Free vibration in underdamped 1 DOF systems with viscous damping

Free 1 degree-of-freedom (DOF) system with viscous damping is represented by the equation of motion (EOM) in the following form

$$m\ddot{x} + c\dot{x} + kx = 0.$$

The trial solution of this EOM is $x(t) = Ae^{st}$, where $s \in \mathbb{C}$. Substitution of the trial solution gives the characteristic equation $ms^2 + cs + k = 0$ and its roots are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

where

$$\omega_0^2 = \frac{k}{m},$$

$$\zeta\omega_0 = \frac{c}{2m} \Rightarrow \zeta = \frac{c}{2m\omega_0} = \frac{c}{2\sqrt{mk}} = \frac{c}{c_{cr}} \Rightarrow c_{cr} = 2\sqrt{mk}.$$

Assuming underdamped vibration, i.e. $0 \le \zeta < 1$, the two roots of this equation are two complex conjugate numbers and these are written in the form

$$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} = -\zeta\omega_0 \pm i\,\omega_0\sqrt{1-\zeta^2} = -\zeta\omega_0 \pm i\,\omega_D$$

where $\omega_D = \omega_0\sqrt{1-\zeta^2} \le \omega_0$ for $0 \le \zeta < 1$.

The total solution is known to be a *linear superposition* of the two individual solutions

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{(-\zeta\omega_0 + i\,\omega_D)t} + A_2 e^{(-\zeta\omega_0 - i\,\omega_D)t}.$$

Function $x(t)$ can be written in various formats, e.g. exponential, harmonic, etc., based on convenience and tradition. We use the above total exponential solution as a starting point

$$\begin{aligned} x(t) &= A_1 e^{(-\zeta\omega_0 + i\,\omega_D)t} + A_2 e^{(-\zeta\omega_0 - i\,\omega_D)t} \\ &= A_1 e^{-\zeta\omega_0 t} e^{i\,\omega_D t} + A_2 e^{-\zeta\omega_0 t} e^{-i\,\omega_D t} \\ &= e^{-\zeta\omega_0 t}(A_1 e^{i\,\omega_D t} + A_2 e^{-i\,\omega_D t}). \end{aligned}$$

The next change is based on the *Euler formula* for complex numbers $e^{\pm i\alpha} = \cos(\alpha) \pm i\sin(\alpha)$

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\begin{aligned}
x(t) &= e^{-\zeta\omega_0 t}(A_1 e^{i\omega_D t} + A_2 e^{-i\omega_D t}) \\
&= e^{-\zeta\omega_0 t}(A_1\{\cos(\omega_D t) + i\sin(\omega_D t)\} + A_2\{\cos(\omega_D t) - i\sin(\omega_D t)\}) \\
&= e^{-\zeta\omega_0 t}(\{A_1 + A_2\}\cos(\omega_D t) + i\{A_1 - A_2\}\sin(\omega_D t)) \\
&= e^{-\zeta\omega_0 t}(B\cos(\omega_D t) + C\sin(\omega_D t)).
\end{aligned}
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It is assumed that $B = A_1 + A_2$ and $C = i(A_1 - A_2)$ are the real numbers and this condition is fulfilled if A_1 and A_2 are complex conjugate, i.e. $A_1 = A_2^*$ or $A_1 = A_{1,R} + i\,A_{1,C} = A_{2,R} - i\,A_{2,C}$, where $A_2 = A_{2,R} + i\,A_{2,C}$. It can be shown that A_1 and A_2 are complex conjugate by using initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0 = v_0$ directly with the total exponential solution presented above.

Further modification of $x(t)$ is achieved by using trigonometric *identity for angle sum*. In this case we use the cosine form $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$. By assuming the transformation $B = X\cos(\phi)$ and $C = X\sin(\phi)$, i.e. $[B,C] \leftrightarrow [X,\phi]$, we arrive at the final exponentially decaying (i.e. $\zeta\omega_0 > 0$) cosine response with the angular frequency ω_D corresponding to free vibration with viscous damping

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\begin{aligned}
x(t) &= e^{-\zeta\omega_0 t}(B\cos(\omega_D t) + C\sin(\omega_D t)) \\
&= e^{-\zeta\omega_0 t}(X\cos(\phi)\cos(\omega_D t) + X\sin(\phi)\sin(\omega_D t)) \\
&= X\,e^{-\zeta\omega_0 t}\cos(\omega_D t - \phi)
\end{aligned}
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