EMAT10100 Engineering Maths I
Lecture 3: Roots and geometry

John Hogan & Alan Champneys

Looking back and forward

- Last lecture:
  - Introduction to $j = \sqrt{-1}$
  - General form $z = x + jy$ of complex numbers
  - Real and imaginary parts x and y
  - Addition, subtraction, multiplication, complex conjugate
  - Division of complex numbers
- This lecture:
  - Solving polynomials
  - More on division & complex conjugate
  - Argand diagram: geometry of complex numbers
  - Polar form of a complex number

Roots of quadratics

- Example Find the (complex) roots of

$$z^2 - z + 1 = 0$$

- We know that if $az^2 + bz + c = 0$ then $z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Hence:

$$\begin{aligned} z &= \frac{1}{2}(1 \pm \sqrt{1-4}) &= \frac{1}{2} \pm \frac{\sqrt{-3}}{2} \\ &= \frac{1}{2} \pm \frac{\sqrt{-1}\sqrt{3}}{2} &= \frac{1}{2} \pm \frac{\sqrt{3}j}{2} \end{aligned}$$

- Exercise Find the (complex) roots of the quadratic

$$2z^2 + 3z + 7 = 0$$

Roots of general polynomials

A *polynomial* is the sum of several terms containing different powers of the same variable:

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

where the constants $a_k, k = 1 \ldots n$ can be complex. Then we know that

- Every polynomial of degree n has n roots if complex roots are allowed and repeated roots are counted in the correct way
- Given any polynomial $P_n(z)$ *with real coefficients*, then its roots occur in complex conjugate pairs

  (because $[z - (A + jB)] \times [z - (A - jB)] = z^2 - 2Az + (A^2 + B^2)$ has real coefficients)

Q. Why does this imply that every real cubic polynomial crosses the axis at least once? What about quartics?

Return to division

- Q. Why does the division trick work?

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(x_1+jy_1)(x_2-jy_2)}{(x_2+jy_2)(x_2-jy_2)}$$
$$= \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right) + j\left(\frac{-x_1y_2+x_2y_1}{x_2^2+y_2^2}\right)$$

- A. Because

$$z_2\overline{z_2} = x_2^2+y_2^2 \quad \text{is a real number!}$$

- More generally, if $z = x+jy$, we say

$$z\overline{z} = x^2+y^2 = |z|^2, \quad \text{where } |z| = \text{“mod } z\text{”}$$

$|z|$ is the modulus (or magnitude) of z . . .

Modulus of a complex number

$$|z| = \sqrt{x^2+y^2}$$

- Exercise

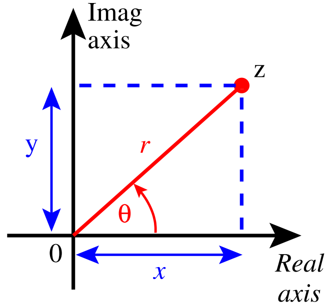
Let $z = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

find $\overline{z}$, $z\overline{z}$, $|z|$, $z/|z|$

- What do you notice?
- Try drawing a triangle with side lengths $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ . . .
- Seems a lot like Pythagoras . . .

The Argand diagram

A neat way of visualising complex numbers



- Trigonometry:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Note: r, θ real

- So

$$z = x + jy$$
$$= r\cos\theta + jr\sin\theta$$
$$z = r(\cos\theta + j\sin\theta) \qquad \text{(polar form)}$$

- The Argand diagram is also known as the complex plane.

Polar form (modulus and argument)

- Polar form of a complex number:

$$z = x + jy \quad = r(\cos\theta + j\sin\theta)$$

- r is called modulus of z
  - it is the same as $|z|$
  - Pythagoras: $|z| = r = \sqrt{x^2+y^2} \geq 0$
- θ is called the argument (or phase) of z
  - written and pronounced $\arg z$
  - from trig: $\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$.

  Q. But which value of arctan?

  A. We want the principal value of arctan such that $-\pi < \theta \leq +\pi$. Written $\theta = \mathrm{Arg}\ z$. (Or, draw a picture! . . .)

