(Lectures 1 and 2 of Ordinary Differential Equations)

Ordinary Differential Equations (ODEs)

Ordinary differential equations (ODEs) describe the rate-of-change of a quantity, x, over time t

$$\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}}_{\text{rate-of-change of } x} = \underbrace{f(x,t)}_{\text{some function of } x \text{ and } t}$$

We say

- x is the dependent variable
- t is the independent variable.
- Independent variables do not depend on any other variables. They usually appear at the bottom of differentials.
- Dependent variables depend on other variables. They usually appear on top of differentials.

general solutions describe all possible solutions of a differential equation for arbitrary *initial conditions*.

If we are interested in the solution for particular initial conditions we obtain a *particular solution*.

Classification of ODE

Linearity

An ODE is *linear* if the *dependent variable* and *its derivatives* do not appear as *products*, *raised to powers*, or as *part of nonlinear functions* (sin, cos, exponents, etc).

The order of a differential equation is the order of the highest derivative (with respect to an independent variable).

We say ODEs that are linear without an “offset” are homogeneous differential equations.

We say ODEs that are linear with an “offset” are non-homogeneous differential equations.

(Lectures 1 and 2 of Ordinary Differential Equations)

- **Solve by inspection**, if you can ...
- ODEs that don't explicitly depend on the dependent variable (and do not contain derivatives of different orders)
  Solve by direct integration

(Lectures 1 and 2 of Ordinary Differential Equations)

Solve by separation of variables

Linear homogeneous first-order ODEs

Nonlinear homogeneous first-order ODEs might be separable or not.

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = g(x)h(t)}$$ separable

Linear homogeneous first-order ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + bxt$$

$$\Downarrow$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(a+bt) = g(x)h(t)$$ separable

Solve by separation of variables

Linear homogeneous first-order ODEs

Nonlinear homogeneous first-order ODEs might be separable or not.

1. Separate the variables (the dependent variable wants to be with the differential operator)
2. Split the differential operator to form two integrals.
3. Solve.

(Note: Separation of variables always results in two integrals, which yield two constants of integration, but we can always drop one of them without loss of generality. We will use this from now on.)

Try a clever substitution

Nonlinear first-order ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x/t)$$

are always separable if we use the substitution $y = x/t$.

It is useful to note that $yt = x$ and hence

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}yt}{\mathrm{d}t} = y + t\frac{\mathrm{d}y}{\mathrm{d}t}$$

- Write as function of x/t.
- Use the substitution.
- Separate
- Solve the integrals
- Find the solution in terms of $x(t)$

(Lectures 4-5 of Ordinary Differential Equations)

Solve by integrating factor

Linear non-homogeneous first-order ODEs

- Bring into the form $\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = r(t)$.
- Identify $p(t)$.
- Compute $\int p(t)\mathrm{d}t$.
- Compute $I = \mathrm{e}^{\int p(t)\mathrm{d}t}$.
- Multiply Eq. by I.
- Apply total derivative backwards to absorb terms.
- Solve by direct integration.

(Lecture 6 of Ordinary Differential Equations)

Higher-order linear homogeneos ODEs

The general procedure for solving a linear homogeneous ODE n-th order ODE with constant coefficients is

- Replace derivatives by powers of m to find the characteristic equation
- Find the roots of the characteristic equation.
- Use the roots to obtain n linearly independent solutions.
- Write the general solution as a linear combination of particular solutions.

(Lecture 7-8 of Ordinary Differential Equations)

Non-homogeneous linear higher order ODEs

1. Find the *complementary function* $x_c(t)$
   - Solution of the homogeneous part
   - Calculate using the *characteristic equation*
   - Gives a solution with arbitrary constants
2. Find the *particular integral* $x_p(t)$
   - Solution of the non-homogeneous part
   - No arbitrary constants

$x_\mathrm{p} + x_\mathrm{c}$ is also a solution to the to the ODE.

Finding the particular Integral

We have no systematic method to find the particular integral, but ...

- Try polynomials when $r(t)$ is polynomial
- Try trigonometric functions when $r(t)$ is a trigonometric function
- Try exponentials when $r(t)$ is exponential
- Try linear combinations of the above when $r(t)$ is a linear combination of polynomials/sinusoids/exponentials

Sometimes the trial solution does not work, this is particularly the case if it has terms in common with the complementary function. In this case multiply the terms by t.

Systems of ODEs

Linear, homogeneous, constant-coefficient systems

If a system of ODEs is linear and homogeneous we can also write it in matrix form.

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \mathbf{A}\vec{x}$$

where $\mathbf{A}$ is a matrix and $\vec{x}$ is a vector of variables.

solution of the form

$$x(t) = \sum_n A_n e^{\lambda_n t} \vec{v}_n \qquad (2)$$

where λ_n and $\vec{v}_n$ are the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ and the A_n are arbitrary constants of the general solution.

Nonlinear / non-constant ODEs

Try to write as a system of first order ODEs and solve numerically