Structural Loads in Beams
Cantilever Beam with a Uniform Load Distribution

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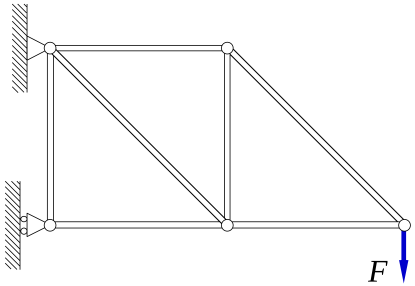
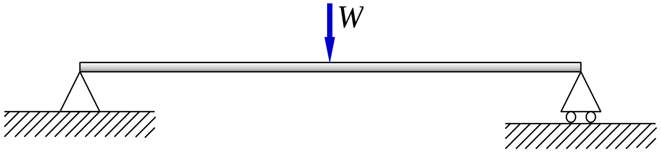
University of BRISTOL

2. Structural Loads in Beams - Contents

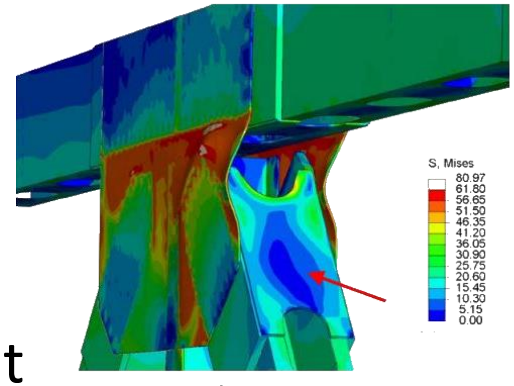
Distributed Loads

- So far we have been dealing with idealised 'point loads'
  - Forces applied at infinitely small areas $\longrightarrow$ not very realistic

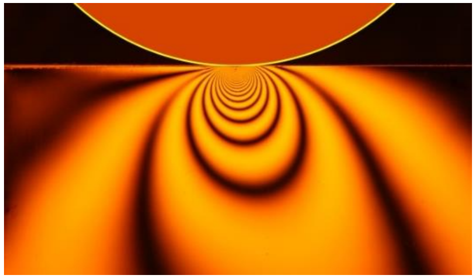




- In reality, loads are applied over a **finite area**, *e.g.*:
  - Stresses (N/m^2) in real 3D joints
  - Pressure (N/m^2) due to contact

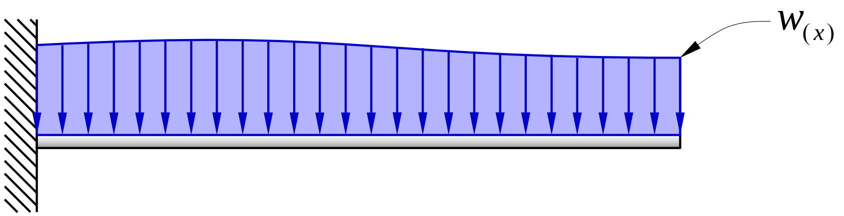


A real truss joint



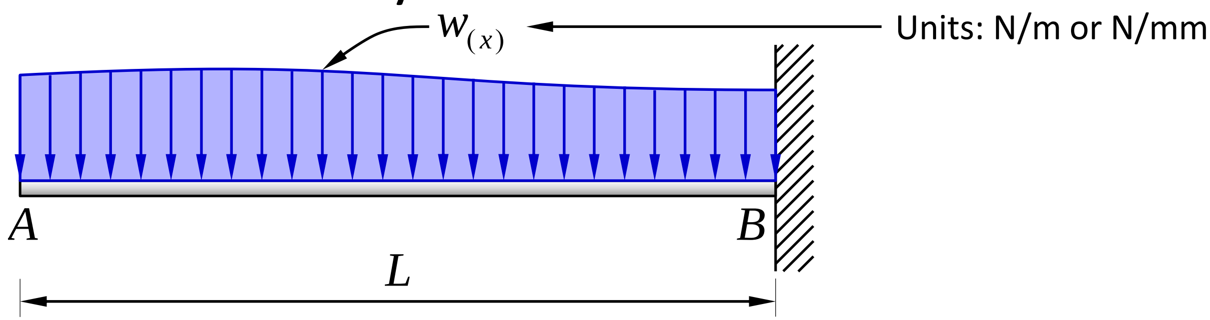
Real contact stresses (photoelastic effect of a cylinder on a flat surface)

- In 2D problems we assume 'unit width', and define distributed loads as <u>**force per unit length**</u> (N/m)

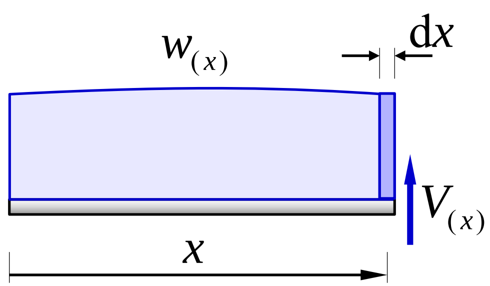


Distributed Loads

- Cantilever beam with an arbitrary load distribution:



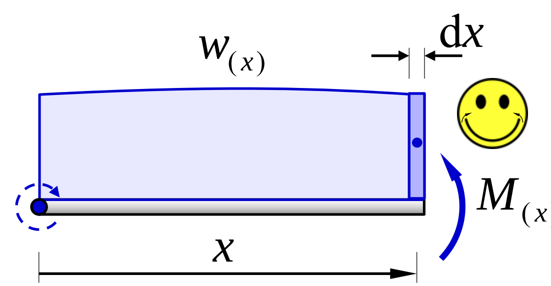
Shear Force



$$V_{(x)} = +\int_0^x w_{(x)}\,dx$$

- Shear force is the **area** under the curve $w(x)$

Bending Moment



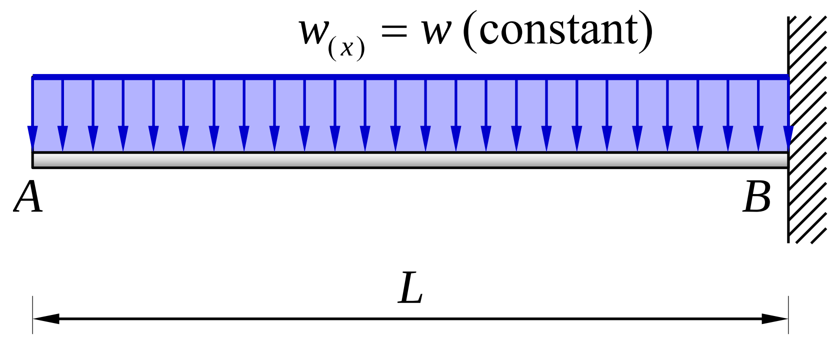
$$M_{(x)} = -\int_0^x w_{(x)}\,x\,dx$$

- Bending moment is the **moment of area** of the curve $w(x)$

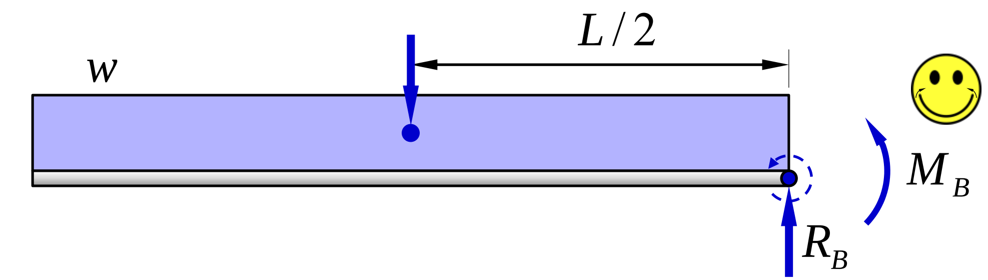
Signs depend on directions of x and $w(x)$

2.5.3 Cantilever Beam with a Distributed Load

- Cantilever beam with constant load distribution:



- Global FBD:

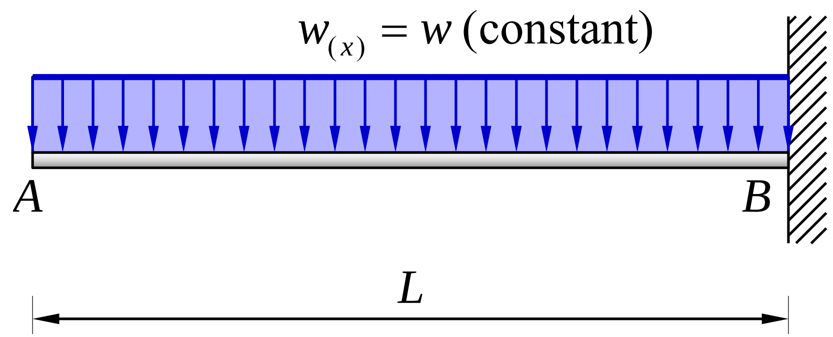


$$\sum F = 0 \quad \therefore \quad R_B - (w)(L) = 0 \quad \therefore \quad \boxed{R_B = wL}$$

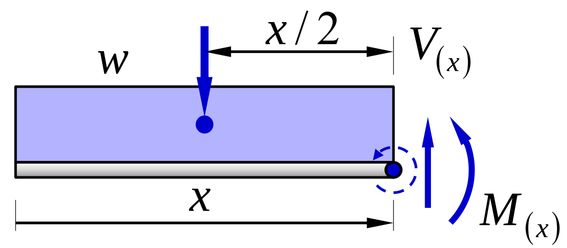
$$\sum M_{@B}^{\circlearrowleft\,\text{ccw}} = 0 \quad \therefore \quad M_B + (wL)\left(\frac{L}{2}\right) = 0 \quad \therefore \quad \boxed{M_B = -\frac{wL^2}{2}}$$

2.5.3 Cantilever Beam with a Distributed Load

- Cantilever beam with constant load distribution:



- Section FBD:



$$\sum F = 0 \quad \therefore \quad V_{(x)} - (w\,x) = 0 \quad \therefore \quad \boxed{V_{(x)} = w\,x}$$

$$\sum M_{@x}^{\text{ccw}} = 0 \quad \therefore \quad M_{(x)} + (w\,x)\left(\frac{x}{2}\right) = 0 \quad \therefore \quad \boxed{M_{(x)} = -\frac{w\,x^2}{2}}$$

Bending Moment *vs.* Shear Force

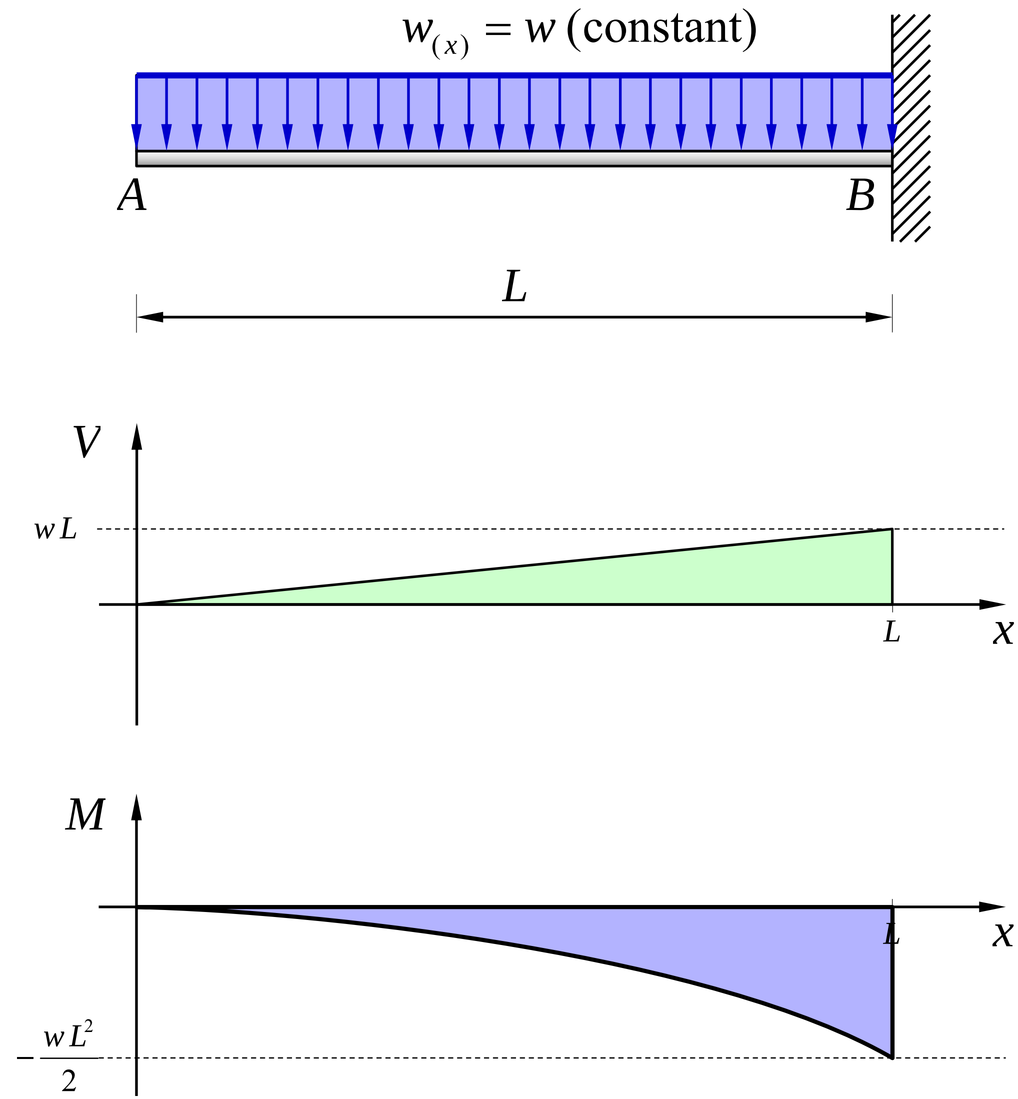
- Finally we can plot the shear force and bending moment diagrams

