Advanced Bending and Torsion

Unsymmetric Bending Example: L-section Beam

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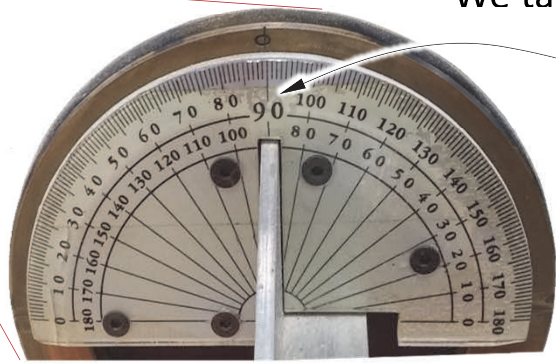
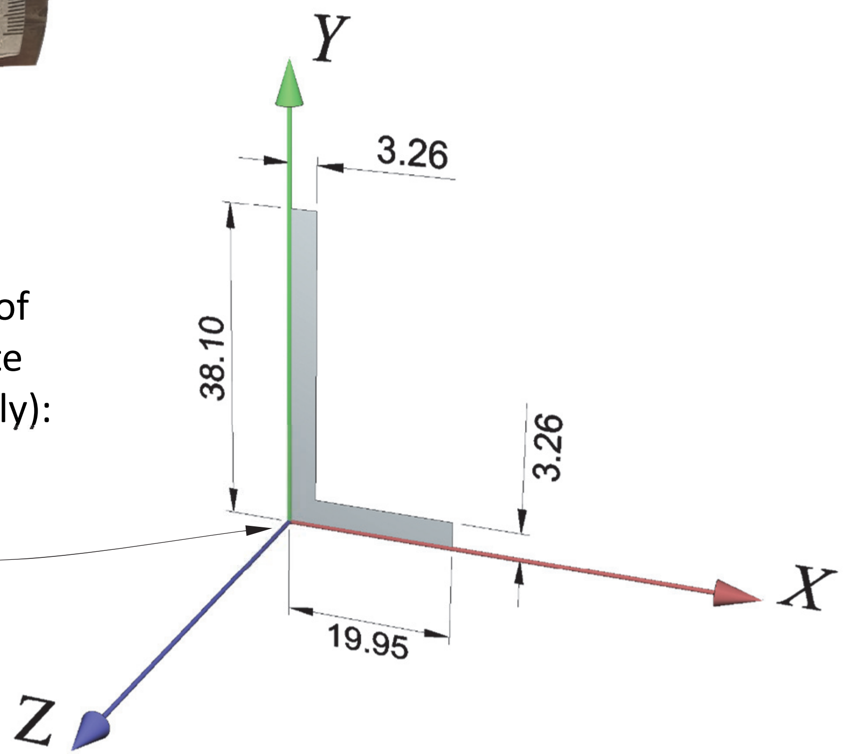
L-section Cantilever Beam

- Let us analyse the L-section beam seen in the Structures Lab:



We take this orientation as a reference (arbitrarily)

We place the origin of the global coordinate system here (arbitrarily):



