Handout 3 – Constitutive Model

After studying the properties of stress and strain, we turn our attention to the relationship *between* stress and strain, to formulate the *constitutive equations*, or *material model*. Many engineering materials are considered:

homogeneous:	properties are the same in all locations;
isotropic:	properties are the same in all directions;
linear-elastic:	there is a linear relationship between strains and stresses.

These assumptions only a cover a subset of potential material properties. For instance, fibre-reinforced composites are not homogeneous (distinct fibres and matrix) or isotropic (material properties can be tailored in different directions) — these aspects will be covered in StM3 Composite Laminate Analysis.

After formulating a generalised Hooke's Law for a linear-elastic material, we formulate the constitutive equations for isotropic materials under plane stress, and derive expressions for a material shear and bulk modulus.

3.1 Generalized Hooke's Law

First, consider a general three-dimensional state of stress and strain, defined by the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}$, and corresponding strain tensor $\bar{\varepsilon}$ (with *mathematical* shear strain $\varepsilon_{xy} = \gamma_{xy}/2$):

$$\bar{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \qquad \bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

Assuming a linear material behaviour (*i.e.* the strains are a linear combination of the stress components) we write a 6×6 compliance matrix $\boldsymbol{S}$:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

with 36 material constants S_{ij}, $i, j = 1 - 6$; this is known as the Voigt notation.

The *Maxwell-Betti reciprocal theorem* shows that the compliance matrix must be symmetric, *i.e.* $S_{ij} = S_{ji}$, which reduces the number of elastic constants to 21. This result follows from a thought experiment.

An applied stress σ_{xx} results in elastic strain energy:

$$U_a = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2}S_{11}\sigma_{xx}^2$$

Applying a second stress σ_{yy}, while *maintaining* the first stress σ_{xx} gives:

$$U_b = \frac{1}{2}S_{22}\sigma_{yy}^2 + \sigma_{xx}S_{12}\sigma_{yy}$$

The total elastic strain energy of both stresses becomes:

$$U_a + U_b = \frac{1}{2}S_{11}\sigma_{xx}^2 + \frac{1}{2}S_{22}\sigma_{yy}^2 + \sigma_{xx}S_{12}\sigma_{yy}$$

Repeating the process, but first applying σ_{yy} followed by σ_{xx}, results in:

$$U = \frac{1}{2}S_{22}\sigma_{yy}^2 + \frac{1}{2}S_{11}\sigma_{xx}^2 + \sigma_{yy}S_{21}\sigma_{xx}$$

The total strain energy must be independent of the order in which the loads are applied; therefore $S_{12} = S_{21}$, and in general $S_{ij} = S_{ji}$. Similarly, the stiffness matrix in Finite Element Analysis must be symmetric.

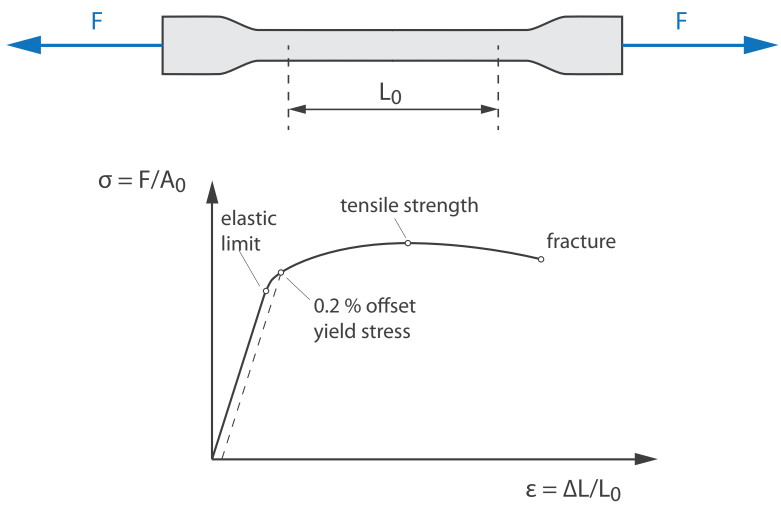
The next step, which is outside the scope of this course, relies on symmetry considerations: for an isotropic material the material properties must be the same in any direction, and therefore any plane is a symmetry plane. Applying those symmetry operations results in the following compliance matrix:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

with only two independent compliances S_{11} and S_{12}. For other material properties, *e.g* orthotropic, the symmetry conditions will give a different format for the compliance matrix.

