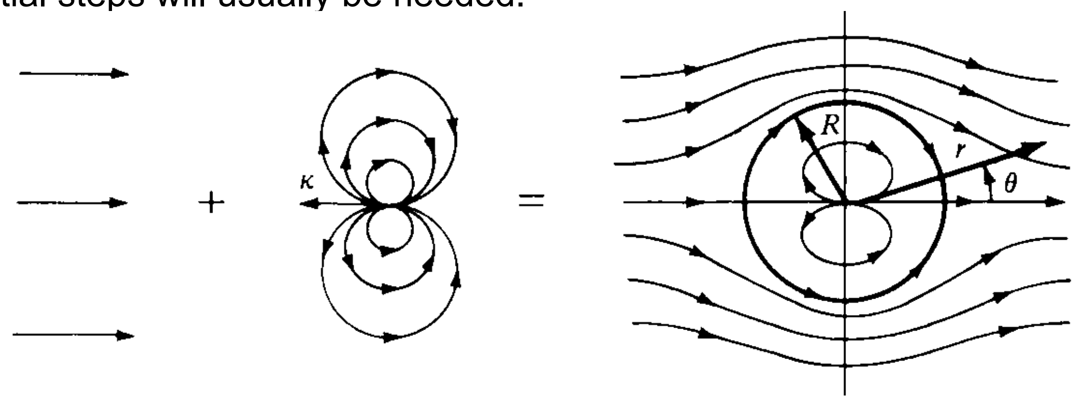
Aims for this lecture

- Look at the detailed analysis of the potential flow around a stationary cylinder
- To consider the impact of introducing a point vortex to the non-lifting cylinder flow.
- To gain some insight into mechanism for the generation of lift in a flow

STEP BY STEP FLOW INVESTIGATION (Flow over Cylinder)

The non-lifting flow over a cylinder is modelled as a free stream+ a doublet. To investigate the model and answer any question the same initial steps will usually be needed.



- STEP 1 Write out the stream function of the combined flow, in consistent coordinates for future calculation of velocities - polar is most suitable.

$$\psi = U_\infty r \sin\theta - \frac{\kappa}{2\pi}\frac{\sin\theta}{r} = U_\infty r \sin\theta \left(1 - \frac{\kappa}{2\pi U_\infty r^2}\right)$$

STEP BY STEP FLOW INVESTIGATION (2)

- STEP 2 Obtain the velocity field by differentiation, or look up components.

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U_\infty \cos\theta - \frac{\kappa \cos\theta}{2\pi r^2} = U_\infty \cos\theta\left(1 - \frac{\kappa}{2U_\infty \pi r^2}\right) \quad v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta\left(1 + \frac{\kappa}{2\pi U_\infty r^2}\right)$$

Notice that when $r = \sqrt{\frac{\kappa}{2\pi U_\infty}}$ the radial velocity is zero and

Then introduce the constant R (radius of cylinder) so that $R^2 = \frac{\kappa}{2\pi U_\infty}$

So when $r = R$:v_r is zero & the circle is shown to be a flow stream line.

the stream function becomes $\psi = U_\infty r \sin\theta\left(1 - \frac{R^2}{r^2}\right)$

Then $\psi = 0$ when $r = R$, i.e. a constant since the circle is a stream line.

Then the velocities at any point become

$$v_r = U_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right) \qquad v_\theta = -U_\infty \sin\theta\left(1 + \frac{R^2}{r^2}\right)$$

STEP BY STEP FLOW INVESTIGATION (3)

STEP 3 Find the pressure coefficients on the surface:

Apply Bernoulli's equation between the freestream & any surface point

$$p + \frac{1}{2}\rho V^2 = p_\infty + \frac{1}{2}\rho U_\infty^2 \quad \rightarrow \quad p - p_\infty = \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho V^2$$

