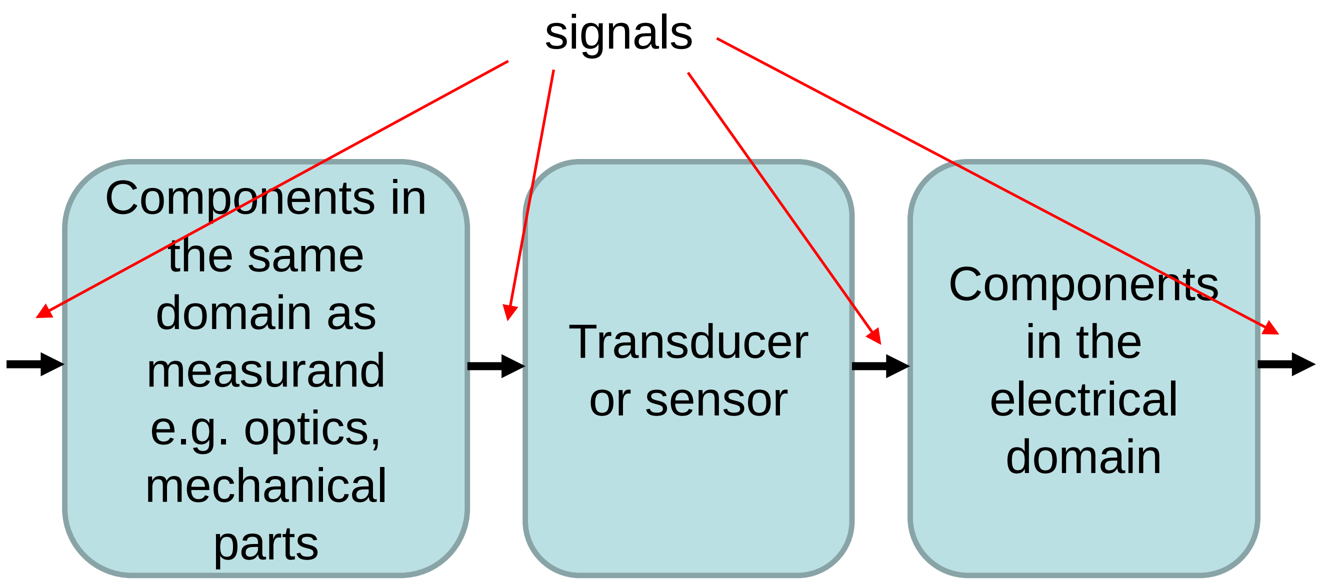
Signals Theory 1.1

An Introduction

What are signals?

- Signals are some phenomenon that can be described quantitatively.
- They carry information; sometimes transfer energy.
- In our block diagram they are the link between system blocks.
- In this lecture you will likely here reference to ‘functions’, ‘wave’, ‘waveforms’ and ‘signals’.

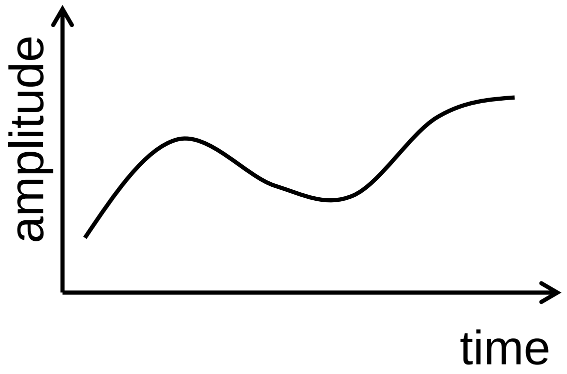


How are they described?

- Theory enables us to describe signals in several domains;
  - Time (t)
  - Discrete time [n]
  - Frequency (ω)
  - Complex frequency (s)
  - Discrete complex frequency (z)
- We have tools to transform our signals (and, as we shall see later in the unit, systems) from one domain to another.
- This is very useful – operations in some domains are much harder to do than in others. We transform between domains to help solve engineering problems.

Continuous signals - x(t)

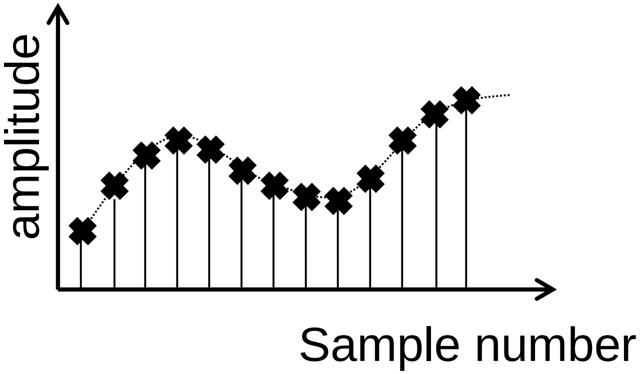
- A continuous signal is defined for all instances of time, over the duration of the signal, and can have any value between maximum and minimum limits.
  - Often described as ‘analogue signals’



- All physical signals that we measure are continuous, however we often discretise signals in time and amplitude….

