Spring Semester 2018

Ordinary Differential Equations

Lecture 7-8: Non-homogeneous higher order ODEs

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Introduction

Higher order ODEs ...

- **Homogeneous higher-order linear ODEs**, use Ansatz e^{mt}
- Non-homogeneous higher order linear ODEs ... today
- Transformation into systems of ODEs ... today

Example

Non-homogeneous ODEs

Lets start with an example

Example

Example

What do we do with the equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} + 3\frac{\mathrm{d} x}{\mathrm{d} t} + 2x = t^2$$

Idea: We can solve the related homogeneous equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} + 3\frac{\mathrm{d} x}{\mathrm{d} t} + 2x = 0$$

Example

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2}+3\frac{\mathrm{d}\, x}{\mathrm{d}\, t}+2x=0$$

We substitute the derivatives to find the characteristic equation

$$m^2+3m+2 = 0 \quad (1)$$

$$(m+1)(m+2) = 0 \quad (2)$$

Therefore the general solution for the homogeneous part is

$$x_{\mathrm{c}}(t)=A\,\mathrm{e}^{-t}+B\,\mathrm{e}^{-2t}.$$

We call such a general solution of the homogeneous part of a non-homogeneous ODE the **complementary function** and denote it by x_{c}.

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Example

But what about the non-homogeneous problem?
There is no systematic method but lets try

$$x(t)=at^2+bt+c$$

We compute the derivatives and

$$\frac{\mathrm{d}\, x}{\mathrm{d}\, t}=2at+b,$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2}=2a.$$

Substituting into

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2}+3\frac{\mathrm{d}\, x}{\mathrm{d}\, t}+2x=t^2$$

we obtain

$$2a+3(2at+b)+2(at^2+bt+c)=t^2$$

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Example

In

$$2a+3(2at+b)+2(at^2+bt+c)=t^2$$

we can now identify the coefficients

$$t^2: \quad 2a=1$$

$$t: \quad 6a+2b=0$$

$$1: \quad 2a+3b+2c=0$$

Hence $a=\frac{1}{2}$, $b=-\frac{3}{2}$ and $c=\frac{7}{4}$ and therefore

$$x_{\mathrm{p}}(t)=\frac{1}{2}t^2-\frac{3}{2}t+\frac{7}{4}$$

We call such a particular solution of the non-homogeneous problem, the **particular integral** and denote it by $x_{\mathrm{p}}(t)$.

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Example

So far we have found the general solution for the homogeneous problem, $x_{\mathrm{c}}(t)$, and a particular solution for the non-homogeneous problem, $x_{\mathrm{p}}(t)$. Lets see what happens when we add these together

$$x_{\mathrm{p}}(t)+x_{\mathrm{c}}(t)=\frac{1}{2}t^2-\frac{3}{2}t+\frac{7}{4}+A\,\mathrm{e}^{-t}+B\,\mathrm{e}^{-2t}$$

Plugging that into $\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2}+3\frac{\mathrm{d}\, x}{\mathrm{d}\, t}+2x=t^2$ we get

$$\underbrace{\frac{\mathrm{d}^2 x_{\mathrm{p}}}{\mathrm{d}\, t^2}+3\frac{\mathrm{d}\, x_{\mathrm{p}}}{\mathrm{d}\, t}+2x_{\mathrm{p}}}_{t^2}+\underbrace{\frac{\mathrm{d}^2 x_{\mathrm{c}}}{\mathrm{d}\, t^2}+3\frac{\mathrm{d}\, x_{\mathrm{c}}}{\mathrm{d}\, t}+2x_{\mathrm{c}}}_{0}=t^2$$

So, $x_{\mathrm{p}}+x_{\mathrm{c}}$ is also a solution to the to the ODE. Because the ODE was second order and we now have a solution that contains 2 constants A, B, it must be the general solution.

