Advanced Bending and Torsion
Composite Beams

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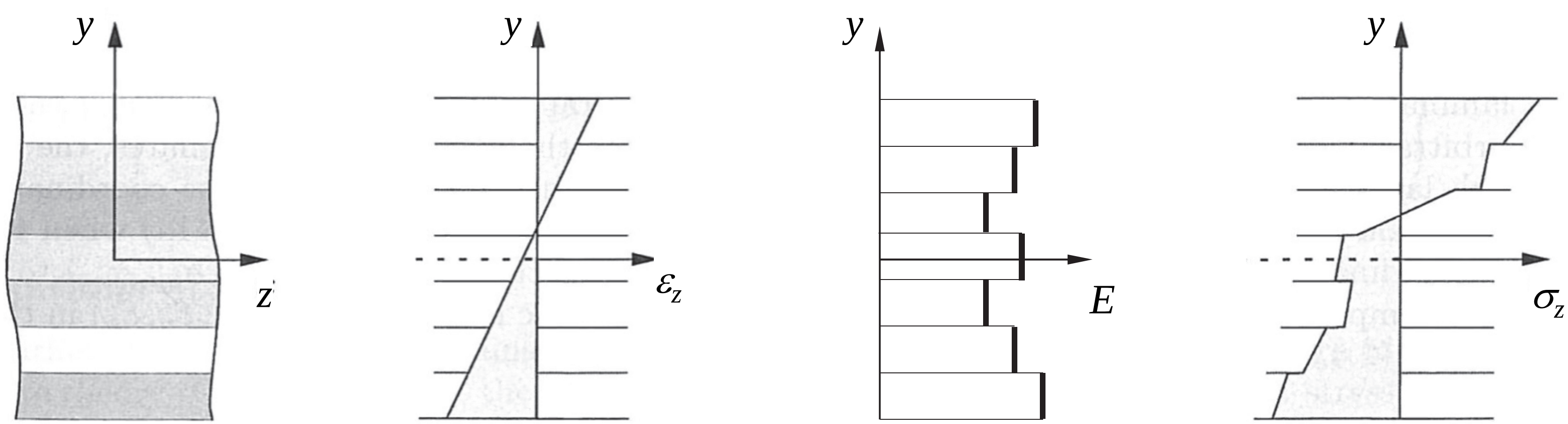
25 October 2018

Composite Beams

- Beams made of two or more different materials (with different Young's moduli)

- Assumptions:
  - 'Plane sections remain plane', *i.e.* linear strain distribution through the thickness (Euler-Bernoulli beams)
  - However, different moduli will cause stress discontinuities or 'jumps'



composite beam **strains** **moduli** **stresses**

Composite Beams – 'Scaled Area' Method

- Method:

1. Choose one Young's modulus to be used as reference: E_{ref}
2. Find the area scaling factors n_i for each material section i: $n_i = \dfrac{E_i}{E_{\text{ref}}}$
3. Conduct the analysis as before, utilising 'effective' (scaled) properties:

$$A_i^{\text{eff}} = A_i\, n_i \qquad I_i^{\text{eff}} = I_i\, n_i$$

4. Finally, calculate global deflections (and hence strains) based on E_{ref} alone

- Note:
  - Only the 'local width' variable (usually denoted b_i) is scaled - limits of integration and centroids **remain unchanged**