$$\boxed{R_B = wL}$$

$$\boxed{V_{(x)} = wx}$$

$$\boxed{M_B = -\frac{wL^2}{2}}$$

$$\boxed{M_{(x)} = -\frac{wx^2}{2}}$$



Structural Loads in Beams

Beams with Non-uniform Load Distributions

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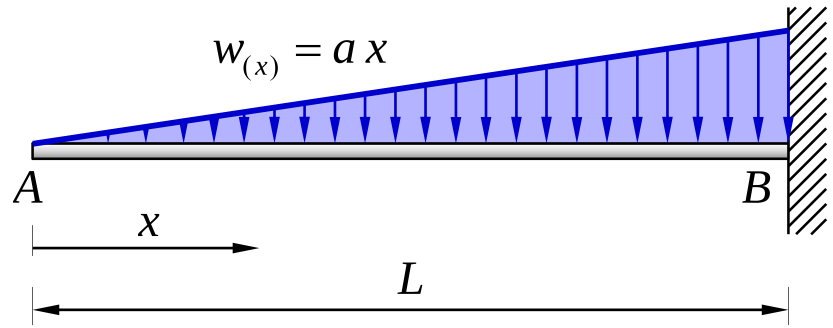
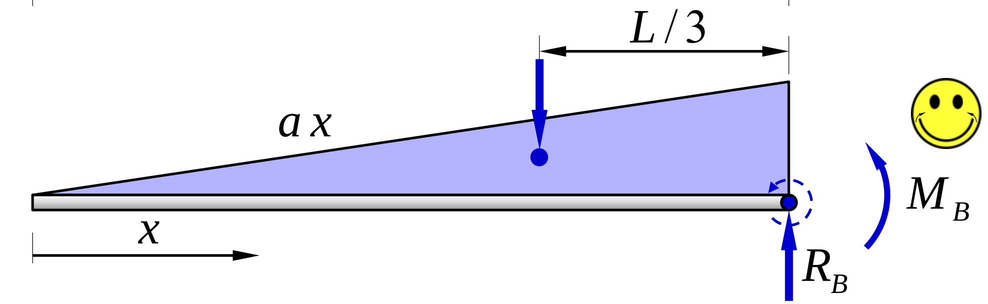
2. Structural Loads in Beams - Contents

2.5.5 Cantilever Beam + Non-Uniform Load Distribution

- Cantilever beam with a linear load distribution:



- Global FBD:

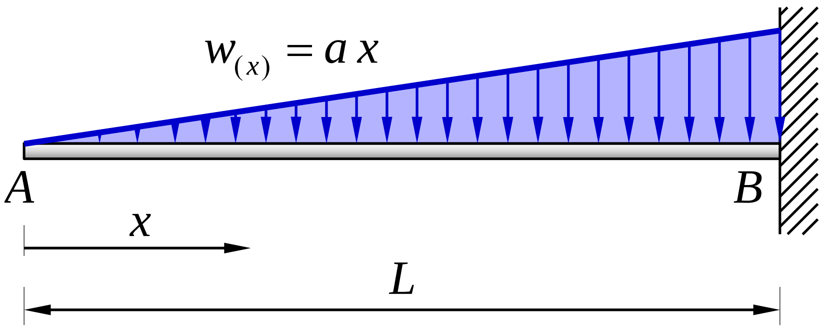


$$\sum F = 0 \quad \therefore \quad R_B - \frac{(a\,L)L}{2} = 0 \quad \therefore \quad \boxed{R_B = \frac{a\,L^2}{2}}$$

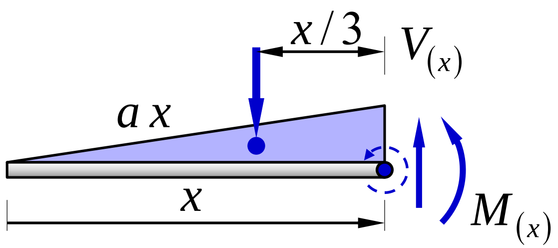
$$\sum M_{@B}^{\text{ccw}} = 0 \quad \therefore \quad M_B + \left(\frac{(a\,L)L}{2}\right)\left(\frac{L}{3}\right) = 0 \quad \therefore \quad \boxed{M_B = -\frac{a\,L^3}{6}}$$

2.5.5 Cantilever Beam + Non-Uniform Load Distribution

- Cantilever beam with constant load distribution:



- Section FBD:



$$\sum F = 0 \quad \therefore \quad V_{(x)} - \frac{(a\,x)x}{2} = 0 \quad \therefore \quad \boxed{V_{(x)} = \frac{a\,x^2}{2}}$$

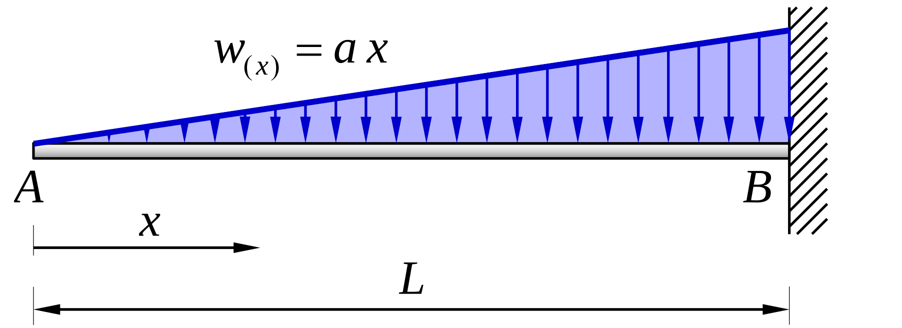
$$\sum M_{@x}^{\text{ccw}} = 0 \quad \therefore \quad M_{(x)} + \frac{(a\,x)x}{2}\left(\frac{x}{3}\right) = 0 \quad \therefore \quad \boxed{M_{(x)} = -\frac{a\,x^3}{6}}$$

