EMAT10100 Engineering Maths I
Lecture 15: More on Rank and Inverse

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Looking back, looking forward

- Last time: Linear systems of equations: $\mathbf{Ax} = \mathbf{b}$
- Solution by (Guassian) row elimination
  - Gives unique solution if det $(\mathbf{A}) \neq 0$ (i.e. $\mathbf{A}$ is nonsingular)
  - If $\mathbf{A}$ is singular,
    - if Rank $(\mathbf{A}) \neq$ Rank $([\mathbf{A}|\mathbf{b}])$ then there is no solution
    - if Rank $(\mathbf{A}) =$ Rank $([\mathbf{A}|\mathbf{b}])$ then there is a family of solutions
- This time: More on row operations
  - reinforcing the idea of rank from the last lecture
  - calculation of inverse via row operations.

Calculating the Rank of a matrix

- Given an arbitrary $n \times m$ matrix $\mathbf{A}$, (not necessarily square).
- Carry out row operations until you obtain an upper triangular form.
- Then: Define: Null $(\mathbf{A})$ is the number of entirely zero rows left at the end of the elimination process.
- Note: Null $(\mathbf{A})$ is the dimension of the solution set of $\mathbf{Ax} = 0$. (Important for next lecture)
- Then: Rank $(\mathbf{A})$ = $n-$ Null $(\mathbf{A})$.
- Example: Note that the matrix below has rank 1 (show it!)

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 3 \\ -3 & 2 & -3 \\ 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

Hence the solution to $\mathbf{Ax} = 0$ has two free parameters.

Exercise

- Perform row eliminations to analyse the solutions of $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & -3 \\ 3 & -2 & 3 \\ 2 & -2 & 0 \end{pmatrix}$$

and we take in turn

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \text{then} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Finding inverse via row operations

- Although we don't usually solve $\mathbf{Ax} = \mathbf{b}$ by finding $\mathbf{A}^{-1}$, sometimes finding inverses is important.
- Example:

$$\text{Let}: \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \quad \text{for which} \quad \mathbf{A}^{-1} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$$

- Alternative method, form $[\mathbf{A}|\mathbf{I}_2]$ and do row operations to make LHS $\mathbf{I}_2$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right) : \quad R_2 \to R_2 + 2R_1 \implies \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right) :$$

$$R_1 \to R_1 - 2R_2 \implies \left(\begin{array}{cc|cc} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right) \text{ so } \mathbf{A}^{-1} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$$

General method

- Given an $n \times n$ matrix $\mathbf{A}$, with det $\mathbf{A} \neq 0$
- Form the augmented matrix $[\mathbf{A}|\mathbf{I}_n]$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix
- Perform row operations (full version) to get zeros everywhere and only 1s on the diagnoals on the LHS (i.e. to get $\mathbf{I}_n$). Do the same to RHS.
- Then the RHS becomes $\mathbf{A}^{-1}$
- Exercise: Use row elimination to find the inverse of

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

- Note This method is not competitive for 2×2 matrices, but is always best for 4×4 or higher. Can use either method for 3×3.

Why does this work?

- For a 2×2 example let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \mathbf{A}^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \text{ and } \mathbf{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$$

- Because $\mathbf{AA}^{-1} = \mathbf{I}_2$, we have (by looking at each column separately):

$$\mathbf{Ab}_1 = \mathbf{e}_1, \quad \mathbf{Ab}_2 = \mathbf{e}_2, \quad \text{where} \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- In other words, when we solve using row operations we get $[\mathbf{A}|\mathbf{e}_1] \to [\mathbf{I}_2|\mathbf{b}_1]$ and $[\mathbf{A}|\mathbf{e}_2] \to [\mathbf{I}_2|\mathbf{b}_2]$.
- So we may as well write: $[\mathbf{A}|\mathbf{e}_1\mathbf{e}_2] \to [\mathbf{I}_2|\mathbf{b}_1\mathbf{b}_2]$
  which is equivalent to $[\mathbf{A}|\mathbf{I}_2] \to [\mathbf{I}_2|\mathbf{A}^{-1}]$.
  Hence row operations lead to $\mathbf{A}^{-1}$ appearing on the RHS.

