Advanced Bending and Torsion
Shear Centre of Composite Thin-Walled Sections

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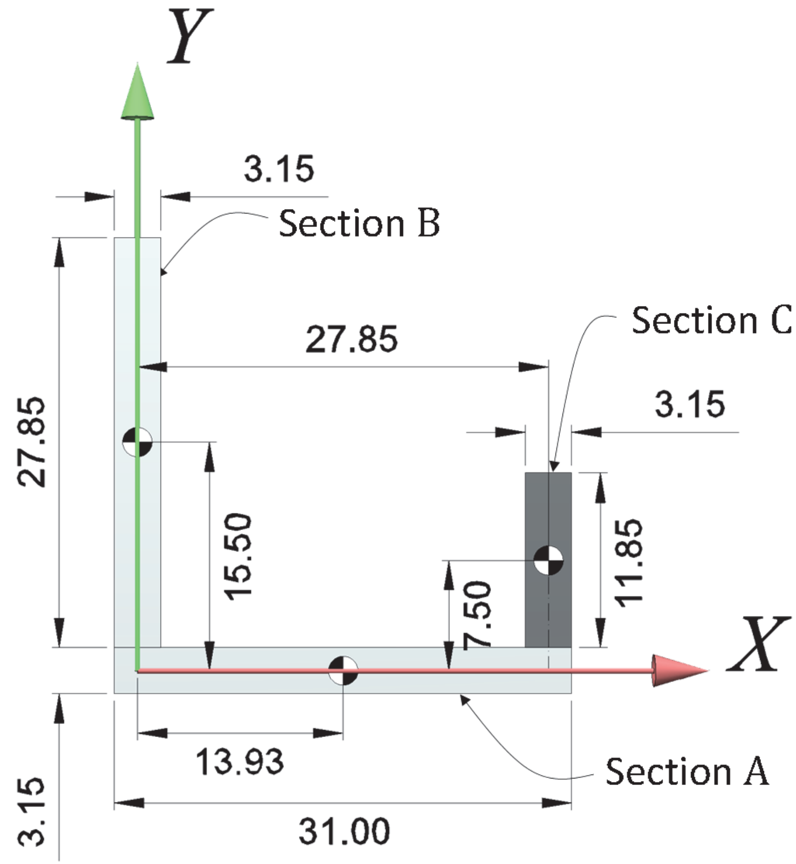
Composite Lipped Section

- Assume section C made of steel:

$$n_C = \frac{G_C}{G_{ref}} = \frac{\dfrac{210 \text{ GPa}}{2(1+0.3)}}{\dfrac{70 \text{ GPa}}{2(1+0.3)}} = \mathbf{3}$$



$A_A = (31.00)(3.15)\ \text{mm}^2$

$A_A = 97.65\ \text{mm}^2$

$\bar{X}_A = 13.925\ \text{mm}$

$\bar{Y}_A = 0$

$A_B = (3.15)(27.85)\ \text{mm}^2$

$A_B = 87.73\ \text{mm}^2$

$\bar{X}_B = 0$

$\bar{Y}_B = 15.50\ \text{mm}$

$A_C = (3.15)(11.85)\ \text{mm}^2$

$A_C = 37.33\ \text{mm}^2$

$\bar{X}_C = 27.85\ \text{mm}$

$\bar{Y}_C = 7.50\ \text{mm}$

Composite Lipped Section

- Centroid of the compound section:

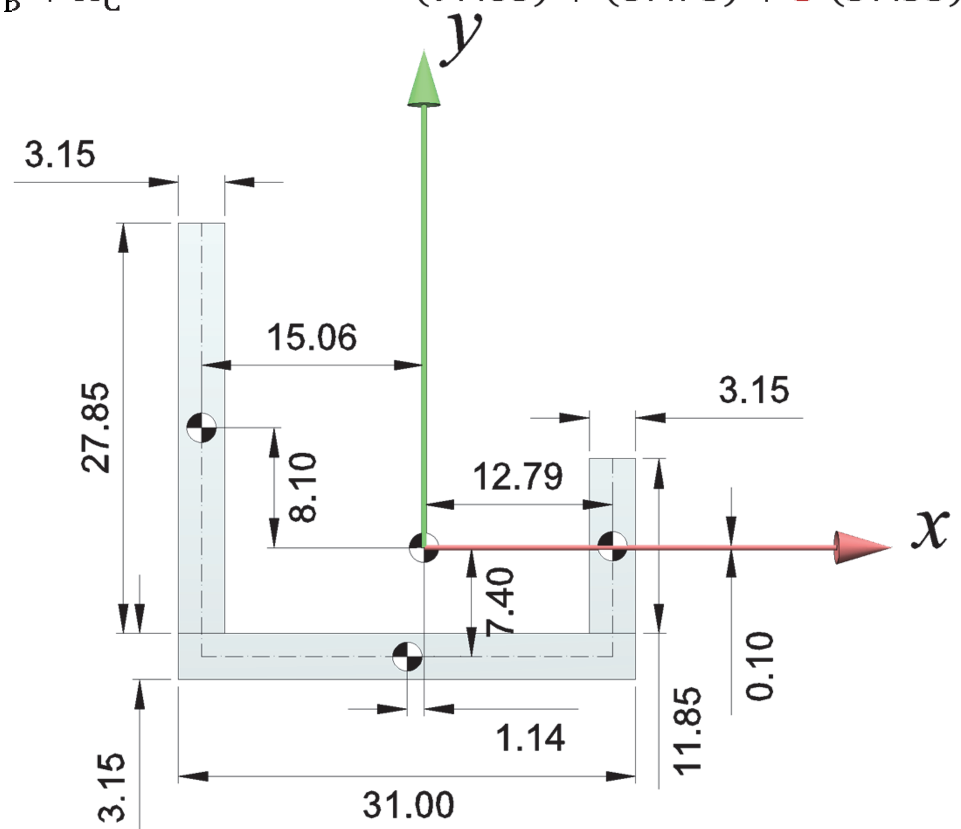
$$\bar{X} = \frac{\sum \bar{X}_i A_i}{\sum A_i} = \frac{\bar{X}_A A_A + \bar{X}_B A_B + \bar{X}_C A_C}{A_A + A_B + A_C} = \frac{(13.925)(97.65) + (0)(87.73) + \mathbf{3}\,(27.85)(37.33)}{(97.65) + (87.73) + \mathbf{3}\,(37.33)}$$

$$\bar{X} = 15.06 \text{ mm}$$

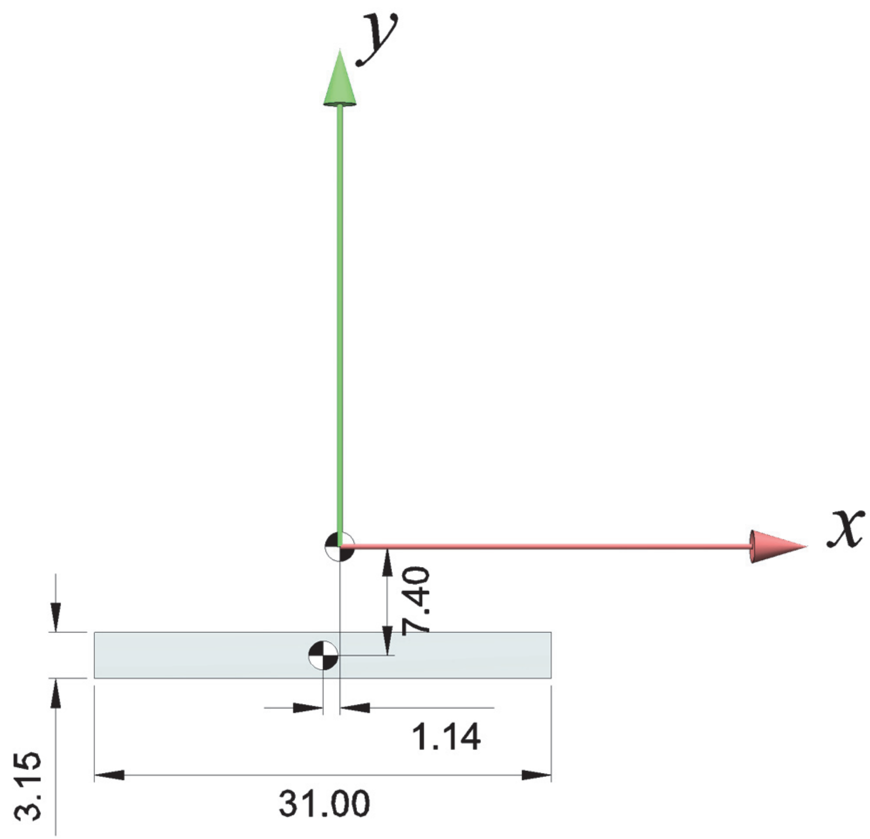
$$\bar{Y} = \frac{\sum \bar{Y}_i A_i}{\sum A_i} = \frac{\bar{Y}_A A_A + \bar{Y}_B A_B + \bar{Y}_C A_C}{A_A + A_B + A_C} = \frac{(0)(97.65) + (15.50)(87.73) + \mathbf{3}\,(7.50)(37.33)}{(97.65) + (87.73) + \mathbf{3}\,(37.33)}$$

$$\bar{Y} = 7.40 \text{ mm}$$

- New coordinates:



- Parallel axis theorem for section A:



$$I_{x_A x_A} = \frac{b\,h^3}{12} = \frac{(31.00)(3.15)^3}{12} = 80.74\ \mathrm{mm}^4$$

$$\bar{y}_A = \bar{Y}_A - \bar{Y} = 0 - 7.40 = -7.40\ \mathrm{mm}$$

$$I_{xx}^{A} = I_{x_A x_A} + A_A(\bar{y}_A)^2$$

$$I_{xx}^{A} = 5{,}424.08\ \mathrm{mm}^4$$

$$I_{y_A y_A} = \frac{b\,h^3}{12} = \frac{(3.15)(31.00)^3}{12} = 7{,}820.14\ \mathrm{mm}^4$$

$$\bar{x}_A = \bar{X}_A - \bar{X} = 13.925 - 15.06 = -1.14\ \mathrm{mm}$$

$$I_{yy}^{A} = I_{y_A y_A} + A_A(\bar{x}_A)^2$$

$$I_{yy}^{A} = 7{,}946.12\ \mathrm{mm}^4$$

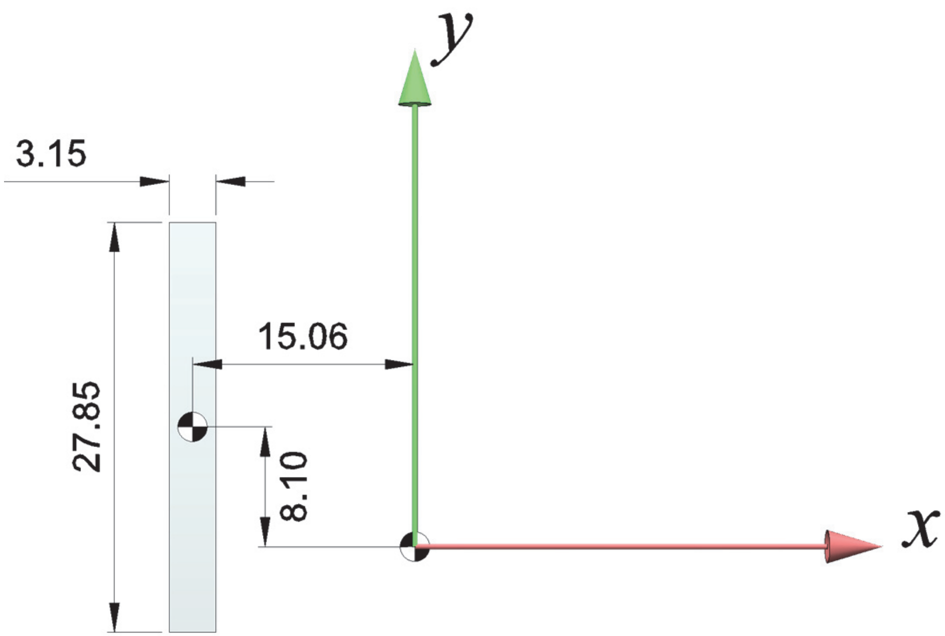
$$I_{x_A\,y_A} = 0\ \text{(symmetric cross-section)}$$

$$I_{xy}^{A} = I_{x_A y_A} + A_A(\bar{x}_A \bar{y}_A)$$

$$I_{xy}^{A} = 0 + (97.65)(-1.14)(-7.40)$$

$$I_{xy}^{A} = 820.46\ \mathrm{mm}^4$$

- Parallel axis theorem for section B:



$$I_{x_B x_B} = \frac{b\,h^3}{12} = \frac{(3.15)(27.85)^3}{12} = 5{,}670.29 \text{ mm}^4$$

