Handout 10 – Angular Momentum of Particles & Rigid Bodies

10.1 Angular Momentum of Particles

Meriam & Kraige, Dynamics: 3/10

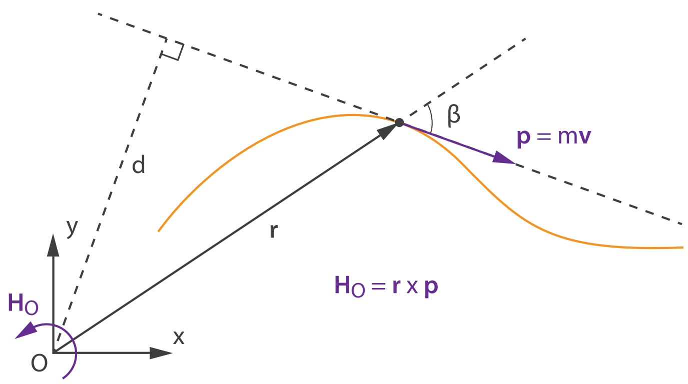
The **angular momentum** of a particle is defined as the *moment of momentum* about a point O:

$$\boldsymbol{H}_O = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v} \tag{10.1}$$

and is a *vector*, with units Nms. For planar problems the angular momentum has axis $\boldsymbol{k}$ and magnitude:

$$H_O = mv\,d = mv\,r\sin\beta \tag{10.2}$$

where r is the distance from O to the particle, and β the angle between the position and momentum vectors.



The angular momentum of a particle depends on the choice of the point O about which it is calculated: it might be positive, negative or zero. This in contrast to linear momentum, whose magnitude does not depend on the choice of the (inertial) reference frame.

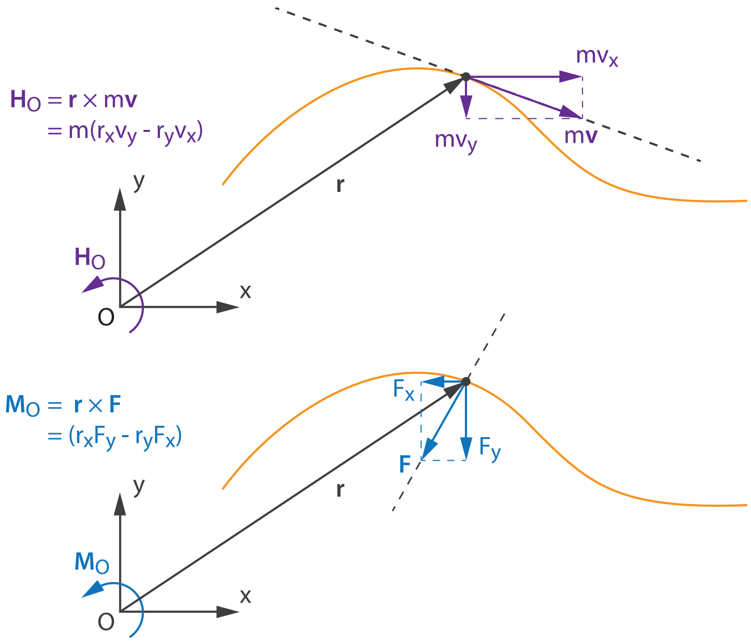
The time derivative of the angular momentum $\dot{\boldsymbol{H}}_O$ is equal to the moment $\boldsymbol{M}_O$ of the resultant force on the particle around the same point O:

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O \tag{10.3}$$

Angular impulse is the time integral of the moment $\boldsymbol{M}_O$, and is equal to the change in angular momentum:

$$\int_{t_1}^{t_2} \boldsymbol{M}_O dt = \boldsymbol{H}_{O_2} - \boldsymbol{H}_{O_1} = \Delta\boldsymbol{H}_O \tag{10.4}$$

If the net force on the particle produces no moment around point O (which can be conveniently chosen), the angular momentum is conserved and $\Delta\boldsymbol{H}_O = 0$. This is a very powerful tool in solving dynamics problems.



