Vibrations 2, Lecture 4
Free damped vibration

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Lecture 3

Initial conditions

$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0 = v_0$$

Free undamped 1 DOF system vibrates harmonically at the frequency ω_0

$$x(t) = A\sin(\underline{\omega_0} t + \varphi)$$

Undamped angular natural frequency

$$\underline{\omega_0 = \sqrt{\frac{k}{m}}}$$

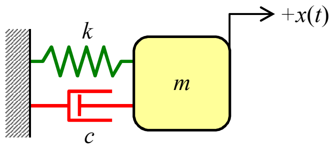
Dynamic equilibrium for rotational motion

$$\sum M_{i,A} + \underline{(-I_A \ddot{\varphi})} = 0$$

Lecture 4

- Free damped vibration of 1 DOF systems
- Classification of vibration responses
- Damping ratio and damped natural frequency
- Logarithmic decrement

1 DOF damped system



Equation of motion … $m\ddot{x} + c\dot{x} + kx = 0$

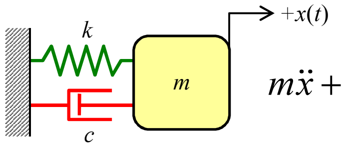
Initial conditions … $x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0 = v_0$

This is a *linear homogeneous differential equation* with constant coefficients.
This equation has the following *exponential solution*:

Trial solution … $x(t) = Ae^{st} = Ae^{(s_R + is_I)t} = Ae^{s_R t}e^{is_I t}$

s … is a complex number, *i* … is the imaginary unit
A … is an unknown constant

1 DOF damped system



$$m\ddot{x} + c\dot{x} + kx = 0$$

The trial solution:

$$x = Ae^{st} \Rightarrow \dot{x} = sAe^{st} \Rightarrow \ddot{x} = s^2 Ae^{st}$$

Substitute the above terms to the EOM and solve the *characteristic equation*:

$$(ms^2 + cs + k)Ae^{st} = 0 \qquad \Rightarrow \underline{ms^2 + cs + k = 0}$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

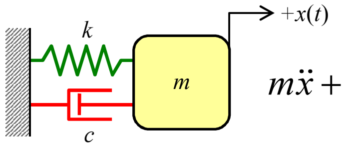
… two real solutions
… two complex solutions
… one (double) real solution

If we have the two distinct roots ($s_1 \neq s_2$) the total solution is:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

A_1,A_2 are two unknown constants

1 DOF damped system



$$m\ddot{x} + c\dot{x} + kx = 0$$

Rearrange $s_{1,2}$ into more convenient form:

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \left(\frac{c^2}{4m^2} - \frac{k}{m}\right)^{1/2}$$

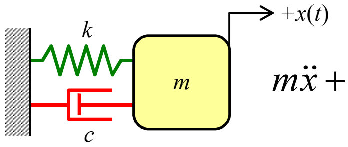
and move ω_0=k/m out of the square root:

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{k}{m}}\left(\frac{c^2}{4mk} - 1\right)^{1/2}$$

… two real solutions
… two complex solutions
… one (double) real solution

The sign of the expression in the square root determines the type of the solution.

1 DOF damped system



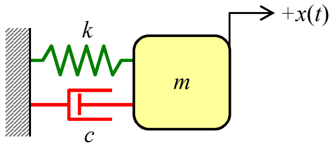
$$m\ddot{x}+c\dot{x}+kx=0$$

$c^2/(4mk)>1$	*two distinct real roots* → **over-damped system**
$c^2/(4mk)<1$	*two complex conjugate roots* → **under-damped system**
$c^2/(4mk)=1$	*one double real root* (s_1=s_2) → **critically damped system**

The ratio $c/2(mk)^{1/2}$ is: (a) dimensionless; (b) the last case, after rearranging, defines the boundary between the oscillatory and non-oscillatory behaviour, when $c=2(mk)^{1/2}$; (c) $2(mk)^{1/2}$ specifies the critical amount of damping c_{cr} at this boundary; and (d) $c/2(mk)^{1/2}=c/c_{cr}$ is the damping ratio ζ. Then, we write:

$$s_{1,2}=-\frac{c}{2m}\pm\sqrt{\frac{k}{m}}\left(\frac{c^2}{4mk}-1\right)^{1/2}=-\zeta\omega_0\pm\omega_0\left(\zeta^2-1\right)^{1/2}$$