$$\bar{y}_B = \bar{Y}_B - \bar{Y} = 15.50 - 7.40 = 8.10 \text{ mm}$$

$$I_{xx}^B = I_{x_B x_B} + A_B(\bar{y}_B)^2$$

$$I_{xx}^B = 11{,}430.00 \text{ mm}^4$$

$$I_{y_B y_B} = \frac{b\,h^3}{12} = \frac{(27.85)(3.15)^3}{12} = 72.54 \text{ mm}^4$$

$$\bar{x}_B = \bar{X}_B - \bar{X} = 0 - 15.06 = -15.06 \text{ mm}$$

$$I_{yy}^B = I_{y_B y_B} + A_B(\bar{x}_B)^2$$

$$I_{yy}^B = 19{,}971.65 \text{ mm}^4$$

$$I_{x_B\, y_B} = 0 \text{ (symmetric cross-section)}$$

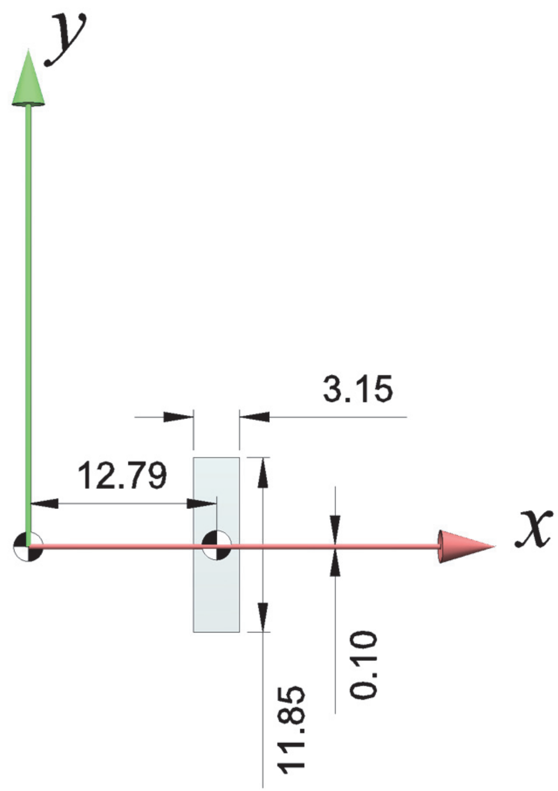
$$I_{xy}^B = I_{x_B\, y_B} + A_B(\bar{x}_B \bar{y}_B)$$

$$I_{xy}^B = 0 + (87.73)(-15.06)(8.10)$$

$$I_{xy}^B = -10{,}705.76 \text{ mm}^4$$

Composite Lipped Section

- Parallel axis theorem for section C:



$$I_{x_C x_C} = \frac{b\,h^3}{12} = \frac{3\,(3.15)(11.85)^3}{12} = 436.80\ \text{mm}^4$$

$$\bar{y}_C = \bar{Y}_C - \bar{Y} = 7.50 - 7.40 = 0.10\ \text{mm}$$

$$I_{xx}^{C} = I_{x_B x_B} + 3\,A_C(\bar{y}_C)^2$$

$$I_{xx}^{C} = 1{,}311.59\ \text{mm}^4$$

$$I_{y_C y_C} = \frac{b\,h^3}{12} = \frac{3\,(11.85)(3.15)^3}{12} = 30.87\ \text{mm}^4$$

$$\bar{x}_C = \bar{X}_C - \bar{X} = 27.85 - 15.06 = 12.79\ \text{mm}$$

$$I_{yy}^{C} = I_{y_C y_C} + 3\,A_C(\bar{x}_C)^2$$

$$I_{yy}^{C} = 18{,}408.77\ \text{mm}^4$$

$I_{x_C\,y_C} = 0$ (symmetric cross-section)

