Stress, Strain and Deformation

Torsion Stress, Strain & Deformation

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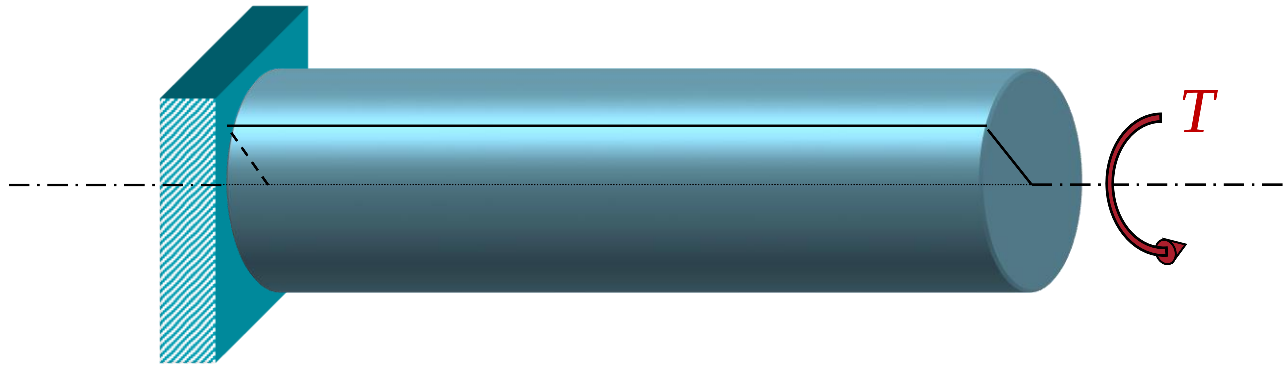
→ for circular shafts

Assumptions (1/2)

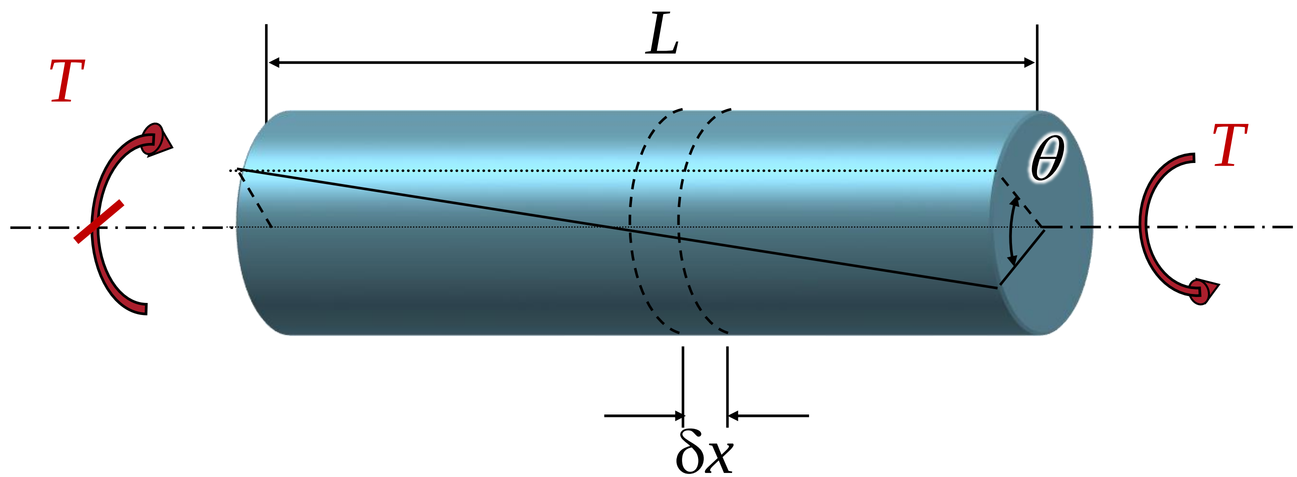
Uniform shaft
length L



Torque T
(constant along length)