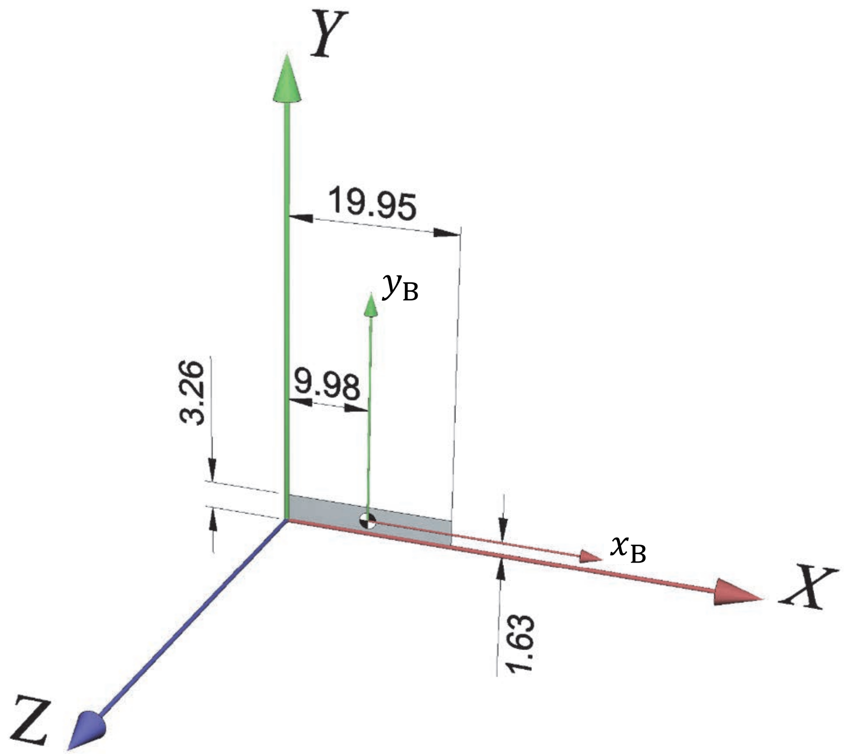
$$A_i^{\text{eff}} = \underbrace{(n_i\, b_i)}_{\text{scaled width}} \int_{-\frac{h_i}{2}}^{+\frac{h_i}{2}} \mathrm{d}y \qquad I_i^{\text{eff}} = \underbrace{(n_i\, b_i)}_{\text{scaled width}} \int_{-\frac{h_i}{2}}^{+\frac{h_i}{2}} y^2\, \mathrm{d}y$$

original height

original height

- We now make the flange (section B) of steel with $E = 210$ GPa

Flange (section B)



$$E_{\text{ref}} = 70 \text{ GPa}$$

$$n_{\text{B}} = \frac{210 \text{ GPa}}{70 \text{ GPa}} = 3$$

$$A_{\text{B}} = n_{\text{B}}(19.95)(3.26) = 195.11 \text{ mm}^2$$

$$\bar{X}_{\text{B}} = 9.98 \text{ mm}$$

$$\bar{Y}_{\text{B}} = 1.63 \text{ mm}$$

Composite (Aluminium-Steel) L-Section Beam

- We can now find the centroid of the compound section:

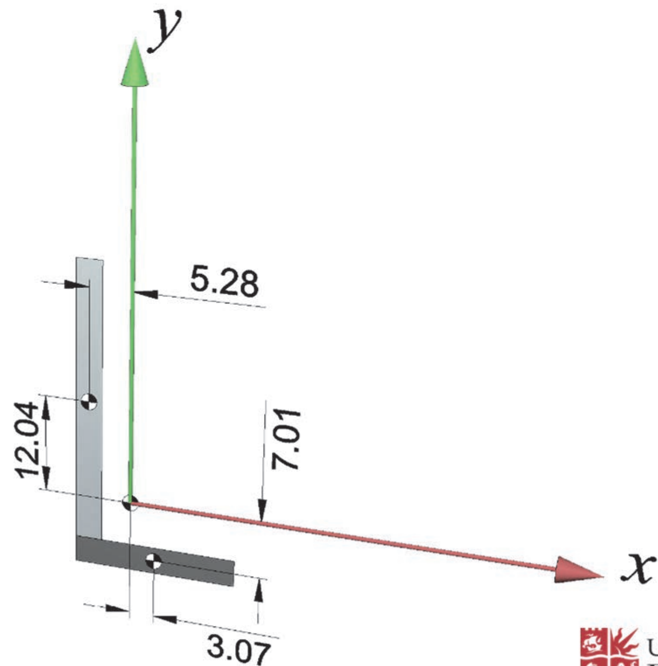
$$\bar{X} = \frac{\sum \bar{X}_i A_i}{\sum A_i} = \frac{\bar{X}_A A_A + \bar{X}_B A_B}{A_A + A_B} = \frac{(1.63)(113.58) + (9.98)(195.11)}{(113.58) + (195.11)}$$

$$\bar{X} = 6.91 \text{ mm}$$

$$\bar{Y} = \frac{\sum \bar{Y}_i A_i}{\sum A_i} = \frac{\bar{Y}_A A_A + \bar{Y}_B A_B}{A_A + A_B} = \frac{(20.68)(113.58) + (1.63)(195.11)}{(113.58) + (195.11)}$$

