Signals recap

1. Signal basics – elementary operations, delay, shift etc.
2. Time domain statistical techniques – moments etc.
3. Signals in the frequency domain
   - Basis function is a rotation, requires 2D description
   - Transform of a real-valued signal produces +ve and –ve spectrum; inverse transform of a +ve spectrum creates complex (analytical) signal.
4. Sampling – discretising the time domain
   - The range over which signals are unique is limited- aliasing
   - Time normalise to the sampling interval
5. Discretising the frequency domain i.e. in the DFT
   - Frequencies normalising to the sequence length.

Signals recap

- Most often you will be working with a sampled, real valued signal.

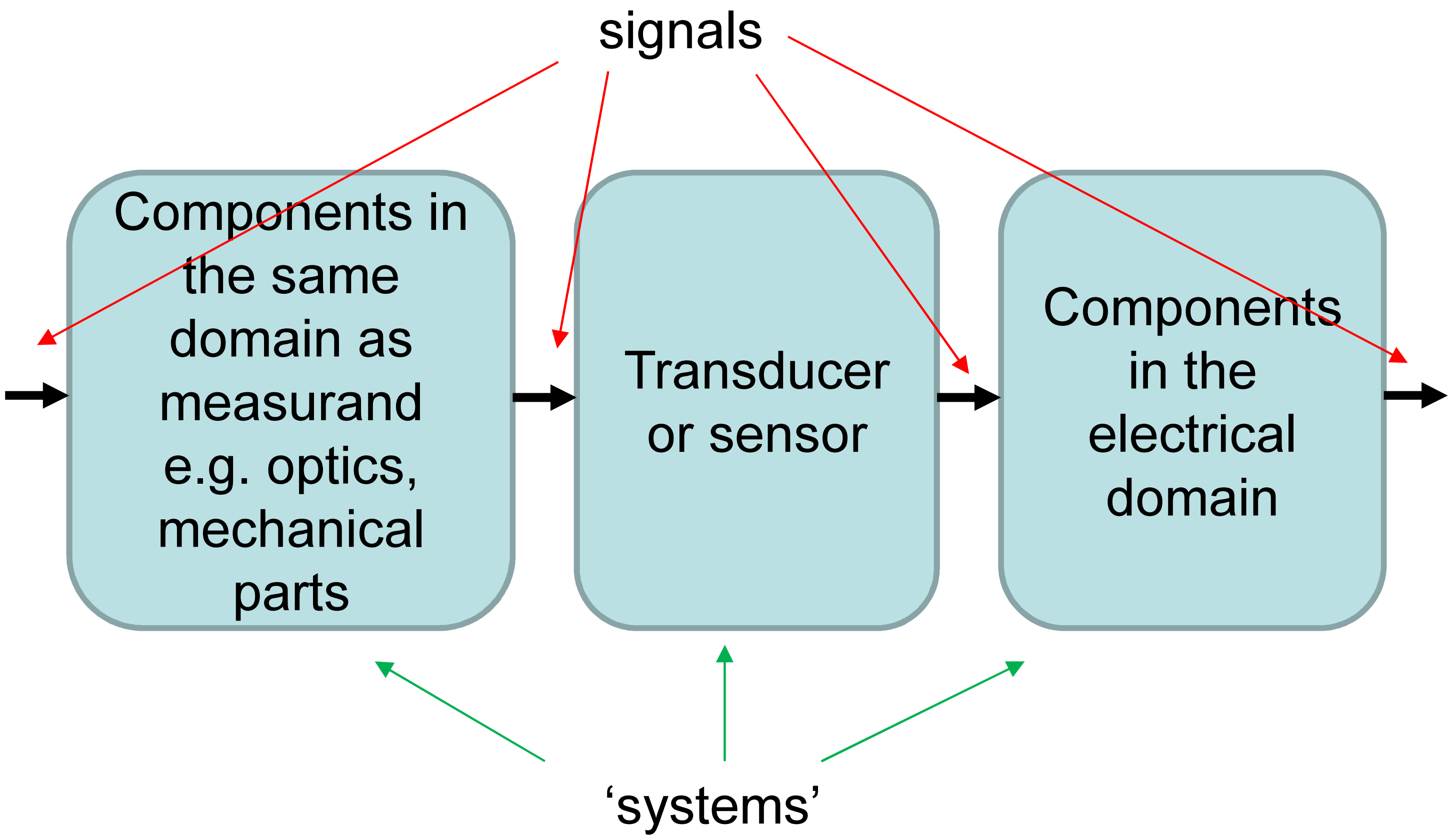
What haven't we covered?