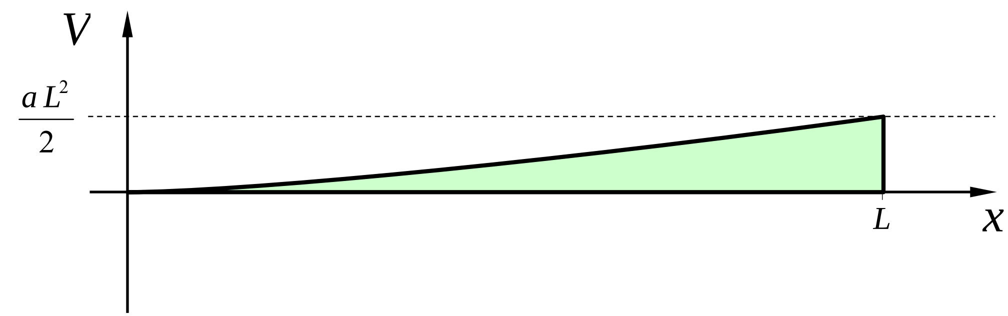
2.5.5 Cantilever Beam + Non-Uniform Load Distribution

- The shear force and bending moment expressions are:

$$\boxed{V_{(x)} = \frac{a\,x^2}{2}}$$

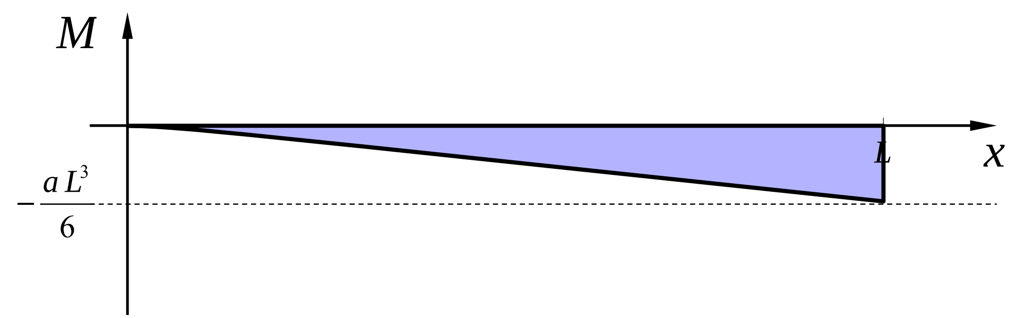
$$\boxed{M_{(x)} = -\frac{a\,x^3}{6}}$$





- At the ‘tip’, A:

$$\boxed{V_{(x=0)} = 0} \qquad \boxed{M_{(x=0)} = 0}$$



- At the ‘root’, B:

$$\boxed{V_{(x=L)} = \frac{a\,L^2}{2}} \qquad \boxed{M_{(x=L)} = -\frac{a\,L^3}{6}}$$

2.5.5 Cantilever Beam + Non-Uniform Load Distribution

- The shear force and bending moment expressions are:

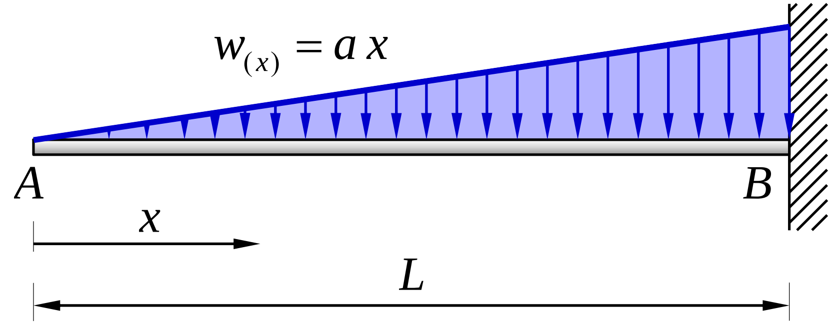
$$\boxed{V_{(x)} = \frac{a\,x^2}{2}}$$

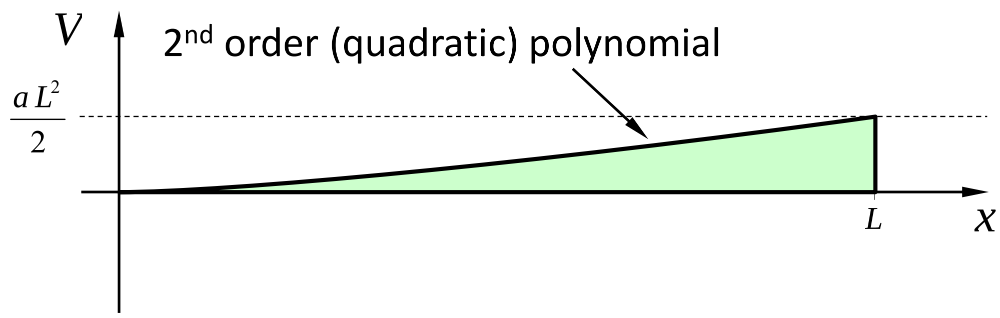
$$\boxed{M_{(x)} = -\frac{a\,x^3}{6}}$$

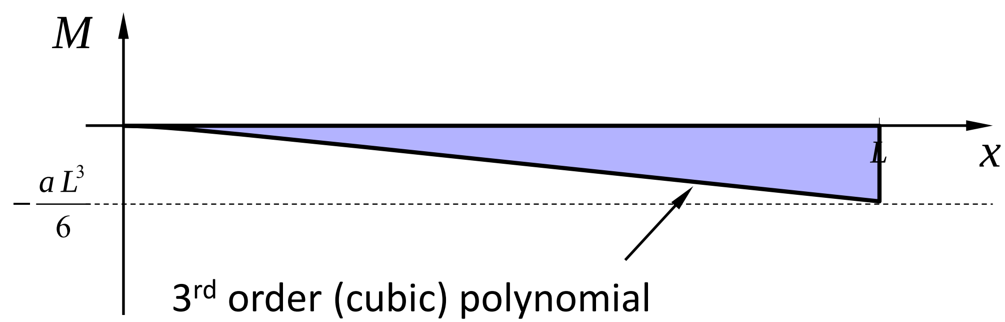
- Note that, as before, the bending moment and shear force diagrams obey the differential relationship:

$$\boxed{V_{(x)} = -\frac{\mathrm{d}\,M_{(x)}}{\mathrm{d}\,x}}$$

$$\boxed{M_{(x)} = -\int_0^x V_{(x)}\,\mathrm{d}x}$$



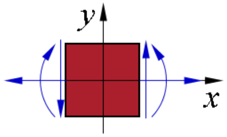
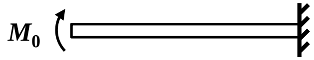
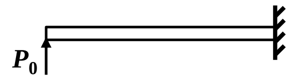
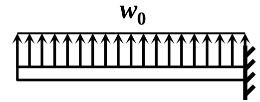
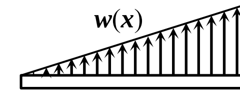
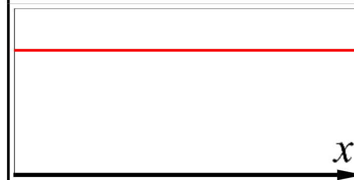
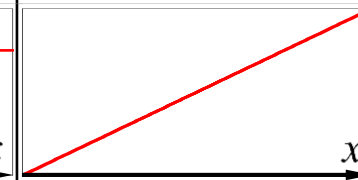
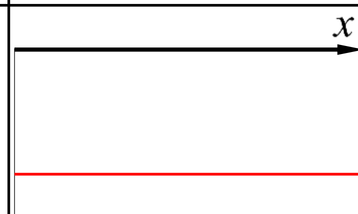
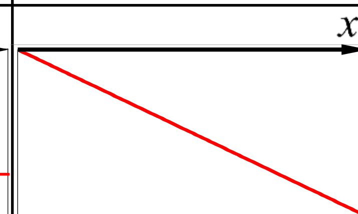
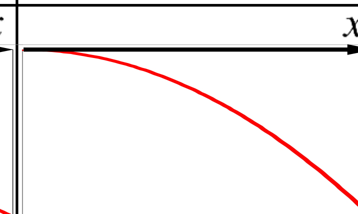
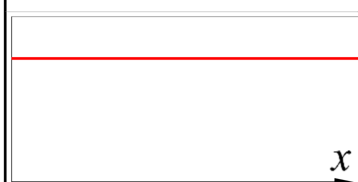
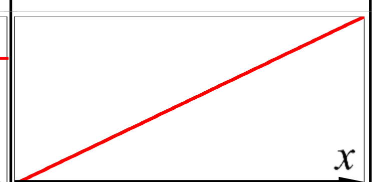
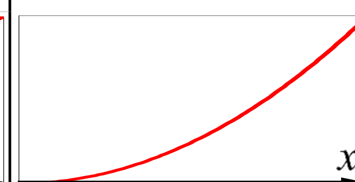
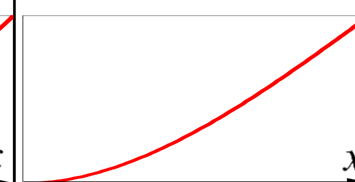
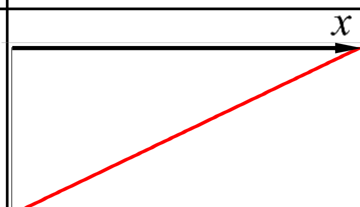
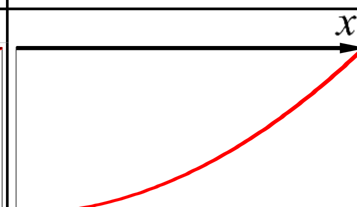
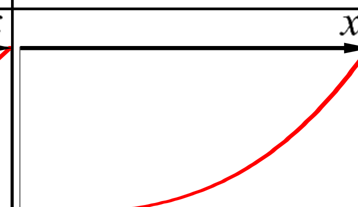
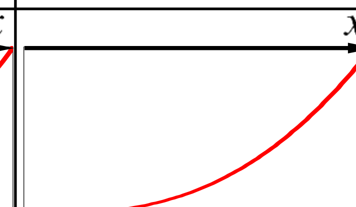
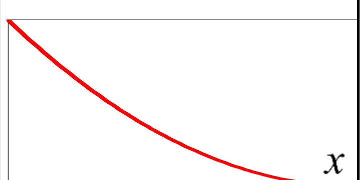
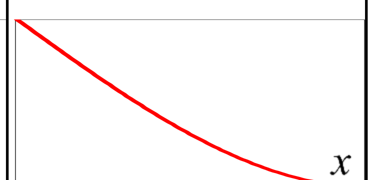
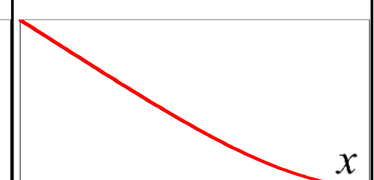
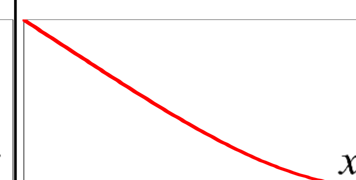




