class 1

1. Introduction

1.1 What is vector calculus?

Vector calculus = calculus grown up.

differentiation & integration for adults!

It answers questions like how to define and measure the variation of temperature, fluid velocity, force, magnetic flux etc. over all three dimensions of space. In the real 3D engineering world, one wants to know things like the stress and strain inside a structure, the vorticity of the air flow over a wing, or the induced electromagnetic field around an aerial. For such questions, it is simply not good enough to deal with dy/dx and $\int f(x)dx$. We must instead know how to integrate and differentiate *vector* quantities with three components (in directions $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$) which depend on three co-ordinates x, y, z.

1-D things

Vector calculus provides the necessary mathematical notation and techniques for dealing with such issues.

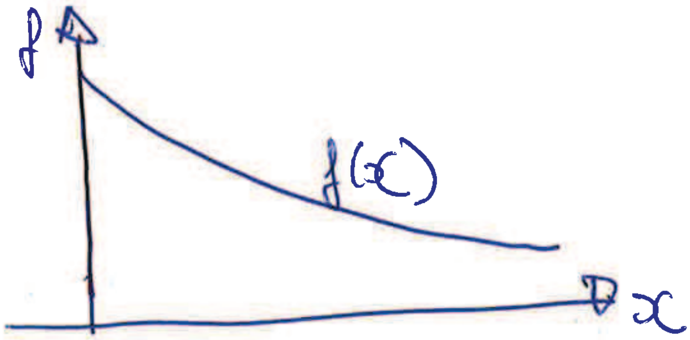
1.2 Scalars, vectors, fields and functions

We will deal with four types of mathematical entities:

1. A **scalar function** (of one variable) $f(x)$ or $f(t)$ is a formula that takes a scalar and returns a scalar.

It might be used to describe the spatial variation of temperature $T(x)$ along a one-dimensional bar heated at one end, or the time variation of the DC current $i(t)$ across a certain component in an electrical circuit.

$$\underset{\text{one real number}}{f : \mathbb{R}} \to \underset{\text{one real number}}{\mathbb{R}}$$



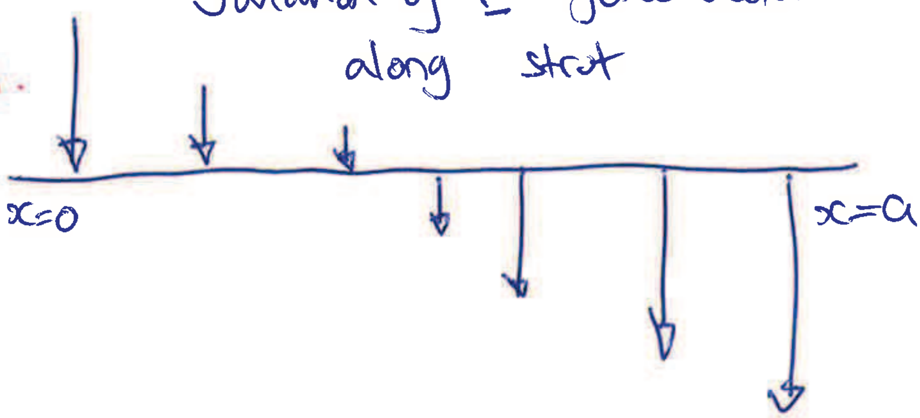
2. A **vector function** (of one variable) $\boldsymbol{v}(x)$ or $\boldsymbol{v}(t)$ takes a scalar and returns a vector:

$$\boldsymbol{v}(t) = v_1(t)\boldsymbol{i} + v_2(t)\boldsymbol{j} + v_3(t)\boldsymbol{k}. \quad = (v_1(t), v_2(t), v_3(t))$$

Such functions might be used to describe the motion of a particle whose position vector is $\boldsymbol{r}(t)$ at time t; or the external forces $\boldsymbol{F}(x)$ acting at distance x along a 1-dimensional strut.

$$\underline{v} : \underset{\text{one real number}}{\overset{\text{scalar}}{\mathbb{R}}} \to \underset{\text{3 real numbers.}}{\overset{\text{3-vector}}{\mathbb{R}^3}}$$

variation of F force vector along strut



3. A **scalar field** ϕ is a scalar quantity defined over a region of space. It takes a vector (of positions) and returns a scalar.

$$\phi = f(x, y, z) = f(\boldsymbol{r}) \qquad \text{(or } f(x, y) \text{ in 2D).}$$

