1

Stiffness Design at Part Level

Axial: $u = \dfrac{FL}{EA}$

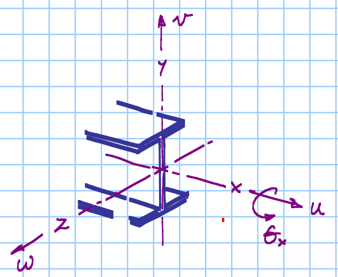
Shear: $v_s = k_s \dfrac{PL}{GA}$

Bending: $v_b = k_b \dfrac{PL^3}{EI}$

Torsion $\theta_x = \dfrac{TL}{GJ}$

These expressions assume prismatic elements and linear elasticity and so are appropriate for simple elements up to limit load. Beyond limit load we can expect non-linearity in material and possibly structural response and an underestimate from these basic equations.

Axial, bending and torsional displacements are covered in detail in StM1/2. Here,we will look at displacement of non-prismatic elements and displacements due to shear in more detail.



2

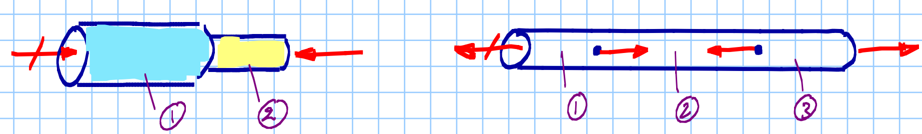
Deflection of Non-Prismatic Elements

Axial deformation

* Discretely varying elements or forces:

i.e. discrete changes of section, material or load along the length

E.g.:



Simply sum the change in length for each element:

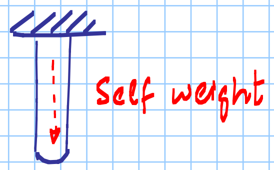
$$u = \sum_{i=1}^{n} \frac{N_i L_i}{E_i A_i}$$

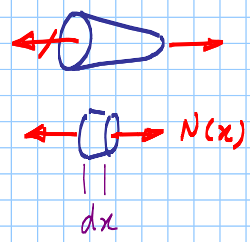
where N_i is the internal axial force

* Continuously varying elements

Consider the change in length of a differential element then integrate over the length of the bar.

i.e. continuosly varying force or cross-sectional area:





E.g., continuosly varying cross-sectional area or force due to self weight or centrifugal force etc.

If the expressions for *N(x)* and *A(x)* are simple we can integrate over the length:

$$u = \int_0^L du = \int_0^L \frac{N(x)}{EA(x)} dx$$

Note errors will increase with taper due to the assumption of uniform stress e.g. for a taper of 20 degrees the error in stress based on *P/A* will be about 3 percent compared with more advanced methods.

If the expressions for *N(x)* and *A(x)* are complex then we will need to use a numerical method.

Bending deformation

*Discretely varying beams

Beams with discretely changing prismatic sections

a) Solving by differential equation of bending (with the Heaviside function)

Allow for different values of EI in each bay, E_iI_i.

$$\frac{d^2v}{dx^2} = \frac{M(x)}{E_iI_i}$$

for bay "i"

integrate: $v'' \Rightarrow v' \Rightarrow v$ SHM1
and use boundary conditions

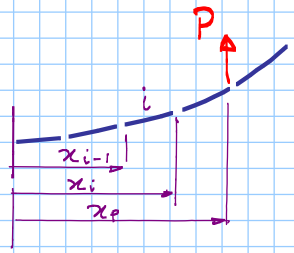
b) Solving by superposition

Consider the beam as a series of connected elements and calculate the deflection components of each elment separately then add the deflection of each element by superposition.

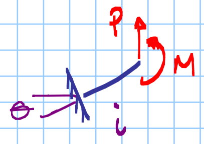
E.g. solution of a non-prismatic cantilever by superposition:

Model beam as a series of incremental cantilevers, i.e.:

E.g.:



for element i:



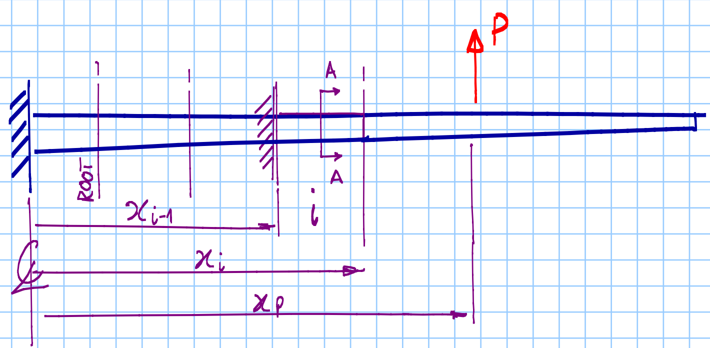
where $M = P.(x_p - x_i)$

Add the components of deflection of each element cumulatively from the root:

