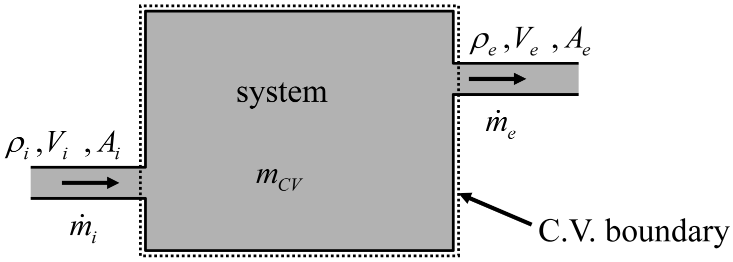
Introduction to Control Volume Analysis

- We have already used control volume analysis in the derivation of:
  - The hydrostatic equation
  - Mass conservation in 1D flow and
  - Bernoulli's equation
- Whilst obviously useful in the derivation of the above, an understanding of control volume analysis allows you to apply your fluids knowledge to a large number of engineering situations.
- What is Control Volume (CV) analysis?
  - A process where we draw an imaginary boundary, associated with a system, that encloses the desired CV. We then consider the rate at which: mass, linear momentum, angular momentum and energy; enter and leave the CV. By careful choice of our CV boundary (surrounding the system except at important "cuts") we may derive useful properties, such as forces or moments, on the system.
- In general we can consider CV boundaries that accelerate and deform but we will only consider: fixed control volumes with zero or constant speeds.
- We will also restrict ourselves to steady incompressible flows

Conservation of Mass for a Control Volume

- The rate of change of mass inside a control volume is equal to the mass flow rate in minus the mass flow rate out.



- For unsteady compressible and incompressible we can write

$$\frac{dm_{CV}}{dt} = \dot{m}_i - \dot{m}_e = \rho_i V_i A_i - \rho_e V_e A_e$$

- Where the inlet and exit areas must me measured perpendicular to the velocities
- For steady problems (values are constant at each location)

$$\frac{dm_{CV}}{dt} = 0 \qquad \therefore \quad \rho_i V_i A_i = \rho_e V_e A_e$$

- For steady incompressible flow

$$V_i A_i = V_e A_e$$

Conservation of Steady Flow Linear Momentum

- Consider Newtons 2nd law applied to a point mass. In vector form we write

$$\underline{\mathbf{F}}_{\text{tot}} = m\frac{d\underline{\mathbf{V}}}{dt} \qquad \underline{\mathbf{F}}_{\text{tot}} = \begin{bmatrix} f_{\text{tot}_x} \\ f_{\text{tot}_y} \\ f_{\text{tot}_z} \end{bmatrix} \qquad \underline{\mathbf{V}} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Generally:
Force=rate of change of momentum
For point mass F=ma

where $\underline{\mathbf{F}}_{\text{tot}}$ is the vector sum of all forces acting on the mass.

- Note how we can treat each coordinate direction separately.

We will consider the momentum change in x then generalise to y & z

- Consider a CV with initial momentum in x of M_x. Over time δt, the masses δm_i and δm_e enter and exit the CV with velocities in the x direction of V_{ix} and V_{ex} respectively.
- For steady control volumes the change of momentum is only due to momentum flux so

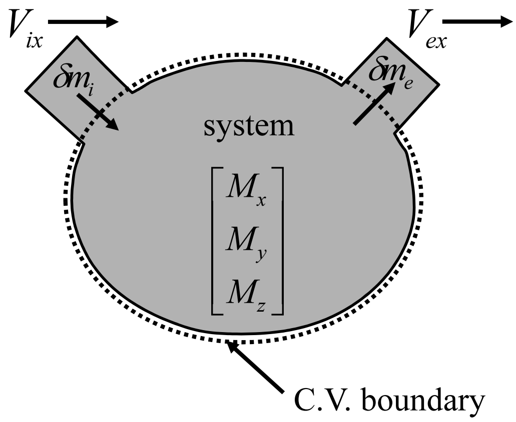
$$\delta M_x = (V_{ix}\delta m_i - V_{ex}\delta m_e)$$

- Further for steady flow $\delta m_i = \delta m_e = \delta m$
- Dividing throughout by δt and taking limits

$$\left.\begin{aligned} f_{\text{tot}_x} &= \dot{M}_x = \dot{m}(V_{ix} - V_{ex}) \\ f_{\text{tot}_y} &= \dot{M}_y = \dot{m}(V_{iy} - V_{ey}) \\ f_{\text{tot}_z} &= \dot{M}_z = \dot{m}(V_{iz} - V_{ez}) \end{aligned}\right\}$$

steady flow through fixed control volume (can be moving at constant speed)