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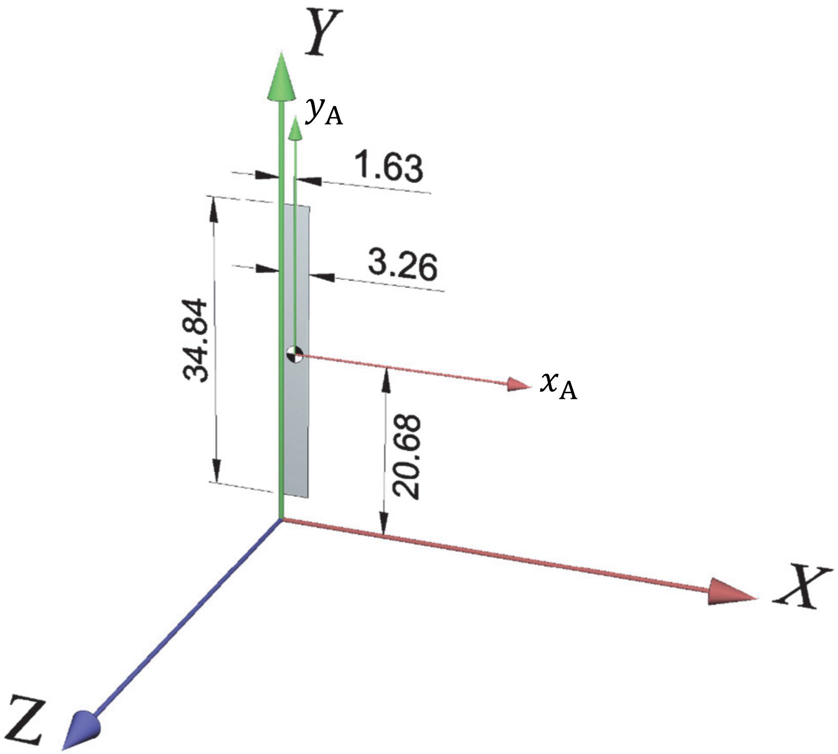
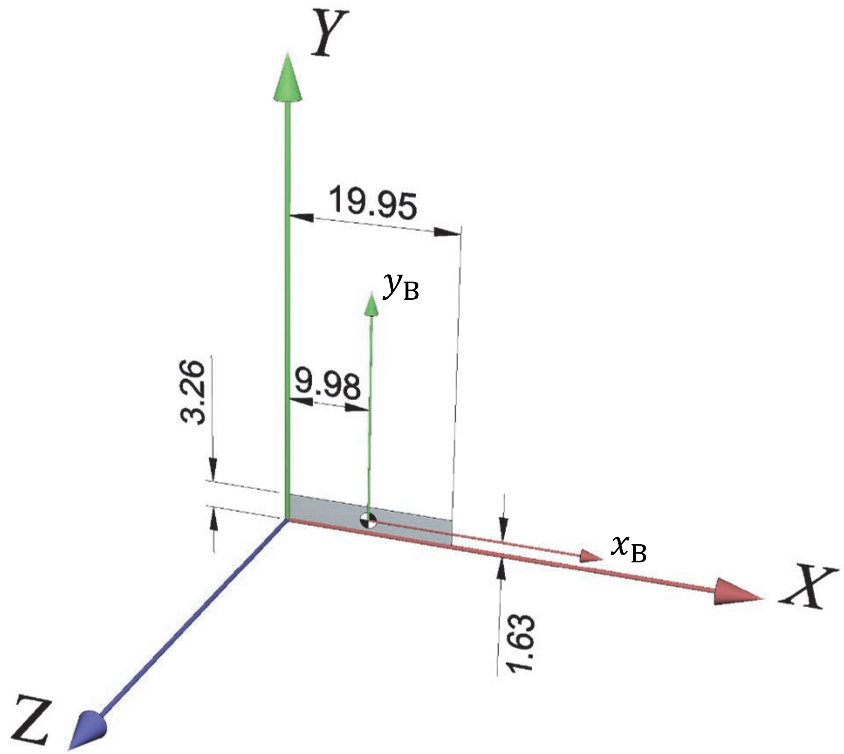
- Dividing the cross section into two simpler sections:

Web (section A)



Flange (section B)



$$A_A = (3.26)(34.84) = 113.58 \text{ mm}^2$$

$$\bar{X}_A = 1.63 \text{ mm}$$

$$\bar{Y}_A = 20.68 \text{ mm}$$

$$A_B = (19.95)(3.26) = 65.04 \text{ mm}^2$$

$$\bar{X}_B = 9.98 \text{ mm}$$

$$\bar{Y}_B = 1.63 \text{ mm}$$

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- We can now find the centroid of the compound section:

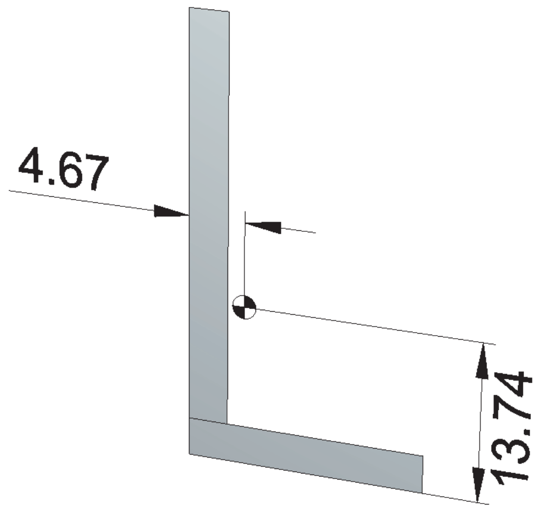
$$\bar{X} = \frac{\sum \bar{X}_i\, A_i}{\sum A_i} = \frac{\bar{X}_\mathrm{A}\, A_\mathrm{A} + \bar{X}_\mathrm{B}\, A_\mathrm{B}}{A_\mathrm{A} + A_\mathrm{B}} = \frac{(1.63)(113.58) + (9.98)(65.04)}{(113.58) + (65.04)}$$

$$\bar{X} = 4.67\ \mathrm{mm}$$

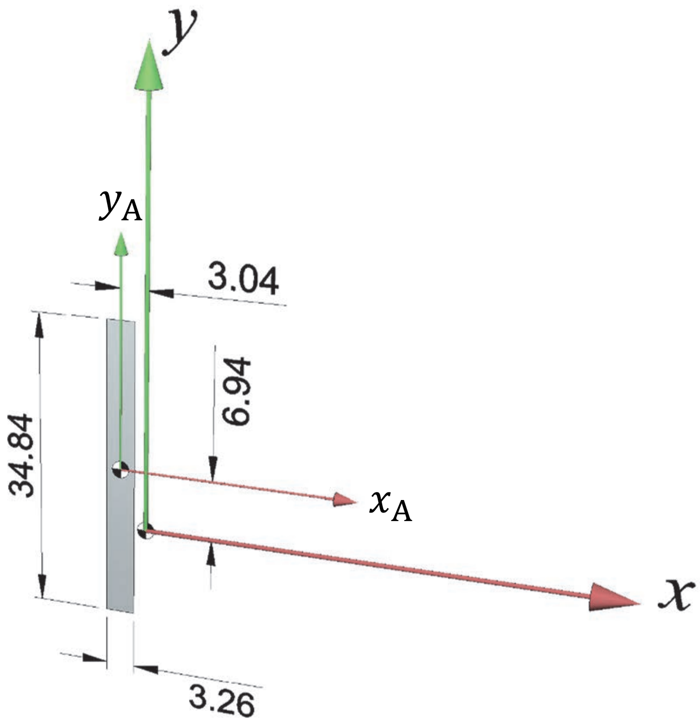
$$\bar{Y} = \frac{\sum \bar{Y}_i\, A_i}{\sum A_i} = \frac{\bar{Y}_\mathrm{A}\, A_\mathrm{A} + \bar{Y}_\mathrm{B}\, A_\mathrm{B}}{A_\mathrm{A} + A_\mathrm{B}} = \frac{(20.68)(113.58) + (1.63)(65.04)}{(113.58) + (65.04)}$$

$$\bar{Y} = 13.74\ \mathrm{mm}$$

- Plotting on the cross-section:



- We now place the origin of (x, y) at the compound centroid and apply the parallel axes theorem for section A:



$$I_{x_A x_A} = \frac{b\,h^3}{12} = \frac{(3.26)(34.84)^3}{12} = 11{,}488.70\ \mathrm{mm}^4$$

$$\bar{y}_A = \bar{Y}_A - \bar{Y} = 20.68 - 13.74 = 6.94\ \mathrm{mm}$$

$$I_{xx}^{A} = I_{x_A x_A} + A_A(\bar{y}_A)^2$$

$$I_{xx}^{A} = (11{,}488.70) + (113.58)(6.94)^2$$

$$I_{xx}^{A} = 16{,}953.43\ \mathrm{mm}^4$$

$$I_{y_A y_A} = \frac{b\,h^3}{12} = \frac{(34.84)(3.26)^3}{12} = 100.58\ \mathrm{mm}^4$$