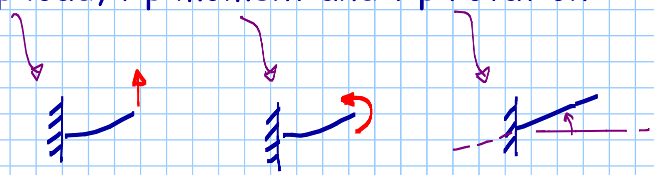
$$\left.\begin{array}{lll} \text{Eg. due to tip load} & & v_{P_i} \\ \text{.. .. ..} & \text{bending moment} & v_{M_i} \\ \text{.. .. ..} & \text{rigid body rotation} & v_{\theta_i} \end{array}\right\} \quad v = \sum \left(v_{P_i} + v_{M_i} + v_{\theta_i} \right)$$

E.g. "Incremental cantilever beam model"

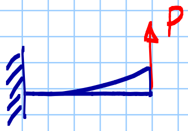
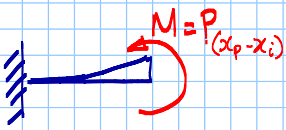
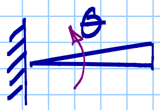
as a system of connected cantilevers with end load and moment.



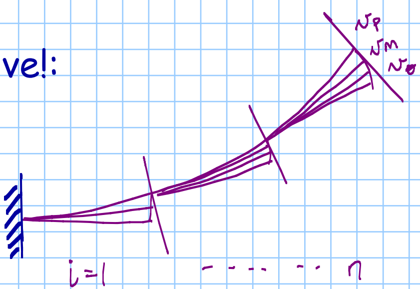
Sum the tip deflections for each cantilever due to:
tip load, tip moment and tip rotation



Cantilever deflections

		Deflection	Rotation (rad)
Tip load:		$v_P = \dfrac{PL_i^3}{3EI_i}$,	$\theta_P = \dfrac{PL_i^2}{2EI_i}$
Tip moment:		$v_M = \dfrac{ML_i^2}{2EI_i}$,	$\theta_M = \dfrac{ML_i}{EI_i}$
Tip rotation:		$v_{\theta_i} = \theta_i L_i$,	$\theta_i = \sum_{i=1}^{i-1} \theta_{P_i} + \theta_{M_i}$

Cumulativel:



*Continuously varying beams

E.g. tapering beams

a) Solving by differential equation of bending (with the Heaviside function)

Here we need to allow for the varying value of I as a function of x

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI(x)}$$

integrate: $v'' \Rightarrow v' \Rightarrow v$
and use boundary conditions

StM1

b) Solving by numerical methods

Solving by integration, above, is only practical for simple cases where integrations are easy to perform but the method soon becomes prohibitive so that numerical methods must be resorted to. An example of a tabulated numerical technique from Roark [] is given below, where tapers in beam width and depth are accounted for.

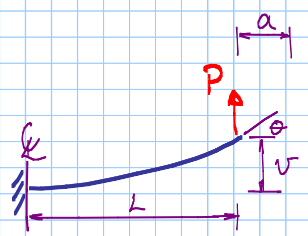
Deflection of a tapered beam by tabulated coefficients

For beams which taper along the length, eg. in width, depth, or both, where the differential equation of bending can be integrated directly, tabulated solutions are available based on deflection coefficients, e.g. Roark 7.8 $\frac{d^2v}{dx^2} = \frac{M(x)}{EI(x)}$

Start by selecting a suitable part model, Ref: Roark Table 3

E.g. Cantilever:

Roark Ia:



Eg. for a=0, tip loaded cantilever:

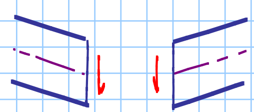
$v = \frac{1}{3}\frac{PL^3}{EI}$, $\theta = \frac{1}{2}\frac{PL^2}{EI}$

Note - only limited cases are supported
- the variable beam model only caters for tapers of I_B/I_A <= 8.0
- taper factors "k_θ" and "k_v" apply to rotation v' and deflection, v, for a prismatic beam of the tapered end A dimensions.

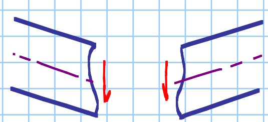
Shear Deflection

In beams of small span depth ratio "L/d" the shear stresses can be relatively high and the resulting deflection due to shear may be significant.

Simplistically we often envisage shear deflection as a straight sided parallelogram deformation: E.g:

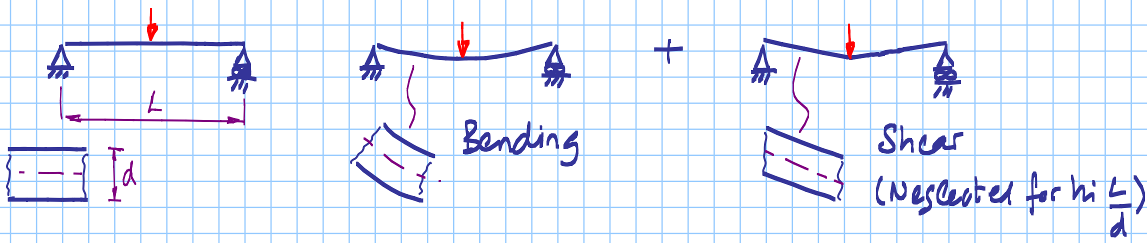


But:
shear stresses and resulting shear strains cause distortion of the cross-section so that cross-section planes are no longer plane as assumed in simple bending.



The simple bending assumption of "plane sections" will be in error when this shear distortion becomes significant.

Shear deflection can be considered additional to bending deflection and added by simple superposition, Eg.:



For L/d ratios down to 3 the deflection due to shear can be found by energy methods:

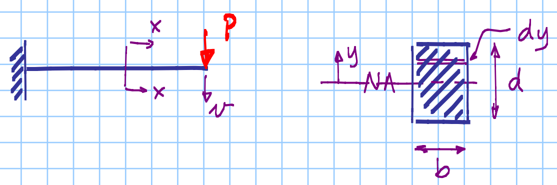
see Stm1

Strain energy due to shear per unit volume: $= \dfrac{\tau^2}{2G}$ — (1)

↳ Integrating for whole beam: $U_s = \dfrac{1}{2G} \iint \tau^2 dA . dx$ — (2)

where dA is an element of cross-section area and dx is an element of length

Eg. for a tip loaded cantilever of rectangular cross-section:



τ = parabolic fn of y

where y = distance from N/A

$A = b.d$

$dA = b dy$

I.e $\tau = \dfrac{6S}{bd^3}\left(\dfrac{d^2}{4} - y^2\right)$

See StM1, 2 notes

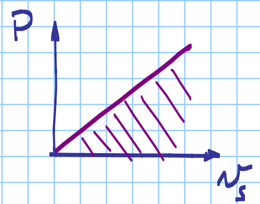
(Also, [2] p 171)

(2) ↳ $U_s = \dfrac{3}{5}\dfrac{P^2 L}{G b d}$ (3)

But for linear behaviour: $U_s = \frac{1}{2} P . v_s$ where v_s = deflection due to shear

↳ $v_s = \dfrac{2 U_s}{P}$

(3) ↳ $v_s = \dfrac{6}{5}\dfrac{PL}{GA}$



Intuitively the factor in this equation can be assumed to apply to the rectangular cross-section area to give the "effective shear area".

E.g. generalising: $v_s = \dfrac{PL}{G\frac{5}{6}A}$

or: $v_s = \dfrac{PL}{GA_s}$ for a tip loaded cantilever

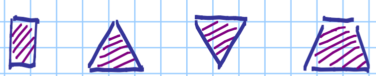
where A_s = effective shear area

I.e.: $A_s = \dfrac{5}{6}A$ for a rectangular cross-section where $A = bd$

Using this intuitive generic form, effective shear areas can be listed for a range of beam cross-sections, e.g. see below.

Beam Cross-sections and Effective Shear Areas

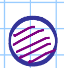
Vertical shear load assumed



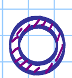
Rectangle, triangle, trapezoid with // sides top + btm

$A_s = \dfrac{5}{6}A$ where A = actual area

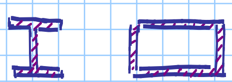
Diamond $A_s = \dfrac{30}{31}A$



Solid circle $A_s = \dfrac{9}{10}A$



Thin wall tube $A_s = \dfrac{1}{2}A$

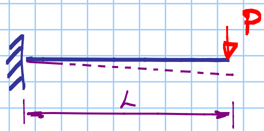


I-section or box-section $A_s = A_w$ where A_w = web area

Similarly, we can list shear deflection expressions for different beam configurations. I.e.:

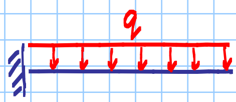
Beam Configurations:

Cantilever with tip load:



$$v_s = \frac{PL}{GA_s}$$

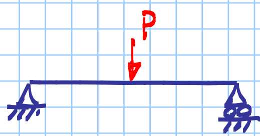
Cantilever with UDL:



$$v_s = \frac{1}{2}\frac{qL^2}{GA_s} = \frac{1}{2}\frac{PL}{GA_s}$$

where $P = qL$ ie total load

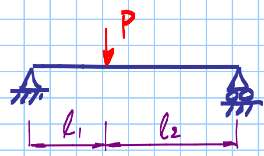
Simply supported beam with central load



$$v_s = \frac{1}{4}\frac{PL}{GA_s}$$

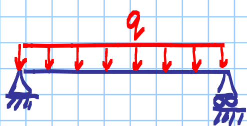
c/w tip loaded cantilever of length $L/2$

Simply supported beam with offset load



$$v_s = \frac{l_1 . l_2}{L}\frac{P}{GA_s}$$

Simply supported beam with UDL



$$v_s = \frac{1}{8}\frac{qL^2}{GA_s} = \frac{1}{8}\frac{PL}{GA_s}$$

c/w UDL cantilever of length $L/2$ and tip load $P/2$ $\quad$ $P = qL$ ie total load

Generalising: For k_s = shear deflection constant $\quad v_s = k_s \frac{PL}{GA_s}$

Further Notes:

- The deflection due to shear will usually be negligible for thick metallic beams, particularly for steel, where shear modulus is relatively high, but for thin wall beams and wooden and composite beams (where shear modulus, G, is low compared with Young's modulus, E) deflection due to shear becomes more important.

- Shear deflection can be inherently accounted for in simple bending calculations by using a value for E obtained from bending tests on beams of similar L/d proportions.

  Alternatively, for L/d ratios in the range $12 < L/d < 24$, using a valve of E 10% less than the value obtained by direct compression will give an inherent account of shear deflection in simple bending deflection calculation.

  For larger L/d ratios > 24 shear deflection will generally be negligible.

- For extremely short deep beams the basic assumption of linear stress distribution underpinning simple bending theory is no longer valid. $\sigma = \frac{My}{I}$ is valid down to $L/d = 3$ but for $L/d < 3$ the stress distribution changes radically.

  Roark tabulates correction factors for direct and shear stresses as obtained from $\sigma = \frac{My}{I}$ and $\tau = \frac{V}{A}$ for low values of $L/d < 3$.

For non-prismatic beams the shear deflection of each element can simply be added linearly or integrated over the beam length, i.e.:

* Discretely varying beams or loading: $v_s = \sum_{i=1}^{n} v_{s_i} = \sum_{i=1}^{n} \frac{V_i L_i}{G_i A_{s_i}}$

* Continuously varying beams or loading: $v_s = \int_0^L v_s \, dx = \int_0^L \frac{V(x)}{G A(x)} \, ds$

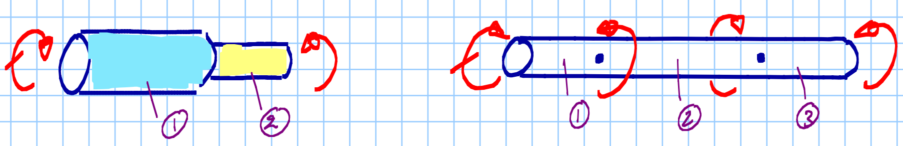
where V_i, or $V(x)$ is the internal shear force

Torsional deflection "twist"

* Discretely varying shafts or loading:

i.e. discrete changes of section, material or load along the length

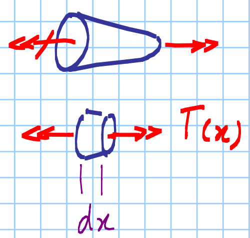
E.g.:



Simply sum the angular twist for each element: $\theta = \sum \theta_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$

* Continuously varying shafts or loading:

Consider the change in angular rotation of a differential element then integrate over the length of the bar.



$$\theta = \int_0^L d\theta = \int_0^L \frac{T(x)}{G J(x)} \, dx$$

i.e. for continuosly varying torque or cross-sectional area:

References:

Roark "Formulas for stress strain" 6th Ed.
McGraw-Hill 1989, ISBN 0-07-100375-8

Gere + Timoshenko "Mechanics of Materials" 4th Ed.
PWs Publishing Co., 1997, ISBN 0-534-93429-3

Ryder "Strength of Materials"