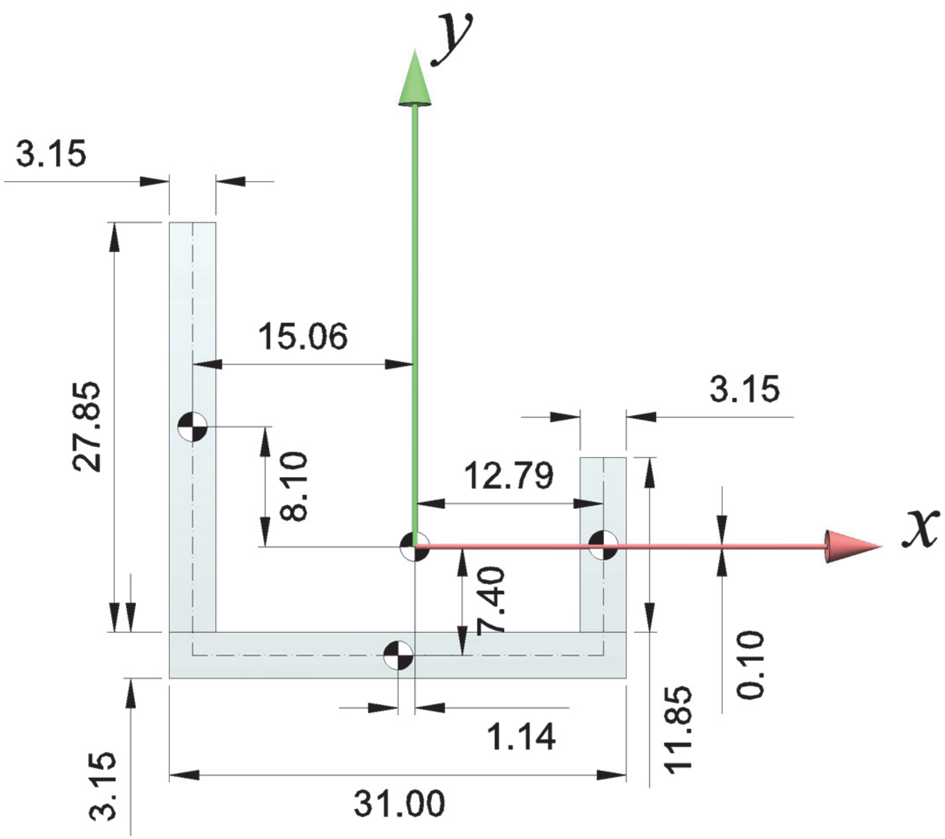
$$I_{xy}^{C} = I_{x_C\,y_C} + 3\,A_C(\bar{x}_C\bar{y}_C)$$

$$I_{xy}^{C} = 0 + 3\,(37.33)(12.79)(0.10)$$

$$I_{xy}^{C} = 147.16\ \text{mm}^4$$

Composite Lipped Section

- Finally, for the compound section:



$$I_{xx} = I_{xx}^{\mathrm{A}} + I_{xx}^{\mathrm{B}} + I_{xx}^{\mathrm{C}}$$

$$I_{xx} = 18{,}165.67 \text{ mm}^4$$

$$I_{yy} = I_{yy}^{\mathrm{A}} + I_{yy}^{\mathrm{B}} + I_{yy}^{\mathrm{C}}$$

$$I_{yy} = 46{,}326.54 \text{ mm}^4$$

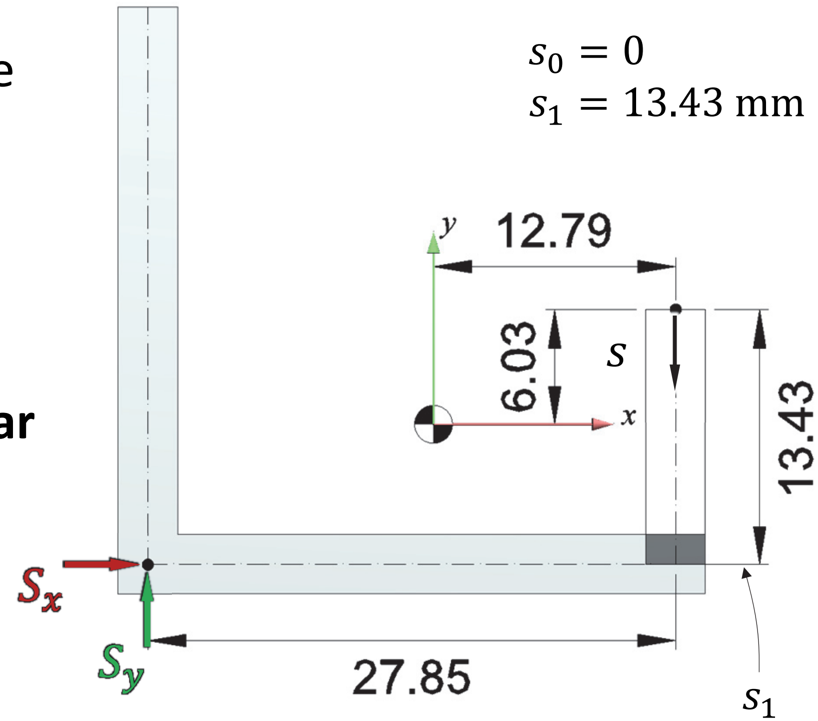
$$I_{xy} = I_{xy}^{\mathrm{A}} + I_{xy}^{\mathrm{B}} + I_{xy}^{\mathrm{C}}$$