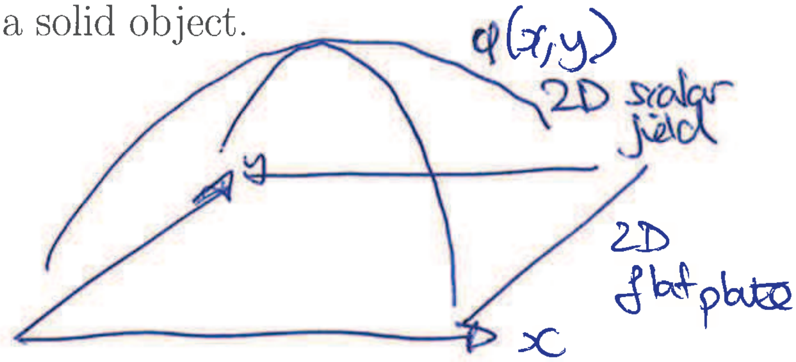
e.g. the variation of temperature $T(x, y, z)$ in this room using Cartesian co-ordinates. We might also think of the variation of density or charge density $\rho(x, y, z)$ inside a solid object.

$\varphi : \mathbb{R}^3 \to \mathbb{R}$

vector input — 3 numbers vector

scalar output — 1 number

e.g. 2D



4. A **vector field** $\boldsymbol{v}(x, y, z)$ is a vector-valued quantity defined over a region of space. It is defined by a ~~function~~ field that takes a vector (of positions) and returns a vector

$= (v_1(x,y,z), v_2(x,y,z), v_3(x,y,z))$

$$\underline{\boldsymbol{v}}(\boldsymbol{r}) = v_1(x, y, z)\boldsymbol{i} + v_2(x, y, z)\boldsymbol{j} + v_3(x, y, z)\boldsymbol{k}$$

$$\text{or } (v_1(x, y)\boldsymbol{i} + v_2(x, y)\boldsymbol{j} \text{ in 2D).}$$

e.g. the spatial variation of fluid velocity $\boldsymbol{v}(x, y, z)$ in a steady flow, or current $\boldsymbol{I}(x, y, z)$ flowing in a conductor.

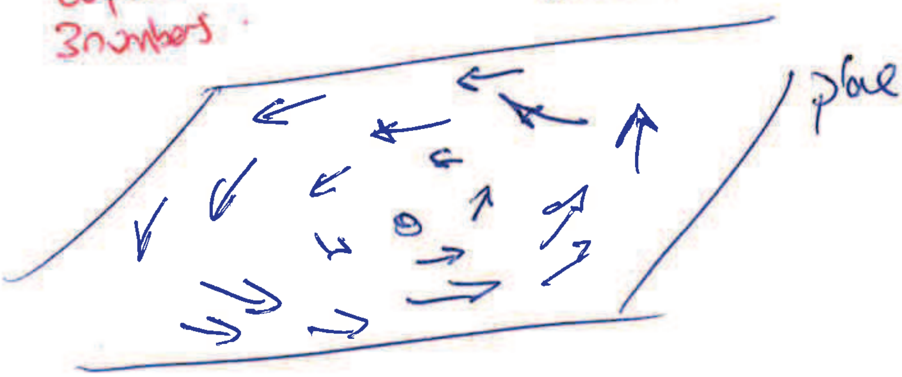
$\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$

vector input — 3 numbers

vector output — 3 numbers

flow of water around a vortex.

e.g 2D



1.3 Vector functions

Differentiation and integration of vector functions are easy! One simply differentiates or integrates the components separately.

$$\frac{d}{dt}\boldsymbol{v}(t) = \frac{d}{dt}v_1(t)\boldsymbol{i} + \frac{d}{dt}v_2(t)\boldsymbol{j} + \frac{d}{dt}v_3(t)\boldsymbol{k}$$

$$= \left(\frac{dv_1}{dt}(t), \frac{dv_2}{dt}(t), \frac{dv_3}{dt}(t)\right)$$

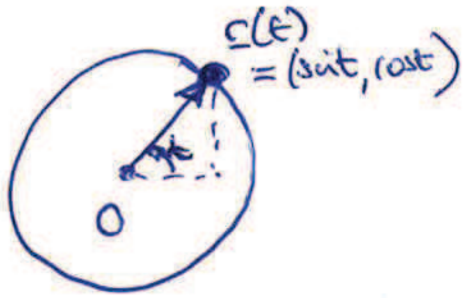
Worked Example 1.1 *A particle moves on a circle of radius 1, such that its position vector is given by*

$$\boldsymbol{r}(t) = \sin t\boldsymbol{i} + \cos t\boldsymbol{j}$$

$\underline{v} = \frac{d}{dt}\underline{r}(t)$ $\underline{a} = \frac{d}{dt}\underline{v}(t)$

Calculate its velocity and acceleration. Show that the velocity and acceleration are orthogonal. Is

$$\left|\frac{d\boldsymbol{r}}{dt}\right| = \frac{d}{dt}(|\boldsymbol{r}|)?$$



Integration works as well. E.g.