- Signals where the frequency content changes with time;
  - Short time Fourier transform – take short sections of the signal and perform FT on each, or
  - Wavelet transform

- Recovering a signal from samples

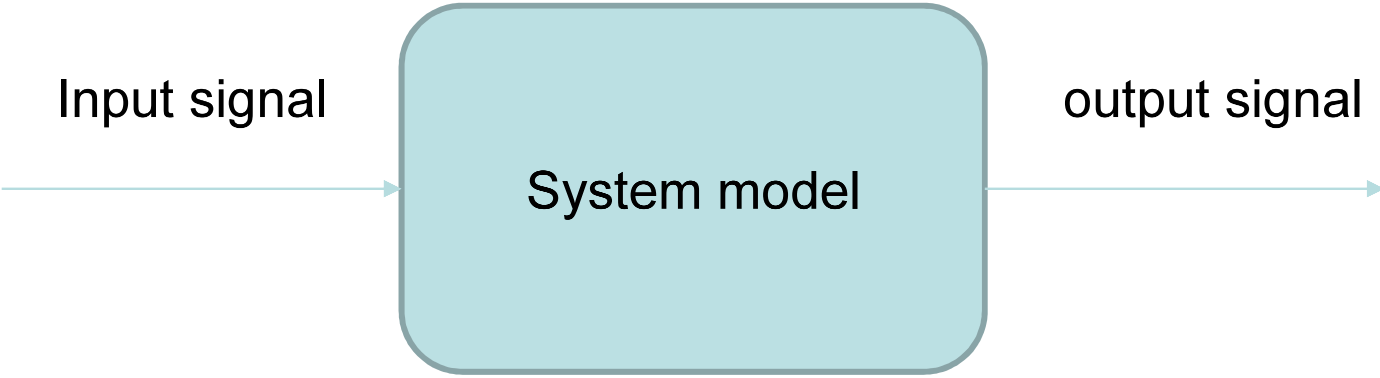
SYSTEMS Pt. 1
LTI, impulse response and convolution

Why model systems?



- So far we have looked at modelling signals, to complete our overall model we need to develop models of the components which act on the signals – the systems.

System model



- What we would like to determine some model for the system that can be used to derive the output for a given input signal.
- It isn't quite as easy as you might think to do this…….

- Similar to signals, we can describe the system model in more than one domain.

LTI systems

- The class of systems we are going to consider are called 'Linear Time Invariant'

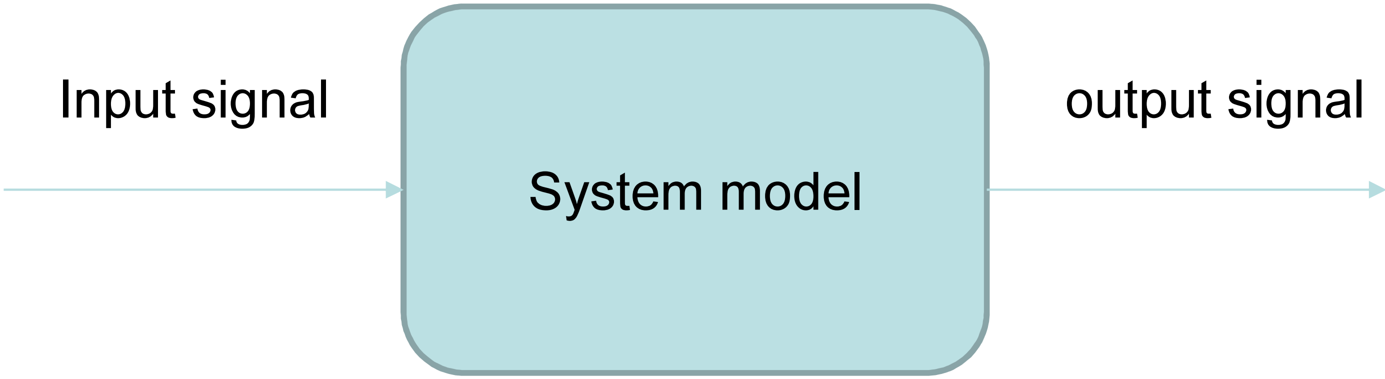
- Linearity
  - If input $x_1(\mathrm{t})$ produces output $y_1(t)$ and input $x_2(\mathrm{t})$, output $y_2(t)$, then input $x_1(\mathrm{t}) + x_2(\mathrm{t})$ produces output $y_1(t) + y_2(t)$
  - This is called *SUPERPOSITION* and is a really useful property.

- Time invariance
  - If input $x_1(\mathrm{t})$ produces $y_1(t)$, then the same input at a different time, $x_1(\mathrm{t} + \Delta t)$ produces the same output, at a corresponding time, $y_1(t + \Delta t)$
  - This is really saying that the system parameters don't change with time, just the input and output.