General Solution

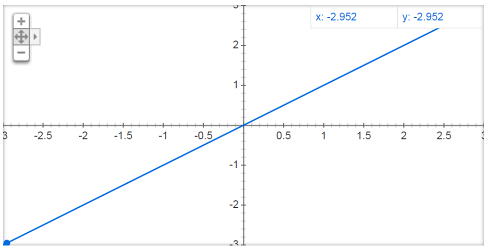
So why did this work

We can explain this using the analogy to functions

General Solution

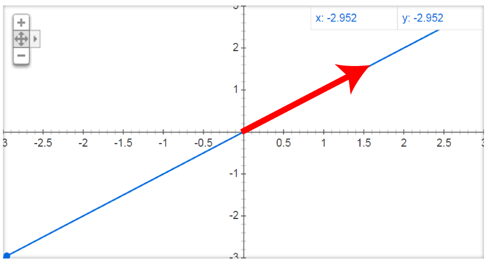
General Solution

why does it work in this way?



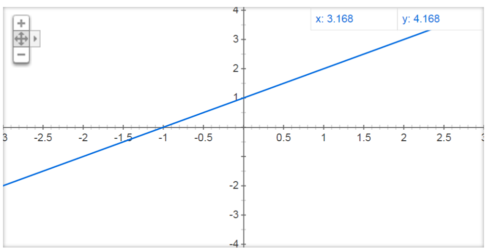
Recall that linear homogeneous ODEs behave like lines through the origin.

General Solution



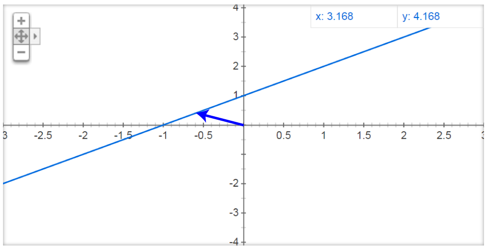
In this picture the general solution to the homogeneous problem is like a variable-length vector pointing in the direction of the line.

General Solution



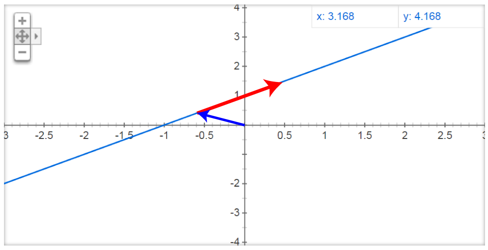
The linear non-homogeneous ODE is like a line with an offset.

General Solution



The particular integral is like a fixed length vector to one point on this line.

General Solution



We obtain a general solution as a sum of the particular integral and the complementary function (one vector onto the line plus a variable-length vector along the line)

General Solution

General Method

We can now formulate a general approach for solving non-homogeneous higher order ODEs

Classification of ODEs

A linear ODE is a linear equation connecting the dependent variable and its derivatives. Standard form of e.g. 1st, 2nd, 3rd order *linear* ODEs

$$\begin{aligned}
\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + a(t)x &= f(t) \\
\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + a(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + b(t)x &= f(t) \\
\frac{\mathrm{d}^3\,x}{\mathrm{d}\,t^3} + a(t)\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + b(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + c(t)x &= f(t)
\end{aligned}$$

The ODE is non-homogeneous if the RHS ($f(t)$) is non-zero.
If the coefficients ($a(t)$ etc.) are constants then we can find the complementary function using the *characteristic equation*.

General solution

Non-homogeneous linear higher order ODEs (with constant coefficients)

1. Find the *complementary function* $x_c(t)$
   - Solution of the homogeneous part
   - Calculate using the *characteristic equation*
   - Gives a solution with arbitrary constants
   - Does not depend on RHS ($f(t)$)
2. Find the *particular integral* $x_p(t)$
   - Solution of the non-homogeneous part
   - No arbitrary constants
   - Depends on the RHS ($f(t)$)

General Solution

Finding the particular Integral

We have no systematic method to find the particular integral, but ...

- Try polynomials when $f(t)$ is polynomial
- Try trigonometric functions when $f(t)$ is a trigonometric function
- Try exponentials when $f(t)$ is exponential
- Try linear combinations of the above when $f(t)$ is a linear combination of polynomials/sinusoids/exponentials

Sometimes the trial solution does not work, this is particularly the case if it has terms in common with the complementary function. In this case multiply the terms by t.

