EMAT10100 Engineering Maths I Lecture 28: Curvature and multi-dimensional Taylor series

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Looking back looking forward

- Partial differentiation: $f_x \equiv \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$
  = "derivative in x direction treating y as a constant"
- gradient vector $\nabla f = f_x \mathbf{i} + f_y \mathbf{j}$
  directional derivative $f_{\hat{\mathbf{v}}} = \nabla f \cdot \hat{\mathbf{v}}$.
- the chain rule for $f(x(s,t), y(s,t))$:
  $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$
- Higher-order derivatives:
  $\frac{\partial}{\partial y}[f_y(x,y)] = f_{yy}(x,y), \quad \frac{\partial}{\partial x}[f_x(x,y)] = f_{xx}(x,y)$
  $\frac{\partial f_x}{\partial y} = f_{xy} =^{(1)} \frac{\partial f_y}{\partial x} = f_{yx}$
- Total differential: & use in error estimation $du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.
  $\Delta u \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$.
- This lecture: higher-dimensional Taylor series and meaning of the second-derivatives

Taylor series

- Recall in 1D for Taylor series for $x = x_0 + \Delta x$:
  $f(x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2!} f''(x_0)(\Delta x)^2 + \dots$
- In similar manner, define Taylor series for function of 2 variables

$$
\begin{aligned}
f(x,y) &= f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0,y=y_0} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{x=x_0,y=y_0} \Delta y \\
&+ \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0,y=y_0} (\Delta x)^2 + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial y^2}\right|_{x=x_0,y=y_0} (\Delta y)^2 + \frac{2}{2!} \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x=x_0,y=y_0} \Delta x \Delta y + \dots
\end{aligned}
$$

- Note that the 2nd-order term can also be written as

$$
\frac{1}{2}(\Delta x, \Delta y) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}
$$

Exercise

1. Compute all first and second partial derivatives of the function

$$f(x,y) = \sin(xy)$$

   and evaluate at the point $(x_0, y_0) = (1, \pi/3)$.

2. Hence write down the Taylor series expansion of $f(x,y)$ about this point (x_0, y_0) up to and including quadratic terms

The Hessian matrix and curvature

- Note: the first-order term in Taylor expansion is like the directional derivative in direction Δx, Δy
- But what does second-order term represent?
- 2nd partial derivatives matrix is called the Hessian

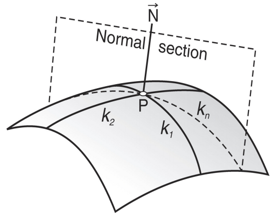
$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

- It represents the curvature of the surface $z = f(x, y)$.
- Its determinant is important; this gives the Gaussian curvature (negative curvature important in design)
- Eigenvalues κ_1, κ_2 of Hessian are called principal curvatures

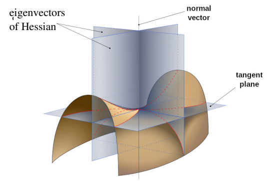
Engineering HOT SPOT:

Curvature and stationary points in 2D

- In the special case that $\nabla f = 0$ (a stationary point)
- Then the determinant of the Hessian matrix is positive $\Rightarrow$ maximum or minimum
- Determinant negative $\Rightarrow$ saddle point



Maximum; both eigenvalues of Hessian are negative



Saddle point; one eigenvalue positive (minimum) and one eigenvalue negative (maximum)

- learn more about this next year in Eng Maths 2

More general notation for Taylor series

- we can write

$$f(x, y) = f(x_0, y_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[(\Delta x) \frac{\partial}{\partial x} + (\Delta y) \frac{\partial}{\partial y} \right]^n f(x, y).$$

- Exercise: show explicitly that this general form gives the earlier-given form for the Taylor series of $f(x, y)$ up to quadratic terms. Explicitly compute an expression for the third order term in terms of f_{xxx}, f_{xxy}, f_{xyy} and f_{yyy} evaluated at $(x, y) = (x_0, y_0)$
- In N dimensions, we can write $\mathbf{x} = (x_1, x_2, \ldots x_N)$ and $f(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{i=1}^{N} (\Delta x_i) \frac{\partial}{\partial x_i} \right]^n f(\mathbf{x})$

Homework

- Please read *James*
  - Section 9.7.1 esp. example 9.36 (4th ed) 9.39 (5th)
  - Section 9.7.2 (not examinable - useful for next year)
- Attempt additional exercises (attached)
  Solutions will be uploaded to blackboard

- Next week
  - Monday: 1st lecture: new material (more properties of functions)
    2nd lecture: revision for Jan exam (with 20% of unit)
  - Thurs: No lecture!
  - drop-in sessions carry on all week