EXERCISE

Worked Example 1.2 *Derive, in vector notation an expression for the position of a projectile at time t which starts at position $(x, y, z) = \mathbf{0}$ at $t = 0$ with initial horizontal velocity v_0.*

under gravity g

Example 1.1

$$\underline{r}(t) = \sin t\,\underline{i} + \cos t\,\underline{j}$$

velocity $\underline{v} = \frac{d}{dt}\underline{r}(t) = \cos t\,\underline{i} - \sin t\,\underline{j}$

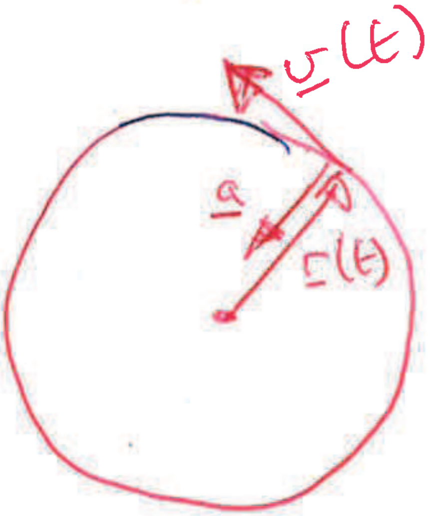
acc^n $\underline{a} = \frac{d}{dt}\underline{v}(t) = -\sin t\,\underline{i} - \cos t\,\underline{j}$

if orthogonal then $\underline{v} \cdot \underline{a} = 0$

ie $\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} = -\cancel{\sin t}\cos t + \cancel{\sin t}\cos t$

$= \underline{\underline{0}}$ DONE!

also $\underline{r} \cdot \underline{v} = 0$ (show yourselves)



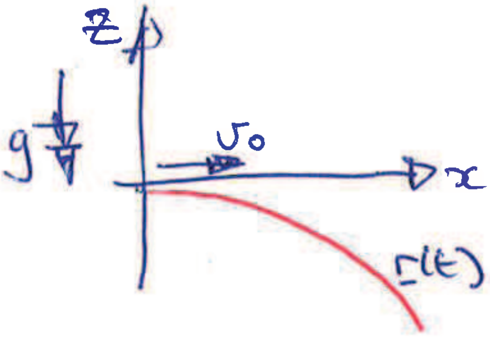
Is $\left| \frac{d\underline{r}}{dt} \right| = \frac{d}{dt} |\underline{r}|$

non zero speed $\left| \frac{d\underline{r}}{dt} \right| = |\underline{v}| = \sqrt{\cos^2 t + \sin^2 t}$
$= 1$

0 because $|\underline{r}| = 1$

Example 1.2

equation for a projectile in gravity



acc^n

$$\underline{a} = \frac{d^2\underline{r}}{dt^2} = -g\,\underline{k} \quad -(*)$$

initial conditions : $\underline{r}(t=0) = \underline{0}$ $\quad -(1)$

$$\frac{d\underline{r}}{dt}(t=0) = v_0\,\underline{i} \quad -(2)$$

to solve :

integrate (*) $\quad \int^t \frac{d^2\underline{r}}{dt^2}\,dt = \int^t -g\,\underline{k}\,dt$

$$\frac{d\underline{r}}{dt} = -g\,\underline{k}\,t + \underline{c}$$

const of integration

init cond^n (2) $\Rightarrow \frac{d\underline{r}}{dt}(t=0) = \underline{c} = v_0\,\underline{i}$

then $\quad \frac{d\underline{r}}{dt} = -g\,\underline{k}\,t + v_0\,\underline{i}$

integrate $\quad \int^t \frac{d\underline{r}}{dt} = \int^t \left(-g\,\underline{k}\,t + v_0\,\underline{i}\right)dt$

$$\underline{r}(t) = -g\,\underline{k}\,\frac{t^2}{2} + v_0\,\underline{i}\,t + \underline{d}$$

const $\int$gration

init cond^n (1) $\Rightarrow \underline{r}(t=0) = \underline{d} = \underline{0}$

$$\therefore \quad \underline{r}(t) = -g\,\underline{k}\,\frac{t^2}{2} + v_0\,\underline{i}\,t$$

