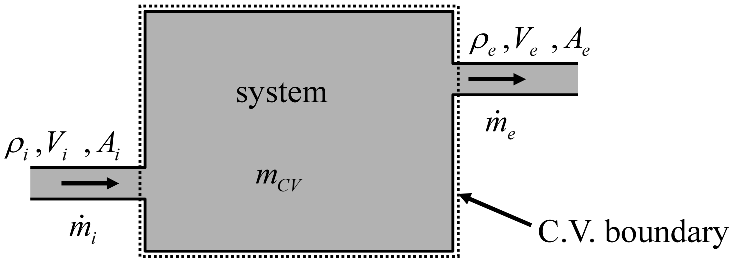

## Introduction to Control Volume Analysis

- We have already used control volume analysis in the derivation of:
  - The hydrostatic equation
  - Mass conservation in 1D flow and
  - Bernoulli's equation
- Whilst obviously useful in the derivation of the above, an understanding of control volume analysis allows you to apply your fluids knowledge to a large number of engineering situations.
- What is Control Volume (CV) analysis?
  - A process where we draw an imaginary boundary, associated with a system, that encloses the desired CV. We then consider the rate at which: mass, linear momentum, angular momentum and energy; enter and leave the CV. By careful choice of our CV boundary (surrounding the system except at important "cuts") we may derive useful properties, such as forces or moments, on the system.
- In general we can consider CV boundaries that accelerate and deform but we will only consider: fixed control volumes with zero or constant speeds.
- We will also restrict ourselves to steady incompressible flows

# Conservation of Mass for a Control Volume

- The rate of change of mass inside a control volume is equal to the mass flow rate in minus the mass flow rate out.




- For unsteady compressible and incompressible we can write

$$\frac{dm_{CV}}{dt} = \dot{m}_i - \dot{m}_e = \rho_i V_i A_i - \rho_e V_e A_e$$

- Where the inlet and exit areas must me measured perpendicular to the velocities
- For steady problems (values are constant at each location )

$$\frac{dm_{CV}}{dt} = 0 \qquad \therefore \quad \rho_i V_i A_i = \rho_e V_e A_e$$

- For steady incompressible flow

$$V_i A_i = V_e A_e$$



# Conservation of Steady Flow Linear Momentum

- Consider Newtons 2nd law applied to a point mass. In vector form we write

$$\underline{\mathbf{F}}_{\text{tot}} = m\frac{d\underline{\mathbf{V}}}{dt} \qquad \underline{\mathbf{F}}_{\text{tot}} = \begin{bmatrix} f_{\text{tot}_x} \\ f_{\text{tot}_y} \\ f_{\text{tot}_z} \end{bmatrix} \qquad \underline{\mathbf{V}} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

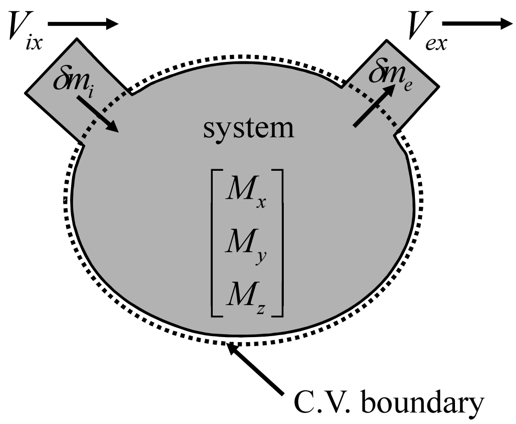
where $\underline{\mathbf{F}}_{\text{tot}}$ is the vector sum of all forces acting on the mass.

- Note how we can treat each coordinate direction separately.
- Consider a CV with initial momentum in $x$ of $M_x$. Over time $\delta t$, the masses $\delta m_i$ and $\delta m_e$ enter and exit the CV with velocities in the x direction of $V_{ix}$ and $V_{ex}$ respectively.
- For steady control volumes the change of momentum is only due to momentum flux so

$$\delta M_x = \left(V_{ix}\delta m_i - V_{ex}\delta m_e\right)$$

- Further for steady flow $\delta m_i = \delta m_e = \delta m$
- Dividing throughout by $\delta t$ and taking limits

$$f_{\text{tot}_x} = \dot{M}_x = \dot{m}\left(V_{ix} - V_{ex}\right)$$

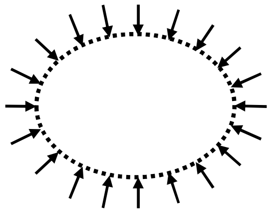


# Conservation of Steady Flow Linear Momentum (2)

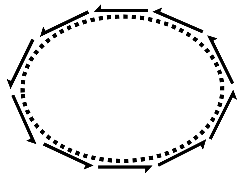
- From previous, the total force exerted **ON** a control volume is given by

$$f_{\text{tot}_x} = -\dot{m}(V_{ix} - V_{ex}) = \dot{m}(V_{ex} - V_{ix})$$
$$f_{\text{tot}_y} = -\dot{m}(V_{iy} - V_{ey}) = \dot{m}(V_{ey} - V_{iy})$$
$$f_{\text{tot}_z} = -\dot{m}(V_{iz} - V_{ez}) = \dot{m}(V_{ez} - V_{iz})$$

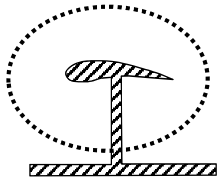
- Each total/net force component can be split into
- Pressure forces acting normally to the external CV boundary. Note that a constant pressure integrates to zero so we can subtract/add a constant from/to the pressure. Eg subtract atmospheric pressure & integrate gauge pressures



- Viscous shear forces acting tangentially to the external control surface boundary. Note that for pressure and viscous forces we are only interested in the boundary.