This is the force exerted **BY** the control volume

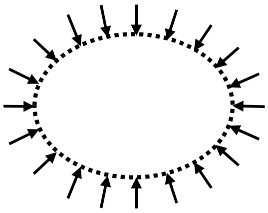


Conservation of Steady Flow Linear Momentum (2)

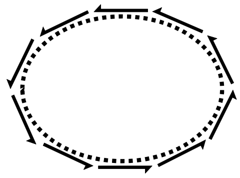
- From previous, the total force exerted **ON** a control volume is given by

$$f_{\text{tot}_x} = -\dot{m}\left(V_{ix} - V_{ex}\right) = \dot{m}\left(V_{ex} - V_{ix}\right)$$
$$f_{\text{tot}_y} = -\dot{m}\left(V_{iy} - V_{ey}\right) = \dot{m}\left(V_{ey} - V_{iy}\right)$$
$$f_{\text{tot}_z} = -\dot{m}\left(V_{iz} - V_{ez}\right) = \dot{m}\left(V_{ez} - V_{iz}\right)$$

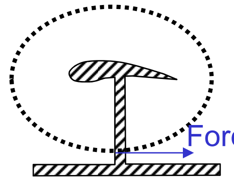
- Each total/net force component can be split into
- Pressure forces acting normally to the external CV boundary. Note that a constant pressure integrates to zero so we can subtract/add a constant from/to the pressure. Eg subtract atmospheric pressure & integrate gauge pressures



- Viscous shear forces acting tangentially to the external control surface boundary. Note that for pressure and viscous forces we are only interested in the boundary.



- Forces exerted on the fluid by solid objects. Whenever a solid surface cuts the CV boundary there is a net resultant force *e.g.* if the drag on body is D, the resultant force on the fluid is –D.



- Gravitational body forces are integrated over the whole control volume. All the other forces act on CV surface only.

Conservation of Energy

See White: section 3.6

- From the 1st Law of Thermodynamics $\delta E = \delta Q - \delta W$

Energy Heat Work done by the CV

- Decompose the work term into "shaft" "viscous" and "pressure" terms

$$\delta W = \delta W_s + \delta W_v + \delta W_p \qquad \delta W_p = p_e \delta m_e / \rho_e - p_i \delta m_i / \rho_i$$

Shaft work isolates the work (pressure & viscous work) done by pumps, fans & pistons

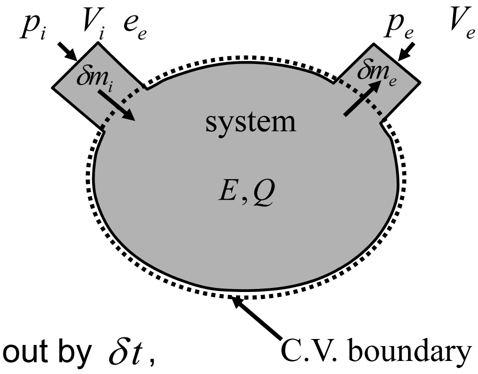
- Consider a CV. Over time δt, the masses δm_i and δm_e enter and exit the CV with velocities pressures and energies (per unit mass) $(V, p, e)_i$ and $(V, p, e)_e$ respectively.

$$\delta E = \delta Q - \delta W_s - \delta W_v - \left(p_e/\rho_e\, \delta m_e - p_i/\rho_i\, \delta m_i\right)$$
$$\delta E = e_e \delta m_e - e_i \delta m_i$$
$$\delta Q - \delta W_s - \delta W_v = \left(p_e/\rho_e + e_e\right)\delta m_e - \left(p_i/\rho_i + e_i\right)\delta m_i$$

$$e = \hat{e} + \tfrac{1}{2}V^2 + gz$$

Internal molecular kinetic gravitational potential

$$\delta Q - \delta W_s - \delta W_v = \left(p_e/\rho_e + \hat{e}_e + \tfrac{1}{2}V_e^2 + gz_e\right)\delta m_e - \left(p_i/\rho_i + \hat{e}_i + \tfrac{1}{2}V_i^2 + gz_i\right)\delta m_i$$