3.2 Isotropic, Linear-Elastic Materials under Plane Stress

We derive the constitutive equations for an isotropic, linear-elastic material under *plane stress* using an engineering approach, based on experimental observations rather than mathematical derivations.



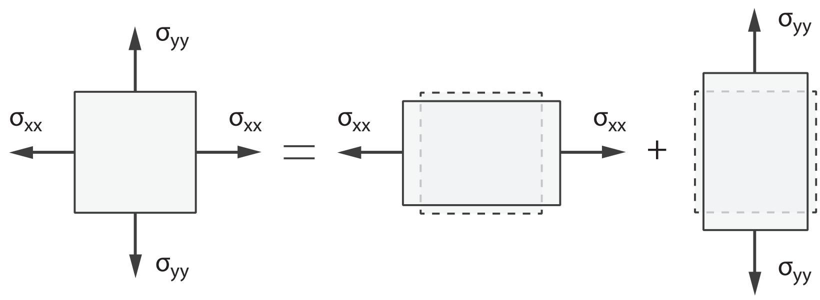
A uni-axial applied direct stress σ_{xx} (with $\sigma_{yy} = \sigma_{zz} = 0$) results in the following strains:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{yy} = -\nu\varepsilon_{xx} = -\nu\frac{\sigma_{xx}}{E}$$

characterised by two material parameters: Young's modulus E and Poisson's ratio ν. The Poisson's ratio is the negative ratio of the transverse and direct strain, for a uniaxial applied stress. For many engineering materials $\nu \approx 0.3$, but we shall derive theoretical bounds later in this handout.

A general bi-axial state of stress can be regarded as a *linear superposition* of two uni-axial stress states.

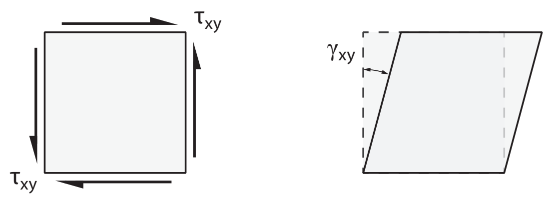


The resulting strains are superimposed:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \\ \varepsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right) \end{aligned} \tag{3.1}$$

and can be inverted to express stress as a function of strain:

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) \end{aligned} \tag{3.2}$$



The shear deformations are determined by the shear modulus G:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{3.3}$$

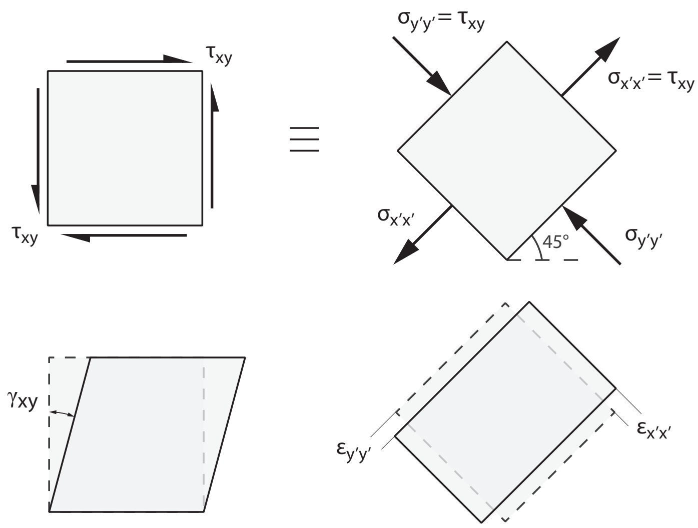
Combining Equations 3.1 and 3.3 into a matrix formulation, gives the compliance matrix:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

It is important to observe that for *isotropic* materials the direct stress σ_{xx} and σ_{yy} do not produce shear strains γ_{xy} with respect to the XY axes. Conversely, a shear stress τ_{xy} will not produce direct strains. This is reflected in the zero coupling terms in the compliance matrix.

3.2.1 Elastic Modulus: Shear Modulus

For isotropic materials the shear modulus G and Young's modulus E are not independent. The relationship between the two elastic moduli is found by observing that pure shear is equivalent to bi-axial tension and compression at 45° to the direction of shear (see Example 1.4). The strains must therefore also be equivalent.