More system properties

- Homogeneity
  - If $x_1(\mathrm{t})$ gives $y_1(t)$, then $nx_1(\mathrm{t})$ gives $ny_1(t)$
  - In practice this means that the constant terms in differential expressions have to be zero
- Causal
  - As system is causal when the current output is a function of current or past inputs only
    - i.e. the system has a memory of past inputs
  - *An acausal system is also a function of future inputs*
- Stability
  - For a Bounded Input, Bounded Output (BIBO)

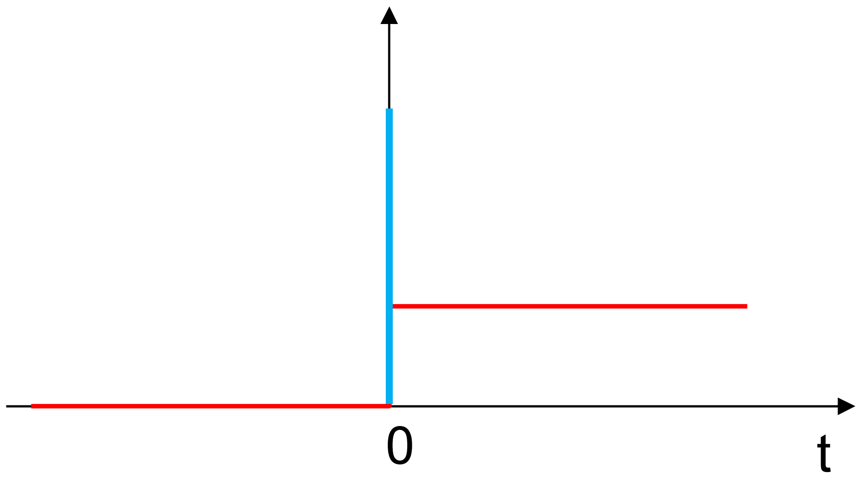
Food for thought



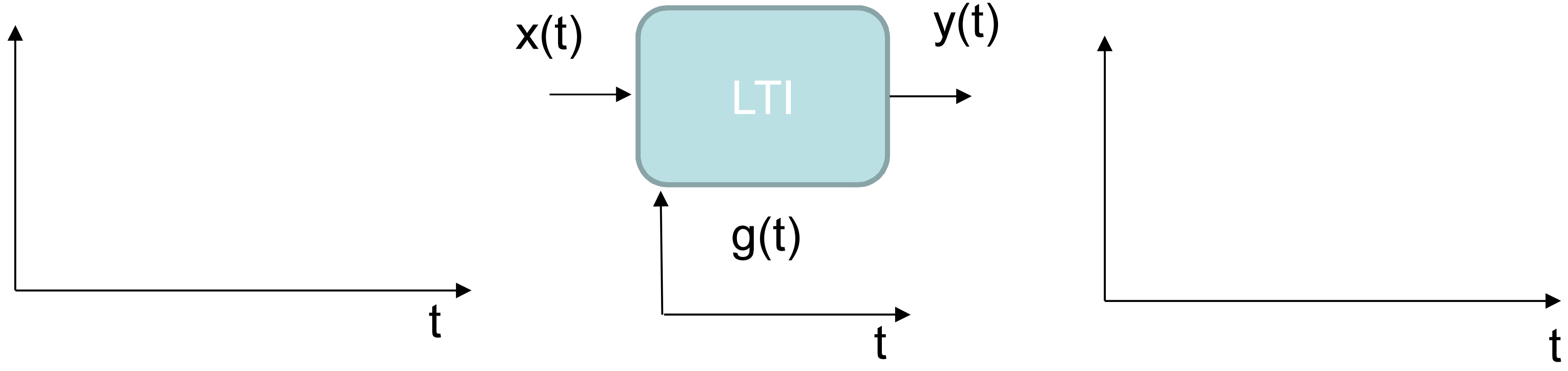
- If our system has no dynamics, i.e. it is just a gain, then the output only depends only on the current input. This case is simple.
- But we want to consider systems with dynamic behaviour – dynamics implies that the action of a particular input extends beyond the instant at which it was applied.
- Thus at any moment our output will be a function of present and past inputs.
- Super-position applies so we can ‘add up’ the contribution of inputs at various times…

Time domain analysis 1: What is an impulse?

- The impulse, or Dirac delta function was introduced by the physicist Paul Dirac.
- It describes a distribution with area of 1, as the width tends to zero.
- Alternatively it is a signal that is zero everywhere except at t=0 where it is infinite.
- It is also the differential of the unit step function.