Discrete signals - x[n]

- A discrete signal is only defined for a particular instance of time.
  - Associated with digital systems
  - Time interval can be uniform or variable, even random.
  - Often produced by sampling a continuous signal

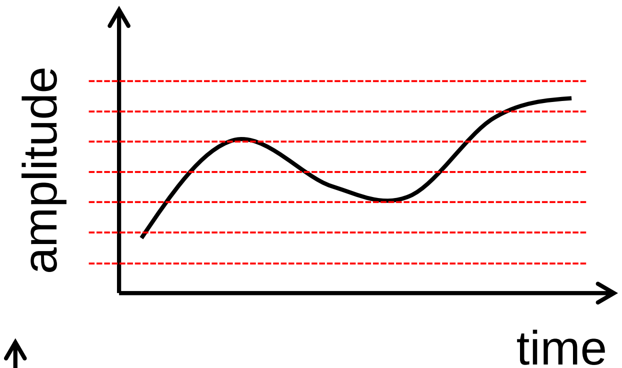


$$x[n] = x(nT),$$

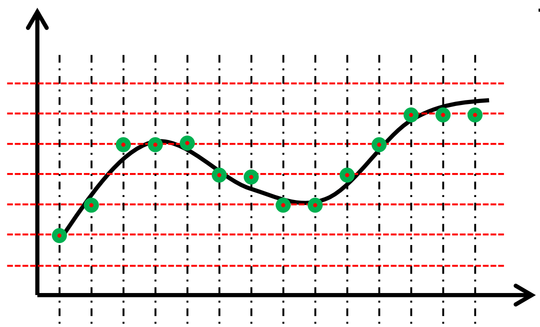
where n is an integer and T is the sampling interval; 1/T is the sampling frequency

Quantised signals

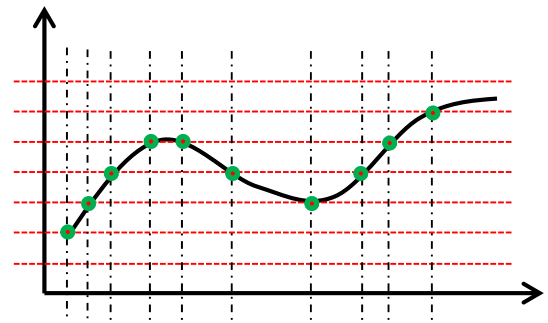
- We can also quantise the magnitude – a requirement for digital systems.



But how do we map the analogue signal onto these fixed levels?



Mapping at fixed time intervals



Mapping when signal passes through threshold

- Discretised and quantised mean the same thing, but to keep our discussions clear let’s think of discretised time and quantised amplitude.

Periodicity

- Periodic signals repeat indefinitely with a fixed period:
  - Mathematically:
    - $x(t) = x(t + T)$ where T is the period
    - $x[n] = x[n + N]$ where N is the number of samples each period
  - Examples;
    - Sine wave
    - Triangular waveform
    - Series of pulses
- Aperiodic signals do not repeat;
    - Exponential decay
    - Impulse function
    - Random signal

Elementary Transformations

- $y(t) = \alpha x(\gamma t + \delta) + \beta$

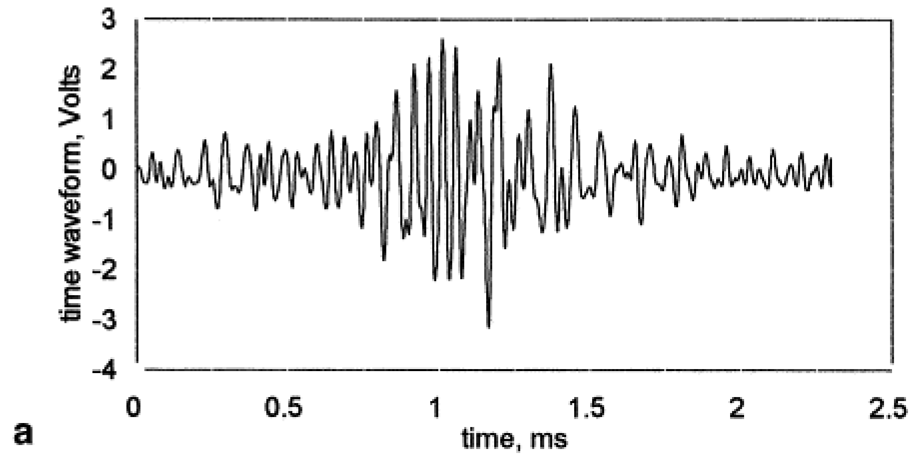
- $y[n] = \alpha x[\gamma n + \delta] + \beta$

- α = Amplitude scaling - makes signal larger or smaller
- β = Amplitude shifting - alters the mean level of the signal
- γ = time scaling - stretches or compresses the signal
- δ = time shifting - signal occurs earlier or later

Signals Theory 1.2

Time domain signal metrics – means and moments

Basic time domain statistical measures



How can we characterise this signal?