Then using definition of pressure coefficient

$$\boxed{c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2}}$$

the velocities on the cylinder are

$$v_r = 0 \qquad v_\theta = -2U_\infty \sin\theta$$

Note: stagnation points located where sin θ =0. Two solutions at θ=0 & θ=π

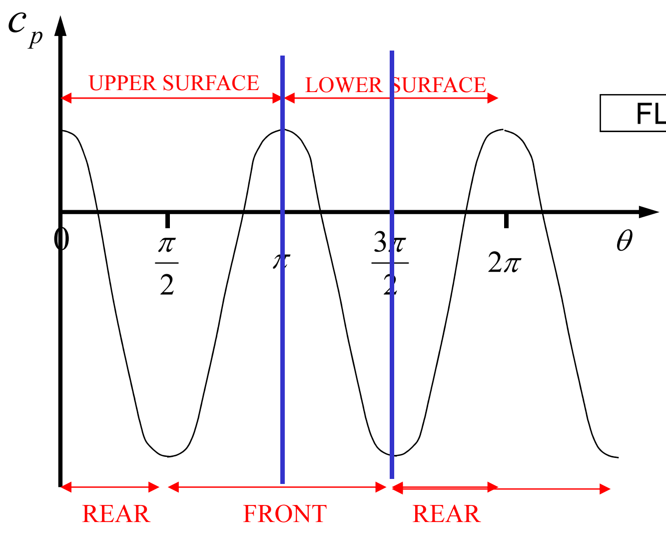
So in this case, on the cylinder $V^2 = v_r^2 + v_\theta^2 = 4U_\infty^2 \sin^2\theta$

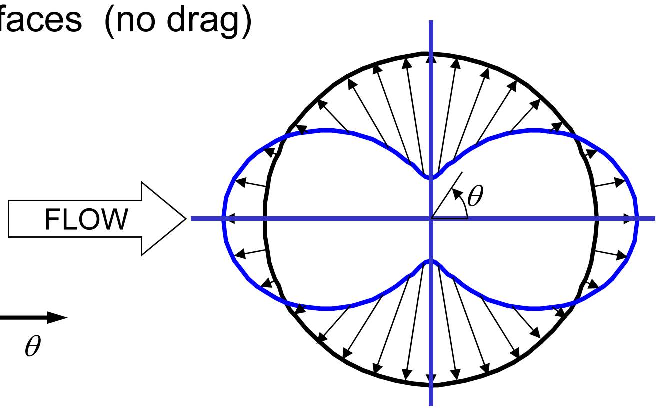
Then the pressure coefficient is given by

$$c_p = 1 - \frac{V^2}{U_\infty^2} = 1 - 4\sin^2\theta$$

Pressure Distribution Around a Cylinder

- Pressure is given by $c_p = 1 - 4\sin^2\theta$
- Symmetry on upper & lower surfaces (no lift)
- Symmetry on front & rear surfaces (no drag)

