Numerical methods

Lecture 9: Sequences and series: series of functions

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Series of functions

- Many computer and data analysis methods approximate functions by series of simpler functions
$$f(x) \approx \sum_{n=0}^{N} c_n g_i(x)$$
where $\{g_i(x)\}$ is a sequence of "simpler" *basis* functions and c_n are typically unknown *coefficients*,
- Two important series of functions are

$$\text{Maclaurin (or Taylor) series:} \quad f(x) \approx \sum_{n=0}^{N} c_n x^n$$

$$\text{Fourier series:} \quad f(x) \approx \sum_{n=0}^{N} a_n \cos(n\pi x/L) + \sum_{n=0}^{N} b_n \sin(n\pi x/L)$$

- This lecture is about the *radius of convergence*

Power series

- Taylor/Maclaurin series are also called **power series**
$$\sum_{n=0}^{\infty} c_n x^n \quad \text{(coefficients } c_n \text{ are constants independent of } x\text{)}$$
- In general we have the Maclaurin series for $f(x)$ given by
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \cdots \tag{1}$$
- For example:
$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$
- Key question: does this series converge?

Convergence of Maclaurin series for e^x

The Maclaurin series for e^x is
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
So this is a series $\sum a_n$ with $a_n = \frac{x^n}{n!}$.

We can check for convergence using the ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ So we have that
$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x^{n+1}}{x^n}\frac{n!}{(n+1)!} = \frac{x}{n+1}$$
So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ for *any* possible value of x.

Exercise: series for $\cos x$

Using (1) find the Maclaurin series for $\cos x$ i.e. find $a, b, \ldots$ for
$$\cos x = a + bx + cx^2 + dx^3 + \ldots$$
Find an expression for the nth term in this sum so that you can write this as
$$\cos x = \sum_{n=0}^{\infty} a_n$$
and hence show that this series converges for all x.

The Maclaurin series for e^x converges for all x but this is not true for all Maclaurin/Taylor series.

Definition (Radius of convergence)

Given a power series $S = \sum_{n=0}^{\infty} c_n x^n$ the series has radius of convergence R if R is the largest number such that S converges for all $|x| < R$.

The Maclaurin series for e^x has an infinite radius of convergence.

What about the Maclaurin series for $(1-x)^{-1}$?

Convergence of Maclaurin series for $(1-x)^{-1}$

We can easily remember this Maclaurin series as it is the formula for the sum of a geometric series:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

Now we have a series $\sum a_n$ with $a_n = x^n$. The ratio test gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x| = |x|$$

Therefore we have convergence if $|x| < 1$ and divergence if $|x| > 1$.

The case $|x| = 1$ needs to be handled specially.

Convergence of Maclaurin series for $(1-x)^{-1}$

We see that

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

converges for $|x| < 1$ and diverges for $|x| > 1$. If $|x| = 1$ then $x = \pm 1$.

If $x = 1$ the series clearly diverges:

$$\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + 1 + \cdots$$

If $x = -1$ it also diverges:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$$

Exercise: convergence for $(1+x)^{-1}$

Given this Maclaurin series

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

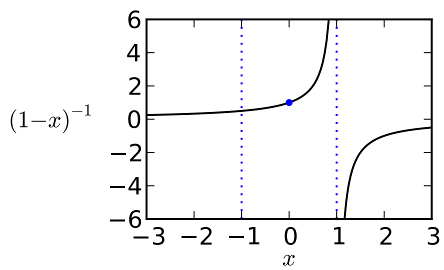
we can substitute $x \to -x$ to find

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

Find the radius of convergence of this series and check what happens when $x = \pm R$.

Visualising the radius of convergence

For $(1-x)^{-1}$ the series converges for $|x| < 1$.



It's easy to see that this cannot converge at $x = 1$.

