Vibrations 2, Lecture 3
Free undamped vibration

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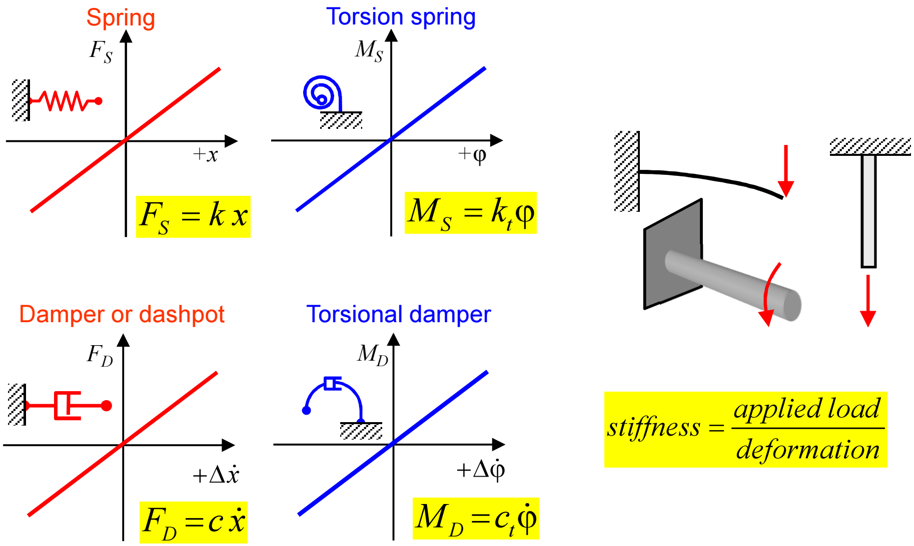
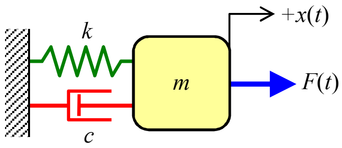
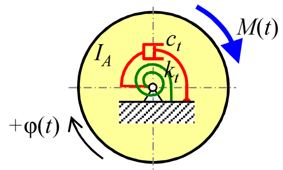
Lecture 2



$$stiffness = \frac{applied\ load}{deformation}$$

Lecture 2





$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

where the **physical parameters**:
- m is the mass [kg],
- c is the damping constant [N.s/m],
- k is the linear stiffness [N/m],

where the **motion**:
- x is the displacement [m],
- $\dot{x}$ is the velocity [m/s],
- $\ddot{x}$ is the acceleration [m/s^2],

F(t) is the external force [N]

$$\boxed{I_A\ddot{\varphi} + c_t\dot{\varphi} + k_t\varphi = M(t)}$$

where the **physical parameters**:
- I_A is the moment of inertia [kg.m^2],
- c_t is the tor. damp. constant [N.m.s/rad],
- k_t is the torsional stiffness [Nm/rad],

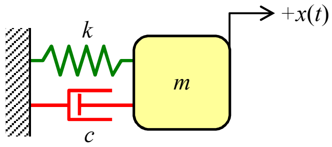
where the **motion**:
- φ is the angular deformation [rad],
- $\dot{\varphi}$ is the angular velocity [rad/s],
- $\ddot{\varphi}$ is the ang. acceleration [rad/s^2],

M(t) is the external moment [N.m]

Lecture 3

- Free undamped vibration
- Undamped natural frequencies
- Solved example

Free vibration F(*t*)=0



$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Initial conditions in t=0

$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0 = v_0$$

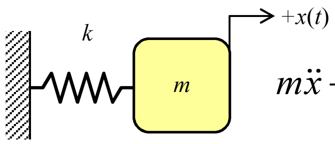
Vibrating systems are dynamic systems. To be able to solve their EOM, we have to know their initial state or **initial conditions** (IC).

