EMAT10100 Engineering Maths I Lectures 7&8, Introduction to Probability: Important Practical Probability Distributions

Rosalyn Moran & Nikolai Bode

Overview

Natural and engineering processes often follow a known distribution. It is important to learn how to recognise important probability distributions:

- Discrete distributions
  - Bernoulli distribution
  - Geometric distribution
  - Binomial distribution
  - Poisson distribution
- Continuous distributions
  - Exponential distribution
  - Normal distribution

We will discuss the discrete distributions today and the continuous distributions next time.

Bernoulli Trial

Random experiment with exactly two possible outcomes.

- Examples:
  - either success or failure,
  - either heads or tails,
  - either win or lose,
  - either even or odd.

We will look at how to characterise mathematically this type of experiments via the so-called Bernoulli distribution, named after the Swiss scientist Jacob Bernoulli (1654-1705)

Bernoulli distribution

- We denote the success probability by p.
- The failure probability must be $1-p$.
- Let X be the random variable that equals 1 if the outcome is a success, 0 if it is a failure.

$\Rightarrow$ The probability function is

$$P_X(1) = P(X=1) = p,$$
$$P_X(0) = P(X=0) = 1-p.$$

This is arguably the simplest conceivable probability distribution for a truly random experiment.

What are its properties?

Mean and variance of the Bernoulli distribution

- Mean:

$$\mu = \mathrm{E}(X) = \sum_{n=0}^{1} [n\, P_X(n)] =$$
$$(0 \times (1-p)) + (1 \times p) = p.$$

- Variance:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = \sum_{n=0}^{1} [n^2\, P_X(n)] - p^2 =$$
$$(0 \times (1-p)) + (1 \times p) - p^2 = p(1-p).$$

Bernoulli process

- A Bernoulli process is a sequence of Bernoulli trials.
- The trials are independent.
- All trials have the same success probability.

Example: 30 trials, $p = 0.5$ (1=success, 0=failure)
1 1 0 1 0 1 0 0 1 0 1 1 0 1 0 1 0 1 1 0 0 1 0 0 0 1 1 0 0 0.

The properties of such random sequences pose many interesting questions. For example,

- How many trials do I need until the first success?
- How many successes do I have in n trials?

Trials until first success

Let the random variable X be the number of trials until (and including) the first success.

If $X = n$, we must have $n-1$ failures followed by 1 success.

All trials are independent, so we can multiply the probabilities.

$$P(X = n) = \underbrace{(1-p) \times (1-p) \times \ldots \times (1-p)}_{n-1 \text{ factors}} \times p$$
$$= (1-p)^{n-1} p.$$

This is called a *geometric distribution*.

Geometric distribution - Formulae

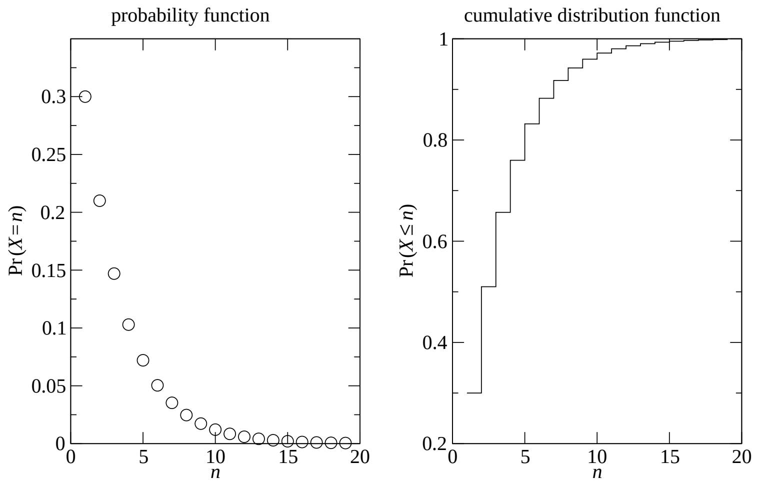
- probability function:
  $P_X(n) = P(X = n) = p(1-p)^{n-1}$

- cumulative distribution function:

$$F_X(n) = P(X \le n) = \sum_{k=1}^{n} P(X = k) = p \sum_{k=1}^{n} (1-p)^{k-1}$$
$$= p \sum_{k=0}^{n-1} (1-p)^k \overset{(1)}{=} p \times \frac{1-(1-p)^n}{1-(1-p)}$$
$$= 1 - (1-p)^n$$

In step (1) we used the formula for the geometric series (p. 476 in James; 5th ed., pp. 482).

Geometric distribution - Plots



Plotted: $p = 0.3$.
For all $p > 0$, the mode (i.e. the most probable value) is $n = 1$.

Exercise: Geometric distribution in quality control

Electrical fuses from a production line fail independently with probability 0.01. The fuses are tested sequentially until the first fuse fails. What is the probability that the first faulty fuse will be found among the first twenty fuses that are tested?

Number of successes in n trials

Consider a Bernoulli process with successive outcomes $x_1, x_2, \ldots, x_n$.
If the i-th outcome is a failure, then $x_i = 0$. Otherwise $x_i = 1$.

What is the probability of exactly k successes?

If the successes must happen in the trials $1, 2, \ldots k$, and all other trials are failures, the probability of this Bernoulli process is

$$\underbrace{p^k}_{k \text{ independent successes}} \times \underbrace{(1-p)^{n-k}}_{n-k \text{ independent failures}} .$$

But in the question above we did not demand that all k successes happen at the beginning. So this is not quite the probability we are looking for.

Counting Bernoulli processes with k successes in n trials

One particular sequence with k successes:

trial	1	2	...	k	k+1	...	n
success?	1	1	...	1	0	...	0

There are $n!$ permutations of the trials.

But not all $n!$ permutations are different sequences. The k successes can be shuffled and still represent the same Bernoulli process.
$\Rightarrow$ We must divide by $k!$ (i.e. the number of permutations of the successes).

And the same is true for the $n-k$ failures.
$\Rightarrow$ We must divide by $(n-k)!$.

$\Rightarrow$ There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ different Bernoulli processes with k successes in n trials.

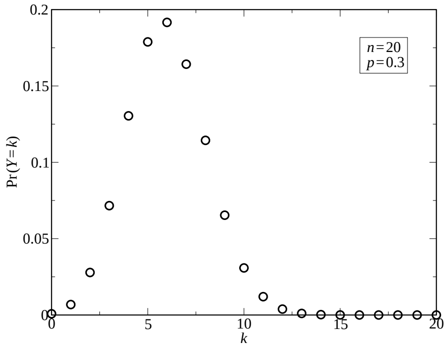
Binomial distribution

Let the random variable Y be the number of successes in n Bernoulli trials.

If $Y = k$, then there are $\binom{n}{k}$ distinct Bernoulli processes, each with a probability $p^k(1-p)^{n-k}$ and we can write:

$P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

This is called a *binomial distribution*.



Binomial distribution - Formulae

- Probability function: $P_Y(k) = P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}$.
- Mean: We could calculate $\mu = \sum_{k=0}^{n} [k\, P_Y(k)]$ directly.[1] But there is a simpler path. Y is the sum of n Bernoulli distributed random variables $X_1, \ldots, X_n$, each with mean p.

$$\Rightarrow \quad \mu = \mathrm{E}(Y) = \mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{E}(X_i) = np.$$

- Variance: $X_1, \ldots, X_n$ are independent and each has variance $p(1-p)$.

$$\Rightarrow \quad \mathrm{Var}(Y) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \stackrel{(*)}{=} \sum_{i=1}^{n} \mathrm{Var}(X_i) = np(1-p).$$

Step $(*)$ is only permitted because the X_i are independent.

[1] Try this calculation as homework!

Exercise: Application of the binomial distribution

Bits are sent over independent digital communication channels. The probability that one bit is corrupted is 2% on all channels. What is the probability that more than two bits are corrupted in a transmission of 16 bits?

Binomial distribution with rare successes and large samples

Suppose the probability of a successful Bernoulli trial is tiny, but we perform a huge number of trials.

Examples:

- Plane accidents in one year.
- Spelling mistakes in a new edition of a dictionary.
- Mutations in a particular stretch of DNA.
- Meteorites hitting the earth.

From binomial to Poisson distribution

Let's start with the Binomial distribution.
We know that $P(Y=k)=\binom{n}{k}p^k(1-p)^{n-k}$

To study what happens when $n\to\infty$ and $p\to 0$, we define $\epsilon=\frac{1}{n}$ and $\lambda=\frac{p}{\epsilon}$, then take the limit $\epsilon\to 0$:

$$P(Y=k)=\lim_{\epsilon\to 0}\left[\binom{n}{k}p^k(1-p)^{n-k}\right]=$$

$$\underbrace{\lim_{\epsilon\to 0}\left(\frac{1(1-\epsilon)\ldots(1-\epsilon(k-1))}{k!}\right)}_{1/k!}\lambda^k\times\underbrace{\lim_{\epsilon\to 0}(1-\epsilon\lambda)^{1/\epsilon}}_{\substack{\exp(-\lambda),\\ \text{James, p. 512}\\ \text{(5th ed., pp. 519)}}}\times\underbrace{\lim_{\epsilon\to 0}(1-\epsilon\lambda)^{-k}}_{1}$$

$$=\frac{\lambda^k}{k!}\exp(-\lambda).$$

This limit is called a *Poisson distribution* with rate parameter λ.

Poisson distribution

- Probability function:
  $P_Y(k)=P(Y=k)=\frac{\lambda^k}{k!}\exp(-\lambda).$
- Mean:

$$\mu=\mathrm{E}(Y)=\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}\exp(-\lambda)=$$

$$\lambda\exp(-\lambda)\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}=$$

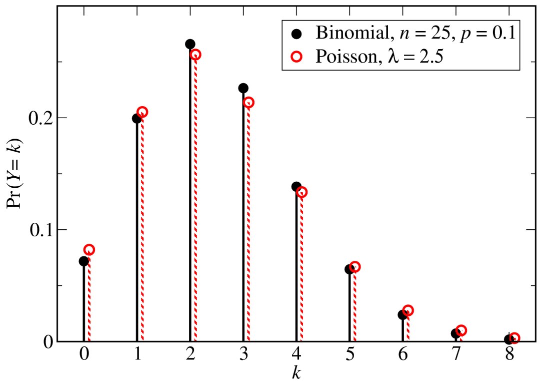
$$\lambda\exp(-\lambda)\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}_{\substack{=\exp(\lambda)\\ \text{James, p. 698 (5th ed., pp. 720)}}}=\lambda$$

Siméon Denis Poisson

- Variance:
  $\mathrm{Var}(Y)=\lambda$ (Calculation left as exercise.)

Poisson approximation to the binomial distribution

- Rule of thumb: a binomial distribution with $n\geq 25$ and $p\leq 0.1$ can be approximated by a Poisson distribution with $\lambda=np$.
- This approximation sometimes simplifies calculations.



Exampercise

Example 13.26 in James, p. 1016 (5th ed., pp. 1042):

If 0.04% of cars break down while driving through a certain tunnel, find the probability that at most two cars break down out of 2000 cars entering the tunnel on a given day

(a) using the exact binomial distribution,

(b) using the Poisson approximation.

Solution

University of BRISTOL
EngMaths I Probability 7&8
Spring Semester 2018

Exercise

Exercise 57 in James, p. 1028 (5th ed., pp. 1054):

A Geiger counter and a source of radioactive particles are so situated that the probability that a particle emanating from the radioactive source will be registered by the counter is $1/10\,000$. Assume that during the time of observation $30\,000$ particles emanated from the source. What is the probability that the number of particles registered was

(a) zero,

(b) three,

(c) more than five.

Compare the result of the exact binomial distribution with the Poisson approximation.

University of BRISTOL
EngMaths I Probability 7&8
Spring Semester 2018

Summary

distribution	parameters	prob. function
Bernoulli	success prob. p	$P(0) = 1 - p$, $P(1) = p$
geometric	success prob. p	$P(k) = p(1-p)^{k-1}$
binomial	success prob. p, number of trials n	$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Poisson	rate λ	$P(k) = \frac{\lambda^k}{k!} \exp(-\lambda)$