Handout 5 – Dynamics of Rigid Bodies

Meriam & Kraige, Dynamics: 6/1 – 6/5, B/1

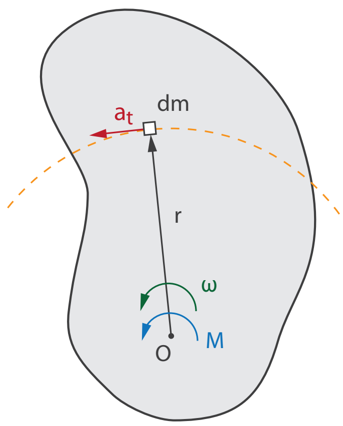
In the analysis of the kinematics and dynamics of rigid bodies, the key difference with a particle is the **rotation** of the body. What constitutes a particle or rigid body can depend on the problem being solved: in orbital mechanics a satellite is considered a particle, but for attitude control (*i.e.* pointing of the satellite) it is a rigid body with a rotational orientation. This handout introduces the concept of moment of inertia, to allow us to describe the dynamics of rotating bodies.

5.1 Moment of Inertia

The **moment of inertia** is the rotational analog of mass for linear motion (*i.e.* inertia), and it quantifies the solid's ability to resist changes in rotational velocity (*i.e.* angular acceleration), as a result of the net applied moments M around an axis of rotation:

$$M = I_O\,\ddot{\theta} \tag{5.1}$$

The moment of inertia I_O depends on the *distribution* of the mass around the axis of rotation O. A large moment of inertia will require a large applied moment to rotate the rigid body.



<u>Proof</u>: consider an element mass dm at a distance r from the fixed point of rotation O, which has a tangential acceleration $a_t = \ddot{\theta}r$. A moment M is required to produce the required accelerations of all the infinitesimal elements of the rigid body. This is found as:

$$\int dM = \int (dm\,a_t)\,r = \ddot{\theta}\int r^2 dm$$

$$M = I_O\,\ddot{\theta}$$

where I_O is defined as the **moment of inertia** around point O.

$$I_O = \int r^2 dm \tag{5.2}$$

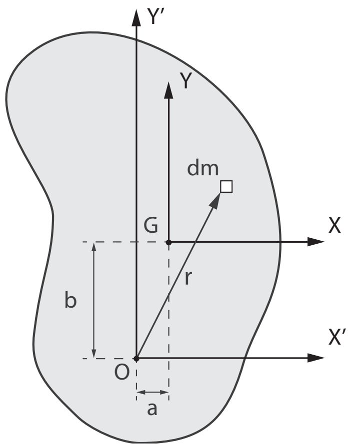
5.1.1 Parallel Axis Theorem

The moment of inertia depends on the point, or axis, of rotation. Using the **parallel axis theorem**, the moment of inertia about any parallel axis can be related to the moment of inertia I_G about the centre of mass, and the distance d to the pivot point.

$$I_O = I_G + d^2 m \tag{5.3}$$

Note that the moment of inertia I_G is a property of the rigid body.

Proof: consider an XY coordinate system with origin at the centre of mass G.



The moment of inertia around a point O can be expressed as:

$$\begin{aligned}
I_O &= \int r^2 dm = \int \left(x'^2 + y'^2\right) dm \\
&= \int \left((x+a)^2 + (y+b)^2\right) dm \\
&= \int \left(x^2 + y^2\right) dm + \int \left(a^2 + b^2\right) dm + 2a\cancel{\int x\, dm} + 2b\cancel{\int y\, dm} \\
&= I_G + \left(a^2 + b^2\right) m \\
&= I_G + d^2 m
\end{aligned}$$

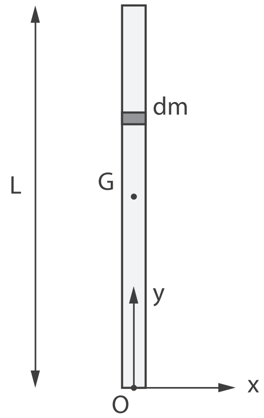
where we make use of the fact that:

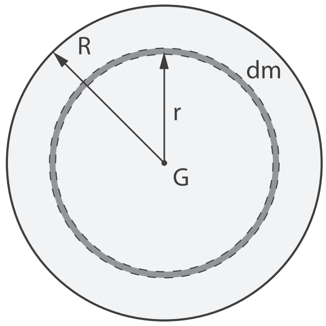
$$\int x\, dm = 0 \qquad\qquad \int y\, dm = 0$$

Why is this true?

5.1.2 Standard Cases

The moment of inertia of many standard shapes can be found analytically by integrating over the surface of the body; we here consider two standard cases.





rod/bar For a slender bar of length L and mass m, the moments of inertia around centre of mass and endpoint are derived as:

$$I_G = \int y^2 dm = \int_{-L/2}^{L/2} y^2 \frac{m}{L} dy = \frac{mL^2}{12}$$

$$I_O = \frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{3}$$

disk/wheel Consider a disk/wheel with radius R and mass m. Using an infinitesimal element of mass dm

$$dm = m\frac{dA}{A} = m\,\frac{2\pi r dr}{\pi R^2} = \frac{2m}{R^2} r dr$$

the moment of inertia is found as follows

$$I_G = \int_0^R r^2 dm = \frac{2m}{R^2}\int_0^R r^3 dr = \frac{mR^2}{2}$$

5.1.3 Radius of Gyration

The radius of gyration k is used to describe the distribution of mass in a rigid body. An equivalent moment of inertia can be achieved by placing a point mass m at a distance k from the axis of rotation.

$$I = \int r^2 dm = m\,k^2$$

and therefore

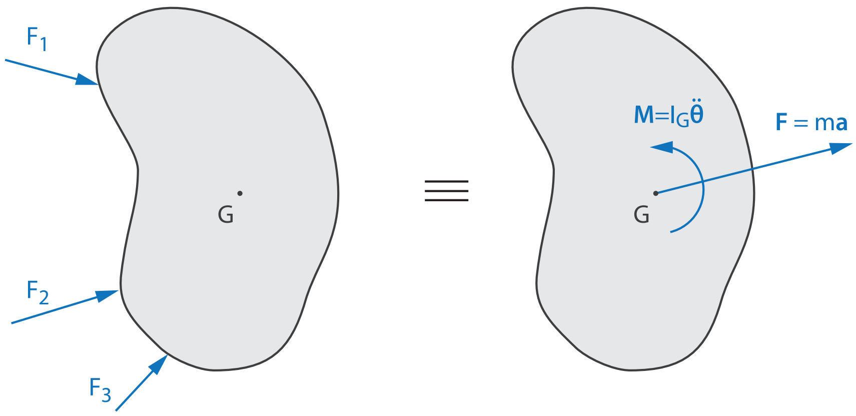
$$k = \sqrt{\frac{I}{m}} \tag{5.4}$$

For a solid disc, the radius of gyration is $k_G = R/\sqrt{2} \approx 0.71R$, while for a spoked wheel $k_G \approx R$.

5.2 General Motion: Translation and Rotation

If **a rigid body** has no fixed axis of rotation, it **will rotate around its centre of mass under an applied couple**. The dynamics of a rigid body under combined loads therefore consists of a linear acceleration $\boldsymbol{a}$ of its centre of mass, and an angular acceleration $\ddot{\theta}$ around its centre of mass.

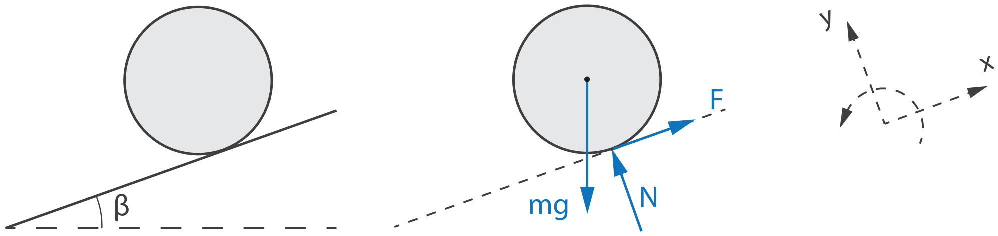
$$\sum \boldsymbol{F} = m\,\boldsymbol{a} \qquad\qquad \sum M_G = I_G\,\ddot{\theta}$$



NB: in setting up the equations of motion for rotation, make sure to calculate the net moments around an inertially fixed point of rotation, or around the centre of mass.

Example 5.1 – Rolling Cylinder

Consider a cylinder with radius R and mass m rolling down a slope at angle β. The cylinder rolls, rather than slips, down the slope due to the presence of friction, with coefficient μ.



From the FBD, apply Newton's 2nd Law:

$$\sum F_y \qquad N - mg\,\cos\beta = m\,\ddot{y} = 0$$

$$\sum F_x \qquad F - mg\,\sin\beta = m\,\ddot{x}$$

$$\sum M_G \qquad F\,R = I_G\,\ddot{\theta}$$

The first equation gives an expression for the (constant) normal force:

$$N = mg\,\cos\beta$$

The second two equations contain three unknowns (F, $\ddot{x}$ and $\ddot{\theta}$), and thus a further equation is required.

The condition of no slip ($|F| \leq \mu N$) gives:

$$x = -\theta\, R \qquad\qquad \rightarrow \qquad\qquad \ddot{x} = -\ddot{\theta}\, R$$

After some algebra, the equation of motion reduces to:

$$\ddot{x} = -\frac{g \sin\beta}{1 + I_G/\left(mR^2\right)}$$

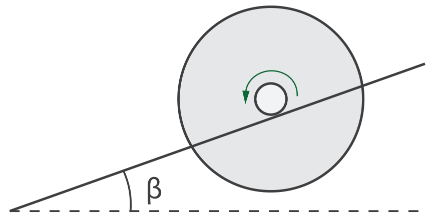
Sliding vs Rolling If the slope is frictionless ($\mu = 0$) the cylinder will not rotate, and would slide down the slope with the same dynamics as a particle:

$$\ddot{x} = -g\, \sin\beta$$

Rolling slows down the motion (due to the moment of inertia) by a factor of $1 + I_G/\left(mR^2\right)$. Comparing a solid and hollow cylinder of equal mass:

solid	$I_G = mR^2/2$	slow down: 3/2
hollow	$I_G \simeq mR^2$	slow down: 2

A specially-constructed cylinder could slow down the motion by an roder of magnitude, and is found, for example, in a yo-yo toy (where $\beta = 90^\circ$).



Sliding and Rolling If the static friction is exceeded, the cylinder would slide *and* roll down the slope simultaneously. The limiting friction is:

$$\frac{F}{N} = \ldots = \frac{\tan\beta}{1 + mR^2/I_G} \leq \mu$$

Therefore, if β or $I_G/\left(mR^2\right)$ is large (a steep slope or high moment of inertia), the static friction may be exceeded and slipping occurs. From the FBD:

$$\sum F_y \qquad N - mg\,\cos\beta = m\,\ddot{y} = 0$$

$$\sum F_x \qquad \mu N - mg\,\sin\beta = m\,\ddot{x}$$

$$\sum M_G \qquad \mu N R = I_G\,\ddot{\theta}$$

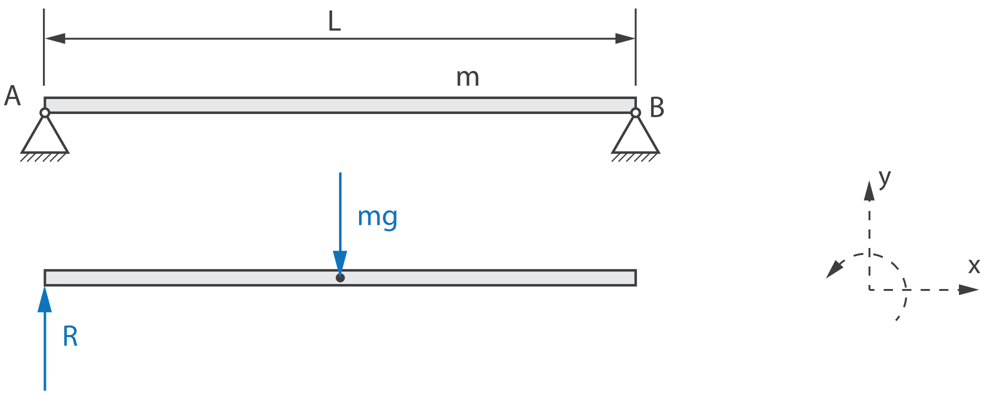
which can be combined to give the equations of motion:

$$\ddot{x} = g\left(\mu\cos\beta - \sin\beta\right)$$

$$\ddot{\theta} = \frac{R\mu mg\cos\beta}{I_G}$$

Example 5.2 – Falling Trapdoor

Consider a trapdoor of length L and mass m, which is simply supported on both ends.



Q: What are the accelerations and reaction forces when the support at B is suddenly removed?

Equations of motion: rotation around point A (which is inertially fixed):

$$\sum M(A): \qquad -mg\frac{L}{2} = I_A\ddot{\theta} = \frac{mL^2}{3}\ddot{\theta} \qquad \rightarrow \qquad \ddot{\theta} = -\frac{3}{2}\frac{g}{L}$$

The acceleration of tip B of the trapdoor is:

$$\ddot{y}_B = \ddot{\theta}L = -\frac{3}{2}g$$

Note that this is greater than the acceleration due to gravity!

Reaction forces: for reaction force R at the left-hand support, consider the net force in y-direction:

$$\sum F_y: \qquad R - mg = m\,\ddot{y}_G$$

where the acceleration of the centre of mass is found as

$$\ddot{y}_G = \ddot{\theta}\frac{L}{2} = -\frac{3}{4}g$$

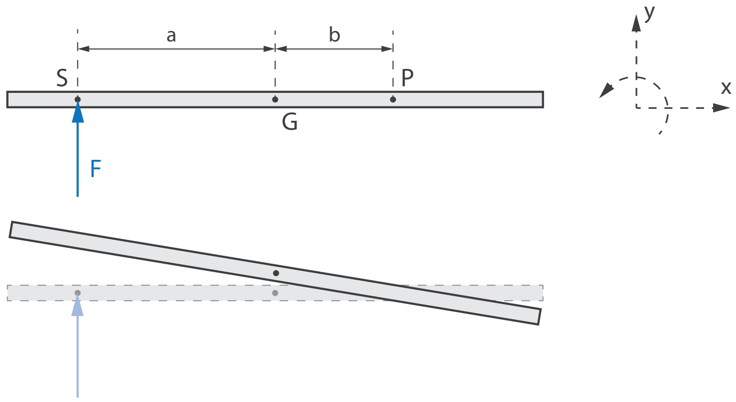
and therefore:

$$R = \frac{mg}{4}$$

Note that this is half the value found by statics, before the support was removed. There is no horizontal reaction force due to centripetal acceleration as $\dot{\theta} = 0$ at the point of release.

5.3 Centre of Percussion

An interesting concept is that of the **centre of percussion**. Consider a uniform bar with mass m.



If an instantaneous force F is applied at a point S at a distance a from the centre of mass G, the equations of motion for the accelerations and rotations are:

$$\sum F_y : \qquad F = m\,\ddot{y}_G$$

$$\sum M_G : \qquad -F\,a = I_G\,\ddot{\theta}$$

The acceleration of any point along the bar consists of two components:

$$\ddot{y}(x) = \ddot{y}_G + \ddot{\theta}\,x$$

where x is measured from the centre of mass. A point at $x = b$ has zero acceleration:

$$\frac{F}{m} - b\,\frac{F\,a}{I_G} = 0$$

which results in:

$$a\,b = \frac{I_G}{m} = k_G^2$$

We define the centre of percussion S as the point that can be struck without causing an acceleration at another point P. This property of rigid body dynamics you will have used in real life without realising it. Consider, for example, the best location to grip a hammer, or the 'sweet spot' on a baseball bat! [1]

[1] In actual fact, there are various interpretations of the sweet spot of a baseball bat:

http://www.acs.psu.edu/drussell/bats/sweetspot.html

However, for our purposes the centre of percussion is close enough!

5.4 3D Dynamics

The moments of inertia are defined *about an axis*. Let us consider a three dimensional body, with a coordinate system XYZ fixed at its centre of mass. The **inertia tensor** describes the dynamics of this body:

$$\boldsymbol{I}_G = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

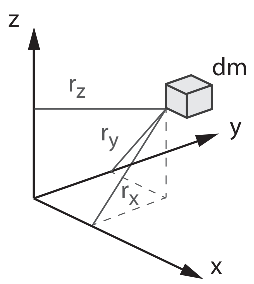
with the *moments* of inertia

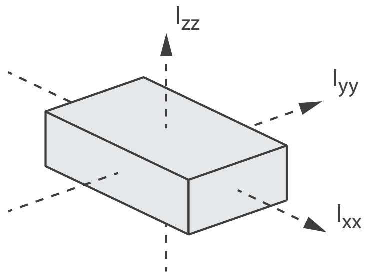
$$I_{xx} = \int r_x^2 dm = \int \left(y^2 + z^2\right) dm$$

$$I_{yy} = \int r_y^2 dm = \int \left(z^2 + x^2\right) dm$$

$$I_{zz} = \int r_z^2 dm = \int \left(x^2 + y^2\right) dm$$

around the three body axes.





An aircraft has different moments of inertia around its pitch, roll and yaw axes. This, in combination with the aerodynamic control surfaces, determines the manoeuvrability of the aircraft around these axes.

The *products* of inertia

$$I_{xy} = -\int xy\, dm$$

$$I_{xz} = -\int xz\, dm$$

$$I_{yz} = -\int yz\, dm$$

are a measure of the *imbalance* in the distribution of the mass with respect to the body axes. It is always possible to find a set of body axes that have zero products of inertia and therefore result in a diagonal inertia tensor; these directions are known as the principal axes.

The 3D dynamics of rigid bodies is a great deal more complicated than planar dynamics, and falls outside the scope of this unit; for the interested reader, more information can be found in Meriam & Kraige.

Revision Objectives Handout 5:

Dynamics of Rigid Bodies

- explain the concept of moment of inertia, and recall its definition ($I_O = \int r^2 dm$)
- apply parallel axis theorem to determine moment of inertia about different axes ($I_O = I_G + md^2$)
- recognise and use the concept of radius of gyration ($I_G = mk_G^2$, or $k_G = \sqrt{I_G/m}$)
- recall moment of inertia for simple geometries, such as:
  - slender rods ($I_G = mL^2/12$, $I_O = mL^2/3$)
  - solid disks ($I_G = mR^2/2$)
- explain the concept of centre of percussion
- set up equations of motion for rigid bodies under applied loads ($\boldsymbol{F} = m\boldsymbol{a}$, $M = I\ddot{\theta}$)
- combine equations of motion with kinematics, for relationship between rotation and translation
- solve/integrate equations of motion for simple cases

Note: the extension of moments of inertia to 3D is **not** examinable.