From the mathematical point of view, our 1 DOF EOM is **2nd order linear ordinary differential equation** (ODE) and to be able to solve it we have to know **2** initial conditions:

(i) initial position, and (ii) initial velocity

1 DOF undamped system

Freely vibrating undamped system ...



$$m\ddot{x} + kx = 0$$

Let us *assume* that this system vibrates harmonically (i.e. trial solution):

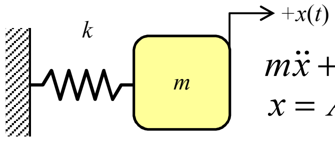
$$x = \underline{A}\sin(\underline{\omega}t + \underline{\varphi}) \Rightarrow \ddot{x} = -\omega^2 A\sin(\omega t + \varphi)$$

Substituting back to the EOM:

$$(-\omega^2 m + k)A\sin(\omega t + \varphi) = 0$$
$$-\omega^2 m + k = 0 \Rightarrow \omega^2 = \frac{k}{m}$$

Result: **Free undamped 1 DOF system** vibrates harmonically with the angular frequency √(k/m) [rad/s] *so* this is the *natural frequency* ω_0.

1 DOF undamped system



$$m\ddot{x} + kx = 0$$
$$x = A\sin(\omega t + \varphi)$$

Consider general nonzero ICs and apply them to *x* and dx/dt to solve for A and φ:

$$x(0): x_0 = A\sin(\omega_0 0 + \varphi) = A\sin(\varphi) \qquad (1)$$
$$\dot{x}(0): v_0 = \omega_0 A\cos(\omega_0 0 + \varphi) = \omega_0 A\cos(\varphi) \qquad (2)$$

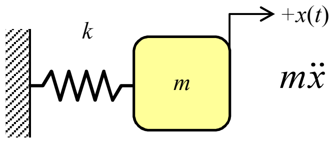
Take the sum of squares of (1) and (2) to get …

Take the ratio of (1) and (2) to get …

$$A = \sqrt{x_0^2 + v_0^2/\omega_0^2}, \varphi = \tan^{-1}(x_0\omega_0/v_0) \qquad x(t) = A\sin(\omega_0 t + \varphi)$$

Try Matlab: » vib2_1dof_freevisc

1 DOF undamped system: natural frequency



$$m\ddot{x} + kx = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

We can obtain information about the natural frequency directly after converting the EOM to its *standard form*. Divide the EOM by the coefficient next to "d^2x/dt^2". After this, the coefficient next to "x" is ω_0^2.

$$\boxed{\ddot{x} + \omega_0^2 x = 0}$$

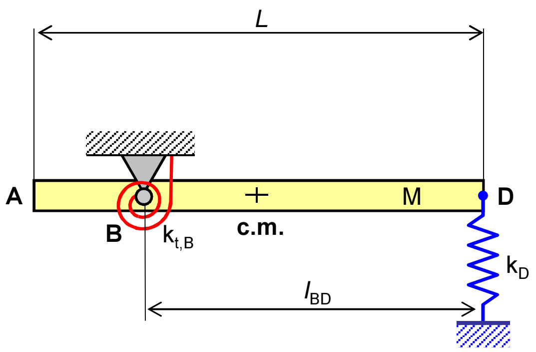
ω_0 is the angular undamped **natural frequency** [rad/s]

f_0 is the undamped **natural frequency** [Hz]

Example 2

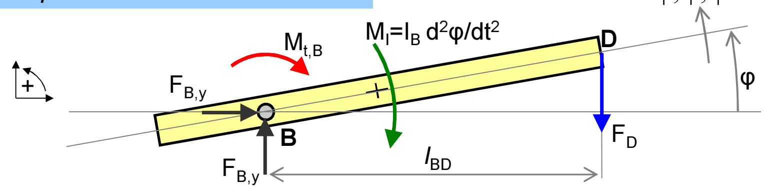
Derive the EOM and determine the natural frequency?

Rigid bar has the moment of inertia with respect to its C.M. $I_{c.m.} = (1/12) \times ML^2$, where M is the mass of bar and L is the length.