- For steady flow control volumes, dividing throughout by δt, taking limits and using $\dot{m}_i = \dot{m}_e = \dot{m}$ for steady flow

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \left(p_e/\rho_e + \hat{e}_e + \tfrac{1}{2}V_e^2 + gz_e - p_i/\rho_i - \hat{e}_i - \tfrac{1}{2}V_i^2 - gz_i\right)\dot{m}$$

Conservation of Energy (2)

- Continuing from previous slide and dividing throughout by $\dot{m}$

$$p_i/\rho_i + \tfrac{1}{2}V_i^2 + gz_i = p_e/\rho_e + \tfrac{1}{2}V_e^2 + gz_e + (\hat{e}_e - \hat{e}_i) - q + w_s + w_v$$

$$q = \frac{\dot{Q}}{\dot{m}} = \frac{dQ}{dm} \qquad w_s = \frac{dW_s}{dm} \qquad w_v = \frac{dW_v}{dm}$$

- Each term has the dimensions of energy per unit mass (m^2/s^2)
- Internal molecular energy is dependant on temperature. Hence for incompressible steady flow we have: ρ constant, $\hat{e}$ constant, adiabatic flow with $q = 0$
- Finally removing the viscous work term we have, for steady inviscid incompressible flow

$$p_i + \tfrac{1}{2}\rho V_i^2 + \rho g z_i = p_e + \tfrac{1}{2}\rho V_e^2 + \rho g z_e + \rho w_s$$

- This is Bernoulli's equation which can be written as

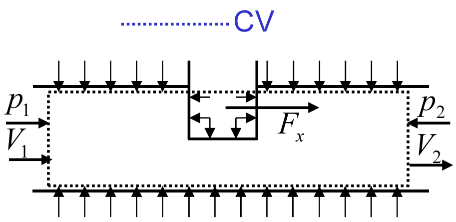
total pressure inlet = total pressure outlet + shaft work done by the CV

Note that we normally consider shaft work as being done TO the fluid so that this term is negative

Do not learn this derivation, just the principle that we have shaft work, heat losses etc that change the total pressure

Example 1: Forces on an Obstruction in a Pipe

- Water at an absolute pressure of 10bar flows steadily along a horizontal pipe of circular cross section and internal diameter 10cm. At some point in the pipe there is a partially closed valve and far downstream the pressure is 9bar. Assuming the flow is inviscid, find the force exerted by the water on the valve.



- Assumptions: Frictionless & incompressible

  Straight streamlines at inlet & outlet (1D approx)

  Horizontal so no hydrostatic terms
- Continuity: $V_1 = V_2$
- Bernoulli's equation (not needed for this Q)

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_1 + \tfrac{1}{2}\rho V_2^2 + \Delta p_{\text{loss}} \qquad \Delta p_{loss} = \rho w_v - \rho q$$

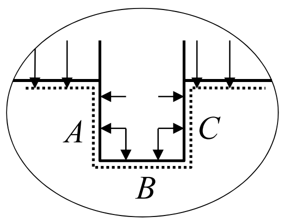
- Steady Flow momentum in x-direction: $(p_1 - p_2)A + F_x = \dot{m}(V_2 - V_1) = 0$

$F_x = -(10\times10^5 - 9\times10^5)\times\pi\times0.05^2 = -785N$ ← Force exerted by valve on CV. Ans=785N

- Notes: The CV cuts the valve. See Slide 5

  Could use a CV that goes around the valve but we would need to know the pressures at A and C

  We have used the pressure drop to find the force but we could find the force and calculate the pressure drop as we do for wind tunnel testing.



Example 2: Turning pipe

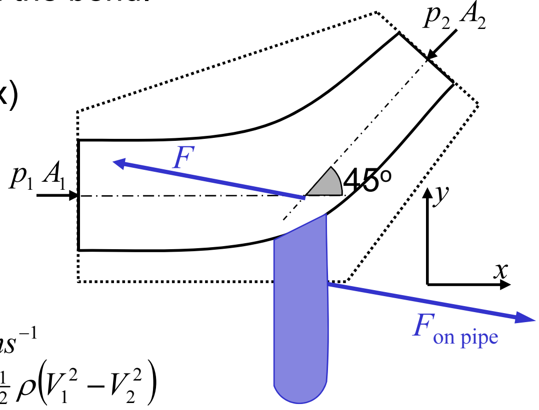
- A 45° reducing pipe-bend (in a horizontal plane) tapers from a 600mm diameter inlet to a 300mm diameter outlet. The gauge pressure at inlet is 140kPa and the rate of flow of water through the bend is 0.425m³/s. Neglecting friction, calculate the net resultant horizontal force exerted by the water on the bend.
- Assumptions: Frictionless & incompressible