For the case of pure shear, the resulting shear strain:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

can be transformed to the 45° $X'Y'$ coordinate system using the strain transformation equations:

$$\begin{aligned}\varepsilon_{x'x'} &= \sin\theta\cos\theta\,\gamma_{xy} \\ &= \frac{\tau_{xy}}{2G}\end{aligned}$$

The equivalent direct stresses, $\sigma_{x'x'} = \tau_{xy}$ and $\sigma_{y'y'} = -\tau_{xy}$, result in the following strain:

$$\begin{aligned}\varepsilon_{x'x'} &= \frac{1}{E}\left(\sigma_{x'x'} - \nu\sigma_{y'y'}\right) \\ &= \frac{\tau_{xy}}{E}\left(1+\nu\right)\end{aligned}$$

Equating both strains expresses the shear modulus G in terms of E and ν:

$$G = \frac{E}{2\left(1+\nu\right)} \tag{3.4}$$

3.2.2 Plane Stress Stiffness and Compliance Matrices

The material law, *i.e.* the relationship between stress and strain, for a linear-elastic, homogeneous and isotropic material under plane stress can now be described using the following **compliance**

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{3.5}$$

and **stiffness** matrices

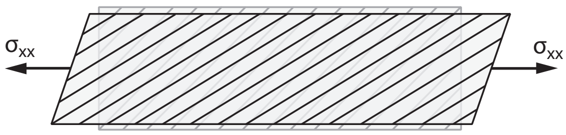
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{3.6}$$

with two material parameters, E and ν.

Other formulations may be found in literature, using the shear modulus G and bulk modulus K as independent parameters (physicists), or the Lamé parameters μ and λ (mathematicans). Most engineering texts, however, will use E and ν as independent parameters.

Note that for isotropic, elastic materials, the principal directions for stress and strain coincide. This is a result of the decoupling between shear and direct strains and is therefore not necessarily the case for anisotropic materials.

Anisotropic Materials For *anisotropic* materials such as fibre reinforced composites, there may exist a coupling between direct and shear effects. For example, consider a composite plate with fibres set an angle to the direction of uni-axial loading.



In this case the direct stresses will result in both direct and shear strains. This means that the compliance matrix will be fully populated:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

where the coupling is described using the $\bar{S}_{16}$ and $\bar{S}_{26}$ matrix terms (the $\bar{S}_{ij}$ notation is introduced in Composite Laminate Analysis in StM3). Such properties are more complicated to characterise, but also provides opportunities for interesting structural behaviour!

Example 3.1 – Plane Stress vs Plane Strain

Using the constitutive equations, we return to a comparison of plane stress and plane strain, and look at the through-thickness strains and stresses.

plane stress: in the case of plane stress $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$. However, the through-thickness strain is non-zero and is given by:

$$\varepsilon_{zz} =$$

$$=$$

In other words, the thickness of the thin-walled structure will change under the applied loads.

plane strain: for plane strain $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$. However, the out-of-plane stresses will be non-zero to satisfy those plane strain conditions:

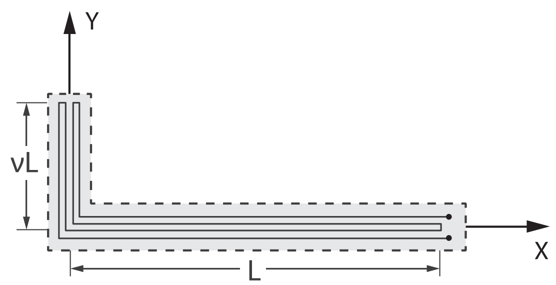
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = 0$$

and therefore:

$$\sigma_{zz} =$$

Example 3.2 – Stress Gauge

Returning to stress and strain measurement, to explore the idea of a *stress* gauge. Consider the following L-shaped strain gauge, where the length of the short leg is νL.



The gauge reading ΔR will be proportional to the change in length, $L\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$. Using Hooke's law:

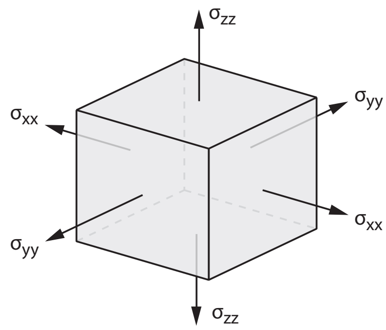
$$\Delta R \propto \frac{L}{E}\left[\sigma_{xx} - \nu\sigma_{yy} + \nu\left(\sigma_{yy} - \nu\sigma_{xx}\right)\right] = \frac{L\left(1-\nu^2\right)}{E}\sigma_{xx}$$

and thus the reading of the stress gauge is proportional to σ_{xx}.

