Stress, Strain and Deformation
Elastic Limit and Elastic-Plastic Behaviour

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10 October 2017

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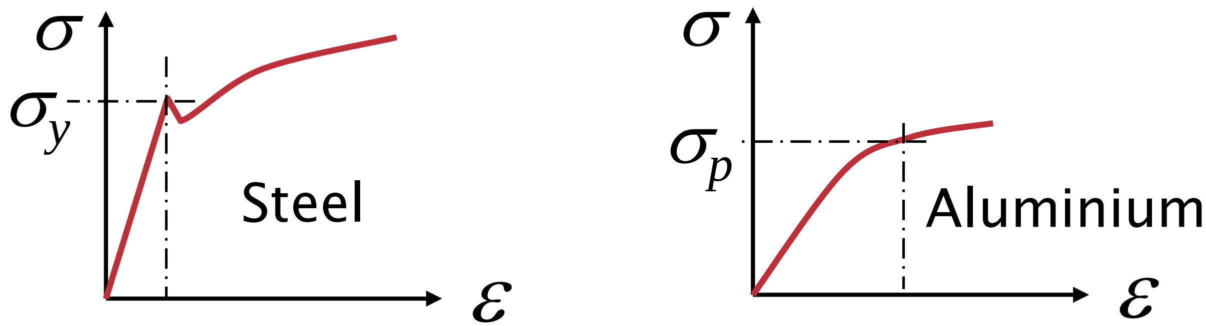
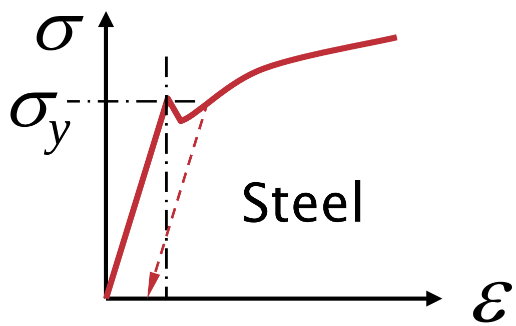
Elastic Limit

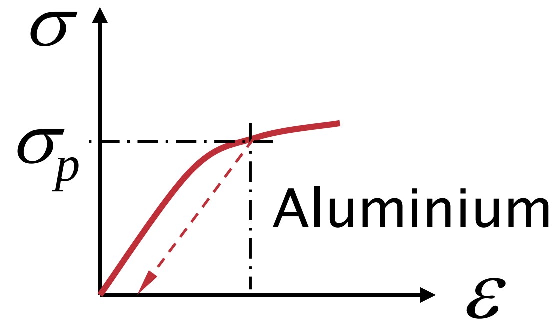
- Real stress-strain curves:



- Typically elastic limit occurs at 0.1% to 0.2% strain
- The stresses associated with the onset of significant yielding (plasticity) are:
  - "yield strength" σ_y
  - "proof stress" σ_p

Plastic Deformation: Yielding

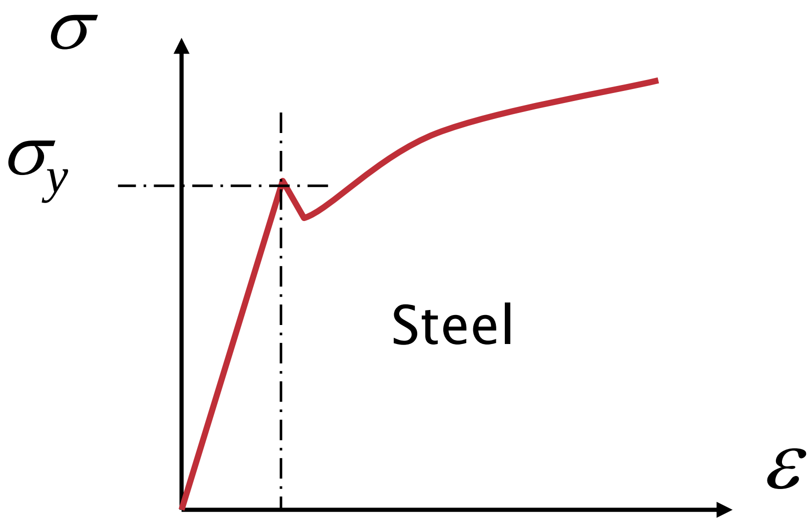




- Below the elastic limit (yield or proof stress or strain) the material **recovers elastically**, *i.e.* with **no permanent strain**
- Above the elastic limit the material retains a **permanent strain** or **plastic deformation**

Yield Stress

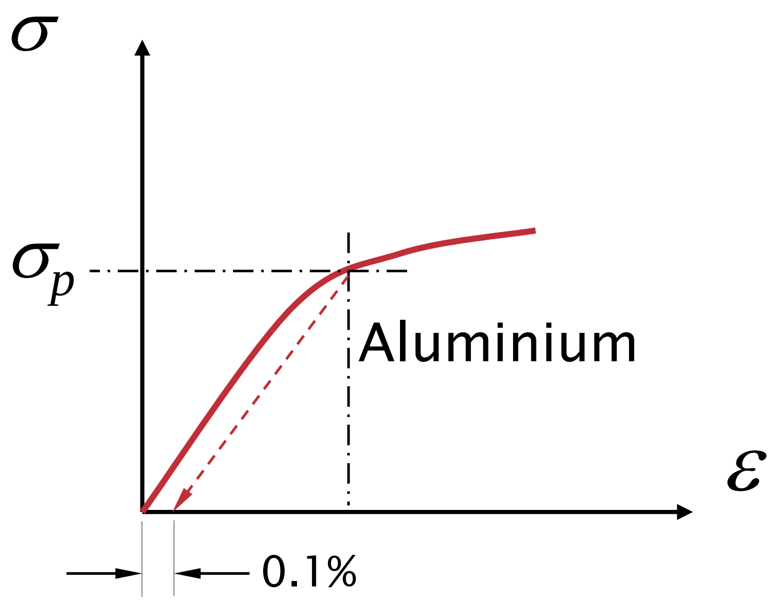
- Yield stress σ_y defines the elastic limit and onset of yielding when the transition is pronounced, *e.g.* typical of some steel alloys