Another Example

Another example (harder)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + 2\frac{\mathrm{d}\, x}{\mathrm{d}\, t} + 4x = \sin(2t) - t + 1$$

To find the complementary function we identify the characteristic equation

$$m^2 + 2m + 4 = 0$$

so $m = -1 \pm \sqrt{3}j$ and we get

$$x_\mathrm{c} = \mathrm{e}^{-t}\left(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)\right)$$

Another Example

Let us now find a particular integral. Since the right hand side is a combination of a sin and a polynomial we try

$$x_p(t) = C\cos(2t) + D\sin(2t) + Et + F$$

this yields

$$4x = 4C\cos(2t) + 4D\sin(2t) + 4Et + 4F$$
$$2\frac{\mathrm{d}\, x}{\mathrm{d}\, t} = -4C\sin(2t) + 4D\cos(2t) + 2E$$
$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} = -4C\cos(2t) - 4D\sin(2t)$$

Another example

Putting these together we arrive at

$$\begin{aligned}&-4C\cos(2t) - 4D\sin(2t) - 4C\sin(2t) + 4D\cos(2t) + 2E\\&+4C\cos(2t) + 4D\sin(2t) + 4Et + 4F = \sin(2t) - t + 1\end{aligned}$$

We take a look at the coefficients

$$\begin{aligned}\sin(2t):&\quad -4D - 4C + 4D = 1\\ \cos(2t):&\quad -4C + 4D + 4C = 0\\ t:&\quad 4E = -1\\ 1:&\quad 2E + 4F = 1\end{aligned}$$

Another example

From the coefficient equations we get $C = -\frac{1}{4}$, $D = 0$, $E = -\frac{1}{4}$, $F = \frac{3}{8}$.
So we have the particular integral

$$x_{\mathrm{p}}(t) = -\frac{1}{4}\cos(2t) - \frac{1}{4}t + \frac{3}{8}$$

and the complementary function

$$x_{\mathrm{c}} = \mathrm{e}^{-t}\left(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)\right)$$

Thus the general solution is

$$\begin{aligned}x(t) &= x_c(t) + x_p(t)\\ &= \mathrm{e}^{-t}\left(A\cos\left(\sqrt{3}t\right) + B\sin\left(\sqrt{3}t\right)\right) - \frac{1}{4}\cos(2t) - \frac{1}{4}t + \frac{3}{8}\end{aligned}$$

Exampercise

Exampercise

Solve
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} - 2x = 3\,\mathrm{e}^{-t}, \qquad x(0) = \dot{x}(0) = 0$$

Exampercise

Resonance

Resonance

Lets see what happens if drive our car down a corrugated road

Resonance

A corrugated road

Resonance

Resonance

Lets consider the mass-spring-damper system again. Suppose our system is described by the equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + \omega^2 x = \Gamma \sin(\omega t)$$

We can compute the complementary function

$$x_c(t) = A\sin(\omega t) + B\cos(\omega t)$$

Resonance

For the particular integral we try

$$x_p(t) = Ct\sin(\omega t) + Dt\cos(\omega t)$$

This leads to

$$C = 0, \quad D = -\frac{\Gamma}{2\omega}$$

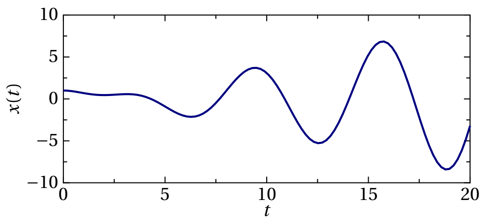
Putting all parts together the general solution is

$$x(t) = A\sin(\omega t) + B\cos(\omega t) - \frac{\Gamma}{2\omega} t\cos(\omega t)$$

Resonance

Lets look at a particular solution with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. In this case we get

$$x(t) = \frac{\Gamma}{2\omega^2}\sin(\omega t) + \cos(\omega t) - \frac{\Gamma}{2\omega}t\cos(\omega t)$$



The system has unbounded oscillations caused by the $t\cos(\omega t)$ term!
In practice, damping will bound the oscillations, but they still become large.