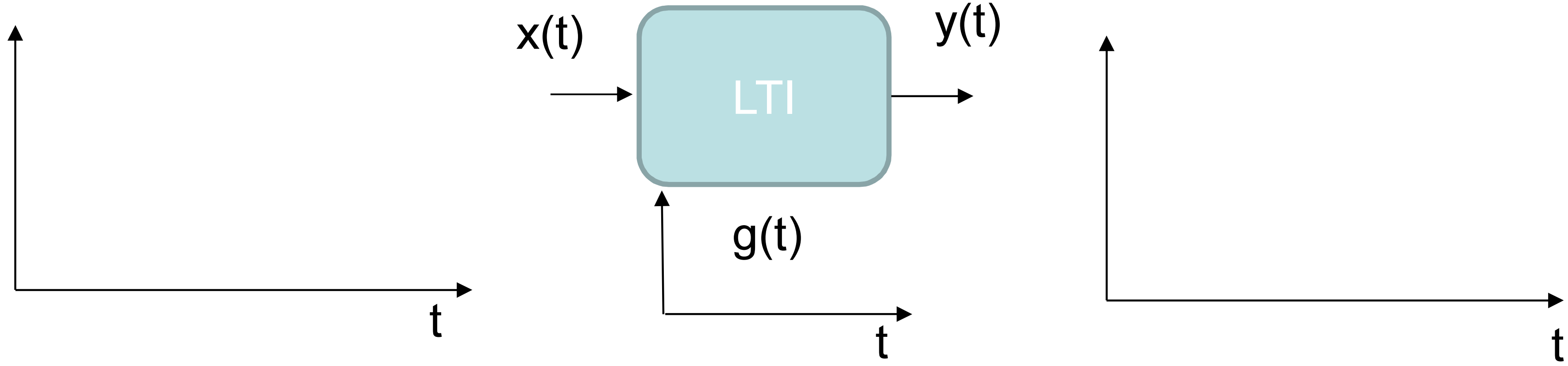


Time domain analysis 2 - Impulse response



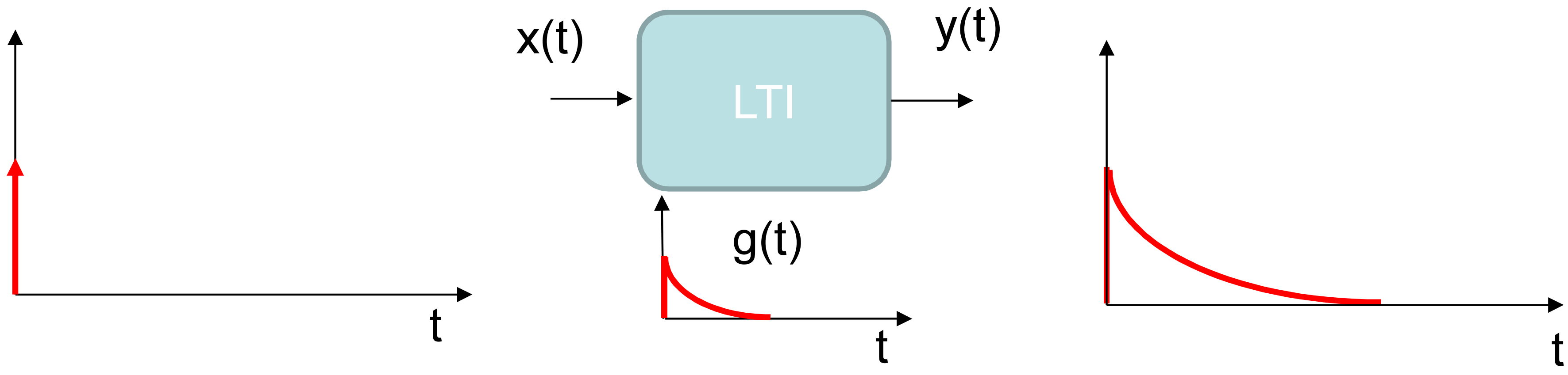
- The first step in time domain analysis is to understand how we approach the problem.

Time domain analysis 2 - Impulse response



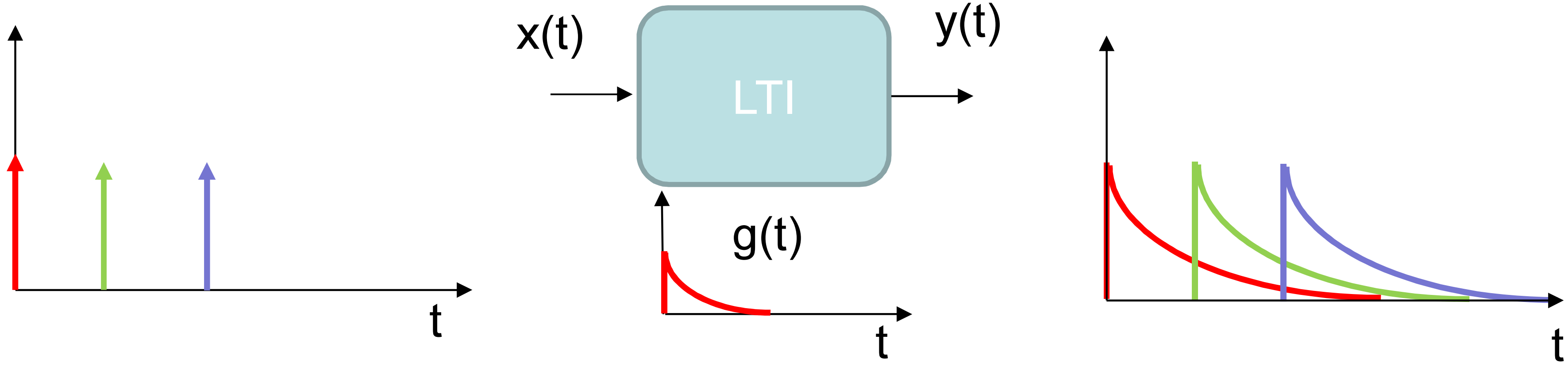
- All systems that have energy storage elements (so everything interesting) have a ‘memory’ – energy input at a particular instant in time is seen at the output over a range of time. So the output at any time instant will be the summation of contributions from the current and past inputs