Proof Stress

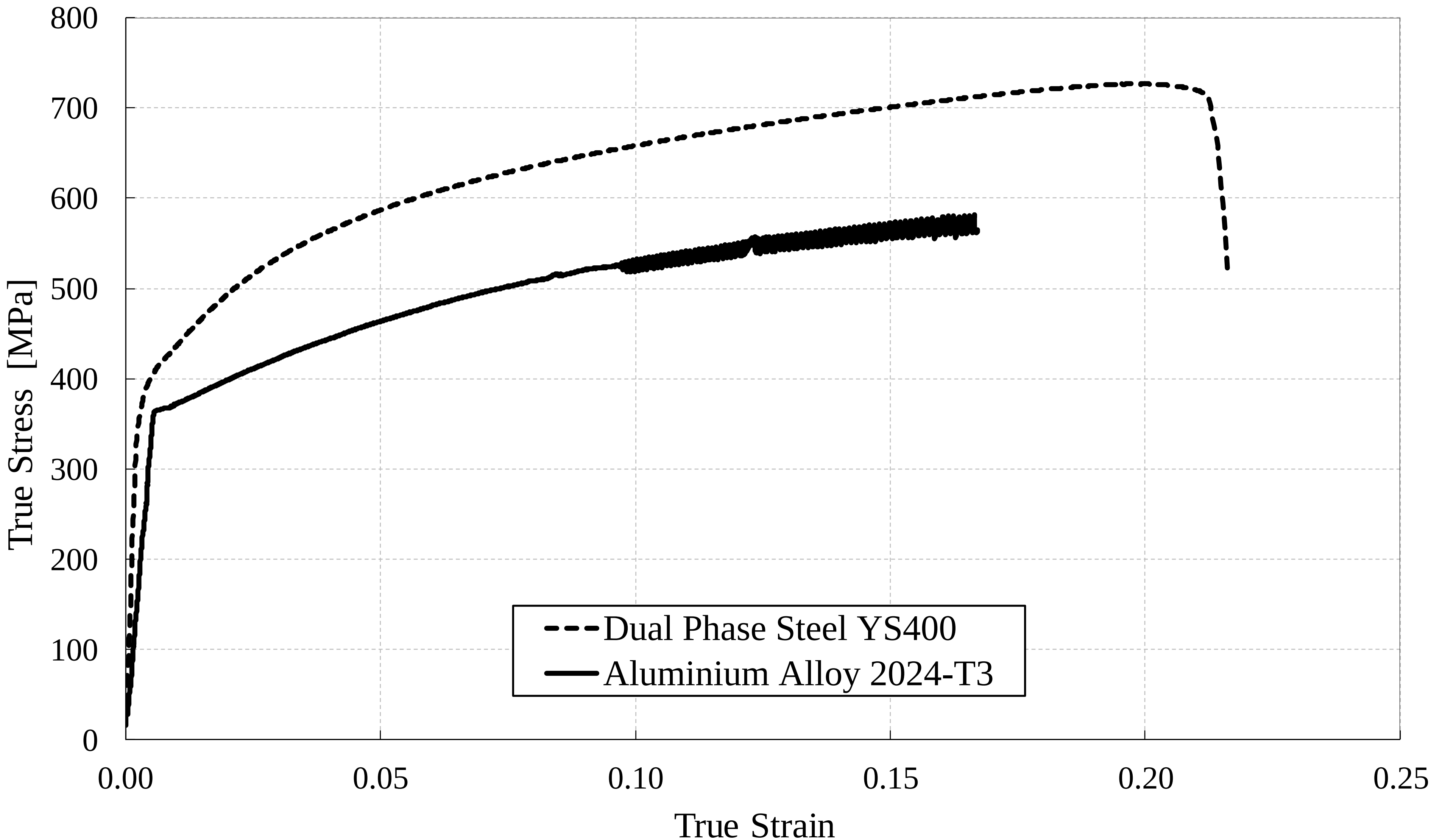
KEY POINTS

- Proof stress σ_p defines the effective elastic limit and onset of yielding when the transition is vague, *e.g.* typical of some aluminium alloys



- For example, a 0.1% proof stress is the stress level which leaves a 0.1% strain when released

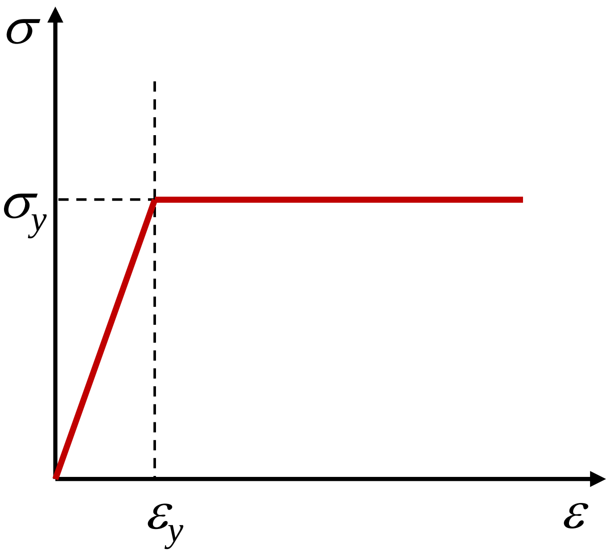
Real True Stress – True Strain Curves



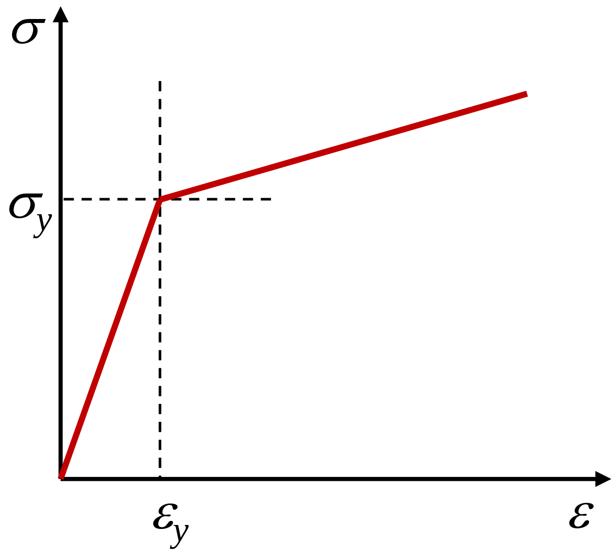
Real True Stress – True Strain Curves

- Common elastic-plastic models for metals:

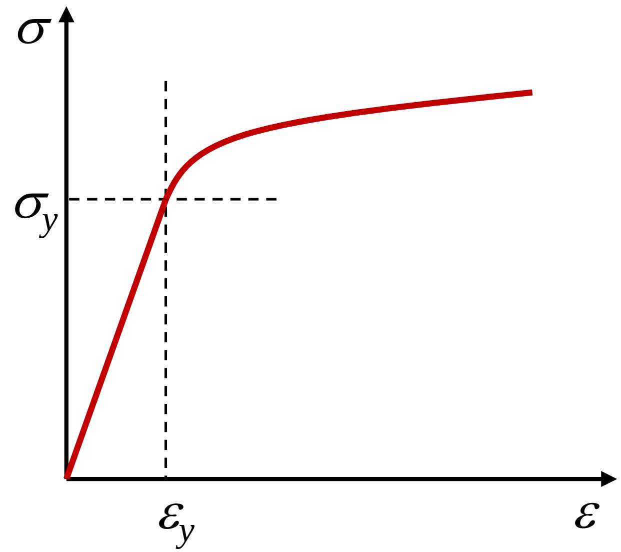
Elastic perfectly-plastic



Linear work-hardening (or bi-linear model)

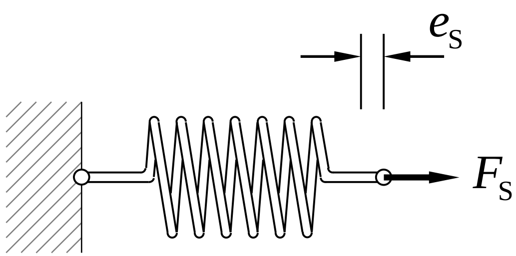


Power-law work-hardening



Spring-Dashpot Models

- Spring: linear relationship between **force** and **displacement**



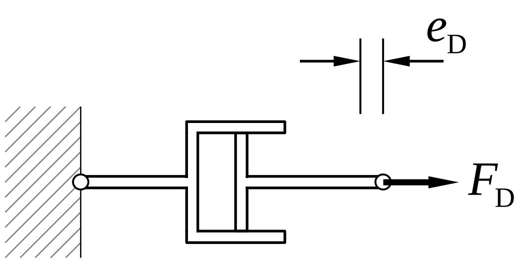
Stiffness

$$F_S = \lambda \cdot e_S$$

Young's Modulus

  - Material elasticity: stress as a function of strain: $\sigma_S = E \cdot \varepsilon_S$

- Dashpot: linear relationship between **force** and **displacement rate**



Damping coefficient

$$F_D = c \cdot \frac{d e_D}{d t}$$

Displacement rate

Viscosity

  - Material viscosity: stress as a function of strain rate: $\sigma_D = \eta \cdot \frac{d \varepsilon_D}{d t}$

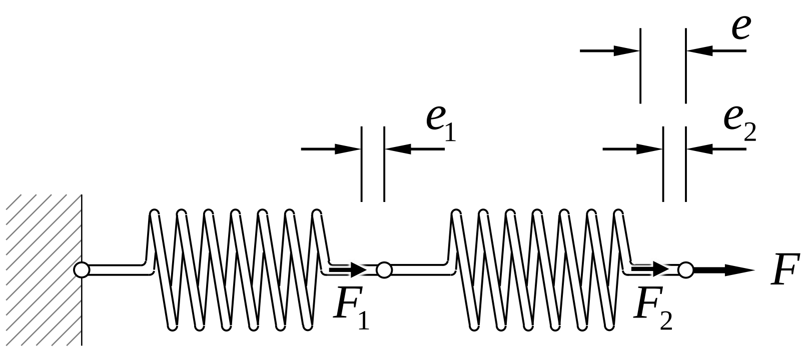
Strain rate

Time-dependent behaviour will be studied later → polymer viscoelasticity

Springs in 'Series' and in 'Parallel'

KEY POINTS

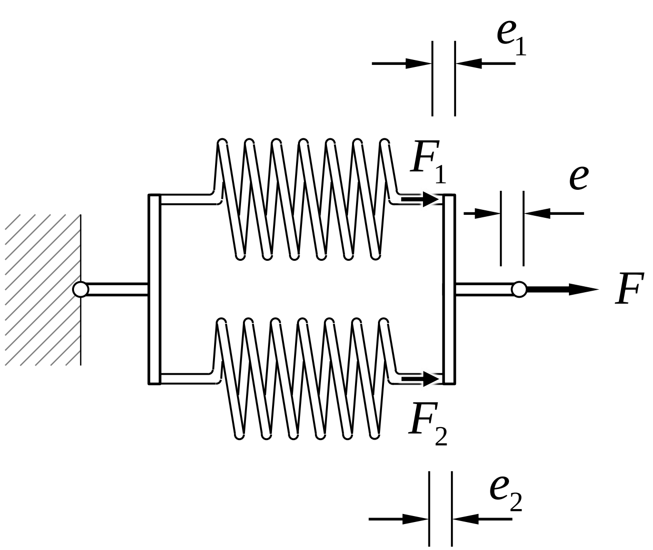
- Springs in **series**



$$\boxed{F = F_1 = F_2}$$

$$\boxed{e = e_1 + e_2}$$

- Springs in **parallel**



$$\boxed{F = F_1 + F_2}$$

$$\boxed{e = e_1 = e_2}$$

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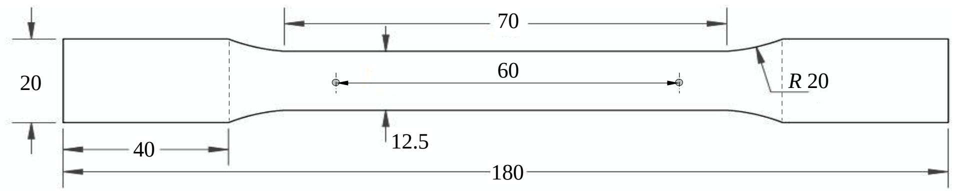
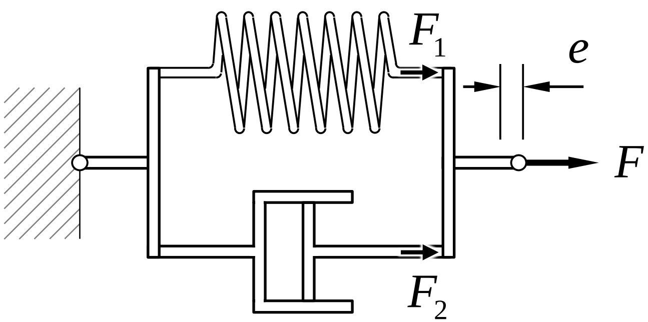
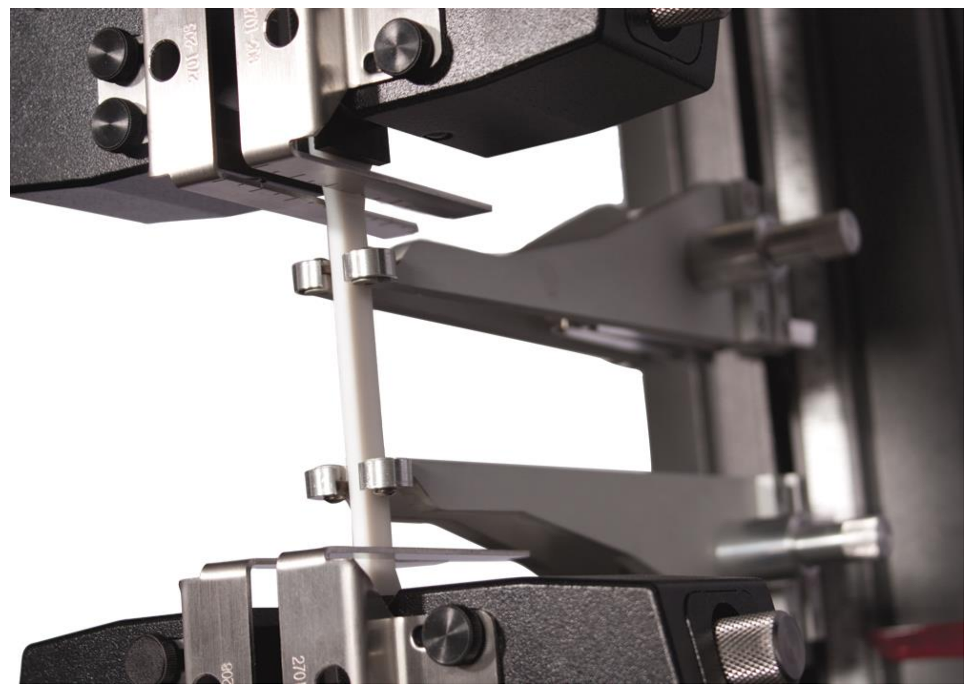
- Tensile tests on three different materials:
  - Aluminium alloy
  - Polyamide 6 (Nylon 6)
  - High-density polyethylene (HDPE)



- You will see that **real** material behaviour is rather complex, especially for polymers:

E.g. Kelvin-Voigt viscoelastic model





Stress, Strain and Deformation
Poisson's Ratio and Biaxial Stress States

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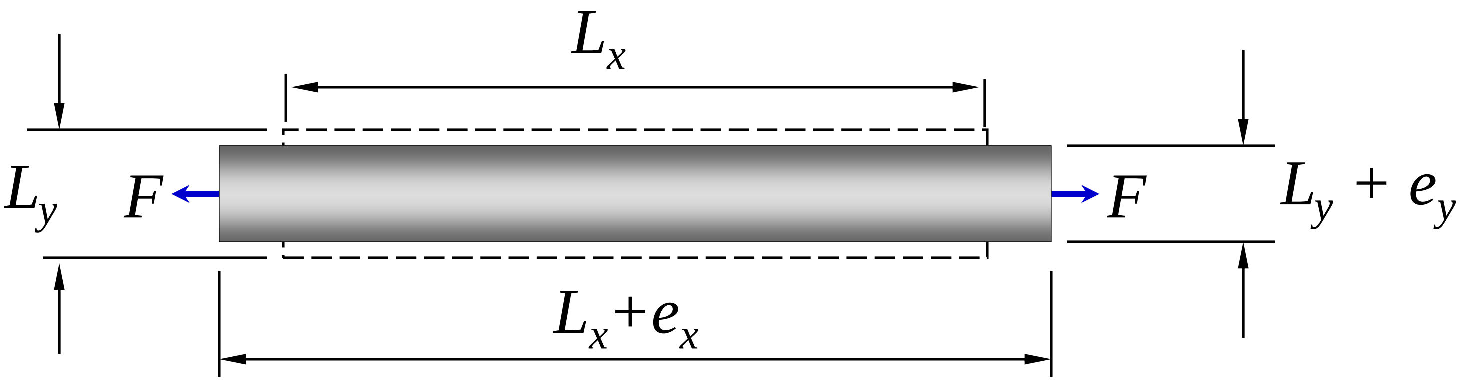
- Lateral as well as longitudinal strains occur under longitudinal loading:



Longitudinal strain: $$\varepsilon_x = \frac{e_x}{L_x}$$

Lateral (transverse) strain: $$\varepsilon_y = \frac{e_y}{L_y} \qquad \therefore$$

Therefore: $$\varepsilon_y = -\nu\,\varepsilon_x$$

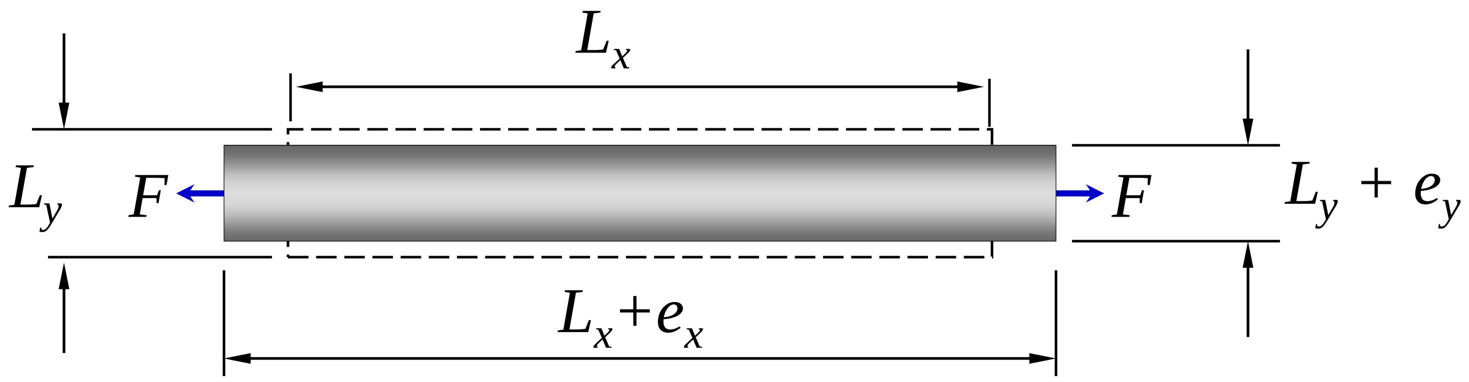
Poisson's Ratio:

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

ν = 'nu' (sounds like 'new')

KEY POINTS

- Lateral as well as longitudinal strains occur under longitudinal loading:



- **Poisson's Ratio** = ratio of transverse strain to longitudinal strain
- **Typically ν = 0.3 for metals**
- Positive, since the 'minus' sign used in the formula compensates for the fact that the longitudinal and transverse strains are in opposite sense

Poisson's Ratio:

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

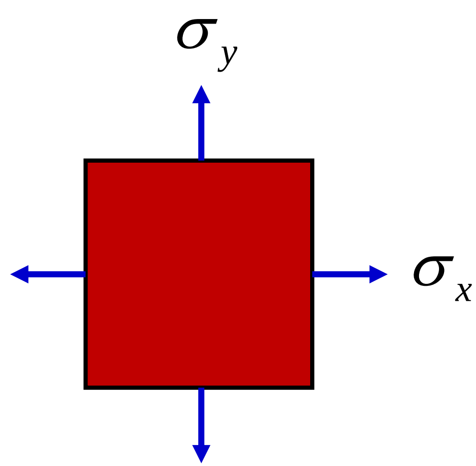
ν ='nu' (sounds like 'new')

2D Stresses & Strains

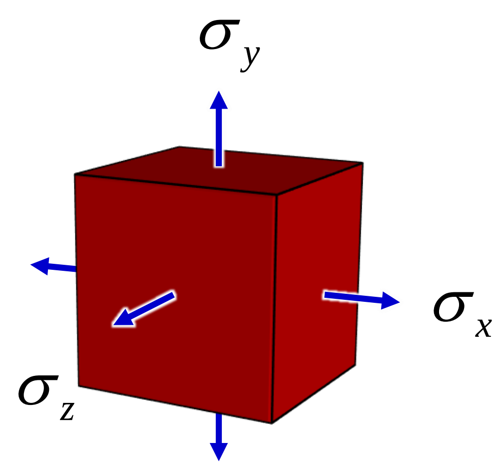
- So far we have only considered one dimension
- However stresses and strains can exist in **two** or **three** dimensions
- We consider here the strains produced by a **bi-axial stress system** for a cube of unit dimensions

One dimension:

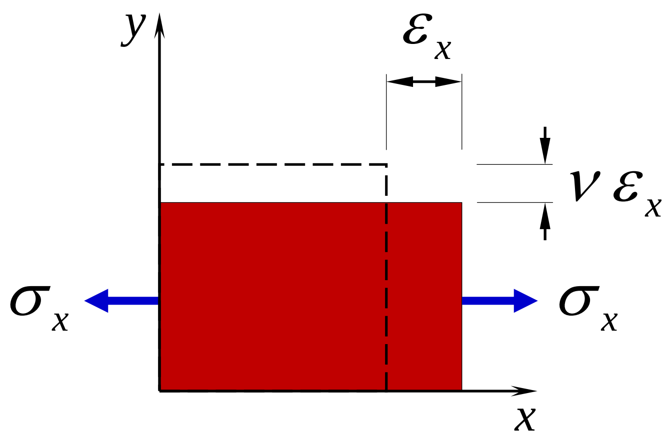
Two dimensions:

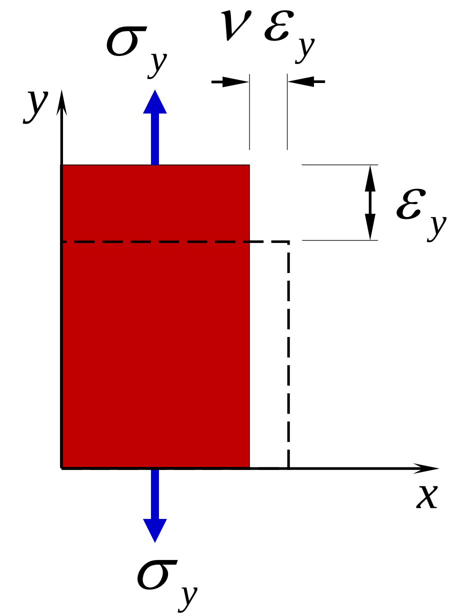


Three dimensions:



- In a biaxial stress state:





$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \\ \varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \end{aligned}}$$

Where $\varepsilon = \dfrac{\sigma}{E}$

Stress, Strain and Deformation

Biaxial Stress States: Thin-Walled Pressure Vessels

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Pressure Vessels

- “Containers designed to maintain internal pressures that are substantially different from the exterior pressure in which the container operates”

Chemical reactors


Gas/Liquid Storage


Autoclaves

- Transport / Defence:


Airliner at cruise altitude (11,000m / 36,000ft) :

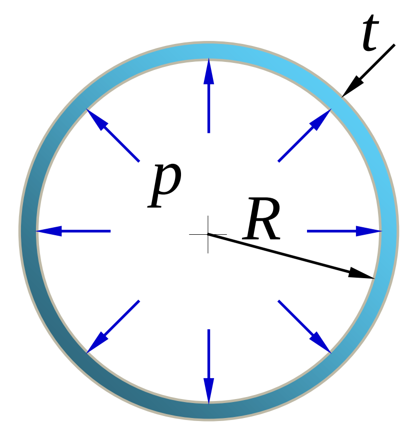
- external pressure:	0.25 bar / 25 kPa / 3.6 psi
- standard internal pressure:	0.77 bar / 77 kPa / 11 psi
- (Boeing 787 / Airbus A350:	0.82 bar / 82 kPa / 12 psi)


Military submarine (400m H_20) :

- external pressure: about	40 bar / 4 MPa / 510 psi
- internal pressure:	1 bar / 100 kPa / 14.6 psi

2D Strains: Cylindrical Pressure Vessel

- Consider a cylindrical vessel of mean radius R and wall thickness t subjected to an internal pressure p :

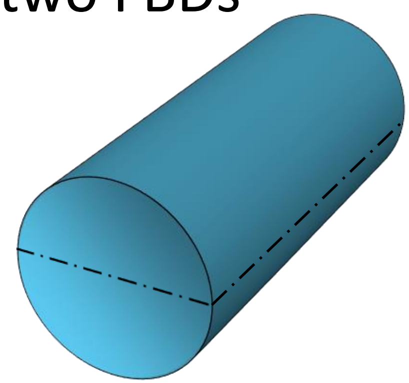


- Assuming a thin-walled cylinder:

$$t << R \qquad \therefore \qquad R_{\text{inner}} \cong R_{\text{outer}}$$

- In this case the stresses are uniform through the thickness, *i.e.* we consider only '**membrane stresses**'

- To reveal these stresses consider the equilibrium of two FBDs obtained by sectioning the cylinder in two halves:



Hoop (or Circumferential) Stress σ_H

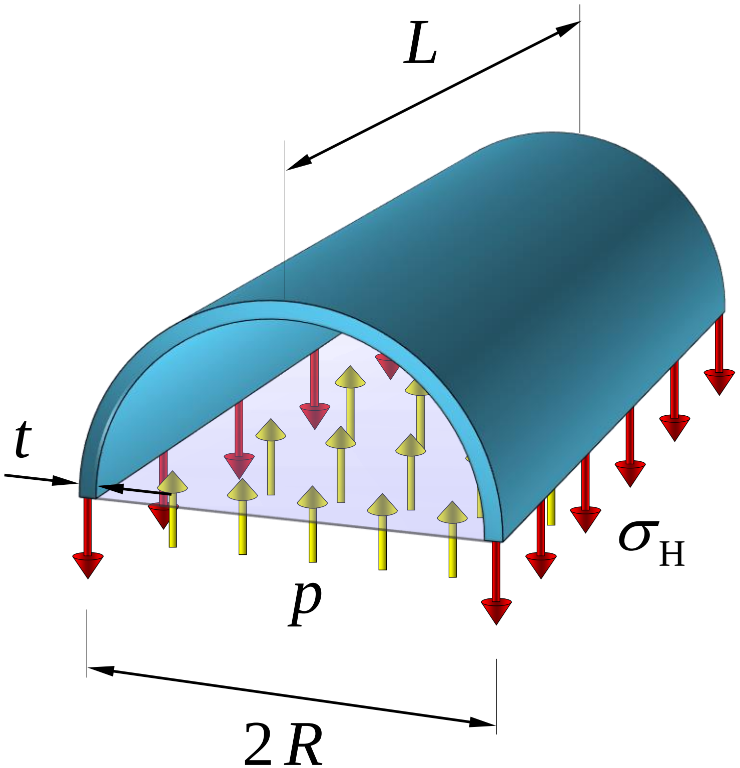
- Equilibrium of FBD:

$$\sum F_y = 0$$

Pressure force - wall membrane force = 0

$$p(2R)(L) - 2\sigma_H(t)(L) = 0$$

$$\boxed{\sigma_H = \frac{pR}{t}}$$

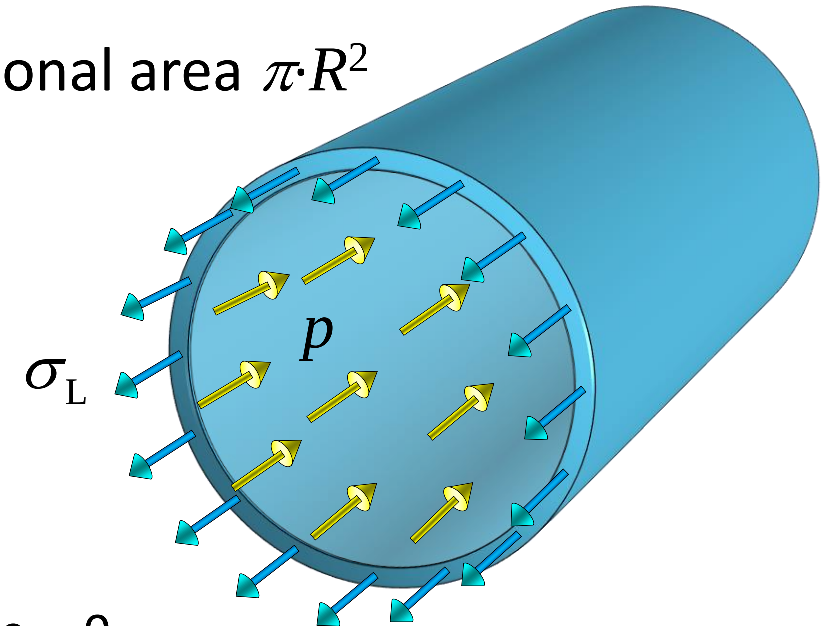


- Note: the force exerted in any direction due to a pressure p is p_A, where A is the projected area normal to the direction of the force

Longitudinal Stress σ_L

Now consider the cylinder sectioned **transversely**:

- The pressure p acts on the cross-sectional area $\pi \cdot R^2$
- Equilibrium of FBD:



$$\sum F_x = 0$$

Pressure force - longitudinal membrane force = 0

$$p\left(\pi R^2\right) - \sigma_L \left(2\pi R\right)\left(t\right) = 0$$

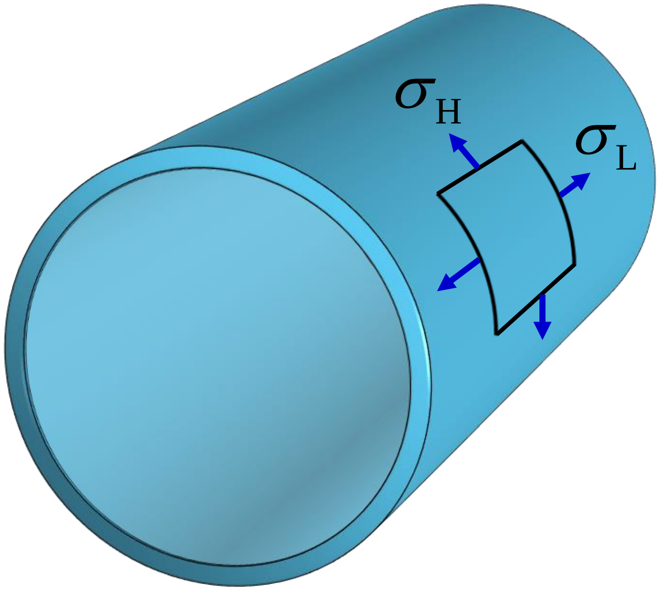
$$\boxed{\sigma_L = \frac{pR}{2t}}$$

Biaxial Stress System

- Therefore the thin-walled cylindrical pressure vessel is in a **biaxial stress state**

- For cylindrical vessels, the **hoop stress is equal to twice the longitudinal stress**:

$$\boxed{\sigma_{\mathrm{H}} = \frac{p\,R}{t}} \qquad \boxed{\sigma_{\mathrm{L}} = \frac{1}{2}\frac{p\,R}{t}}$$

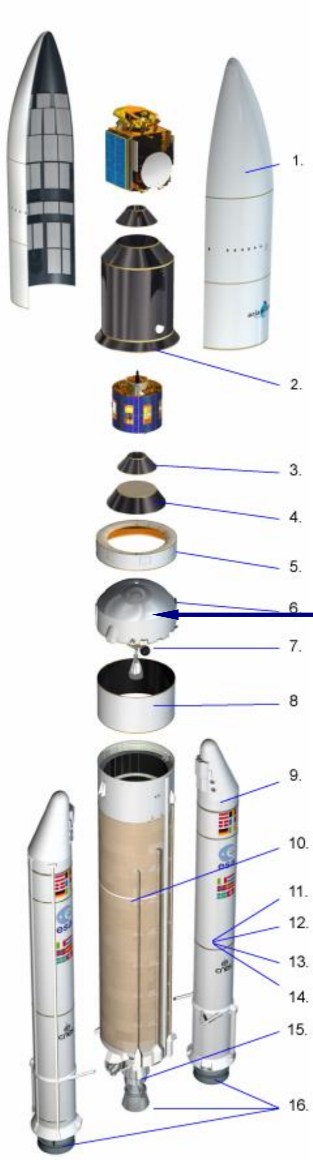


- This is why cylindrical vessels burst along the length rather than at the caps!

Pressure Vessels

- What is the most 'efficient' shape for a pressure vessel, in terms of the stress state of its walls?
  - Answer: a **sphere**!

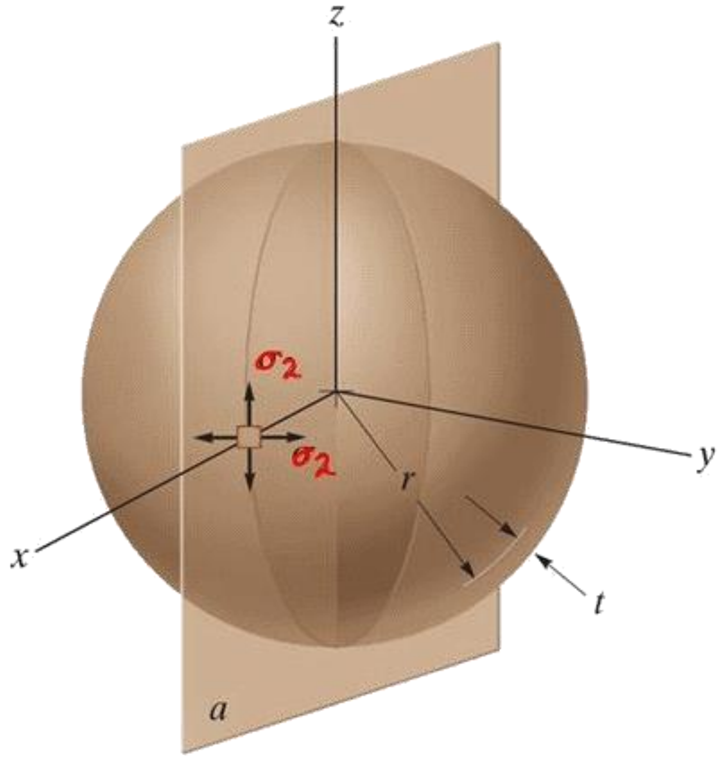
Gas storage

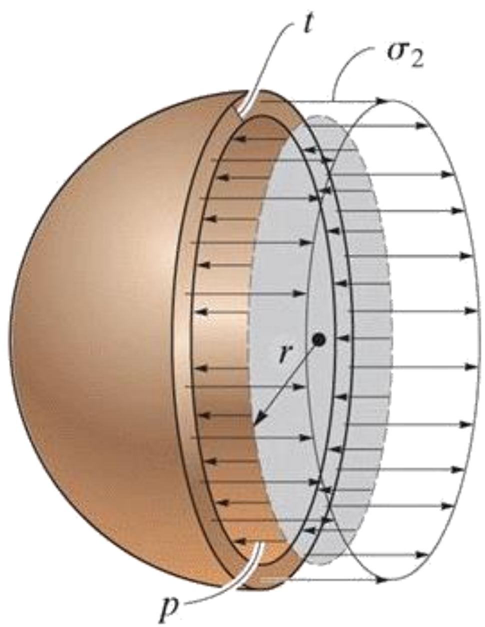


Ariane 5 cryogenic fuel tank

Spherical Pressure Vessels

- Spheres have infinite numbers of planes (or axes) of symmetry:





- Therefore the two components of membrane stresses are always identical:

$$\boxed{\sigma_1 = \sigma_2 = \frac{1}{2}\frac{p\,R}{t}}$$

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Stress, Strain and Deformation
Constitutive Relations

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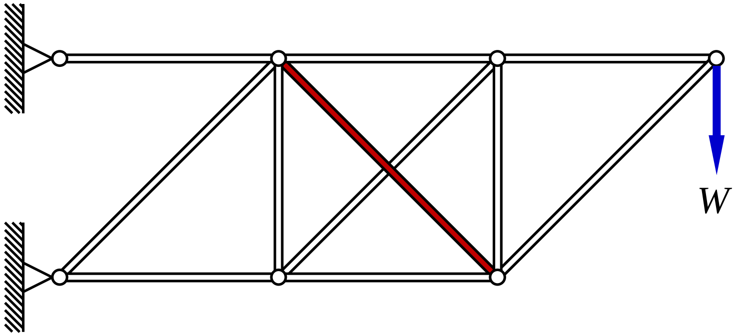
Statically Indeterminate Problems

- Problems where there are more unknowns than equations of static equilibrium. Consider our previous pin-jointed truss with an ‘extra’ (redundant) member:



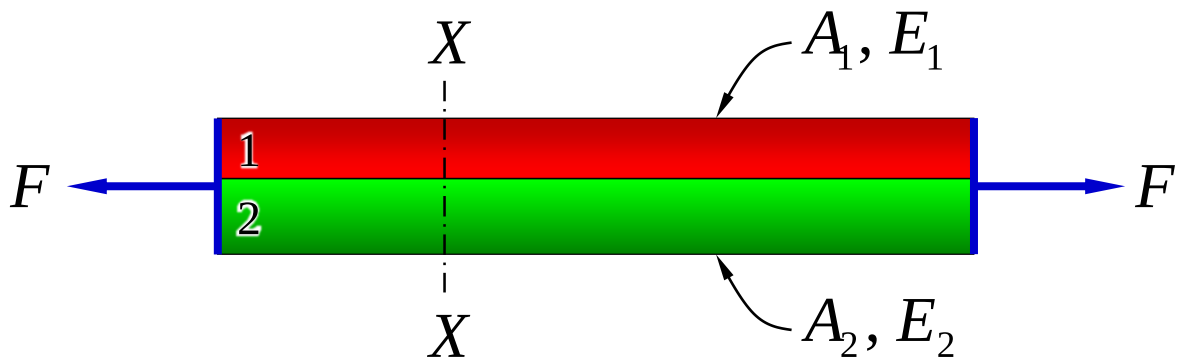
- There is one more load path than needed for equilibrium
- The load taken by each member will then depend on the relative stiffness and accuracy of fit of each member
- To solve this we need more equations!

There are 3 arguments available to solve a statically indeterminate structural problem:

- Static equilibrium of forces and moments
- Compatibility of displacements and rotations (constraints)
- Constitutive relations:
  - Force-displacement: $\boxed{F = \lambda\, e}$
  - Stress-strain: $\boxed{\sigma = E\, \varepsilon}$

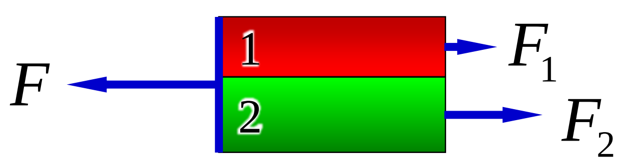
Example: Loaded Composite Bar

- Two dissimilar bars constrained to carry an axial load together:



  - These behave as two springs in parallel !

- Let us analyse the FBD of the left-hand-side of section X-X:

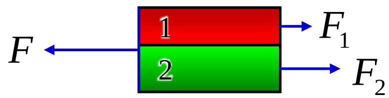


Example: Loaded Composite Bar

There are 3 unknowns: $F_1, \; F_2, \; e$

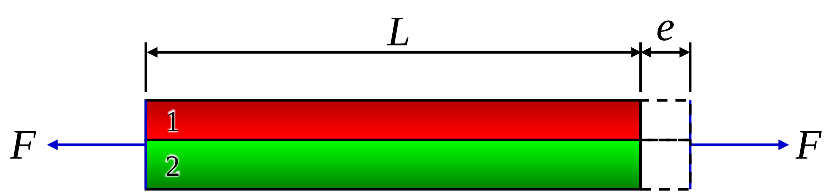
- Static equilibrium

$$\sum F_x = 0 \quad \therefore \quad -F + F_1 + F_2 = 0$$



- Compatibility of displacements

$$e_1 = e_2 = e$$



- Constitutive relations

$$F_1 = \lambda_1 \, e_1$$
$$F_2 = \lambda_2 \, e_2$$

where
$$\begin{cases} \lambda_1 = \dfrac{A_1 \, E_1}{L_1} \\[2ex] \lambda_2 = \dfrac{A_2 \, E_2}{L_2} \end{cases}$$

Stress, Strain and Deformation
Thermal Stresses and Strains

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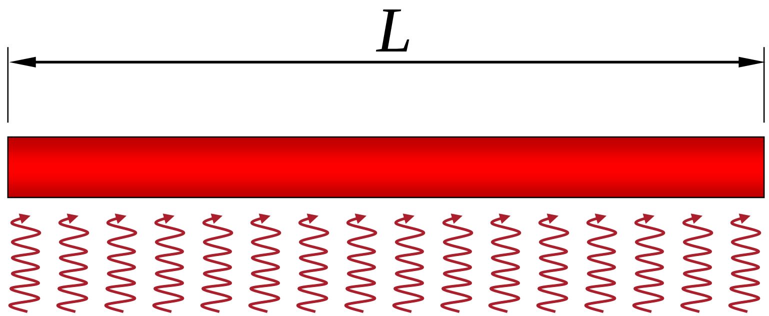
Thermal Stresses & Strains

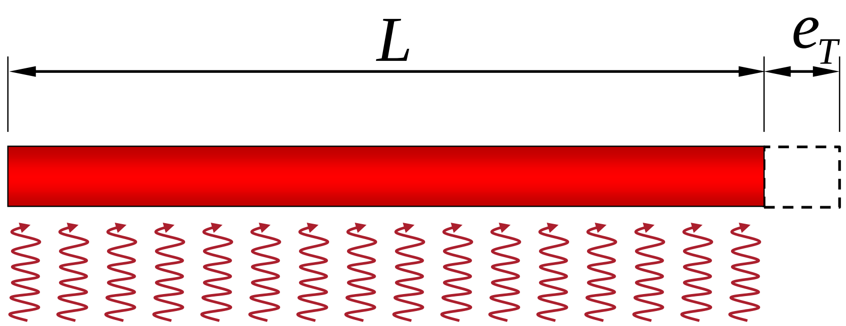
- Consider a metal bar heated to a new increased temperature:



- Original length, L [m or mm]
- Change in temperature, ΔT [°C or K]
- Expansion coefficient, α [°C^{-1} or K^{-1}] α =‘alpha’
  - units of strain (non-dimensional) per °C or K
- For metals: typically $\alpha = 10^{-5}$ K^{-1}

Thermal Strains

- Considering 1D linear expansion and assuming no thermal distortion (i.e. no shear strains):



- For an unrestrained bar:
  - Free thermal strain: $$\varepsilon_T = \frac{e_T}{L} = \alpha \, \Delta T$$
  - Change in length: $$L \, \varepsilon_T = L \, \alpha \, \Delta T$$

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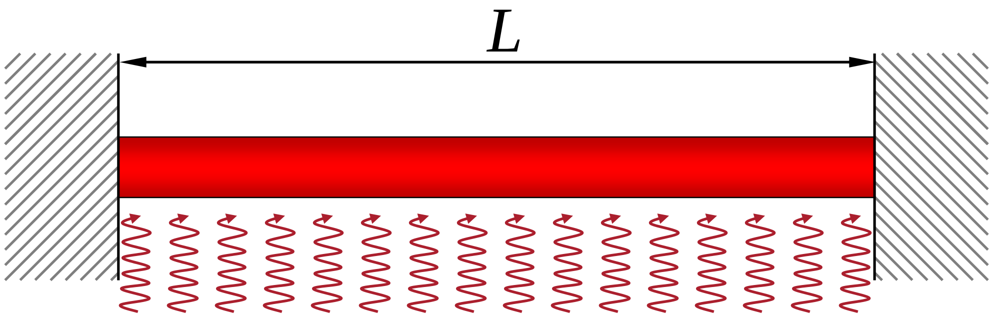
Thermal Stresses