$$I_{xy} = -9{,}738.14 \text{ mm}^4$$

- **Shear centre**:
  - Most convenient reference point is the bottom left corner (origin of X, Y)
  - Only stresses along section C will generate moments about this reference point (!)
  - We integrate shear flow from $s = 0$ to $s = s_1 = 13.425 \text{ mm}$
  - <u>**Important**</u>: shear stresses and **shear flow** are defined in terms of $\boldsymbol{x}, \boldsymbol{y}$ while the **shear centre** is defined in terms of $\boldsymbol{X}, \boldsymbol{Y}$

$(X_0, Y_0) = (27.85 \text{ mm}, 13.425 \text{ mm})$
$(x_0, y_0) = (12.79 \text{ mm}, 6.03 \text{ mm})$

$s_0 = 0$
$s_1 = 13.43 \text{ mm}$



Equations:

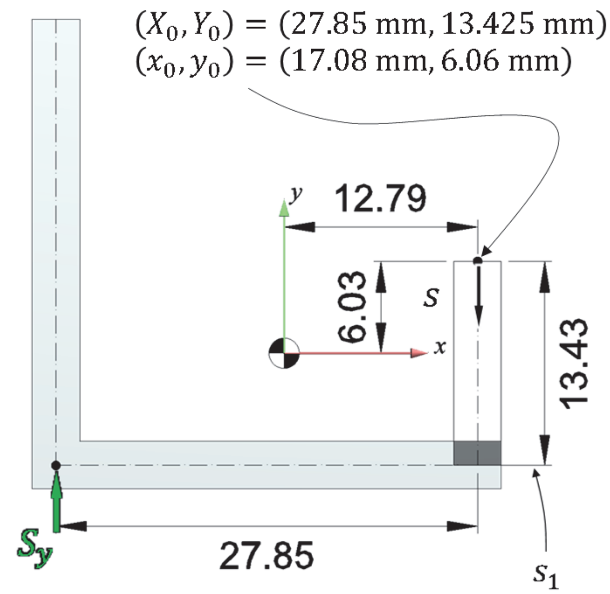
Shear centre:

$$S_y\, e_x = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{ds} \qquad S_x\, e_y = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{ds}$$

Shear flow:

$$-q_s = \left(\frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{ds} + \left(\frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{ds}$$

Composite Lipped Section – Shear Centre e_x



$$-q_s = \left(\frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{d}s$$

To find e_x we apply S_y, make $S_x = 0$ and therefore:

$$-q_s = \left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s y\, t\, \mathrm{d}s$$

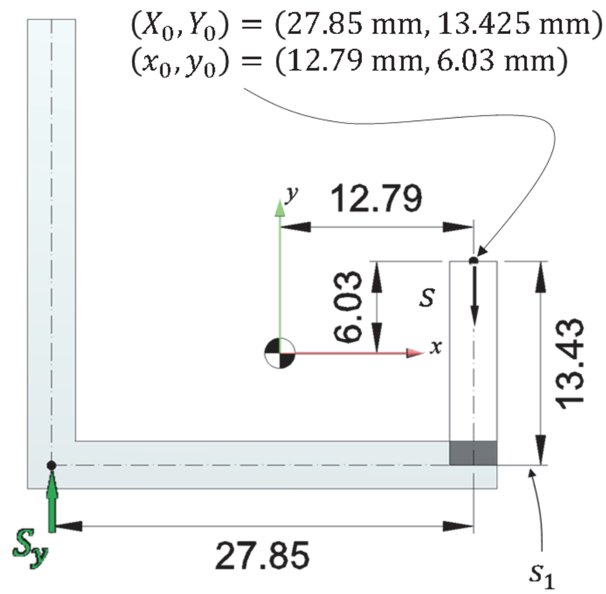
Note that here $x_{(s)} = x_0 = 17.08\ \text{mm}$, while $y_{(s)} = y_0 - s$ and therefore:

$$-q_s = \left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) \int_0^s x\, t\, \mathrm{d}s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \int_0^s (y_0 - s)\, t\, \mathrm{d}s$$

$$-q_s = \left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, t\, s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right) \left(y_0\, s - \frac{s^2}{2}\right) t$$

