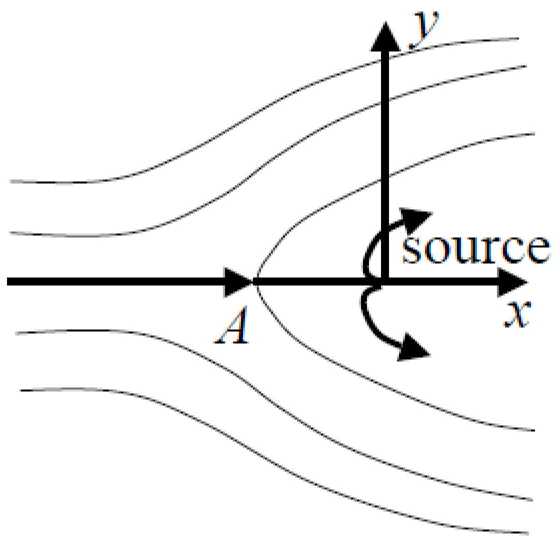
FLUIDS I
Example sheet 6: Potential Flow - Solutions

1.

(a)



The stream function for the combined uniform freestream and point source is

$$\psi = U_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

To find the location of the stagnation point at the nose the <u>first step</u> is to determine the velocity components. The stream function is in terms of Cartesian coordinates so use $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Also the derivative $d(\tan^{-1}(x))/dx=1/(1+x^2)$ is a standard result available in SMP tables. Thus

$$u = \frac{\partial\psi}{\partial y} = U_\infty + \frac{\Lambda}{2\pi}\frac{1}{(1+(y/x)^2)}\left(\frac{1}{x}\right) = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2}$$

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\Lambda}{2\pi}\frac{1}{(1+(y/x)^2)}\left(\frac{-y}{x^2}\right) = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2}$$

Note the differentiation of the $\tan^{-1}$ term requires differentiation of a function of a function.

<u>Second step</u> is to find the stagnation point when $u=v=0$

$v = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2} = 0$ when $y=0$ i.e. on the centre line.

$u = \frac{\partial\psi}{\partial y} = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2} = U_\infty + \frac{\Lambda}{2\pi}\frac{1}{x}$ when $y=0$

Therefore u=0 when $x = -\Lambda/(2\pi U_\infty)$. The stagnation point is on the centre line a distance $\Lambda/(2\pi U_\infty)$ forward of the source.

The stagnation is located on the dividing streamline so the value of the streamfunction on the dividing streamline can now be evaluated. Substituting y=0, $x = -\Lambda/(2\pi U_\infty)$ into the equation for the streamfunction gives

$$\psi_{DS} = \frac{\Lambda}{2\pi}\tan^{-1}(0)$$

<u>CAUTION</u> $\tan^{-1}(0)=0$ or π, but since the stagnation point is forward of the source (which is at the origin of the Cartesian coordinate system $x=y=0$) so π is the correct value to use and thus

$$\psi_{DS} = \frac{\Lambda}{2\pi}\pi = \frac{\Lambda}{2}$$

Finally to determine the body "width" look at the flow field behaviour far downstream as $x \to \infty$. The simplest method is to use the stream function.

$$\psi = U_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) \to U_\infty y \quad \text{as } x \to \infty$$

since as $x \to \infty$ the $\tan^{-1}$ term $\to 0$.

Then it follows that the dividing streamline is in fact

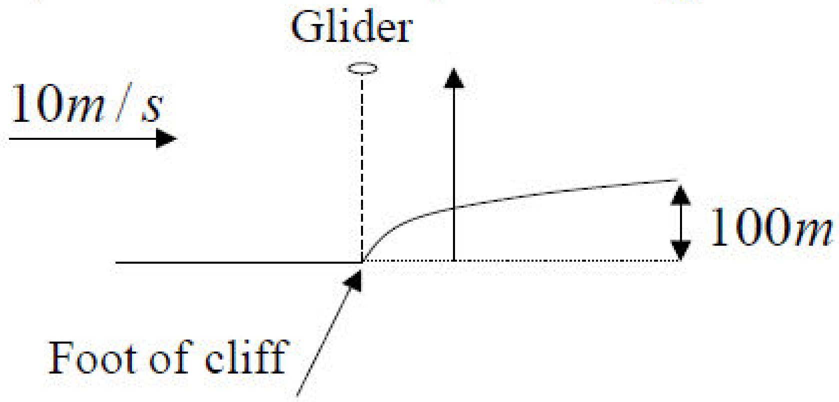
$$\psi_{DS} = \frac{\Lambda}{2} \to U_\infty y \text{ as } x \to \infty$$

.

$$y_{DS} \to \frac{\Lambda}{2U_\infty} \text{ as } x \to \infty$$

The ultimate body "width" is thus $2\,y_{DS}^{U} = \Lambda / U_\infty$

(b) This part of the question makes use of the results derived in part (a). Information given is $U_\infty = 10m/s$ and the ultimate height of the cliff is $y_{DS} = 100m$.