Homework

- Read the rest of *James* section 5.5.2
- Read *James* Sec. 5.6
- Do exercises (4th edition) or (5th edition):
  - 5.5.3 Q.72 and 74
  - 5.6.1 Q.87, 89 and 90
- Don't get behind, there is a lot of material in this matrices section!
- Don't forget the QMP questions too.

EMAT10100 Engineering Maths I
Lecture 16: Eigenvalues and Eigenvectors (part 1)

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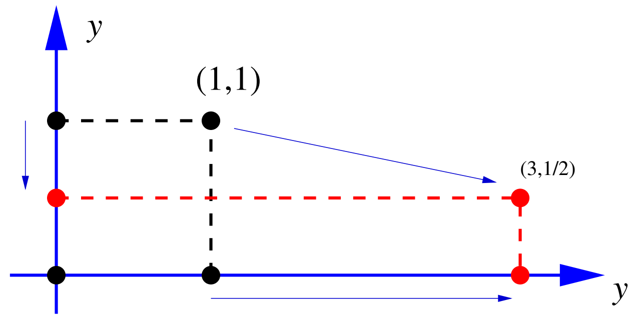
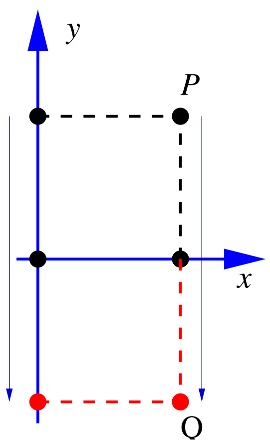
Recall matrix transforms in 2D



- $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$
- x-direction gets stretched by factor of 3
- y-direction gets condensed by factor of $1/2$
- Area of square is scaled by $3/2$

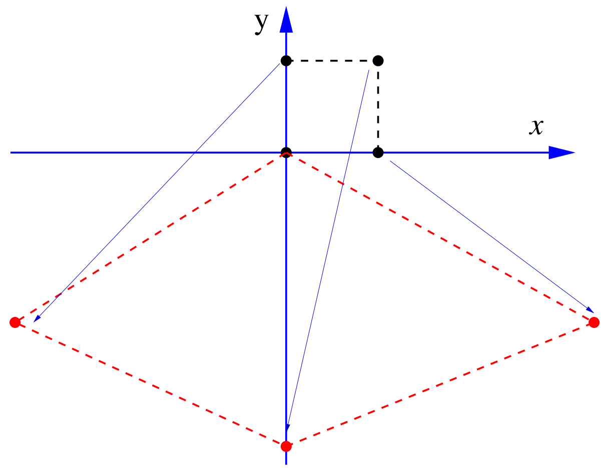
Another example

- $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection in x-axis
- Area of square unchanged

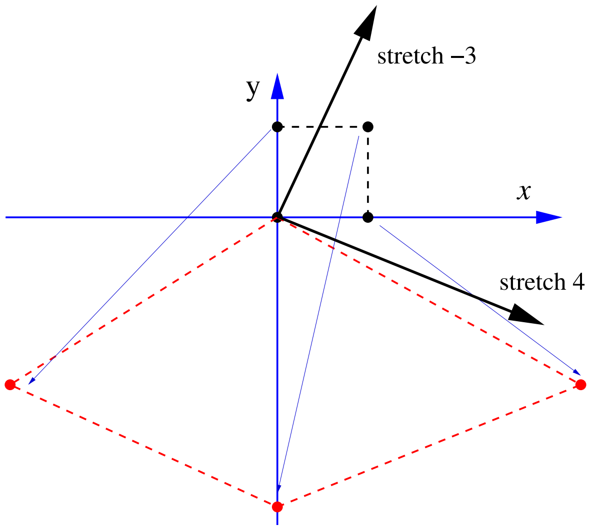


What about general transformations?