proof: from the vector formulation for angular momentum

$$\boldsymbol{H}_O = \boldsymbol{r} \times m\boldsymbol{v}$$

take the time derivative:

$$\dot{\boldsymbol{H}}_O = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \cancel{\boldsymbol{v} \times m\boldsymbol{v}} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \boldsymbol{r} \times m\dot{\boldsymbol{v}}$$

and note that $\boldsymbol{v} \times m\boldsymbol{v}$ is zero, as both vectors are parallel. Consider the moment due to the resultant force:

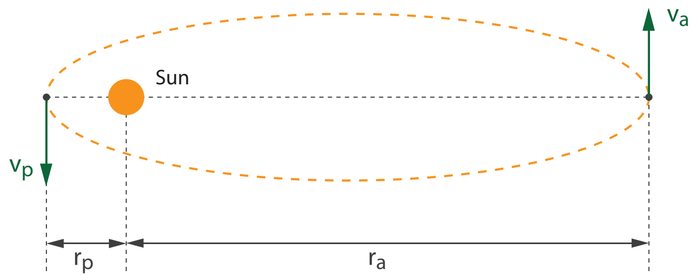
$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times m\dot{\boldsymbol{v}}$$

and substitute $\boldsymbol{F} = m\boldsymbol{a} = m\dot{\boldsymbol{v}}$. It is now evident that:

$$\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O$$

Example 10.1 – Orbital Mechanics

Consider a comet in a highly eccentric orbit around the Sun. The distance from the Sun to *aphelion* (furthest point from the Sun) is $r_a = 6 \cdot 10^9$ km, and distance to *perihelion* (closest point) $r_p = 75 \cdot 10^6$ km. Its speed at aphelion is $v_a = 740$ m/s.



Q: What is the velocity v_p at perihelion?

The only significant force acting on the comet is the gravitational attraction, which is central (and thus passes through the centre of the sun). Thus, the angular moment H_O about the centre of the Sun is conserved:

$$mv_a r_a = mv_p r_p$$

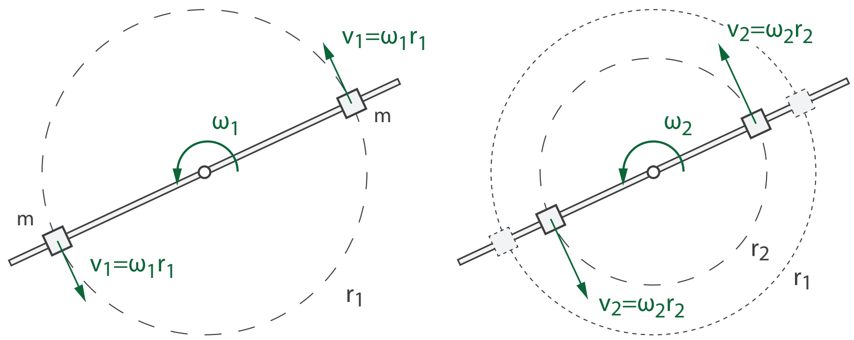
and therefore

$$v_p = \frac{r_a}{r_p} v_a = 59.200 \text{ m/s}$$

Example 10.2 – Spinning Desk Chair / Figure Skater

A first-year aerospace student, during a brief moment of procrastination from revision, is swivelling on his desk chair. He notices that his angular velocity increases when he tucks in his legs and decreases when he swings them out. After taking the mechanics module, he is confident that he can explain what is happening!

For convenience, the problem is simplified to a massless rod with two equal masses m at a distance r_1, spinning at an angular velocity ω_1. As the masses are moved inwards to radius r_2, the angular velocity to ω_2 increasess.



The centripetal forces on the masses produce no net moment around the point of rotation. Thus, the angular momentum is conserved ($\Delta H = 0$) between r_1 and r_2

$$H_{O_1} = H_{O_2}$$
$$2m(\omega_1 r_1) r_1 = 2m(\omega_2 r_2) r_2$$

to give the angular velocity:

$$\omega_2 = \left(\frac{r_1}{r_2}\right)^2 \omega_1$$

The change in kinetic energy:

$$\Delta T = T_2 - T_1 = \frac{1}{2} m\omega_2^2 r_2^2 - \frac{1}{2} m\omega_1^2 r_1^2$$
$$= \frac{1}{2} m\omega_1^2 r_1^4 \left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)$$

is positive for $r_1 > r_2$, and the kinetic energy has increased. Where did the required energy come from?