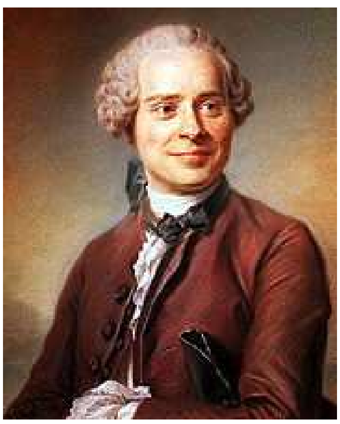


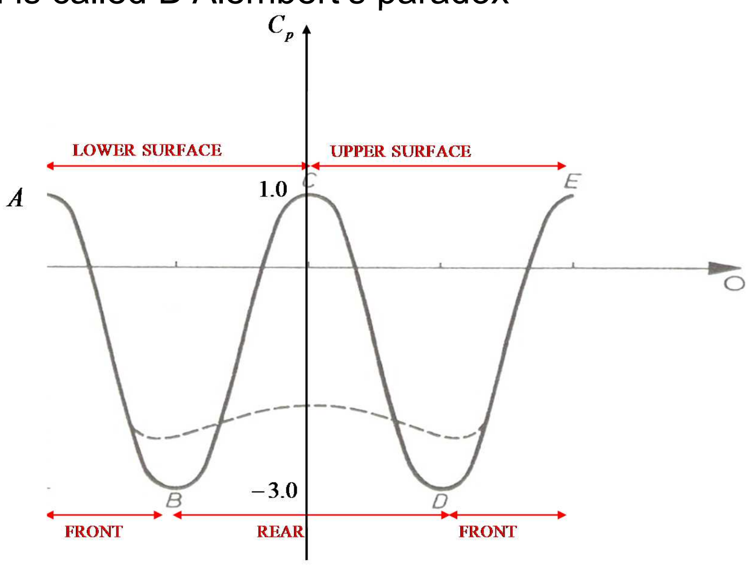


c_p distribution around a cylinder

Forces on Body and D'Alembert's paradox

- pressure distribution is symmetrical top to bottom
  - 'lift coefficient' $C_L = 0$ (particular to this flow)
- pressure distribution is symmetrical left to right
  - 'drag coefficient' $C_D = 0$ (general result)
- Zero drag is in fact, a general result for **2D** potential (or irrotational) flow around closed bodies and is called D'Alembert's paradox





INTEGRATING PRESSURES FOR LIFT COEFFICIENTS

To get lift on cylinder mathematically, consider the forces acting on a small element of its surface. Any other shape follows the same process

DON'T WORRY ABOUT DETAILS AS ALL GIVEN IN EXAMPLE 2 OF HANDOUT

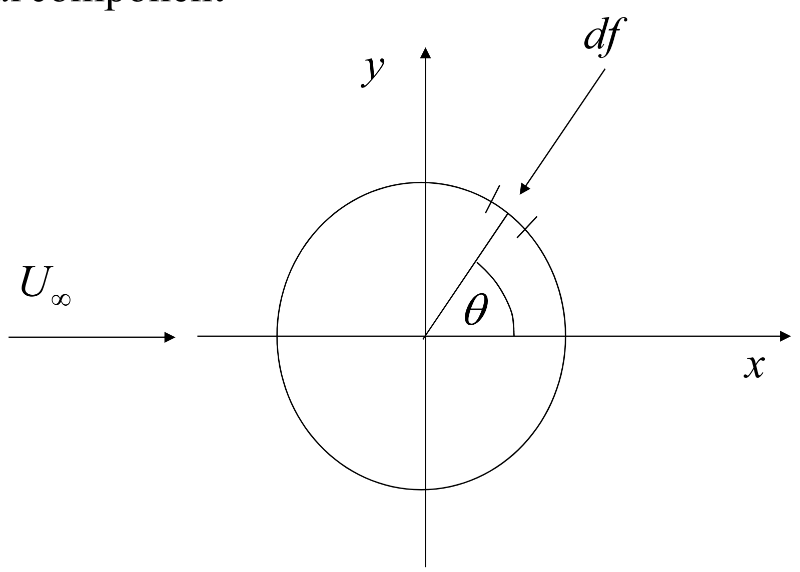
$$df = p\,ds$$

resolving into vertical component

$$dL = -df\sin\theta$$

but as $\quad ds = R\,d\theta$

$$dL = -p\sin\theta\, R\,d\theta$$



INTEGRATING PRESSURES FOR LIFT COEFFICIENTS(2)

Then the total lift on cylinder is obtained by integration

$$L = -\int_0^{2\pi} p\sin\theta\, R\,d\theta \quad = -\int_0^{2\pi}(p-p_\infty)\sin\theta\, R\,d\theta - \int_0^{2\pi} p_\infty \sin\theta\, R\,d\theta$$

$$L = -\frac{1}{2}\rho_\infty U_\infty^2 R\int_0^{2\pi} c_p \sin\theta\, d\theta$$

Second integral easily shown to be zero

Then substituting in for the pressure coefficient the total lift on a cylinder can be shown to be (see Example 2 in previous Handout for detail)

