for the (vector) velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ of viscous fluid flow

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla p = \nu \nabla^2 \boldsymbol{u}$$

It's remarkably accurate at describing *Newtonian* fluid flow, both laminar and turbulent.

Schrödinger equation: the fundamental equation of quantum mechanics, for the wavefunction $\psi(x, t)$

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(x, t) + V(x, t)\psi(x, t)$$

Because of the rich variety of PDEs and the nature of their solutions, in these lectures, we will give only the briefest overview of the theory of PDEs:

- what they are used for in Engineering,
- how they can be classified,
- how they can be solved.
- We shall focus exclusively on single PDEs for a scalar dependent variable u that depends on two (or occasionally three) independent variables.

PDEs are not like ODEs

Given suitable boundary or initial conditions, the solutions to these PDE's are functions, u - typically illustrated as 3D graphs of the dependent variable u against the independent variables – (x, t) for the heat and wave equations, (x, y) for the Laplace equation.

Q. *So, are PDEs just like ODEs grown up? We just need to learn some more advanced, but nevertheless similar techniques for their solution?*

A. *Not really! Because ...*

- The behaviour of a 'solution' to a PDE depends strongly on the type of PDE; as can the method of finding a 'solution'.

↳ both analytic & computational.

PDEs are not like ODEs (cont)

affects both behaviour & method.

- The 'solution' to a PDE depends crucially on the domain (range of the independent and dependent variables) in which we solve it. For example, we shall see that the solution of the wave equation on a finite domain e.g. $0 < x < L$, $0 < t < \infty$, is very different from the solution on an infinite domain, e.g. $0 < x < \infty$, $0 < t < \infty$.
- The 'solution' also depends greatly on the boundary and initial conditions. For example the initial condition of the wave equation can be a function of x, e.g. $u(x, 0) = f(x)$, and the boundary conditions can be a function of t, e.g. $u(0, t) = h(t)$,
- The number of boundary or initial conditions we require in order to specify the solution uniquely depends crucially on the classification of the terms with the highest-order derivatives.

initialisation also affects behaviour & method

depends on the type of PDE.

Classification of PDEs

tells us about
- how the PDE behaves.
- how we can try to solve it
- how to formulate the problem.

PDEs come in all shapes and sizes

what we're solving for
"the unknown"

- **Definition:** *A* **linear** *PDE is one in which the dependent variables and their derivatives appear only in linear combinations.* For example:

usually, on the top of the derivatives.

$u(x,y)$ is the dependent variable: a function of the independent variables.

on the bottom of derivatives

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y)$$

independent variables. x, y

- **Definition:** *A* **linear homogeneous** *PDE is a linear PDE in which all terms contains a dependent variable or its derivative.* e.g. as above but with $g = 0$. → every term contains u

Classification (cont)

→ highest derivative terms linear in the dependent variable u ⇒

we can deal with them

- **Definition:** *A* **semilinear** *PDE is one in which is linear if you only consider the terms in the equation with the highest derivatives.* For example:

$$\boxed{a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2}} = g\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

where g may be a nonlinear function.

Too hard for us!

- **Definition:** **Nonlinear** *equations are not linear!* e.g. → or semilinear!

$$\underbrace{u\frac{\partial^2 u}{\partial x^2}}_{} + \frac{\partial^2 u}{\partial y^2} + u^2 = 0$$

↳ nonlinear highest derivative term:

⇒ NONLINEAR PDE

<u>Linear PDE</u> has terms that look like $au + b$

where "a" could be - constants
- x, y's (independent variable)
- derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, etc..

- Terms that are <u>linear</u> in u are

$$5u, \quad 5u + x, \quad 5(x^2+y^2)u, \quad x\frac{\partial u}{\partial x}$$

(all ok in a linear PDE)

- Terms that are <u>not</u> linear in u are

$$u^2, \quad u\frac{\partial u}{\partial x}, \quad \frac{1}{u}, \quad \frac{x^2+y^2}{u}, \quad e^u, \quad \sin u$$

(all <u>not</u> allowed in linear PDEs)

eg.

$$\underbrace{\frac{\partial^3 u}{\partial x^3} + x\frac{\partial^3 u}{\partial x^2 \partial y}} = \underbrace{\left(\frac{\partial u}{\partial y}\right)^2} + x^2$$