- Forces exerted on the fluid by solid objects. Whenever a solid surface cuts the CV boundary there is a net resultant force *e.g.* if the drag on body is D, the resultant force on the fluid is –D.



- Gravitational body forces are integrated over the whole control volume. All the other forces act on CV surface only.



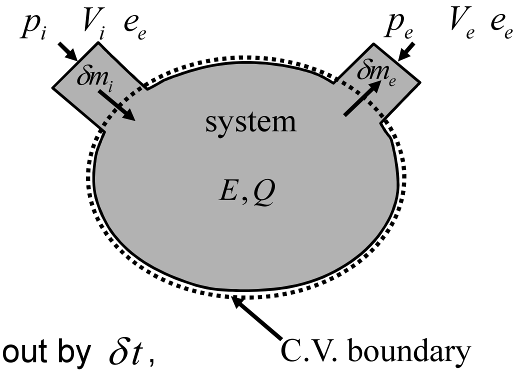
# Conservation of Energy

- From the 1st Law of Thermodynamics $\delta E = \delta Q - \delta W$
- Decompose the work term into "shaft" "viscous" and "pressure" terms

$$\delta W = \delta W_s + \delta W_v + \delta W_p \qquad \delta W_p = p_e \delta m_e / \rho_e - p_i \delta m_i / \rho_i$$

- Consider a CV. Over time $\delta t$, the masses $\delta m_i$ and $\delta m_e$ enter and exit the CV with velocities pressures and energies (per unit mass) $(V, p, e)_i$ and $(V, p, e)_e$ respectively.

$$\delta E = \delta Q - \delta W_s - \delta W_v - \left(p_e/\rho_e\, \delta m_e - p_i/\rho_i\, \delta m_i\right)$$
$$\delta E = e_e \delta m_e - e_i \delta m_i$$
$$\delta Q - \delta W_s - \delta W_v = \left(p_e/\rho_e + e_e\right)\delta m_e - \left(p_i/\rho_i + e_i\right)\delta m_i$$




$$\delta Q - \delta W_s - \delta W_v = \left(p_e/\rho_e + \hat{e}_e + \tfrac{1}{2}V_e^2 + gz_e\right)\delta m_e - \left(p_i/\rho_i + \hat{e}_i + \tfrac{1}{2}V_i^2 + gz_i\right)\delta m_i$$

- For steady flow control volumes, dividing throughout by $\delta t$, taking limits and using $\dot{m}_i = \dot{m}_e = \dot{m}$ for steady flow

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \left(p_e/\rho_e + \hat{e}_e + \tfrac{1}{2}V_e^2 + gz_e - p_i/\rho_i - \hat{e}_i - \tfrac{1}{2}V_i^2 - gz_i\right)\dot{m}$$

# Conservation of Energy (2)

- Continuing from previous slide and dividing throughout by $\dot{m}$

$$p_i/\rho_i + \tfrac{1}{2}V_i^2 + gz_i = p_e/\rho_e + \tfrac{1}{2}V_e^2 + gz_e + (\hat{e}_e - \hat{e}_i) - q + w_s + w_v$$

- Each term has the dimensions of energy per unit mass (m²/s²)
- Internal molecular energy is dependant on temperature. Hence for incompressible steady flow we have: $\rho$ constant, $\hat{e}$ constant, adiabatic flow with $q = 0$
- Finally removing the viscous work term we have, for steady inviscid incompressible flow

$$p_i + \tfrac{1}{2}\rho V_i^2 + \rho g z_i = p_e + \tfrac{1}{2}\rho V_e^2 + \rho g z_e + \rho w_s$$

- This is Bernoulli's equation which can be written as

total pressure inlet = total pressure outlet + shaft work done by the CV

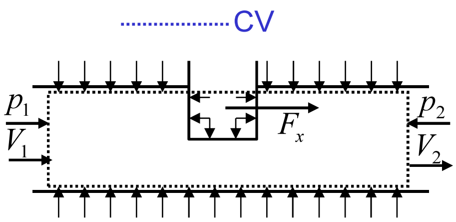


# Example 1: Forces on an Obstruction in a Pipe

- Water at an absolute pressure of 10bar flows steadily along a horizontal pipe of circular cross section and internal diameter 10cm. At some point in the pipe there is a partially closed valve and far downstream the pressure is 9bar. Assuming the flow is inviscid, find the force exerted by the water on the valve.
- Assumptions: Frictionless & incompressible

  Straight streamlines at inlet & outlet (1D approx)

  Horizontal so no hydrostatic terms




- Continuity: $V_1 = V_2$
- Bernoulli's equation (not needed for this Q)

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_1 + \tfrac{1}{2}\rho V_2^2 + \Delta p_{\text{loss}}$$

- Steady Flow momentum in x-direction: $(p_1 - p_2)A + F_x = \dot{m}(V_2 - V_1) = 0$

$$F_x = -(10\times10^5 - 9\times10^5)\times\pi\times0.05^2 = -785N$$

- Notes: The CV cuts the valve.

  Could use a CV that goes around the valve but we would need to know the pressures at A and C

  We have used the pressure drop to find the force but we could find the force and calculate the pressure drop as we do for wind tunnel testing.