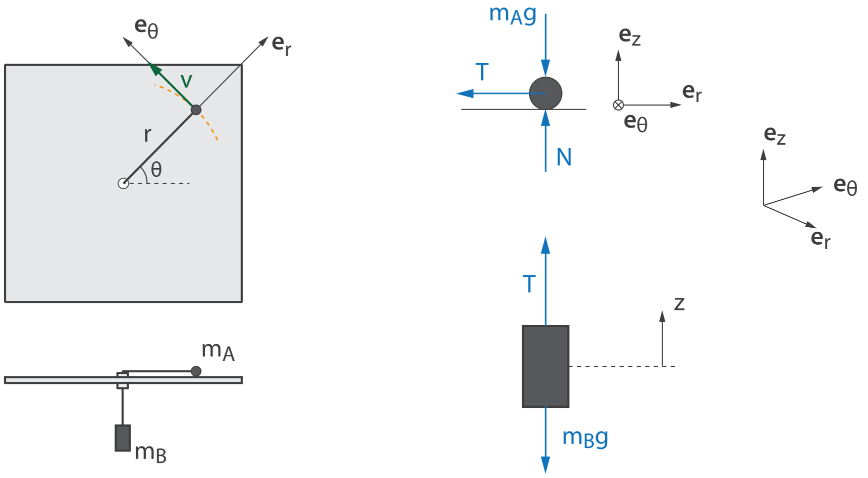
A centripetal force $F_n = ma_n$ is required to maintain the masses at a radius r, where $a_n = \omega^2 r$. The work done by the centripetal force over the change in radius:

$$
\begin{aligned}
U_{12} &= -2 \int_{r_1}^{r_2} F_n dr = -2 \int_{r_1}^{r_2} m\omega^2 r dr \\
&= -2 \int_{r_1}^{r_2} m \left(\frac{r_1^2}{r^2} \omega_1 \right)^2 r dr = -2m\omega_1^2 r_1^4 \int_{r_1}^{r_2} \frac{1}{r^3} dr \\
&= m\omega_1^2 r_1^4 \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) = \Delta T
\end{aligned}
$$

is equal to the change in kinetic energy.

Example 10.3 – Spinning and Dropping Mass

Particle A, with mass $m_A = 1$ kg, slides on a smooth horizontal surface and is connected by a weightless string to particle B, with mass $m_B = 2$ kg. Particle B is released from rest at $t = 0$, when A has a circumferential velocity of $v_0 = 6$ m/s at a radius $r_0 = 0.5$ m.



Q: What are the equations of motion of the system?

As the angular momentum around the central hole is conserved:

$$
\boldsymbol{H}_O = \boldsymbol{r} \times m_A \boldsymbol{v} = r\boldsymbol{e}_r \times m_A \left(\dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \right) = m_A r \left(r\dot{\theta} \right) \boldsymbol{e}_z
$$

the angular velocity $\dot{\theta}$ at radius r is:

$$
\dot{\theta} = \left(\frac{r_0}{r} \right)^2 \dot{\theta}_0
$$

where $\dot{\theta}_0 = v_0 / r_0$.

From the FBD of particle A:

$$\sum F_r : \qquad -T = m_A\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$\sum F_\theta : \qquad 0 = m_A\left(\ddot{\theta}r + 2\dot{r}\dot{\theta}\right)$$

and particle B:

$$\sum F_z : \qquad T - m_B g = m_B a$$

The motion of particles A and B is linked:

$$a = \ddot{r}$$

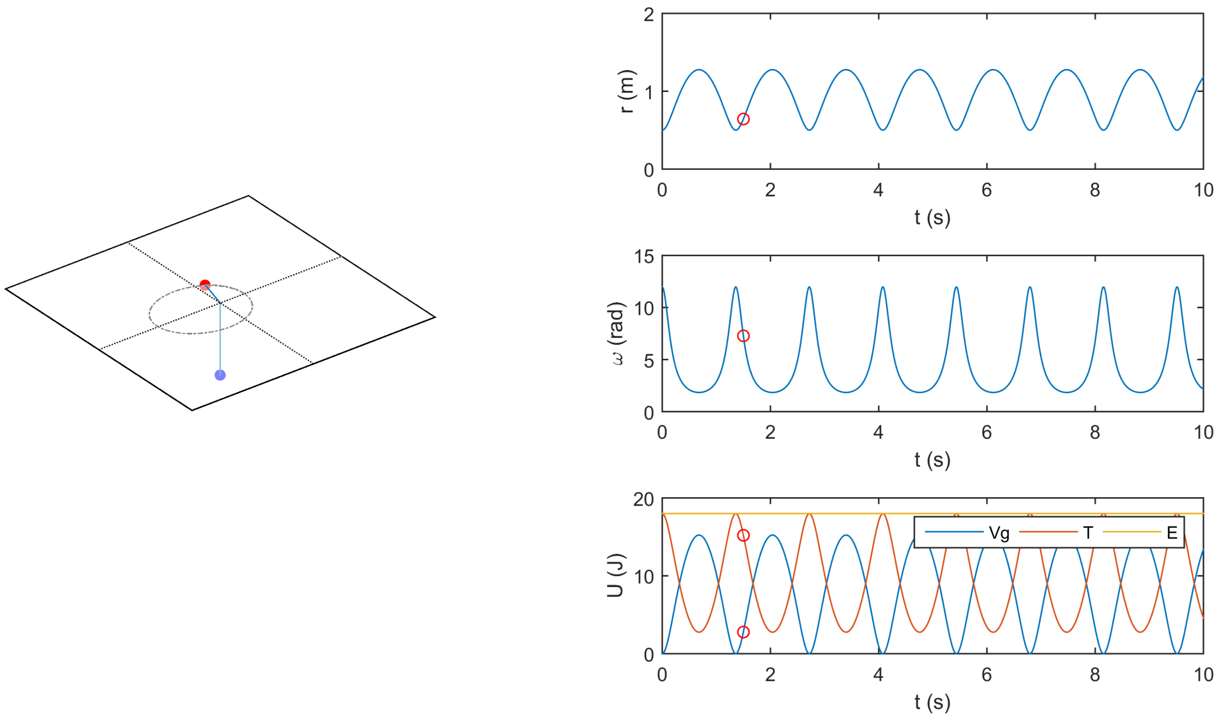
Combine equations to eliminate T and substitute expression for angular velocity:

$$\ddot{r}\left(m_A + m_B\right) - \frac{1}{r^3} m_A r_0^2 v_0^2 + m_B g = 0$$

Substitute values to give:

$$\ddot{r} - 3\frac{1}{r^3} + \frac{2}{3}g = 0$$

This differential equation is solved numerically, to find the motion of the particle. The angular velocity can be found from the conservation of angular momentum, and is integrated for angular position.



10.2 Angular Momentum of Rigid Bodies

Meriam & Kraige, Dynamics: 4/4, 6/2, 6/5, 6/8

10.2.1 Linear Momentum

As derived in Handout 9, the linear momentum of a rigid body is:

$$\boldsymbol{p} = m\boldsymbol{v}_G \tag{10.5}$$

where $\boldsymbol{v}_G$ is the velocity of the centre of mass.

The resultant force $\boldsymbol{F}$ is found as:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} = m\boldsymbol{a}_G \tag{10.6}$$

10.2.2 Angular Momentum

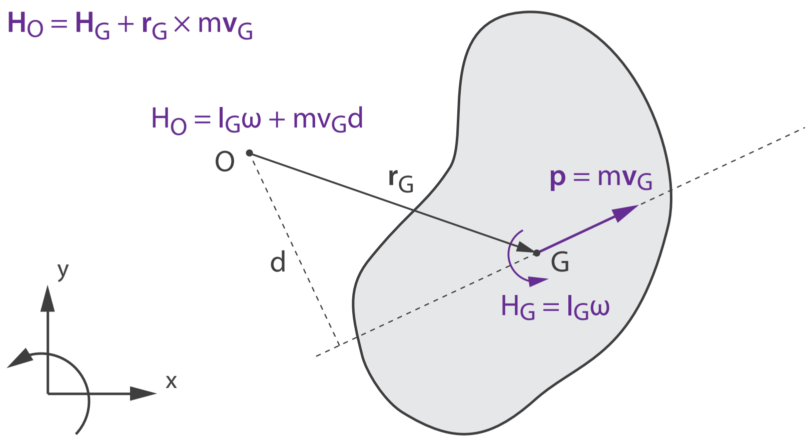
The angular momentum of a rigid body around a general point O is calculated as:

$$\begin{aligned}\boldsymbol{H}_O &= \boldsymbol{H}_G + \boldsymbol{r}_G \times \boldsymbol{p} \\ &= \boldsymbol{H}_G + \boldsymbol{r}_G \times m\boldsymbol{v}_G\end{aligned} \tag{10.7}$$