  Straight streamlines at inlet & outlet (1D approx)

  Horizontal so no hydrostatic terms

  Only pressure forces acting on pipe walls

  p_a acts on entire CV (so no net force)



- Continuity: $V_1A_1 = V_2A_2$

  $V_1 = Q/A_1 = 0.425/\left(\pi\times 0.3^2\right) = 1.503ms^{-1} \quad V_2 = 6.01ms^{-1}$
- Bernoulli's equation between 1 & 2: $p_2 = p_1 + \frac{1}{2}\rho\left(V_1^2 - V_2^2\right)$

  $p_2 = 1.4\times10^5 + \frac{1}{2}\times 1000\times\left(1.503^2 - 6.01^2\right) = 1.231\times10^5\,\text{pa (gauge)}$
- Steady Flow momentum in x-direction: $p_1A_1 - p_2A_2\cos 45^o + F_x = \rho Q\left(V_2\cos 45^o - V_1\right)$

  $F_x = -1.4\times10^5\times\pi\times0.3^2 + 1.231\times10^5\times\pi\times0.15^2\cos 45^o + 1000\times0.425\left(6.01\cos 45^o - 1.503\right) = -32264\,N$
- Steady flow momentum in y-direction: $-p_2A_2\sin 45^o + F_y = \rho Q\left(V_2\sin 45^o - 0\right)$

  $F_y = 1.231\times10^5\times\pi\times0.3^2\sin 45^o + 1000\times0.425\times6.01\sin 45^o = 7959\,N$

$$F = \sqrt{F_x^2 + F_y^2} = 33231\,N \qquad \tan\theta = \frac{F_x}{F_y} \rightarrow \theta = 180^o - 13.86^o$$

Force exerted on pipe is opposite θ= -13.86°

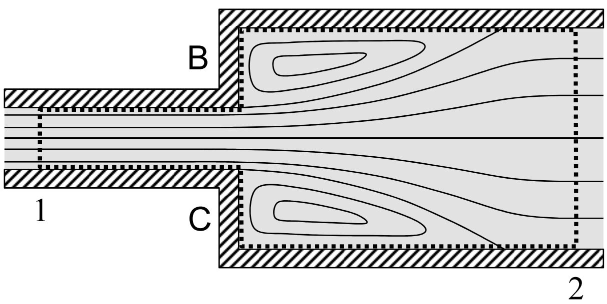
Application 1: Loss at an abrupt enlargement

- Assumptions: Straight streamlines at inlet & outlet so uniform conditions at 1 & 2

  Steady Frictionless & incompressible

  $p_C = p_B \approx p_1$

  Low curvature at the enlargement and low velocity flow in the recirculation regions.



- Continuity: $Q = A_1V_1 = A_2V_2$
- Bernoulli's equation between 1 & 2:

  $p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 + \Delta p_{loss}$
- Steady Flow momentum in x-direction:

  $p_1A_1 + p_1\left(A_2 - A_1\right) - p_2A_2 = \rho Q\left(V_2 - V_1\right) \quad \rightarrow \quad \left(p_1 - p_2\right)A_2 = \rho Q\left(V_2 - V_1\right)$
- Rearranging these 3 equations

  $\Delta p_{loss} = p_1 - p_2 + \frac{1}{2}\rho\left(V_1^2 - V_2^2\right) \quad \rightarrow \quad \left(p_1 - p_2\right)A_2 = \rho V_1A_1\left(V_1\frac{A_1}{A_2} - V_1\right)$

  $\Delta p_{loss} = \rho V_1^2\frac{A_1}{A_2}\left(\frac{A_1}{A_2} - 1\right) + \frac{1}{2}\rho V_1^2\left(1 - \left(\frac{A_1}{A_2}\right)^2\right) \quad = \frac{1}{2}\rho V_1^2\left[2\frac{A_1}{A_2}\left(\frac{A_1}{A_2} - 1\right) + \left(1 - \frac{A_1}{A_2}\right)\left(1 + \frac{A_1}{A_2}\right)\right]$