Time domain analysis 2 - Impulse response



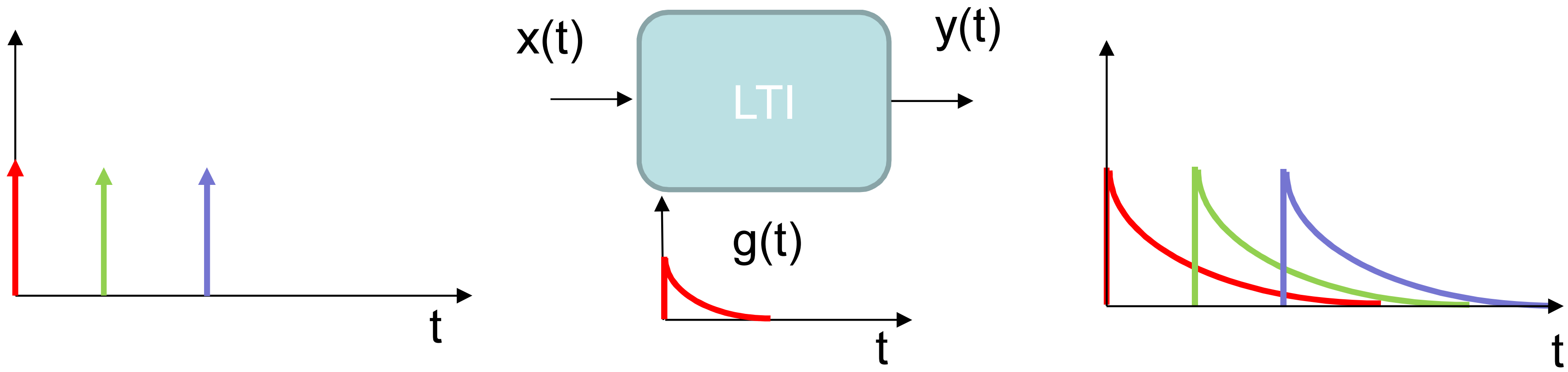
- To ‘tease’ out individual contributions we characterise the system for a solitary, brief, time input – the impulse. The LTI system is described by its response to this impulse, g(t).

Time domain analysis 2 - Impulse response



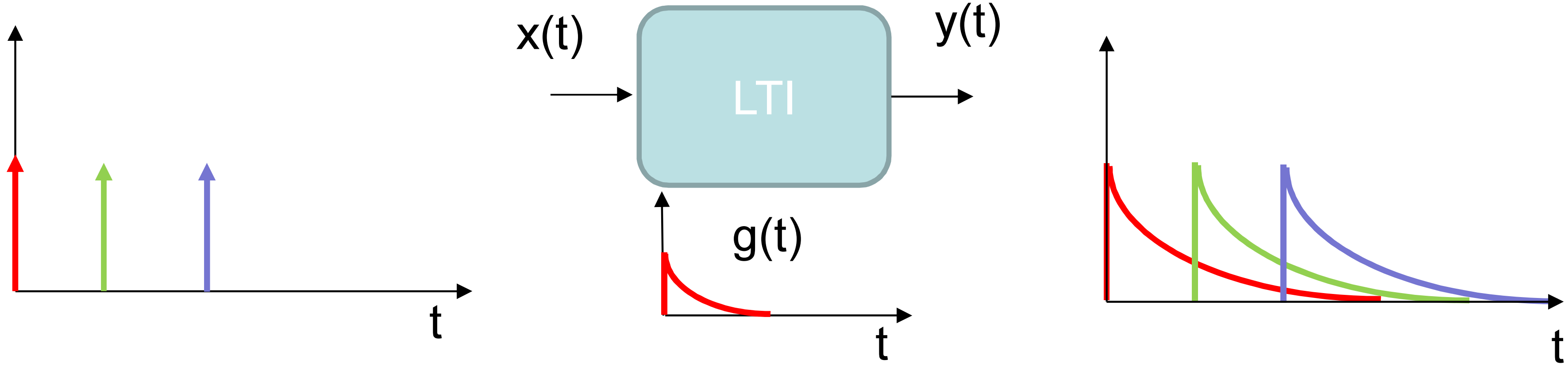
- If we present the system with a train of input impulses, the output is a train of the responses to the individual impulses.

