Lecture 14
Trimming & Linearisation

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Generalised force equations, symmetric aircraft (from Lectures 6-7)

$$\boxed{\begin{aligned} m\left(\dot{U}-rV+qW\right)&=X \\ m\left(\dot{V}-pW+rU\right)&=Y \\ m\left(\dot{W}-qU+pV\right)&=Z \end{aligned}} \quad (9)$$

$$\boxed{\begin{aligned} I_{xx}\dot{p}-\left(I_{yy}-I_{zz}\right)qr-I_{xz}\left(pq-\dot{r}\right)&=L \\ I_{yy}\dot{q}-\left(I_{xx}-I_{zz}\right)pr-I_{xz}\left(p^2-r^2\right)&=M \\ I_{zz}\dot{r}-\left(I_{xx}-I_{yy}\right)pq+I_{xz}\left(qr-\dot{p}\right)&=N \end{aligned}} \quad (17)$$

$$\boxed{\begin{aligned} \dot{\phi}&=p+\tan\theta\left(q\sin\phi+r\cos\phi\right) \\ \dot{\theta}&=q\cos\phi-r\sin\phi \\ \dot{\psi}&=\left(q\sin\phi+r\cos\phi\right)/\cos\theta \end{aligned}} \quad (18)$$

$$\boxed{\begin{aligned} \dot{x}_E &= U\cos\theta\cos\psi+V\left(\sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi\right)+ \\ &\quad W\left(\cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi\right) \\ \dot{y}_E &= U\cos\theta\sin\psi+V\left(\sin\phi\sin\theta\sin\psi+\cos\phi\cos\psi\right)+ \\ &\quad W\left(\cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi\right) \\ \dot{z}_E &= -U\sin\theta+V\sin\phi\cos\theta+W\cos\phi\cos\theta \end{aligned}} \quad (19)$$

6DOF Equations of Motion (from Lectures 6-7)

Equations (9), (17), (18) and (19) form the 12 equations of motion for general atmospheric 6 degree-of-freedom flight.

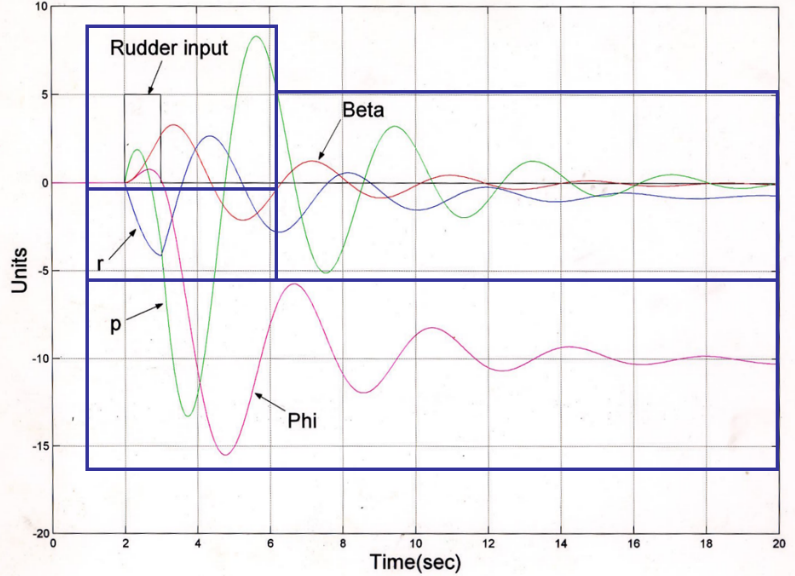
They are all first-order differential equations that can be expressed in nonlinear state-space form as:

$$\dot{x} = f\left(x, \delta\right)$$

where $x = \left[U, V, W, p, q, r, \psi, \theta, \phi, x_E, y_E, z_E\right]'$ (state vector)

and δ is the vector of input parameters of interest. These are often aerodynamic or propulsion system inputs such as aileron, elevator, rudder deflection or thrust (upon which the formulation of the forces X, Y and Z will depend).

Example: nonlinear equations for simulation (from Lectures 6-7)



Trimming & linearisation

$$\dot{x} = f(x, \delta)$$

Numerical trimming
e.g. Matlab 'trim' function

$\dot{x} = ?$ e.g. find the control inputs required for a specified trim condition

Simplify analytically

Numerical linearisation – e.g.

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.068 & -0.011 & -0.049 & -9.81 \\ 0.023 & -2.10 & 366.0 & 0 \\ 0.011 & -0.160 & -9.52 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} -0.41 \\ -77.0 \\ -61.0 \\ 0 \end{bmatrix} \eta$$

Small perturbation equations – e.g.

$$\begin{bmatrix} ms - X_u & -X_w & mg - X_q s \\ -Z_u & ms - Z_w & -mUs - Z_q s \\ -M_u & -M_{\dot{w}} s - M_w & I_{yy} s^2 - M_q s \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix} = \begin{bmatrix} X_\eta & X_\delta \\ Z_\eta & Z_\delta \\ M_\eta & M_\delta \end{bmatrix} \begin{bmatrix} \eta \\ \delta \end{bmatrix} - w_g \begin{bmatrix} X_w \\ Z_w \\ M_w \end{bmatrix}$$

Analysis of aircraft modes
and handling qualities
e.g. eigenvalues & eigenvectors

The longitudinal equations of motion for a fighter type aircraft are given below in a State-Space form. Use these within Matlab to find the eigenvalues and eigenvectors associated with the aircraft modes, and state the aircraft modes that are described. (Note: for these equations, *radians* & *meters* are used). Comment on your results.

10 marks

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.068 & -0.011 & -0.049 & -9.81 \\ 0.023 & -2.10 & 366.0 & 0 \\ 0.011 & -0.160 & -9.52 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} -0.41 \\ -77.0 \\ -61.0 \\ 0 \end{bmatrix} \eta$$

Aircraft 1

(a) Provide a full description of all five of the classical aircraft modes and indicate where they can be found on the complex plane for a typical passenger aircraft configuration. Relate these positions where possible to conventional handling qualities.

(8 Marks)

The lateral dynamics for an aircraft are given below for flight at Mach 0.44 and at an altitude of 15000ft.

$$\begin{bmatrix} s+0.1008 & -32.2 & 468.2 \\ 0.00579 & s^2+1.232s & 0.397 \\ -0.00278 & 0.0346s & s+0.257 \end{bmatrix} \begin{bmatrix} v \\ \phi \\ r \end{bmatrix} = \begin{bmatrix} 0 & 13.48 \\ -1.62 & 0.392 \\ -0.0188 & -0.864 \end{bmatrix} \begin{bmatrix} \xi \\ \zeta \end{bmatrix}$$

The longitudinal small-perturbation equations are as given below for a conventional aircraft. These can be used to find an approximate short-period quadratic equation from which the appropriate pair of roots could be found.

$$\begin{bmatrix} ms - X_u & -X_w & mg - X_q s \\ -Z_u & ms - Z_w & -mUs - Z_q s \\ -M_u & -M_{\dot{w}} s - M_w & I_{yy} s^2 - M_q s \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix} = \begin{bmatrix} X_\eta & X_\delta \\ Z_\eta & Z_\delta \\ M_\eta & M_\delta \end{bmatrix} \begin{bmatrix} \eta \\ \delta \end{bmatrix} - w_g \begin{bmatrix} X_w \\ Z_w \\ M_w \end{bmatrix}$$

(a) For the equations above, explain the meaning of the matrices on the right-hand side. Discuss the relative magnitudes and signs for three individual elements using diagrams where appropriate.

(5 marks)

(b) By simplifying the equations given, find the approximate quadratic equation from which the short period roots can be found.

(10 marks)

Small Disturbance Equations

Longitudinal Small-Disturbance Equations

Introduction

We stated the full 6-DOF equations of motion for a rigid aircraft at the start of the course. For longitudinal motion only, these are reduced to:

$$
\begin{aligned}
\textit{fore/aft}: &\quad X - mg\sin\theta = m(\dot{U} + qW) \\
\textit{transverse}: &\quad Z + mg\cos\theta = m(\dot{W} - qU) \\
\textit{pitch}: &\quad M = I_{yy}\dot{q}
\end{aligned}
\qquad (1)
$$

Longitudinal Small-Disturbance Equations

- Nonlinearities exist in the:
  - trigonometric terms
  - products of the variables such as qW and qU
  - aerodynamic forces X, Z and M

- If we assume that motions will be confined to small disturbances only, from some steady trim, it is possible to derive a set of linear differential equations and often these are sufficient for a study of most flight situations.

- Any aerobatics or high-incidence flight would not be adequately represented by the linearization.

- Such linear equations are certainly easier to solve!

Longitudinal Small-Disturbance Equations

Reference condition

- For trimmed flight implies that initially we have:

$$\dot{U} = \dot{W} = q = \dot{q} = 0$$

(and if equations are in wind or stability axes, $W = 0$ too).

- In other words, there are no accelerations in the three longitudinal directions and indeed there are no changes from the steady rates in those directions, i.e. freestream speed components remain U and W.

Longitudinal Small-Disturbance Equations

- The aerodynamic forces and moment which exist during the trimmed flight are

$$X = X_0, \quad Z = Z_0, \quad M = M_0$$

- Initial pitch orientation:

$$\theta = \theta_0 .$$

- The balanced condition in trimmed flight then follows from Eqn. (1) as:

fore / aft :	$X - mg\sin\theta = m(\dot{U} + qW)$
transverse :	$Z + mg\cos\theta = m(\dot{W} - qU)$
pitch :	$M = I_{yy}\dot{q}$

$$\begin{aligned} X_0 - mg\sin\theta_0 &= 0 \\ Z_0 + mg\cos\theta_0 &= 0 \\ M_0 &= 0 . \end{aligned} \quad (2)$$

Longitudinal Small-Disturbance Equations

- In the disturbed state, we write the variables as their reference values in the balanced condition, plus a small additional value, i.e.

$$\begin{aligned} &total\ X \to X_0 + X \quad && total\ Z \to Z_0 + Z \\ &total\ M \to M_0 + M \quad && total\ \theta \to \theta_0 + \theta \\ &total\ w \to w_0 + w \quad && total\ u \to u_0 + u. \end{aligned} \tag{3}$$

- For these small disturbances we neglect products of the disturbances. If we also use:

$$\begin{aligned} \sin(\theta_0 + \theta) &= \sin\theta_0 + \theta\cos\theta_0 \\ \cos(\theta_0 + \theta) &= \cos\theta_0 - \theta\sin\theta_0 \end{aligned} \tag{4}$$

Longitudinal Small-Disturbance Equations

- Rewriting Eqn. (1):

$$
\begin{aligned}
(X_0 + X) - mg(\sin\theta_0 + \theta\cos\theta_0) &= m[\dot{u} + q(w_0 + w)] \\
(Z_0 + Z) + mg(\cos\theta_0 - \theta\sin\theta_0) &= m[\dot{w} - q(u_0 + u)] \\
M_0 + M &= I_{yy}\dot{q}.
\end{aligned}
\tag{5}
$$

- If we then remove the trim balance from each of these (using Eqn. (2)) and also remove products of small quantities we obtain:

$$
\begin{aligned}
X - mg\,\theta\cos\theta_0 &= m\dot{u} + mqw_0 \\
Z - mg\,\theta\sin\theta_0 &= m\dot{w} - mqu_0 \\
M &= I_{yy}\dot{q}.
\end{aligned}
\tag{6}
$$

- Further, by carefully choosing the direction of the body axis x_B (before a disturbance) we have $\theta_0 = w_0 = 0$ and $u_0 = U$ (fwd. speed).

Longitudinal Small-Disturbance Equations

- The consequence is that Eqn. (5) can be reduced to:

$$
\begin{aligned}
m\dot{u} + mg\,\theta &= X \\
m\dot{w} - mq\,U &= Z \\
I_{yy}\,\dot{q} &= M
\end{aligned}
\qquad (7)
$$

- The forces and moment X, Z, M, could be a consequence of:
  - a. any of the disturbance variables u, w, q, or perhaps their derivatives,
  - b. a deflection of either of the two longitudinal control surfaces: elevator η, or spoiler δ,
  - c. a vertical gust w_g.

Longitudinal Small-Disturbance Equations

The external aerodynamic forces

- Consider a transverse gust of strength w_g which is positive when in the positive z direction, i.e. downward, the velocity of the vehicle relative to the local fluid is therefore $(w - w_g)$.
- Using the expression for a partial derivative, e.g. $X_u = \partial X / \partial u$
- Using the positive directions for $X, Z, M.$

We therefore have the following three general expressions:

$$
\begin{aligned}
X &= X_u u + X_w w + X_q q + X_\eta \eta + X_\delta \delta - X_w w_g \\
Z &= Z_u u + Z_w w + Z_q q + Z_\eta \eta + Z_\delta \delta - Z_w w_g \\
M &= M_u u + M_{\dot{w}} \dot{w} + M_w w + M_q q + M_\eta \eta + M_\delta \delta - M_w w_g.
\end{aligned}
\tag{8}
$$

Total Change in X

All assumed to be independent (not necessarily the case!)

Partial Derivatives

- Relative size?
- Sense? e.g. –ve or +ve

$$X_u = \frac{\partial X}{\partial u}$$

- Forward force due to perturbation velocity (u)
- Essentially extra drag with positive $u \rightarrow$ negative X

Partial Derivatives

$$Z_w = \frac{\partial Z}{\partial w}$$

- Positive w is downward velocity of the vehicle which creates upward fluid flow

  → greater incidence on aerofoils

  → greater upward lift, but lift is negative Z
- Overall: $w>0$ leads to $Z<0$ so $Z_w<0$

$$M_\eta = \frac{\partial M}{\partial \eta}$$

- Pitching moment due to elevator (t.e. down)
- This is what the elevator is for → large M
- Positive η leads to greater upward force on the tail → strong negative pitching moment
- So $\frac{\partial M}{\partial \eta} < 0$

An alternative set of motion variables

- If we recognise that q (pitch rate) and θ (pitch angle) are related as $\theta = \int q \, dt$
- we can then rewrite the equations in terms of u, w, θ in place of u, w, q.
- Hence in matrix form:

$$\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \begin{bmatrix} u \\ w \\ q \end{bmatrix} \Rightarrow \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}$$

The Longitudinal Equations

- Replacing the three external forces X, Z, M in Eqn. (7) with the expanded form

$$\underbrace{\begin{bmatrix} ms - X_u & -X_w & mg - X_q s \\ -Z_u & ms - Z_w & -mU\,s - Z_q s \\ -M_u & -M_{\dot{w}} s - M_w & I_{yy} s^2 - M_q s \end{bmatrix}}_{\text{aircraft dynamics}} \underbrace{\begin{bmatrix} u \\ w \\ \theta \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} X_\eta & X_\delta \\ Z_\eta & Z_\delta \\ M_\eta & M_\delta \end{bmatrix}}_{\substack{\text{control} \\ \text{surface} \\ \text{derivatives}}} \begin{bmatrix} \eta \\ \delta \end{bmatrix} - w_g \underbrace{\begin{bmatrix} X_w \\ Z_w \\ M_w \end{bmatrix}}_{\substack{\text{gust} \\ \text{terms}}}$$

- The aerodynamic derivatives which are associated with the motion variables (u, w, q) have been taken to the LHS and now appear with negative signs, whereas those still on the RHS have retained their original signs.

The Longitudinal Equations

- We could use numerical integration to find the aircraft responses $u(t)$, $u(t)$ and $u(t)$ to specified functions of $w_g(t)$ or $\eta(t)$ (*time solution*).
- Alternatively we can deduce stability characteristics directly from the set of equations given on the previous slide.
- No need to derive these equations.
- Requirement to recognise, understand and use.
- Aerodynamic derivatives.

Next Lecture

Small Perturbation Equations