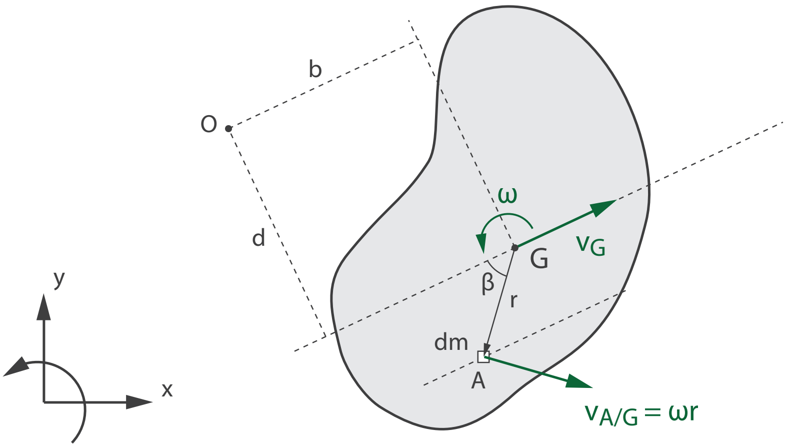
where $\boldsymbol{r}_G$ is the position vector of the centre of mass, and $\boldsymbol{p}$ the linear momentum of the rigid body.

For planar problems:

$$H_O = I_G\omega + mv_Gd \tag{10.8}$$



proof: consider the rigid body as a system of infinitesimal elements with mass dm, and take the vector sum of their angular momentum around point O.



Taking the direction of velocity v_G as reference axis, the moment of momentum H_O about O is equal to:

$$
\begin{aligned}
H_P &= \int (v_G + \omega r \sin\beta)(d + r\sin\beta)\, dm - \int (\omega r \cos\beta)(b - r\cos\beta)\, dm \\
&= \int \left[v_G d + v_G r \sin\beta + \omega r d \sin\beta + \omega r^2 \sin^2\beta - \omega r b \cos\beta + \omega r^2 \cos^2\beta \right] dm \\
&= \int \left[v_G d + \omega r^2 \right] dm + (v_G + \omega d) \cancel{\int r \sin\beta dm} - \omega b \cancel{\int r\cos\beta dm} \\
&= I_G \omega + m v_G d
\end{aligned}
$$

where the last two terms cancel out by definition of the centre of mass, and $I_G = \int r^2 dm$.

10.2.3 Moment Equations

The **resultant moment** (as a result of the external forces on the body) about a point can be calculated from the angular momentum about that point. The form these equations take depends on the choice of reference point about which the angular momentum is calculated.

centre of mass: taking the centre of mass G, where $d = 0$, results in:

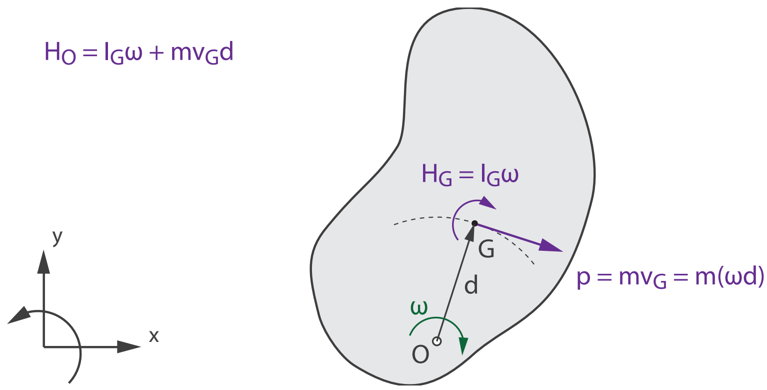
$$M_G = \dot{H}_G = \frac{d}{dt}(I_G\omega) = I_G\ddot{\theta} \tag{10.9}$$