Now from above

$$y_{DS}^{U} = \frac{\Lambda}{2U_\infty} = \frac{\Lambda}{2\times 10} = 100 \text{ therefore } \Lambda = 2000m^2/s$$

The sinking speed of the glider is 1.5m/s, but it is maintaining level flight thus the local fluid vertical velocity v must be +1.5m/s i.e. upwards

$$v = \frac{\Lambda}{2\pi}\frac{y}{x^2 + y^2} = 1.5$$

The glider is flying above the foot of the cliff i.e. the x coordinate of the glider is the same as that of the stagnation point i.e $x = -\Lambda/(2\pi U_\infty) = -31.8\,\text{m}$

Substituting for Λ, x gives a quadratic equation in y

$1.5y^2 - 318.3y + 1519.8 = 0$

which has two solutions y =207m or y=5m (rounded to the nearest metre). Only one of these is physically sensible, y=207m. This is because the cliff rises to a height of 5m approximately quarter of a metre from its base. Thus it would not be possible to fly at this altitude.

Returning to the equation for an aircraft which is maintaining level flight with a sinking speed of 1.5m/s

$v = \frac{\Lambda}{2\pi}\frac{y}{x^2 + y^2} = 1.5$ becomes $x^2 + y^2 = \frac{2000y}{3\pi}$

this links x and y ie. $y=f(x)$ and the maximum height at which the glider could fly is determined by the location where $dy/dx=0$ so differentiating with respect to x gives

$$2x + 2y\frac{dy}{dx} = \frac{2000}{3\pi}\frac{dy}{dx}$$

Thus $dy/dx = 0$ when $x = 0$ i.e. directly above the source. So the maximum height, y, is found by substituting into the above equation when $v = 1.5$

$$0 + y^2 = \frac{2000y}{3\pi}$$

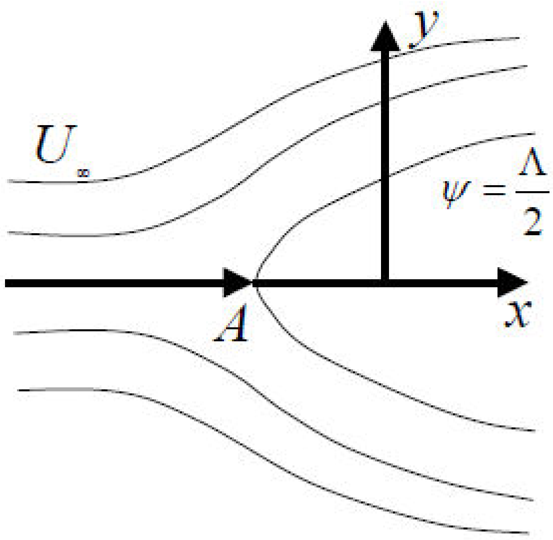
and is given by

$$y_{\max} = \frac{2000}{3\pi} = 212.2m$$

This is approximately 5m higher than above the base of the cliff.

2.

The leading edge of the wing is modelled as a combination of a point source and a freestream



The stream function is given by

$\psi = U_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$ and $\psi_{surface} = \frac{\Lambda}{2}$

the wing geometry is defined in terms of the maximum thickness, t, of the wing and the position of the stagnation point.

The velocity components are again given by the expressions derived in the solution for 1(a) above

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2 + y^2}$$

$$v = \frac{\Lambda}{2\pi}\frac{y}{x^2 + y^2}$$

Then proceeding in a similar manner to 1(a) above yields the stagnation point as

$y = 0, x = -\frac{\Lambda}{2\pi U_\infty}$ and the maximum thickness $t = \frac{\Lambda}{U_\infty} = 0.3$

Thus $x_{nose} = -\frac{\Lambda}{2\pi U_\infty} = -0.048m$.