The following **rules of differentiation** are obtained by applying the corresponding rules to the separate components. Here $\boldsymbol{u} = \boldsymbol{u}(t)$, $\boldsymbol{v} = \boldsymbol{v}(t)$, $' = d/dt$ and c is a constant

linearity

product rule for vector fields

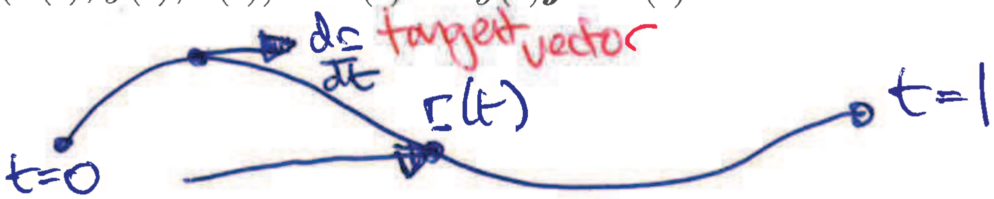
$$\begin{aligned} \frac{d}{dt}(\underline{u}+\underline{v}) = (\boldsymbol{u}+\boldsymbol{v})' &= \boldsymbol{u}' + \boldsymbol{v}' = \frac{d\underline{u}}{dt} + \frac{d\underline{v}}{dt}. \\ (c\boldsymbol{u})' &= c\boldsymbol{u}' \\ (\boldsymbol{u}\cdot\boldsymbol{v})' &= \boldsymbol{u}'\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{v}' \\ (\boldsymbol{u}\times\boldsymbol{v})' &= \boldsymbol{u}'\times\boldsymbol{v} + \boldsymbol{u}\times\boldsymbol{v}' \end{aligned}$$

can prove yourselves by writing in components.

version of with scalar functions $(fg)' = f'g + fg'$

One important type of vector function is the **intrinsic definition of a curve r(t)** in 3 dimensions.

$$\boldsymbol{r}(t) = (x(t), y(t), z(t)) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$



e.g.

- straight line $\boldsymbol{r} = \underset{\text{starting point}}{\boldsymbol{a}} + t\underset{\text{direction}}{\underline{\boldsymbol{b}}}$
- circle $\boldsymbol{r} = (a\cos t, a\sin t, 0)$ in the (x, y)-plane.

Tangent vector to the curve:

$$\frac{d\underline{r}}{dt} = \boldsymbol{r}'(t) = (x'(t), y'(t), z'(t)) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

does change magnitude of tangent vector

There are many ways of choosing the parametrisation variable t (replacing t by t^2 does not change the curve, nor the direction of the tangent vector!) An important choice is the so-called distance, or arclength s, along the curve. We will return to this in Chapter 4 of these notes, on Line Integrals.

3D $\varphi : \mathbb{R}^3 \to \mathbb{R}$

2D $\varphi : \mathbb{R}^2 \to \mathbb{R}$

1.4 Scalar fields

How do we visualise scalar fields $\phi = f(\boldsymbol{r})$? One way is to sketch **level sets** (= level curves, level surfaces, iso-surfaces)

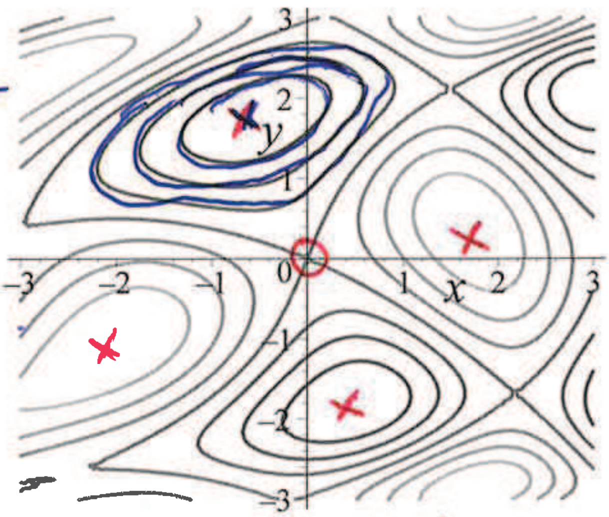
$$\phi = const.$$

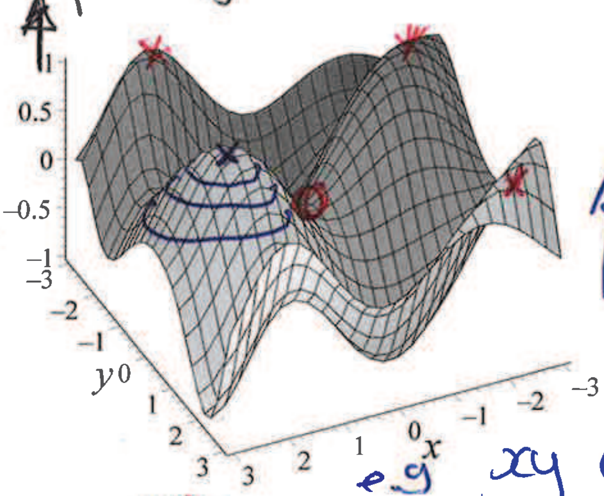
In 2D where $\phi(\boldsymbol{r}) = \phi(x, y)$, this has a natural interpretation:

- Map contours = lines of constant height

lines φ = const ||| level sets

φ = height(z)





height = temperature

e.g. xy coords on 2D sheet

x – maxima minima (points)   o – saddle.