Divergence for well-behaved functions

We already have

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

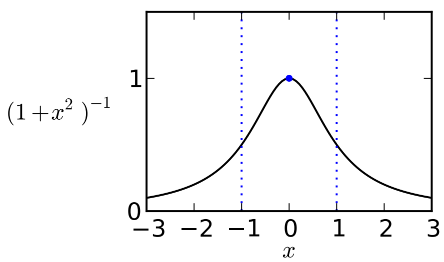
Now if we do $x \to x^2$ we can find that

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

This is also a geometric series with $r = -x^2$ and so converges if $|x| < 1$ but this function is always well behaved.

Visualising the radius of convergence

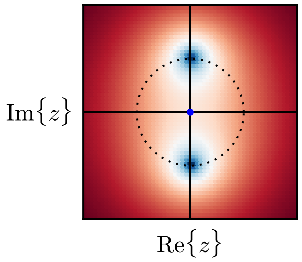
For $(1+x^2)^{-1}$ the series converges for $|x| < 1$.



This function is well-behaved for all $x \in \mathbb{R}$ though. . .

Convergence in the complex plane

Consider $f(z) = (1+z^2)^{-1} = \frac{1}{1+z^2} = \frac{1}{(z+j)(z-j)}$



The function is singular at $z = \pm j$ and so is well-behaved for $|z| < 1$.

Power series

- A common power series: **the binomial series**

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

- Valid for any $p \in \mathbb{R}$,
  - p positive integer $\Rightarrow$ finite series (coefficients satisfy Pascal's triangle)
  - otherwise, get infinite series

Binomial series exercises

Write down the first four terms and, where appropriate, the general term of the Binomial expansions of

1. $(1+x)^4$
2. $(1+x)^{\frac{1}{2}}$
3. $(1+x)^{-1} = \frac{1}{1+x}$

Radius of convergence exercises

Find the radius of convergence R of the following power series. In each case, find what happens when $x = \pm R$:

1. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{x^n}$$
2. $$\sum_{n=0}^{\infty} \frac{x^n}{n}$$

Homework:

- Read Section 7.7
- Do Exercises 7.7.3 Qns 50, 51

Fourier series: an introduction (not examinable)

- Suppose that $f(x)$ is a function defined on an interval $-L \leqslant x \leqslant L$.
- The Fourier series of $f(x)$ is then given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

- With the coefficients a_0, $\{a_n\}$ and $\{b_n\}$ chosen correctly the infinite series *converges* to $f(x)$.

Fourier series example

- Suppose $f(x)$ is defined by:

$$f(x) = \begin{cases} 1 & \text{if } -1 \leqslant x < 0 \\ 0 & \text{if } 0 \leqslant x < 1 \end{cases} \quad \text{with periodic extension}$$

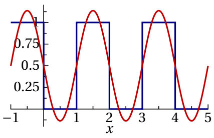
- In this case the Fourier coefficients are given by:

$$a_0 = 1,$$
$$a_n = 0$$
$$b_n = \frac{(-1)^n - 1}{n\pi}$$

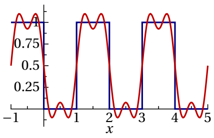
Fourier series example (continued)

- Notice that all the a_n coefficients $(n > 0)$ are zero!
- Furthermore, notice that all even b_n coefficients are zero!
- So we have

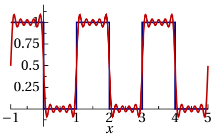
$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{(2n-1)\pi} \sin n\pi x$$



1 term



3 terms



9 terms

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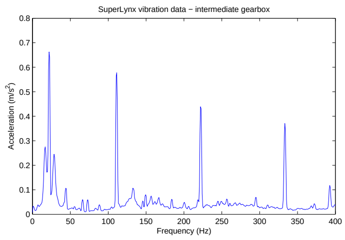
Engineering Hotspot: Fourier series

A Fourier series can be used to provide *frequency domain* information about a system. Damaged components can be detected by monitoring the frequency signature of a device.



We will learn more about Fourier series in Eng Maths II, next year.