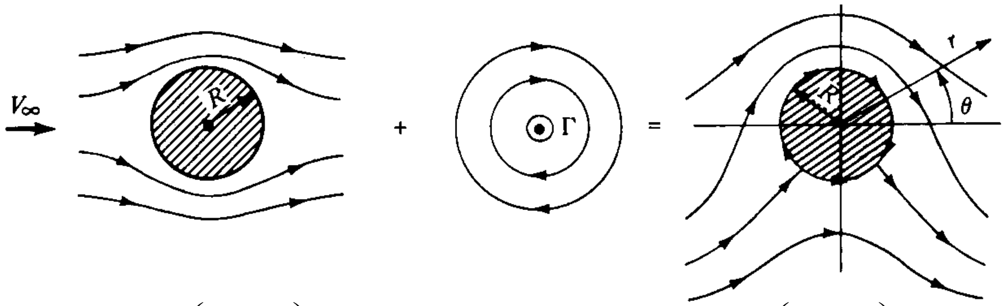
$$L = -\frac{1}{2}\rho_\infty U_\infty^2 R\int_0^{2\pi}\left(1-4\sin^2\theta\right)\sin\theta\, d\theta = -\frac{1}{2}\rho_\infty U_\infty^2 R\left(\underbrace{\int_0^{2\pi}\sin\theta\, d\theta}_{=0} - 4\underbrace{\int_0^{2\pi}\sin^3\theta\, d\theta}_{=0}\right)$$

$$L = 0$$

Remember the definition of Lift Coefficient $\quad C_L = \dfrac{L}{Area\times\frac{1}{2}\rho_\infty U_\infty^2}$

Lifting Cylinder Flow

- Only mechanisms for transmitting forces (& hence lift) to a body is via pressure and shear stress on the body surface
- The potential flow over a rotating cylinder will give us some insight
- A point vortex is added to the previous model of flow over a non-lifting cylinder i.e. the model consists of a uniform stream + doublet + point vortex
- Streamline of vortex is a circle so boundary unchanged



$$\psi = \underbrace{U_\infty r\sin\theta\left(1-\frac{R^2}{r^2}\right)}_{\text{Freestream \& doublet}} + \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\text{vortex}} - \underbrace{\frac{\Gamma}{2\pi}\ln R}_{\text{arbitrary constant}} = U_\infty r\sin\theta\left(1-\frac{R^2}{r^2}\right)+\frac{\Gamma}{2\pi}\ln\frac{r}{R}$$

- Term added to match streamfunction value at surface to non-lifting case

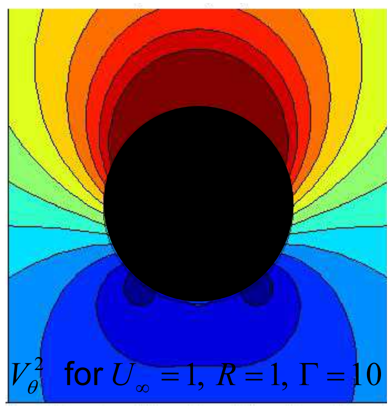
Velocities & Pressure

- Differentiation gives the velocity terms

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left(1-\frac{R^2}{r^2}\right)U_\infty\cos\theta, \qquad V_\theta = -\frac{\partial\psi}{\partial r} = -\left(1+\frac{R^2}{r^2}\right)U_\infty\sin\theta - \frac{\Gamma}{2\pi r}$$

- On the 'surface' of the cylinder (i.e. $r = R$)

$$V_r = 0, \qquad V_\theta = -\underbrace{2U_\infty\sin\theta}_{\text{freestream \& doublet}} - \underbrace{\frac{\Gamma}{2\pi R}}_{\text{vortex}}$$



- Surface pressure found using

$$c_p = 1 - \frac{V^2}{U_\infty^2} = 1 - \frac{V_\theta^2}{U_\infty^2}$$

$$C_p = \underbrace{1-4\sin^2\theta}_{\text{basic cylinder flow}} - \underbrace{\frac{2\Gamma\sin\theta}{\pi R U_\infty}}_{\text{asymmetric swirl component}} - \underbrace{\left(\frac{\Gamma}{2\pi R U_\infty}\right)^2}_{\text{constant swirl component}}$$