↳ not linear in u

⇒ PDE is not linear

highest derivative terms: linear in u ⇒ PDE is semilinear

$$\frac{\partial^3 u}{\partial x^3} + \underbrace{u\frac{\partial^3 u}{\partial x^2 \partial y}} = \left(\frac{\partial u}{\partial y}\right)^2 + x^2$$

highest derivative term not linear in u ⇒ PDE is not semilinear (or linear)

Order of a PDE

Definition: *The* **order** *of PDE is the number of derivatives of the highest derivative term*

e.g. the KdV equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \underbrace{\frac{3}{2}u\frac{\partial u}{\partial x}}_{\text{not linear in } u} + \underbrace{\frac{\partial^3 u}{\partial x^3}}_{\text{highest derivative is 3rd order}}$$

is a third-order semilinear PDE.

highest derivative is 3rd order
$\Rightarrow$ 3rd order PDE

linear in u } semilinear PDE.

not linear in u

Worked Example 4.1

Classify the following PDEs by deciding if each is (I) first-order, second-order, or higher-order; (II) linear homogeneous, linear inhomogeneous, semi-linear or nonlinear. In each case identify the independent variables *and the* dependent variable

→ highest deriv. term that's not linear in u.

(a) $$\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y} + 3u = 0$$

(b) $$r\theta^2\frac{\partial^2 V}{\partial \theta^2} + r^2\sin\theta\frac{\partial^2 V}{\partial \theta^2} + r\frac{\partial^2 V}{\partial \theta \partial r} = 0$$

(c) $$\boxed{\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 x}{\partial s^2}} + \sin x = 0$$

→ highest deriv. terms linear in x

→ not linear in x.

(d) $$y^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} + \sin x = 0$$

Ex 4.1

	dependent	independent	order	type.
(a)	u	x, y	1st	nonlinear
(b)	V	θ, r	2nd.	linear homogeneous.
(c)	x	t, s	2nd.	semilinear
(d)	u	x, y	2nd	linear non-homogeneous

2nd-order semilinear PDEs

Consider a semilinear second-order PDE for $u(x, y)$: $\frac{\partial u}{\partial x}$

functions of x, y

$$\underbrace{A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2}} = \underbrace{f(x, y, u, u_x, u_y)}$$

→ any combination of x, y, u, u_x, u_y

They can be classified into three different types:

- **parabolic** equations for which $B^2 - 4AC = 0$ → the heat eqn.
- **hyperbolic** equations for which $B^2 - 4AC > 0$ → the wave eqn.
- **elliptic** equations for which $B^2 - 4AC < 0$ → Laplace's eqn.

The importance of this classification comes in the nature of the solutions that the different classes of PDE give rise to, and what kinds of boundary conditions are required.

the discriminant

Worked Example 4.2

Classify the following 2nd order semilinear *equations as being parabolic, hyperbolic or elliptic*

(a)	$u_t = \alpha^2 u_{xx}$	*(the heat equation)*
(b)	$u_{tt} = c^2 u_{xx}$	*(the wave equation)* $c \neq 0$
(c)	$u_{xx} + u_{yy} = 0$	*(the Laplace equation)*
(d)	$u_{xx} - 3u_{xy} + u_{yy} = 0$	

Ex 4.2

$$Au_{xx} + Bu_{xy} + Cu_{yy} = \text{Stuff}.$$

(a) $A = \alpha^2, \quad B = 0, \quad C = 0$

$B^2 - 4AC = 0^2 - 4.\alpha^2.0 = 0 \Rightarrow$ PARABOLIC.

(b) $c^2 u_{xx} - u_{tt} = 0$

$A = c^2, \quad B = 0, \quad C = -1$

$B^2 - 4AC = 0 - 4.c^2.-1 = 4c^2 > 0 \Rightarrow$ HYPERBOLIC.

(c) $A = 1, \quad B = 0, \quad C = 1$

$B^2 - 4AC = 0 - 4.1.1 = -4 < 0 \Rightarrow$ ELLIPTIC

(d) $A = 1, \quad B = -3, \quad C = 1$

$B^2 - 4AC = 9 - 4.1.1 = 5 > 0 \Rightarrow$ HYPERBOLIC.