  $\Delta p_{loss} = \frac{1}{2}\rho V_1^2\left(1 - \frac{A_1}{A_2}\right)^2$
- Define $\text{Loss Coefficient} = \dfrac{\Delta p_{loss}}{\frac{1}{2}\rho V_1^2} = \left(1 - \frac{A_1}{A_2}\right)^2$

A_1/A_2	Theory	Exp't
0.25	0.56	0.60
0.174	0.68	0.74

Table of loss coefficients

Application 2: Loss at an abrupt contraction

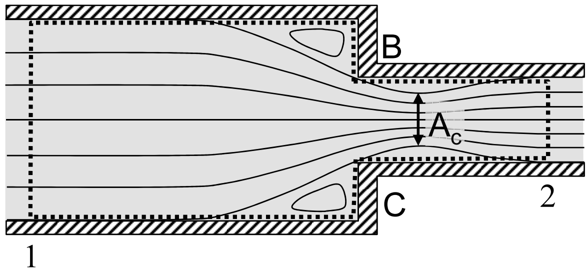
- Assumptions: Straight streamlines at inlet & outlet so uniform conditions at 1 & 2

  Steady Frictionless & incompressible

  But unknown pressures p_C , p_B

  Analysis as before but applied

  between the Vena-Contracta & 2



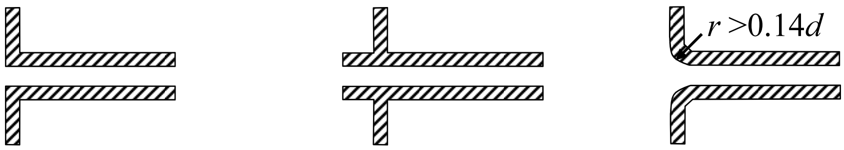
- So

$$\Delta p_{loss} = \tfrac{1}{2}\rho V_2^2\left(1-\tfrac{A_2}{A_c}\right)^2 = \tfrac{1}{2}\rho V_2^2 k$$

- A_c and therefore the loss coefficient k, are functions of the area ratio A_2/A_1
- Must find these from experiment, for Circular ducts with diameters d_1 & d_2

d_2/d_1	0	0.2	0.4	0.6	0.8	1
k	0.5	0.45	0.38	0.28	0.14	0

- Entry loss dependant on geometry (Don't memorise the numbers)



Sharp edged k=0.5 | Protruded k=1.0 | Rounded k=0

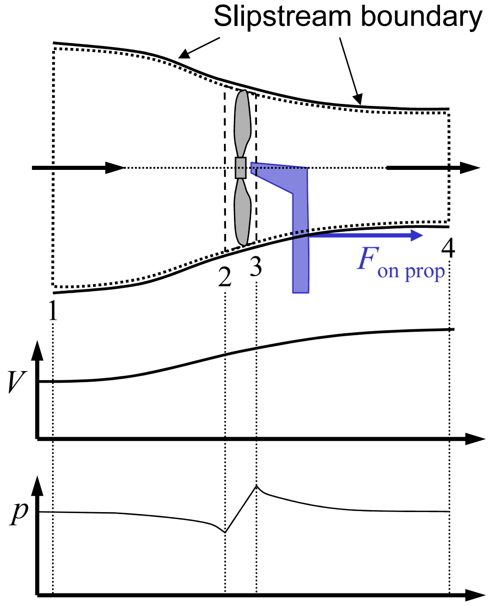
Application 3: Actuator Disc Theory: Propeller

- Propeller does work on the fluid (shaft work) and the increase in momentum gives thrust to the disc.
- Assumptions: Frictionless & incompressible

  Steady 1D flow (neglect rotation and variation across the disc radius)

  Actuator disc is thin so $A_2=A_3=A_d$ & $V_2=V_3=V_d$

  $p=p_a$ at all points on slipstream boundary & 1 & 4



- Continuity: $Q = V_d A_d$
- Bernoulli's equation for CV 1-2 & CV 3-4

$$\left.\begin{aligned} p_1 + \tfrac{1}{2}\rho V_1^2 &= p_2 + \tfrac{1}{2}\rho V_d^2 \\ p_3 + \tfrac{1}{2}\rho V_d^2 &= p_4 + \tfrac{1}{2}\rho V_4^2 \end{aligned}\right. \rightarrow \quad p_3 - p_2 = \tfrac{1}{2}\rho\left(V_4^2 - V_1^2\right)$$