(a) For a velocity sensor mounted on the centreline $y=0$, $v=0$ and the detected velocity is therefore given by

$$u = U_\infty + \frac{\Lambda}{2\pi}\frac{1}{x}$$

Now the actual freestream velocity is U_∞ and the error in the measured velocity is thus given by $U_\infty - u = -\frac{\Lambda}{2\pi}\frac{1}{x}$ and the relative error is given by $\frac{U_\infty - u}{U_\infty} = -\frac{\Lambda}{2\pi}\frac{1}{xU_\infty}$.

<u>Position for a relative error of 1%</u>

$$0.01 = \frac{U_\infty - u}{U_\infty} = -\frac{\Lambda}{2\pi}\frac{1}{xU_\infty}$$

which gives $x = -4.775m$ or $(4.775\text{-}0.048)m = 4.73m$ in front of the nose.

Position for a relative error of 10%

$$0.1 = \frac{U_\infty - u}{U_\infty} = -\frac{\Lambda}{2\pi}\frac{1}{xU_\infty}$$

which gives $x = -0.4775m$ or $(0.4775\text{-}0.048)m = 0.43m$ in front of the nose.

(b) First need to determine where the position of "half maximum thickness" is located.
Maximum thickness found where $y_{\max} = \frac{\Lambda}{2U_\infty}$ and thus wing has half maximum thickness when

$$y_{0.5\max} = \frac{\Lambda}{4U_\infty}$$

The dividing streamline is given by

$$\psi = U_\infty y + \frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = \frac{\Lambda}{2}$$

so substituting yields $\frac{\Lambda}{2\pi}\tan^{-1}\left(\frac{\Lambda}{4U_\infty x_{0.5\max}}\right) = \frac{\Lambda}{4}$ or $\tan^{-1}\left(\frac{\Lambda}{4U_\infty x_{0.5\max}}\right) = \frac{\pi}{2}$ which means that since $\tan\left(\frac{\pi}{2}\right) = \infty$ so $x_{0.5\max} = 0$. i.e. directly above the source.

Now the x location of the sensor is known need to find the speed at this location. For $x=0$ it can be shown that $u = U_\infty$ and $v = \frac{\Lambda}{2\pi}\frac{1}{y}$ and thus the velocity magnitude is given by

$$V = \sqrt{u^2 + v^2} = \sqrt{U_\infty^2 + (\Lambda/(2\pi y))^2}$$

Note that this is always greater than U_∞ and decreases as y increases.

Position for a relative error of 1%

$0.01 = \frac{V - U_\infty}{U_\infty}$ or $V = 1.01U_\infty$ which on substituting becomes

$$U_\infty^2 + \left(\frac{\Lambda}{2\pi y}\right)^2 = 1.01^2 U_\infty^2$$

then

$$\left(\frac{\Lambda}{2\pi y U_\infty}\right)^2 = 0.0201$$

and since $\Lambda / U_\infty = 0.3m$ it follows that y=0.337m.

(c) The nose pressure tapping is at the stagnation point (i.e. u=v=0) . Therefore from Bernoulli the static pressure at the nose $p_{nose} = p_0$. The second pressure tapping is at x=0 and we have already shown that
the velocity is given by $V = \sqrt{u^2 + v^2} = \sqrt{U_\infty^2 + (\Lambda/(2\pi y))^2}$ at this x location . On the surface $y_{0.5\max} = \dfrac{\Lambda}{4U_\infty}$ so that $V^2 = U_\infty^2\left(1+\left(\dfrac{2}{\pi}\right)^2\right) = 1.405U_\infty^2$

Therefore on the surface using Bernoulli's equation
$p_0 = p_{surface} + \frac{1}{2}\rho(1.405U_\infty^2)$
but $p_{nose} = p_0$ so
$p_{nose} - p_{surface} = \frac{1}{2}\rho(1.405U_\infty^2)$ is the pressure differential reading.

The freestream dynamic pressure is $q_\infty = \frac{1}{2}\rho U_\infty^2$. The pressure differential reading would need to be scaled by the following factor 1/1.405=0.711 to give the freestream dynamic pressure.

3.

(a) Given the stream function

$\psi = U_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)$

in terms of cylindrical polar coordinates the velocity components are given by

$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, v_\theta = -\frac{\partial \psi}{\partial r}$

Now the cylinder is defined by $r = R$ s and $v_r = 0$ on the cylinder (no need to prove must be true for cylinder to be a streamline of the flow) and the tangential component is given by

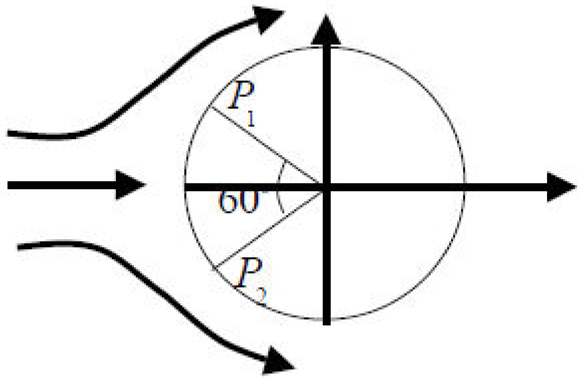
$v_\theta = -U_\infty \sin\theta \left(1 - \frac{R^2}{r^2}\right) - U_\infty r \sin\theta \left(2\frac{R^2}{r^3}\right) = -U_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right)$

and on the surface

$v_\theta = -2U_\infty \sin\theta$

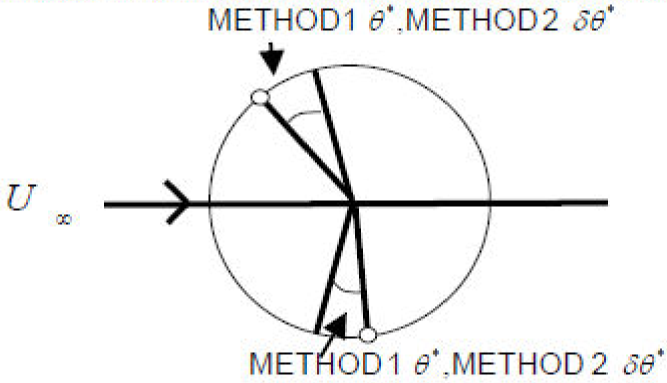
For potential flow $C_p = 1 - \left(\frac{V}{U_\infty}\right)^2$ and thus on the surface $C_{p,surface} = 1 - 4\sin^2\theta$

(b)



$C_{p,surface} = 1 - 4\sin^2\theta$

Change in yaw angle is equivalent to a change in pitch angle



<u>METHOD 1</u>

The pressure difference between the two locations of the holes labelled 1 and 2 is zero when aligned with the flow. Assume the cylinder rotates by a small angle θ^* then

$$\Delta C_p = C_{p1} - C_{p2} = -4\sin^2(\theta+\theta^*) + 4\sin^2(\theta^* - \theta\ )$$

Then the sensitivity scaled by dynamic pressure is given by k

$$k = \frac{d\Delta C_p}{d\theta^*} = -8\sin(\theta+\theta^*)\cos(\theta+\theta^*) + 8\sin(\theta^*-\theta\ )\cos(\theta^*-\theta) \quad \text{when } \theta^* = 0 \text{ so that}$$

$$k = -16\sin\theta\cos\theta = -8\sin 2\theta$$

METHOD 2

For a small yaw angle $\delta\theta^*$ approximately it can be assumed that the pressure at one tapping will rise by an amount $\delta p = \left(\dfrac{dp}{d\theta}\right)_{\delta\theta^*=0} \delta\theta^*$ and by symmetry the pressure at the other tapping will decrease by the same amount. The sensitivity of the yaw meter to small changes in pressure is thus defined as (difference in pressure between holes/change in angle)= $2dp/d\theta$. Then the sensitivity scaled by dynamic pressure equals

$$k = 2\frac{dC_p}{d\theta} = --16\sin\theta\cos\theta = -8\sin 2\theta$$

Note the sign of the answer is immaterial since it depends on our definition of yaw angle and the sign of the pressure difference.

When the tappings are 60° apart then $\theta = 150°$ and the sensitivity is thus

k=6.928 per radian or 0.121 per degree.