Parabolic equations

"Hmmm"

like the heat equation
all have solutions that "diffuse"
information propagates infinitely fast

These have solutions that *evolve* in a 'time-like' independent variable. Generally speaking a single initial condition is required in the time-like variable, with a pair of boundary conditions in the other variable.

PDE domain

e.g. for the heat equation: $\boxed{u_t = \alpha^2 u_{xx}}$, $\boxed{0 \leq x \leq L}$ we might have

+ two BOUNDARY conds.

$$\boxed{\underset{x=0}{u(0,t)} = c_1, \qquad \underset{x=L}{u(L,t)} = c_2}$$

for all time
known temps

which for the heated bar example means that the ends are held at fixed temperature values. (OR $\boxed{\underset{x=0}{u_x(0,t)} = d_1, \ \underset{x=L}{u_x(L,t)} = d_2}$ if given heat flux at each end.)

know heat fluxes.

We also need a single initial condition

+ one INITIAL cond.

$$\boxed{\underset{t=0}{u(x,0)} = f(x)}$$

NOTE, no condition on $u_t(x,0)$.

for all x in the domain

Hyperbolic equations

"Nasty"

like the wave equation
all have wave-like solutions
eg. standing, travelling : depends on domain
information flows at fixed, finite, speed.

These have solutions that *travel* without decay along characteristic directions in the domain. Care has to be taken to specify boundary conditions that do not contradict each-other. In such cases, the solutions can develop discontinuities (*shock waves* or steep *fronts*).

e.g. for the wave equation PDE

domain

$$\boxed{u_{tt} = c^2 u_{xx}} \quad + \quad \boxed{0 \leq x \leq L}$$

we should normally specify two boundary conditions and two initial conditions, e.g.

known slope $\partial u/\partial x$

+ two BOUNDARY conditions

$$\boxed{\underset{x=0}{u(0,t)} = c_1, \qquad \underset{x=L}{u(L,t)} = c_2,}$$

for all time
known displacement u

and

+ two INITIAL conditions

$$\boxed{\underset{t=0}{u(x,0)} = f(x), \qquad \underset{t=0}{u_t(x,0)} = g(x).}$$

~~known slope~~

for all x (in the domain)

initial displacement AND initial velocity

Elliptic equations

all like Laplace's equation
there is no time coordinate
(hence no information flow)
solutions are at rest
"Nice"

These have solutions that *rest* at equilibrium. There is no distinguished direction that is 'time-like' and a single condition should be specified at every boundary point of the domain. The solutions of elliptic equations are always smooth.

e.g. for the Laplace equation

PDE
DOMAIN (a region in space)

$$\boxed{u_{xx} + u_{yy} = 0,} \quad \boxed{0 \le x \le L_x, \qquad 0 \le y \le L_y}$$

we might have constant values for u, e.g.

+ ONE condition on each BOUNDARY

$x=0$ (left) $x = L_x$ (right)

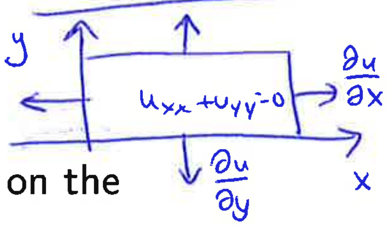
$$\boxed{\underbrace{u(x,0) = a}_{y=0,\ \text{bottom}}, \quad \underbrace{u(x,L_y) = b}_{y=L_y,\ \text{top}}, \quad u(0,y) = c, \quad u(L_x,y) = d}$$

(known as Dirichlet boundary conditions).

known u

Elliptic equations (2)

OR known gradient normal to the boundary. of u



OR any one of these could be replaced by a condition on the *derivative* of u normal to the boundary, e.g.

$$\boxed{u_y(x,0) = a}$$

(known as a Neumann boundary condition).

Definition: *Boundary conditions are called* **homogeneous** *if they are of the form that a function of the dependent variable is zero (e.g. $u(0,t) = 0$). Otherwise they are called* **inhomogeneous**

How to solve PDEs?

Several methods are available for finding the solution of PDEs.

- Analytical methods || we'll do this
  - separation of variables
  - method of characteristics (e.g. d'Alembert's method)
  - solution by Laplace or Fourier transform
  - *ad hoc* 'similarity solution' methods
- Numerical methods || we won't do this !
  - finite differences
  - finite elements

Summary

!!

