EMAT10100 Engineering Maths I
Lecture 4: De Moivre's Theorem and Exponential form

John Hogan & Alan Champneys

Multiplication and division in polar form

- Let $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$
- then $z_1 z_2 = r_1(\cos\theta_1 + j\sin\theta_1)r_2(\cos\theta_2 + j\sin\theta_2)$

$$= r_1 r_2\big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\big]$$
$$= r_1 r_2\big[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)\big]$$

- thus: $|z_1 z_2| = |z_1|\,|z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- Also, (proof is similar) for division we have :

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

- Conclusion:
  - Multiplication, division are nice in polar form
  - BUT: addition, subtraction are nasty in polar form

De Moivre's theorem

- Recall: $|z_1 z_2| = |z_1|\,|z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- Let $z = r(\cos\theta + j\sin\theta)$. Then

$$|z^2| = |z|\,|z| = r^2, \qquad \arg(z^2) = \arg(z) + \arg(z) = \theta + \theta = 2\theta,$$
$$|z^3| = |z^2|\,|z| = r^3, \qquad \arg(z^3) = \arg(z^2) + \arg(z) = 2\theta + \theta = 3\theta,$$
$$|z^4| = |z^3|\,|z| = r^4, \qquad \arg(z^4) = \arg(z^3) + \arg(z) = 3\theta + \theta = 4\theta$$

- De Moivre's theorem:

$$\big[r(\cos\theta + j\sin\theta)\big]^n = r^n\big[\cos(n\theta) + j\sin(n\theta)\big]$$

  - also works for negative integers $-n$
  - $\simeq$ works for all rational powers $\pm m/n$
  - gives methods for evaluating powers and roots

Examples

- Exercise 1: Given $z_1 = 2[\cos(\pi/2) + j\sin(\pi/2)]$
  and $z_2 = 3[\cos(\pi/6) + j\sin(\pi/6)]$,
  find (a) $z_1 z_2$, (b) z_2/z_1, (c) z_2^3
  put answer to (a) in Cartesian form $x + jy$
- Example: Find z_3^6/z_4 for $z_3 = 1 + j\sqrt{3}, \quad z_4 = 1 - j$
- A. Convert to polar form: $z_3 = 2[\cos(\pi/3) + j\sin(\pi/3)]$,
  $z_4 = \sqrt{2}[\cos(-\pi/4) + j\sin(-\pi/4)]$
  then the answer is easy:

$$\frac{z_3^6}{z_4} = \frac{2^6}{\sqrt{2}}\left[\cos\frac{9\pi}{4} + j\sin\frac{9\pi}{4}\right] = 2^5\sqrt{2}\left[\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right]$$

Another way of writing polar form

- It turns out we can write

$$z = r(\cos\theta + j\sin\theta) = r\exp(j\theta) \quad \text{(or } re^{j\theta}\text{)}$$

  which is much simpler to write . . .
- and De Moivre's theorem written this way is trivial

$$z^n = (re^{j\theta})^n = r^n(e^{j\theta})^n = r^n e^{jn\theta}$$

- "What on earth does it mean to raise something to an imaginary power?!"
- two options:
  - either just accept that $e^{j\theta} = \cos\theta + j\sin\theta$
  - or I will try to persuade you (one of two ways) . . .

Justification of exponential form

- Let $w = r(\cos\theta + j\sin\theta)$. Then

$$\frac{\mathrm{d}w}{\mathrm{d}\theta} = r(-\sin\theta + j\cos\theta), \\ = jr(\cos\theta + j\sin\theta), \quad = jw$$

- Therefore, w solves the differential equation

$$\frac{\mathrm{d}w(\theta)}{\mathrm{d}\theta} = jw(\theta), \qquad w(0) = r$$

- But, from theory of ODEs (next term), solution is

$$w(\theta) = re^{j\theta}$$

- therefore $r(\cos\theta + j\sin\theta) = w(\theta) = re^{j\theta}$ Q.E.D.

An alternative justification

- Consider Maclaurin series expansion (later this term)

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots$$
$$j\sin\theta \approx j\theta - j\frac{\theta^3}{6} + j\frac{\theta^5}{120} + \dots$$

- adding, we get

$$\cos(\theta) + j\sin(\theta) \approx 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{6} + \frac{\theta^4}{24} + j\frac{\theta^5}{120} + \dots$$

- But $e^z \approx 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \dots$, so

$$e^{j\theta} \approx 1 + j\theta + j^2\frac{\theta^2}{2} + j^3\frac{\theta^3}{6} + j^4\frac{\theta^4}{24} + j^5\frac{\theta^5}{120} + \dots$$

  and two expressions are equal because $j^2 = -1$

Logarithms of complex numbers

- Suppose z given in exponential form $z = re^{j\theta}$.
  What is $\ln z = \log_e z$?
-

$$\ln z = \ln\left(re^{j\theta}\right) \\ = \ln r + \ln\left(e^{j\theta}\right) \\ = \ln r + j\theta \qquad \text{plus arbitrary integer multiple of } 2\pi j$$