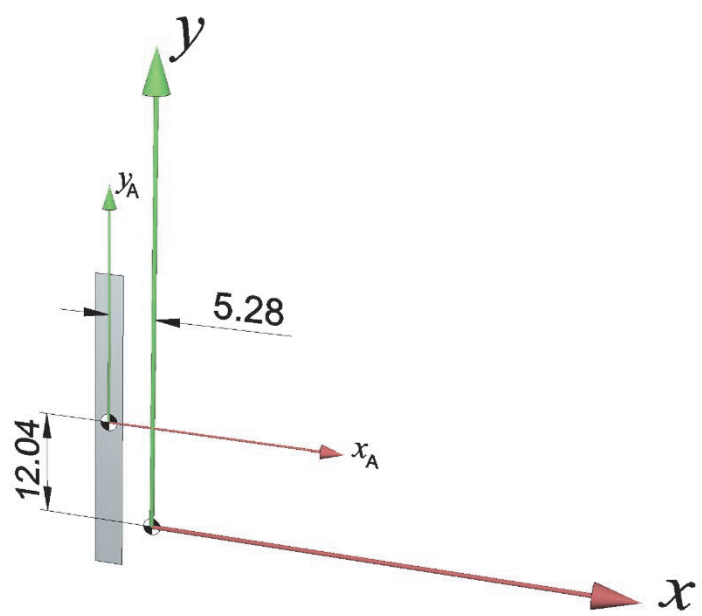
$$\bar{Y} = 8.64 \text{ mm}$$

- Plotting on the cross-section:



Composite (Aluminium-Steel) L-Section Beam

- We now place the origin of (x, y) at the compound centroid and apply the parallel axes theorem for section A:



$$I_{x_A x_A} = \frac{b\,h^3}{12} = \frac{(3.26)(34.84)^3}{12} = 11{,}488.70 \text{ mm}^4$$

$$\bar{y}_A = \bar{Y}_A - \bar{Y} = 20.68 - 8.64 = 12.04 \text{ mm}$$

$$I_{xx}^{A} = I_{x_A x_A} + A_A(\bar{y}_A)^2$$

$$I_{xx}^{A} = (11{,}488.70) + (113.58)(12.04)^2$$

$$I_{xx}^{A} = 27{,}955.37 \text{ mm}^4$$

$$I_{y_A y_A} = \frac{b\,h^3}{12} = \frac{(34.84)(3.26)^3}{12} = 100.58 \text{ mm}^4$$

$$\bar{x}_A = \bar{X}_A - \bar{X} = 1.63 - 6.91 = -5.28 \text{ mm}$$

$$I_{yy}^{A} = I_{y_A y_A} + A_A(\bar{x}_A)^2$$

$$I_{yy}^{A} = (100.58) + (113.58)(-5.28)^2$$

$$I_{yy}^{A} = 3{,}264.24 \text{ mm}^4$$

$I_{x_A\,y_A} = 0$ (symmetric cross-section)

