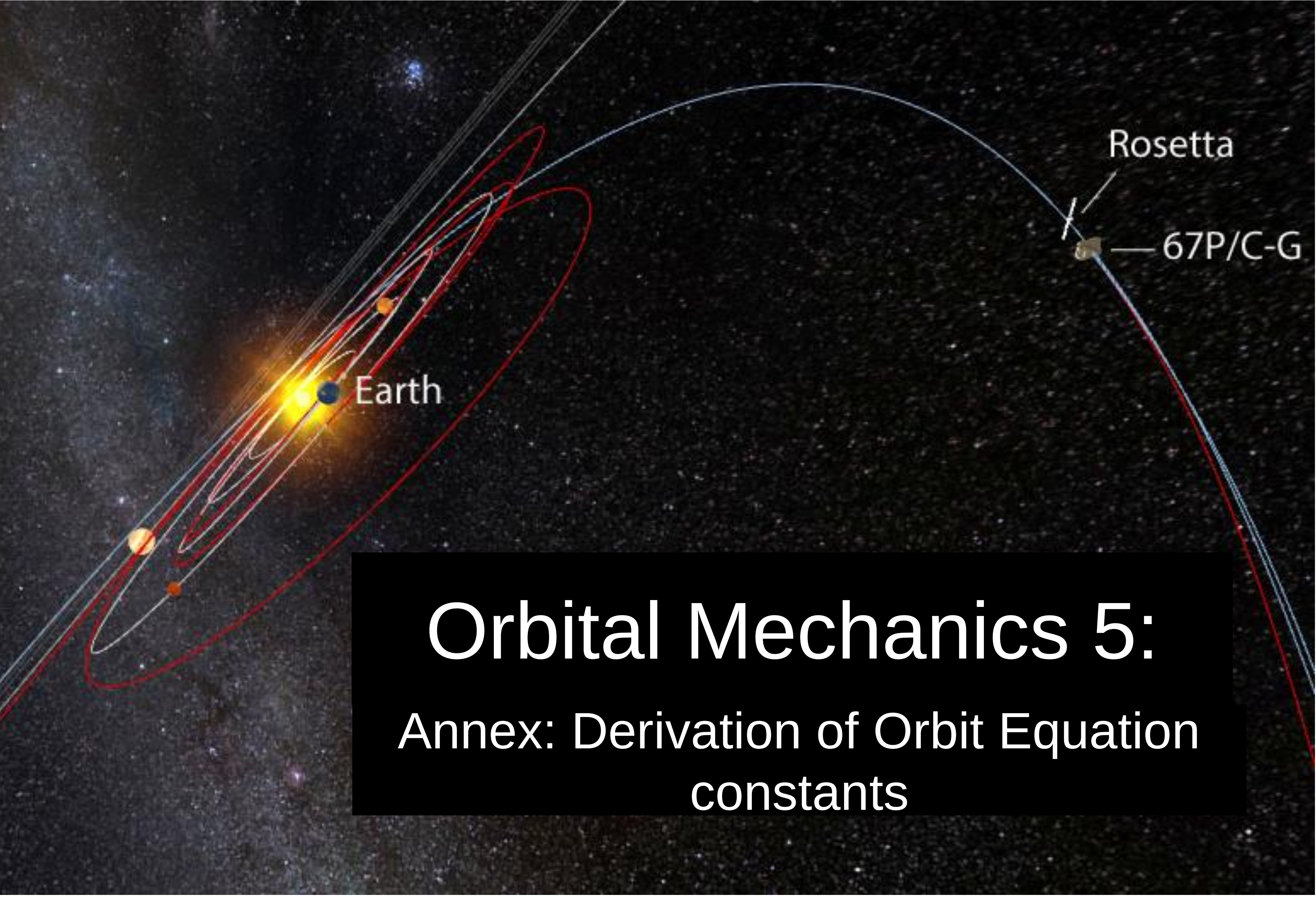


Orbital Mechanics 5:

Annex: Derivation of Orbit Equation constants

Conservation of Energy and the Orbit Equation

- So how can we use the conservation of energy to solve (5-9)?