- Alternative notation: $\ln z = \ln|z| + j(\arg z + 2n\pi)$
- Exercise 2: find
  - $\ln(-1)$
  - $\ln(j)$
  - j^j

  will give the answers after the break

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Lecture 5: Applications of de Moivre's Theorem

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Recap

- three equivalent forms:
  - $z = x + jy$ (Cartesian form)
  - $z = r(\cos\theta + j\sin\theta)$ (polar form)
  - $z = r\exp(j\theta)$ (exponential form)
- which to use depends on question
  - e.g. Cartesian is good for addition/subtraction
- tip: when converting to or from polar form, draw the Argand diagram
- De Moivre's theorem is obvious in exponential form

$$z^n = (re^{j\theta})^n = r^n e^{jn\theta} = r^n[\cos(n\theta) + j\sin(n\theta)]$$

- exponential form is good for taking roots . . .

Roots of complex numbers

- A positive real number x has two square roots $\pm\sqrt{x}$
- Negative numbers also have two square roots

e.g. $y_1 = j$ and $y_2 = -j$ both solve $y^2 = -1$

- We know any polynomial $z^2 - c = 0$, (where c is any complex number) must have two roots.
- More generally, $z^n = c$ must have n roots.
- but how do we find these n^{th} roots?

example

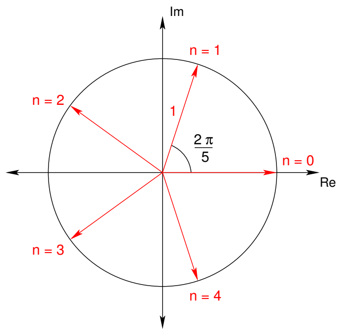
- find the "fifth roots of unity" $z = 1^{1/5}$
- Use exponential form and a trick to add $2n\pi$ to argument

$$1 = 1e^0 = e^{0+2n\pi j}, \qquad n \text{ arbitrary integer}$$

- hence $z = 1^{1/5} = (e^{2n\pi j})^{1/5} = e^{2n\pi j/5}$
  - so there are 5 distinct solutions (with different arguments) for $n = 0, 1, 2, -1, -2$ (or $n = 0, 1, 2, 3, 4$)

$$z = 1,\ e^{2\pi j/5},\ e^{4\pi j/5},\ e^{-2\pi j/5},\ e^{-4\pi j/5}$$

  - Argand diagram: equally spaced



Exercise

- (a) Find all solutions to $z = (1 + j)^{1/3}$

  Hint: first write $1 + j$ in polar form.

  & then (trick) add $2n\pi j$ to the argument . . .

- (b) Plot the solutions on the Argand diagram

Engineering HOT SPOT - AC circuits

- Suppose circuit with current $i = A\sin(\omega t)$ ampères
- how find voltages in circuit? (Ohm's law is for DC)
- the voltages across certain components are
  - resistor R ohms: $v_R = RA\sin(\omega t)$
  - capacitor C farads: $v_C = \frac{A}{\omega C}\sin(\omega t - \frac{\pi}{2})$
  - inductor L henries: $v_L = \omega LA\sin(\omega t + \frac{\pi}{2})$
- Idea: write everything in terms of complex numbers

$$i = \text{Im}(Ae^{j\omega t}), \qquad v_R = \text{Im}\left(RAe^{j\omega t}\right)$$

$$v_C = \text{Im}\left(\frac{A}{\omega C}e^{j\omega t - j\pi/2}\right) = \text{Im}\left(\frac{-j}{\omega C}Ae^{j\omega t}\right)$$

$$v_L = \text{Im}\left(\omega LAe^{j\omega t + j\pi/2}\right) = \text{Im}\left(j\omega LAe^{j\omega t}\right)$$

Using de Moivre for multi-angle formulae

- Example: use De Moivre's theorem to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.
- A. Write $\cos(3\theta) = \text{Re}(e^{3j\theta})$, $\sin(3\theta) = \text{Im}(e^{3j\theta})$

$$\begin{aligned}\cos(3\theta) =& \text{Re}\left[(e^{j\theta})^3\right] = \text{Re}\left[(\cos\theta + j\sin\theta)^3\right] \\ =& \text{Re}\left[\cos^3\theta + 3j\cos^2\theta\sin\theta \right. \\ & \left. -3\cos\theta\sin^2\theta - j\sin^3\theta\right] \\ =& \quad \cos^3\theta - 3\cos\theta\sin^2\theta = \quad 4\cos^3\theta - 3\cos\theta\end{aligned}$$

- similarly $\sin 3\theta = -4\sin^3\theta + 3\sin\theta$
- Exercise: express $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$

Homework

- read *James* Sect. 3.2.7–3.2.10, Sect. 3.3.1
- attempt *James* exercises: 3.2.8 Qs.19, 24,277

  & exercises 3.3.3 Qs. 28

Next lecture: (final one on complex numbers):

- hyperbolic functions
- links with trig functions
- more homework (try not to get behind)