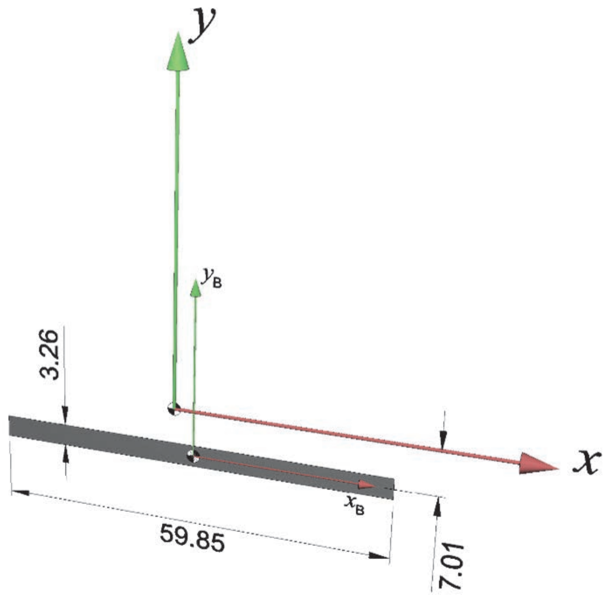
$$I_{xy}^{A} = I_{x_A y_A} + A_A(\bar{x}_A \bar{y}_A)$$

$$I_{xy}^{A} = 0 + (113.58)(-5.28)(12.04)$$

$$I_{xy}^{A} = -7{,}217.67 \text{ mm}^4$$

Composite (Aluminium-Steel) L-Section Beam

- Now applying the parallel axis theorem for section B:



$$I_{x_B x_B} = \frac{b\,h^3}{12} = \frac{(3)(19.95)(3.26)^3}{12} = 172.80\ \mathrm{mm}^4$$

$$\bar{y}_B = \bar{Y}_B - \bar{Y} = 1.63 - 8.64 = -7.01\ \mathrm{mm}$$

$$I_{xx}^{B} = I_{x_B x_B} + A_B(\bar{y}_B)^2$$

$$I_{xx}^{B} = (172.80) + (3)(65.04)(-7.01)^2$$

$$I_{xx}^{B} = 9{,}758.41\ \mathrm{mm}^4$$

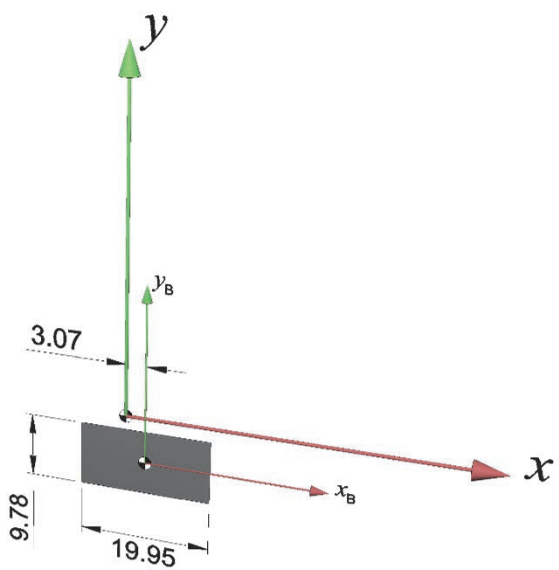
$$I_{y_B y_B} = \frac{b\,h^3}{12} = \frac{(3)(3.26)(19.95)^3}{12} = 6{,}471{,}22\ \mathrm{mm}^4$$

$$\bar{x}_B = \bar{X}_B - \bar{X} = 9.98 - 6.91 = 3.07\ \mathrm{mm}$$

$$I_{yy}^{B} = I_{y_B y_B} + A_B(\bar{x}_B)^2$$

$$I_{yy}^{B} = (6{,}471{,}22) + (3)(65.04)(3.07)^2$$

$$I_{yy}^{B} = 8{,}312.85\ \mathrm{mm}^4$$



$I_{x_B\,y_B} = 0$ (symmetric cross-section)