$$u(\theta) = \frac{\mu}{h^2} + A\cos(\theta - \theta_0) \qquad \text{(5-9)}$$

- It will be useful later to have an expression for $\frac{du}{d\theta}$, so we will define it now:

$$\frac{du}{d\theta} = -A\sin(\theta - \theta_0) \qquad \text{(5A-1)}$$

Conservation of Energy and the Orbit Equation

- From the conservation of energy we have:

$$E = \frac{1}{2}mv^2 - \frac{\mu m}{r}$$

- Now, let us consider the energy $'\varepsilon'$: of a unit mass moving in a gravitational potential field:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v^2}{2} - \mu u \qquad \text{(5A-2)}$$

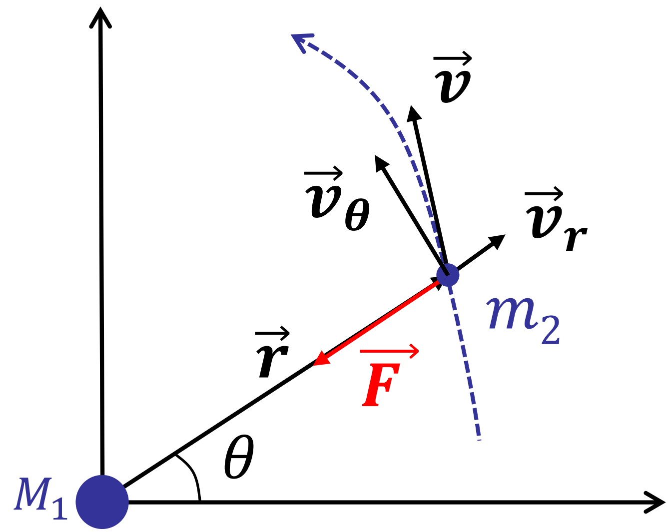
- Substituting (5-9) for u, gives:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v^2}{2} - \frac{\mu^2}{h^2} - \mu A\cos(\theta - \theta_0), \qquad \text{(5A-3)}$$

which we will use later, after we've found an expression for v^2…

Conservation of Energy and the Orbit Equation

- Remember the problem we are solving:



- Thus, we can write:

$$v^2 = v_\theta^2 + v_r^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 \qquad \text{(5A-4)}$$

Conservation of Energy and the Orbit Equation

$$v^2 = v_\theta^2 + v_r^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 \quad \text{(5A-4)}$$

- Substituting in for $\dot{r}$ (5-6) and $\dot{\theta}$ (5-5) and $r = 1/u$:

$$v^2 = \left(-h\frac{du}{d\theta}\right)^2 + \left(\frac{hu^2}{u}\right)^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] \quad \text{(5A-5)}$$

- We have already seen that:

$$\frac{du}{d\theta} = -A\sin(\theta - \theta_0) \quad \text{and} \quad u = \frac{\mu}{h^2} + A\cos(\theta - \theta_0) \quad \text{(5-9)}$$

- So:

$$\left(\frac{du}{d\theta}\right)^2 = A^2\sin^2(\theta - \theta_0) \quad \text{(5A-6)}$$

$$u^2 = \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}A\cos(\theta - \theta_0) + A^2\cos^2(\theta - \theta_0) \quad \text{(5A-7)}$$

Conservation of Energy and the Orbit Equation

- Substituting back into (5A-5):

$$\boxed{\left(\frac{du}{d\theta}\right)^2 = A^2\sin^2(\theta-\theta_0)} \qquad \text{(5A-6)}$$

$$\boxed{u^2 = \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}A\cos(\theta-\theta_0) + A^2\cos^2(\theta-\theta_0)} \qquad \text{(5A-7)}$$

$$v^2 = h^2\left[\boxed{\left(\frac{du}{d\theta}\right)^2} + \boxed{u^2}\right] \qquad \text{(5A-5)}$$

$$\frac{v^2}{h^2} = A^2\sin^2(\theta-\theta_0) + \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}A\cos(\theta-\theta_0) + A^2\cos^2(\theta-\theta_0)$$

$$\frac{v^2}{h^2} = \frac{\mu^2}{h^4} + \frac{2\mu}{h^2}A\cos(\theta-\theta_0) + A^2\,\underbrace{\left(\cancel{\sin^2}(\theta-\theta_0) + \cancel{\cos^2}(\theta-\theta_0)\right)}_{=1}$$

$$v^2 = \frac{\mu^2}{h^2} + 2\mu A\cos(\theta-\theta_0) + A^2h^2 \qquad \text{(5A-8)}$$