Terms of Pressure Coefficient

(1) Basic cylinder flow $1 - 4\sin^2\theta$

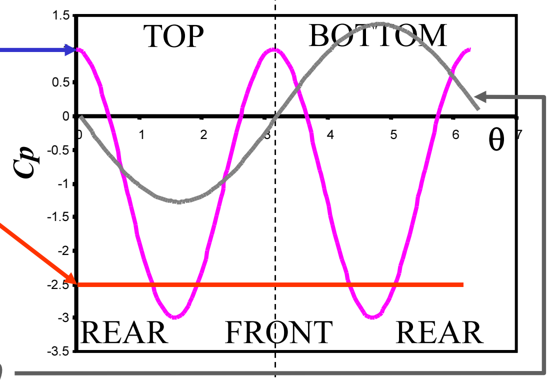
- lift and drag contribution are zero

(2) Constant swirl term $-\left(\Gamma/(2\pi R U_\infty)\right)^2$

- lift and drag contribution are zero

(note a constant suction contribution)

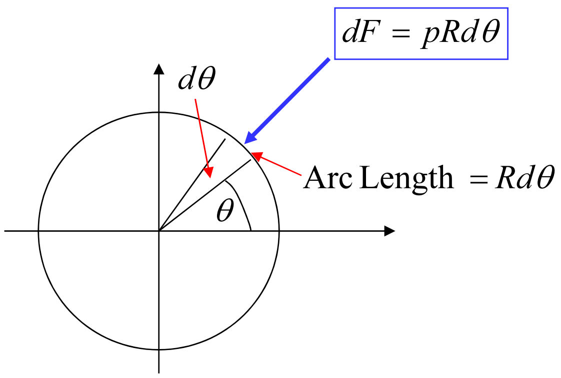
(3) Asymmetric swirl term $-\left(\dfrac{2\Gamma}{\pi R U_\infty}\right)\sin\theta$



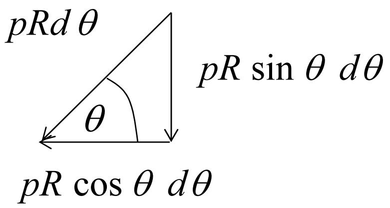
- lift contribution non-zero due to the **combination** of vortex and freestream
- Positive lift from a suction on the upper surface and a positive pressure on the lower surface
- Drag force zero because the pressure distributions on the front and rear halves of cylinder have equal areas of +ve and –ve pressure, therefore everything cancels out

Lift Generation

To evaluate the lift it is easiest to work it out from first principles



Resolve this force to get components



So the expression for lift force in the upwards direction is $l = -\int_0^{2\pi} p(\theta) R\sin\theta\, d\theta$

See handout for derivation

$$\boxed{l = \rho_\infty U_\infty \Gamma}$$

<u>Kutta-Joukowski Theorem</u> (1902/06)

general result for 2D body with 'bound' circulation Γ

Fundamental relation of Fluid Mechanics

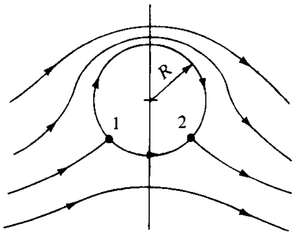
In non-dimensional terms (radius R is reference length) $C_l = \dfrac{l}{1/2\rho_\infty U_\infty^2 R} = \dfrac{\Gamma}{RU_\infty}$

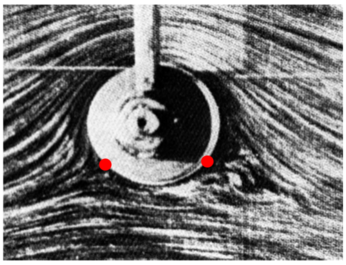
Stagnation Points

$$v_r = 0 \quad \Rightarrow \quad r = R \quad or \quad \cos\theta = 0$$

$$v_\theta = 0 \quad \Rightarrow \quad \sin\theta = \left(\frac{-\Gamma}{2\pi r U_\infty (1 + R^2/r^2)}\right) \qquad \Gamma > 0 \quad \rightarrow \quad \sin\theta < 0 \quad \rightarrow \quad y_{\text{stagnation}} < 0$$