Beams - Differential Relations

Signs depend on assumed x-direction

differentiate ↑	Rate of change of the rate of change of curvature (!)	$\dfrac{d^2\kappa_{(x)}}{dx^2} = \dfrac{d^4 y}{dx^4} = (\pm)\dfrac{1}{EI} w_{(x)}$	External load distribution
	'Rate of change' of curvature	$\dfrac{d\kappa_{(x)}}{dx} = \dfrac{d^3 y}{dx^3} = (\pm)\dfrac{1}{EI} V_{(x)}$	Shear force
Starting point (usually) →	Curvature	$\kappa_{(x)} = \dfrac{d^2 y}{dx^2} = \dfrac{1}{EI} M_{(x)}$	Moment
	Slope	$\theta_{(x)} = \dfrac{dy}{dx}$	
integrate ↓	Deflection	$y_{(x)} = y$	

					
$-EI\dfrac{d^4y}{dx^4} = w_{(x)}$ **4th order ODE**	**External load distribution**	**0**	**0**	 **constant**	 **1st order polynomial**
$-EI\dfrac{d^3y}{dx^3} = V_{(x)}$ **3rd order ODE**	**Shear force**	**0**	 **constant**	 **1st order polynomial**	 **2nd order polynomial**
$EI\dfrac{d^2y}{dx^2} = M_{(x)}$ **2nd order ODE**	**Moment**	 **constant**	 **1st order polynomial**	 **2nd order polynomial**	 **3rd order polynomial**
$\dfrac{dy}{dx} = \theta_{(x)}$ **1st order ODE**	**Slope**	 **1st order polynomial**	 **2nd order polynomial**	 **3rd order polynomial**	 **4th order polynomial**
$y = y_{(x)}$ **polynomial**	**Deflection**	 **2nd order polynomial (arc)**	 **3rd order polynomial**	 **4th order polynomial**	 **5th order polynomial**

Structural Loads in Beams
The Principle of Superposition

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24 October 2017

2. Structural Loads in Beams - Contents

2.6 Principle of Superposition

- Consider a beam subjected to multiple loads:
  - point forces, distributed loads, applied couples etc.

- Assume that we want to find its 'final' configuration:
  - axial force diagram, shear force diagram, bending moment diagram, deflection profile

- The principle states that we can analyse each load separately (considering the same BCs) and then sum the individual contributions

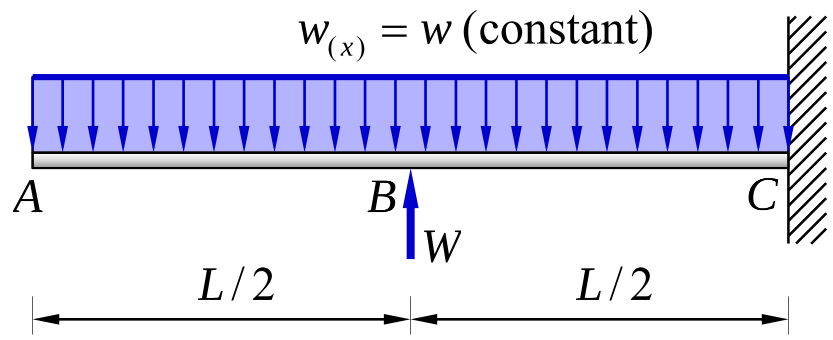
- Graphically this means that the final diagrams can be found by adding/subtracting curves

- Algebraically this means that expressions for the various quantities can be found (on a span-to-span basis only) by summing the respective expressions obtained with each individual load

