Numerical Methods - revision

Summary of numerical methods for ODEs

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Numerical integration of ODEs

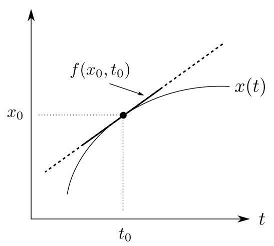
Key points:

- General way to solve ODEs (when analytic methods not possible)
- Most basic: Euler's method (need to know)
- Other methods - need to understand but not memorise.
- Solutions are always approximate: local/global error.
- Order of methods
- Stability

Initial value problem

We want to solve

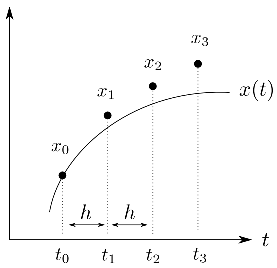
$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = f(x,t), \quad x(t_0) = x_0$$



We have the first point (x_0, t_0) and want to find other points x_1, x_2, etc.

Numerical solutions are discrete

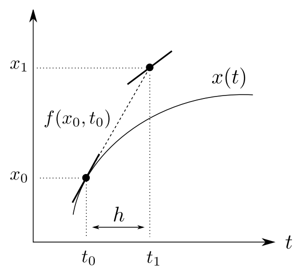
We represent the continuous solutions $x(t)$ using a discrete set of values x_0, x_1, x_2 . . . which are estimates of the true solution at discrete times t_1, t_2 . . .



Solution generated at points t_n where $t_{n+1} = t_n + h$ and h is the stepsize. Hopefully x_n will be close to $x(t_n)$. . .

Euler's method

$$x_{n+1} = x_n + hf(x_n, t_n)$$



Assume the solution continues in a straight line with the same gradient that it must have at t_0.

Euler's method: example

Consider $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + t$ with $x(0) = 0$.

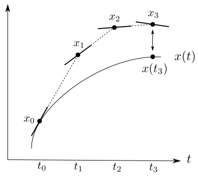
Let's estimate the solution with Euler's method and step-size $h = 0.1$. We start with $x_0 = 0$ and $t_0 = 0$.

New points are generated with $t_{n+1} = t_n + h$ and $x_{n+1} = x_n + hf(x_n, t_n)$ where $f(x_n, t_n) = x_n + t_n$.

n	t_n	x_n	$f(x_n, t_n)$
0	0.0	0	0
1	0.1	0.0	0.1
2	0.2	0.01	0.21
3	0.3	0.031	0.331
4	0.4	0.0641	0.4641
5	0.5	0.11051	0.61051

Global error

Global error after n steps. $\quad \text{error} = x_n - x(t_n)$



The global error after n steps is the difference between our estimate x_n and the true solution $x(t_n)$.
Hopefully the global error gets smaller when the local error gets smaller - but this is not always the case (e.g. for stiff problems).

Errors example

Consider $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + t$ with $x(0) = 1$ (again). True solution is $x(t) = e^t - t - 1$.

We estimated the solution with Euler's method and step-size $h = 0.1$. We can now compute the true solution $x(t_n) = e^{t_n} - t_n - 1$ and error $x_n - x(t_n)$.

n	t_n	x_n	$x(t_n)$	$x_n - x(t_n)$
0	0.0	0	0.000000	0.000000
1	0.1	0.0	0.005171	-0.005171
2	0.2	0.01	0.021403	-0.011403
3	0.3	0.031	0.049859	-0.018859
4	0.4	0.0641	0.091825	-0.027725
5	0.5	0.11051	0.148721	-0.038211

So error at $t = 0.5$ (after 5 steps) is approximately -0.04 (clearly not very accurate).

Methods summary

- Euler method (EM): explicit, 1st order
$$x_{n+1} = x_n + f(x_n, t_n)$$
- Backward Euler (BEM): implicit, 1st order.
$$x_{n+1} = x_n + f(x_{n+1}, t_{n+1})$$
- Explicit midpoint (EMM): explicit, 2nd order
$$x_{n+\frac{1}{2}} = x_n + \frac{h}{2} f(x_n, t_n)$$
$$x_{n+1} = x_n + hf\left(x_{n+\frac{1}{2}}, t_n + \frac{h}{2}\right)$$
- Implicit midpoint (IMM): implicit, 2nd order
$$x_{n+1} = x_n + f\left(\frac{x_n + x_{n+1}}{2}, t_n + \frac{h}{2}\right)$$

Methods summary continued

Runge-Kutta (RK4), explicit 4th order:

$$\begin{aligned}
k_1 &= hf(x_n, t_n) \\
k_2 &= hf(x_n + k_1/2, t_n + h/2) \\
k_3 &= hf(x_n + k_2/2, t_n + h/2) \\
k_4 &= hf(x_n + k_3, t_n + h) \\
x_{n+1} &= x_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

You don't need to memorise these methods but you should understand how to apply them from the definitions shown here.

Methods example

Consider $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + t$ with $x(0) = 1$ (again).

We'll show the result for each of the methods considered:

n	t_n	EM	BEM	EMM	IMM	RK4
0	0.0	0.000000	0.000000	0.000000	0.000000	0.000000
1	0.1	0.000000	0.011111	0.005000	0.005263	0.005171
2	0.2	0.010000	0.034568	0.021025	0.021607	0.021403
3	0.3	0.031000	0.071742	0.049233	0.050197	0.049858
4	0.4	0.064100	0.124158	0.090902	0.092323	0.091824
5	0.5	0.110510	0.193509	0.147447	0.149409	0.148721

Formula for Backwards Euler can be worked out from f as

$$x_{n+1} = \frac{x_n + h(t_n + h)}{1 - h}$$

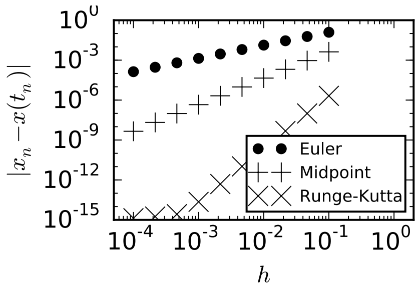
For implicit midpoint we have

$$x_{n+1} = \frac{x_n(1 + \frac{h}{2}) + h(t_n + \frac{h}{2})}{1 - \frac{h}{2}}$$

Global error using different methods to solve for $x(1)$ given

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x, \quad x(0) = 1$$

(True answer is e.)



Order of error

- Local error is the error after 1 step of size h.
- Global error is the total error integrating from t_1 to t_2 in many ($\frac{t_2-t_1}{h}$) steps of size h.
- We say a method has order n if the local error is $O(h^{n+1})$.
- Hopefully a method of order n will have global error that is $O(h^n)$.
- Global error may not scale as expected if e.g. the problem is stiff or has conserved quantities.
- Stiff problems: need a stable (implicit) method.
- Conserved quantities: need a geometric integrator (e.g. implicit midpoint).