This supports the assertion in Handout 5 that the general motion of a rigid body could be described by the translation of the centre of mass, and rotation about the centre of mass.

fixed axis of rotation: for a rigid body with inertially fixed axis of rotation:

$$\begin{aligned} M_O &= I_G\ddot{\theta} + m a_G d \\ &= I_G\ddot{\theta} + m\left(\ddot{\theta}d\right)d \\ &= \left(I_G + md^2\right)\ddot{\theta} \\ &= I_O\ddot{\theta} \end{aligned} \tag{10.10}$$

where I_O is the moment of inertia around point O, which is found using the parallel axis theorem.



general point: for a general point P, with velocity and acceleration, the moment equation becomes:

$$M_P = I_G\ddot{\theta} + \boldsymbol{r}_G \times m\boldsymbol{a}_G$$

where $\boldsymbol{r}_G$ and $\boldsymbol{a}_G$ are the position and acceleration of the centre of mass.

Alternatively, if point P is fixed to the body, it can be rewritten as:

$$M_P = I_P\ddot{\theta} + \boldsymbol{r}_G \times m\boldsymbol{a}_P$$

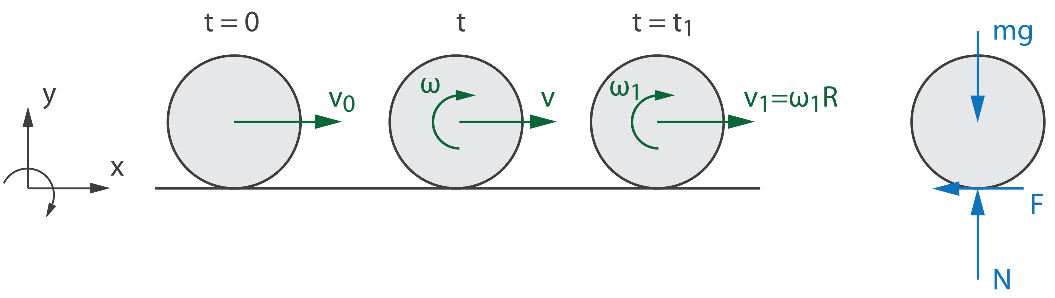
where I_P is the moment of inertia around point P, and $\boldsymbol{a}_P$ is the acceleration of point P. For details of derivation, the interested reader is referred to Meriam & Kraige (section 4/4 and 6/2).

This discussion highlights that in dynamics the point about which moments are taken is crucial in the correct formulation of the equations of motion. This is unlike statics, where moments can be taken about any point!

note: *in this unit only moment equations about centre of mass or a fixed axis of rotation are considered.*

Example 10.4 – Billiard Ball: Slide and Roll

A billiard ball is struck with the cue and gains velocity v_0. Initially the ball slides, before the friction with the table initiates a rotation. After a while, the rotation has increased so that the ball is no longer sliding and is purely rolling.



Q: At what time t_1 does the ball stop sliding?

The billiard ball has radius R and mass m, and the moment of inertia for a sphere is $I_G = \frac{2}{5}mR^2$.

From the FBD, write the equations of linear impulse/momentum:

$$-\mu N t_1 = \Delta \boldsymbol{p} = mv_1 - mv_0$$

and angular impulse/momentum:

$$\mu N R t_1 = \Delta H_G = I_G \omega_1 - 0$$

At time $t = t_1$ the ball is purely rolling:

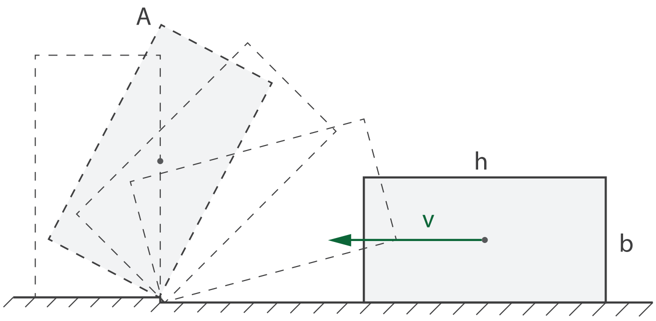
$$v_1 = \omega_1 R_1$$

The three equations are solved simultaneously:

$$v_1 = \frac{v_0}{1 + \frac{I_G}{mR^2}}$$
$$\omega_1 = \frac{v_1}{R}$$
$$t_1 = \frac{v_0}{\mu g \left(1 + \frac{mR^2}{I_G}\right)}$$

Example 10.5 – Tilting Block

A uniform rectangular block of mass m slides on a smooth horizontal surface with velocity v before hitting a small step in the surface.