- Steady Flow momentum for CV 1-4:

$$0 + F = \rho Q\left(V_4 - V_1\right)$$

- Steady Flow momentum for CV 2-3: $\left(p_2 - p_3\right)A_d + F = \rho Q\left(V_d - V_d\right) = 0 \rightarrow F = \left(p_3 - p_2\right)A_d$
- From momentum & continuity $\left(p_3 - p_2\right) = \rho V_d\left(V_4 - V_1\right)$
- Eliminating p_3-p_2 using Bernoulli's equation above $V_d = \tfrac{1}{2}\left(V_1 + V_4\right)$

F = force of disc on air in flow direction &

force of air on disc in direction of travel

Application 3: Actuator Disc Theory: Propeller(2)

- For a stationary rotor (fixed fan or helicopter in hover)

$$V_1 = 0 \quad \rightarrow \quad V_4 = 2V_d$$

$$F = \rho A_d V_d (2V_d - 0) = 2\rho A_d V_d^2$$

$$V_d = \sqrt{F/2\rho A_d}$$ So we can relate the required force to the disc velocity

- For a propeller with a forward velocity v into still air, and a final air velocity relative to the disc of V_4 we have Galilean transformation

$$V_1 = v \quad \rightarrow \quad V_d = \tfrac{1}{2}(V_4 + v)$$

$$F = \rho A_d V_d (V_4 - v) = \tfrac{1}{2}\rho A_d (V_4^2 - v^2) = \tfrac{1}{2}\rho A_d v^2 a(1+a) \qquad \text{where} \quad a = \frac{V_d - v}{v} \quad \text{is the "inflow factor"}$$

- The power supplied to the disc to produce the thrust ("ideal" power input) is

$$FV_d = \rho Q(V_4 - V_1)V_d = \tfrac{1}{2}\rho Q(V_4 - V_1)(V_4 + V_1) = \tfrac{1}{2}\rho Q(V_4 - V_1)(V_4 - V_1) + \rho Q(V_4 - V_1)V_1$$

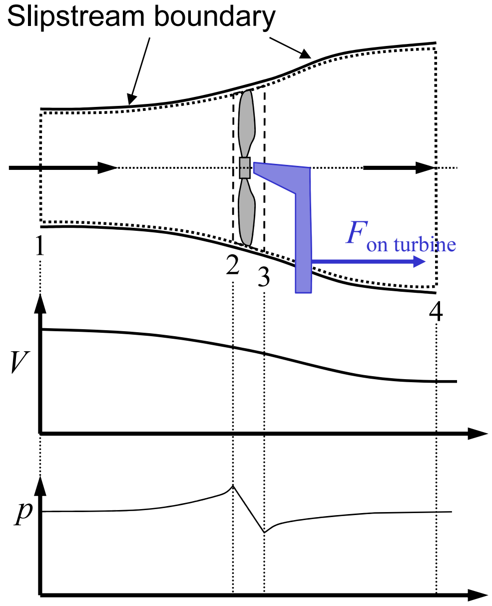
- For a hovering rotor $FV_d = F\sqrt{F/2\rho A_d} = F^{\frac{3}{2}}\big/\sqrt{2\rho A_d}$
- The power put into the air (effective power output) is given by

$$FV_1 = \rho Q(V_4 - V_1)V_1 \qquad = Fv \qquad \text{for forward flight}$$

- Propulsive efficiency (for forward flight) defined as the ratio of power input & output

$$\eta = \frac{\rho Q(V_4 - V_1)}{\rho Q(V_4 - V_1)}\frac{V_1}{\tfrac{1}{2}(V_4 - V_1) + V_1} = \frac{2V_1}{V_4 + V_1}$$

Application 4: Vertical axis Turbine Actuator disc



- Fluid does work on the turbine (shaft work) and the decrease in momentum gives thrust to the disc.
- Assumptions: Frictionless & incompressible
  Steady 1D flow (neglect rotation and variation across the disc radius)
  Actuator disc is thin so $A_2 = A_3 = A_d$ & $V_2 = V_3 = V_d$
  $p = p_a$ at all points on slipstream boundary & 1 & 4
- Continuity: $Q = V_d A_d$
- Bernoulli's equation for CV 1-2 & CV 3-4

$$\begin{aligned} p_1 + \tfrac{1}{2}\rho V_1^2 &= p_2 + \tfrac{1}{2}\rho V_d^2 \\ p_3 + \tfrac{1}{2}\rho V_d^2 &= p_4 + \tfrac{1}{2}\rho V_4^2 \end{aligned} \quad \rightarrow \quad p_3 - p_2 = \tfrac{1}{2}\rho(V_4^2 - V_1^2)$$