Time domain analysis 2 - Impulse response



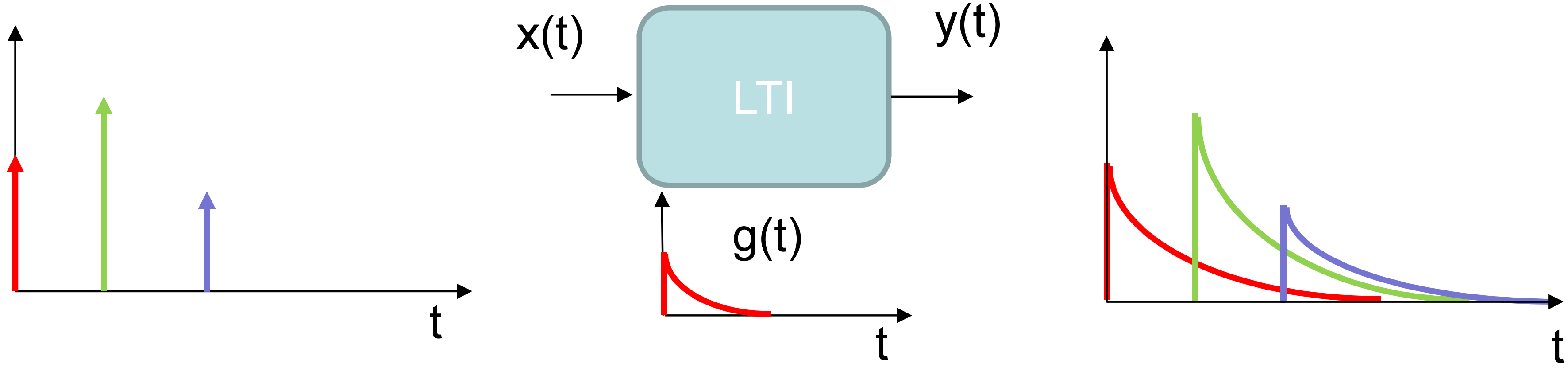
- Here’s the first mind-bending bit: time for the input and output is just ‘normal’ linear time from t=0 when we start to wherever we are now. But time from the system’s point of view ‘slides’ so that the current time is always t=0. If system t ≠0 we are looking back (or forward) in time.

Time domain analysis 2 - Impulse response



- In calculations we need to introduce a second time variable (say τ) to take this time sliding into account.

Time domain analysis 3 - Impulse response



- It is (hopefully) only a short stretch to see how if the magnitude of the input impulses change, then so do the output components – remember this is an LTI system!
- The question that the quicker of you will now be asking is – 'but the input signal has a bounded magnitude at any moment and the impulse (or dirac delta function) has an infinite amplitude, so how can the input signal be the sum of lots of *weighted* and delayed impulses?'
- There are two options;
  1. Take it on trust – you don't need to *understand* it for this course
  2. Listen to the following…

Time domain analysis 2a - Sifting property

- The impulse function has a particular property called the sifting (or sampling) property defined as;

$$x(t) = \int x(\tau)\delta(t-\tau)d\tau$$

- Remember ‘t’ is our time, as we would observe at the input or output, and ‘τ’ is our sliding time – looking backwards or forwards from where we are now.

Time domain analysis 2a - Sifting property

- The impulse function has a particular property called the sifting (or sampling) property defined as;

$$x(t) = \int x(\tau)\delta(t-\tau)d\tau$$

- At first it looks quite pointless: why express the value as the sum of weighted impulse functions when we could just directly substitute 't' into 'x'? – we need to know 'x' to solve the integral anyway. But remember, all we wanted to show was that a signal can be represented as the sum of impulse functions, not for it to be any use *per*-*se*. (note that is it useful for other things!).

Time domain analysis 2a - Sifting property

- The impulse function has a particular property called the sifting (or sampling) property defined as;

$$x(t) = \int x(\tau)\delta(t-\tau)d\tau$$

- To work out what is going on you need to remember that:
  - The impulse (δ) function has a value that is zero everywhere except at 0, hence the above integral is zero everywhere except when $t=\tau$
  - The impulse function has the property that its integral is 1. $\int_{-\infty}^{\infty} \delta(t)dt = 1$
  - So when $t=\tau$ the integral returns the value of the signal at ‘t’

Time domain analysis 2a - Sifting property

- The impulse function has a particular property called the sifting (or sampling) property defined as;

$$x(t) = \int x(\tau)\delta(t - \tau)d\tau$$

- It may seem circular to start a description of convolution by using the integral above, but if you can see how we are representing a current value ($x(t)$) as the sum of a weighted ($x(\tau)$) and delayed function ($\delta(t-\tau)$), then you are most of the way to understanding convolution…..