- PDEs are much more complicated than ODEs. They come in a variety of forms: **linear homogeneous**, **linear inhomogeneous**, **semilinear** and **nonlinear**.
- Their solution depends significantly on the domain of the independent variables, and the boundary conditions on the dependent variables. Boundary conditions can be **homogeneous** or **inhomogeneous**.
- Only the very simplest PDEs can be solved with analytical methods
- Here we consider linear, constant coefficient 2nd-order PDEs, which again come in three types. **parabolic**, **hyperbolic** and **elliptic**
- We have one important example of each type, the **heat equation**, **wave equation** and **Laplace equation**.

5. The separation of variables method

A 'try it and see' technique to solve PDEs

- Separating the variables: PDE $\rightarrow$ ODEs
- Satisfying the homogeneous boundary conditions
- Solution process for wave, heat, and Laplace equations
- How to satisfy the inhomogenous boundary condition? Use Fourier series!
- Linear superposition principle.
- Inhomogeneous equations or multiple inhomogeneous boundary conditions.

[James Advanced MEM (4th Edn) §9.3.2, 9.4.1, 9.5.1]

Outline of the method

1. **Separate the variables**
   Assume, for example, that $u(x,t) = X(x)T(t)$. Substitute this into the PDE to get 2 *separate* ODEs for X and T.
2. **Decide on the sign of the separation constant**
   The constant arises when you separate the variables. More on this later.
3. **Solve the separated ODEs**
   You get, for example, ODEs to solve for $X(x)$ and $T(t)$ that depend on the constant in Step 2.
4. **Solve the (homogeneous) boundary conditions**, so that you know what $X(t)$ and $T(t)$ are, and reconstruct the funtion, for example $u(x,t)$ that you need, using $u(x,t) = X(x)T(t)$.
5. **Check** that your $u(x,t)$ actually solves the problem.

5.1. The wave equation

$u(x,t)$ displacement of a stretched string.

Consider the wave equation on a finite domain DOMAIN.

PDE

$$\boxed{u_{tt} = c^2 u_{xx},} \qquad \boxed{0 \leqslant x \leqslant L, \quad t \geqslant 0,} \qquad (1)$$

subject to homogeneous boundary conditions and a simple initial condition. 2 boundary conds. 2 initial conds.

+ LH end, RH end, displacement, velocity

$$\underbrace{u(0,t) = 0}_{x=0}, \quad \underbrace{u(L,t) = 0}_{x=L}, \quad \underbrace{u(x,0) = f(x)}_{t=0}, \quad \underbrace{u_t(x,0) = 0}_{t=0} \qquad (2)$$

for all $t \geqslant 0$

for some given (non-zero) function $f(x)$. $f \neq 0$ for all $0 \leqslant x \leqslant L$

HOMOGENEOUS.

[Note that this means that the solution $u(x,t) \neq 0$, since it's non-zero at $t = 0$]

Separating the variables

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}[X(x)T(t)] = X(x)\frac{dT}{dt}(t)$$

T'

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The basic idea is to *try* to find a solution that is a function of x times a function of t. That is, we write

a guess!

$$\boxed{u(x,t) = X(x)T(t),}$$

Substituting this form into the PDE we get

$$\boxed{\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)} = \boxed{c^2X''(x)T(t) = c^2\frac{\partial^2 u}{\partial x^2}}$$

which is equivalent to

$\div c^2 X(x)T(t)$

$$\underbrace{\frac{1}{c^2}\frac{T''(t)}{T(t)}}_{\text{function of time } t \text{ only}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{function of space } x \text{ only.}} = \underbrace{\text{constant}}_{\text{because } x,t \text{ are independent coordinates.}} = \mu \qquad (3)$$

$\Rightarrow$ we have two <u>separate</u> equations for T, X

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \mu \quad \& \quad \frac{X''(x)}{X(x)} = \mu$$

$$\Downarrow$$

$$\boxed{T''(t) = \mu c^2 T(t) \quad \& \quad X''(x) = \mu X(x)}$$

The separation constant

Now, the left-hand side of (3) is a function of time t, while the right-hand side is a function of space x. The only way that this can be true for all x and t (which are independent variables) is if both functions are actually equal to a constant. Hence

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \text{const.} \tag{4}$$

This constant is called the **separation constant**. The question remains what sign this constant should have. We proceed by trial and error to see what fits the boundary and initial conditions.

