EMAT10100 Engineering Maths I Lecture 21: Integration - area under the curve

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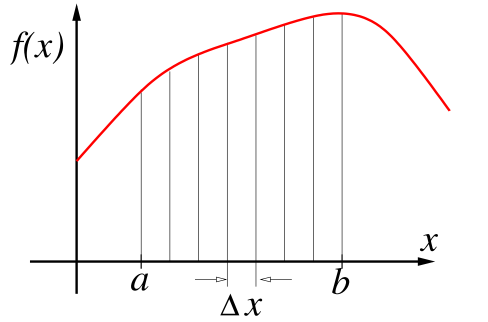
Formal definition of the integral

simplified definition – called the Riemann integral



$$\int_a^b f(x)dx = \lim_{N\to\infty} \frac{(b-a)}{N} \sum_{n=1}^{N} f(a+n\Delta x),$$

where $\Delta x = \frac{b-a}{N}$

An example from first principles

- Consider:

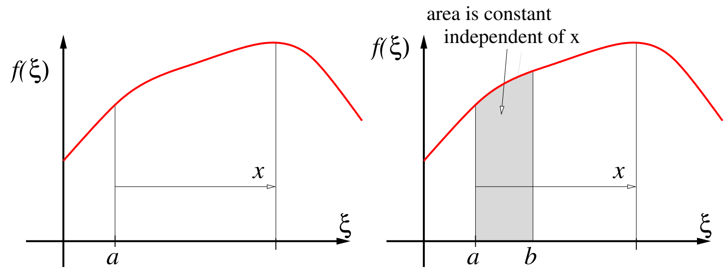
$$\begin{aligned}
\int_0^1 x^2 dx &= \lim_{N\to\infty} (1/N) \sum_{n=1}^{N} (n/N)^2 \\
&= \lim_{N\to\infty} (1/N^3) \sum_{n=1}^{N} n^2 \\
&= \lim_{N\to\infty} (1/N^3)[(1/6)N(N+1)(2N+1)] \\
&= \lim_{N\to\infty} \frac{2N^3+3N^2+N}{6N^3} = \frac{1}{3}
\end{aligned}$$

where we used $\sum_{n=1}^{N} n^2 = (1/6)N(N+1)(2N+1)$ (formulae sheet)

- Exercise Similarly, compute $\int_0^1 x\, dx$ from first principles
- $\Rightarrow$ Compared with differentiation, evaluating integrals from first principles is hard.

Integral as a function - the indefinite integral

- Let $F(x) = \int_a^x f(\xi)d\xi$ then we can think of $F(x)$ as being a function. Note the need for a *dummy argument* (Greek letter ξ)



- Also, given constants a and b, $\int_a^x f(\xi)d\xi = \int_b^x f(\xi)d\xi +$ constant
- So, we often write $F(x) = \int^x f(\xi)d\xi$ where $F(x)$ is called the indefinite integral and is defined only up to an arbitrary constant c

The fundamental theorem of calculus

given sufficiently smooth functions, etc. (always read the small print)

- integration is the inverse of differentiation! i.e.

$$\int f'(x)dx = f(x) + c$$

- or, alternatively $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$,
  where $F(x) = \int^x f(\xi)d\xi$ is the indefinite integral of F
- $F(x)$ is sometimes called the primitive of $f(x)$
  just like $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ is called the derivative
- definite integrals: evaluate $F(x)$ at the two end points:
  $\int_a^b f(x)dx = [F(x)]_a^b = [F(x)]_b - [F(x)]_a = F(b) - F(a)$

Some common integrals

Some simple functions (more given on formulae sheet)

$f(x)$	$F(x) = \int^x f(\xi)d\xi$
const.	x
x	$x^2/2$
$x^n \quad (n \neq -1)$	$x^{n+1}/(n+1)$
$1/x$	$\ln(x)$
$\sin(ax)$	$-(1/a)\cos(ax)$
$\cos(ax)$	$(1/a)\sin(ax)$
$\exp(ax)$	$(1/a)\exp(ax)$
$\frac{a}{a^2+x^2}$	$\arctan(x/a)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$

Nb. Integration is a signed measure of area under the curve, so
$\int_a^b f(x)dx = -\int_b^a f(x)dx$

Integration: the bad news

- For differentiation, we had the product, quotient and chain rule. There are no direct equivalents for integration.
- Even some integrals of simple looking functions do not have an expression in terms of known functions e.g.
  - $\int e^{-x^2}dx$, (the normal distribution)
  - $\int \frac{1}{\sqrt{1-\sin^2(x)}}dx$ (period of a pendulum)
  - $\int \frac{\sin(x)}{x}$ (used in curve fitting))
- next week: important tricks (partial fractions, by parts etc.)
- but first let us give one useful trick . . .

The substitution method

- Given a function of a function $g(u(x))$:

$$\int_a^b g(u(x))dx = \int_{x=a}^{x=b} g(u)\frac{\mathrm{d}x}{\mathrm{d}u}du = \int_{x=a}^{x=b} \frac{g(u)}{u'(x)}du$$

- sometimes the new function $g(u)/u'(x)$ is a simple function of u that can be integrated
- Example. Consider $\int \sin(7x+1)dx$,
  Let $u = (7x+1)$. Then $\frac{du}{dx} = 7$. So
  $\int \sin(7x+1)dx = \int \frac{1}{7}\sin(u)du = -(1/7)\cos(u) + c$
  $= -(1/7)\cos(7x+1) + c$
- Exercise: use the substitution method to evaluate

$$1. \int_1^2 xe^{-x^2}dx, \qquad 2. \int \frac{3x^2+2x+1}{x^3+x^2+x}dx$$

- No homework! But lots last Monday. & Don't forget qmp.bris.ac.uk