Time domain analysis 3: Convolution

$$y(t) = \int x(\tau)g(t-\tau)d\tau$$

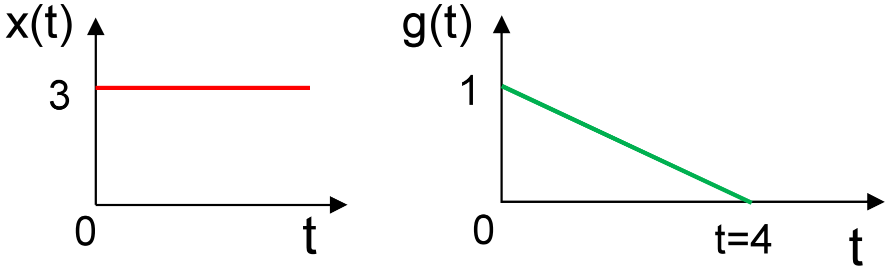
$$(xconv.g)(t) = \int x(\tau)g(t-\tau)d\tau$$

If we extend our individual pulse approach into a continuous expression we can derive the convolution integral above. It describes the convolution of ‘x’ with ‘g’.

This expresses the system output at any moment in time as the continuous sum of the current and past inputs weighted by the impulse response.

T and τ are both variables that act independently along the time axis

Time domain analysis 3: Visualising Convolution

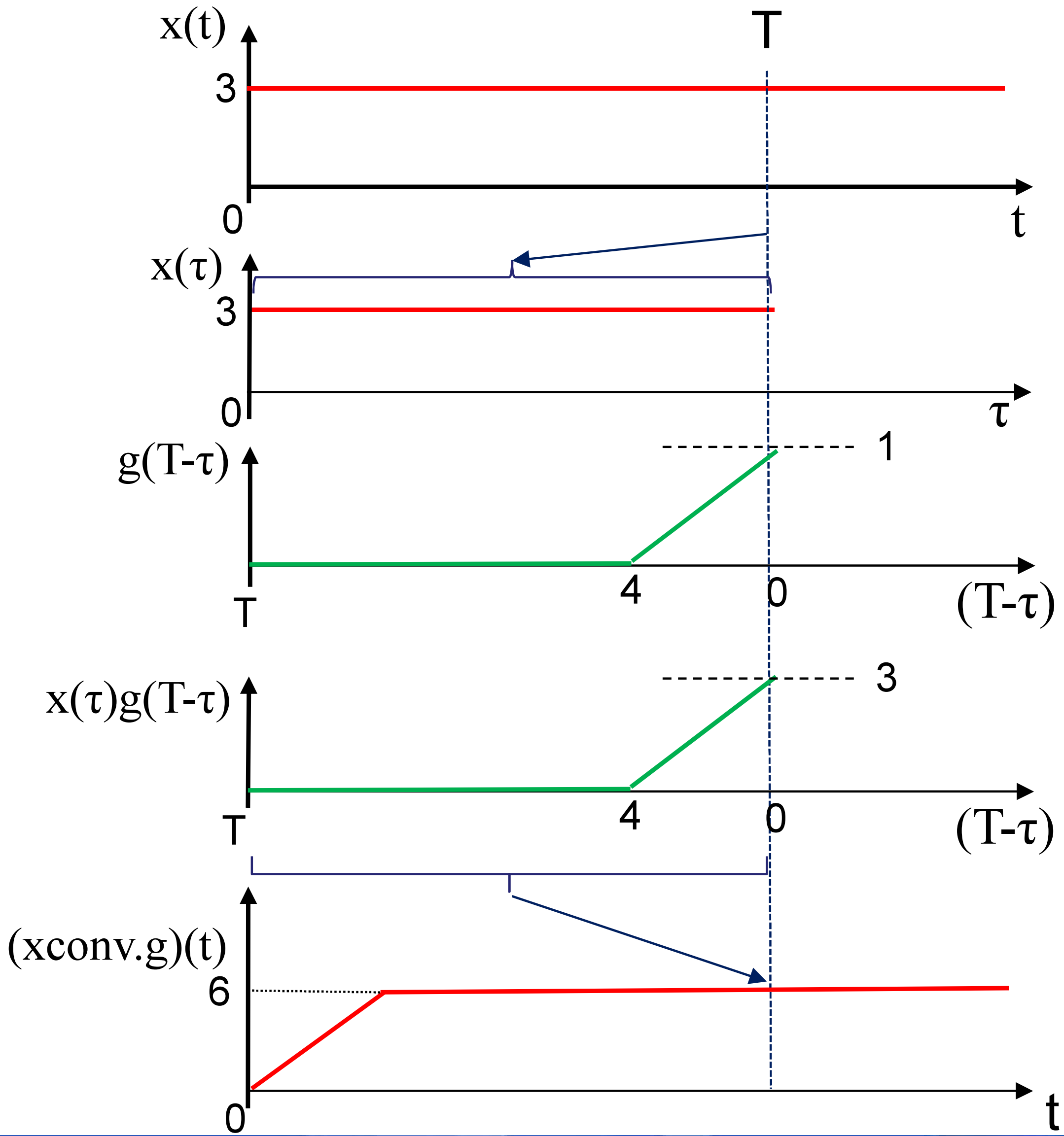


Imagine an input signal of constant value ‘3’ that we are applying to a system with a impulse response, g(t)=1-0.25t for 0<t<4.

