EMAT10100 Engineering Maths I
Lecture 13: Solving linear systems of equations

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Looking back looking forward

Last time:

- Determinants:
  - 3×3 and 4×4 examples
  - Minors and cofactors and general procedure
  - Cross product as a 3×3 example
  - Simplifying procedures and examples
- Matrix inverse: $\mathbf{A}^{-1}$
  - Cramer's rule formula using matrix of cofactors adj**A**
  - Special case of 2×2 inverse

This time:

- Using matrices to solve linear systems of equations

Linear systems: $n = 2$

- Main engineering use of of matrices is solving (very large) linear systems of equations
- Start with $n = 2$ $\Rightarrow$ two equations in two unknowns
- Linear $\Rightarrow$ intersection of two straight lines
- Exercise solve

$$\begin{aligned} 2x + 3y &= 4 \\ -2x + y &= -8 \end{aligned}$$

- Rewrite as matrix equation:

$$\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}.$$

- We write this as $\mathbf{A}\mathbf{x} = \mathbf{b}$, to be solved for $\mathbf{x} = (x, y)^{\mathrm{T}}$

Linear systems: $n = 3$

- $n = 3$ $\Rightarrow$ three equations, three unknowns
- Linear $\Rightarrow$ intersection of 3 planes (see next lecture)

$$\begin{aligned} x + y + 2z &= 3 \\ x - 2y + 3z &= -5 \\ 2x + 2y + 2z &= 4 \end{aligned}$$

- Can be written as a matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

to be solved for $\mathbf{x} = (x, y, z)^{\mathrm{T}}$

Linear systems and inverse (general case)

- We want to solve $n \times n$ system

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \text{for } \mathbf{x}.$$

- Pre-multiply by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}\mathbf{b},$$
$$\text{so} \quad (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{I}\mathbf{x} = \mathbf{x}$, we have

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

(formula for solving linear system)

Two golden rules

1. Cramer's rule (minors, cofactors etc.):
   - is bad way of finding inverse of matrix bigger than 3×3
   - (Will show you another method in next week)
2. To solve $\mathbf{A}\mathbf{x} = \mathbf{b}$:
   - you wouldn't normally use $\mathbf{A}^{-1}$ if this were a system of two equations in two unknowns
   - you would instead start subtracting or adding multiples of one equation to another
   - matrices give us a way of formalising this for arbitrary $(n \times n)$-dimensional systems.

- INSTEAD: use row elimination (also known as Gaussian elimination)

Row elimination: basic idea (I)

- Re-write

$$\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}.$$

in so-called augmented form $[\mathbf{A}|\mathbf{b}]$:

$$\left(\begin{array}{cc:c} 2 & 3 & 4 \\ -2 & 1 & -8 \end{array}\right) \quad \text{and perform row operations}$$

- $R_2 \to R_2 + R_1$:

$$\left(\begin{array}{cc:c} 2 & 3 & 4 \\ 0 & 4 & -4 \end{array}\right)$$

Row elimination: basic idea (II)

- We had

$$\left(\begin{array}{cc:c} 2 & 3 & 4 \\ 0 & 4 & -4 \end{array}\right) \qquad \text{(upper triangular form)}$$

- $R_2 \to R_2/4$, then back substitution: $R_1 \to R_1 - 3R_2$

$$\left(\begin{array}{cc:c} 2 & 3 & 4 \\ 0 & 1 & -1 \end{array}\right), \mapsto \left(\begin{array}{cc:c} 2 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right)$$

- $R_1 \to R_1/2$:

$$\left(\begin{array}{cc:c} 1 & 0 & 7/2 \\ 0 & 1 & -1 \end{array}\right) \qquad \text{which gives } x = 7/2,\ y = -1.$$

Row operations: general principles

- You can:
  1. multiply a row by a scalar
  2. add a multiple of any row to any other row
  3. swap rows (but not columns)
- General method:
  - obtain upper triangular (or echelon) form
    - by making elements below diagonal $= 0$
  - perform back substitution, either:
    - solve the equations successively starting from the bottom row
  - or:
    - make the elements above the diagonal $= 0$.
    - read off the answer from the diagonals

3×3 example (I)