1. Min/Max values
2. Peak-to-peak value
3. Mean value
4. Period of dominant components
5. Envelope – rise times, fall times
6. RMS – *good indication of 'energy'*
7. Moments

Statistical moments

$$m_i = \frac{1}{T}\int_0^T [x(t)]^i dt$$

Raw moment

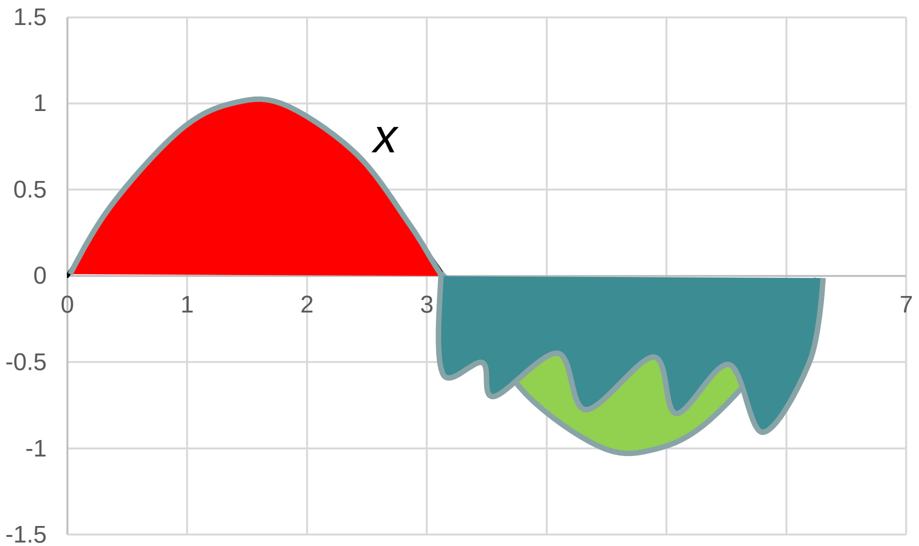
$$m_i = \frac{1}{T}\int_0^T [x(t) - \bar{x}]^i dt$$

Central moment

The first moments are familiar: from the 1st moment we derive mean, from the 2nd moment, variance.

Higher order moments indicate properties of waveform shape – e.g. ‘skewness’ and ‘peakyness’

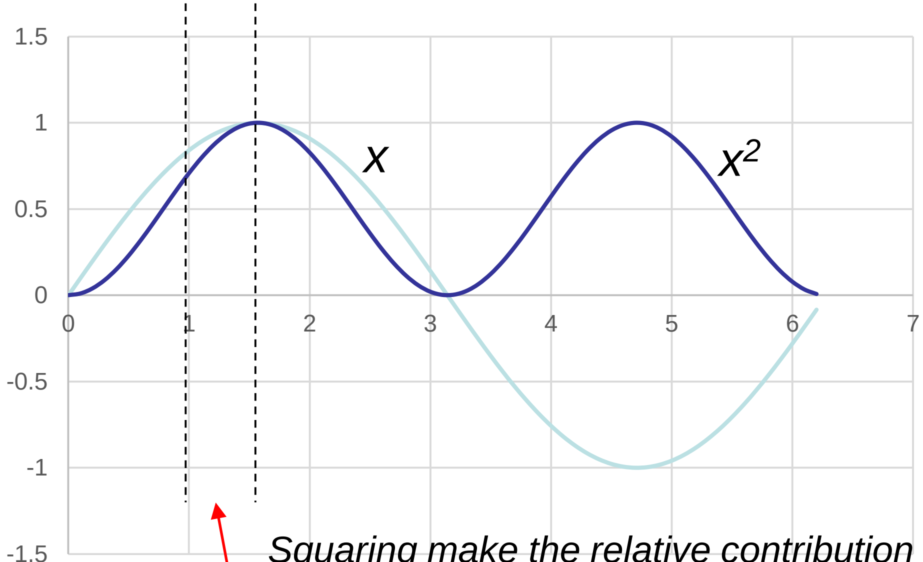
What is the physical significance of i = 1?



The moments give us an average value over time, and from this average value we can determine something about the shape of the waveform.

- When i = 1 we are comparing the area on each side of zero (or the expected value) .
- All that matters is area – this area could be any shape.

What is the physical significance of i = 2?

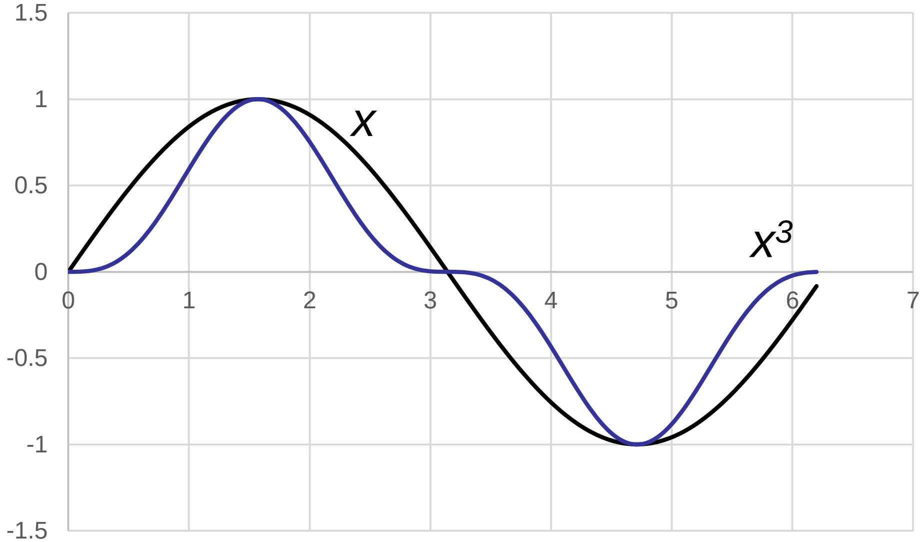


Squaring make the relative contribution to the average from t=1 less than t =1.5

Consider the graph of x and x^2. The 2nd moment (x^2) produces a non-zero average, even with an expected value of 0. This is should be familiar as the variance.