$$\bar{x}_A = \bar{X}_A - \bar{X} = 1.63 - 4.67 = -3.04\ \mathrm{mm}$$

$$I_{yy}^{A} = I_{y_A y_A} + A_A(\bar{x}_A)^2$$

$$I_{yy}^{A} = (100.58) + (113.58)(-3.04)^2$$

$$I_{yy}^{A} = 1{,}150.50\ \mathrm{mm}^4$$

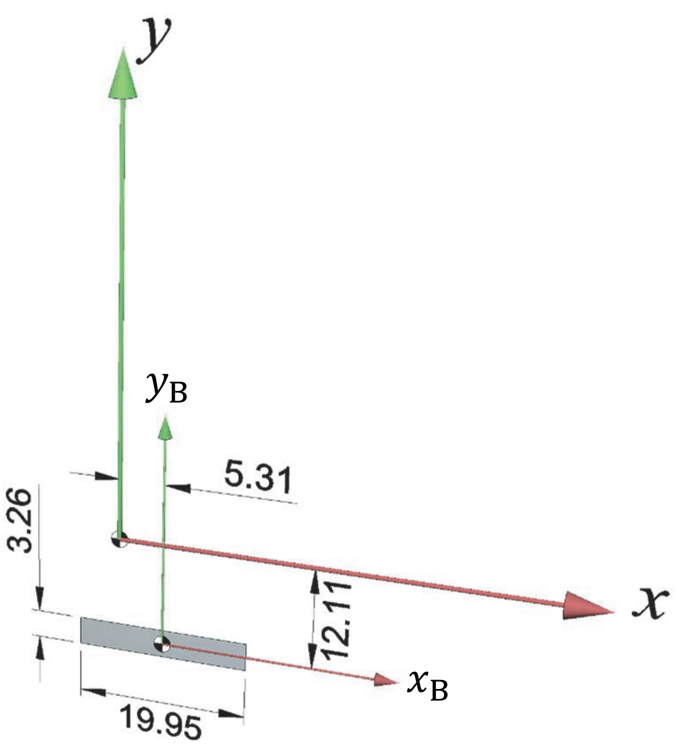
$I_{x_A\, y_A} = 0$ (symmetric cross-section)

$$I_{xy}^{A} = I_{x_A y_A} + A_A(\bar{x}_A \bar{y}_A)$$

$$I_{xy}^{A} = 0 + (113.58)(-3.04)(6.94)$$

$$I_{xy}^{A} = -2{,}395.30\ \mathrm{mm}^4$$

- Now applying the parallel axis theorem for section B:



$$I_{x_B x_B} = \frac{b\,h^3}{12} = \frac{(19.95)(3.26)^3}{12} = 57.60\ \text{mm}^4$$

$$\bar{y}_B = \bar{Y}_B - \bar{Y} = 1.63 - 13.74 = -12.11\ \text{mm}$$

$$I_{xx}^{B} = I_{x_B x_B} + A_B(\bar{y}_B)^2$$

$$I_{xx}^{B} = (57.60) + (65.04)(-12.11)^2$$

$$I_{xx}^{B} = 9{,}601.02\ \text{mm}^4$$

$$I_{y_B y_B} = \frac{b\,h^3}{12} = \frac{(3.26)(19.95)^3}{12} = 2{,}157.07\ \text{mm}^4$$