- Solve

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{b}, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 3 & 3 & 1 \\ 2 & -1 & 0 \\ 3 & 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- Step 1: form augmented matrix: $[\mathbf{A}|\mathbf{b}]$
- Step 2: get zeros in first column below diagonal:

$$\left(\begin{array}{ccc|c} 3 & 3 & 1 & 1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{array}\right) \quad \begin{array}{c} R_2 \to R_2 - \frac{2}{3}R_1 \\ R_3 \to R_3 - R_1 \end{array} \quad \left(\begin{array}{ccc|c} 3 & 3 & 1 & 1 \\ 0 & -3 & -2/3 & 1/3 \\ 0 & -2 & -2 & 0 \end{array}\right)$$

- Step 3: get zeros in second column below diagonal:

$$R_3 \to R_3 - \frac{2}{3}R_2 \quad \left(\begin{array}{ccc|c} 3 & 3 & 1 & 1 \\ 0 & -3 & -2/3 & 1/3 \\ 0 & 0 & -14/9 & -2/9 \end{array}\right)$$

Now the matrix has been reduced to upper triangular or echelon form.

3×3 example (II)

- Back-substitution Step (1st method) read off the answer

$$\begin{array}{cccl} 3x & +3y & +z & = 1 \\ & -3y & -\frac{2}{3}z & = \frac{1}{3} \\ & & -\frac{14}{9}z & = \frac{-2}{9} \end{array} \quad \Rightarrow z = \frac{1}{7}$$

$$\Rightarrow -3y = \frac{2}{21} + \frac{7}{21} \Rightarrow y = -\frac{1}{7} \Rightarrow 3x = \frac{3}{7} - \frac{1}{7} + 1 \Rightarrow x = \frac{3}{7}$$

- OR Back-substitution Step (2nd method) try to get zeros above the diagonal:

$$\begin{array}{c} R_1 \to R_1 + \frac{9}{14}R_3 \\ R_2 \to R_2 - \frac{3}{7}R_3 \end{array} \left(\begin{array}{ccc|c} 3 & 3 & 0 & 6/7 \\ 0 & -3 & 0 & 3/7 \\ 0 & 0 & -14/9 & -2/9 \end{array}\right)$$

$$R_1 \to R_1 + R_2 \left(\begin{array}{ccc|c} 3 & 0 & 0 & 9/7 \\ 0 & -3 & 0 & 3/7 \\ 0 & 0 & -14/9 & -2/9 \end{array}\right) \Rightarrow \begin{pmatrix} 3x \\ -3y \\ -14z \end{pmatrix} = \begin{pmatrix} 9/7 \\ 3/7 \\ -2 \end{pmatrix}$$

- $\Rightarrow \quad (x, y, z) = (\frac{3}{7}, -\frac{1}{7}, \frac{1}{7})$

Exercise

- Exercise: Use row elimination to try to solve the following

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

- A useful trick (not in the text books!!!) to avoid algebraic slips
  keep track of the sum of each row

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & -2 & 3 & -5 \\ 2 & 2 & 2 & 4 \end{array}\right) \quad \begin{array}{c} 7 \\ -3 \\ 10 \end{array}$$

$$\begin{array}{c} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ & & & -8 \\ & & & -2 \end{array}\right) \quad \begin{array}{c} 7 \\ -10 \\ -4 \end{array}$$

Engineering HOT SPOT

- This trick uses over-specification of data to avoid arithmetic slips.
- Same principle is used in communications systems to avoid read/write errors
- E.g. the encoding of information in file formats such as MP3
- Simplest (and oldest) example: ISBN numbers on books.
  - How does a bar-code reader know if it has misread the ISBN?
  - The number is contained in 9 digits. The 10th digit is the sum of the first 9 (mod 10).
  - Reject the read operation if 10th is not sum of first 9.

Another exercise

- Use row elimination to solve

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

- What goes wrong?
- Row elimination (usually) doesn't work if the matrix $\mathbf{A}$ is singular i.e. det$(\mathbf{A}) = 0$ (see the next lecture).

Homework

- Read *James* section 5.5.2
  (but stop before "Thomas algorithm").
- Do Exercises (4th edition) or (5th edition):
  - exercises 5.5.1 Q. 60
  - exercises 5.5.3 Q. 72
- Make sure you practice doing these row operations this week.
- NOW 10 minute break, then the test!