- When i = 2 the average captures deviation from the mean, irrespective of the sign.
- However squaring places higher weigh on larger deviations, hence std. dev. is the square root of the 2nd moment

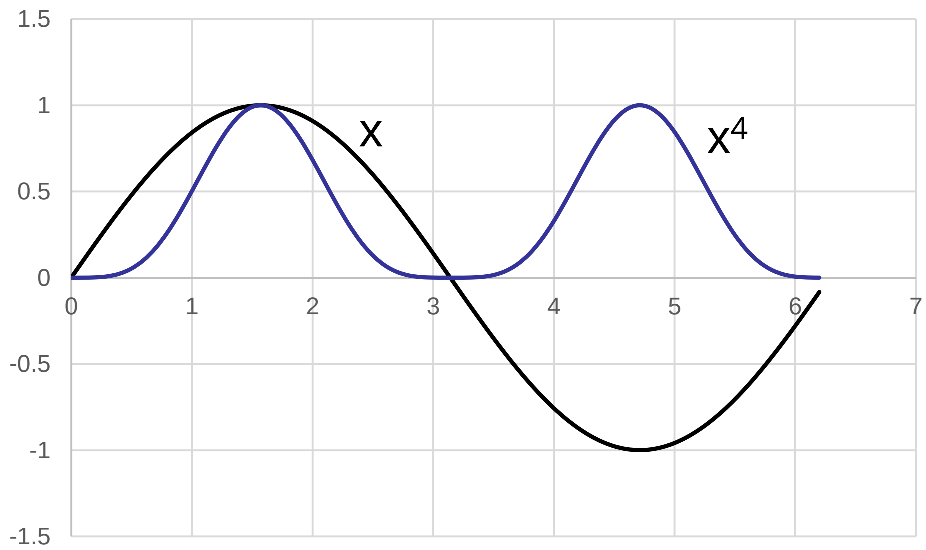
What is the physical significance of i = 3?



The 3rd moment further accentuates the weighting on higher values and but can produce a zero value if the weighted positive and negative excursions are similar.

- When i = 3 the symmetry of the waveform is captured .
- For a mean of 0, it is not just sufficient to have equal areas (equal areas only produce zero 3rd moment if the shape is identical).

What is the physical significance of i = 4?



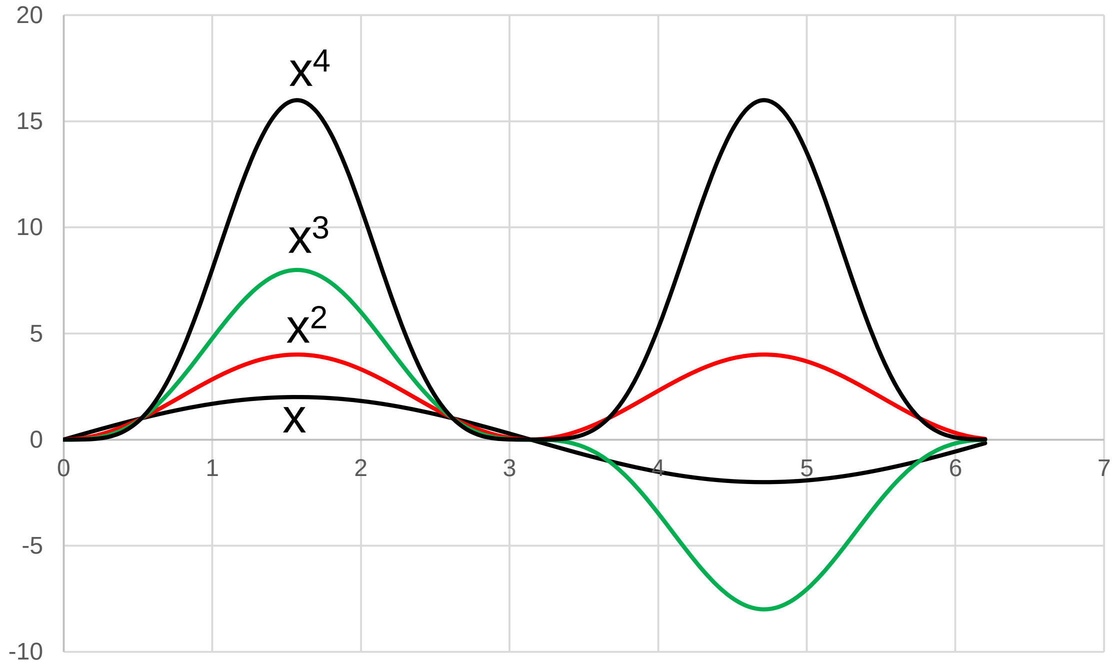
Like variance, the 4th moment has a non-zero average for an expected value (mean) of 0. However the high values are even more accentuated

- When i = 4, similar to i = 2, the mean is no longer zero for a non-zero valued sequence.
- The 4th moment further accentuates the weight of the higher values in the average.