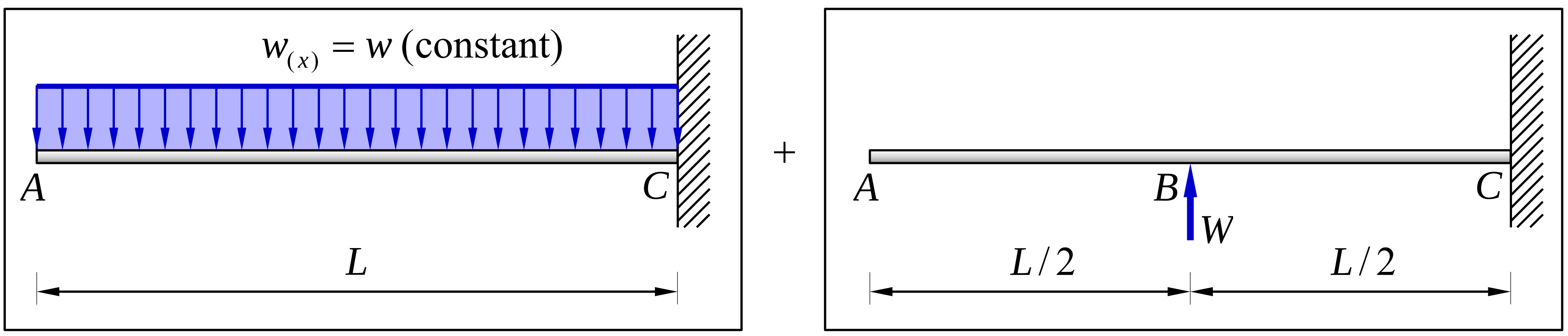
- However full expressions for the entire beam (written in a common x-coordinate) require the use of the Heaviside 'step' function

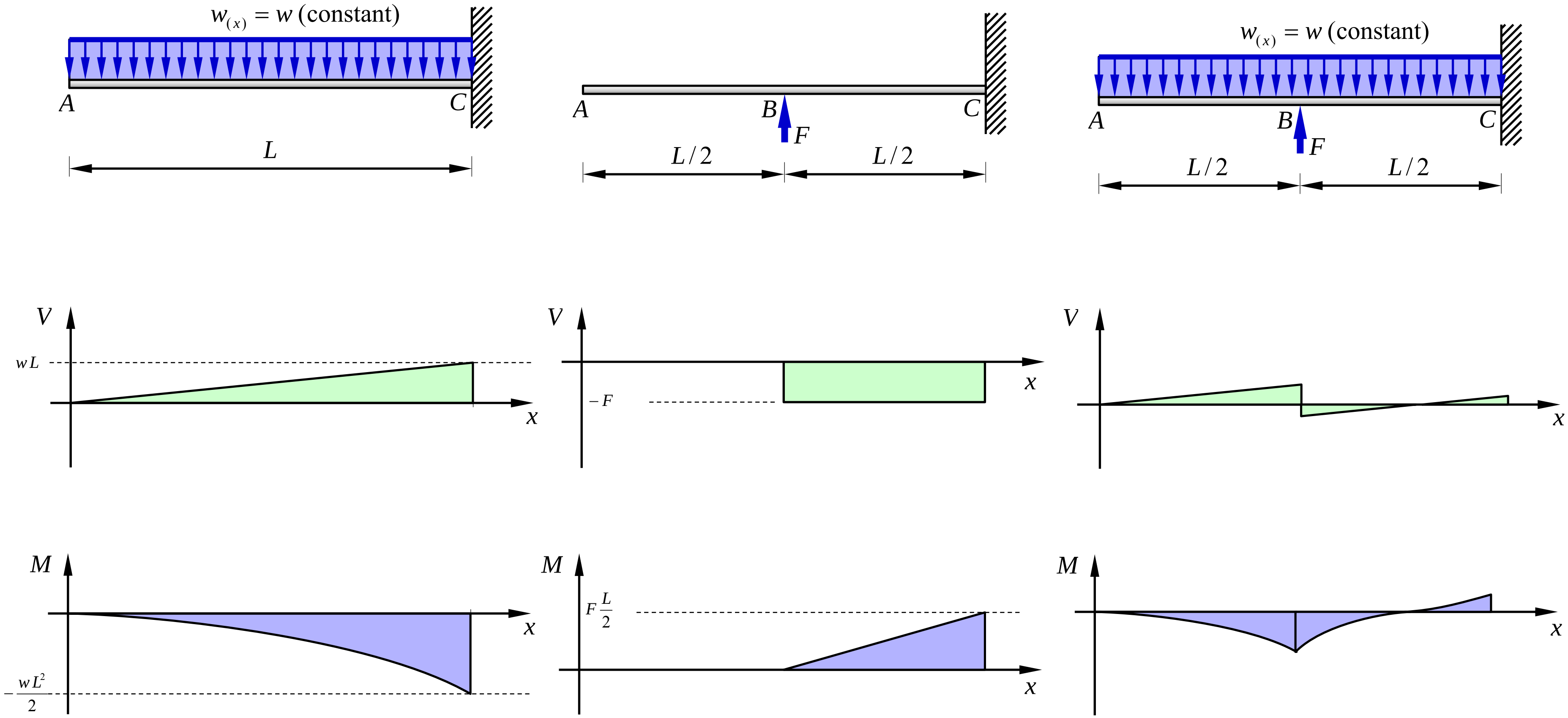
2.6 Principle of Superposition

- Consider the following cantilever beam with two loads:



- The easiest way to analyse this is to split it into two problems:





$w_{(x)} = w$ (constant)
A
C
L
A
B
F
C
L / 2
L / 2
$w_{(x)} = w$ (constant)
A
B
F
C
L / 2
L / 2
V
wL
x
V
−F
x
V
x
M
$-\frac{wL^2}{2}$
x
M
$F\frac{L}{2}$
x
M
x