1 DOF damped system



Underdamped motion

$$0<\zeta<1\Rightarrow s_{1,2}=-\zeta\omega_0\pm i\,\omega_0\sqrt{1-\zeta^2}$$

… use Euler formulas

$$x(t)=A_1e^{(-\zeta\omega_0+i\,\omega_D)t}+A_2e^{(-\zeta\omega_0-i\,\omega_D)t}=\ldots=Ae^{-\zeta\omega_0 t}\cos(\omega_D t-\phi)$$

$$\omega_D=\omega_0\sqrt{1-\zeta^2}$$

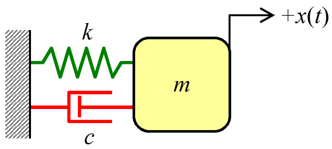
Exponentially decaying function due to damping

Harmonic motion with the frequency ω_D

Phase lag due to damping

Try Matlab script: » vib2_1dof_freevisc

1 DOF damped system



Overdamped motion

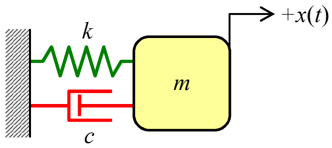
$$\zeta > 1 \Rightarrow s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$$x = A_1 e^{(-\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1})t} + A_2 e^{(-\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1})t} = e^{-\zeta\omega_0 t}\left(A_1 e^{\omega_0\sqrt{\zeta^2-1}\,t} + A_2 e^{-\omega_0\sqrt{\zeta^2-1}\,t}\right)$$

Exponentially decaying *aperiodic* motion!

Real non-oscillatory function.

1 DOF damped system



Critically damped motion

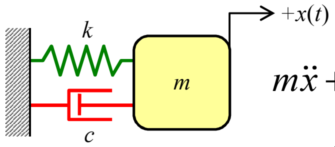
$$\zeta = 1 \Rightarrow s_{1,2} = -\omega_0$$

This is the special solution type ($a_1e^{st}+ta_2e^{st}$) due to having one "double" real root

$$x = (A_1 + t\,A_2)\,e^{-\omega_0 t}$$

Exponentially decaying response.

1 DOF damped system summary



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

Dynamic parameters:

$\boxed{\omega_D = \omega_0\sqrt{1-\zeta^2}}$ is the angular damped *natural frequency* [rad/s].

$\boxed{\zeta = \frac{c}{2\sqrt{mk}}}$ is the *damping ratio* [-].

Vibrations and Matlab simulations

- Matlab solves systems of 1st order ODEs
- We have to transform our 2nd order ODE to two 1st order ODEs

$$m\ddot{x} + c\dot{x} + kx = F \quad \Rightarrow \ddot{x} = -(c/m)\dot{x} - (k/m)x + F/m$$

$$dx/dt = \dot{x}$$
$$d\dot{x}/dt = -(c/m)\dot{x} - (k/m)x + F/m$$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ F(t)/m \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{f}(t)$$

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>> % For numerical integration see in Matlab's ode23, ode45, ode23s, …
>> vib2_1dof_freevisc
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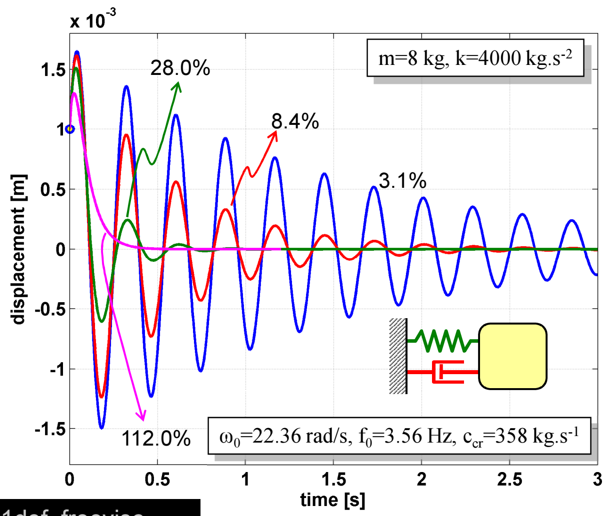
Matlab simulations

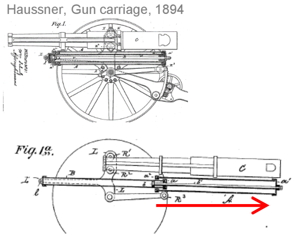


» vib2_1dof_freevisc

From "Gun Carriage" to vertical landing

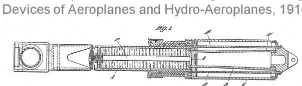
History

Haussner, Gun carriage, 1894

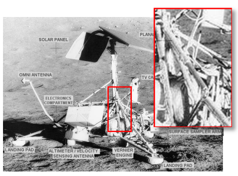


aviationancestry.co.uk

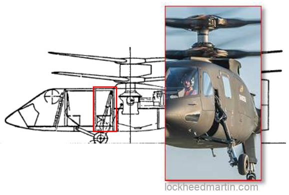
Duncan, Improvements in or relating to the Landing Devices of Aeroplanes and Hydro-Aeroplanes, 1916