What is the general effect

- If ‘i’ is odd – we get some measure of symmetry of a waveform.
  - If ‘i’ is even, symmetry is irrelevant
- As ‘i’ increases we get increasing sensitivity to the high values of a waveform or the ‘peakyness’

What's the downside of moments?



- In the previous examples we masked a key feature of moments by having a maximum signal value of 1.
- Increasing higher moments 'amplify' the signal making comparison difficult – the units of the i^{th} moment are the (signal units)^i

What's the downside of moments?

$$i^{th} powermean = \left(\frac{1}{T} \int_0^T x^i(t) dt \right)^{\frac{1}{i}}$$

- The power mean partially addresses this – by taking the i^{th} root of the moment.
  - You will notice that the standard deviation is the 2^{nd} power mean when the expected value is zero.
- Hence the power mean has the same units as the original signal.
- We can go a stage further to focus on shape irrespective of magnitude…….

Standardised moments

$$K_i = \frac{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^i \, dt}{\sigma^i}$$

$$K_4 = \frac{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^4 \, dt}{\sigma^4} = \frac{\frac{1}{T}\int_0^T [x(t) - \bar{x}]^4 \, dt}{x_{rms}^4}$$

By dividing the i^{th} moment by the standard deviation (or RMS if the expect value is zero) to the 'i', we standardise the moment and it is a dimensionless measure of shape.

The 4^{th} standardised moment is known as 'Kurtosis' and is widely used in fault detection.

Moments in probability distribution

- The probability density function of a sample provides the *relative likelihood* that the value of a random variable is equal to that sample.

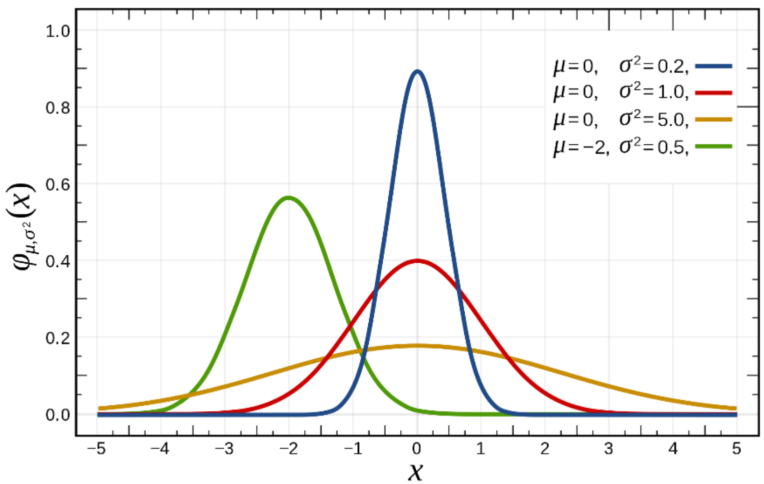
- In probability theory, the probability density function often exhibits a Gaussian distribution.

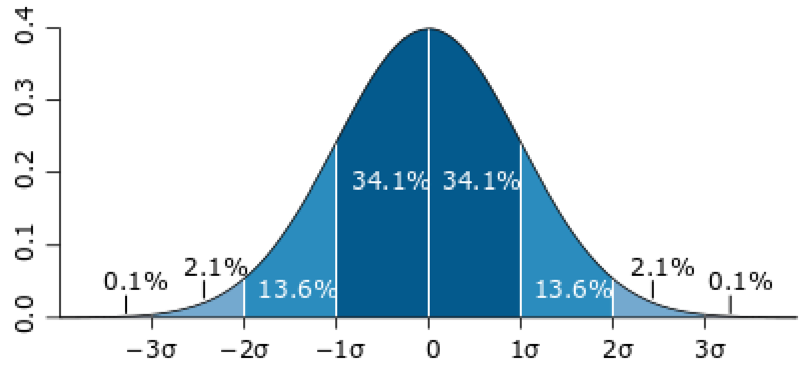
$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}((x-\mu)/\sigma)^2}$$

where μ is the mean (first moment) and σ is the standard deviation (second moment).

- A Gaussian distribution is observed for instance in metrology, where in absence of systematic errors the measurand coincides with the mean value, and the standard deviation is used to estimate the uncertainty.

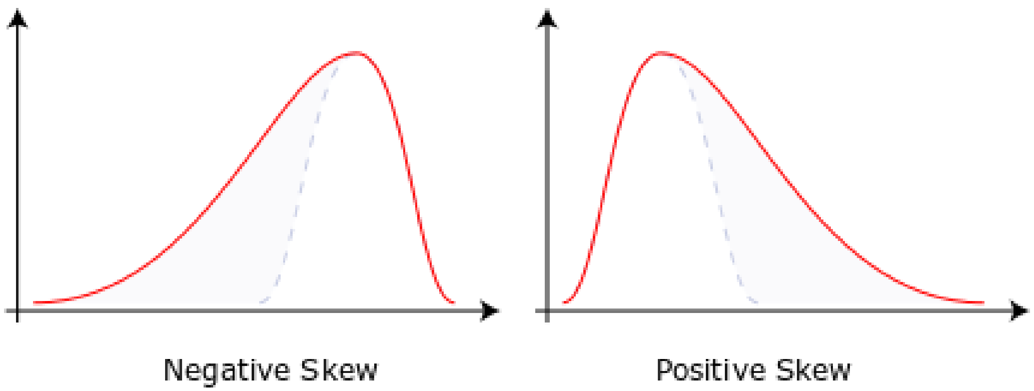
Moments in probability distribution





- The uncertainty is expressed in relation to the confidence interval.
- For example, the uncertainty for a confidence interval of 68% is plus/minus the standard deviation $\pm\sigma$, while for a confidence interval of 95%, the uncertainty is $\pm 2\sigma$.