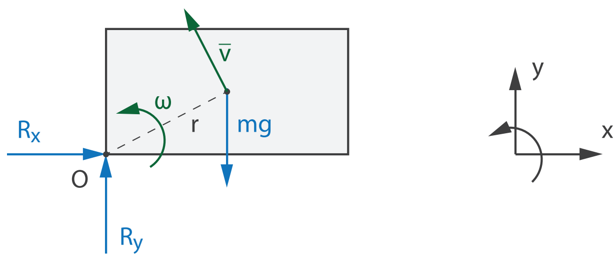


Q: assuming neglible rebound, (a) calculate the minimum value of v required to allow the block to pivot around the edge and reach standing position with no velocity; (b) calculate the percentage energy loss $\Delta E/E$ for the case of $b = h$.

(a) It is assumed that the block pivots around point O, and that the height of the step is neglible compared to dimensions of the block. Furthermore, the angular impulse due to the weight of the block is neglected due to the short time of impact. Thus, the conservation of momentum is taken to be valid.

The angular momentum of the block around O before impact is

$$H_O = mv\frac{b}{2}$$



The velocity of the centre of mass immediately after impact is $\bar{v}$, with angular velocity $\omega = \bar{v}/r$. The angular velocity directly after impact is given from the angular momentum:

$$H_O = I_O\omega$$

where $I_O = I_G + mr^2$, and thus

$$\begin{aligned} H_O &= \left[\frac{1}{12}m\left(b^2+h^2\right) + m\left(\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right)\right]\omega \\ &= \frac{m}{3}\left(b^2+h^2\right)\omega \end{aligned}$$

From conservation of angular momentum, $\Delta H_O = 0$:

$$mv\frac{b}{2} = \frac{m}{3}\left(b^2+h^2\right)\omega$$

and thus

$$\omega = \frac{3vb}{2\left(b^2 + h^2\right)}$$

In order to tilt the block upright, the kinetic energy after impact must be as great as gain in potential energy:

$$\Delta T + \Delta V_g = 0$$

$$\left(0 - \frac{1}{2} I_O \omega^2\right) + mg\left[\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2} - \frac{b}{2}\right] = 0$$

This allows to solve for required initial velocity v:

$$v = 2\sqrt{\frac{g}{3}\left(1 + \frac{h^2}{b^2}\right)\left(\sqrt{b^2 + h^2} - b\right)}$$

(b) the percentage loss in energy:

$$\frac{\Delta E}{E} = \frac{\frac{1}{2}mv^2 - \frac{1}{2}I_O\omega^2}{\frac{1}{2}mv^2} = \ldots = 1 - \frac{3}{4\left(1 + \frac{h^2}{b^2}\right)}$$

which for $b = h$ results in an energy loss of 62.5%.

Revision Objectives Handout 10:

Angular Impulse and Momentum of Particles

- calculate angular momentum of particle ($\boldsymbol{H}_O = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$, $H_O = mv\,d$)
- recall that change in angular momentum around and axis through point O equals the moment the resultant forces on the particle produce around that axis ($\dot{\boldsymbol{H}}_O = \boldsymbol{M}_O$)
- recall that angular impulse results in a change in angular momentum

Angular Impulse and Momentum of Rigid Bodies

- calculate angular momentum of a rigid body ($\boldsymbol{H}_O = \boldsymbol{H}_G + \boldsymbol{r}_G \times m\boldsymbol{v}_G$, $H_O = \omega I_G + mv_G d$)
- use dynamic moment equations for rotation around:
  - centre of mass of the body ($M_G = I_G\ddot{\theta}$)
  - inertially fixed rotation axis ($M_O = I_O\ddot{\theta}$)
- recognise that, in contrast to moment equilibrium equations in statics, the axis about which a moment is taken is of critical importance in dynamics

apply the equations for angular momentum to solve problems in dynamics of particles and rigid bodies