- Steady Flow momentum for CV 1-4:

$$0 + F = \rho Q(V_4 - V_1)$$

- Steady Flow momentum for CV 2-3: $(p_2 - p_3)A_d + F = \rho Q(V_d - V_d) = 0 \rightarrow F = (p_3 - p_2)A_d$
- From momentum & continuity $(p_3 - p_2) = \rho V_d (V_4 - V_1)$
- Eliminating $p_3 - p_2$ using Bernoulli's equation above $V_d = \tfrac{1}{2}(V_1 + V_4)$

Analysis identical to previous but force on fluid has changed sign.

Application 4: Vertical axis Turbine(2)

- The power drawn from the air by the disc is

$$P_{\text{disc}} = -FV_d = -\rho Q(V_4 - V_1)V_d = \rho A_d V_d (V_1 - V_4)V_d = \tfrac{1}{4}\rho A_d (V_4 + V_1)(V_1^2 - V_4^2)$$

- Power in the wind passing through the disc area, if the disc were not present.

$$P_{\text{wind}} = \tfrac{1}{2}\dot{m}V_1^2 = \tfrac{1}{2}\rho A_d V_1\, V_1^2 = \tfrac{1}{2}\rho A_d V_1^3$$

- The efficiency of the turbine is therefore

$$\eta = \frac{P_{\text{disc}}}{P_{\text{wind}}} = \frac{\tfrac{1}{4}\rho A_d (V_4 + V_1)(V_1^2 - V_4^2)}{\tfrac{1}{2}\rho A_d V_1^3} = \frac{(V_4 + V_1)(V_1^2 - V_4^2)}{2V_1^3}$$

- Differentiating w.r.t V_4 and equating to zero defines the minima. From this we find efficiency is a maximum when

$$\frac{V_4}{V_1} = \frac{1}{3}$$

$$\eta_{\max} = \frac{V_1^3}{V_1^3}\frac{\left(\tfrac{1}{3}+1\right)\left(1-\tfrac{1}{9}\right)}{2} = 0.59$$

$$\frac{\partial \eta}{\partial V_4} = \frac{(V_1^2 - V_4^2) - 2V_4(V_1 + V_4)}{2V_1^3} = 0$$

$$3V_4^2 + 2V_4V_1 - V_1^2 = 0$$

$$V_4 = \frac{-2V_1 \pm \sqrt{4V_1^2 + 12V_1^2}}{6}$$

$$\frac{V_4}{V_1} = \frac{-2 \pm 4}{6} \quad \rightarrow \quad \frac{V_4}{V_1} = \begin{cases} \tfrac{1}{3} & \text{max} \\ -1 & \text{min} \end{cases}$$

- In reality $\eta \approx 0.15$, efficiency not the design driver

$$\eta \rightarrow 0.3$$

Example 3: Windmill

- An ideal windmill, 12m diameter, operates at a theoretical efficiency of 50% in a 14m/s wind. If the air density is 1.235 kg/m³ determine the thrust on the windmill, the air velocity through the disc, the mean gauge pressures immediately in front of and behind the disc, and the shaft power developed.
- Use previous results for vertical axis turbine & assume same figure labeling

$$\eta = 0.5 = \frac{(V_4 + V_1)(V_1^2 - V_4^2)}{2V_1^3} = \frac{(V_4 + 14)(14^2 - V_4^2)}{2\times 14^3}$$

$$14^2 V_4 - 14V_4^2 - V_4^3 = 0$$

$$(14^2 - 14V_4 - V_4^2)V_4 = 0$$

- Solving (neglect negative root and zero root) $V_4 = 8.65ms^{-1}$

$$V_d = \tfrac{1}{2}(V_1 + V_4) = 11.13ms^{-1}$$

- Steady Flow momentum : $F = \rho Q(V_4 - V_1)$ $\quad F = \rho A_d V_d (V_4 - V_1) = \tfrac{1}{2}\rho A_d (V_4^2 - V_1^2)$

$$F = \tfrac{1}{2}1.235\times\pi\times 6^2(8.65^2 - 14^2) = -8463N$$

Thrust on windmill =8463N

- Bernoulli's equation for CV 1-2 $\quad p_2 - p_1 = \tfrac{1}{2}\rho(V_1^2 - V_2^2)$

$$p_2 = \tfrac{1}{2}\times 1.235\times(14^2 - 11.33^2) = 41.8\text{pa} \qquad (\text{gauge as } p_1 \text{ atmosperic})$$