Separating the boundary/initial conditions

We can also separate the boundary and initial conditions, but **only if they are homogeneous** (e.g. function value equal to zero).

For example, we have **for all** $t > 0$ that

$\boxed{X(0)=0}$

L.H. end. $x=0$

or

$$0 = u(0,t) = X(0)T(t) \quad \Rightarrow \quad \cancel{T(t) = 0}. \text{ for all } t.$$

else $u=0$ for all t.

Therefore either $X(0) = 0$ or $T(t) = 0$ for all $t > 0$. The latter implies that $u(x,t) = 0$ for all $t > 0$, which can't be true, so we must have $X(0) = 0$.

We separate all the homogeneous boundary & initial conditions similarly, to get

$x=0$ $\quad x=L$ $\quad t=0$

$$\boxed{X(0) = X(L) = 0} \quad \boxed{T'(0) = 0} \tag{5}$$

Separated boundary conditions $\quad$ initial conditions

We can't say anything about the non-homogeneous condition.

Sign of the separation constant $\Rightarrow$ μ must be negative

always $-$ve for the wave eqn.
heat eqn.

First guess Try a *positive* constant. Hence we write (4) as

$\mu = k^2$
$k > 0$.

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = k^2 > 0$$

separate

Then we get two separate linear ODEs to solve:

$$T''(t) = (kc)^2 T(t) \qquad (6)$$
$$X''(x) = k^2 X(x) \qquad (7)$$

solve

Both these ODEs are easy to solve

$$T(t) = A\,\mathrm{e}^{-kct} + B\,\mathrm{e}^{kct}$$
$$\boxed{X(x) = Ce^{-kx} + De^{kx} =} \; 0.$$

for arbitrary constants A, B, C, D. NB: solution isn't wave like!
+ $k>0$

Applying the homogeneous boundary/initial conditions

to find A, B, C, D, k.

Applying the separated boundary conditions (5) for X we get

$x=0$

$$0 = X(0) = C + D$$

which means that $C = -D$. In addition

$x = L$

$$0 = X(L) = Ce^{-kL} + De^{kL} = D(e^{kL} - e^{-kL}) = 2D\sinh(kL)$$

$D=0$

$\neq 0$ as $k \neq 0$ or $L \neq 0$.

Since $\sinh(kL) \neq 0$ for $kL \neq 0$, we must have $D = 0$, which implies that $C = 0$, and so $X(x) = 0$. This is just the trivial solution, so **a positive separation constant is the wrong choice.**

Same thing for $\mu > 0$ ✗ $\mu = 0$. ✗

must have $\mu < 0$.

Negative separation constant

Hence we should take the original separation constant to be *negative*. That is we write (4) in the form

$\mu = -k^2$

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -k^2 < 0$$

separated.

Thus we get the two separate linear ODEs:

$$T''(t) = -(kc)^2 T(t) \tag{8}$$
$$X''(x) = -k^2 X(x) \tag{9}$$

solve.

which are easy to solve and *do* give wave-like solutions

$$T(t) = A\cos(kct) + B\cancel{\sin}(kct)$$
$$\boxed{X(x) = C\cancel{\cos}(kx) + D\sin(kx)}$$

for constants A, B, C, D, k.

Applying the homogeneous boundary conditions

$x=0$

Once again, we must try to satisfy $X(0) = X(L) = 0$. Now

$x=L$

$x=0$: $$0 = X(0) = C$$

which means that $X(x) = D\sin(kx)$. Furthermore

$x=L$: $$0 = X(L) = \cancel{D}\sin(kL)$$

$k > 0$.
$L > 0$.