- $\mathbf{A} = \begin{pmatrix} 3 & -3 \\ -2 & -2 \end{pmatrix}$
- how to express as two separate stretches?



- stretch +4 in direction 3**i** - **j**
- stretch -3 in direction **i** + 2**j**



How do I know this?

- Can all 2D transformations be expressed as combination of two separate stretches about different axes?
  - Yes!
- What defines the special directions of these axes?
  - Eigenvectors
- What defines the stretch factors?
  - Eigenvalues

Eigenvalues and eigenvectors

- Eigenvectors are special vectors $\mathbf{v}$ whose *direction* is not altered under Matrix multiplication:

$$\mathbf{A}\mathbf{v} \quad \text{has the same direction as} \quad \mathbf{v}$$

- That is,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \quad \text{for some scalar} \lambda$$

- Such a λ (which gives the amount of stretch) is called an eigenvalue
- Eigenvectors are also sometimes called *modes*
- Calculation of eigenvalues and eigenvectors plays a crucial role in stability analysis, vibration engineering and optimisation

Calculation of eigenvalues I

- Let $\mathbf{A}$ be an $n \times n$ matrix and $\mathbf{v}$ be an eigenvector with eigenvalue λ:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

- This can be rewritten

$$\begin{aligned} \mathbf{A}\mathbf{v} &= \lambda\mathbf{I}_n\mathbf{v} \\ \mathbf{A}\mathbf{v} - \lambda\mathbf{I}_n\mathbf{v} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I}_n)\mathbf{v} &= \mathbf{0} \end{aligned}$$

where $\mathbf{A}_n$ is the $n \times n$ identity matrix and $\mathbf{0}$ is the $n \times 1$ zero vector.

Calculation of eigenvalues II

- But, in order to have a non-trivial solution to

$$(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{v} = \mathbf{0},$$

then the matrix

$$(\mathbf{A} - \lambda \mathbf{I}_n)$$

must not have full rank (see last lecture)

- Which gives a formula for calculating eigenvalues:

$$\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$$

Example

- Find the eigenvalues of the following matrix:

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

- Solution: We obtain this by evaluating the determinant $|\mathbf{A} - \lambda \mathbf{I}_2|$ which gives:

$$0 = |\mathbf{A} - \lambda \mathbf{I}_2| = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - 1.$$

- Expanding this out we obtain the characteristic equation:

$$\lambda^2 + 4\lambda + 3 = 0,$$

whose roots $\lambda = -1$ and $\lambda = -3$ are the eigenvalues.

What about the eigenvectors?

- We calculate these after we have calculated the eigenvalues
- Example: Calculate the eigenvectors of the following matrix:

$$\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

- Solution:
  - Luckily we already know $\lambda = -1$ and $\lambda = -3$
  - So we can substitute these values into $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$
  - And solve the resulting systems of linear equations for $\mathbf{v}$

Case $\lambda = -1$:

- Substituting this value of λ we obtain:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- This equation does not have full rank (you have gone wrong if it does)
- So we expect an infinite family of solutions (which all point in the same direction)
- Choosing $v_{11} = \alpha$, we find $v_{21} = \alpha$
- By convention, we *often* choose a unit vector, so:

$$\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

also known as a normalised eigenvector.

Case $\lambda = -3$:

- Substituting this value of λ we obtain:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Choosing $v_{11} = \alpha$, we find $v_{21} = -\alpha$
- So, the unit eigenvector is

$$\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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Exercise

- Return to the matrix

$$2\mathbf{A} = \begin{pmatrix} 3 & -3 \\ -2 & -2 \end{pmatrix}$$

- Show that its eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -3$
- With corresponding eigenvectors (not normalised)

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

More examples in the next lecture.