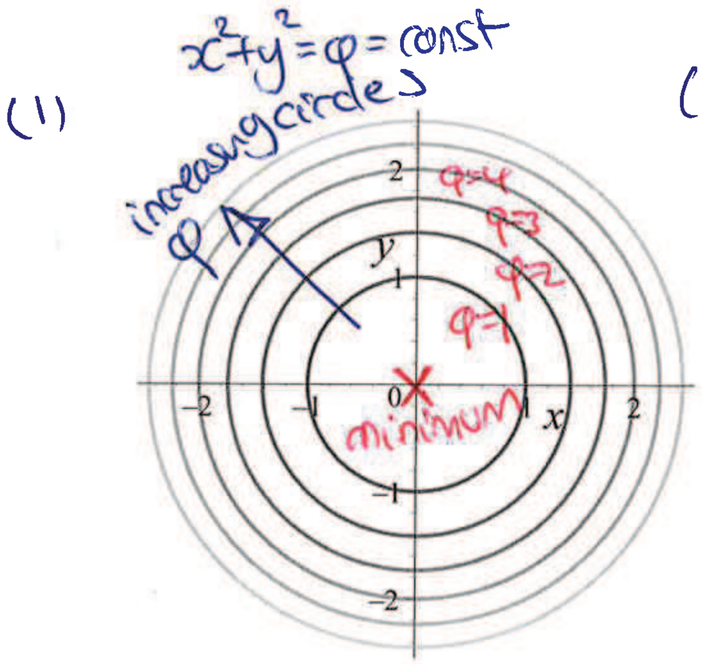
- Weather map (iso-bars) = lines of constant pressure

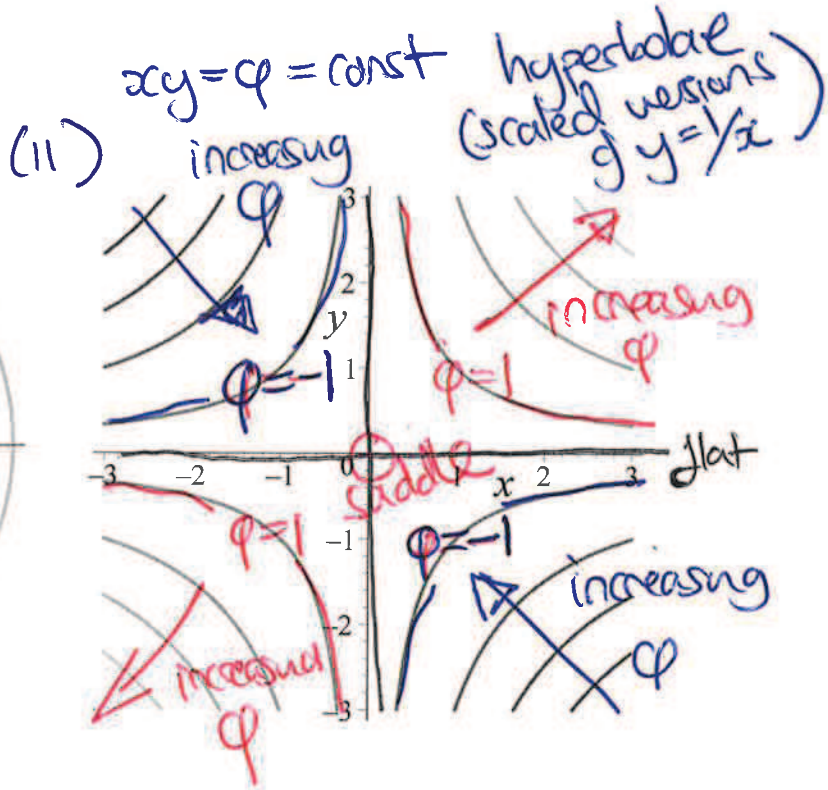
x – maxima minima   o – saddle.

Example 1.3: *Sketch the level curves of the ~~functions~~ scalar fields.*

$$(i) \quad \phi = x^2 + y^2, \qquad (ii) \quad \phi = xy.$$

Thinking of ϕ as the height above the contour plot, observe from the graph where the ~~function~~ field is (i) flat (ii) steepest. Which direction defines 'uphill'? Where do the maxima/minima lie?





Moral: contours close together $\Rightarrow$ large variation of height ϕ = steep (isobars close together $\Rightarrow$ large variation in fluid pressure = windy!)