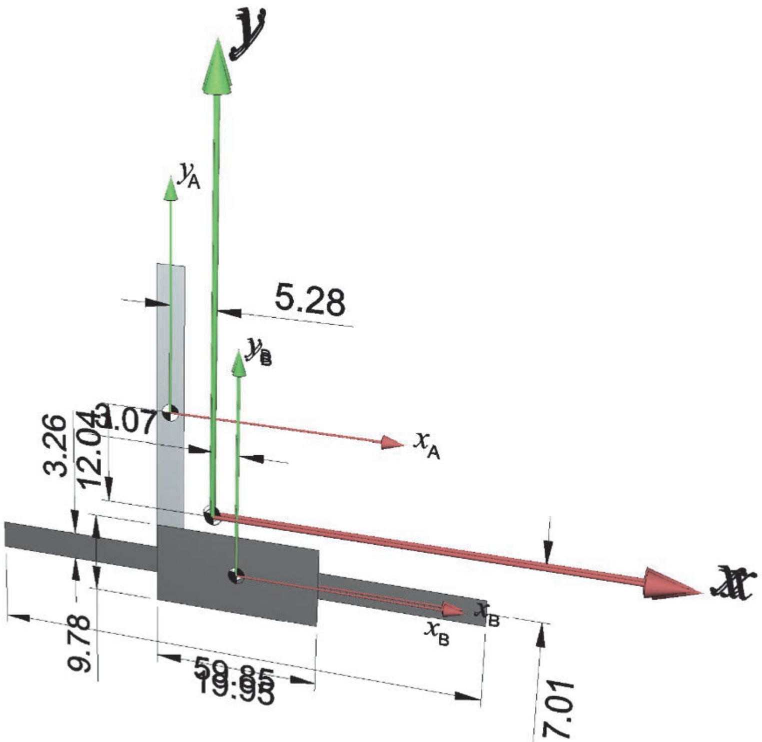
$$I_{xy}^{B} = I_{x_B\,y_B} + A_B(\bar{x}_B\bar{y}_B)$$

$$I_{xy}^{B} = 0 + (3)(65.04)(3.07)(-7.01)$$

$$I_{xy}^{B} = -4{,}201.57\ \mathrm{mm}^4$$

Composite (Aluminium-Steel) L-Section Beam

- Finally, for the **compound composite** section:



$$I_{xx} = I_{xx}^{\mathrm{A}} + I_{xx}^{\mathrm{B}}$$

$$I_{xx} = (27{,}955.37) + (9{,}758.41)$$

$$I_{xx} = 37{,}713.78 \text{ mm}^4$$

$$I_{yy} = I_{yy}^{\mathrm{A}} + I_{yy}^{\mathrm{B}}$$

$$I_{yy} = (3{,}264.24) + (8{,}312.85)$$

$$I_{yy} = 11{,}577.10 \text{ mm}^4$$

$$I_{xy} = I_{xy}^{\mathrm{A}} + I_{xy}^{\mathrm{B}}$$

$$I_{xy} = (-7{,}217.67) + (-4{,}201.57)$$

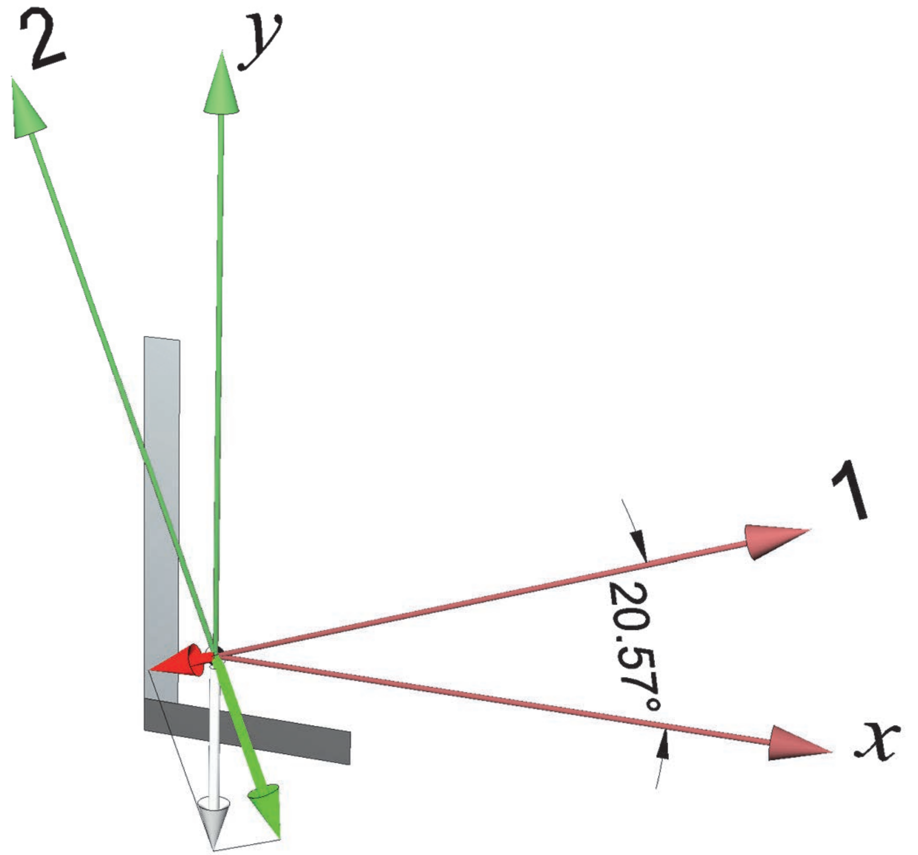
$$I_{xy} = -11{,}419.24 \text{ mm}^4$$

- We can now find the **principal axes**:

$$\theta_{\mathrm{p}} = \frac{1}{2}\arctan\left(\frac{2\,I_{xy}}{I_{yy} - I_{xx}}\right)$$

$$\theta_{\mathrm{p}} = \frac{1}{2}\arctan\left[\frac{2\,(-11{,}419.24)}{(11{,}577.10) - (37{,}713.78)}\right]$$

$$\theta_{\mathrm{p}} = 20.57°$$



- And the **principal 2nd moments of area** are:

$$\begin{Bmatrix} I_{11} \\ I_{22} \\ I_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2\,m\,n \\ n^2 & m^2 & 2\,m\,n \\ m\,n & -m\,n & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} I_{xx} \\ I_{yy} \\ I_{xy} \end{Bmatrix}$$

- where: $m = \cos\theta_{\mathrm{p}} = 0.936$

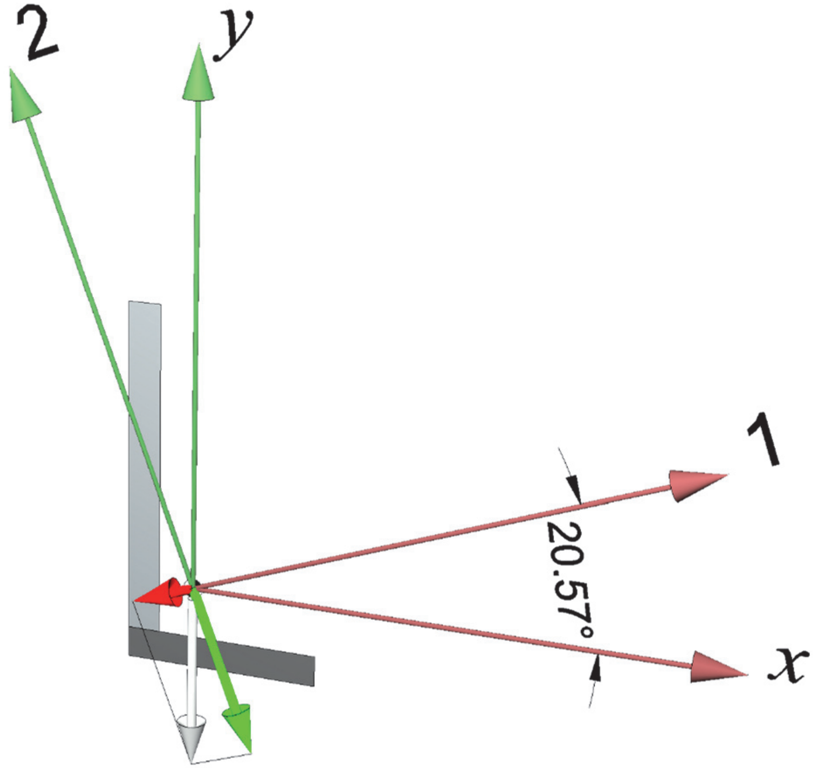
$$n = \sin\theta_{\mathrm{p}} = 0.351$$

- Using an Excel spreadsheet (see Blackboard) we get:

$$\begin{Bmatrix} I_{11} \\ I_{22} \\ I_{12} \end{Bmatrix} = \begin{Bmatrix} 41{,}999.99 \\ 7{,}290.89 \\ \sim 0 \end{Bmatrix} \mathrm{mm}^4$$

Composite (Aluminium-Steel) L-Section Beam

- Now we can project the applied load onto our principal axes:



$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} 0.962 & 0.272 \\ -0.272 & 0.962 \end{bmatrix} \begin{Bmatrix} 0 \\ -19.62 \end{Bmatrix} \mathrm{N}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} -6.90 \\ -18.37 \end{Bmatrix} \mathrm{N}$$

- Applying the tip deflection formula:

$$\boxed{\delta = \frac{P\,L^3}{3\,EI}}$$

$$\delta_1 = \frac{(-6.90)(1000)^3}{3\,(70{,}000)(7{,}290.88)} \qquad \delta_2 = \frac{(-18.37)(1000)^3}{3\,(70{,}000)(41{,}999.99)}$$

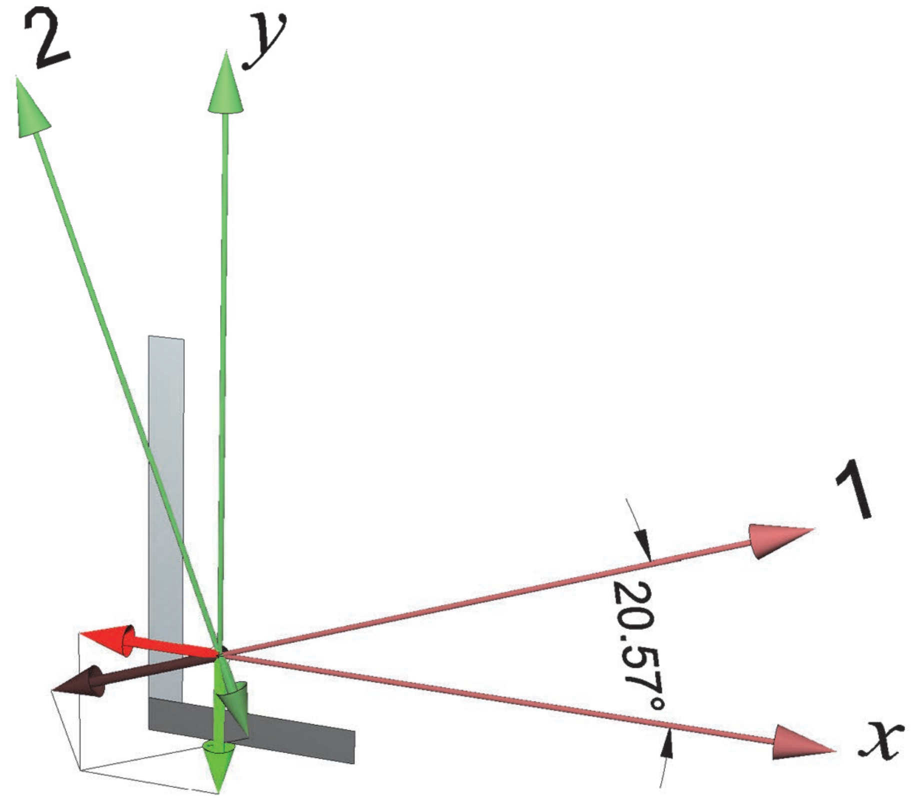
$$\delta_1 = -4.50\ \mathrm{mm} \qquad \delta_2 = -2.08\ \mathrm{mm}$$

- Finally, transform deflections from principal axes $(1,2)$ back to our reference axes (x, y) through rotation by $(-\theta_\mathrm{p})$:

$$\begin{Bmatrix}\delta_x\\ \delta_y\end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m\end{bmatrix}\begin{Bmatrix}\delta_1\\ \delta_2\end{Bmatrix} \qquad \begin{matrix} m = \cos(-\theta_\mathrm{p}) \\ n = \sin(-\theta_\mathrm{p})\end{matrix}$$