Moments in probability distribution



- The skewness represents a deviation from the Gaussian (or *normal*) distribution.
- The kurtosis of a normal distribution is equal to 3. Values of the kurtosis lower than 3 are characterised by fewer and less extreme events than in the normal distribution, and vice-versa.

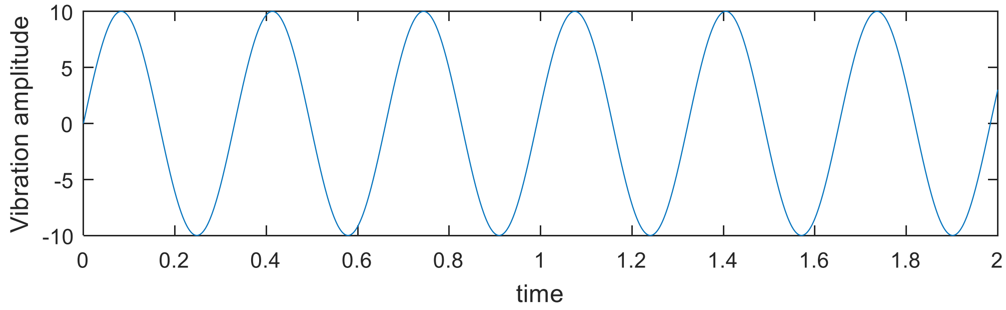
Means and moments - units

- Power means have the same units as the original signal
  - The rms of a voltage signal has units of volts.

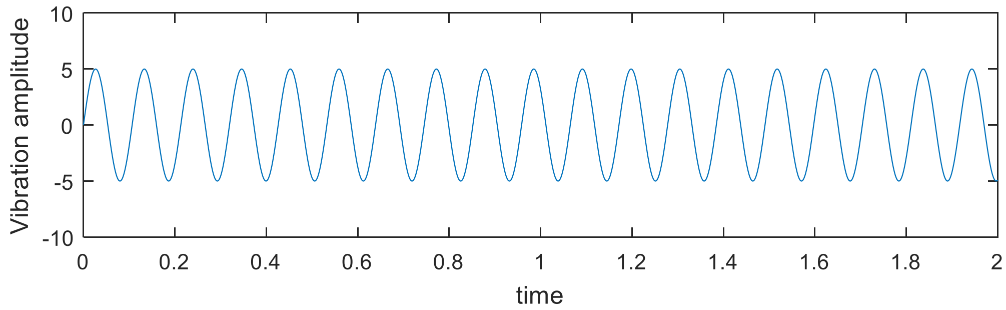
- The ‘i^{th}’ moment or central moment has units of the variable to the power ‘i’
  - A voltage signal has moments with units of v^i.

- A standardised moment is dimensionless
  - The value is an indication of shape only