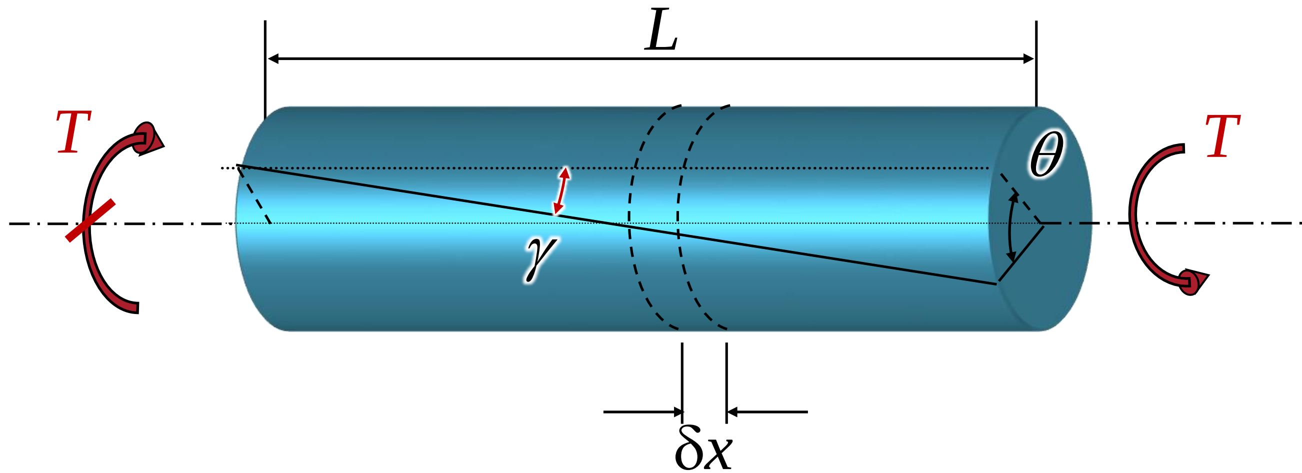
Twist θ over length L

FBD:



Assumptions (2/2)

- Linear elastic material
- "Free torsion" *i.e.* no warping restraint
- Cross-sections remain plane
- Radii remain straight
- Constant shear γ along length:



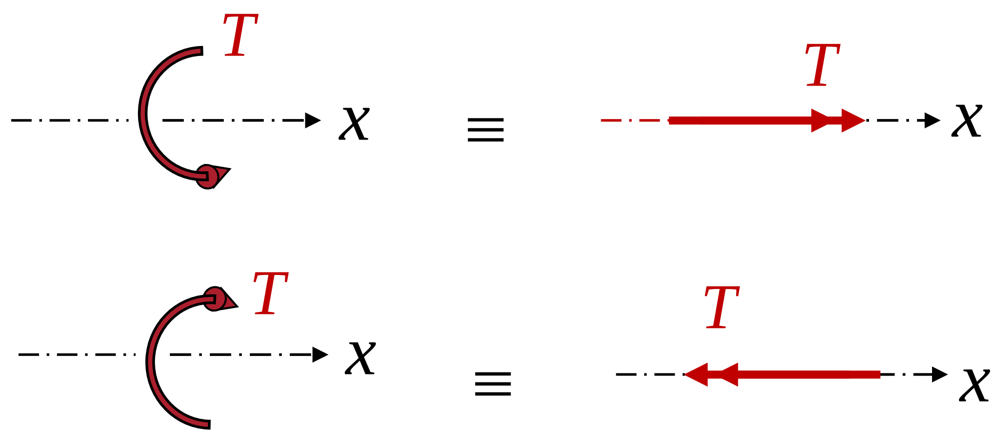
Sign Convention

Torsion sign convention: "right hand rule"

E.g. torque about the *x*-axis:

+ve torque

-ve torque

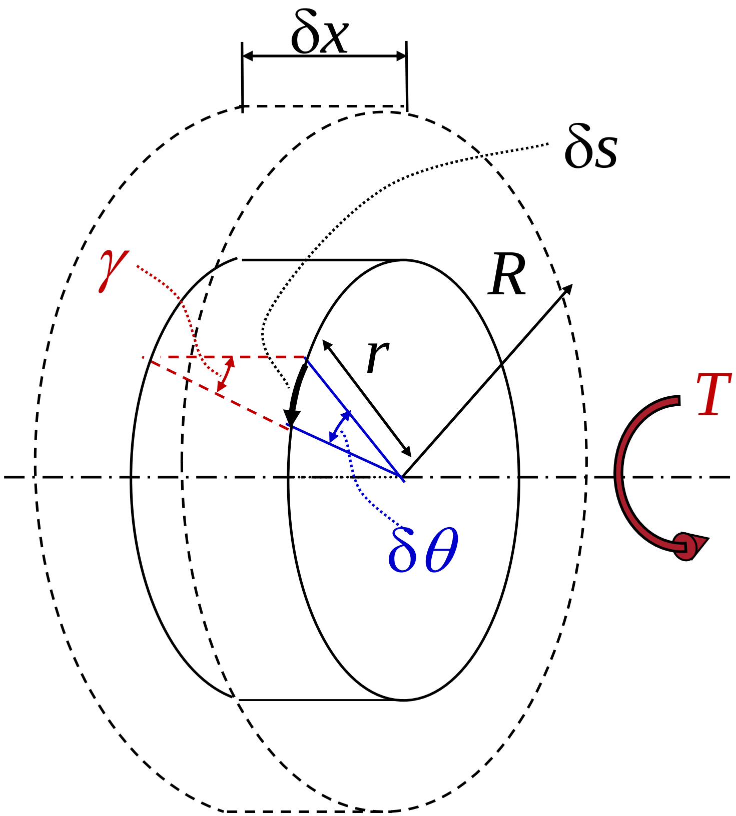


Applies to torque (*i.e.* torsional moments) and twist

Consider element of length δx and point at radius r which moves through δs under torque T relative to LHS of element

$$\delta s = \gamma\,\delta x \quad \text{and} \quad \delta s = r\,\delta\theta$$

$$\Rightarrow \quad \gamma\,\delta x = r\,\delta\theta \;\Rightarrow\; \gamma = r\,\frac{\delta\theta}{\delta x}$$



$$\text{but} \quad \frac{\delta\theta}{\delta x} = \text{constant} = \frac{\theta}{L} \;\Rightarrow\; \gamma = r\,\frac{\theta}{L}$$

$$\text{and} \quad \tau = G\,\gamma \quad \textit{i.e.}\ \text{elastic} \;\Rightarrow\; \frac{\tau}{G} = r\,\frac{\theta}{L}$$

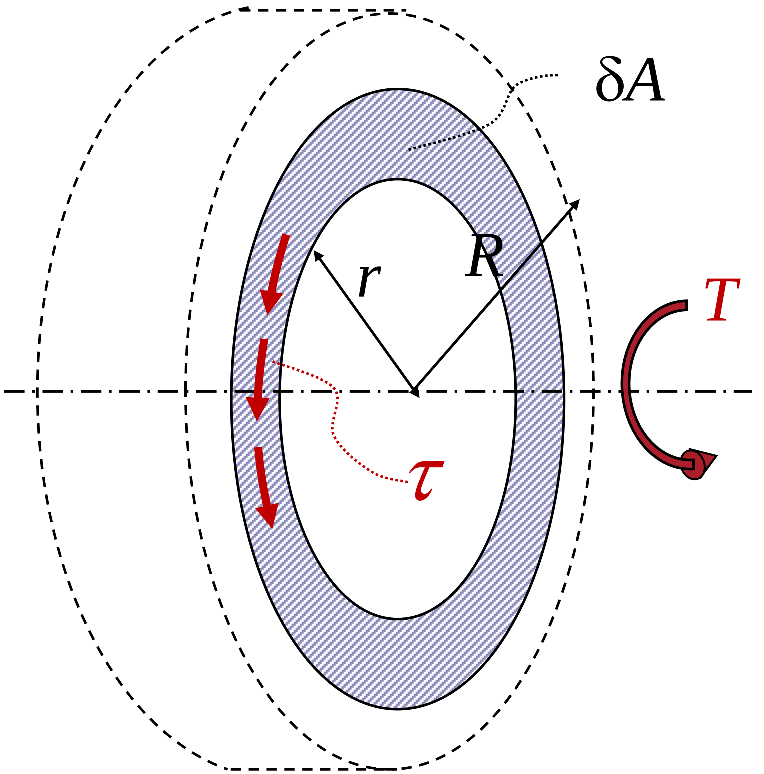
$$\Rightarrow \quad \frac{\tau}{r} = G\,\frac{\theta}{L} \qquad (1)$$