- Case(1): The stagnation points are on the cylinder surface, $r=R$ then
- If $\frac{\Gamma}{(4\pi U_\infty R)} < 1$ then there are two values of θ, symmetric about the y-axis



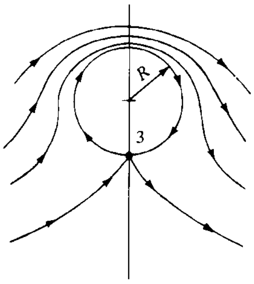


$$\sin\theta = \left(\frac{-\Gamma}{4\pi R U_\infty}\right)$$

- Case(2):

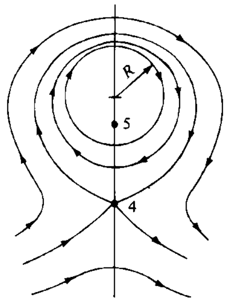
$$\frac{\Gamma}{(4\pi U_\infty R)} = 1$$

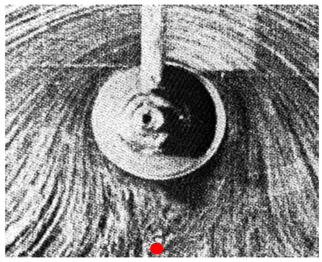
$$\sin\theta = -1$$



- Case(3):

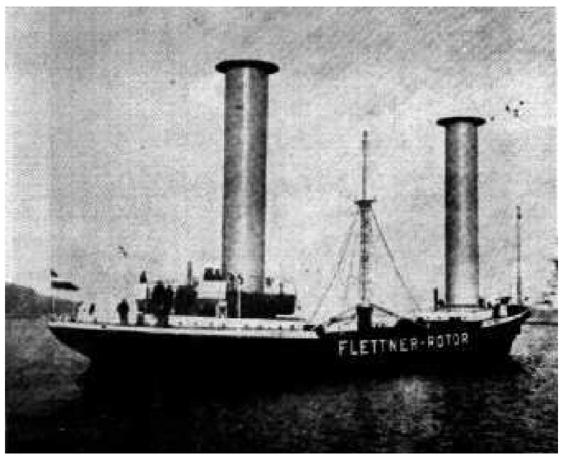
$$\frac{\Gamma}{(4\pi U_\infty R)} > 1$$





Example of real life use of "Lifting Cylinders"

- applied to ship propulsion by Flettner (1920)
  - twin vertical cylinders
  - highly manoeuvrable
  - mechanically over-complex (cost, reliability)



Where Does the Lift Come From .. ?

- Model shows that circulation on its own does NOT lead to lift
- Model indicates that changes in the pressure distribution which lead to lift occur if there is a combination of

  a. circulation (ie flow ***rotation***) and

  b. freestream (ie ***cross-flow***) velocity

  – increased velocity *magnitude* on upper surface

  – reduced velocity magnitude on lower

- Be cautious when reading text books or looking at websites as many give seemingly convincing, but wrong explanations of lift !!
- Arguments and controversy over lift generation have arisen because people misapply Bernoulli's or Newton's equations. Both sets of equations do apply, but not in the way that is frequently described in many discussions of lift!!

Learning Outcomes:
"What you should have learnt so far"

- Use suitable potential models to investigate velocities, pressures and forces for a uniform flow and a doublet.
- Explain D'Alembert's paradox
- Derive the velocity and pressure distributions from the stream function for a lifting cylinder
- Derive the Kutta-Joukowski theorem relating lift and circulation
- Distinguish between the elements of the cylinder velocity distribution and hence explain qualitatively how lift is generated