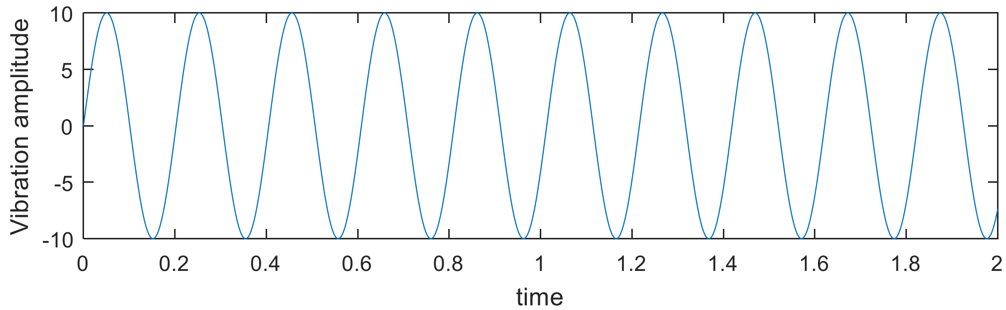
Example time domain metrics



Mean = 0
RMS = 7.07
Skewness = 0
Kurtosis = 1.5

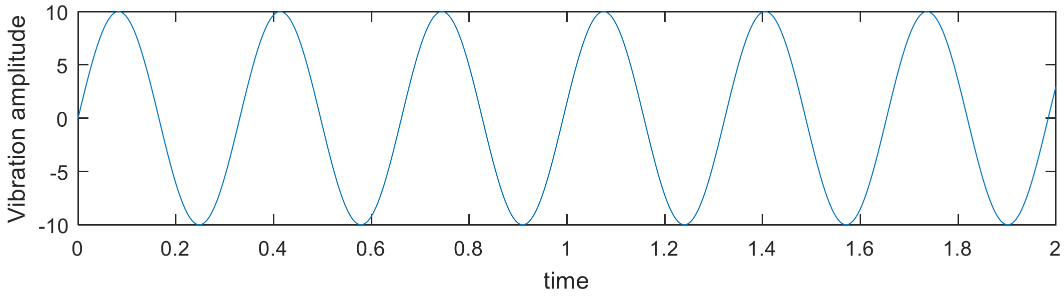


Mean = 0
RMS = 3.53
Skewness = 0
Kurtosis = 1.5

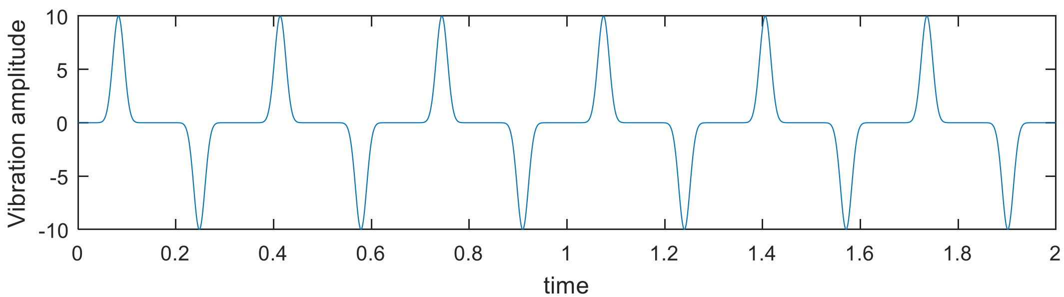


Mean = 0
RMS = 7.07
Skewness = 0
Kurtosis = 1.5

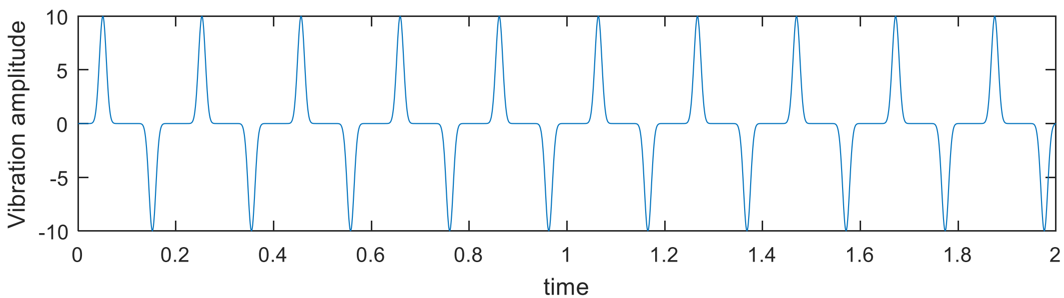
Example time domain metrics



Mean = 0
RMS = 7.07
Skewness = 0
Kurtosis = 1.5

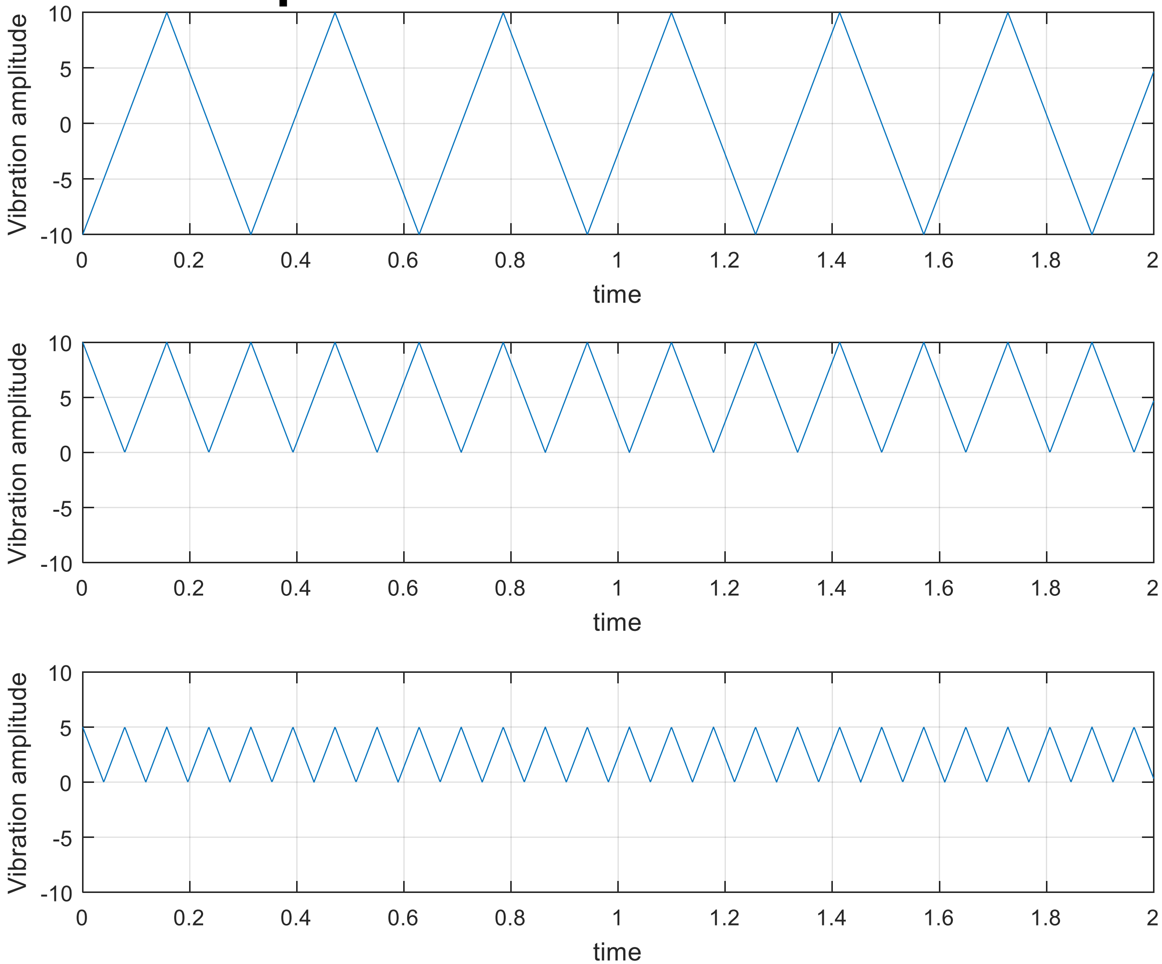


Mean = 0
RMS = 3.2
Skewness = 0
Kurtosis = 5.8



Mean = 0
RMS = 3.2
Skewness = 0
Kurtosis = 5.8

Example time domain metrics



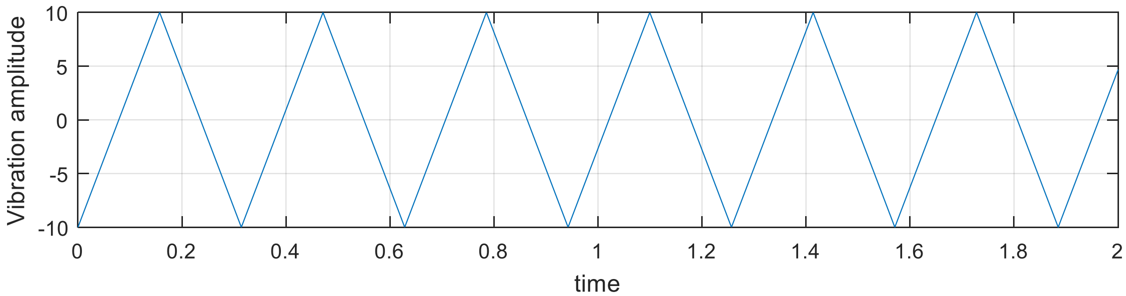
Mean = 0
RMS = 5.77
Skewness = 0