Consider element δA swept by δr at radius r
and shear stress τ produced by torque T for equilibrium:

$$T = \int_0^R \tau\, r \,\mathrm{d}A \quad \text{where} \quad \tau = G\, r\, \frac{\theta}{L} \quad (1)$$

$$\Rightarrow T = G\frac{\theta}{L}\int_0^R r^2\,\mathrm{d}A \quad \text{where} \quad \underbrace{\int_0^R r^2\,\mathrm{d}A}_{\text{“Polar 2nd moment of area”}} = J$$

$$\Rightarrow T = G\frac{\theta}{L}J \quad \Rightarrow \quad \frac{T}{J} = G\frac{\theta}{L} \quad (2)$$



Simple torsion equation

① = ② $\Rightarrow$ $$\boxed{\frac{\tau}{r} = \frac{T}{J} = G\frac{\theta}{L}}$$ *

Note "Rate of twist" : $$\frac{\theta}{L} = \frac{T}{GJ}$$

Where:
GJ = "Torsional rigidity"

Compare with axial loading and bending equations:

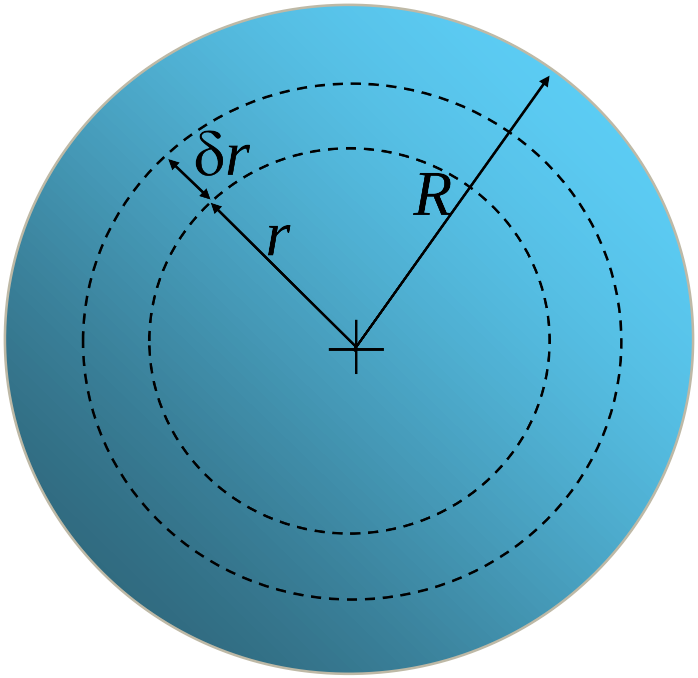
Axial: $$\frac{e}{L} = \frac{F}{AE}$$ AE = "Axial stiffness"

Bending: $$\kappa = \frac{1}{R} = \frac{M}{EI}$$ EI = "Bending stiffness"

For a solid circular shaft:

$$J = \int_0^R r^2 \, \mathrm{d}A \qquad \text{where } \mathrm{d}A = 2\pi r \, \mathrm{d}r$$

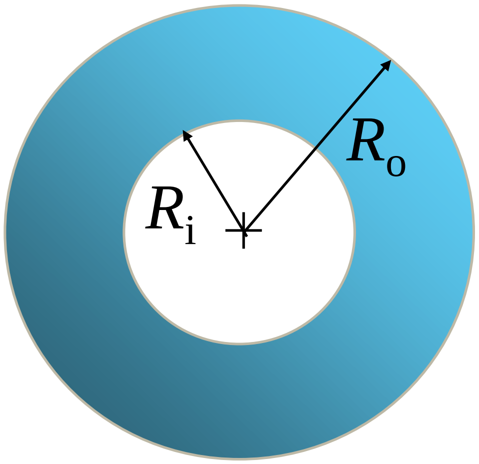
$$J = \int_0^R 2\pi r^3 \, \mathrm{d}r \quad = \frac{\pi R^4}{2} \quad = \frac{\pi D^4}{32} \; *$$



Polar 2nd moment of area

For a thick hollow circular shaft:

$$J = \int_{R_i}^{R_o} 2\pi r^3\, \mathrm{d}r \qquad = \frac{\pi}{2}(R_o^{\,4} - R_i^{\,4})$$ *



For a thin wall circular shaft ($t \ll R$) :

$$J \approx Ar^2 \qquad \text{where } A = 2\pi R\, t$$

$$J \approx 2\pi R^3\, t$$ *