Considering the k as a function of angular tapping position θ, then

$$k = -8\sin 2\theta$$

and the maximum sensitivity can be found when

$$\frac{dk}{d\theta} = \frac{d}{d\theta}(-8\sin 2\theta) = -16\cos 2\theta\ = 0$$

Therefore $dk/d\theta = 0$ when $\cos 2\theta = 0$ i.e. $2\theta = \pi/2, 3\pi/2$ so $\theta = \pi/4, 3\pi/4$

For a tapping on the forward face of the cylinder $\theta > \pi/2$ so $\theta = 3\pi/4$ and the tappings are 90° apart and the sensitivity is

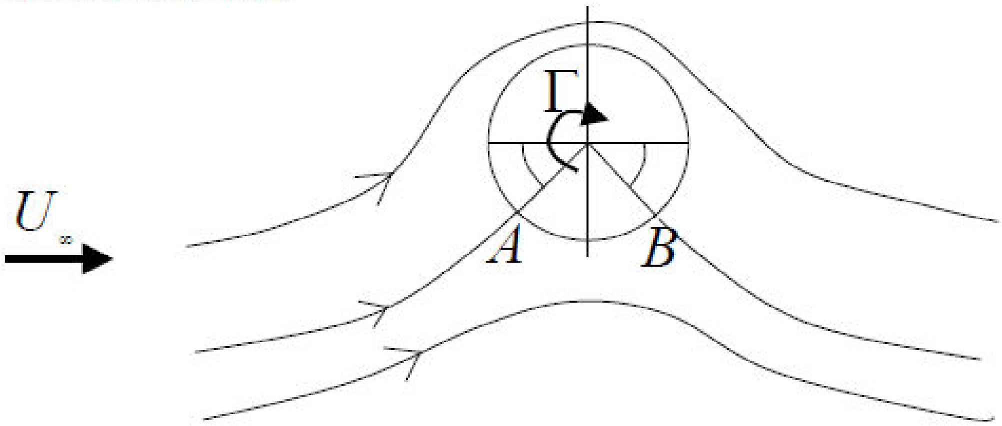
k=8 per radian or 0.140 per degree

4.

(a) The stream function for a cylinder in a uniform flow with circulation is

$$\psi = U_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{R}\right)$$

Sketch of streamlines



Points to notice in sketch are

(1) L and R symmetry
(2) Displacement of stagnation points compared to no circulation case
(3) Closer spacing of streamlines in high velocity regions (if stream lines are equi-spaced in ψ)

To find the surface velocity distribution work in cylindrical polar coordinates since streamfunction given in appropriate form and remember that on the surface of the cylinder the radial component of velocity is zero (no need to prove this) and since

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U_\infty \sin\theta\left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi}\frac{R}{r}\frac{1}{R}$$

then on the surface $r = R$ and

$$v_{\theta,surface} = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

At the stagnation point $v_{\theta,surface} = 0$ so than

$$\sin\theta_{stag} = -\frac{\Gamma}{4\pi R U_\infty}$$

(b) The angular displacement of the stagnation point effectively "fixes" Γ

when $\theta_{stag} = -30^o$ then $-0.5 = -\frac{\Gamma}{4\pi R U_\infty}$ and so $\frac{\Gamma}{2\pi R} = U_\infty$ so on

substituting into the velocity equation gives

$$v_{\theta,surface} = -2U_\infty \sin\theta - \frac{\Gamma}{2\pi R} = -U_\infty(2\sin\theta + 1)$$

The top and bottom of the cylinder correspond to $\theta = +\pi/2$ and $\theta = -\pi/2$ respectively. Hence

$$v_{\theta,top} = -U_\infty(2\sin(\pi/2)+1) = -3U_\infty$$

$$v_{\theta,bottom} = -U_\infty(2\sin(-\pi/2)+1) = U_\infty$$

Note the sign behaviour.. v_θ is positive **anticlockwise** therefore a negative velocity on the cylinder top is directed L to R (i.e. in the same direction as U_∞) and similarly a positive velocity on the cylinder directed bottom is directed L to R.

The local pressure coefficient is given by $C_p = 1 - \left(\frac{V}{U_\infty}\right)^2$ and

$C_{p,top} = -8$ and $C_{p,bottom} = 0$.

Finally the lift is given by the Kutta-Joukowski law/theorem as

$l = \rho_\infty U_\infty \Gamma$

Therefore the lift coefficient based on projected chord *2R is*

$$c_l = \frac{l}{1/2\rho_\infty U_\infty^2 2R} = \frac{l}{\rho_\infty U_\infty^2 R} = \frac{\Gamma}{U_\infty R}$$

but for this case it is known that $\Gamma/(2\pi R) = U_\infty$ so that

$$c_l = 2\pi$$