Kurtosis = 1.8

Mean = 5
RMS = 5.77
Skewness = 0
Kurtosis = 1.8

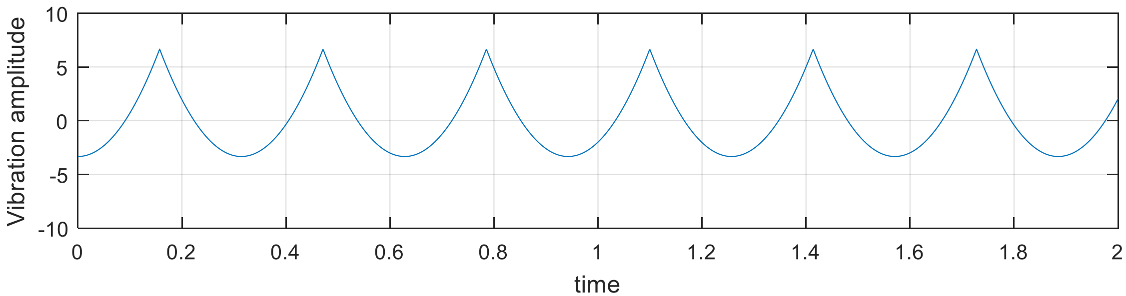
Mean = 2.5
RMS = 2.88
Skewness = 0
Kurtosis = 1.8

Because the standardised moments are taken around the expected value (mean), these waveforms all have 0 skewness (note we could set the expected value to something other than the mean and we would see skewness)

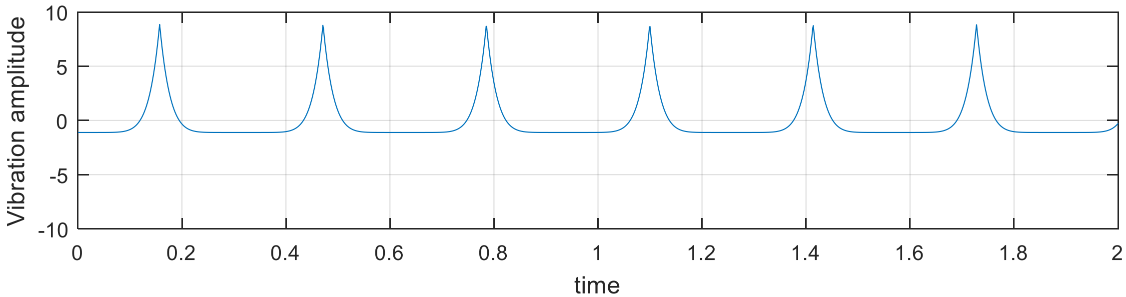
Example time domain metrics



Mean = 0
RMS = 5.77
Skewness = 0
Kurtosis = 1.8



Mean = 0
RMS = 2.98
Skewness = 0.64
Kurtosis = 2.14



Mean = 0
RMS = 2.15
Skewness = 2.3
Kurtosis = 7.6

These waveforms have non-symmetry around the mean, hence produce skewness.