Composite Lipped Section – Shear Centre e_x

$$S_y\, e_x = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{d}s$$



Note that here we only consider the vertical shear flow ($q_{s,y}$) while the 'moment arm' is constant and equal to X_0, therefore:

$$S_y\, e_x = \int_0^{s_1} \left(-X_0\, q_s\right) \mathrm{d}s$$

$$S_y\, e_x = X_0\, t \int_0^{s_1} \left[\left(\frac{S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, s + \left(\frac{S_y\, I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, s - \frac{s^2}{2}\right)\right] \mathrm{d}s$$

$$e_x = X_0\, t \left[\left(\frac{I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s^2}{2} + \left(\frac{I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, \frac{s^2}{2} - \frac{s^3}{6}\right)\right]_0^{s_1}$$

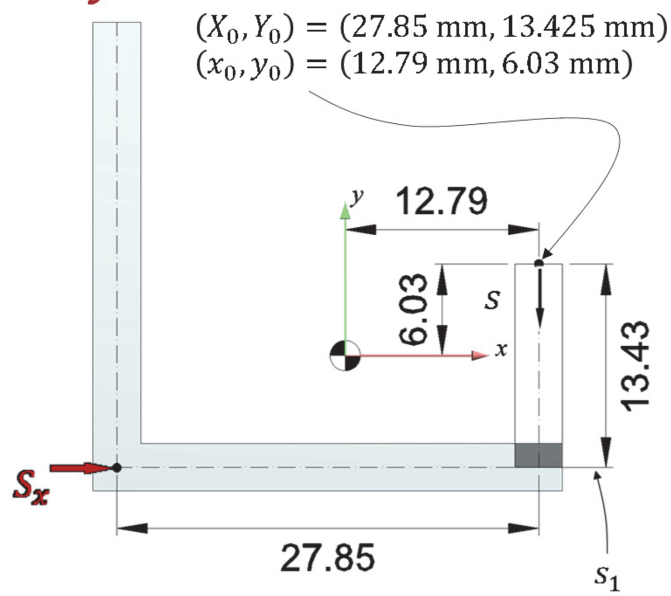
$$e_x = X_0\, t \left[\left(\frac{I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s_1^2}{2} + \left(\frac{I_{yy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, \frac{s_1^2}{2} - \frac{s_1^3}{6}\right)\right]$$

$$e_x = (27.85)\, (3)\, (3.15) \left\{\left(\frac{-9{,}738.14}{-746{,}721{,}139}\right)(12.79)\frac{(13.425)^2}{2} + \left(\frac{49{,}326.54}{746{,}721{,}139}\right)\left[(6.03)\frac{(13.425)^2}{2} - \frac{(13.425)^3}{6}\right]\right\}$$

$$e_x = 6.24\ \mathrm{mm}$$

Composite Lipped Section – Shear Centre e_y

$$-q_s = \left(\frac{S_x\, I_{xx} + S_y\, I_{xy}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right)\int_0^s x\, t\, \mathrm{ds} + \left(\frac{S_y\, I_{yy} + S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\int_0^s y\, t\, \mathrm{ds}$$



To find e_y we apply S_x, make $S_y = 0$ and therefore:

$$-q_s = \left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right)\int_0^s x\, t\, \mathrm{ds} + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\int_0^s y\, t\, \mathrm{ds}$$

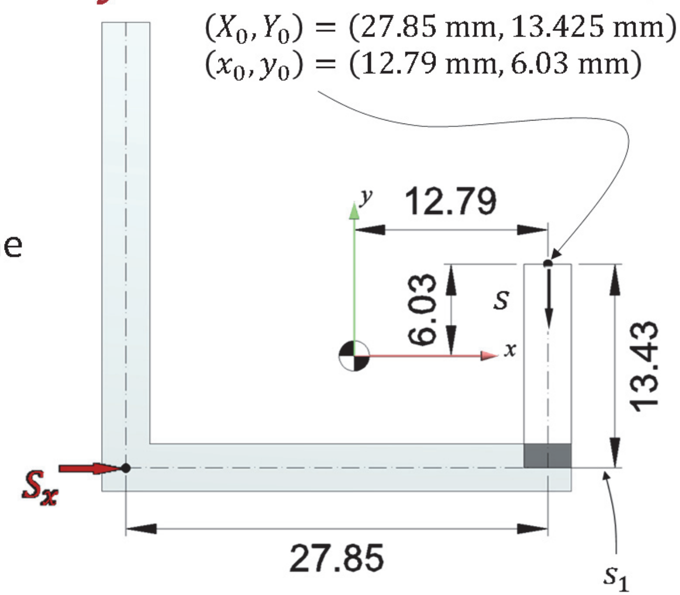
Note that here $x_{(s)} = x_0 = 17.08\ \mathrm{mm}$, while $y_{(s)} = y_0 - s$ and therefore:

$$-q_s = \left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right)\int_0^s x\, t\, \mathrm{ds} + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\int_0^s (y_0 - s)\, t\, \mathrm{ds}$$

$$-q_s = \left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, t\, s + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, s - \frac{s^2}{2}\right) t$$