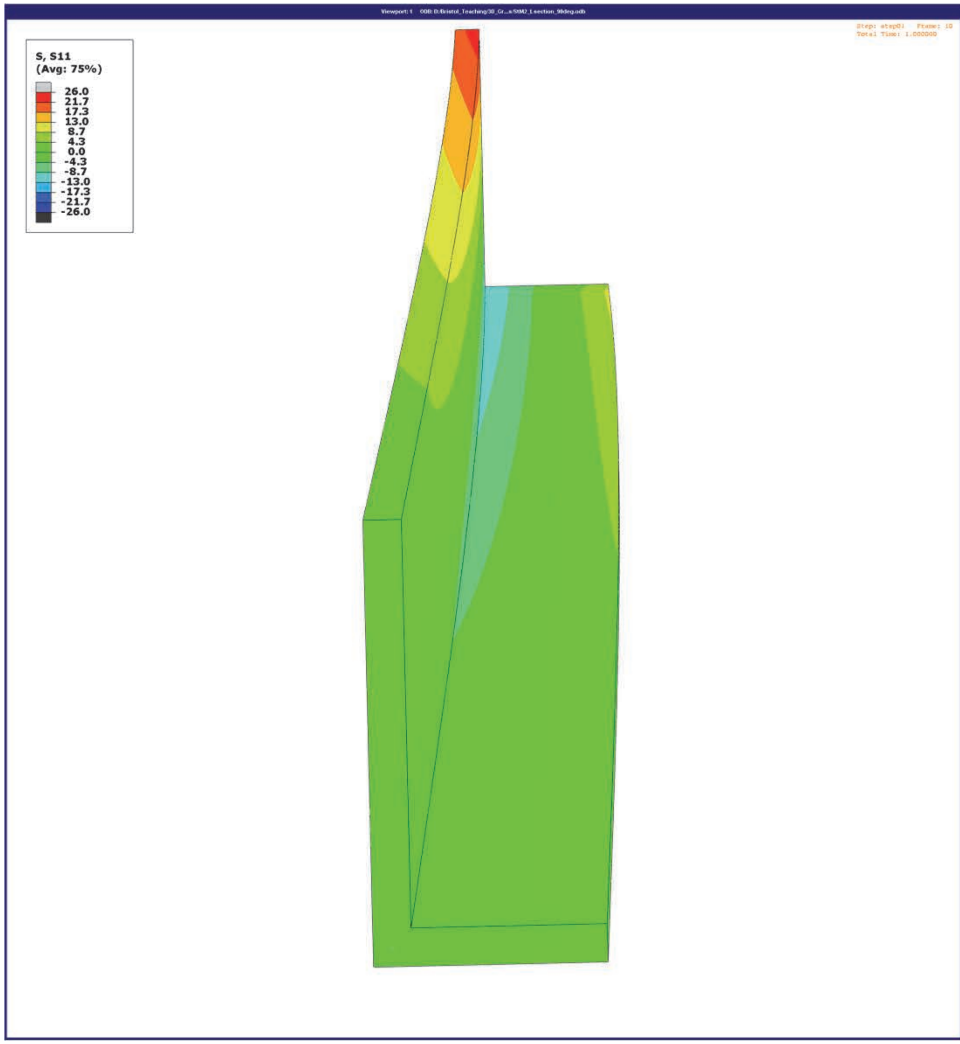
$$\begin{Bmatrix}\delta_x\\ \delta_y\end{Bmatrix} = \begin{bmatrix} 0.962 & -0.272 \\ 0.272 & 0.962\end{bmatrix}\begin{Bmatrix}-4.50\\ -2.08\end{Bmatrix} \mathrm{mm}$$

$$\begin{Bmatrix}\delta_x\\ \delta_y\end{Bmatrix} = \begin{Bmatrix}-3.48\\ -3.53\end{Bmatrix} \mathrm{mm}$$



L-Section Beam Example – FE Analysis

Aluminium Alloy



Aluminium Alloy + Steel