Example

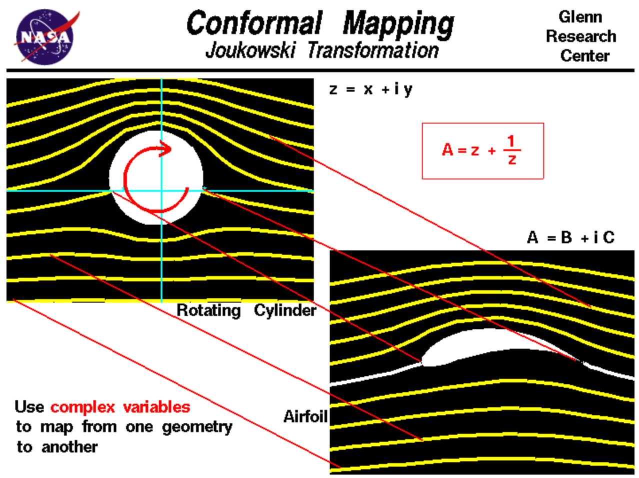
- Put the complex numbers $z_1 = j$ and $z_2 = -j$ into polar form and sketch them in the Argand diagram.
- Solution: First, compute the modulus and argument

$$
\begin{aligned}
z_1 &= 0 + 1j, & z_2 &= 0 + (-1)j \\
|z_1|^2 &= 0^2 + 1^2 = 1, & |z_2|^2 &= 0^2 + 1^2 = 1 \\
\arg z_1 &= \arctan(\infty) = \frac{\pi}{2} + 2n\pi, & \arg z_2 &= \arctan(-\infty) = -\frac{\pi}{2} + 2n\pi
\end{aligned}
$$

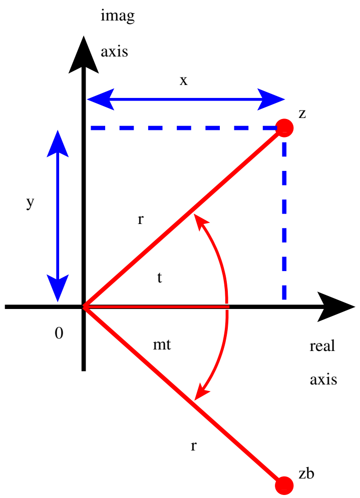
- To find which value of n ensures that $-\pi < \arg z_1, \arg z_2 \le +\pi$, plot on Argand diagram (do it!) $\Rightarrow$ $\mathrm{Arg}\ z_1 = \pi/2$ and $\mathrm{Arg}\ z_2 = -\pi/2$
- Hence $z_1 = j = \cos(\pi/2) + j\sin(\pi/2)$,
  $z_2 = -j = \cos(-\pi/2) + j\sin(-\pi/2)$
- Exercise put $z_3 = \sqrt{3} + j$ and $z_4 = -1 - j$ into polar form

Engineering HOT SPOT

Using the complex plane for Conformal mapping:



Geometry of complex conjugate



- Take
  $$z = x + jy, \ = r(\cos\theta + j\sin\theta)$$
- Then
  $$\overline{z} = x - jy, \ = r(\cos\theta - j\sin\theta), \quad = r\left[\cos(-\theta) + j\sin(-\theta)\right]$$
- So:
  - $|\overline{z}| = |z|$
  - $\mathrm{Arg}(\overline{z}) = -\mathrm{Arg}(z)$
- Q. How are modulus and argument of $\overline{z}$ and $1/z$ related to those of z?

Homework

- read *James* Sect. 3.2.3–3.2.6

  go through *James* worked ex. 3.7–3.11
- attempt *James* exercises: 3.2.5 Qns. 2, 7-11, 14
- take the online *Questionmark* Week 1 test
  go to `qmp.bris.ac.uk` and take complex1
  can take as many attempts as you like! (test different each time)
- if you get stuck:

  go to at least one of the drop-in sessions: