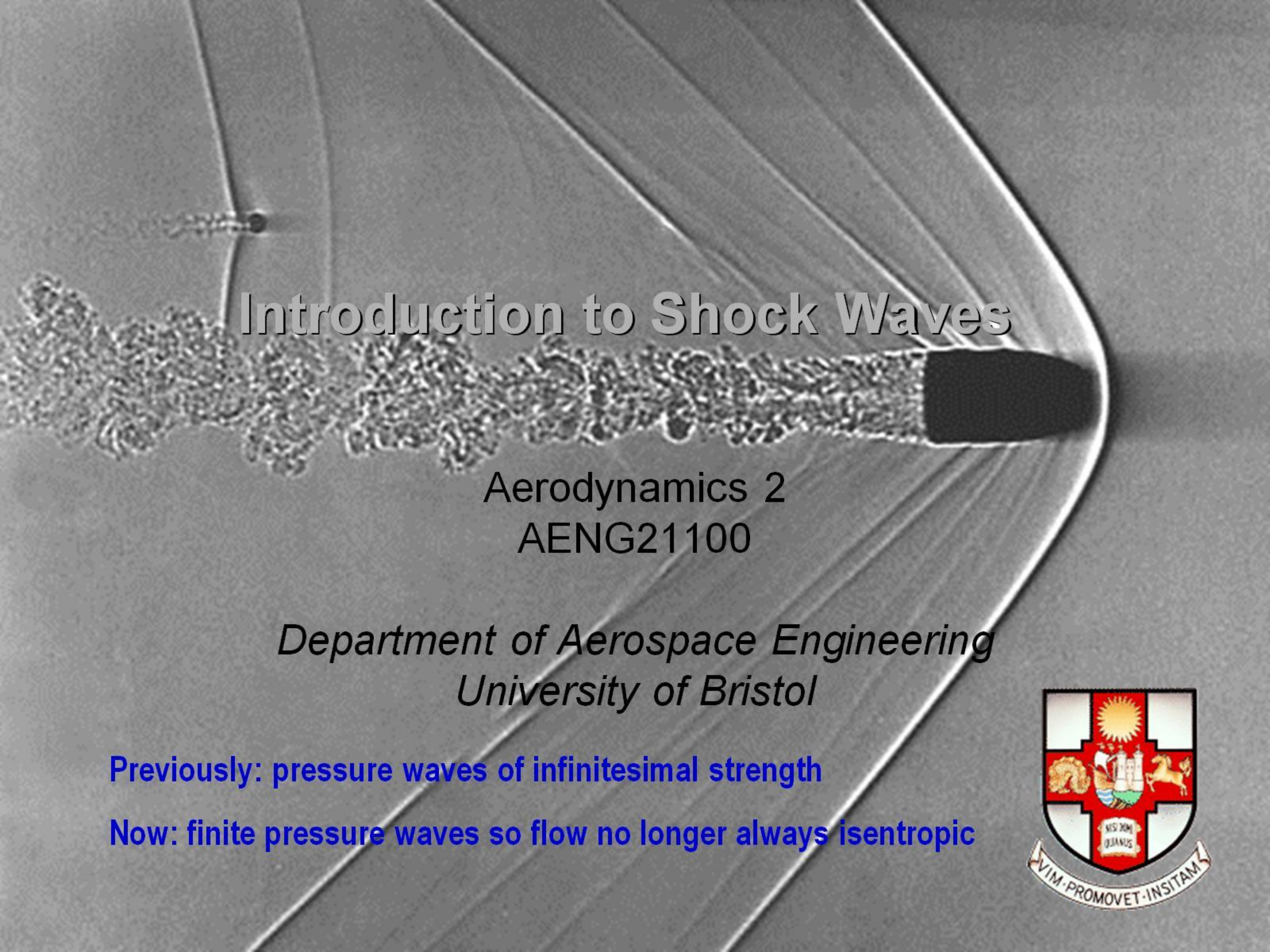


Introduction to Shock Waves

Aerodynamics 2
AENG21100

Department of Aerospace Engineering
University of Bristol

Previously: pressure waves of infinitesimal strength

Now: finite pressure waves so flow no longer always isentropic

VIM PROMOVET INSITAM

Map so far

Isentropic flow along streamlines and nozzles → Non isentropic and isentropic flow meeting corners → Applications to aerofoils

Done

Next 3 lectures

Future

Next 3 lectures

- Normal shocks ← Today
- Oblique shocks
- Expansion fans

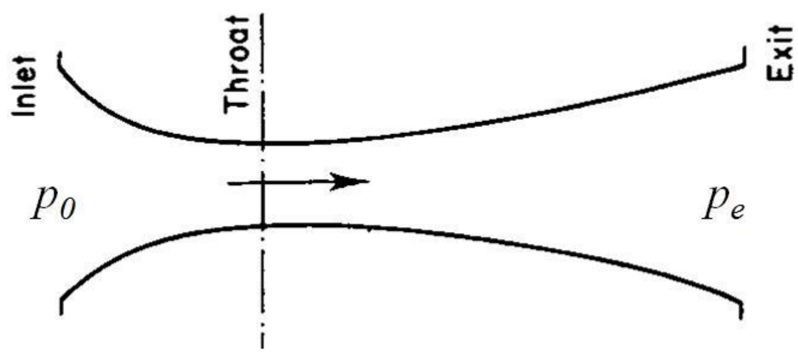
[redacted]

What is a shockwave?

- It is a discontinuity in velocity, temperature, pressure, total pressure, density and total density
- But not in total temperature – why?
- It takes the flow irreversibly from > Mach 1 to < Mach 1
- It is the line along which downstream perturbations coalesce when they meet a supersonic upstream zone
- It is ~ 60 molecular mean free paths (MFP) wide so <<< thinner than anything else. We do not need to use models on this scale – the Euler equations are still satisfactory
- Similar to a breaking wave on a beach - what we are interested in is the pressure behind the shock (height of water) related to its Mach number (speed of wave hitting beach). Not interested in the frothing white region near the front (since it is ~ 60MFP only)

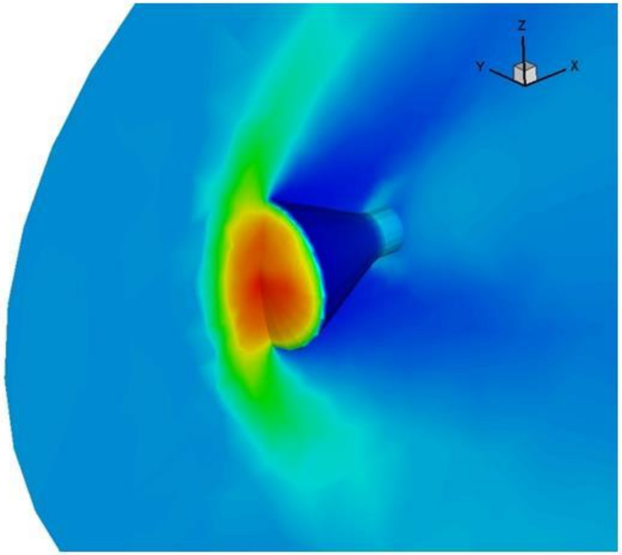
Hydraulic analogy

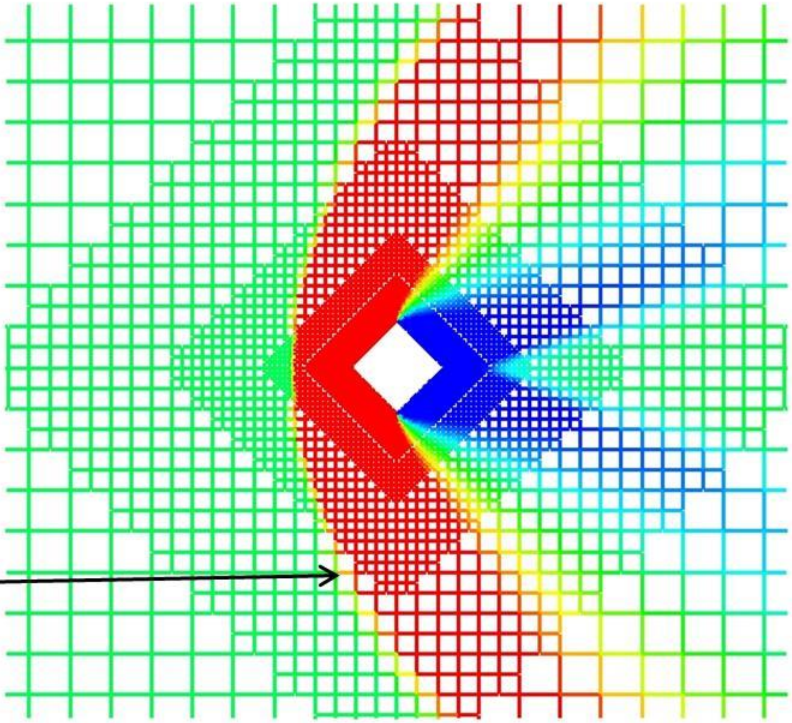
- Surface water waves travel at a set speed – the Froude number (Fr) is the ratio between the speed of the water and the speed of these waves. It is analogous to the Mach number
- So if we get some water moving at Fr>1 we should see behaviour similar to shockwaves



Other examples

Last year - Chelyabinsk meteor
'Mach 60' - ?





Pressure jump through shock breaks windows…

Sonic boom rocks large part of Britain

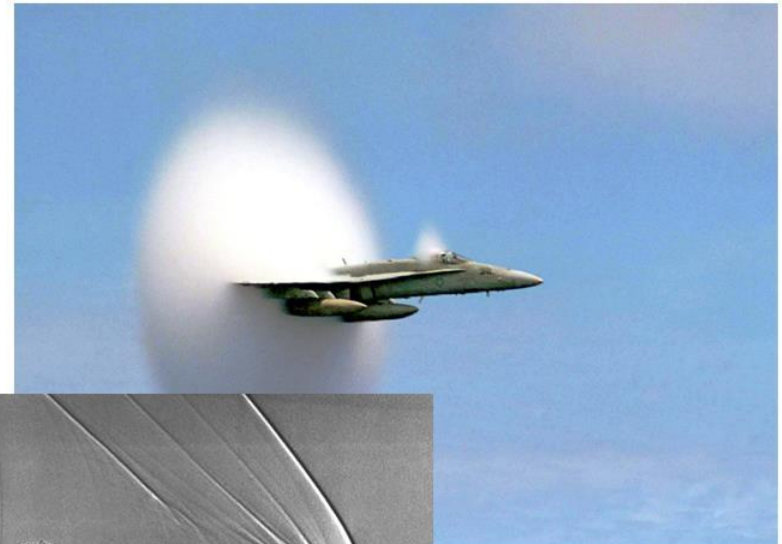
Daily Mail 13 April 2012

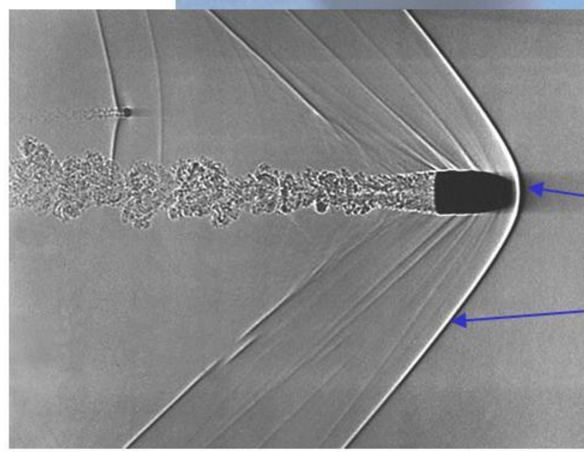
'A sonic boom is the sound associated with the shockwaves created when an object travels through the air and breaks the sound barrier…'

'The noise contains large amounts of sound energy, meaning sonic booms are often mistaken for explosions…'

Shock Waves etc

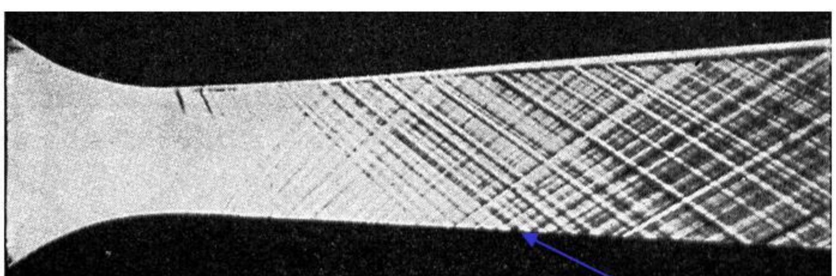
- normal shock waves
  - extension of sound wave analysis
  - change in properties through shock
- 2D supersonic flow
  - Mach waves
  - Prandtl-Meyer expansion
  - oblique shock waves
- shocks in ducts
  - intakes & nozzles
  - supersonic pitot-static

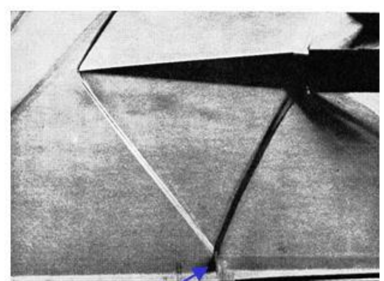


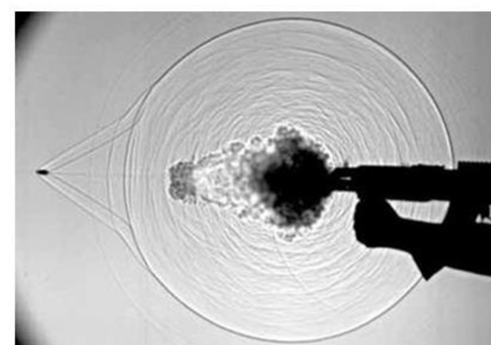


Normal (bow) shock

Oblique shock







Shock “reflections”

Shock Spotting

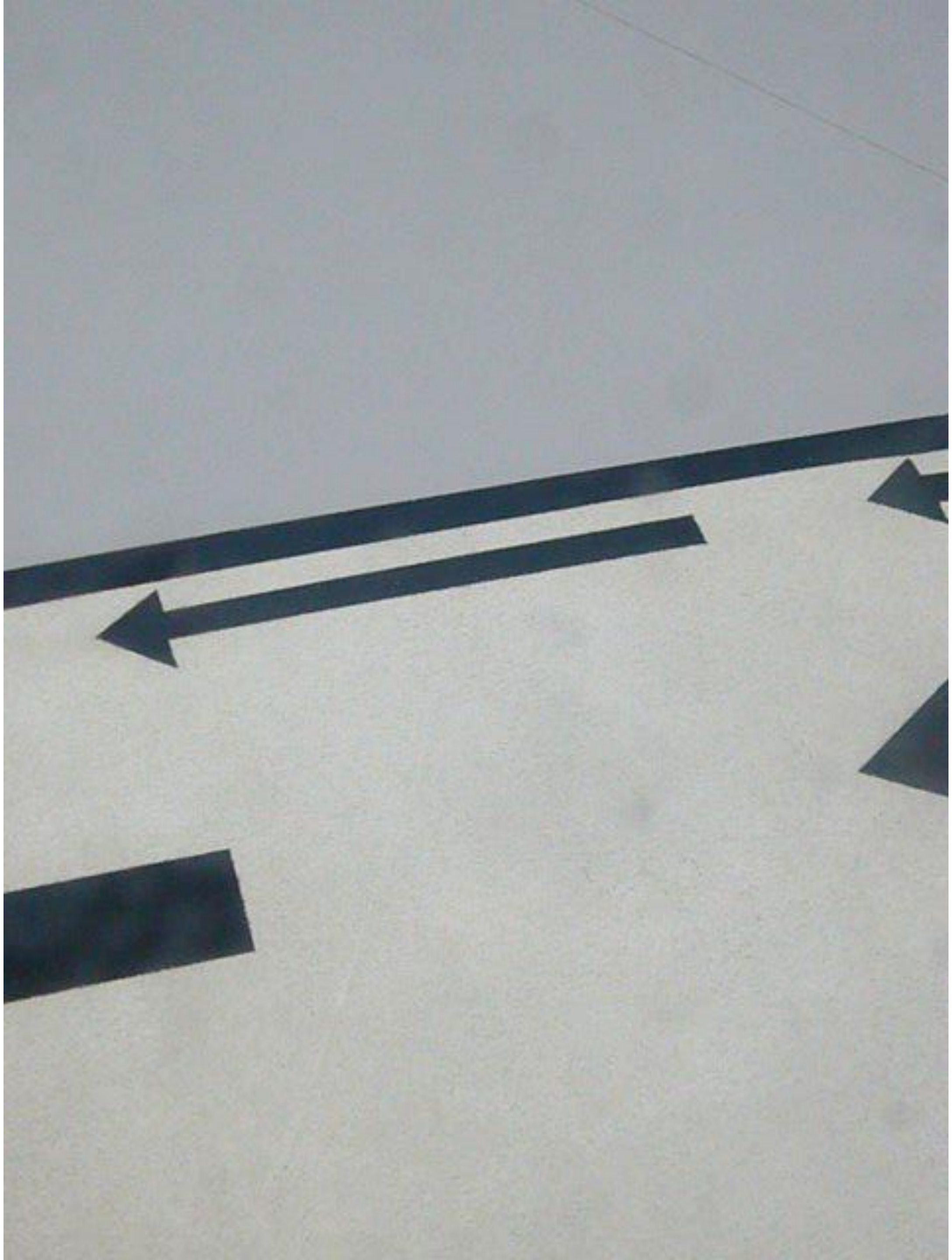
- Wait for clear, bright sunlight
- Best from directly above, so close to midday
- Keep looking. Depending on aircraft weight and Mach number, the shock will move in and out of the light (forwards with lower weight or lower Mach number)
- Turbulence makes them more obvious as they move back and forth a few cm. The incidence is changing in the gusts, altering lift coefficient and shock position
- If you get a good photo, send it to me!

Shocks

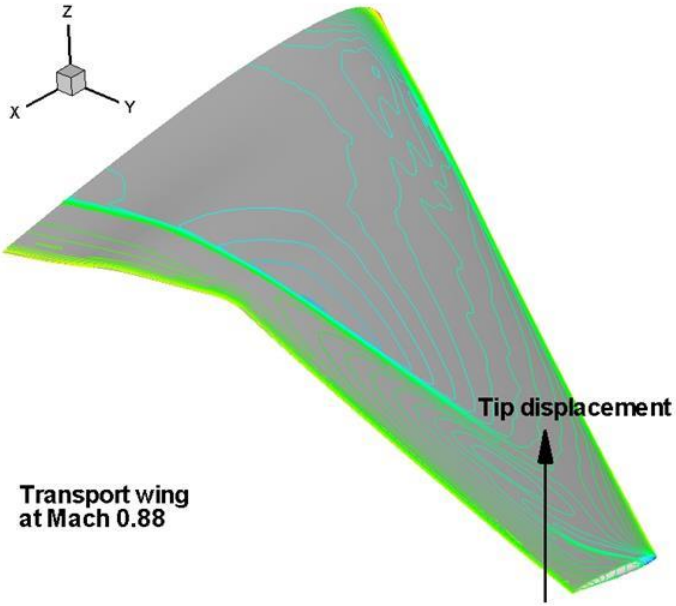
Shocks

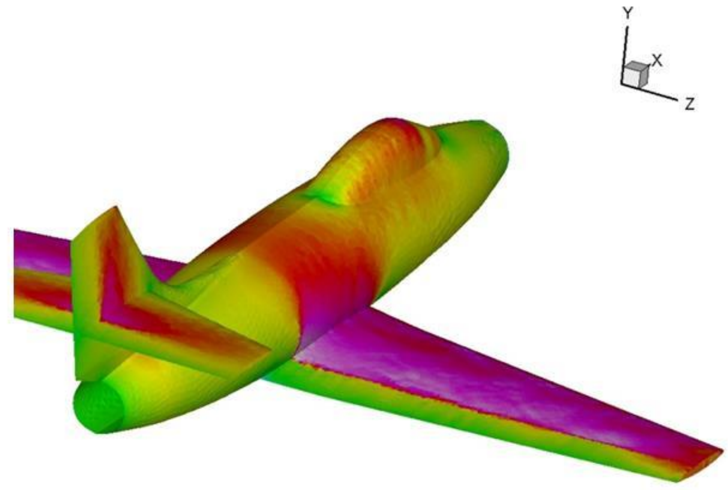
Shocks

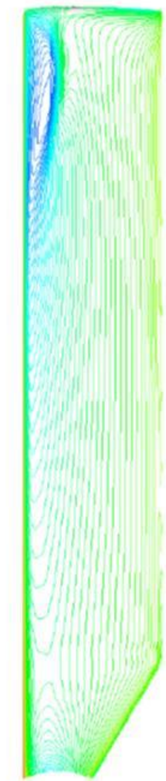

easyJet

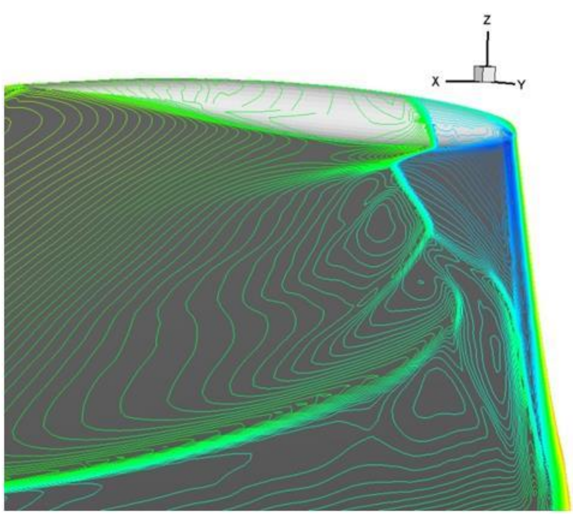


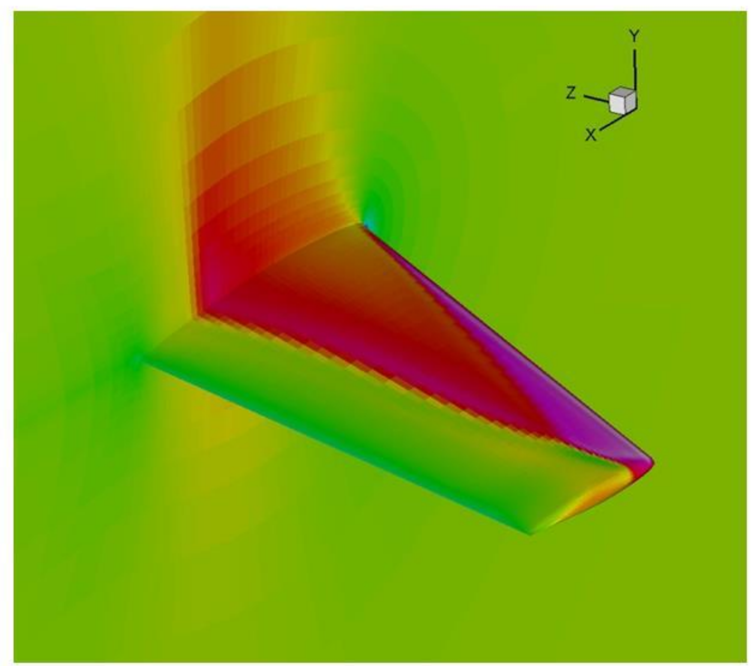
Some more examples…



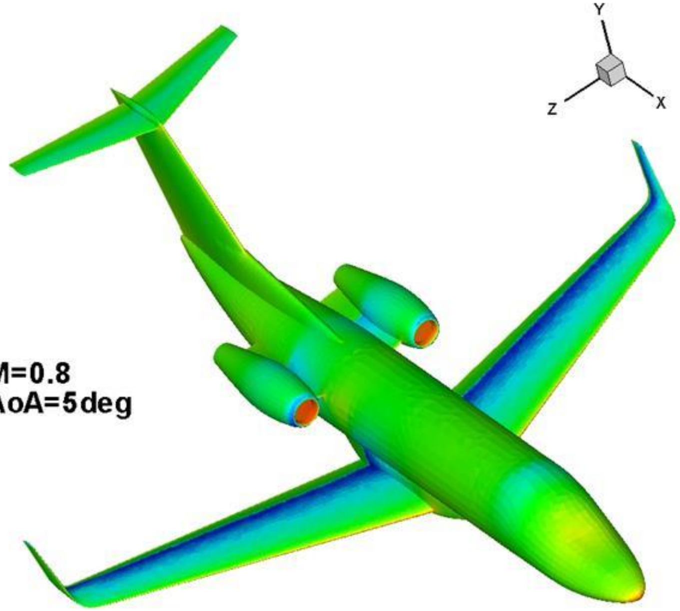


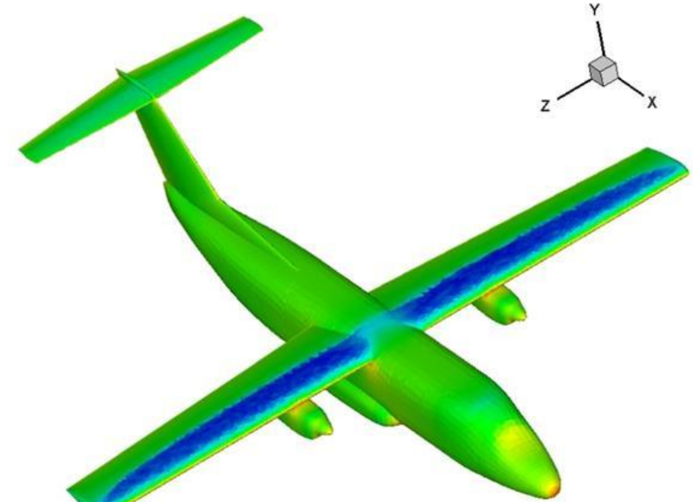






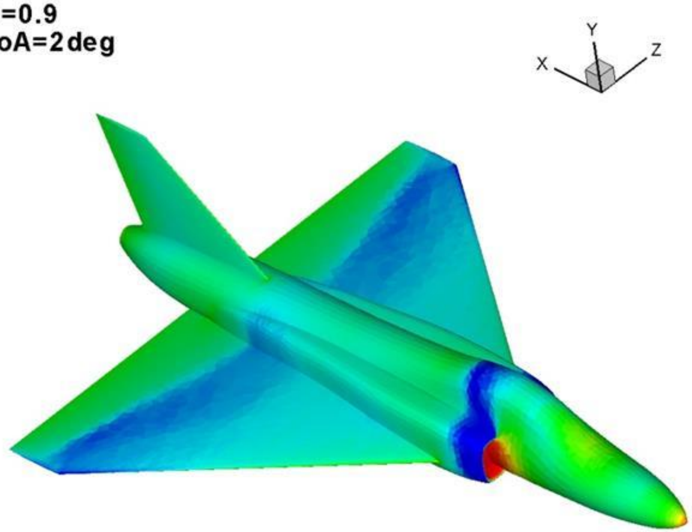
M=0.8
AoA=5deg



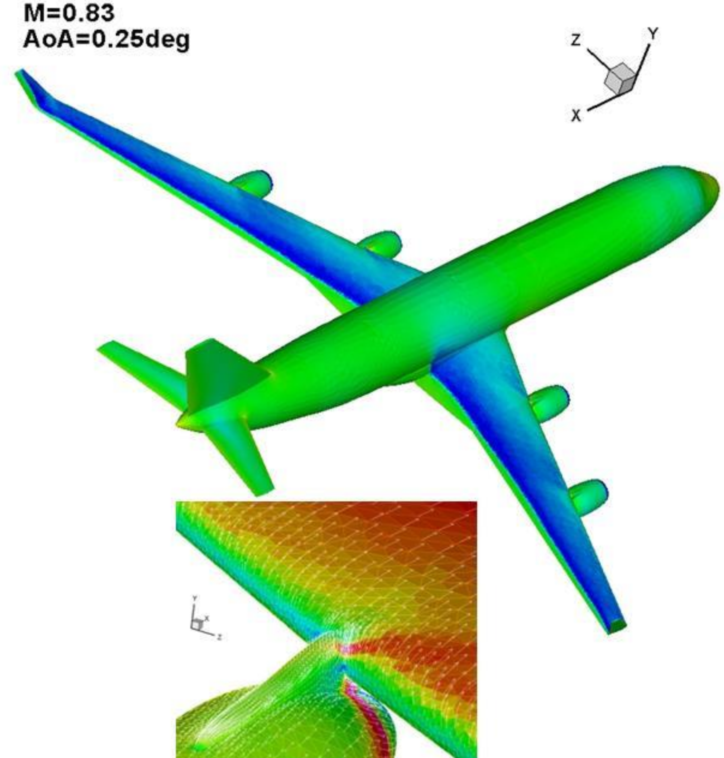


M=0.725
AoA=5deg

M=0.9
AoA=2deg

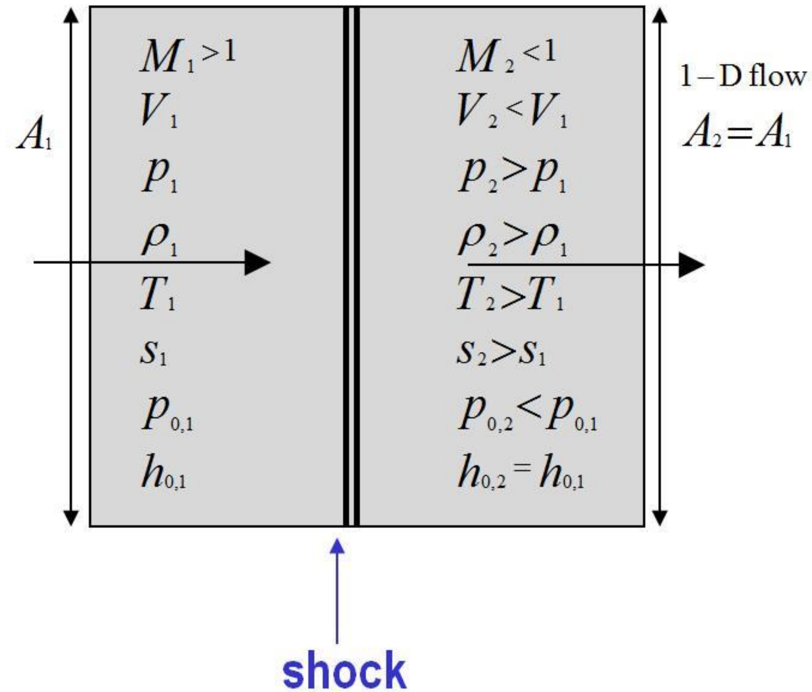


M=0.83
AoA=0.25deg



Normal Shock Waves (1)

- discontinuity in supersonic flow
  - due to 'large' disturbance
  - Compression & deceleration
  - width ~ 6x10^{-8}m
  - Subsonic flow allows upstream propagation: flow adjusts, no shocks
- similar approach to sound wave
  - control volume moving with the shock
  - 'station numbers': 1 & 2 up-/down-stream
- shock has finite strength, hence
  - adiabatic process- changes too rapid for heat transfer
  - ***nonisentropic process*** - but still inviscid (externally)
  - changes in fluid properties p, ρ and u are 'large'
- apply momentum, continuity and energy equations to flow through control volume
  - rewrite in terms of Mach Number M and speed of sound a



Normal Shock Waves (2)
Mach Number Variation Through Shock

- Continuity $\quad \dot{m} = \rho_1 A_1 V_1 \; = \; \rho_2 A_2 V_2 \rightarrow \rho_1 a_1 M_1 \; = \; \rho_2 a_2 M_2$

- Momentum **See previous analysis of infinitesimal (sound) waves**

pressure force=rate of change of momentum

$$p_1 A_1 - p_2 A_2 = \rho_1 A_1 V_1 \left(V_2 - V_1\right)$$

apply continuity and eqn. of state $\quad a^2 = \gamma RT = \dfrac{\gamma p}{\rho} \quad p = \dfrac{\rho a^2}{\gamma}$

$$p_1 - p_2 = \rho_1 V_1 (V_2 - V_1) \;\Rightarrow\; \frac{\rho_1 a_1^2}{\gamma} - \frac{\rho_2 a_2^2}{\gamma} = \rho_1 a_1 M_1 (a_2 M_2 - a_1 M_1)$$

rearrange $\quad \rho_1 a_1^2\left(1 + \gamma M_1^2\right) = \rho_2 a_2^2\left(1 + \gamma M_2^2\right)$

- Energy – adiabatic flow so use temperature energy eqn. $\dfrac{T_0}{T} = 1 + \dfrac{\gamma - 1}{2} M^2$ **BC18**

$$\left(\frac{T_1}{T_2}\right) = \left(\frac{T_0}{T_2}\frac{T_1}{T_0}\right) = \left(1 + \frac{\gamma-1}{2}M_2^2\right)\Big/\left(1 + \frac{\gamma-1}{2}M_1^2\right) = \left(\frac{a_1^2}{\gamma R}\right)\Big/\left(\frac{a_2^2}{\gamma R}\right) = \left(\frac{a_1^2}{a_2^2}\right) \quad \text{using} \quad a = \sqrt{\gamma RT}$$

$$a_1^2\left(1 + \frac{\gamma-1}{2}M_1^2\right) = a_2^2\left(1 + \frac{\gamma-1}{2}M_2^2\right)$$

Algebraic Interlude (not examinable)

$$\rho_1 a_1 M_1 = \rho_2 a_2 M_2$$

$$\rho_1 a_1^2\left(1+\gamma M_1^2\right) = \rho_2 a_2^2\left(1+\gamma M_2^2\right)$$

$$\frac{a_1}{M_1}\left(1+\gamma M_1^2\right) = \frac{a_2}{M_2}\left(1+\gamma M_2^2\right)$$

$$\left(\frac{a_1}{a_2}\right)^2 = \left(\frac{M_1}{M_2}\frac{\left(1+\gamma M_2^2\right)}{\left(1+\gamma M_1^2\right)}\right)^2$$

$$\frac{\left(2+(\gamma-1)M_2^2\right)}{\left(2+(\gamma-1)M_1^2\right)} = \left(\frac{M_1}{M_2}\frac{\left(1+\gamma M_2^2\right)}{\left(1+\gamma M_1^2\right)}\right)^2$$

$$a_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right) = a_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

$$\frac{a_1^2}{a_2^2} = \frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)} = \frac{\left(2+(\gamma-1)M_2^2\right)}{\left(2+(\gamma-1)M_1^2\right)}$$

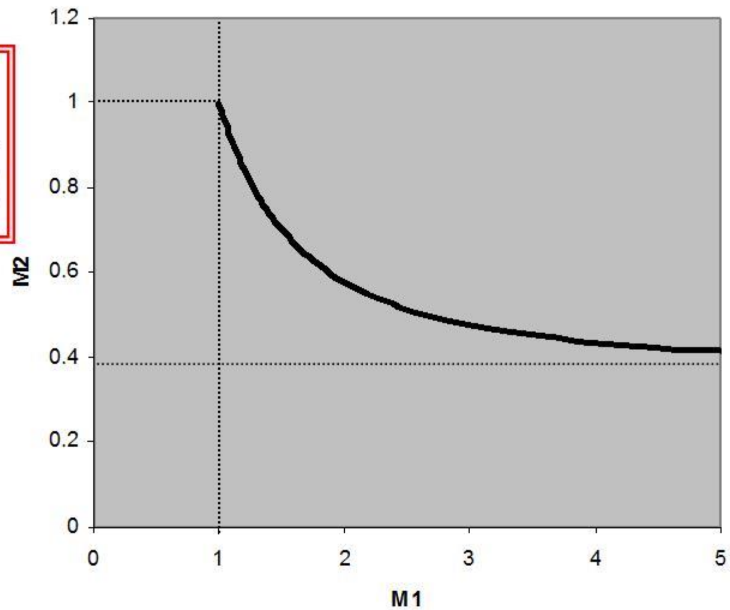
Normal Shock Waves (3)
Mach Number Variation Through Shock

- Eliminate ρ and a from previous continuity momentum and energy equations

$$\left(M_1^2 - M_2^2\right)\left\{2 + (\gamma - 1)\left(M_1^2 + M_2^2\right) - 2\gamma M_1^2 M_2^2\right\} = 0$$

- Solutions:
  - $M_1 = M_2$ no shock
  - symmetric in M – but entropy change determines direction

$$\boxed{M_2^2 = \frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)}} \qquad \boxed{M_1^2 = \frac{2 + (\gamma - 1)M_2^2}{2\gamma M_2^2 - (\gamma - 1)}}$$



- upstream $M_1 \geq 1 \rightarrow$ downstream $M_2 \leq 1$
  - only occurs in *supersonic* flow
  - downstream flow *subsonic* (but M_2 > 0.378)

Normal Shock Waves (4)
Flow Properties: Variation Through Shock

- pressure ratio
  - *compression* process
- temperature ratio
  - Temperature rise
- density ratio

compression process

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

$$\frac{T_2}{T_1} = \frac{\left(2 + (\gamma - 1)M_1^2\right)\left(2\gamma M_1^2 - (\gamma - 1)\right)}{(\gamma + 1)^2 M_1^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \rightarrow 6 \text{ as } M_1 \rightarrow \infty$$

see standard texts for derivation

>1

$M_1>1$

- total pressure ratio
  - NB some authors use p_t for p_0
  - p_0 ***reduces*** across shock
  - direct link between entropy gain and loss in total pressure
  - in practice the ratio p_{02}/p_1 is more useful
  - $(p_{02}/p_1)=(p_{02}/p_{01})(p_{01}/p_1)$ where (p_{01}/p_1) from compressible Bernoulli

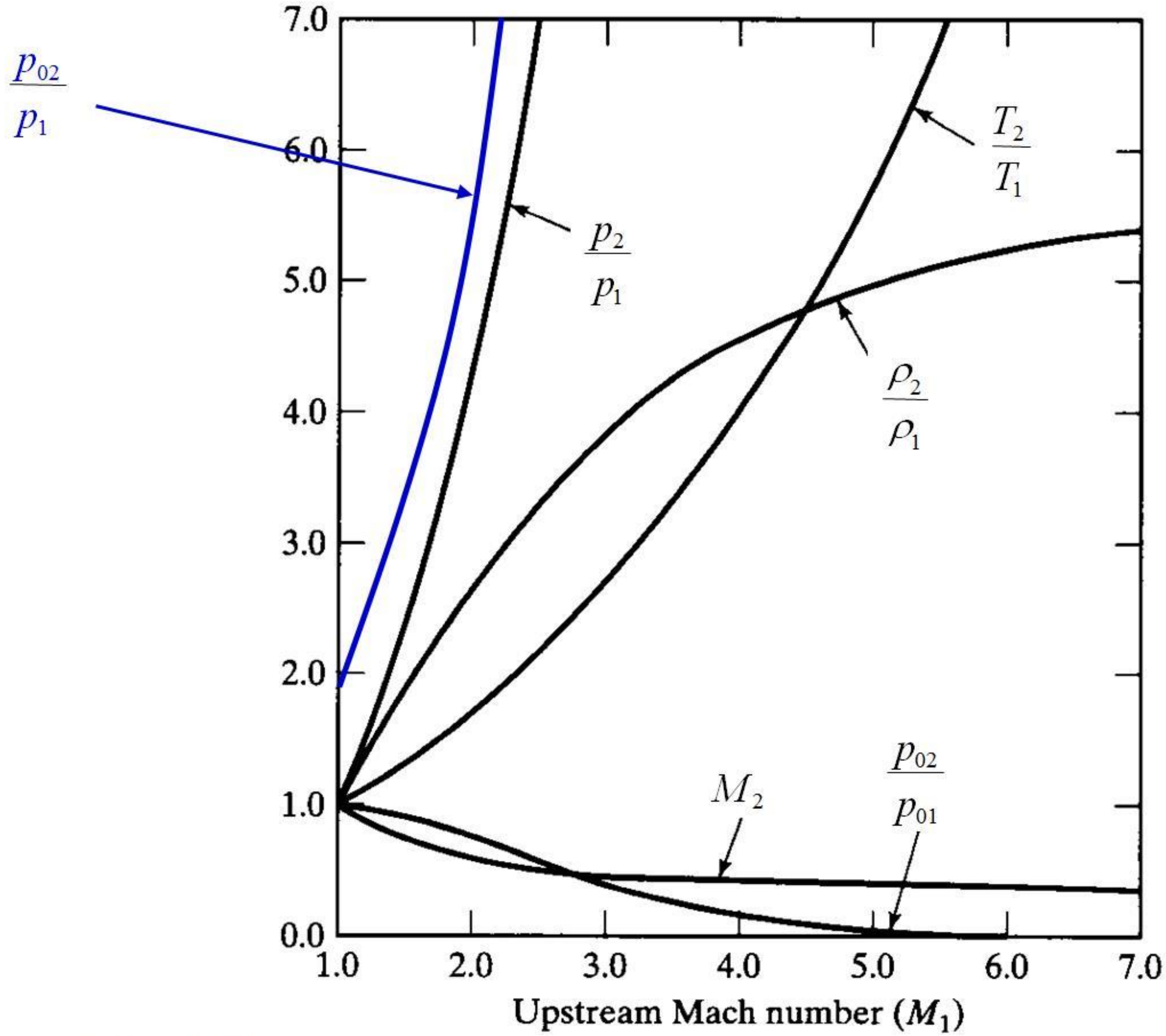
$$-R\ln\left(\frac{p_{02}}{p_{01}}\right) = s_2 - s_1 = fn(M_1)$$

from BC1.12 $$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = fn(M_1) - fn(M_1)$$

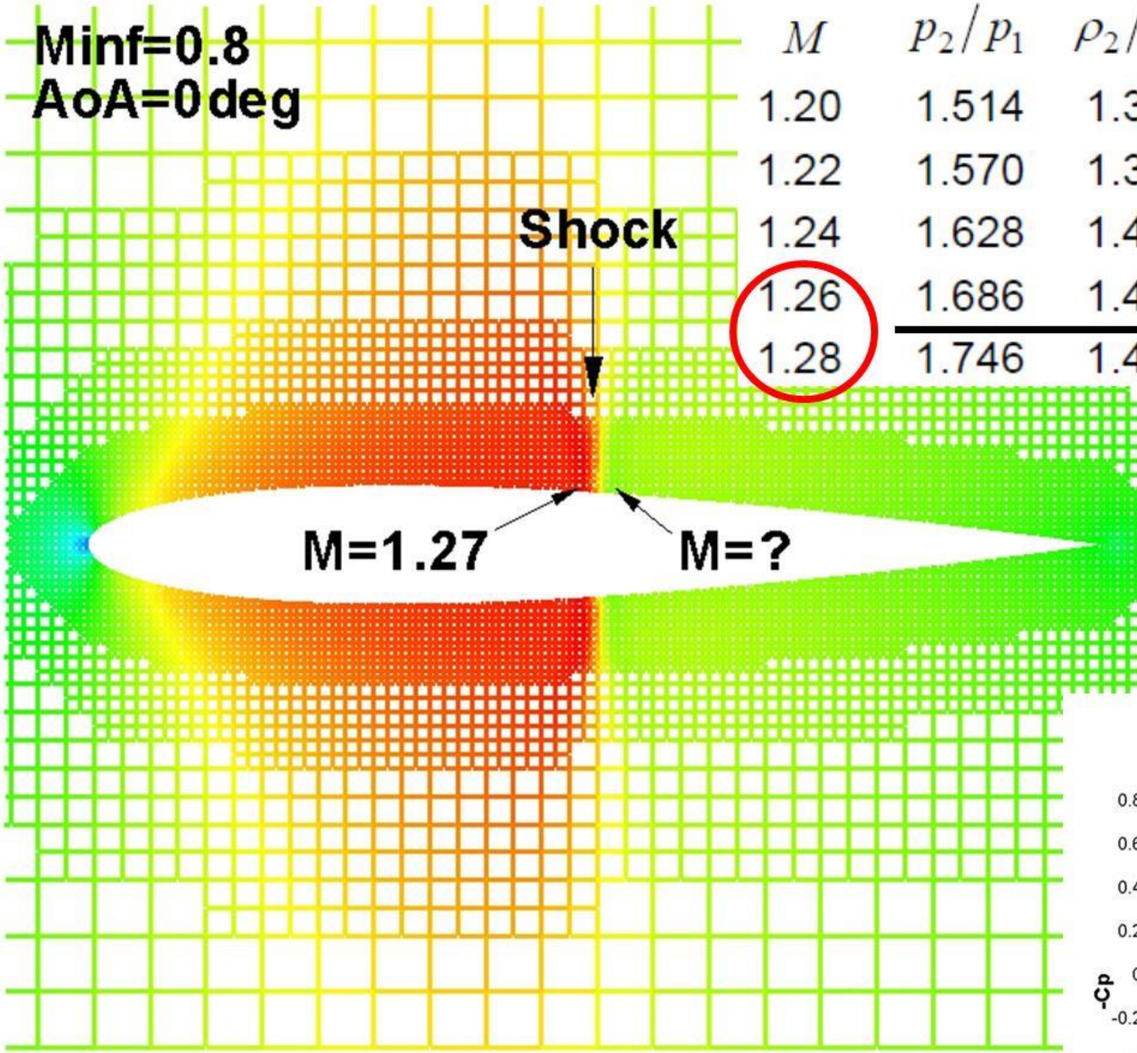
- ratios p_2/p_1, T_2/T_1, ρ_2/ρ_1 and p_{02}/p_1 tabulated vs M_1

formulae too complex for routine use so use tables when γ=1.403

Normal Shock Relations

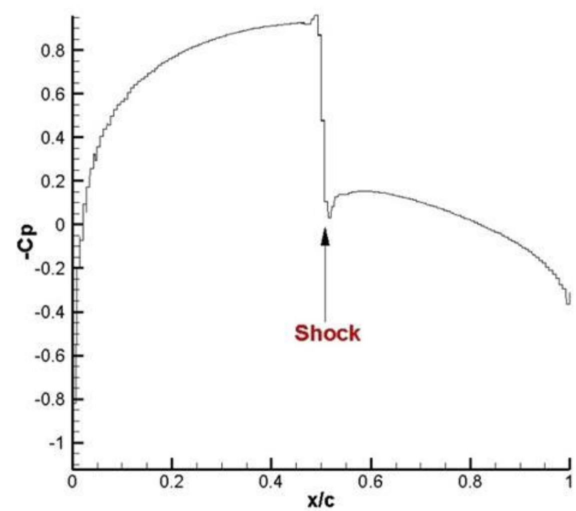


Example 1



M	p_2/p_1	ρ_2/ρ_1	T_2/T_1	M_2	p_{02}/p_1
1.20	1.514	1.341	1.129	0.842	2.410
1.22	1.570	1.376	1.141	0.830	2.469
1.24	1.628	1.410	1.154	0.818	2.529
1.26	1.686	1.445	1.167	0.807	2.591
1.28	1.746	1.480	1.179	0.796	2.653

Linear interp.
gives M=0.802

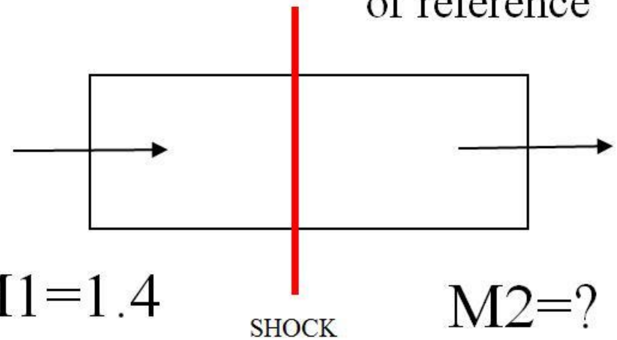


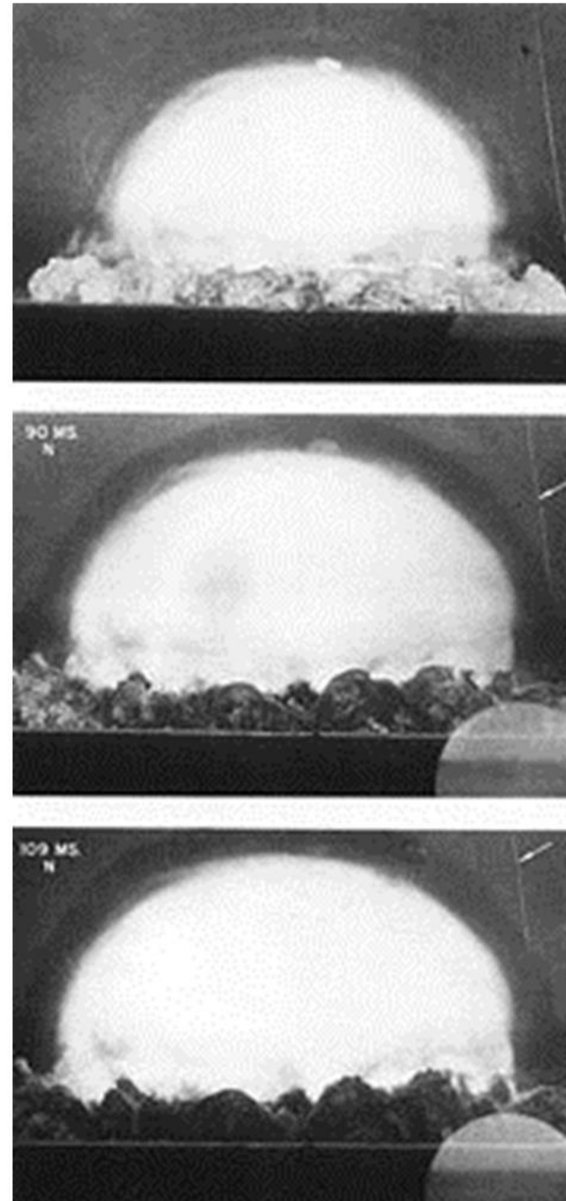
Example 2

Mach number consistent with average value
10s after 1MT explosion

Moving frame
of reference

Shock moving at
M=1.4 into
air at STP





From tables M2=0.74, T2/T1=1.256

So air exits at 0.74*340*sqrt(1.256)=282m/s
(in the moving frame)

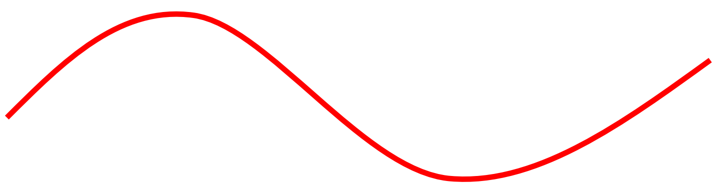
Frame of reference moving at 1.4*340=476m/s

So air behind shock has
velocity of 476-282=194m/s=698km/h=433mph

T2 =1.256*288=362K=89C

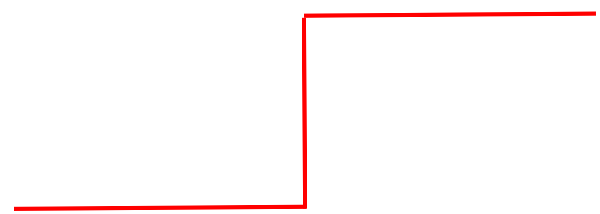
P2=2.121atm=16PSI above atmospheric

Sound wave



Speed of air behind wave ~0mph

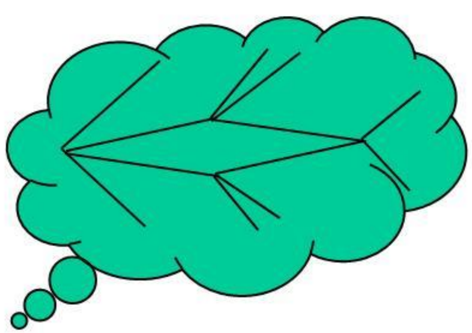
Shockwave

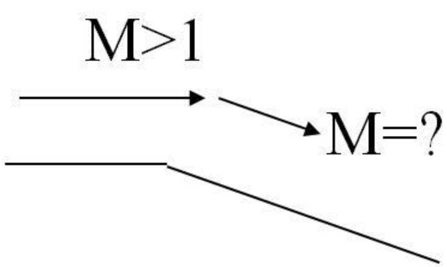


Speed of air behind wave ~100s mph

Today

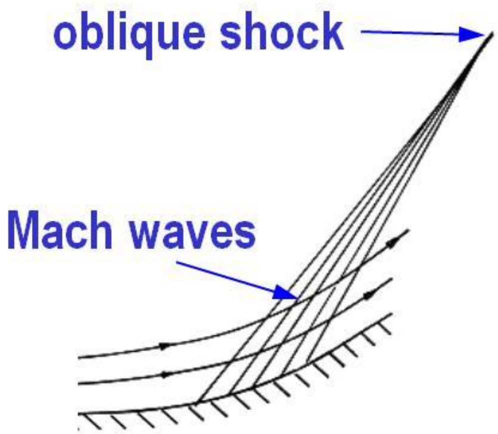
- Oblique shocks – normal shocks at an angle, with some trigonometry, and some unexpected complications
- Detached shocks – a cross between oblique and normal shocks (for these you only need to consider the normal part, anything else is beyond this course)
- Next week – expansions!

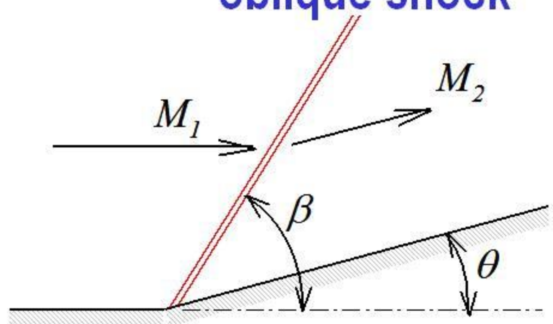


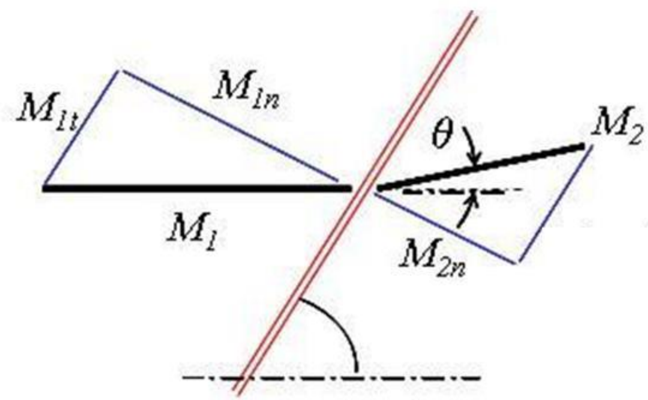


2D Supersonic Flow
Oblique Shock

- for ‘large’ negative deflection angle Mach waves coalesce to form an *oblique shock wave*
  - a nonisentropic compression wave





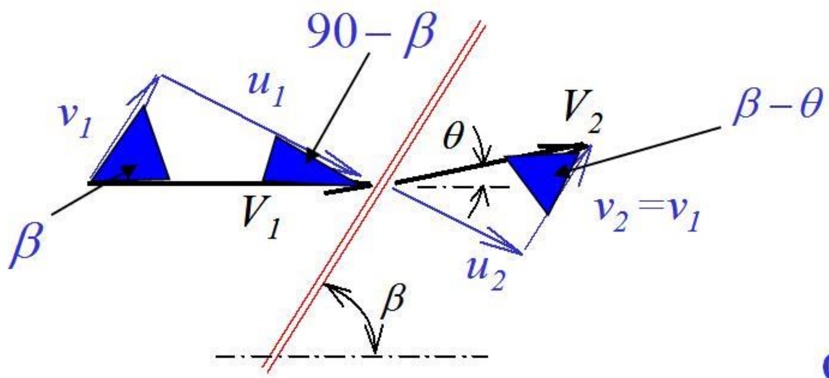


- use normal shock relations for normal velocity component
  - no change in tangential velocities

$$M_{1n} = M_1 \sin\beta, \quad M_{2n} = M_2 \sin(\beta - \theta)$$

- M_{2n} and ratios p_2/p_1, T_2/T_1, ρ_2/ρ_1 tabulated vs M_{1n}
- β still unknown

2D Supersonic Flow
Oblique Shock angle



$$\tan(\beta)=\frac{u_1}{v_1}$$

$$\tan(\beta-\theta)=\frac{u_2}{v_2}$$

consider 1D flow normal to the shock

- From continuity

$$\rho_1 u_1 = \rho_2 u_2 \quad \rightarrow \quad \frac{u_1}{u_2}=\frac{\rho_2}{\rho_1} \qquad v_1 = v_2 \qquad \frac{u_1/v_1}{u_2/v_2}=\frac{\rho_2}{\rho_1}$$

$$\frac{\tan(\beta)}{\tan(\beta-\theta)}=\frac{\rho_2}{\rho_1}$$

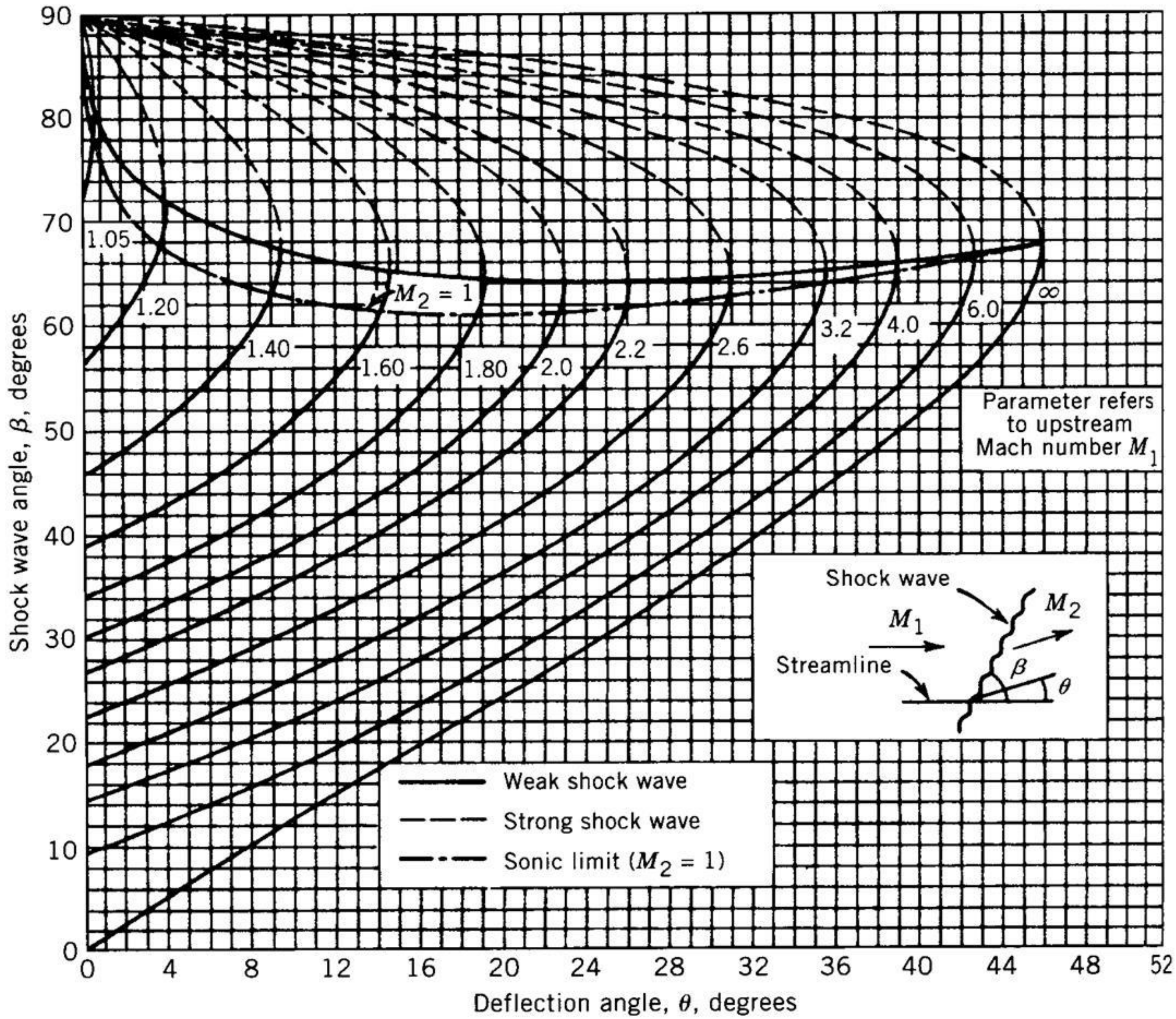
- use normal shock relations and some algebra to give

$$\tan(\theta)=\frac{\cot(\beta)\left\{(M_1\sin\beta)^2-1\right\}}{\left(\gamma+1/2\right)M_1^2-\left\{(M_1\sin\beta)^2-1\right\}}$$

$$\boxed{\begin{aligned}\frac{p_2}{p_1}&=\frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}\\ \frac{T_2}{T_1}&=\frac{\left(2+(\gamma-1)M_1^2\right)\left(2\gamma M_1^2-(\gamma-1)\right)}{(\gamma+1)^2 M_1^2}\\ \frac{\rho_2}{\rho_1}&=\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\end{aligned}}$$

- Graphical solution: variation of β with θ for upstream Mach No

2D Supersonic Flow
Oblique Shock Angle



Finding the shock angle

- Given the shock angle, the wedge angle may be found explicitly
- However, finding the shock angle from the wedge angle is not easy (though it can be done through solution of a cubic with much algebra – see NASA Report 187173 'Exact and Approximate Solutions to the Oblique Shock Equations for Real-Time Applications', 1991)
- It is often done using an iterative procedure

  (or graphically if desired)
- Could use bisection, direct iteration (below) or a Newton-Raphson procedure
- Similar ideas can be used to give the maximum shock angle
- Rearrange the method below and start from ~90degrees to get the strong shock

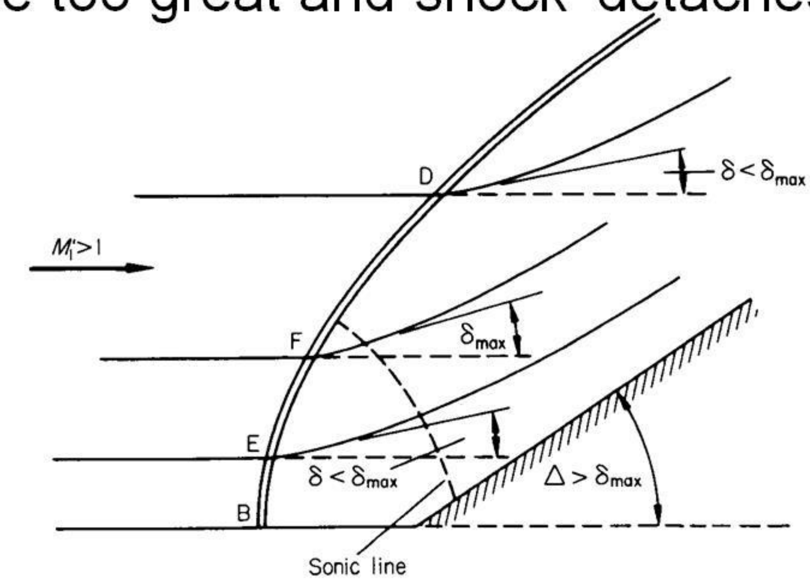
```
(start beta at some small value, give gamma, M1 and theta)
do i=1,100 ! Usually ~ 10 iterations would be enough
  bot=0.5*(gam+1)*M1**2-((M1*sin(beta))**2-1)
  top=asin((sqrt(tan(theta)*bot*tan(beta)+1))/M1)
  beta=top
enddo
```

For M1=2, theta=10, beta_initial=0.1deg

30.02954
37.60822
39.03094
39.27807
39.32045
39.32770
39.32894
39.32916
39.32919
39.32920
39.32920
39.32920(12)

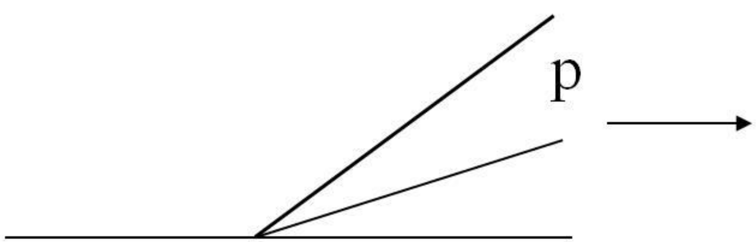
Strong, Weak & Detached Shocks

- for given wedge angle θ and onset Mach Number M_1 there are two solutions to the equations
  - 'weak' shock occurs in external flows ($\beta_{weak} < \beta_{strong}$)
  - 'strong' shock can occur in internal flows with high back pressure
- M_2 supersonic for weak shocks except for small region near θ_{max}
- for $\theta > \theta_{max}$ pressure rise too great and shock 'detaches'
  - curved shock 'stands off' ahead of wedge
  - effectively 'normal' near wedge – subsonic region behind shock
  - curved shock generates vorticity

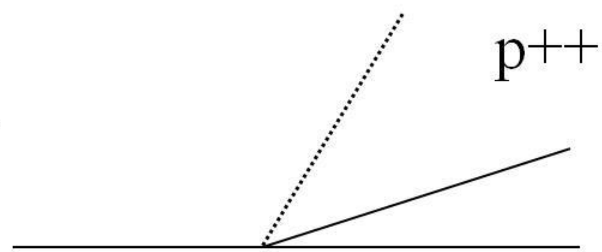


Weak/Strong shock

- If a weak shock exists, and then the downstream pressure is increased (perhaps by a gauze or some other external mechanism), the solution can switch to the strong shock
- Unfortunately some abuse of terminology takes place – weak shocks are sometimes called ‘strong’ if they have a big change in pressure across them

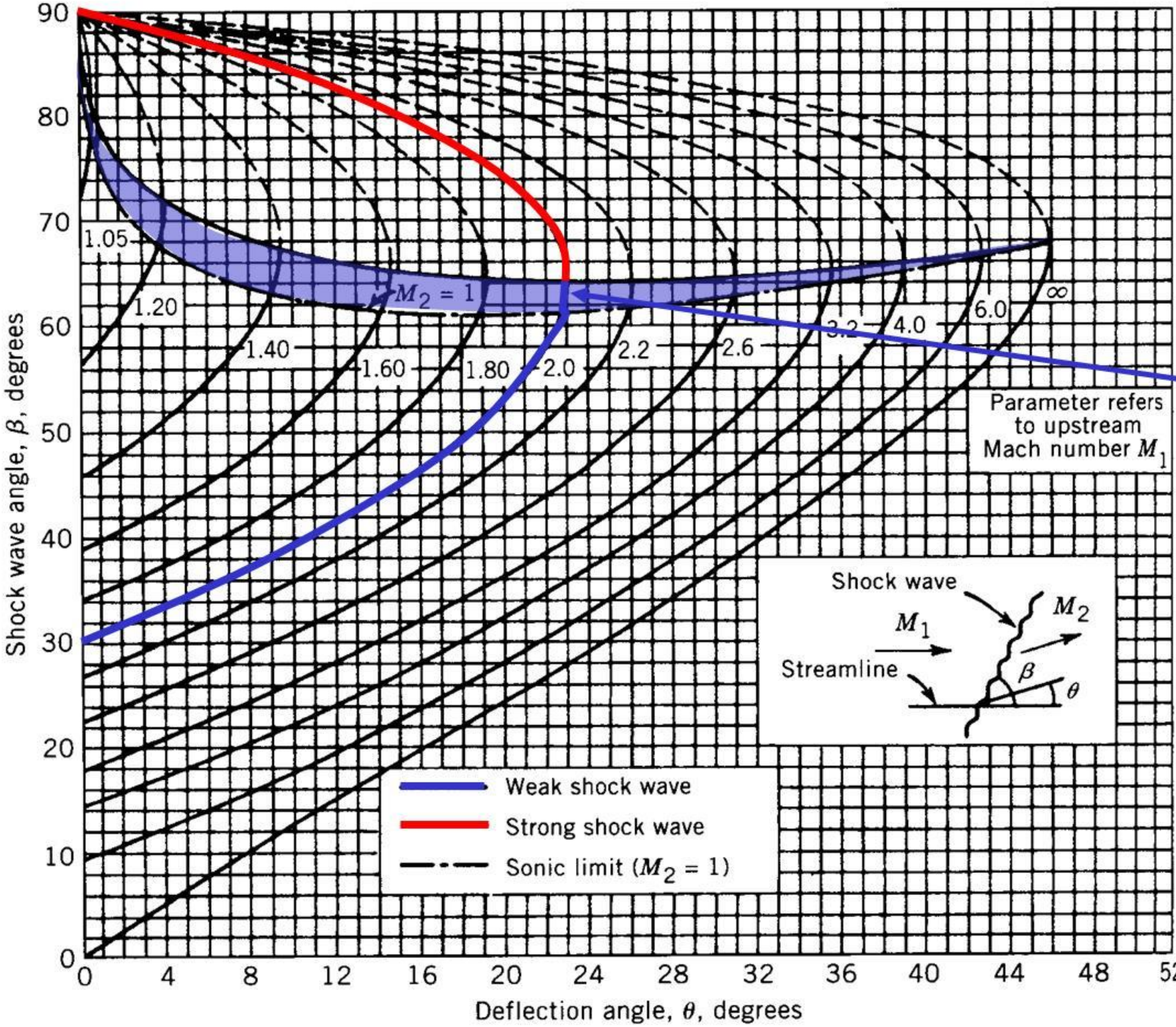


Weak shock



Strong shock

2D Supersonic Flow
Oblique Shock Angle

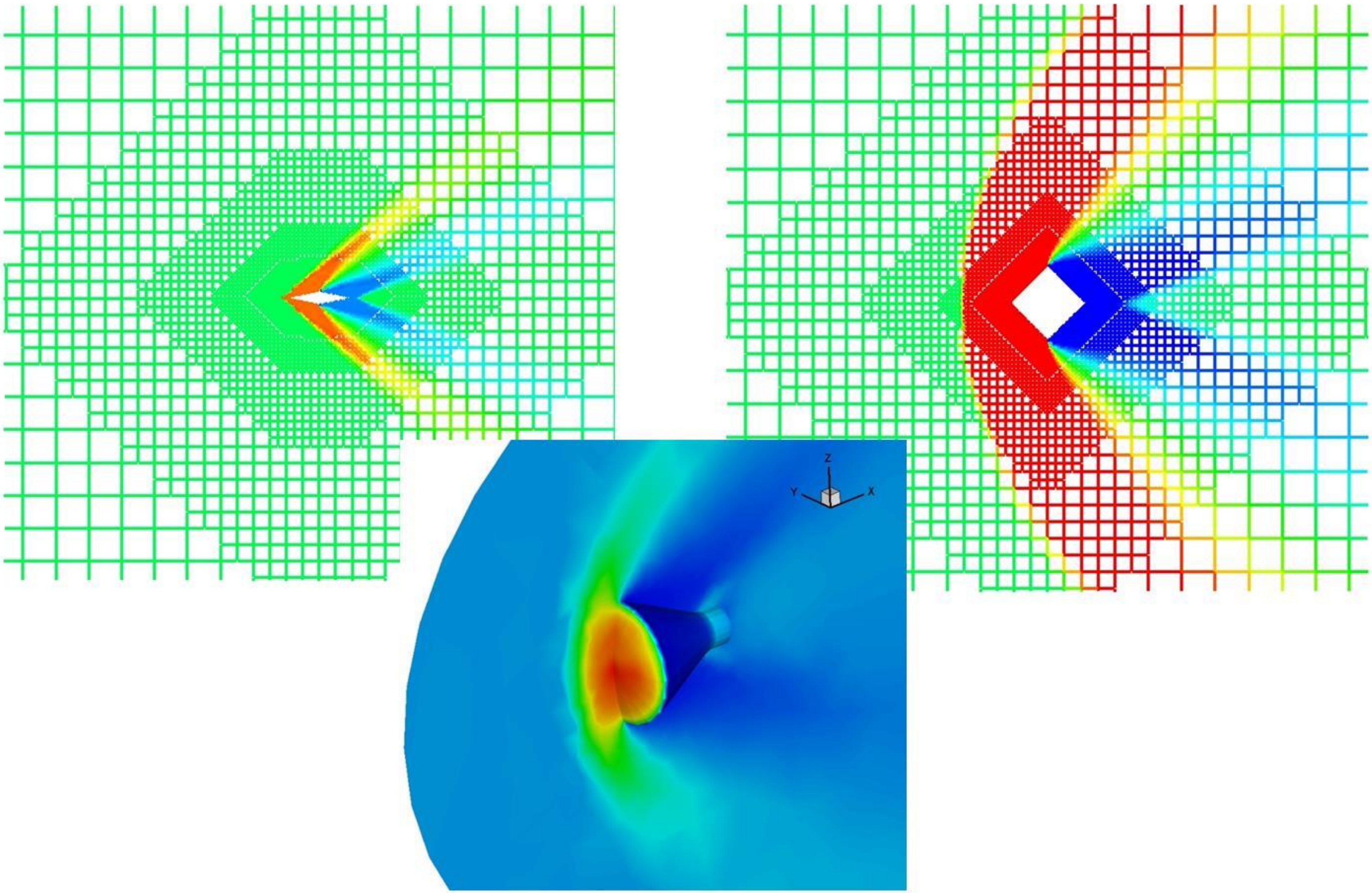


consider Mach 2 line
M_1=2

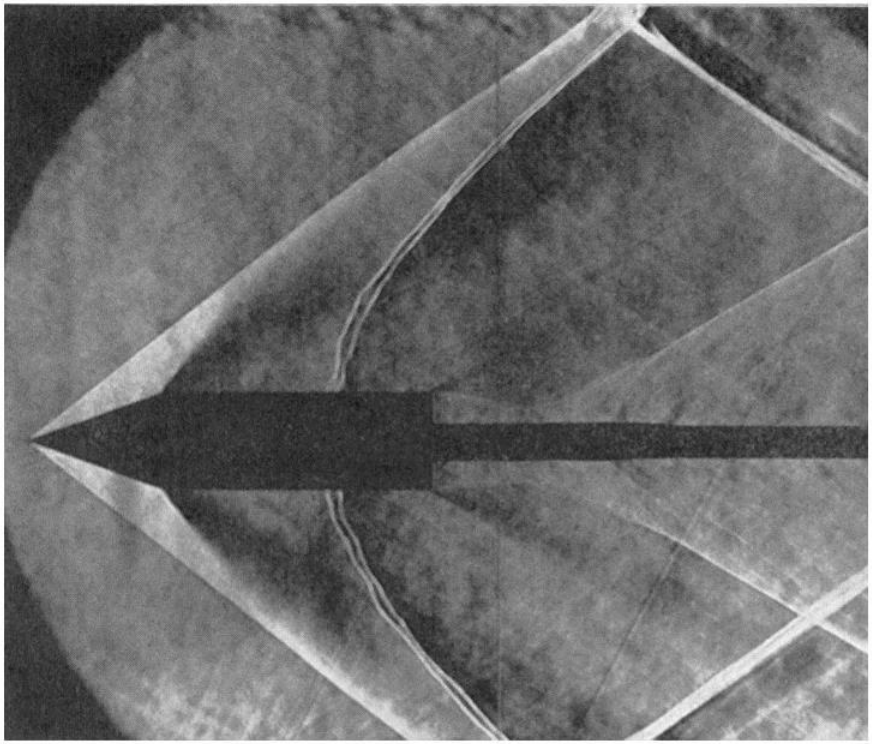
Subsonic flow downstream of oblique shock If θ greater than 22.45°

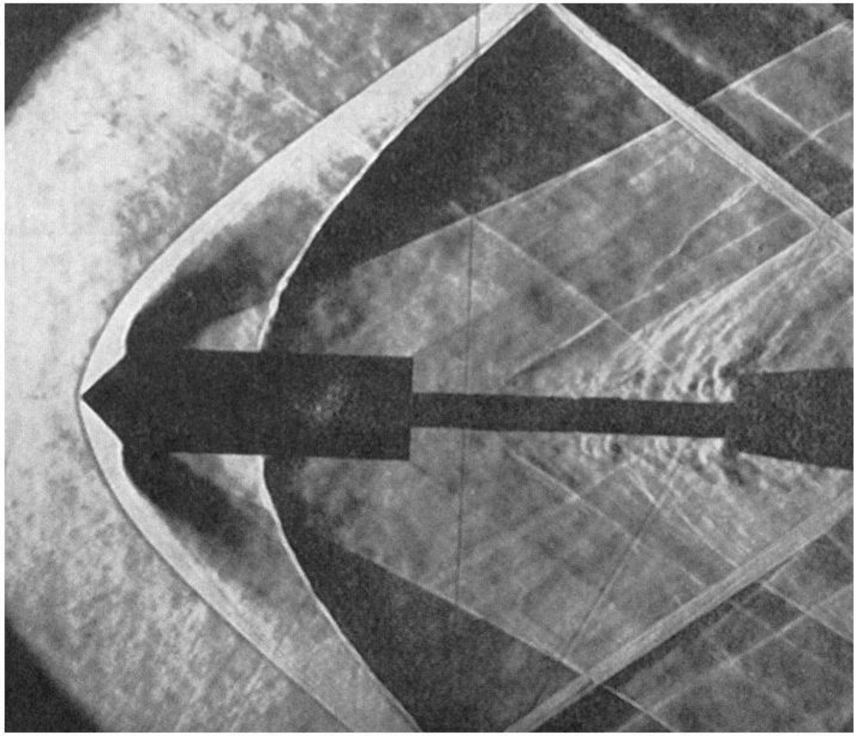
For M_1=2: if θ greater than 22.93° oblique shock detaches

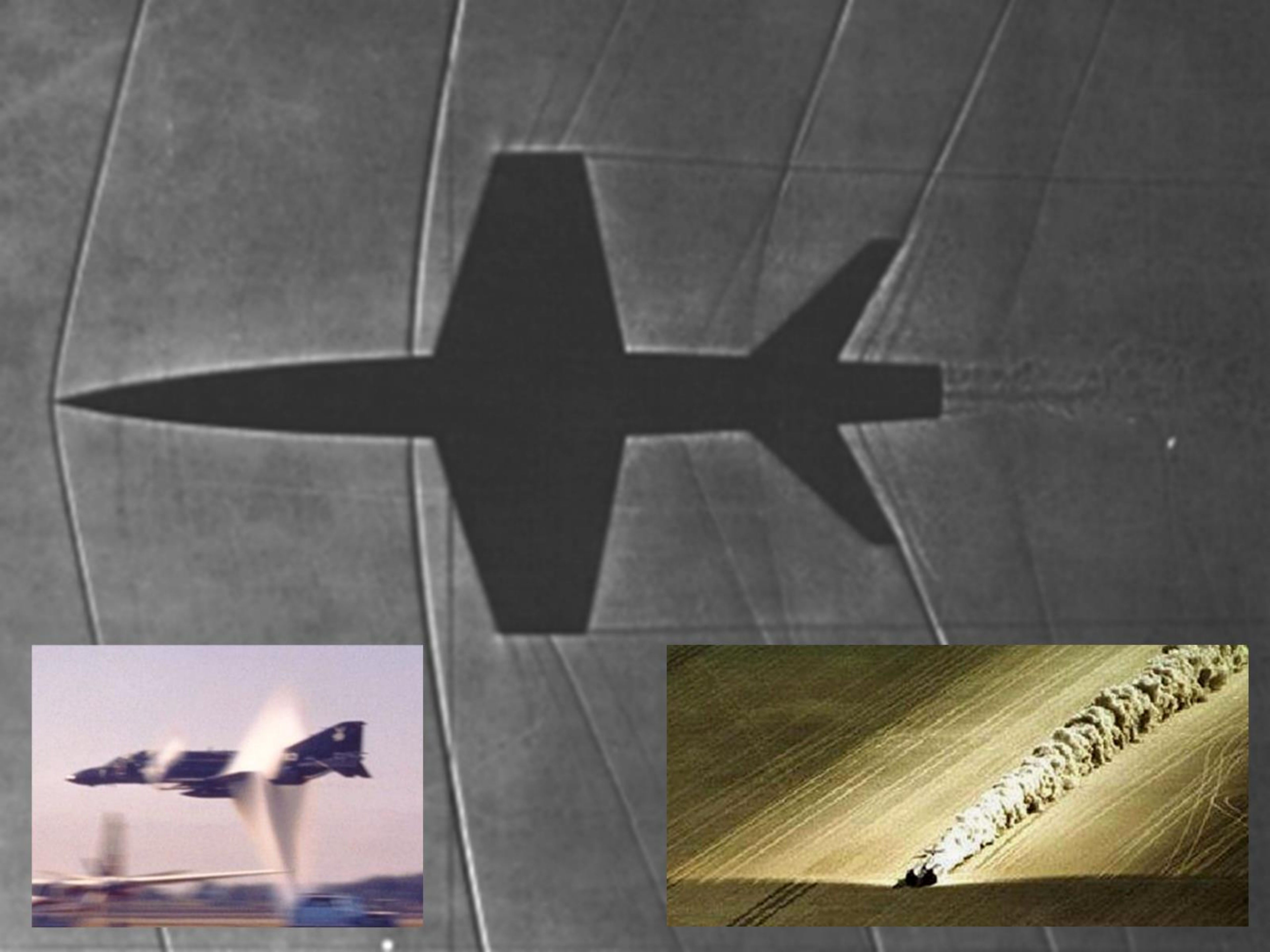
Oblique shock and a detached bow shock for M=2



Schlieren Images of Cone Shock System







Oblique shock example

- Mach number usually known to be one of the values in the tables (increment of 0.1)
- Deflections given in increments of 2 degrees up to the maximum
- Usual to interpolate in terms of deflection, and possibly Mach number too – although unlikely in practice
- Try finding M2 for M1=1.87, deflection=14.7 degrees

For 1.8 have 1.288+(14.7-14)*(1.194-1.288)/(16-14)=1.2551

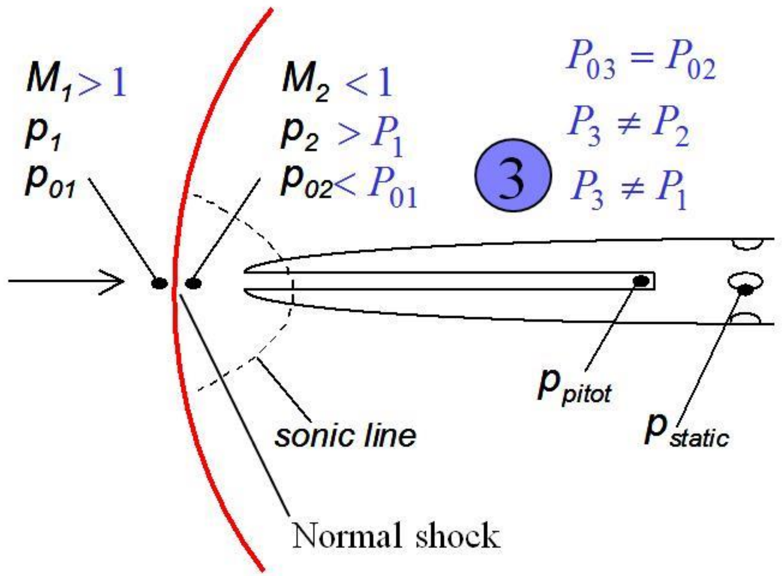
For 1.9 have 1.39+(14.7-14)*(1.304-1.39)/(16-14)=1.3599

Then for 1.87 have 1.2551+(1.87-1.8)*(1.3599-1.2551)/(1.9-1.8)=1.33

Detached Shock Example: Supersonic Pitot-Static

- detached shock ahead of probe
  - $p_{static} \neq p_2$
  - $p_{pitot} = p_{02}$
- can relate p_{02} to p_1 and M_1 only using: previous derivations, some algebra &

$$\frac{p_{02}}{p_1} = \left(\frac{p_{02}}{p_2}\right)\left(\frac{p_2}{p_1}\right)$$

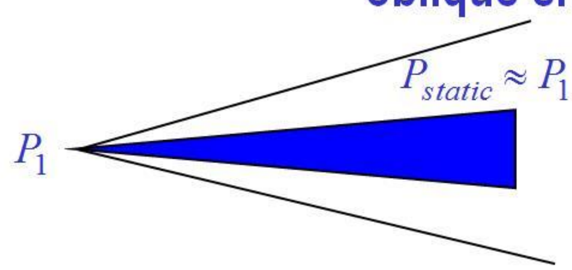


- Rayleigh's Supersonic Pitot Equation

$$\boxed{\frac{p_{02}}{p_1} = \frac{\left(\frac{\gamma+1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}\right)^{\frac{1}{\gamma-1}}}}$$

tabulated in normal shock tables

- given in compressible flow tables
- How do we calculate upstream static, p_1?



The double wedge aerofoil (M=2) – part 1

$$\theta = \tan^{-1}\left(\frac{0.1}{0.5}\right) = 11.310\,\text{deg}$$

Get angle – note this is the 'half angle'

$$\frac{p_2}{p_1} = 1.708 + \frac{1.891 - 1.708}{12 - 10}(11.310 - 10) = 1.828$$

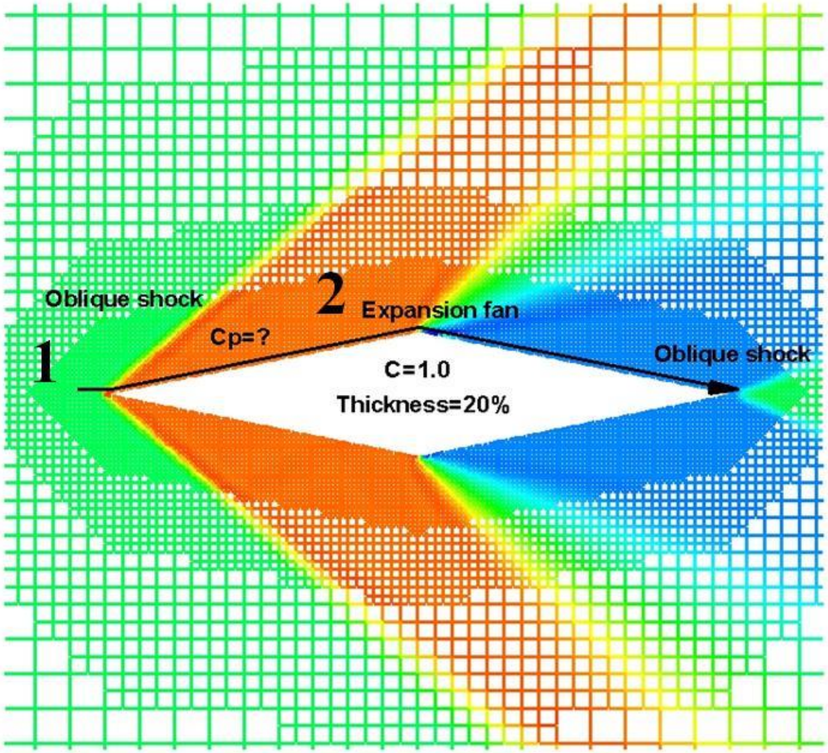
Interpolate linearly to find pressure ratio across oblique shock

$$C_p = \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p}{p_\infty} - 1\right)$$

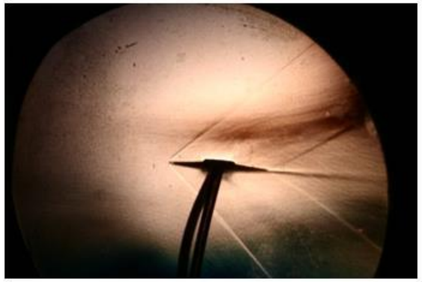
$$= \frac{2}{\gamma M_\infty^{\ 2}}\left(\frac{p_2}{p_1} - 1\right)$$

$$= \frac{2}{1.4 \times 2^2}(1.828 - 1) = 0.296$$

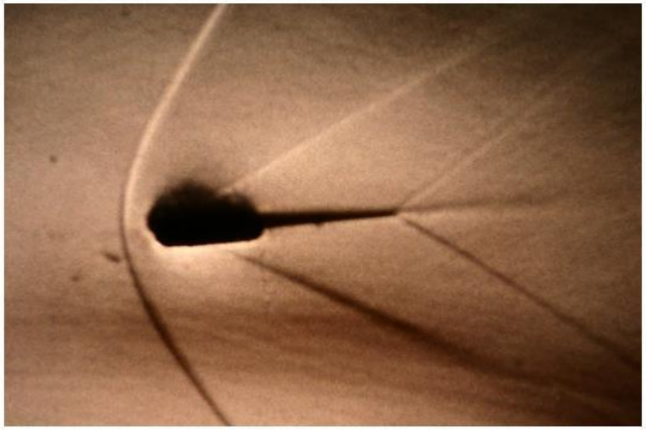
p1=p_inf for this case (only true here because this is a shock into the freestream flow)

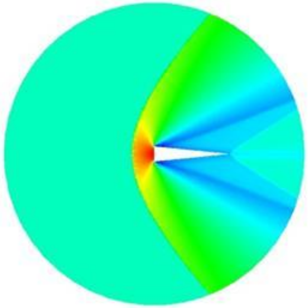


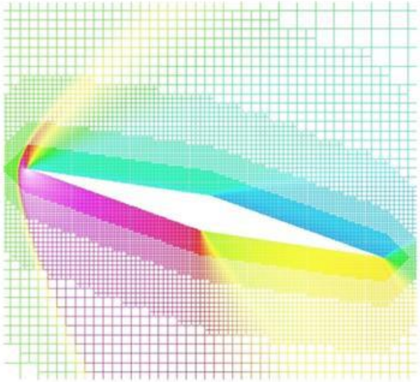
Compressible Flow Lab (next term)

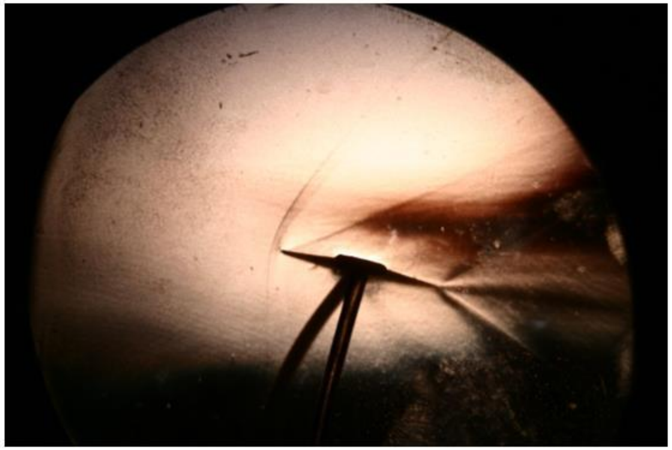


- Opportunity to see oblique shocks and detached bow shocks in practice









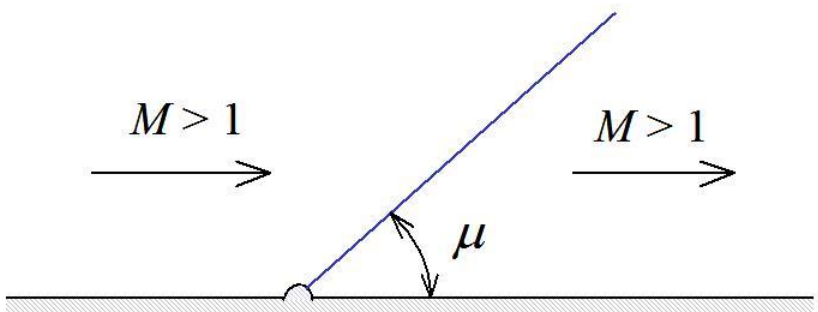
Today

- Expansion fans and the Prandtl-Meyer function
- Expansion example
- Wave reflections

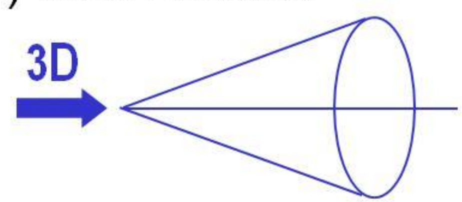
2D Supersonic Flow

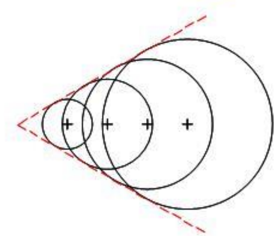
Mach Waves building blocks for larger disturbances

- start with supersonic flow along a *straight* surface
- isolated perturbations on surface generate disturbances in flow
  - eg scratches, rivets, panel edges etc
  - straight surface hence upstream and downstream conditions equal
  - infinitesimal pressure wave propagating at Mach Angle μ
  - point perturbations generate conical (3D) Mach Waves



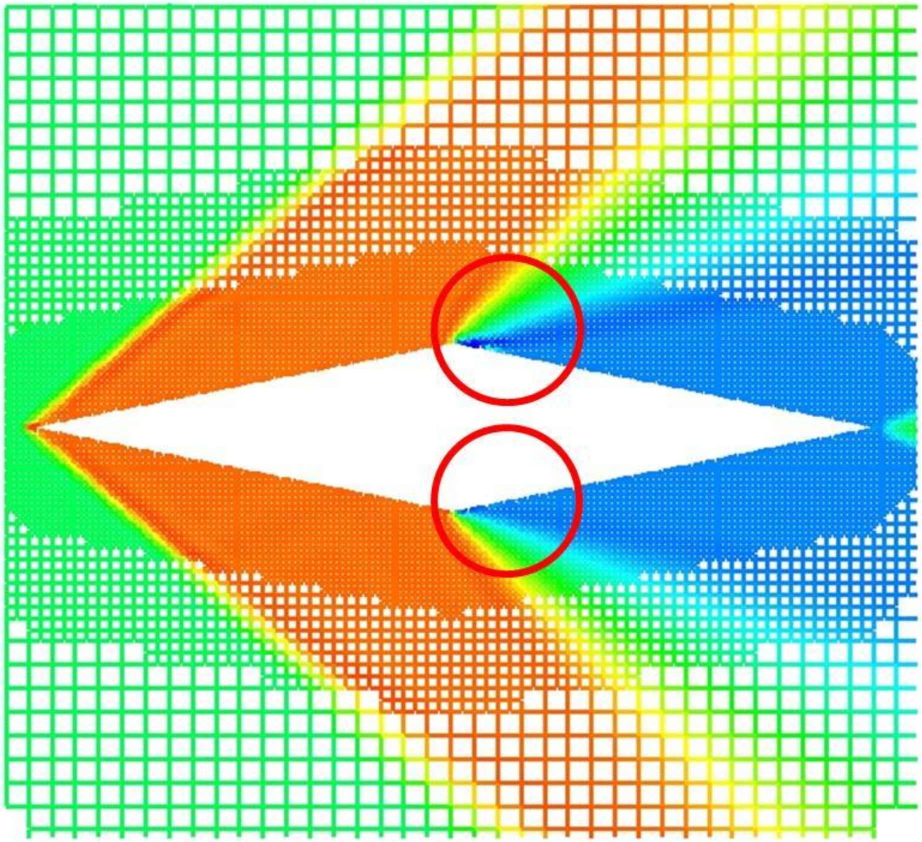
$$\sin \mu = \frac{1}{M}$$





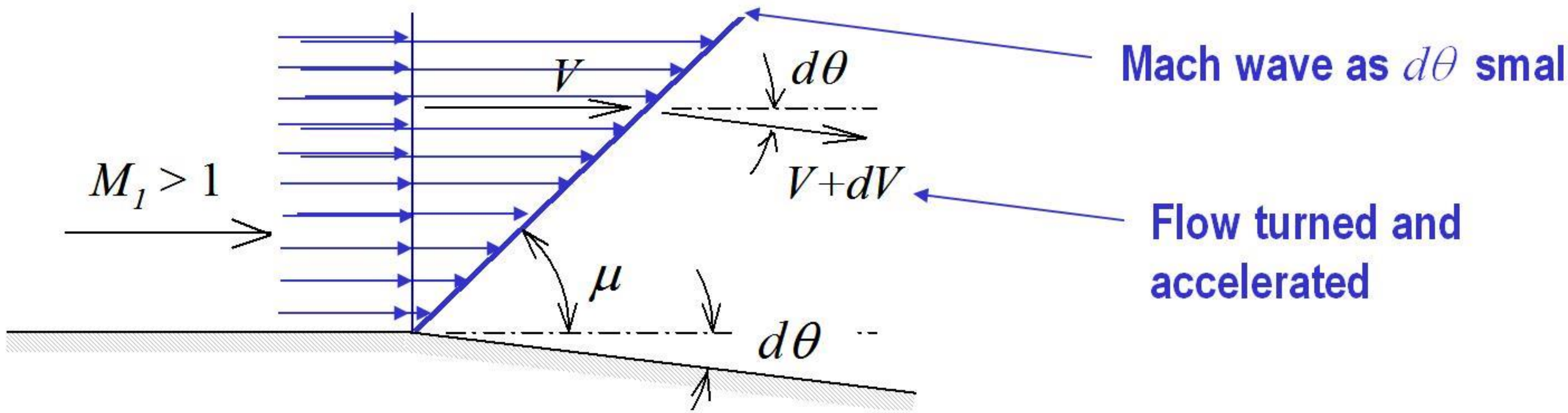
- slightly different situation if perturbation is a 2D corner …

The double wedge aerofoil – part 2



2D Supersonic Flow
'Small' Flow Deflection (1)

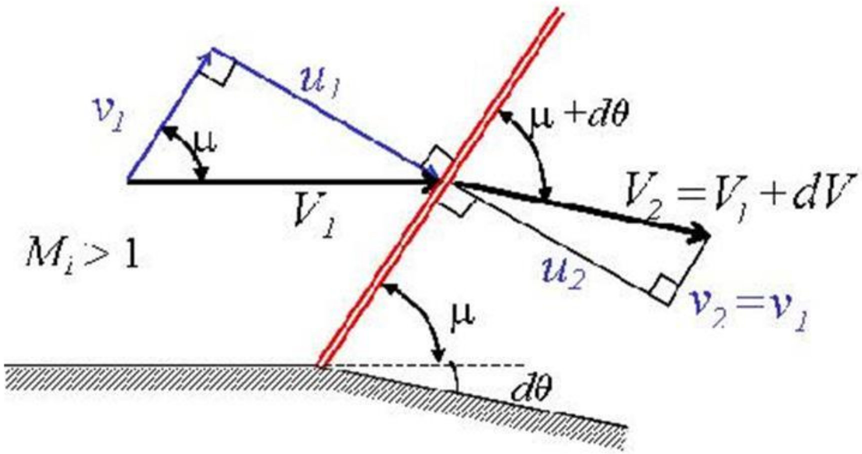
- surface (and hence flow) deflected by small angle $d\theta$



- no effect of deflection felt upstream of Mach wave
- 2D flow $\rightarrow$ flow properties ***constant*** along the wave
  - therefore no pressure gradient parallel (tangential) to the wave
- no change in ***tangential*** velocity component across the wave
  - only need to deal with changes in ***normal*** velocity component
  - also applicable to oblique shock wave analysis

2D Supersonic Flow
‘Small’ Flow Deflection (2)

- equate upstream and downstream tangential velocities
  - make small angle approximations & ignore 2nd order terms



$$v_1 = V_1 \cos(\mu) \qquad v_2 = (V_1 + dV)\cos(\mu + d\theta)$$
$$u_1 = V_1 \sin(\mu) \qquad u_2 = (V_1 + dV)\sin(\mu + d\theta)$$

tangential velocity unchanged

use cos(A+B)=cos(A)cos(B)-sin(A)sin(B)

$$v_1 = v_2$$
$$V_1 \cos(\mu) = (V_1 + dV)\begin{pmatrix} \cos(\mu)\cos(d\theta) - \\ \sin(\mu)\sin(d\theta) \end{pmatrix}$$

$d\theta$ **small** $\Rightarrow \cos(d\theta) \approx 1 \quad \sin(d\theta) \approx d\theta$

$$V_1 \cos(\mu) = (V_1 + dV)(\cos(\mu) - \sin(\mu) d\theta)$$
$$dV d\theta \approx 0$$
$$\cancel{V_1 \cos(\mu)} = \cancel{V_1 \cos(\mu)} - V_1 \sin(\mu) d\theta + dV \cos(\mu)$$
$$\sin(\mu) = \frac{1}{M} \quad \therefore \quad \cos(\mu) = \sqrt{1 - \sin^2(\mu)} = \frac{\sqrt{M^2 - 1}}{M}$$

$$\boxed{\frac{dV}{d\theta} = \frac{V}{\sqrt{M^2 - 1}}}$$

- positive for $M > 1$ $\rightarrow$ *expansion* = *acceleration*

2D Supersonic Flow
'Small' Flow Deflection (3)

- apply 1D Euler Equation + Newton's speed of sound equation

$$dp = -\rho V dV \qquad\qquad a^2 = \gamma RT = \frac{\gamma p}{\rho}$$

$$dp = -\frac{\gamma p}{a^2} V dV \quad = -\frac{\gamma p V^2}{a^2}\frac{dV}{V} \quad = -\gamma p M^2 \frac{dV}{V}$$

$$\frac{dV}{d\theta} = \frac{V}{\sqrt{M^2-1}} \Rightarrow \frac{dV}{V} = \frac{d\theta}{\sqrt{M^2-1}} \Rightarrow dp = \frac{-\gamma M^2 d\theta\, p}{\sqrt{M^2-1}}$$

$$\boxed{\frac{dp}{d\theta} = \frac{-\gamma M^2}{\sqrt{M^2-1}} p}$$

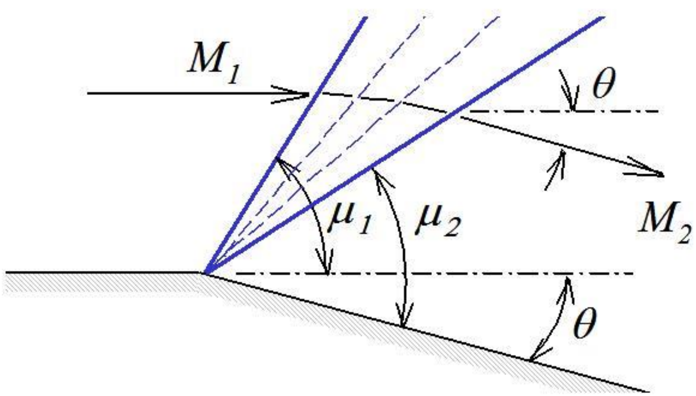
- negative for $M > 1$ $\rightarrow$ *expansion* = *pressure drop*
- need to integrate for finite deflection angle (Prandtl-Meyer expansion)

The integration

- For your information – no need to memorise or reproduce, but you should understand the final line

$$\frac{dV}{d\theta} = \frac{V}{\sqrt{M^2 - 1}} \qquad \text{so} \qquad \int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{V} dV$$

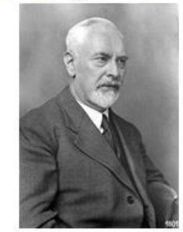
Fortunately Messrs. Prandtl and Meyer have done this for you – so no integration heroics required. In return, we humbly refer to the integral of the RHS as the Prandtl-Meyer function ν



$$\boxed{\theta_2 - \theta_1 = \theta = \nu(M_2) - \nu(M_1)}$$

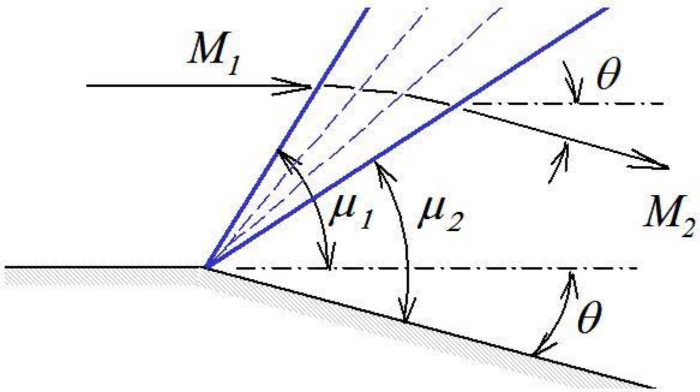
2D Supersonic Flow
Prandtl-Meyer Expansion

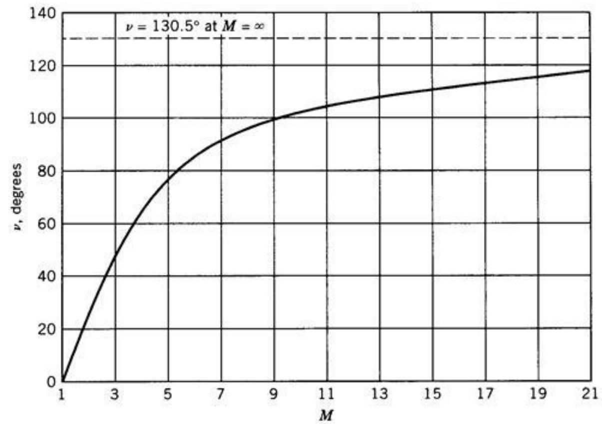
Ludwig Prandtl



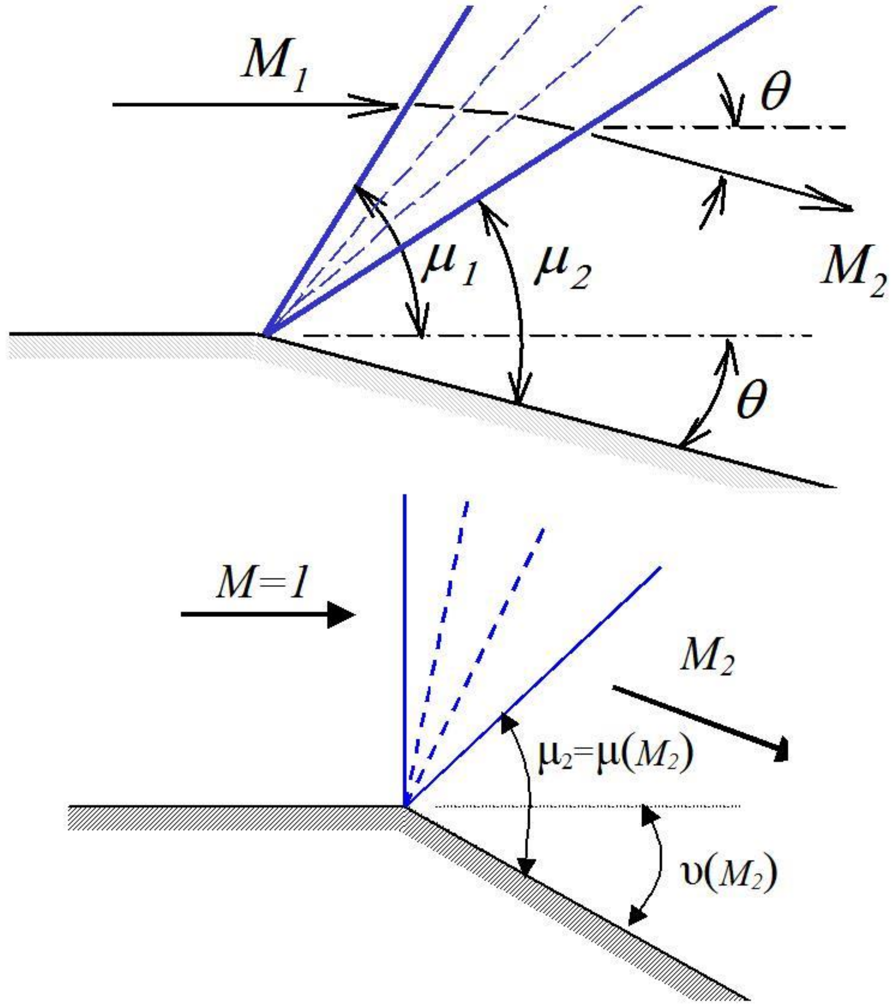
Theodor Meyer

- consider 'large' positive deflection angle θ
  - flow expands through series of infinitesimal expansions $d\theta$
  - isentropic *'expansion fan'*





- integral starts from M_1 =1 to give *Prandtl-Meyer angle* ν as a function of downstream Mach Number M_2 only
  - ie $\nu(M) \equiv \theta(M_1=1, M_2=M)$
  - a 'fictitious' angle through which sonic flow (ie at a Mach Number of 1) would have to be turned to give downstream Mach Number M
- For $M_1>1$, $\theta(M_1,M_2)$ *is difference of expansion angles for* M_1 *and* M_2



- For $M_1>1$, $\theta\,(M_1,M_2)$ *is difference of expansion angles for* M_1 *and* M_2
- $\theta = \nu\,(M_2) - \nu(M_1)$

What is that function?

- It is this

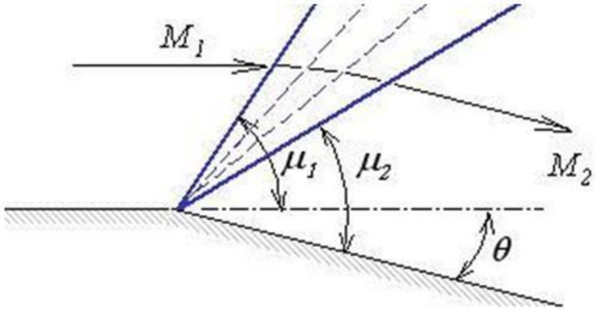
$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right)$$

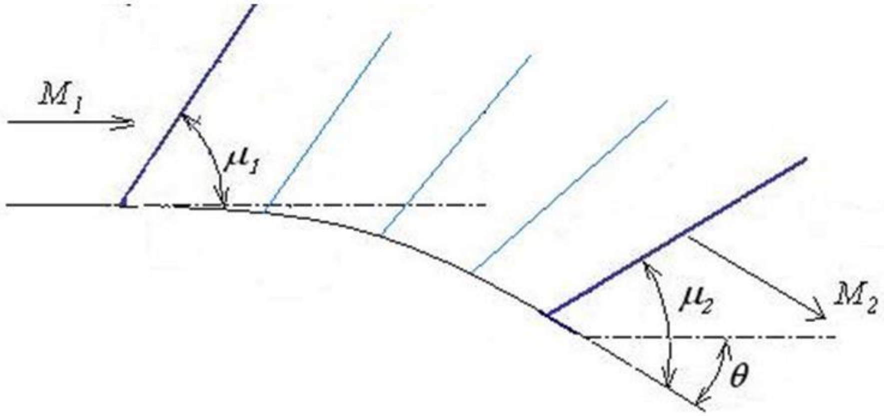
…and it is unwieldy and tricky to solve for M given ν

…so we don't. Instead use tables and trusty linear interpolation!

2D Supersonic Flow
Prandtl-Meyer Expansion (2)

- Given the upstream Mach number M_1 and the expansion angle, how do we find the downstream Mach number M_2
  - $\theta = \nu(M_2) - \nu(M_1)$
  - We know θ and can calculate $\nu(M_1)$
  - Therefore must find M_2 such that $\nu(M_2) = \theta + \nu(M_1)$
- As long as flow remains isentropic expansion path can be ignored





Prandtl-Meyer expansion angle tabulated in supersonic isentropic tables – hurrah!

Example for trailing part of the wedge

- M1=1.59 for which the PM function=14.535deg
- Deflection=22.6 degrees from the geometry

We find the second PM value…

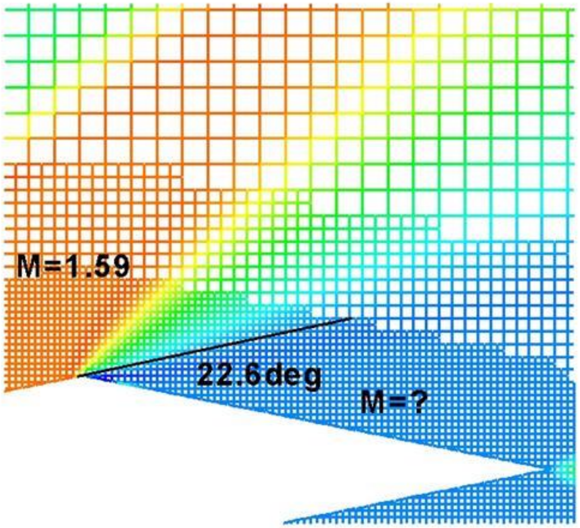
$$\nu(M_2) = 22.6 + \nu(M_1) = 22.6 + 14.535 = 37.135$$

Then look up the Mach number for that value of the PM function, to get…

$$M_2 = 2.42$$

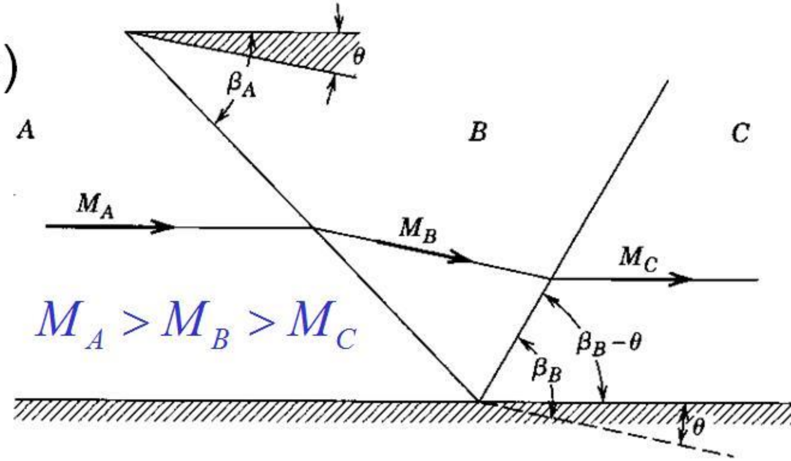
Same for pressure ratios etc.

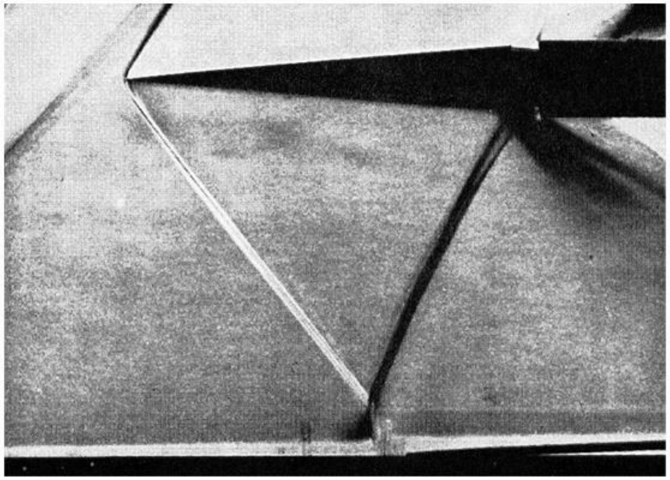
Everything in degrees, not radians!

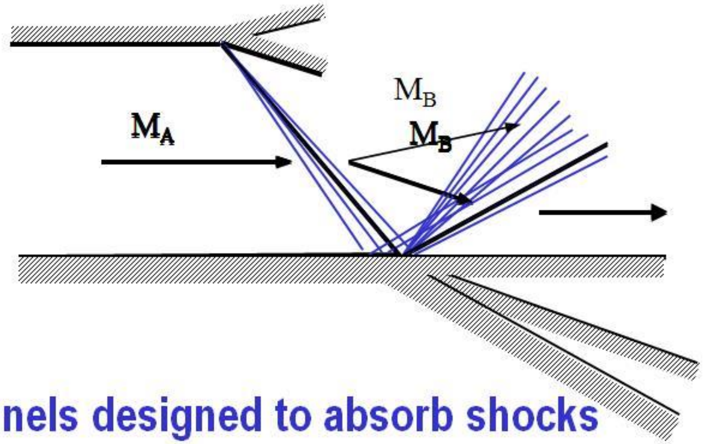


Wave Reflections

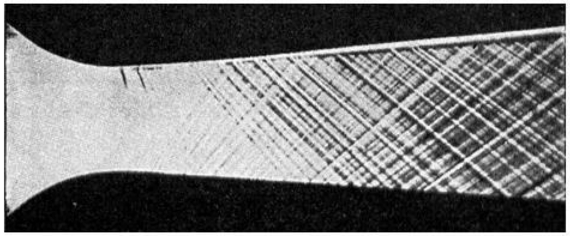
- Mach Waves (expansion & compression) and oblique shock waves are “reflected” at flow boundaries
  - flow tangency for solid boundaries
  - pressure equilibrium for jet boundaries
- for solid walls the type of reflection depends on the wall inclination. eg oblique shock can be reflected as a shock, as an expansion fan or be absorbed







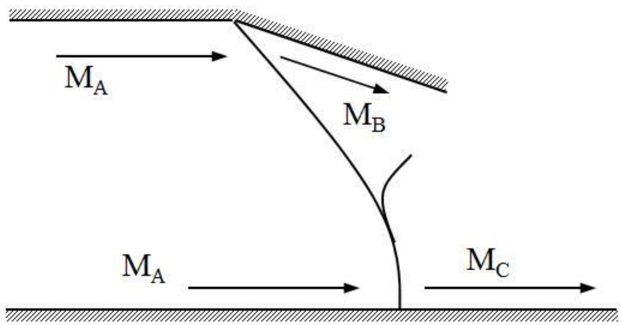
wind tunnels designed to absorb shocks

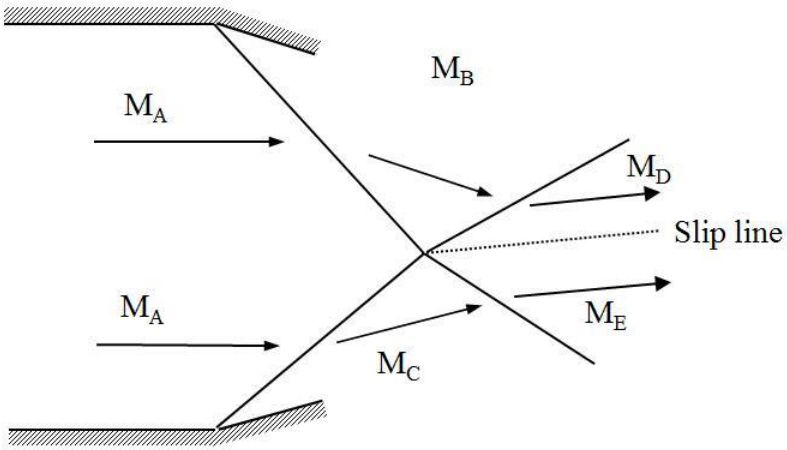


Wave Reflections (2)

Solid surface reflections

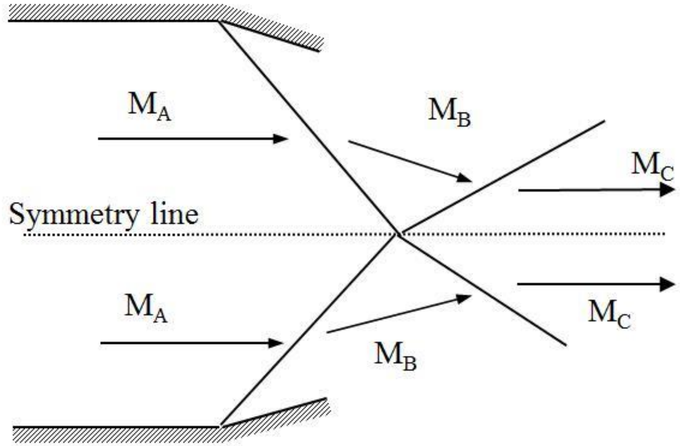
Mach wave reflection



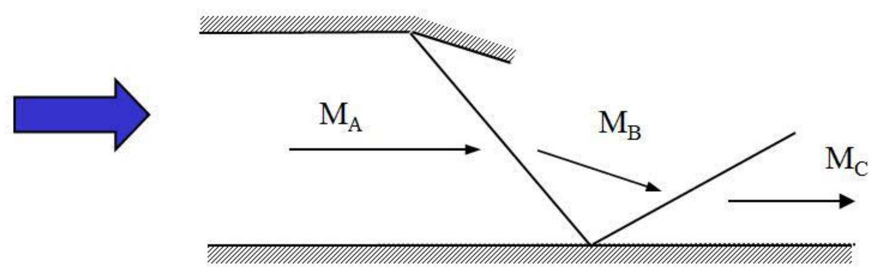


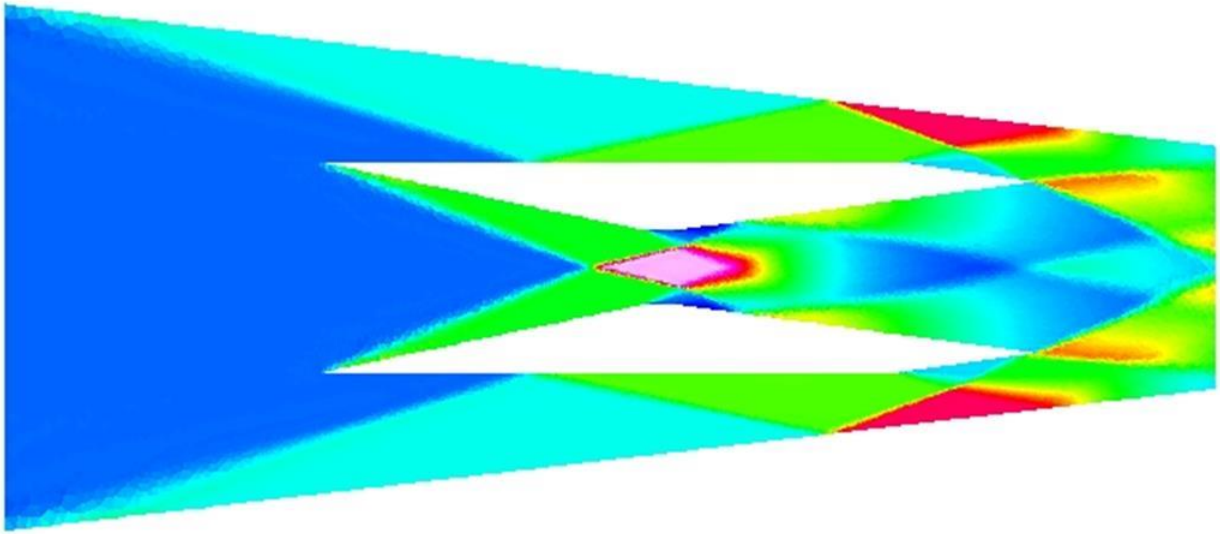
If M_c insufficient for 2nd Oblique shock

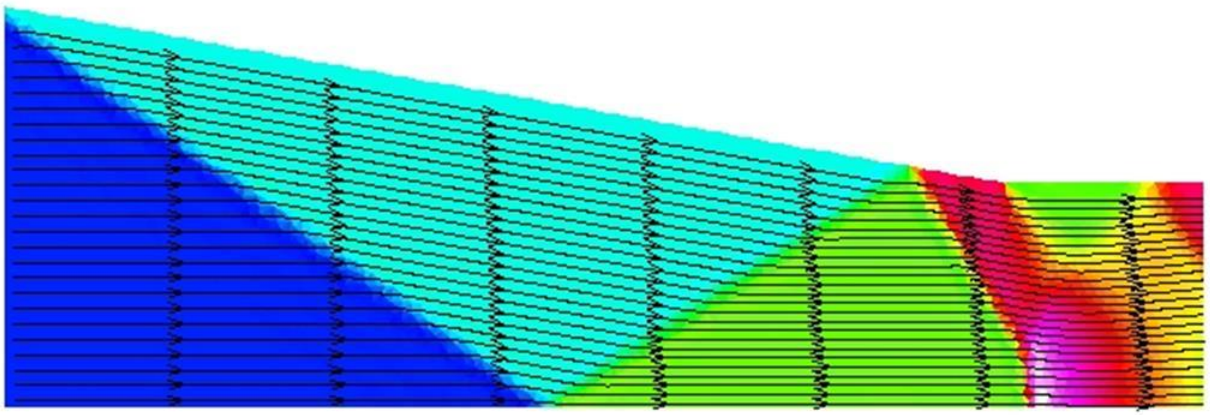
Slip line : direction defined by equality of pressure P_D=P_E

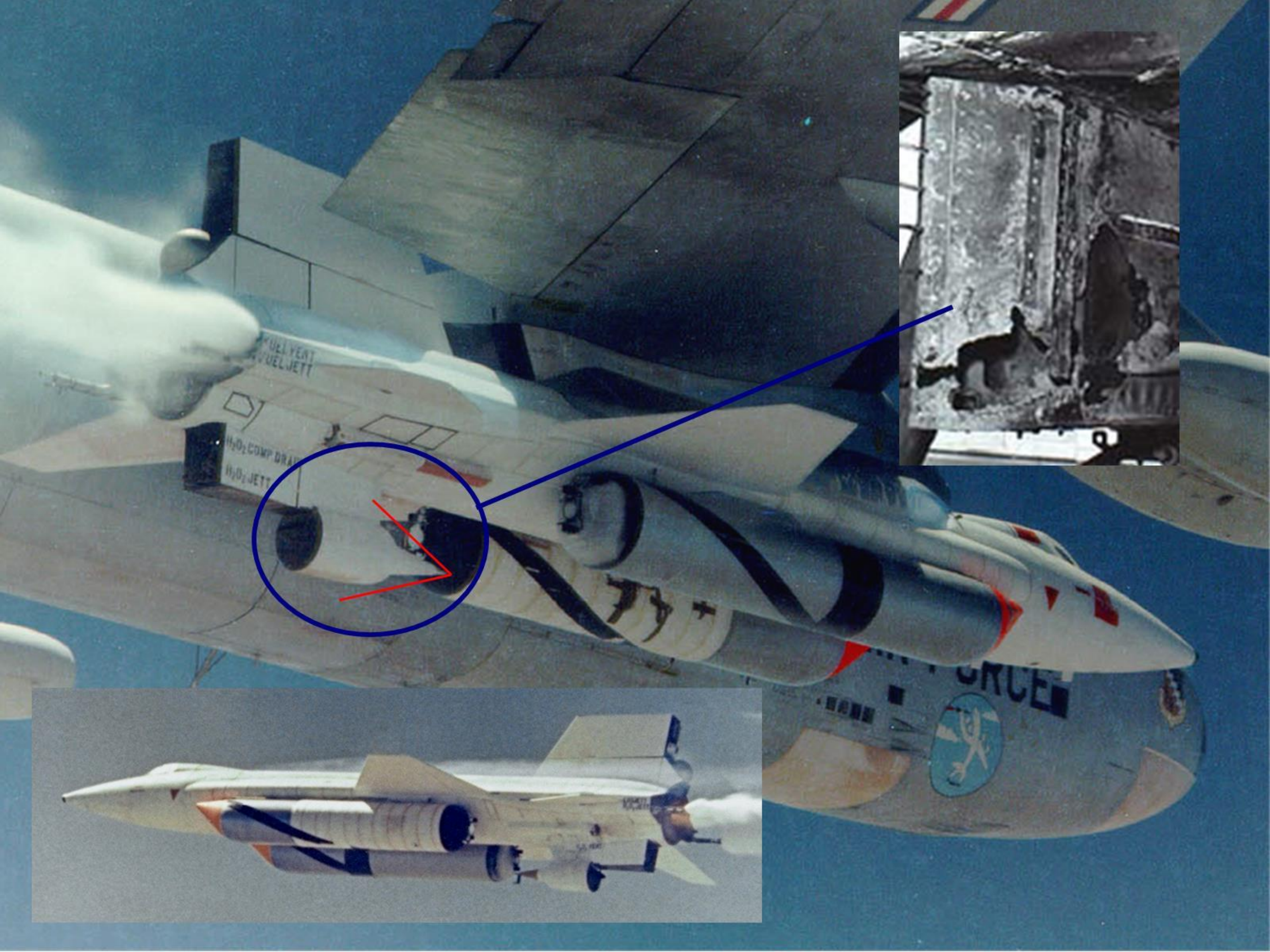


Equality of symmetry plane and solid wall





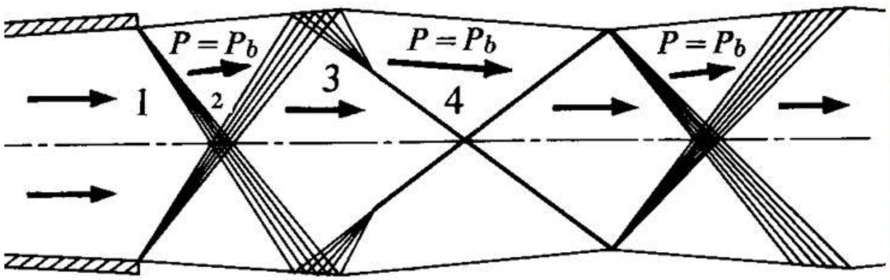



FUEL JETT
H2O2 JETT

Wave Reflections (3)

Fluid boundary reflections

- for fluid boundary (ie edge of jet) pressure equilibrium gives alternating compression and expansion waves
  - shock diamonds
  - viscosity smears out effect



Underexpanded supersonic nozzle

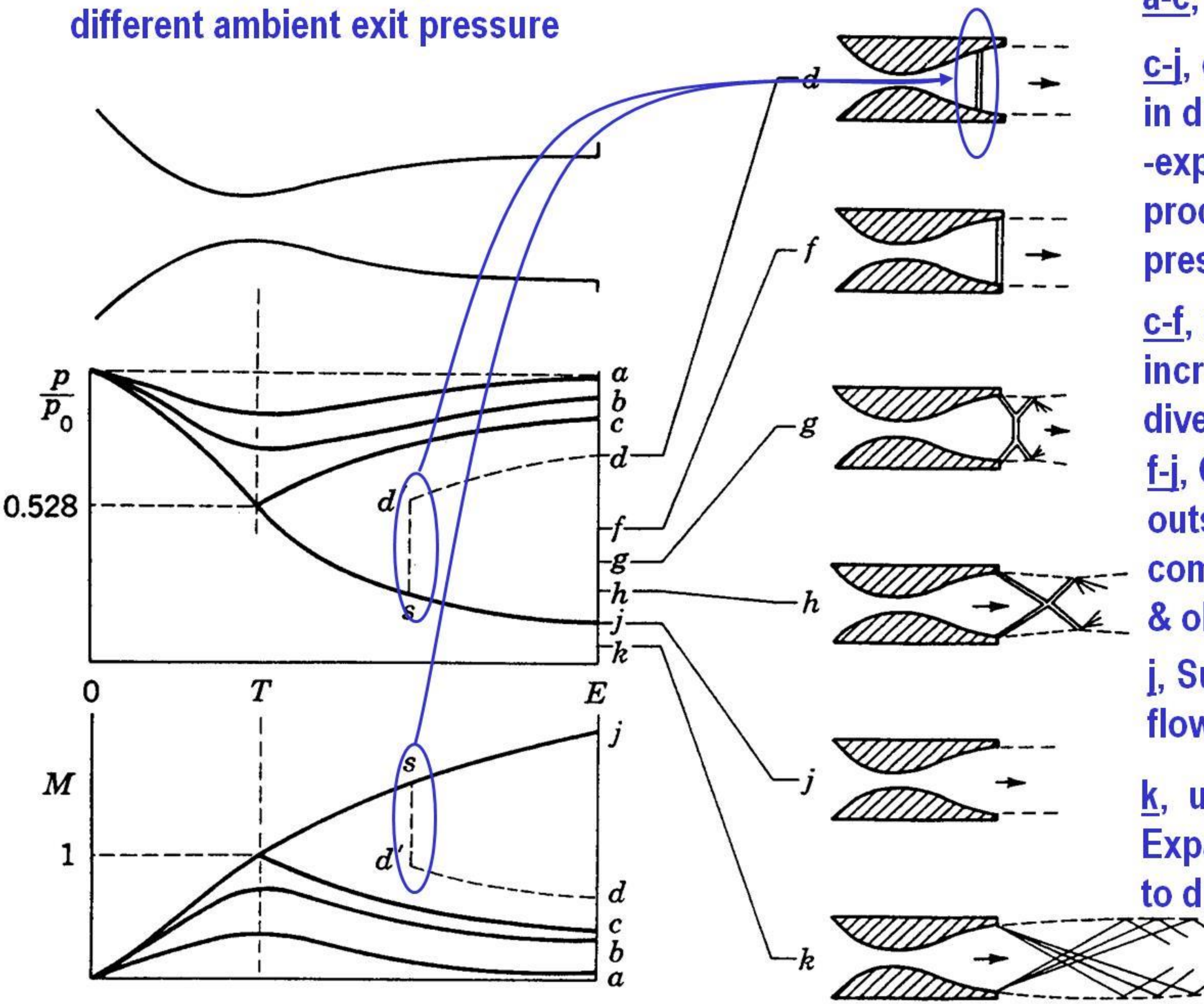
- Shock or expansion at exit dependant on pressure

Under-expanded nozzle has $P_1 > P_b$, Over-expanded nozzle has $P_1 < P_b$, what will this look like?

Low Mach numbers & viscosity can greatly complicate the picture.

Shocks in Convergent-Divergent Nozzles

each case, a-k, corresponds to a different ambient exit pressure



<u>a-c</u>, subsonic isentropic

<u>c-j</u>, exit pressure > pressure in divergent nozzle (over--expanded). Compression process needed to raise pressure

<u>c-f</u>, normal shock in nozzle increases pressure in divergent duct

<u>f-j</u>, Compression in jet outside duct. Can be complex mix of normal & oblique shocks.

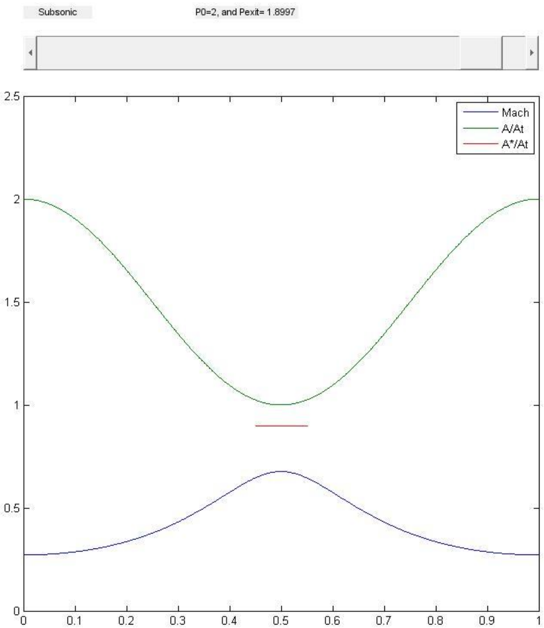
<u>j</u>, Supersonic isentropic flow. (ideally expanded)

<u>k</u>, underexpanded jet. Expansion process needed to drop jet pressure at exit.

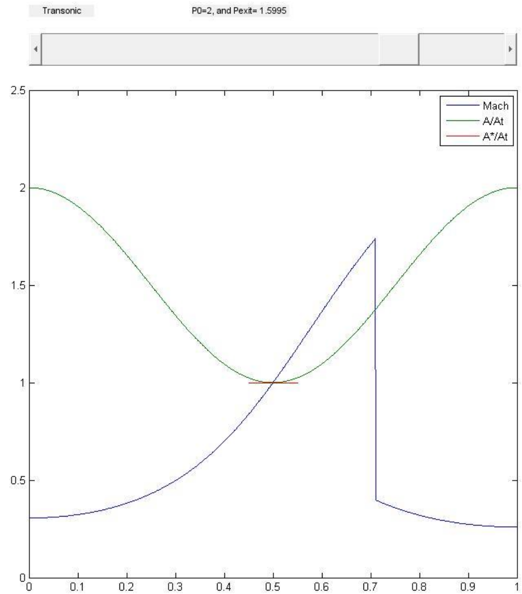
$$\frac{A}{A_t} = 2 - \sin^2(\pi x)$$

P0_start=2b
Pexit varies
with slider

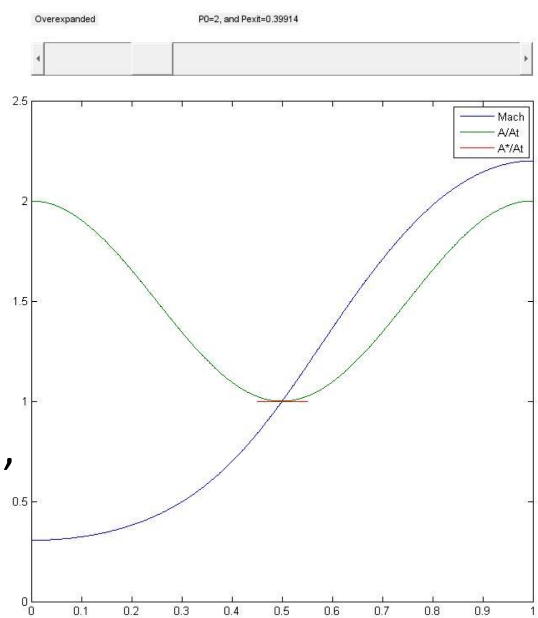
Subsonic

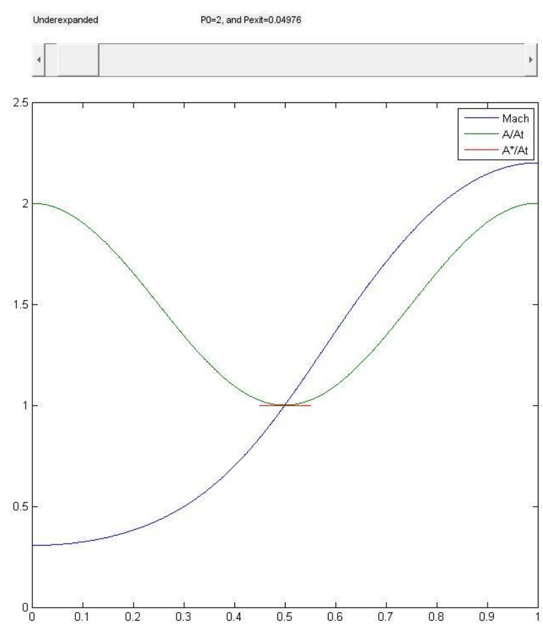


Transonic



Supersonic – overexpanded, shock outside nozzle





Supersonic – underexpanded, expansion outside nozzle

Case 1 – subsonic, Pe=1.9b

Subsonic

A/At	A/A*	M
2.000000	2.227262	0.271405
1.975480	2.199955	0.275103
1.904324	2.120714	0.286461
1.793511	1.997309	0.306276
1.653910	1.841846	0.335861
1.499214	1.669571	0.376924
1.344595	1.497382	0.431105
1.205218	1.342168	0.498667
1.094753	1.219151	0.575202
1.024036	1.140399	0.644801
1.000002	1.113634	0.674986
1.025009	1.141482	0.643668
1.096603	1.221212	0.573659
1.207764	1.345003	0.497200
1.347587	1.500714	0.429886
1.502359	1.673073	0.375979
1.656899	1.845174	0.335163
1.796051	2.000138	0.305789
1.906166	2.122765	0.286154
1.976443	2.201028	0.274955
2.000000	2.227262	0.271405

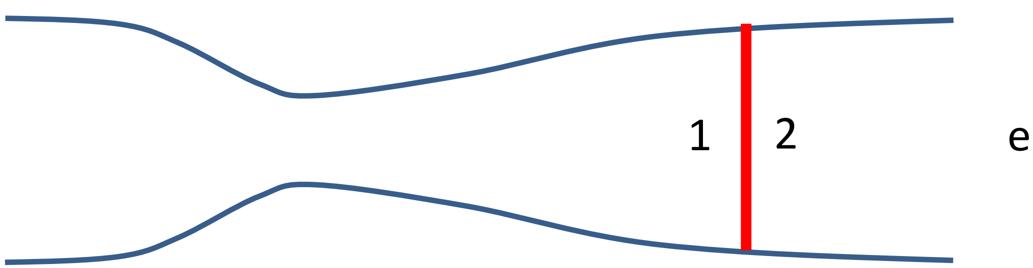
$$\frac{p_0}{p_e} = \frac{2}{1.9} = 1.0526$$

Mach number from pressure ratio

$$M = 0.26 + 0.02 \times \frac{1.0526 - 1.048}{1.056 - 1.048} = 0.2715$$

Important – the static pressure of the air at exit is equal to the downstream static pressure. This is only true for a subsonic jet discharging

Nomenclature – note the shock is not present for this calculation



Note p02=p0e

Case 2 – transonic, Pe=1.5b

Choked - transonic
Shock location | A/At at shock | A*_beforeshock/A*_aftershock | P0_aftershock/P_beforeshock
----
0.717860 1.399713 0.828832 4.512082

M_before | M_after | P0/P_before | P0/P_after | P0/P_exit

1.763973 0.470818 5.443907 1.164310 1.105108

(1/(P0/P_exit))*(P0_aftershock/P_beforeshock)*(1/(P0/P_before))= 1.500001

A/At	A/A*	M
2.000000	2.000000	0.305813
1.975480	1.975480	0.310090
1.904324	1.904324	0.323260
1.793511	1.793511	0.346367
1.653910	1.653910	0.381217
1.499214	1.499214	0.430434
1.344595	1.344595	0.497411
1.205218	1.205218	0.585953
1.094753	1.094753	0.699361
1.024036	1.024036	0.838842
1.000002	1.000002	1.005006
1.025009	1.025009	1.180775
1.096603	1.096603	1.365608
1.207764	1.207764	1.545454
1.347587	1.347587	1.711442
1.502359	1.245202	0.556472
1.656899	1.373290	0.483123
1.796051	1.488624	0.434346
1.906166	1.579890	0.403086
1.976443	1.638138	0.385651
2.000000	1.657663	0.380178

Before we can answer this problem, we need to know the downstream pressure for two other cases…

1. Choked, **but with** subsonic flow after throat (done on last slide)
2. Ideally expanded
3. Choked, with supersonic flow after throat **followed by a normal shock exactly at the nozzle exit**

These results will tell us what's happening in the nozzle.

Option 1 – choked/subsonic

For this case we know At=A*. So pressure at exit for this would be

$$\frac{p_0}{p} = 1.065 + (2 - 2.035)\frac{1.074 - 1.065}{1.921 - 2.035} = 1.0678$$

Pressure ratio from A/A*

$$p = \frac{2}{1.0678} = 1.8731$$

Lower than this, and there will certainly be a supersonic region in the nozzle, **but we don't know about the shock position yet**

Options 2 and 3

Ideally expanded

Pressure from area

$$\frac{p_0}{p} = 10.366 + (2 - 1.966)\frac{10.695 - 10.366}{2.001 - 1.966} = 10.6856$$

$$p = \frac{2}{10.6856} = 0.1872$$

This is pretty low. Below this the nozzle would be underexpanded (expansions outside nozzle), and above this it is overexpanded (shocks at exit or outside nozzle).

Exit Mach for ideal case is $M = 2.18 + 0.02 \times \frac{2 - 1.966}{2.001 - 1.966} = 2.199$

An important case is “ideal+normal shock”. So compute a normal shock from here

$$\frac{p_{02}}{p_1} = 6.614 + (2.199 - 2.18)\frac{6.726 - 6.614}{0.02} = 6.7204$$ Using p02/p1 column

$$M = 0.55 + (2.199 - 2.18)\frac{0.548 - 0.55}{0.02} = 0.5481$$ Downstream (exit) M

$$\frac{p_0}{p} = 1.22 + (0.5481 - 0.54)\frac{1.238 - 1.22}{0.02} = 1.2273$$ Downstream (exit) total pressure ratio

$$\frac{p_e}{p_{01}} = \frac{p_e}{p_{0e}}\frac{p_{0e}}{p_1}\frac{p_1}{p_{01}} = \frac{1}{1.2273}6.7204 \times \frac{1}{10.6856} = 0.5124 \rightarrow \boxed{p_e = 2 \times 0.5124 = 1.0248}$$

Transonic summary

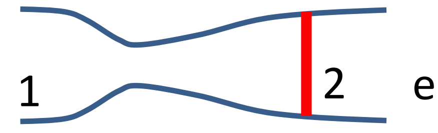
- If Pe below 1.0248, then shocks/expansions are outside nozzle, and nozzle fully supersonic
- If Pe above 1.0248, then shock is **inside** divergent part of nozzle - transonic
- If Pe above 1.8731, then fully subsonic

Case 2 – transonic, Pe=1.5b

We now know there must be a shock in the divergent part.

It is actually quite hard to work out (analytically) exactly where. However, we know everything we need to from the relationship

$$p_e = p_{01} \frac{p_e}{p_{0e}} \frac{p_{0e}}{p_1} \frac{p_1}{p_{01}}$$



Note p02=p0e

We can easily iterate (bisect) on the shock position, until the calculated exit pressure matches the specified exit pressure. This is how the matlab tool works.

Alternatively, it is quite easy to find the exact exit pressure for a specified shock location, by using the same procedure as on the previous slide, but with the shock at any other location. This is the type of question you should be able to solve – see tutorial sheet for more examples.

Ie. You would not be expected to find the shock position from the exit pressure, but you should be able to find the exit pressure given the shock position.

Case 3 – supersonic, Pe=1b

Choked - supersonic

A/At	A/A*	M
2.000000	2.000000	0.305813
1.975480	1.975480	0.310090
1.904324	1.904324	0.323260
1.793511	1.793511	0.346367
1.653910	1.653910	0.381217
1.499214	1.499214	0.430434
1.344595	1.344595	0.497411
1.205218	1.205218	0.585953
1.094753	1.094753	0.699361
1.024036	1.024036	0.838842
1.000002	1.000002	1.005006
1.025009	1.025009	1.180775
1.096603	1.096603	1.365608
1.207764	1.207764	1.545454
1.347587	1.347587	1.711442
1.502359	1.502359	1.857397
1.656899	1.656899	1.979508
1.796051	1.796051	2.075629
1.906166	1.906166	2.144632
1.976443	1.976443	2.185966
2.000000	2.000000	2.199399

$$\cancel{\frac{p_0}{p_e} = \frac{2}{1} = 2}$$

Not true, because we know this is supersonic discharge

Just find M from A/A*. We know At=A*, so

$$M = 2.18 + 0.02 \times \frac{2 - 1.966}{2.001 - 1.966} = 2.199$$ Supersonic

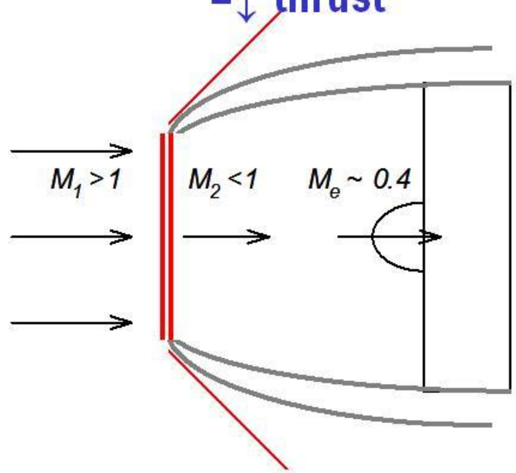
$$M = 0.3 + 0.02 \times \frac{2-2.035}{1.921-2.035} = 0.3061$$ Subsonic

Supersonic Intakes

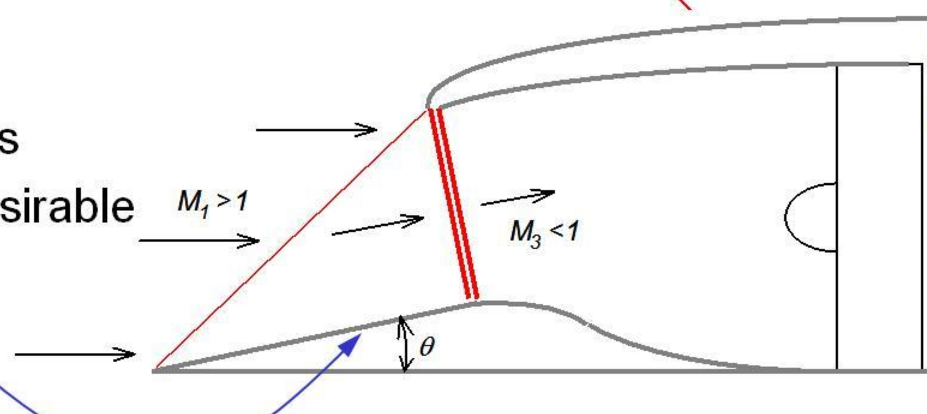
- conventional engines require subsonic flow at fan face (<0.4M)
  - implies shock system at supersonic speeds
  - shock provides compression – but at a price
- basic Pitot intake
  - normal shock at lip
  - 2-stage compression
  - inefficient above 1.5M
- external compression intake
  - external oblique shock
  - normal shock at lip
  - 3-stage compression
  - reduced pressure loss
  - variable geometry desirable

Shock means:compression, pressure rise but total pressure drop.

P_0 ↓=↓ available mechanical energy.
=↓ thrust



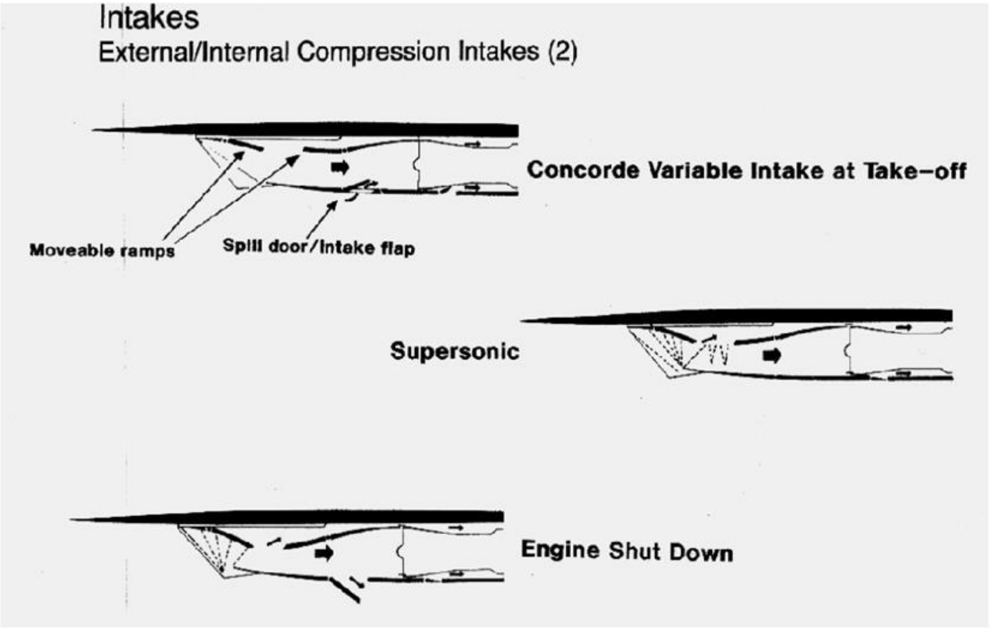
Intake design: minimise drop In P_0



Oblique shock has less P_0 ↓for same M↓

Normal shock losses decrease with M

Concorde



B-58…22.4%
lost in accidents!

Force coefficients

Argh! I hate summations!

$$\mathbf{F} = \int p\mathbf{n}ds$$ **n** is inward normal

$$p = C_p q_\infty + p_\infty$$

$$\mathbf{F} = \int C_p q_\infty \mathbf{n} ds$$

$$\mathbf{C}_F = \frac{\mathbf{F}}{q_\infty c} = \int C_p \mathbf{n} d\left(\frac{s}{c}\right)$$

$$C_X = \int C_p \mathbf{n}\cdot\mathbf{i}\, d\left(\frac{s}{c}\right)$$

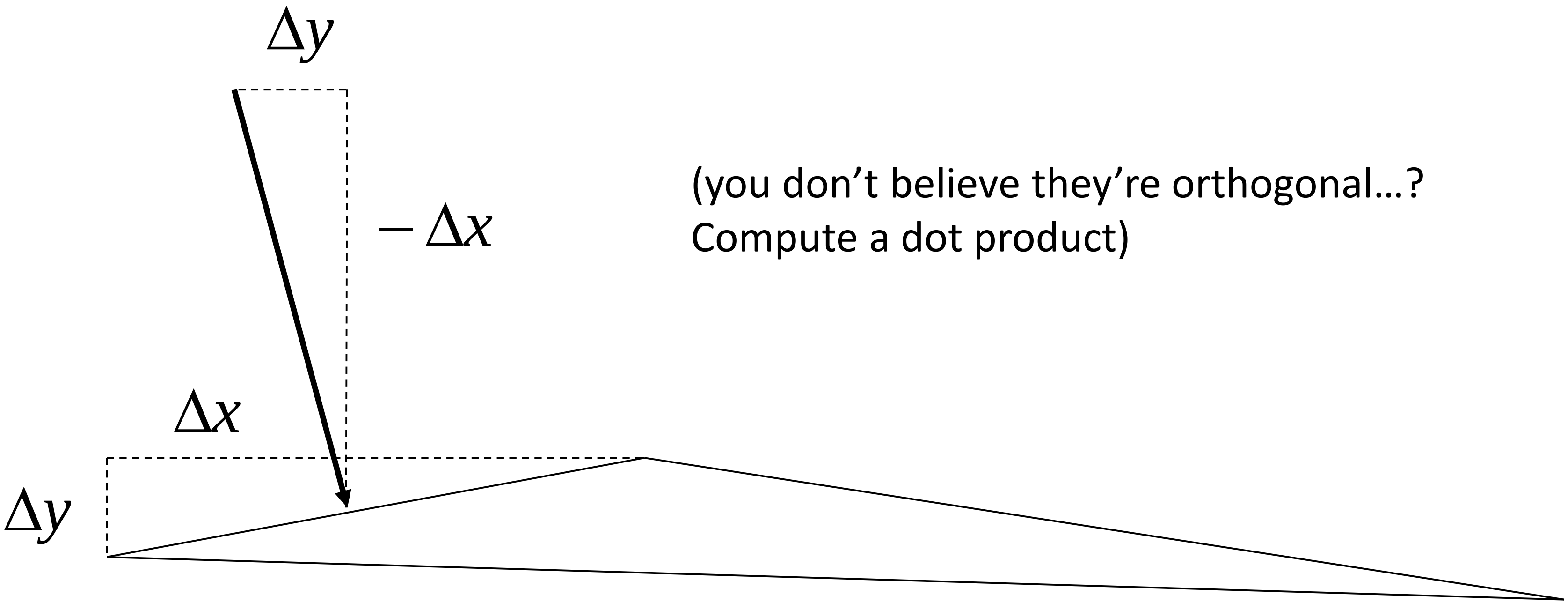
$$C_Y = \int C_p \mathbf{n}\cdot\mathbf{j}\, d\left(\frac{s}{c}\right)$$

$$\mathbf{n}' = \begin{pmatrix} n_x \\ n_y \end{pmatrix}\frac{s}{c} = \begin{pmatrix} \Delta y \\ -\Delta x \end{pmatrix}$$

$$C_X = \sum_{i=1}^{i=N} C_{p_i}\mathbf{n}'_{x_i} = \sum_{i=1}^{i=N} C_{p_i}\Delta y_i$$

$$C_Y = \sum_{i=1}^{i=N} C_{p_i}\mathbf{n}'_{y_i} = -\sum_{i=1}^{i=N} C_{p_i}\Delta x_i$$

(inward!) Normals

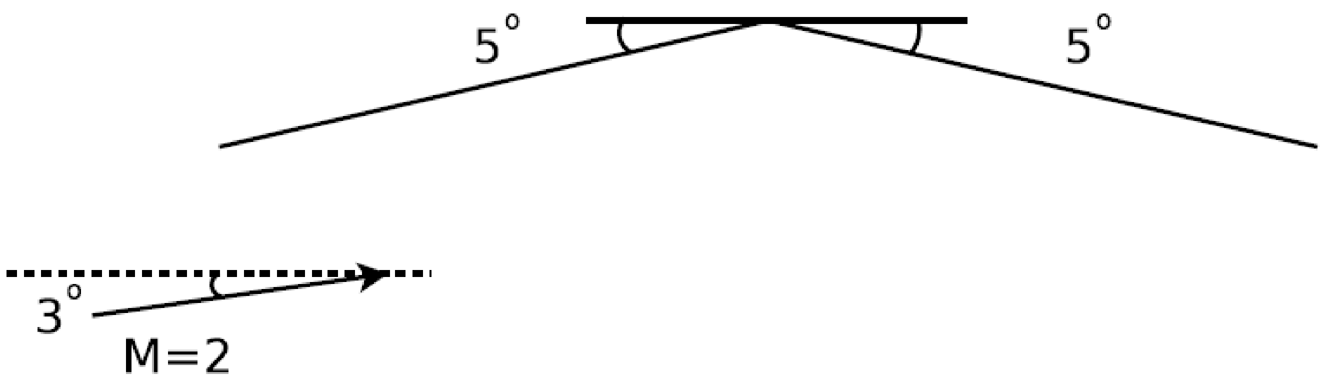


(you don't believe they're orthogonal…? Compute a dot product)

Best approach is to work out the signs on each contribution by **<u>common sense</u>**

Or, go for the doctrinaire approach of memorising a system for doing it, but wave goodbye to an element of your sanity.

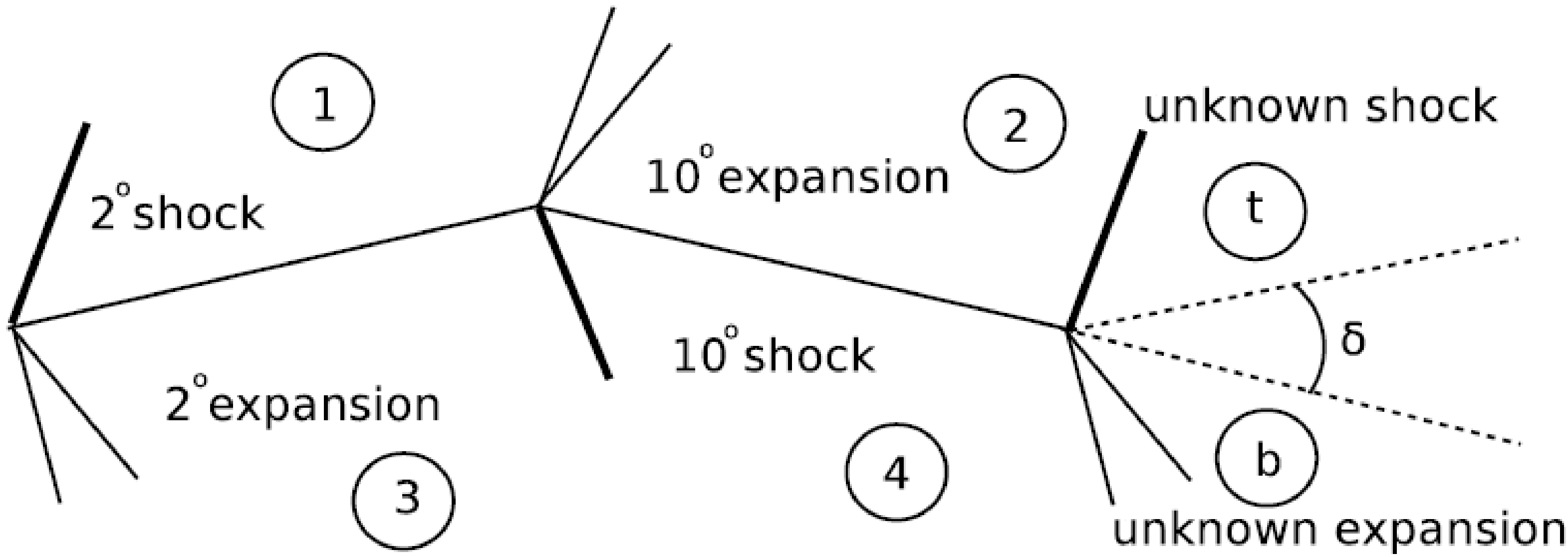
Full aerofoil example

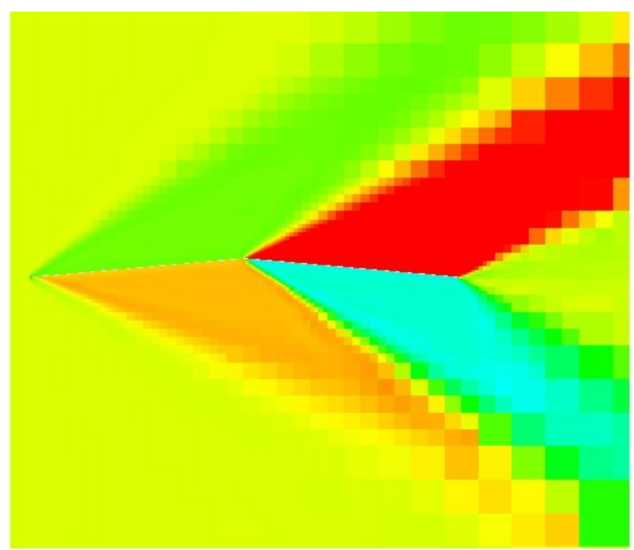


Picture
Strategise
Calculate
Integrate

<u>P</u>edantic <u>s</u>tudents <u>c</u>ash <u>i</u>n

Picture


1
2
unknown shock
2°shock
10°expansion
t
δ
10°shock
2°expansion
b
3
4
unknown expansion



Strategise

Surface 1 requires only $\frac{p_1}{p_\infty}$

Surface 2 requires $\frac{p_2}{p_\infty} = \frac{p_2}{p_{02}} \frac{p_{01}}{p_1} \frac{p_1}{p_\infty}$

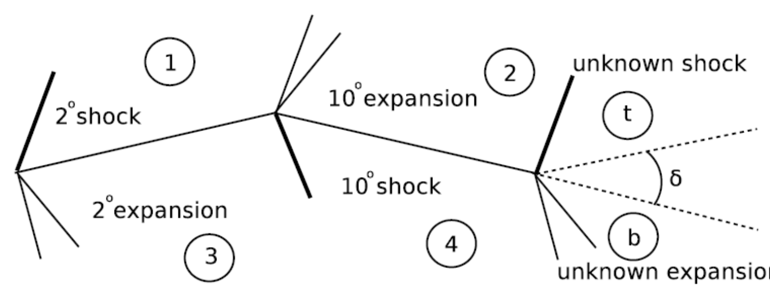
Surface 3 requires only $\frac{p_3}{p_\infty} = \frac{p_3}{p_{03}} \frac{p_{0\infty}}{p_\infty}$

Surface 4 requires $\frac{p_4}{p_\infty} = \frac{p_4}{p_3} \frac{p_3}{p_{03}} \frac{p_{0\infty}}{p_\infty}$

Surface 1

$$C_{p_1} = \frac{2}{\gamma M_\infty^2}(1.118 - 1) = 0.0421$$

Remember to use M_freestream for finding Cp values, NOT M_local



Surface 3 $\dfrac{p3}{p\infty} = \dfrac{p3}{p03}\dfrac{p0\infty}{p\infty}$

Moving to surface 3, which is an expansion of 2°

$$\nu(2) = 26.32$$

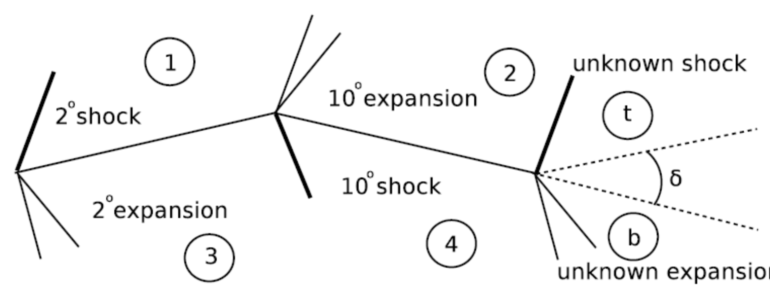
$$\nu^{-1}(28.32) = \mathcal{L}\left(8.596, 8.868, 27.95, 28.49, 28.32\right) = 8.782$$

and we already know $\frac{p0\infty}{p\infty} = 7.830$ so

$$C_{p3} = \frac{2}{\gamma M_\infty^2}\left(\frac{7.830}{8.782} - 1\right) = -0.0386$$

We now need the Mach number on surface 3, which is

$$\nu^{-1}(26.320 + 2) = \mathcal{L}\left(2.06, 2.08, 27.95, 28.49, 28.32\right) = 2.0737$$



Surface 2 $\frac{p2}{p\infty} = \frac{p2}{p02}\frac{p01}{p1}\frac{p1}{p\infty}$

For surface 2, we need ν for the flow from surface 1 (where M=1.928 downstream of the shock). also need the total pressure ratio for surface 1

$$\mathcal{L}\,(6.916, 7.134, 1.92, 1.94, 1.928) = 7.0032$$

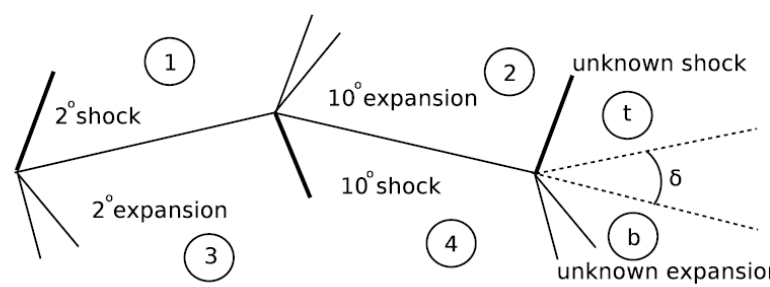
now ν for surface 1

$$\nu(1.928) = \mathcal{L}\,(24.090, 24.650, 1.92, 1.94, 1.928) = 24.314$$

What we actually want is the total pressure ratio, not the Mach number, so

$$\nu^{-1}(34.314) = \mathcal{L}\,(12.504, 12.901, 36.690, 34.314) = 12.6025$$

$$C_{p_2} = \frac{2}{\gamma M_\infty^2}\left(1.118\frac{7.0032}{12.6025} - 1\right) = -0.1350$$



Surface 4 $\frac{p_4}{p_\infty} = \frac{p_4}{p_3}\frac{p_3}{p_{03}}\frac{p_{0\infty}}{p_\infty}$

For surface 4 there is a shock through 10°. Interpolating in terms of Mach number for the shock for pressure ratio

$$\mathcal{L}\left(1.708, 1.736, 2.0, 2.1, 2.0737\right) = 1.7286 \tag{101}$$

and for downstream Mach

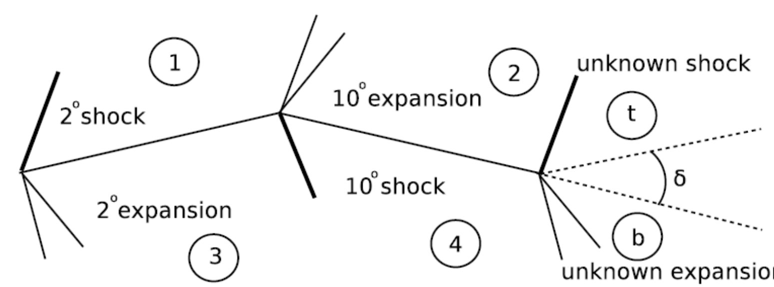
$$\mathcal{L}\left(1.64, 1.731, 2.0, 2.1, 2.0737\right) = 1.7071 \tag{102}$$

We will need the total pressure ratio here later for the slip line, so

$$\mathcal{L}\left(4.941, 5.093, 1.70, 1.72, 1.7071\right) = 4.9950 \tag{103}$$

Finally

$$C_{p_2} = \frac{2}{\gamma M_\infty^2}\left(1.7286\frac{7.830}{8.782} - 1\right) = 0.1929 \tag{104}$$



Integrate

$$C_X = 0.5\tan(5)(0.0421 + 0.0386 + 0.1350 + 0.1929) = 0.01787$$

$$C_Y = 0.5(-0.0421 - 0.0386 + 0.1350 + 0.1929) = 0.1236$$

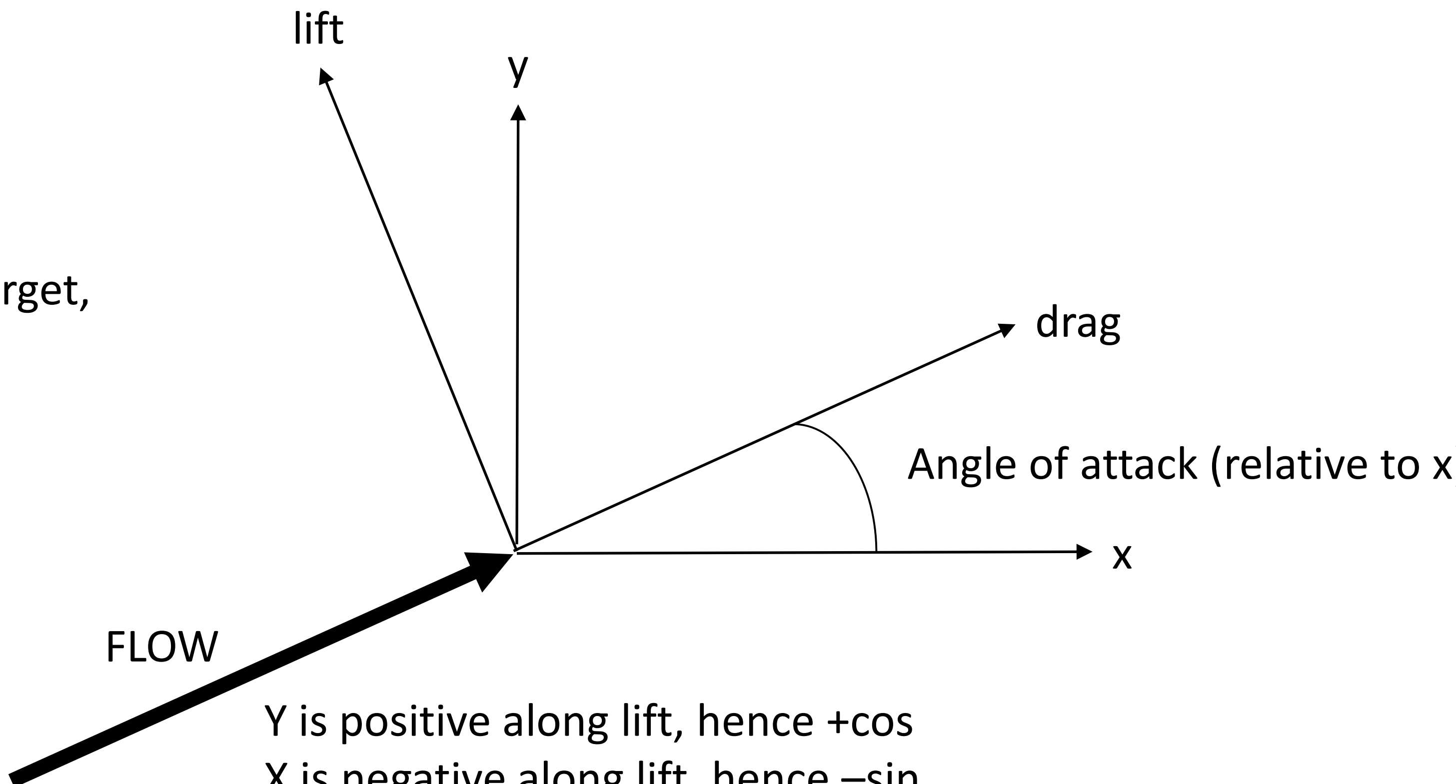
$$C_L = C_Y\cos(3) - C_X\sin(3) = 0.1225$$

…huh?

$$C_D = C_Y\sin(3) + C_X\cos(3) = 0.0243$$

$$C_M = 0.5(0.0421 \times 0.25 + 0.0386 \times 0.25 - 0.1350 \times 0.75 - 0.1929 \times 0.75) + 0.5^3\tan^2(5)(0.0421 + 0.0386 + 0.1350 - 0.1929) = -0.1129$$

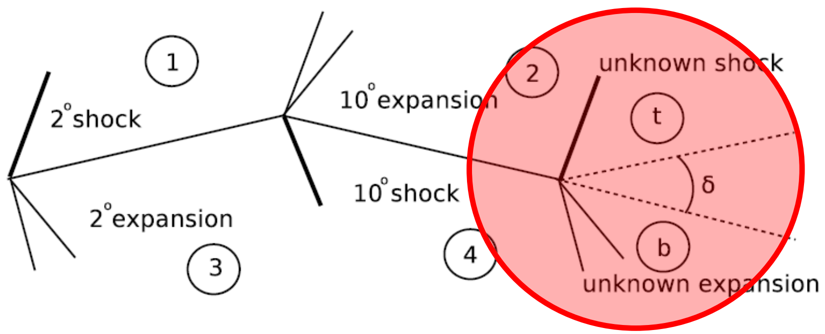
Rotation



If you forget,
draw
the
picture!

Y is positive along lift, hence +cos
X is negative along lift, hence –sin
Y is positive along drag, hence +sin
X is positive along drag, hence +cos

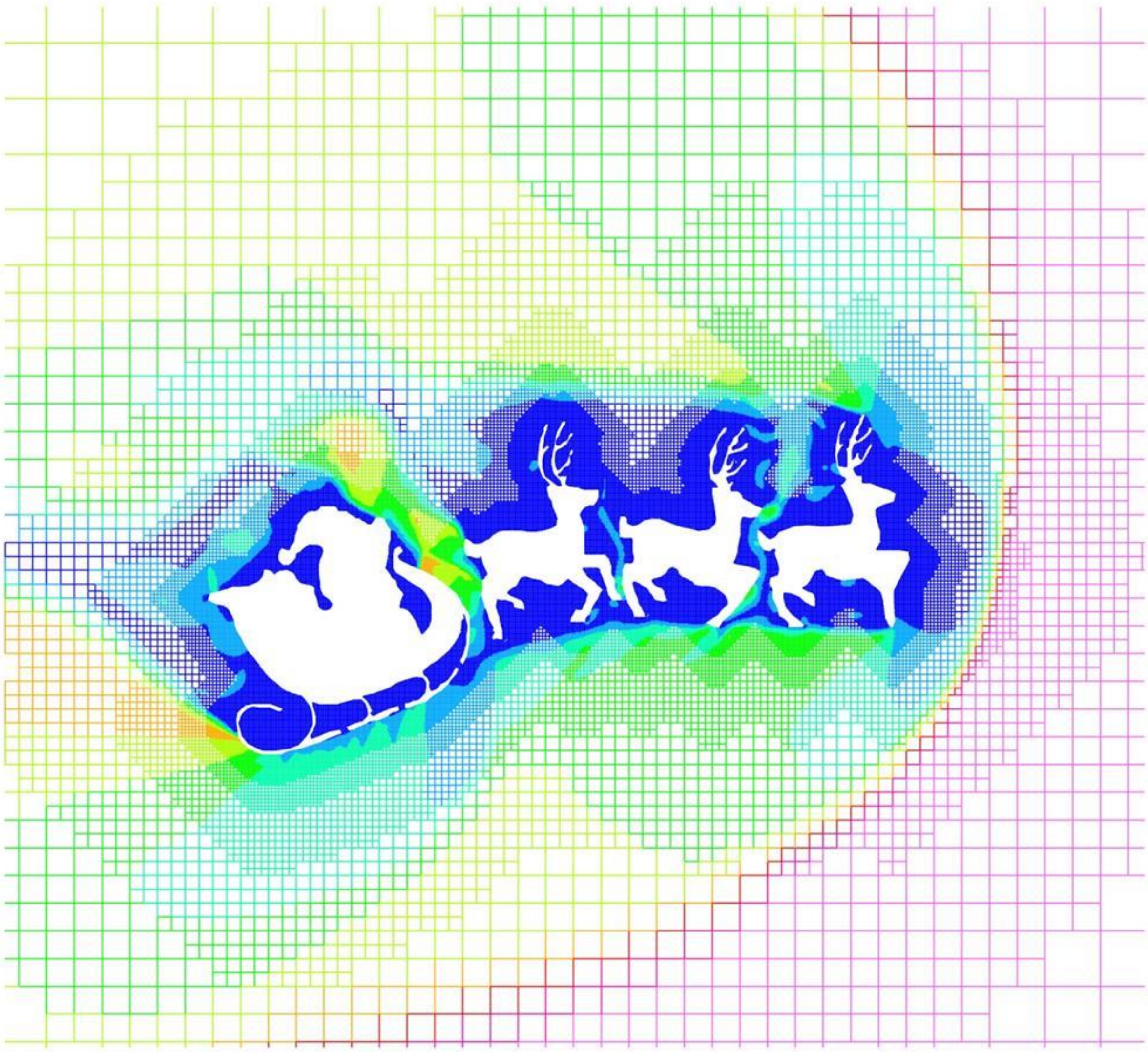
Slip line



Angle will be nearly aligned with freestream.

Compute by making pressure at top equal to pressure at bottom of slip line.

Full slip line calculation is in the tips and tricks document…



Aerod


Happy
Xmas