Conservation of Energy and the Orbit Equation

- Substituting (5A-8) back into our expression for ε, (5A-3):

$$v^2 = \frac{\mu^2}{h^2} + 2\mu A\cos(\theta-\theta_0) + A^2h^2 \qquad \text{(5A-8)}$$

$$\varepsilon = \frac{v^2}{2} - \mu\frac{\mu}{h^2} - \mu A\cos(\theta-\theta_0) \qquad \text{(5A-3)}$$

$$\varepsilon = \frac{\frac{\mu^2}{h^2} + 2\mu A\cos(\theta-\theta_0) + A^2h^2}{2} - \frac{\mu^2}{h^2} - \mu A\cos(\theta-\theta_0) \qquad \text{(5A-9)}$$

$$\varepsilon = \frac{\mu^2}{2h^2} + \cancel{\mu A\cos(\theta-\theta_0)}^{\,c} + \frac{A^2h^2}{2} - \frac{\mu^2}{h^2} - \cancel{\mu A\cos(\theta-\theta_0)}^{\,c} \qquad \text{(5A-10)}$$

$$\varepsilon = \frac{A^2h^2}{2} - \frac{\mu^2}{2h^2} \qquad \text{(5A-11)}$$

Conservation of Energy and the Orbit Equation

- Now we can solve (5A-11) for A:

$$\varepsilon = \frac{A^2h^2}{2} - \frac{\mu^2}{2h^2} \longrightarrow A = \frac{\mu}{h^2}\sqrt{1 + 2\varepsilon\frac{h^2}{\mu^2}} \qquad \text{(5A-12)}$$

- It can also be shown that eccentricity, e, can be expressed as:

$$e = \sqrt{1 + 2\varepsilon\frac{h^2}{\mu^2}} \qquad \text{(5A-13)}$$

- So: $A = e\dfrac{\mu}{h^2}$ and so… $\varepsilon = \dfrac{\mu^2 e^2}{2h^2} - \dfrac{\mu^2}{2h^2}$ (5A-14)

$$\varepsilon = (e^2 - 1)\frac{\mu^2}{2h^2} \qquad \text{(5A-15)}$$

Conservation of Energy and the Orbit Equation

$$\varepsilon = (e^2 - 1)\frac{\mu^2}{2h^2} \qquad \text{(5A-15)}$$

- This is an important result, relating the eccentricity of an orbit to its total energy, we will return to this relationship shortly.
- First, we will substitute our new expression for A, back into our equation of motion (5-9):

$$A = e\frac{\mu}{h^2} \qquad \text{(5A-14)}$$

$$u = \frac{\mu}{h^2} + A\cos(\theta - \theta_0) \qquad \text{(5-36)}$$

$$u = \frac{\mu}{h^2} + e\frac{\mu}{h^2}\cos(\theta - \theta_0) = \frac{\mu}{h^2}[1 + e\ \cos(\theta - \theta_0)] \qquad \text{(5A-16)}$$

Conservation of Energy and the Orbit Equation

$$u = \frac{\mu}{h^2}[1 + e\ \cos(\theta - \theta_0)] \qquad \text{(5A-16)}$$

- Finally, we can substitute back for r:

$$r = \frac{h^2/\mu}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-17)}$$

i.e. we now have an expression for r as a function of θ!

- Also, remember the expression for a conic section:

$$r = \frac{p}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-18)}$$

i.e. the motion of r is a conic section with “parameter p”

$$p = h^2/\mu \qquad \text{(5A-19)}$$

Conservation of Energy and the Orbit Equation

$$r = \frac{p}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-18)}$$

If we substitute in for θ=0 and π, and we know for an ellipse:

$$2a = r_{\theta=0} + r_{\theta=\pi} = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2}$$

Rearranging we have: $p = a(1 - e^2)$ (5A-19)

So for an ellipse:

$$r = \frac{a(1-e^2)}{1 + e\ \cos(\theta - \theta_0)} \qquad \text{(5A-20)}$$