HINT: for sketching level curves. Recall equations for

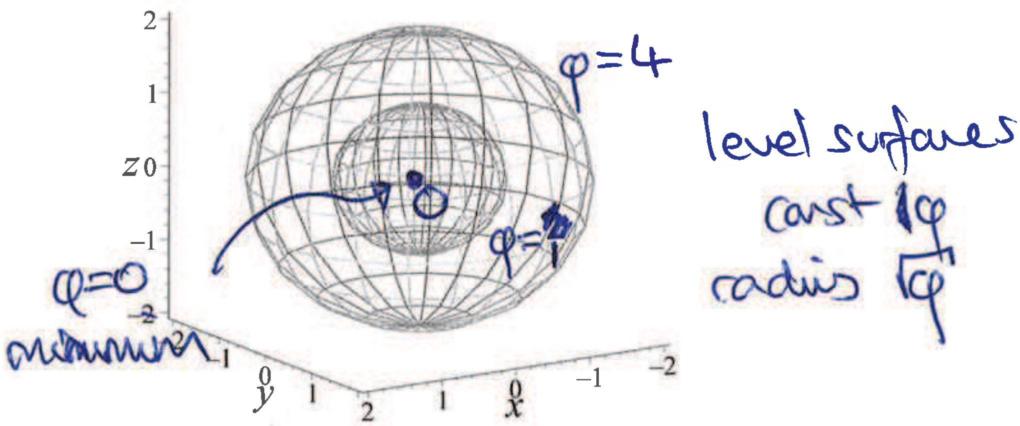
$$\text{ellipse} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{and hyperbola} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

In 3D, where $\phi(\boldsymbol{r}) = \phi(x, y, z)$, level sets ϕ =const. are surfaces.

Examples

$= r^2$

- for $\phi = x^2 + y^2 + z^2$ the condition $\phi =$ const. defines the equation for a sphere, hence the level surfaces are a set of nested spheres.



- Imagine surfaces of equal temperature in the thermal boundary layer around a body in a cold bath.

3D : $\underline{v} : \mathbb{R}^3 \to \mathbb{R}^3$

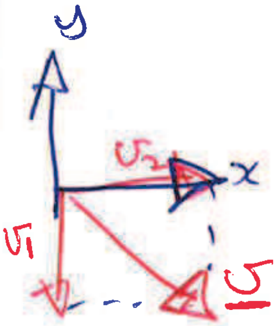
2D : $\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$

1.7 Vector fields

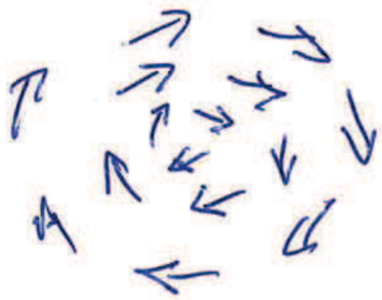
Vector fields are harder to visualise. One needs to represent vectors (with magnitude and direction) at each point in space. One way to do this is by using arrows of varying length or thickness.

In 2D where $\boxed{\boldsymbol{v} = v_1(x,y)\boldsymbol{i} + v_2(x,y)\boldsymbol{j},}$ can interpret this as

- The wind field plots on TV weather forecasts (at given altitude).



- A vertical view of a partially flattened field of corn (crop circles!)



The plot can be constructed by considering at a grid of points x_i, y_j

$$\text{magnitude} \quad |\boldsymbol{v}| = \sqrt{v_1^2(x_i,y_j) + v_2^2(x_i,y_j)}$$

$$\text{\& direction} \quad \theta = \tan^{-1}(v_2(x_i,y_j)/v_1(x_i,y_j))$$

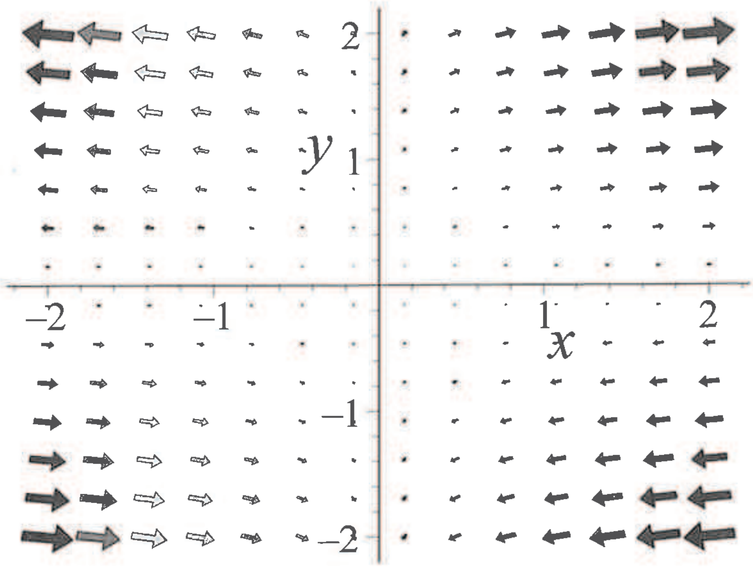
It is not practical (or examinable) to produce such pictures by hand. Computer algebra packages like MATLAB can produce such pictures and they give invaluable insight.