The obvious downside of a stress gauge is that the Poisson's ratio of the material you are measuring must be known accurately *a priori*, making these not much more than an interesting exercise.

3.2.3 Elastic Modulus: Bulk Modulus

The material *bulk modulus* describes how much a material will compress under an external pressure. Consider an infinitesimal element (dimensions $dx \times dy \times dz$) with direct stresses σ_{xx}, σ_{yy} and σ_{zz}.



The element will deform under the applied loads, and its deformed volume V is:

$$\begin{aligned} V &= (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\; dx\,dy\,dz \\ &= (1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}+\text{higher order terms})\; dx\,dy\,dz \\ &= (1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})\; dx\,dy\,dz \end{aligned}$$

Note that shear stresses (and thus shear strains) will not result in a change of volume. The volumetric, or dilational, strain describes the change in volume of the infinitesimal element:

$$\begin{aligned} \frac{\Delta V}{V_0} = \frac{V - V_0}{V_0} &= \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} \\ &= \frac{(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})(1-2\nu)}{E} \end{aligned}$$

where we substituted Hooke's law:

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy}+\sigma_{zz}\right)\right]; \qquad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx}+\sigma_{zz}\right)\right]; \qquad \varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx}+\sigma_{yy}\right)\right]$$

For a state of *spherical stress* ($\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$) the volumetric strain is:

$$\frac{\Delta V}{V_0} = \frac{3\sigma(1-2\nu)}{E}$$

The bulk modulus K relates the spherical stress to the volumetric strain:

$$\sigma = K\frac{\Delta V}{V_0}$$

which gives the following expression

$$K = \frac{E}{3(1-2\nu)} \tag{3.7}$$

for the bulk modulus, in terms of E and ν.

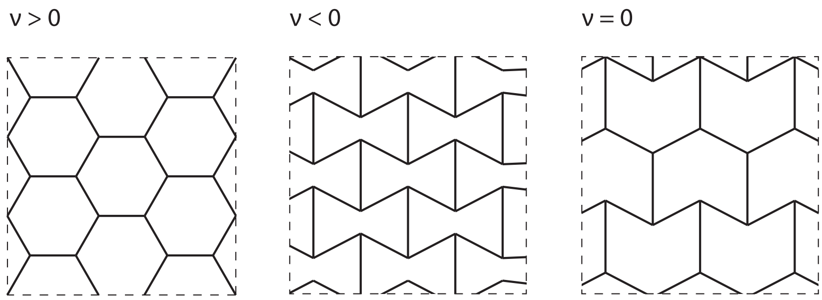
3.2.4 Poisson's ratio

The expressions for the shear modulus (Equation 3.4) and bulk modulus (Equation 3.7) of an isotropic material provide insight into the allowable range of values for Poisson's ratio. The shear and bulk modulus must both be positive and finite, which respectively provide a lower and upper bound:

$$\nu \in \langle -1, 0.5 \rangle$$

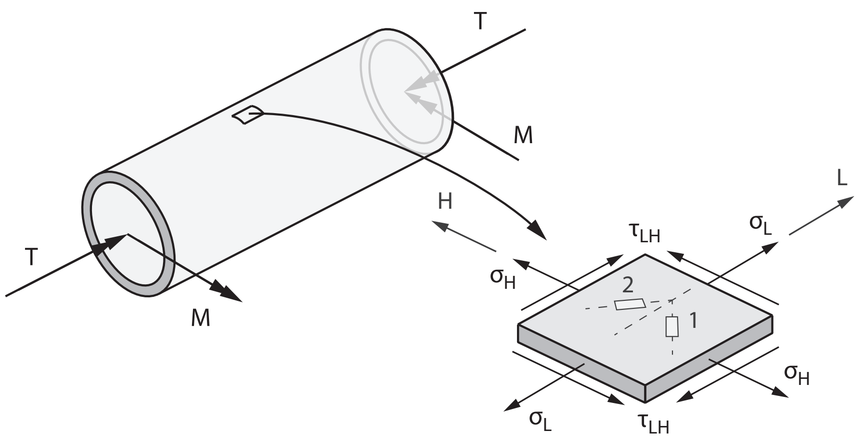
In fact, for most engineering materials $\nu \in [0.2, 0.5\rangle$, and for most metallics $\nu \approx 0.3$. For many rubbery materials $\nu \approx 0.5$, and these are considered effectively incompressible due to their high bulk modulus.

Materials known as *auxetics* have a negative Poisson's ratio (and thus expand transversely under uni-axial tension) as a result of their micro-structure.



Example 3.3 – Cylindrical shaft under combined loading

A uniform, cylindrical, thin-walled tube is subjected to a torque T and a pure bending moment M. The strain is measured at angles of $\pm 45°$ to the longitudinal axis, and the strain gauges lie in the plane of the applied bending moments. The tube has a mean radius $R = 125$ mm and a wall thickness $t = 1.63$ mm. The tube material is aluminium, with $E = 70$ GPa and $\nu = 0.3$.



Q: What are the values of the applied torque T and bending moment M when the gauges give tensile strain readings of 823 $\mu\varepsilon$ and 242 $\mu\varepsilon$ respectively?

A: Two strain gauges are not sufficient to uniquely determine the state of stress at a point, unless extra information can be derived from the loading conditions. The thin-walled element where the strain gauges are attached is subjected to a longitudinal stress σ_L due to M, and a shear stress τ_{LH} as a result of T, but no hoop stresses are applied.

The strain transformation equation can therefore be written as:

$$\begin{aligned}\varepsilon_\theta &= \varepsilon_L \cos^2\theta + \varepsilon_H \sin^2\theta + \gamma_{LH}\sin\theta\cos\theta \\ &= \varepsilon_L\left(\cos^2\theta - \nu\sin^2\theta\right) + \gamma_{LH}\sin\theta\cos\theta\end{aligned}$$

where it was observed that:

$$\varepsilon_H =$$

Substitute the strain gauge measurements ($\theta_1 = 45°$, $\varepsilon_1 = 823\cdot 10^{-6}$; $\theta_2 = -45°$ $\varepsilon_2 = 242\cdot 10^{-6}$):

$$\begin{aligned}\varepsilon_1 &= \varepsilon_L\,(0.5 - 0.5\,\nu) + 0.5\,\gamma_{LH} = 823\mu\varepsilon \\ &= 0.35\,\varepsilon_L + 0.5\,\gamma_{LH} \\ \varepsilon_2 &= \varepsilon_L\,(0.5 - 0.5\,\nu) - 0.5\,\gamma_{LH} = 242\mu\varepsilon \\ &= 0.35\,\varepsilon_L - 0.5\,\gamma_{LH}\end{aligned}$$

to obtain two linear equations, which are solved to find:

$$\begin{aligned}\varepsilon_L &= \\ \varepsilon_H &= \\ \gamma_{LH} &= \end{aligned}$$

Using Hooke's law:

$$\begin{bmatrix}\sigma_L\\ \sigma_H\\ \tau_{LH}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_L\\ \varepsilon_H\\ \gamma_{LH}\end{bmatrix}$$

provides the stresses:

$$\begin{aligned}\sigma_L &= \\ \sigma_H &= 0 \text{ MPa} \\ \tau_{LH} &= \end{aligned}$$

The applied loads can now be back-calculated from the stresses (using $I = \frac{J}{2} \approx \pi R^3 t$) to give:

$$\begin{aligned}M &= \frac{\sigma_L\, I}{R} = \\ T &= \frac{\tau_{LH}\, J}{R} = \end{aligned}$$

3.3 Summary

In this handout the Generalized Hooke's Law was derived for a homogeneous, isotropic, linear-elastic material under a state of plane stress. The shear modulus G was derived in terms of two independent elastic properties E and ν, by equating the elastic strain energy for a state of pure shear, and that for the equivalent bi-axial tensile/compressive state. The bulk modulus K, which relates the change in material volume to the spherical stress, was derived. The equations for the shear and bulk modulus provide bounds on possible values for Poisson's ratio ν in isotropic materials.

This handout completes our description of plane stress: stress transformation equations (Handout 1), strain transformation equations (Handout 2), and the constitutive equations that relate stress and strain for homogeneous, isotropic, linear-elastic materials (Handout 3). The next handout will focus on using this information to consider material failure criteria.

Revision Objectives Handout 3:

- derive and recall the constitutive equations for isotropic, linear-elastic materials under plane stress

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

- explain why direct strains and shear strains are decoupled for isotropic materials;
- calculate strains due to applied stresses, and calculate stresses from measured strains;
- derive and recall expressions for shear modulus G and bulk modulus K in terms of E and ν

$$G = \frac{E}{2\,(1+\nu)} \qquad K = \frac{E}{3\,(1-2\nu)}$$