Composite Lipped Section – Shear Centre e_y

$$S_x\, e_y = \int \left(Y\, q_{s,x} - X\, q_{s,y}\right) \mathrm{d}s$$



Note that here we only consider the vertical shear flow ($q_{s,y}$) while the 'moment arm' is constant and equal to X_0, therefore:

$$S_x\, e_y = \int_0^{s_1} \left(-X_0\, q_s\right) \mathrm{d}s$$

$$S_x\, e_y = X_0\, t \int_0^{s_1} \left[\left(\frac{S_x\, I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x\, s + \left(\frac{S_x\, I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0\, s - \frac{s^2}{2}\right)\right] \mathrm{d}s$$

$$e_y = X_0\, t \left[\left(\frac{I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s^2}{2} + \left(\frac{I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0 \frac{s^2}{2} - \frac{s^3}{6}\right)\right]_0^{s_1}$$

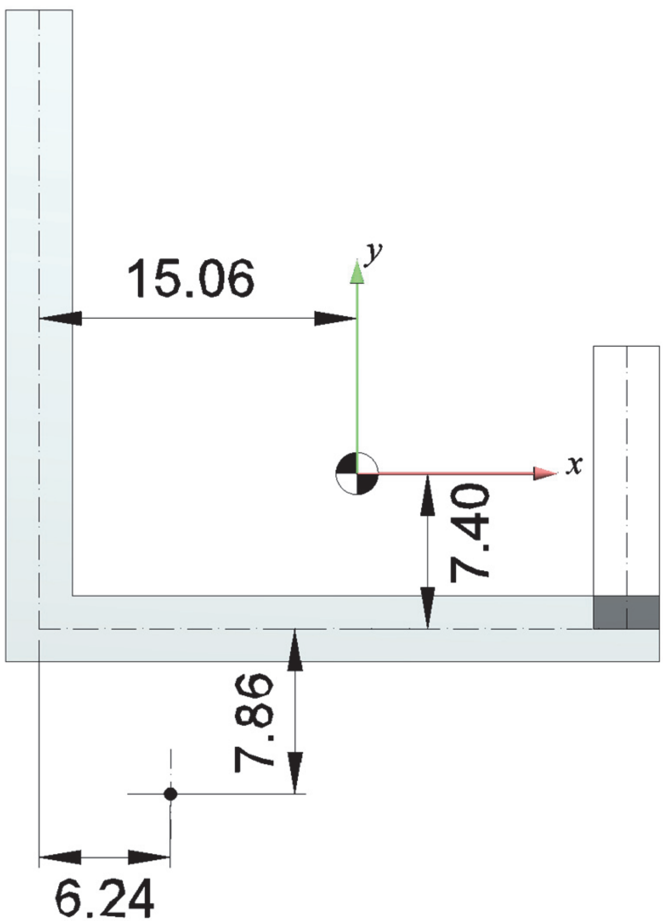
$$e_y = X_0\, t \left[\left(\frac{I_{xx}}{I_{xy}^2 - I_{xx}\, I_{yy}}\right) x \frac{s_1^2}{2} + \left(\frac{I_{xy}}{I_{xx}\, I_{yy} - I_{xy}^2}\right)\left(y_0 \frac{s_1^2}{2} - \frac{s_1^3}{6}\right)\right]$$

$$e_y = (27.85)\,(3)\,(3.15)\left\{\left(\frac{18{,}165.67}{-746{,}721{,}139}\right)(12.79)\frac{(13.425)^2}{2} + \left(\frac{-9{,}738.14}{746{,}721{,}139}\right)\left[(6.03)\frac{(13.425)^2}{2} - \frac{(13.425)^3}{6}\right]\right\}$$

$$e_y = -7.86\ \mathrm{mm}$$

Composite Lipped Section – Shear Centre

Thin wall (analytical) solution



Full 2D (numerical) solution