$v_1(x,y)$ $v_2(x,y)$

Example $\boldsymbol{v}(x, y) = 4xy\boldsymbol{i} + y\boldsymbol{j}$

$$\Rightarrow |\boldsymbol{v}| = y\sqrt{1 + 16x^2}; \qquad \tan\theta = 1/(4x) = v_2/v_1$$

$\sqrt{v_1^2 + v_2}$



here does not depend upon y

$\therefore$ all parallel at same x-co-ordt

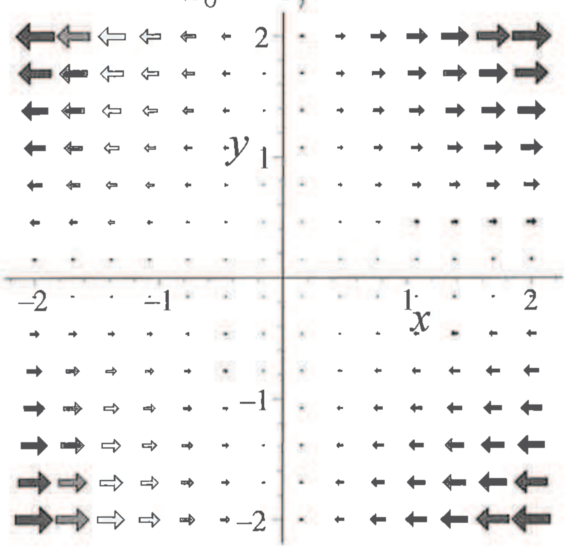
In 3D, where $\boldsymbol{v} = v_1(x, y, z)\boldsymbol{i} + v_2(x, , zy)\boldsymbol{j} + v_3(x, y, z)\boldsymbol{k}$, a good way to visualise the field is to take 2D sections.

Example $\boldsymbol{v}(x, y, z) = 4xy\boldsymbol{i} + yz\boldsymbol{j} + x\boldsymbol{k}$.

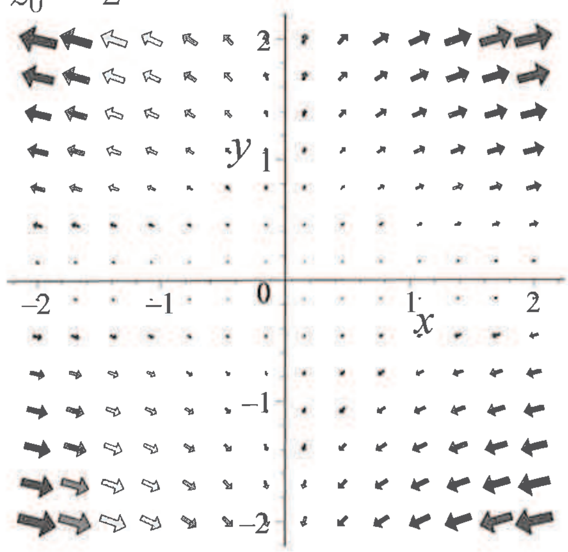
e.g., take sections $z = z_0$ and draw only x, y components