$$\bar{x}_B = \bar{X}_B - \bar{X} = 9.98 - 4.67 = 5.31\ \text{mm}$$

$$I_{yy}^{B} = I_{y_B y_B} + A_B(\bar{x}_B)^2$$

$$I_{yy}^{B} = (2{,}157.07) + (65.04)(5.31)^2$$

$$I_{yy}^{B} = 3{,}990.60\ \text{mm}^4$$

$$I_{x_B\,y_B} = 0\ \text{(symmetric cross-section)}$$

$$I_{xy}^{B} = I_{x_B\,y_B} + A_B(\bar{x}_B\bar{y}_B)$$

$$I_{xy}^{B} = 0 + (65.04)(5.31)(-12.11)$$

$$I_{xy}^{B} = -4{,}183.07\ \text{mm}^4$$

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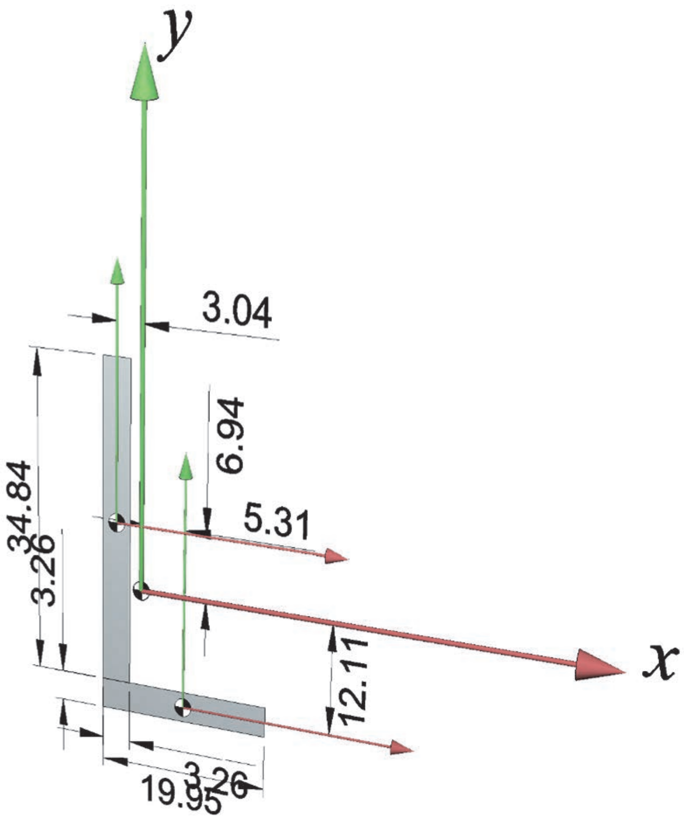
- Finally, for the compound section:



$$I_{xx} = I_{xx}^{\mathrm{A}} + I_{xx}^{\mathrm{B}}$$

$$I_{xx} = (16{,}953.43) + (9{,}601.02)$$

$$I_{xx} = 26{,}554.45 \text{ mm}^4$$

$$I_{yy} = I_{yy}^{\mathrm{A}} + I_{yy}^{\mathrm{B}}$$

$$I_{yy} = (1{,}150.50) + (3{,}990.60)$$

$$I_{yy} = 5{,}141.10 \text{ mm}^4$$

$$I_{xy} = I_{xy}^{\mathrm{A}} + I_{xy}^{\mathrm{B}}$$

$$I_{xy} = (-2{,}395.30) + (-4{,}183.07)$$

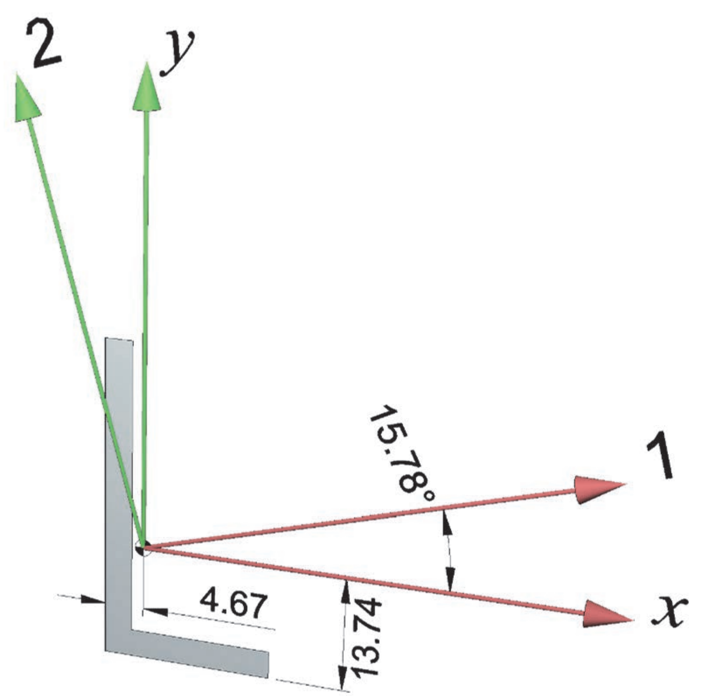
$$I_{xy} = -6{,}578.38 \text{ mm}^4$$

- We can now find the **principal axes**:

$$\theta_p = \frac{1}{2}\arctan\left(\frac{2\, I_{xy}}{I_{yy} - I_{xx}}\right)$$

$$\theta_p = \frac{1}{2}\arctan\left[\frac{2\,(-6{,}578.38)}{(5{,}141.10) - (26{,}554.45)}\right]$$

$$\theta_p = 15.78°$$



- **Physical meaning**: Loading the beam in any direction <u>other than</u> the principal directions (1 or 2) will cause the beam to deflect along <u>two different directions</u> (*i.e.* it will display bending-bending coupling behaviour)

- We can now find the **principal 2nd moments of area**:

$$\begin{Bmatrix} I_{11} \\ I_{22} \\ I_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2\,m\,n \\ n^2 & m^2 & 2\,m\,n \\ m\,n & -m\,n & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} I_{xx} \\ I_{yy} \\ I_{xy} \end{Bmatrix}$$

- where: $m = \cos\theta_{\mathrm{p}} = 0.9623$

  $n = \sin\theta_{\mathrm{p}} = 0.272$

- Using *e.g.* an Excel spreadsheet (see Blackboard) we get:

$$\begin{Bmatrix} I_{11} \\ I_{22} \\ I_{12} \end{Bmatrix} = \begin{Bmatrix} 28{,}413.92 \\ 3{,}281.63 \\ \sim 0 \end{Bmatrix} \mathrm{mm}^4$$

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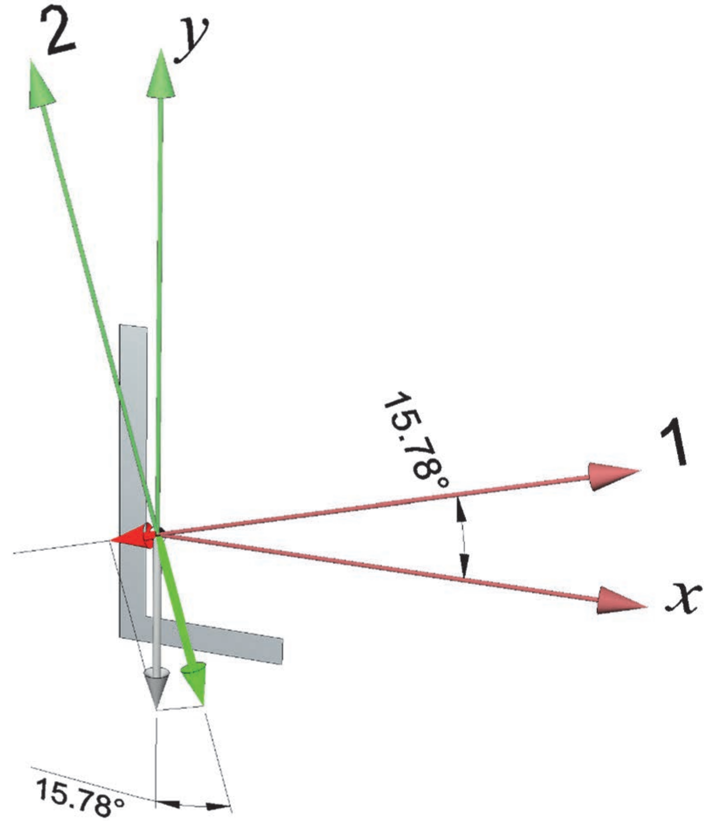
- Now we can project the applied load onto our principal axes:



$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} 0.962 & 0.272 \\ -0.272 & 0.962 \end{bmatrix} \begin{Bmatrix} 0 \\ -19.62 \end{Bmatrix} \mathrm{N}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} -5.34 \\ -18.88 \end{Bmatrix} \mathrm{N}$$

- Applying the tip deflection formula:

$$\boxed{\delta = \frac{P\,L^3}{3\,EI}}$$

$$\delta_1 = \frac{(-5.34)(1000)^3}{3\,(70{,}000)(3{,}281.63)} \qquad \delta_2 = \frac{(-18.88)(1000)^3}{3\,(70{,}000)(28{,}413.92)}$$

$$\delta_1 = -7.74\ \mathrm{mm} \qquad \delta_2 = -5.15\ \mathrm{mm}$$

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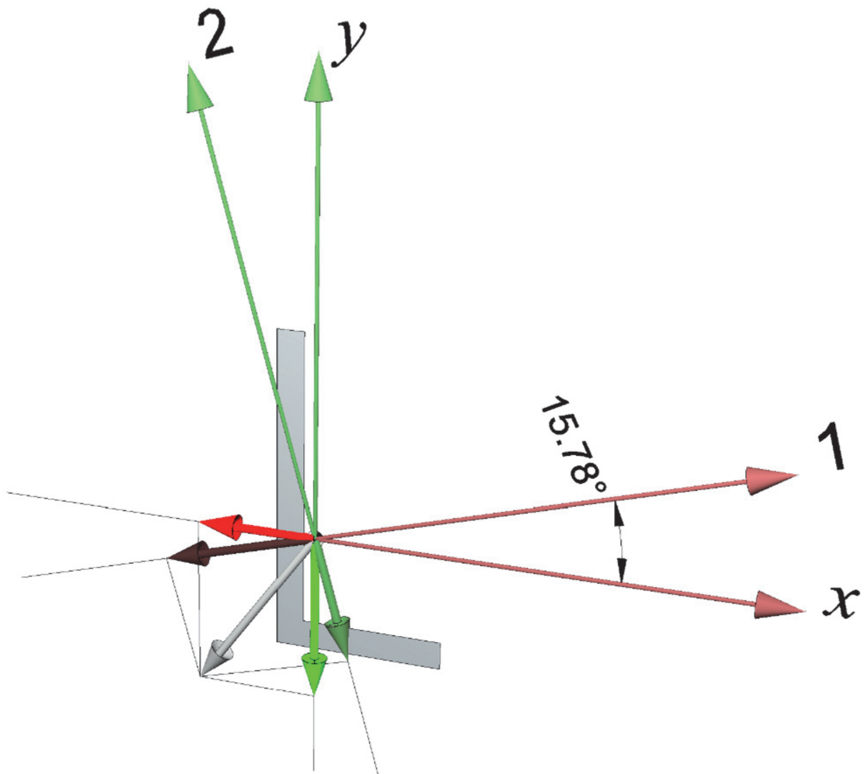
- Finally, transform deflections from principal axes $(1,2)$ back to our reference axes (x, y) through rotation by $(-\theta_\mathrm{p})$:

$$\begin{Bmatrix}\delta_x\\ \delta_y\end{Bmatrix} = \begin{bmatrix} m & n\\ -n & m\end{bmatrix}\begin{Bmatrix}\delta_1\\ \delta_2\end{Bmatrix} \qquad \begin{aligned} m &= \cos(-\theta_\mathrm{p})\\ n &= \sin(-\theta_\mathrm{p})\end{aligned}$$

$$\begin{Bmatrix}\delta_x\\ \delta_y\end{Bmatrix} = \begin{bmatrix} 0.962 & -0.272\\ 0.272 & 0.962\end{bmatrix}\begin{Bmatrix}-7.74\\ -5.15\end{Bmatrix}\ \mathrm{mm}$$

$$\begin{Bmatrix}\delta_x\\ \delta_y\end{Bmatrix} = \begin{Bmatrix}-6.59\\ -5.15\end{Bmatrix}\ \mathrm{mm}$$



- which <u>might be</u> similar to the deflections measured in the lab