Example 2

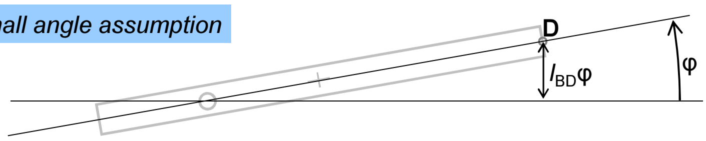
Step 1: 'positive' deformation and FBD



***Note**: The effect of gravity is not included in this example. It is already "included" in the assumption that the system vibrates about the equilibrium position. For more info see future lectures; Thomson, Ex. 2.2-4; revision lecture; example sheets; …*

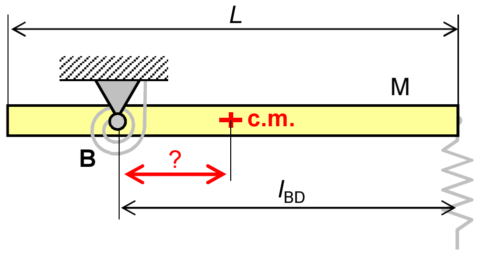
$$\sum M_{i,B} = 0: \; -M_I - M_{t,B} - F_D \, l_{BD} = 0$$

Step 2: small angle assumption



Example 2

Step 3: moment of inertia about B



General statement of the *Parallel Axis Theorem*: $I_A = I_{c.m.} + m\,l_{A,c.m.}^2$

$$I_B = I_{c.m.} + M\,l_{B,c.m.}^2 = \frac{1}{12}ML^2 + M\left(l_{BD} - \frac{L}{2}\right)^2$$

… mass moment of inertia

Example 2

Step 4: equation of motion

$$I_B\,\ddot{\varphi} + k_{t,B}\varphi + (l_{BD}\varphi k_D)\,l_{BD} = 0$$

$$I_B\,\ddot{\varphi} + (k_{t,B} + l_{BD}^2\,k_D)\,\varphi = 0$$

Step 5: undamped natural frequency

Converting to a standard form …

$$\ddot{\varphi} + \frac{k_{t,B} + l_{BD}^2\,k_D}{I_B}\varphi = 0$$

$$\omega_0^2 = \frac{k_{t,B} + l_{BD}^2\,k_D}{ML^2/12 + M\left(l_{BD} - L/2\right)^2}$$

Rotational vs. rectilinear motion in 2D

	Rotational (fixed axis of rotation)	**Rectilinear**
Reference	Point in 2D, e.g. **A**	Direction in 2D, e.g. *X* or *Y*
Inertial property	Moment of inertia $I_A \; [kg.m^2]$	Mass $m \; [kg]$
Motion (kinematics)	$\varphi, \dot{\varphi}, \ddot{\varphi} \; [rad, rad/s, rad/s^2]$	$x, \dot{x}, \ddot{x} \; [m, m/s, m/s^2]$
Loads	Moments, torques, force couples $M \; [N.m]$	Forces $F \; [N]$
Dynamic equilibrium (d'Alembert)	$\sum M_{i,A} - I_A \ddot{\varphi} = 0$	$\sum F_{i,X} - m\ddot{x} = 0$
Accelerating forces (Newton)	$I_A \ddot{\varphi} = \sum M_{i,A}$	$m\ddot{x} = \sum F_{i,X}$
Theorems	Parallel Axis Theorem (P.A.T.) $I_A = I_{c.m.} + m l_{A,c.m.}^2$	

Summary

- The frequency of free vibration of undamped 1 DOF system is equal to the *undamped natural frequency* of this system
  - Free vibration is harmonic vibration
- The undamped natural frequency is equal to the square root of the "total effective stiffness" divided by the "total effective mass"
  - Increasing the mass decreases the natural frequency
  - Increasing the stiffness increases the natural frequency
- *Moment of inertia* represents the resistance to angular acceleration. It is always defined relative to a reference point.