$$|\boldsymbol{v}| = y\sqrt{z_0^2 + 16x^2}; \qquad \tan\theta = z_0/4x$$

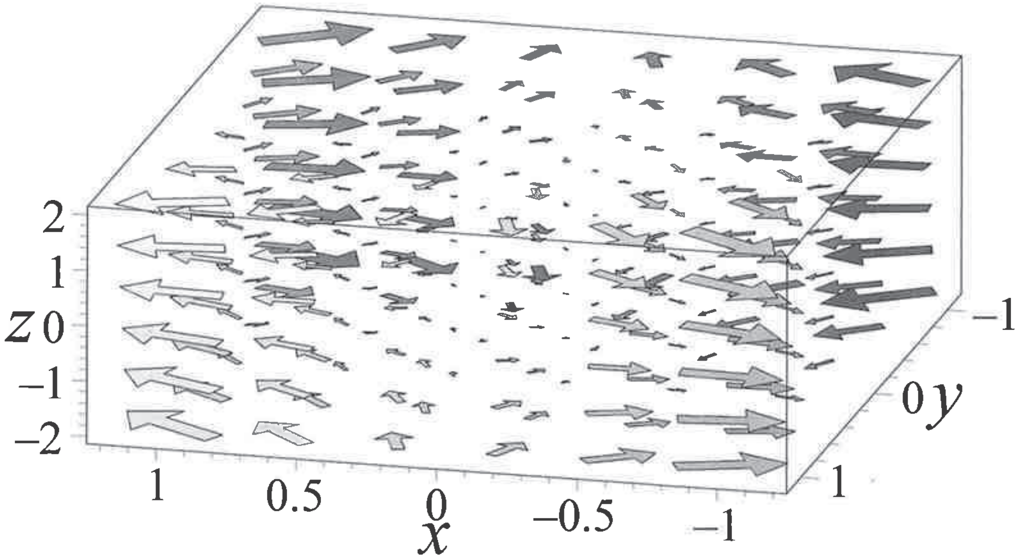
$z_0 = 0$,



$z_0 = 2$



Alternatively by using rotation to view a 3D image, computer graphics (e.g. MATLAB) can plot the true arrows (not just their components in plane slices) with thickness or length representing magnitude.



Maple commands for figures in this Chapter:

```
> with(plots):
> contourplot(sin(x^2/8+(x/2)+y)*sin(x-y/3-y^3/12),x=-3..3,y=-3..3);
> plot3d(sin(x^2/8+(x/2)+y)*sin(x-y/3-y^3/12),x=-3..3,y=-3..3);
> contourplot(x^2+y^2,x=-3..3,y=-3..3);
> contourplot(x*y,x=-3..3,y=-3..3);
>sp1 := [seq([ seq([sin(i*Pi/10)*cos(j*Pi/10),sin(i*Pi/10)*sin(j*Pi/10),
cos(i*Pi/10)], i=0..20)], j=0..20)]:
>sp2 := [seq([ seq([2*sin(i*Pi/10)*cos(j*Pi/10),2*sin(i*Pi/10)*sin(j*Pi/10),
2*cos(i*Pi/10)], i=0..20)],j=0..20)]:
surfdata({sp1,sp2},style=WIREFRAME, axes=framed)
> fieldplot( [4*x*y,y],x=-2..2,y=-2..2,arrows=THICK,grid=[14,14]);
> fieldplot3d([4*x*y,z*y,x],x=-1..1,y=-1..1,z=-2..2,grid=[6,6,6],arrows=THICK);
```

Matlab commands for figures in this Chapter:

```
> x=-3:0.1:3; y = -3:0.1:3; [X,Y] = meshgrid(x,y);
> contour(X,Y,sin(X.^2/8+(X/2)+Y).*sin(X-Y/3-Y.^3/12))
> plot3(X,Y,sin(X.^2/8+(X/2)+Y).*sin(X-Y/3-Y.^3/12))
> contour(X,Y,X.^2+Y.^2)
> contour(X,Y,X.*Y)
> x=-2:0.1:2; y = -2:0.1:2; [X,Y] = meshgrid(x,y);
> plot3(X,Y,sqrt(9-X.^2-Y.^2), X,Y,-sqrt(9-X.^2-Y.^2)); hold on
> plot3(0.5*X,0.5*Y,0.5*sqrt(9-X.^2-Y.^2), 0.5*X,0.5*Y,-0.5*sqrt(9-X.^2-Y.^2))
> x=-2:0.4:2; y = -2:0.4:2; [X,Y] = meshgrid(x,y);
> quiver(X,Y,4*X.*Y,Y)
> x=-1:0.4:1; y=-1:0.4:1;z=-2:0.5:1; [X,Y,Z]=meshgrid(x,y,z);
> quiver3(X,Y,Z,4.*X.*Y,Z.*Y,X)
```

1.8 Summary

- Scalar functions $f(x)$ or $f(t)$;
  vector functions $\boldsymbol{v}(x)$ or $\boldsymbol{v}(t)$;
  scalar fields $\phi(\boldsymbol{r})$;
  vector fields $\boldsymbol{v}(\boldsymbol{r})$
- Vector fields: differentiation

$$\frac{d}{dt}\boldsymbol{v}(t) = \frac{d}{dt}v_1(t)\boldsymbol{i} + \frac{d}{dt}v_2(t)\boldsymbol{j} + \frac{d}{dt}v_3(t)\boldsymbol{k}$$

- Rules of vector field differentiation:

$$\begin{aligned}
(\boldsymbol{u}+\boldsymbol{v})' &= \boldsymbol{u}'+\boldsymbol{v}' \\
(c\boldsymbol{u})' &= c\boldsymbol{u}' \\
(\boldsymbol{u}\cdot\boldsymbol{v})' &= \boldsymbol{u}'\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{v}' \\
(\boldsymbol{u}\times\boldsymbol{v})' &= \boldsymbol{u}'\times\boldsymbol{v}+\boldsymbol{u}\times\boldsymbol{v}'
\end{aligned}$$

- Visualization: scalar field level curves $\phi(\boldsymbol{r})$ =const.
- Visualization: vector field quiver plot