- If unrestrained then thermal stress will be zero
- If constrained then a thermal stress will result

“Fully restrained”



- ‘Residual strain’ = prevented thermal strain:

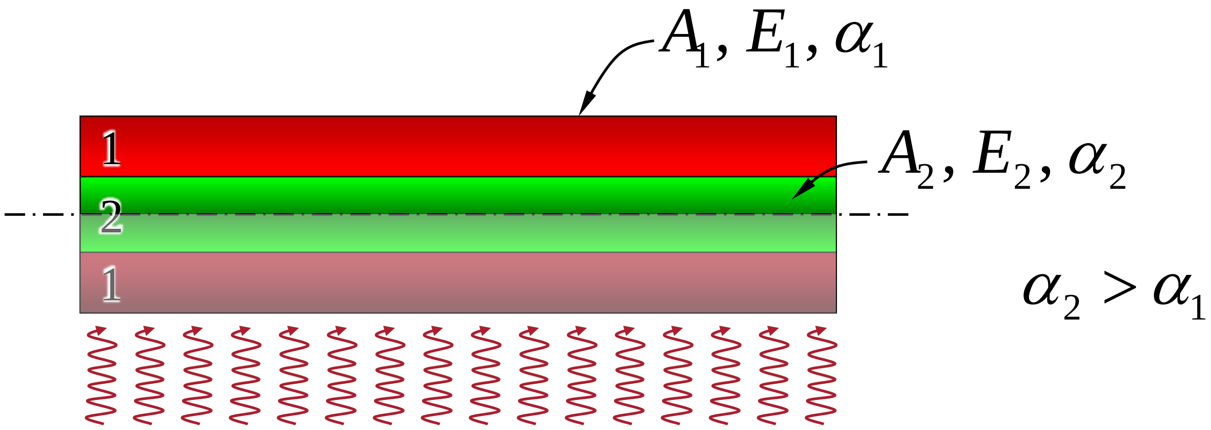
$$\varepsilon = \alpha \, \Delta T = \varepsilon_{\mathrm{R}}$$

- ‘Residual stress’ = thermal stress:
  - (assuming linear elastic behaviour, i.e. Hooke’s law)

$$\sigma = E \, \varepsilon_{\mathrm{R}} = \sigma_{\mathrm{R}}$$

Example 2: Heated Composite Bar

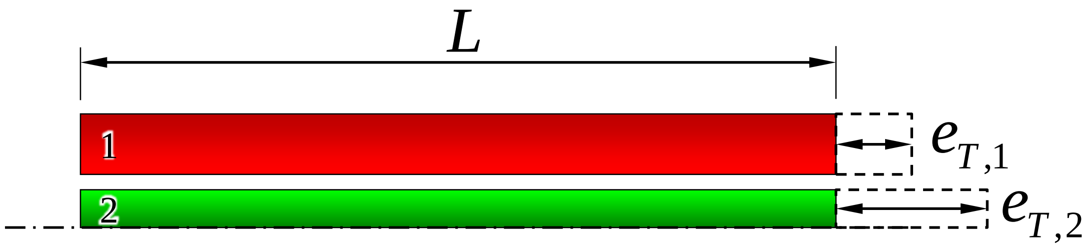
- Consider a composite bar which is symmetric through the thickness being subjected to a temperature rise of ΔT:



- Because of symmetry we only need to focus on the upper half of the beam as shown

Free & Constrained Thermal Expansion

- Free thermal expansion:
  - No attachment between bars

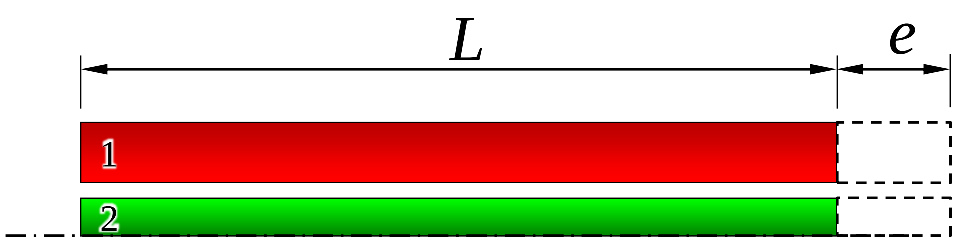


$$e_{T,1} = \alpha_1 \, \Delta T \, L$$

$$e_{T,2} = \alpha_2 \, \Delta T \, L$$

  - **No internal forces generated**

- Constrained thermal expansion:
  - Bars bonded together

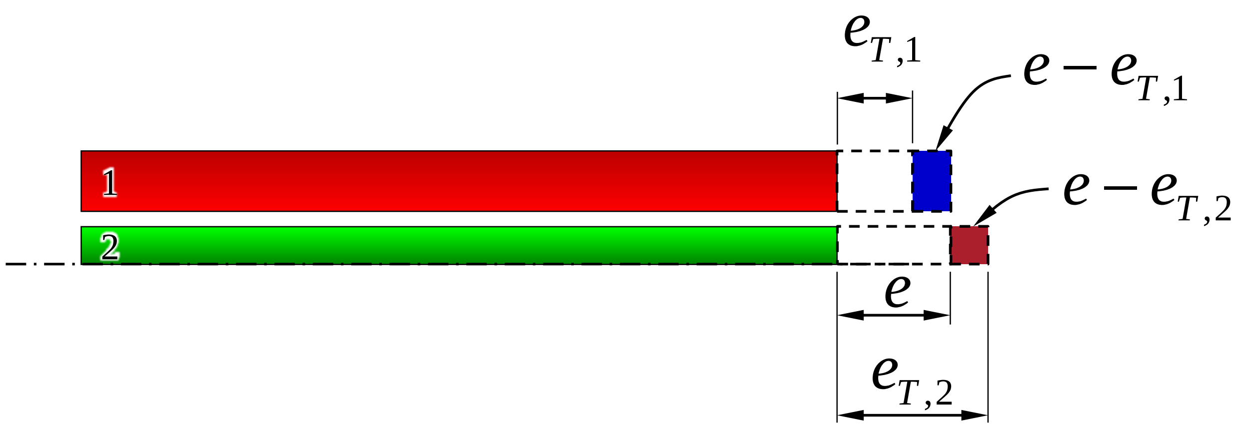


Common extension e

$$e_{T,1} < e < e_{T,2}$$

  - **Internal forces generated**

Internal Forces



- Prevented extension = $e - e_{T,i}$
  - (*i.e.* common extension minus free thermal extension)
- This generates internal forces:
  - element *1* "pulled on" to common extension by element *2*
  - element *2* "held back" to common extension by element *1*

Equations

- Static equilibrium

$$\sum F_x = 0 \qquad \therefore \qquad F_1 + F_2 = 0$$

  - Note: zero net external forces

- Compatibility of displacements

$$e_1 = e_2 = e$$

  - Note: forces relate to <u>prevented</u> strains

- Constitutive relations

$$F_1 = \lambda_1 \left(e - e_{T,1}\right) \qquad F_2 = \lambda_2 \left(e - e_{T,2}\right)$$

  - Note: here F_1 is positive (tension) and F_2 is negative (compression)

Stress, Strain and Deformation
Shear Stresses and Strains

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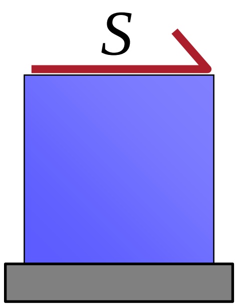
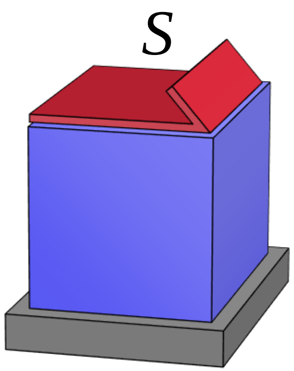
- Consider a cube element of material subject to a 'sliding force' (*i.e.* a force tangential to the surface) of intensity S



- The **shear stress** τ is a measure of 'force per unit area' where the force is tangential to the surface

- It is a field property like the 'direct stress' σ; it can vary continuously within a body and can be considered at a point:

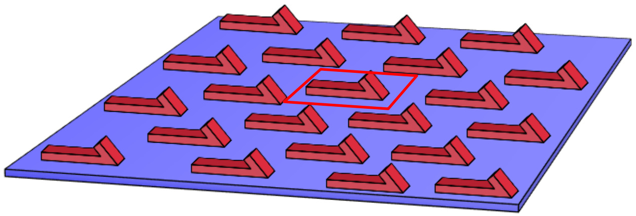


$$\tau = \lim_{\delta A \to 0} \frac{\delta S}{\delta A}$$

τ = 'tau'

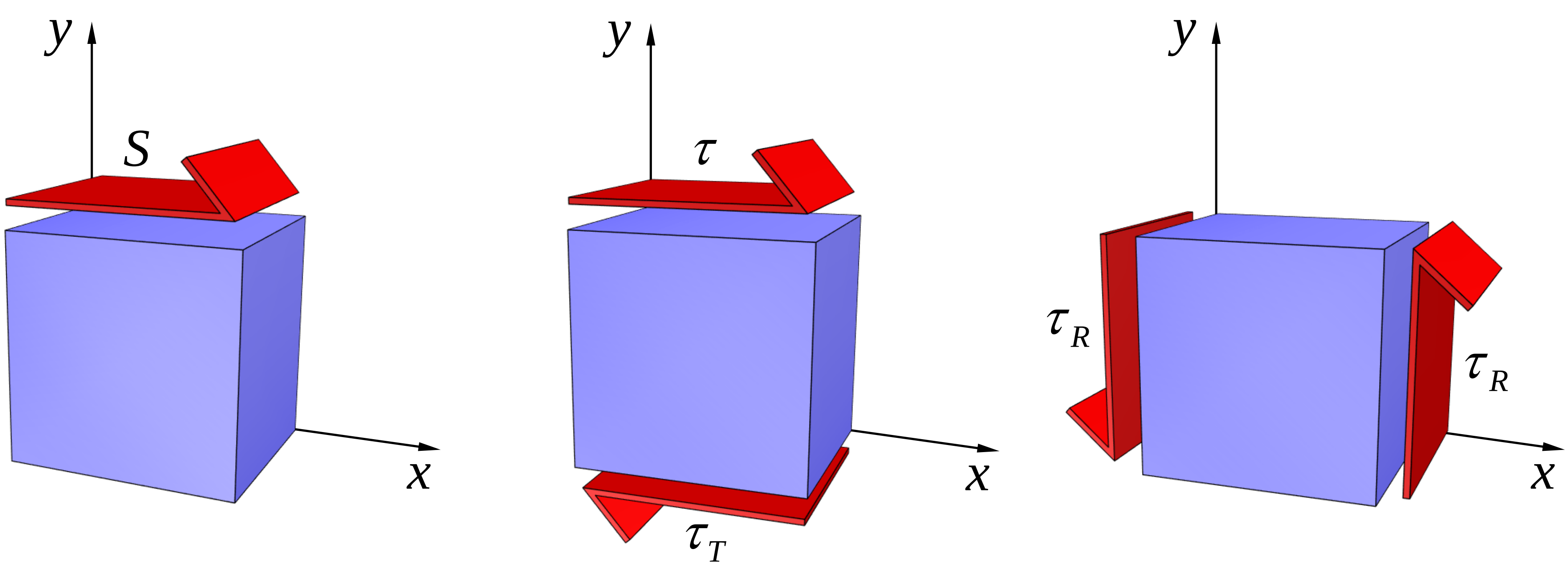
- Same units as for other stresses, *i.e.*

$$1\,\mathrm{Pa} = 1\frac{\mathrm{N}}{\mathrm{m}^2} = 10^{-6}\,\mathrm{MPa} = 10^{-6}\frac{\mathrm{N}}{\mathrm{mm}^2}$$



Complementary Shear Stresses

- For equilibrium, **complementary shear stresses** must exist to balance translational and rotational tendencies



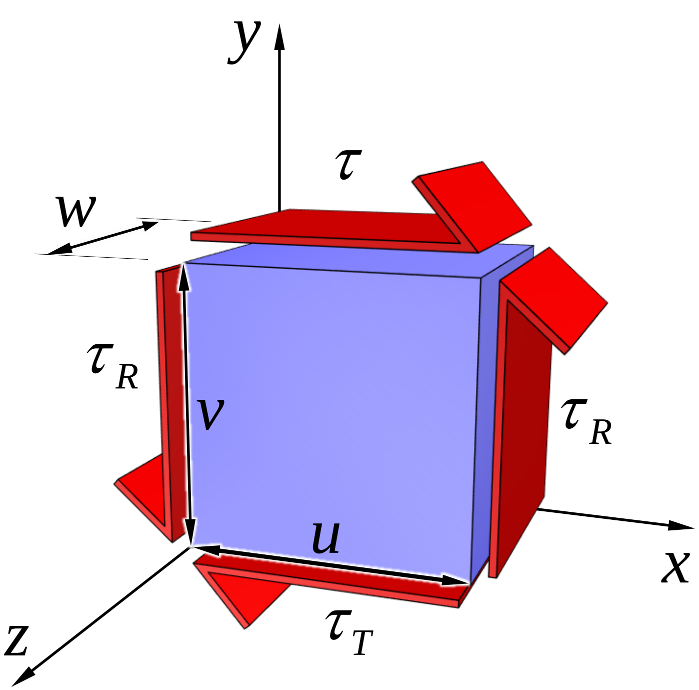
Applied shear force

Shear stresses for translational balance

Shear stresses for rotational balance

Complementary Shear Stresses

- In order to balance translational and rotational tendencies the magnitudes of the shear stress components are related:



$$\sum F_x = 0$$ ← Translational equilibrium along axis x

$$\tau\,(u\,w) - \tau_T\,(u\,w) = 0$$

$$\boxed{\tau = \tau_T}$$

$$\sum M_z = 0$$ ← Rotational equilibrium about axis z

$$\tau\,(u\,w)(v) - \tau_R\,(v\,w)(u) = 0$$

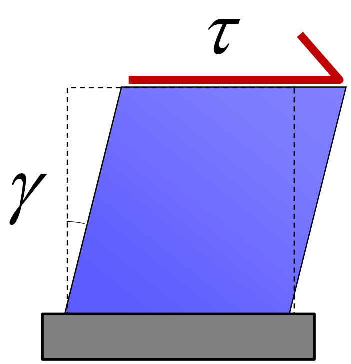
$$\boxed{\tau = \tau_R}$$

i.e. all complementary stresses are equal!

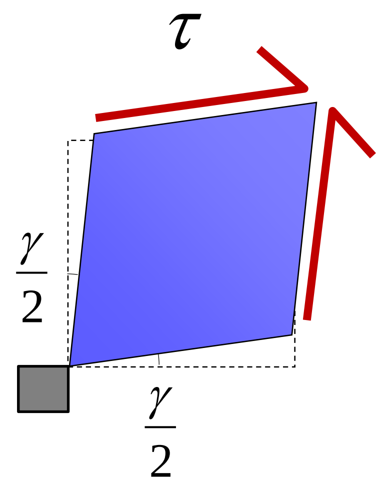
Shear Strain

- **Shear strain γ**: angular rotation in radians (non-dimensional)

Simple Shear
'Element fixed along an edge'

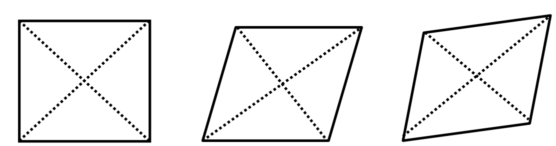


Pure Shear
'Element fixed at a corner'



γ = 'gamma'

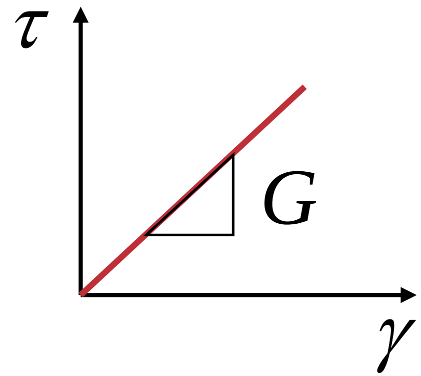
- Element 'edges' (or 'shear planes') do not change length but simply translate or rotate
- Element 'diagonals' <u>**do**</u> change length:
  - *i.e.* shear = diagonal 'tension' and 'compression'



Shear Modulus

- For linear elastic behaviour shear stress is proportional to shear strain

$$\boxed{\tau = G\,\gamma}$$



- Where the proportional constant G is the **Shear Modulus**
  – This is a **material property** like Young's modulus E or Poisson's ratio ν
  – In fact, for **isotropic materials** these three properties obey a very simple relationship:

$$\boxed{G = \frac{E}{2(1+\nu)}}$$