- Bernoulli's equation for CV 3-4 $\quad p_3 - p_4 = \tfrac{1}{2}\rho(V_4^2 - V_3^2)$

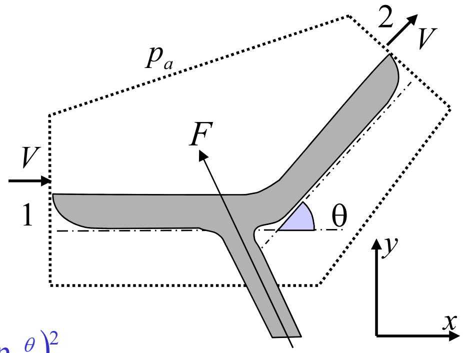
$$p_3 = \tfrac{1}{2}\times 1.235\times(8.65^2 - 11.33^2) = -33.1\text{pa} \qquad (\text{gauge as } p_4 \text{ atmosperic})$$

- Power $\quad P_{\text{disc}} = -FV_d = \tfrac{1}{2}\rho A_d V_d^2 (V_1 - V_4)$

$$p_3 = \tfrac{1}{2}\times 1.235\times\pi\times 6^2\times 11.33^2(14^2 - 8.65^2) = 95.9kW$$

Example 4: Turning vane

- A fixed vane turns a water jet of area A through an angle θ without changing the magnitude of its velocity The flow is steady, pressure is p_a everywhere and friction on the vane is negligible. (a) Find the component of force F_x and F_y that the flow applies to the vane (b) Find the expression for the force magnitude F and the angle ϕ between F and the horizontal.



- (a) Continuity $\dot{m} = \rho A V$
- Steady Flow momentum in x :

$$F_x = \dot{m}(V\cos\theta - V) = \dot{m}V(\cos\theta - 1)$$

- Steady Flow momentum in y :

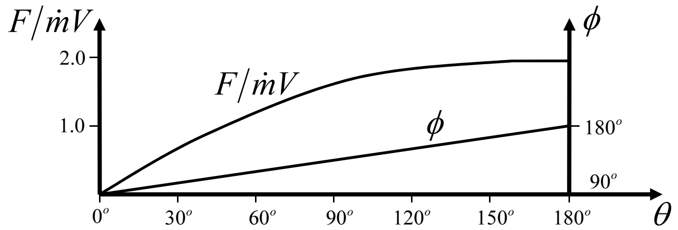
$$F_y = \dot{m}(V\sin\theta - 0) = \dot{m}V\sin\theta$$

- (b) $F = \sqrt{F_x^2 + F_y^2} = \dot{m}V\left(\sin^2\theta + (\cos\theta - 1)^2\right)^{\frac{1}{2}}$

$$F = \dot{m}V(2 - 2\cos\theta)^{\frac{1}{2}} = 2\dot{m}V\sin\tfrac{\theta}{2} \qquad 2(1-\cos\theta) = 4\left(\sin\tfrac{\theta}{2}\right)^2$$

$$\phi = \pi - \tan^{-1}\left|\frac{F_y}{F_x}\right| = \pi - \tan^{-1}\left|\frac{\sin\theta}{\cos\theta - 1}\right| = \pi - \tan^{-1}\left|\cot\frac{\theta}{2}\right| = \pi - \tan^{-1}\left|\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right|$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}$$



Learning Outcomes: "What you should have learnt"

- How to apply the principle of mass conservation to a range of examples
- How to apply the linear momentum conservation principle to a range of examples

  You will need to remember that $f_{\text{tot}_x} = \dot{m}(V_{ex} - V_{ix})$ with similar for y & z
- Understanding Bernoulli's equation as a conservation of energy, and how terms representing viscous work, shaft work and heat losses for the CV can be included as pressure loss terms.
- How to apply the conservation of energy to a range of examples.
- You should be able to reproduce the derivations involved in the analysis of: the sudden expansion; the actuator disc theory of propellers and axial turbines.
- You should understand: the nature of flow in a sudden expansion; how to apply CV analysis to this situation and the effects of inlet design on pressure losses.

Remember: Being able to apply these principles to simple systems is the most important part of this section. You do not need to memorise the derivations for energy momentum and mass conservation.