Systems of ODEs

Systems of ODEs

We can turn higher order differential equations into systems of first order equation

Systems of ODEs

Consider for instance the ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} - 3x + 5 = 0$$

We can define

$$y = \frac{\mathrm{d}x}{\mathrm{d}t}$$

This allows us to write the higher order ODE as a system of first order ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 5y - 3x + 5 = 0$$

Systems of ODEs

The rewritten ODE together with the definition of y forms a system of first order equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad (3)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -5y + 3x - 5 \qquad (4)$$

Note that in systems of ODEs we typically write all the derivatives on the left-hand side and everything else on the right-hand side.

Systems of ODEs

Exampercizes

Put the following equations into first order form

1. $\dfrac{d^2 x}{dt^2} + 4\dfrac{dx}{dt} + \dfrac{1}{2}x = 0$
2. $\dfrac{d^3 x}{dt^3} + 2\dfrac{d^2 x}{dt^2} + x = 0$
3. $\dfrac{d^2 x}{dt^2} + 4\dfrac{dx}{dt} + \dfrac{1}{2}x = y, \quad \dfrac{d^2 y}{dt^2} + 2\dfrac{dy}{dt} + \dfrac{3}{2}y = x$

Systems of ODEs

If a system of ODEs is linear and homogeneous we can also write it in matrix form. For instance

$$\frac{dx}{dt} = y \tag{5}$$

$$\frac{dy}{dt} = -5y + 3x \tag{6}$$

becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Systems of ODEs

Solving systems of equations

Suppose we have a first order ODE system of the form

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$$

where $\mathbf{A}$ is a matrix and $\vec{x}$ is a vector of variables.

How do we solve systems like this?

Whenever you encounter a matrix it is good think about eigenvectors.

First-order linear homogeneous ODE systems

So we have an ODE system of the form

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$$

This looks a bit like

$$\frac{dx}{dt} = \lambda x$$

which we can solve with the *ansatz*

$$x = Ce^{\lambda t}$$

But $\vec{x}$ is a *vector*. So we try the ansatz

$$\vec{x} = \vec{v}e^{\lambda t}$$

where $\vec{v}$ is a constant vector and λ is a constant scalar.

First-order linear homogeneous ODE systems

So substituting our ansatz into the ODE we have

$$\frac{\mathrm{d}\left(\vec{v}\mathrm{e}^{\lambda t}\right)}{\mathrm{d}\,t} = \mathbf{A}\left(\vec{v}\mathrm{e}^{\lambda t}\right)$$

Since $\vec{v}$ is constant and $\mathrm{e}^{\lambda t}$ is scalar we have

$$\vec{v}\lambda\mathrm{e}^{\lambda t} = (\mathbf{A}\vec{v})\,\mathrm{e}^{\lambda t}$$

Dividing by $\mathrm{e}^{\lambda t}$ we finally have

$$\mathbf{A}\vec{v} = \vec{v}\lambda$$

So our ansatz $\vec{x} = \vec{v}\mathrm{e}^{\lambda t}$ works provided $\vec{v}$ is an eigenvector of $\mathbf{A}$ and λ is the corresponding eigenvalue.

First-order linear homogeneous ODE systems

So given an ODE system of the form

$$\frac{\mathrm{d}\,\vec{x}}{\mathrm{d}\,t} = \mathbf{A}\vec{x}$$

we can make solutions of the form

$$\vec{x}_i = \vec{v}_i\mathrm{e}^{\lambda_i t}$$

where $\vec{v}_i, \lambda_i$ are the eigenvectors/values.

Bonus Levels

Solve

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 3\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 2x = \mathrm{e}^{-2t}$$

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x = \mathrm{e}^{-t}$$

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 3x = \mathrm{e}^{t}$$

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 3x = 4\cos(3t) + 9t^2$$

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x = t$$

$$\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} - 5\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 4x = \mathrm{e}^{t}$$

Homework

James 5th edition
Read section 10.9.3
Solve exercise 62 from 10.9.4

James 4th edition
Read section 10.9.3
Solve exercise 62 from 10.9.4