We can't have $D = 0$ (else $u(x,t) = 0$), and so it must be that

$$\sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{L} \text{ for some } n \in \mathbb{Z}$$

n integer. Separation const.

$$\boxed{\text{So } X(x) = D\sin\left(\frac{n\pi x}{L}\right)}$$

Applying the homogeneous initial condition

zero initial velocity

$t=0 \quad T'(t) = -kc.A\sin(kct) + kc.B\cos(kct)$

We also have from (5) that $T'(0) = 0$. Thus

$$0 = -kcA\underbrace{\sin(0)}_{0} + Bkc\underbrace{\cos(0)}_{1} = Bkc$$

$t=0$, $k>0$, $c>0$.

which gives us $B = 0$. Hence $\boxed{T(t) = A\cos(kct)}$ for some arbitrary constant A.

But we know that $k = \dfrac{n\pi}{L}$ for some $n \in \mathbb{Z}$, so

$$\boxed{T(t) = A\cos\left(\frac{n\pi ct}{L}\right)}$$

$u(x,t) = X(x)\,T(t)$

$= A.D \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$

b_n for any n integer

Putting it all together

Hence we have the solution

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right),$$

infinitely many solutions of

where $b_n = AD$, and n can be any integer (we still need to decide which value of n to take).

At this stage we should check that we satisfy the PDE and the homogen. boundary + initial conditions. We have got a function that meets

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u_t(x,0) = 0 \qquad (10)$$

but NOT $u(x,0) = f(x)$.

So how do we do it? Well, we've still got b_n and n in our $u(x,t)$. So we have some flexibility left to meet $u(x,0) = f(x)$.

Linearity

Let

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

ie the solution that we found.

KEY POINT: If $u_1(x,t)$ and $u_2(x,t)$ meet the conditions (10), then so does $u_1(x,t) + u_2(x,t)$. Furthermore, because the original PDE (1) is linear, $u_1(x,t) + u_2(x,t)$ is still a solution of the wave equation too. Thus, so is any sum of the u_ns!

So, the general solution to this PDE satisfying the homogenous boundary and initial conditions is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right),}$$

general solution

What b_n's do we need to get u at $t=0$

$u(x,0) = f(x)$

Applying the non-homogeneous initial condition

$t=0$

Now if we try to satisfy the initial condition $u(x,0) = f(x)$ we get

finding the b_n's is a Fourier ½-range series question

$$\boxed{u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)} \qquad (11)$$

given initial displacement.

If we could find a set of b_ns to solve this equation we'd be done. **But we already know how to do this... using Fourier series!**

So, the general solution to the PDE (1) satisfying the boundary and initial conditions (2) is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right),$$

$$\text{where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

particular solution.

this is the solution!

Worked example 5.1

Solve the wave equation on the finite domain

$$u_{tt} = c^2 u_{xx}, \qquad 0 \leqslant x \leqslant L, \quad t \geqslant 0,$$

subject to

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0$$

for the specific case $L = 4$ *and*

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 2 \\ 4 - x, & 2 < x \leqslant 4 \end{cases} \tag{12}$$

Changing the boundary/initial conditions

Subtle changes in the boundary conditions lead to different forms of solution. For example, it is easy to replace the pinned-end boundary conditions $u(0,t) = u(L,t) = 0$ with simply supported ends $u_x(0,t) = u_x(L,t) = 0$. We can also specify an initial velocity $u_t(x,0) = g(x)$ rather than (or as well as) an initial profile $u(x,0) = f(x)$ at every point along the string, without any significant extra complication.

Non-homogenous boundary conditions (e.g. $u(0,t) = 0$ and $u(L,t) = 1$) require a little more work... see example sheet.

Ex 5.1 //

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{n\pi ct}{4}\right) \qquad L=4$$

solves the wave equation

+ $u=0$ at $x=0,4$

+ $\frac{\partial u}{\partial t}=0$ at $t=0$

to apply the initial condition $t=0$.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) \cdot 1 = f(x)$$

Fourier ½-range sin series.

$$b_n = \frac{2}{4}\int_0^4 f(x) \sin\left(\frac{n\pi x}{4}\right) dx = \frac{16}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

We've done this before! Worked Ex 1.5 in Fourier series.

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \underbrace{\frac{16}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)}_{b_n} \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{n\pi ct}{4}\right)}$$

particular solution.

What does this tell us?

$$u = \sum$$

$n=1 \quad b_1 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi c t}{L}\right)$

$+$

$n=2 \quad b_2 \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi c t}{L}\right)$

$+$

$n=3 \quad b_3 \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{3\pi c t}{L}\right)$

$+$

$\vdots$