Vertical landing applications



en.wikipedia.org/wiki/Surveyor_program

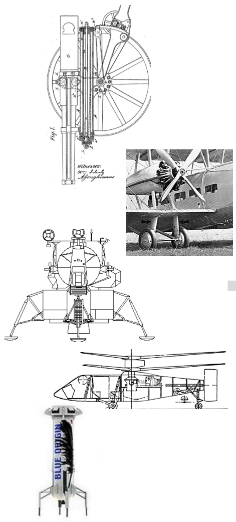


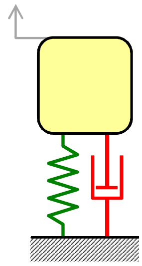
lockheedmartin.com

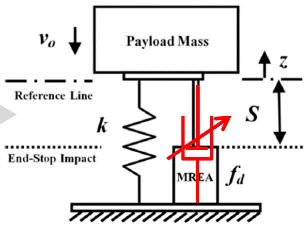
i.redd.it/7o8urj8zgalx.jpg

historicspacecraft.com

Models for initial conceptual analysis







Singh, Wereley, Adaptive magnetorheological shock isolation mounts for drop-induced impacts, 2013

Experimental identification of damping

We can *identify* damping from the *free response* using *Logarithmic Decrement* (LogDec). Consider the *free response* of 1 DOF damped system in t_1 and t_1+T_D, $T_D=2\pi/\omega_D$ is the period of damped motion:

$$\frac{x(t_1)}{x(t_1+T_D)} = \frac{X\,e^{-\zeta\omega_0 t_1}\sin(\omega_D t_1+\varphi)}{X\,e^{-\zeta\omega_0(t_1+T_D)}\sin(\omega_D(t_1+T_D)+\varphi)} = e^{\zeta\omega_0 T_D}$$

$$\Lambda = \ln\left(\frac{x(t_1)}{x(t_1+T_D)}\right) = \zeta\omega_0 T_D = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta \qquad \boxed{\zeta_{\exp} \approx \frac{1}{2\pi}\ln\left(\frac{x(t_1)}{x(t_1+T_D)}\right)}$$

Alternatively, consider the two displacements separated by N periods T_D:

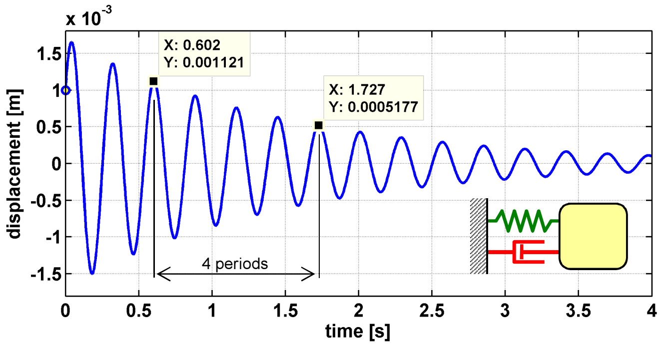
$$\frac{x(t_1)}{x(t_1+NT_D)} = \frac{x(t_1)}{x(t_1+T_D)}\frac{x(t_1+T_D)}{x(t_1+2T_D)}\cdots\frac{x(t_1+(N-1)T_D)}{x(t_1+NT_D)} = e^{N\zeta\omega_0 T_D}$$

$$\Lambda_N = \frac{1}{N}\ln\left(\frac{x(t_1)}{x(t_1+NT_D)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta \qquad \boxed{\zeta_{\exp} \approx \frac{1}{2\pi N}\ln\left(\frac{x(t_1)}{x(t_1+NT_D)}\right)}$$

Example: damping identification

Use vib2_1dof_freevisc to calculate the free response:

m=8 kg, c=11 N.s/m, k=4000 N/m
(ω_0=22.36 rad/s, ω_D=22.35 rad/s, T_D=0.28 s, ζ=3.07 %, c_{cr}=357.8 N.s/m)



Use the free response to "identify" the parameters:

$T_{D,e}=(X_{t+N.T}-X_t)/N=(1.727-0.602)/4\approx 0.28$ s
$\omega_{D,e}=2\pi/T_D=2\pi/0.28=22.34$ rad/s, $f_{D,e}=\dots$
$\zeta_e=(1/(2\pi\times N))\times\ln(Y_t/Y_{t+N.T})=(1/(2\pi\times 4))\times\ln(0.001121/0.000518)\approx 0.0307=3.07\ \%$

Summary

- Vibrating systems can be:
  - overdamped
  - critically damped
  - *underdamped*
- New parameters:
  - damped natural frequency
  - damping ratio
  - critical viscous damping
- LogDec is used to identify the damping ratio