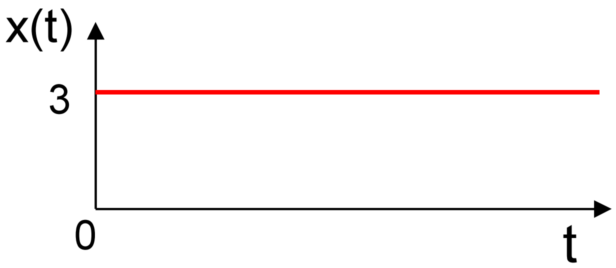
Consider the convolution at a moment in time, T. Inside the integral t is now constant T and our variable τ, has a range 0<τ<T.

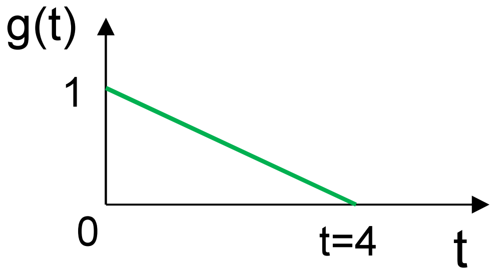
The argument of x inside the integral is τ (this looks just like x(t)). The argument of g is T-τ, which is zero at the present instant and increases towards T at τ=0 – this has the effect of ‘reversing’ the system response.

The convolution of x and g at time t=T is given by the integral of x(τ).g(T-τ)



Time domain analysis 3: Visualising Convolution

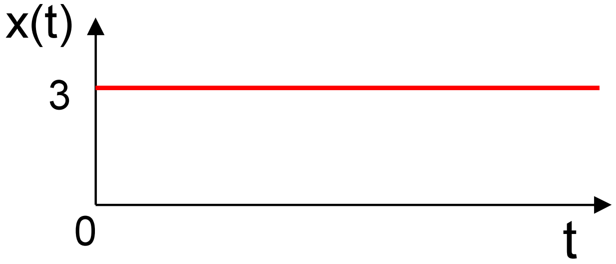


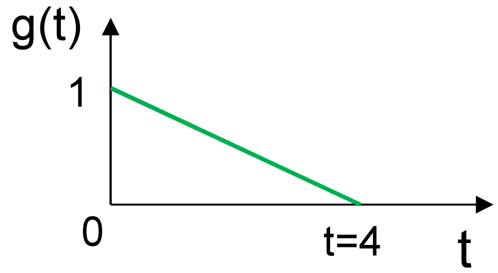


We can also describe convolution in words:

The system response above can be seen as saying: the component of the output coming from the input signal right now is 1; The output component coming from the input 2 seconds ago is ½; or we could generalise and say the output component coming from the input x seconds ago is 1-0.25x, for 0<x<4. Any component from more than 4 seconds ago has died away.

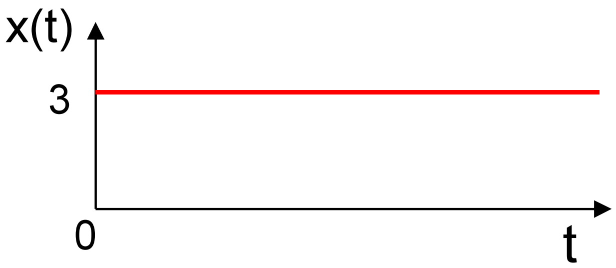
Time domain analysis 3: Visualising Convolution

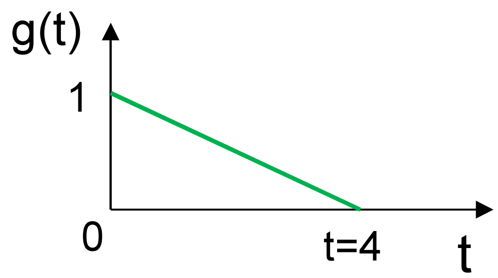




An important property to realise when trying to understand convolution is that the impulse response is reversed in time because of how the argument is expressed. Some examples you will see, for example in online resources, use symmetrical functions which mask this.

Time domain analysis 3: Visualising Convolution





Some times convolution is described a ‘windowing function’ where the input signal is ‘viewed’ through the moving ‘window’ of the impulse response. If you want to push the analogy, think of sitting on a train, facing backwards with a window adjacent to you. The scenery you pass is the input signal, the window somewhat like the system response. Out of your window you will see the scenery you are passing right now plus, into the distance, scenery that you past a few seconds ago. The convolution is the aggregate picture.

Comparison of Convolution and Correlation

- Both correlation and correlation use two variables operating on the time axis, we are using ‘t’ for our normal time, and ‘τ’ for our ‘sliding’, or ‘offset’ into past/future time.

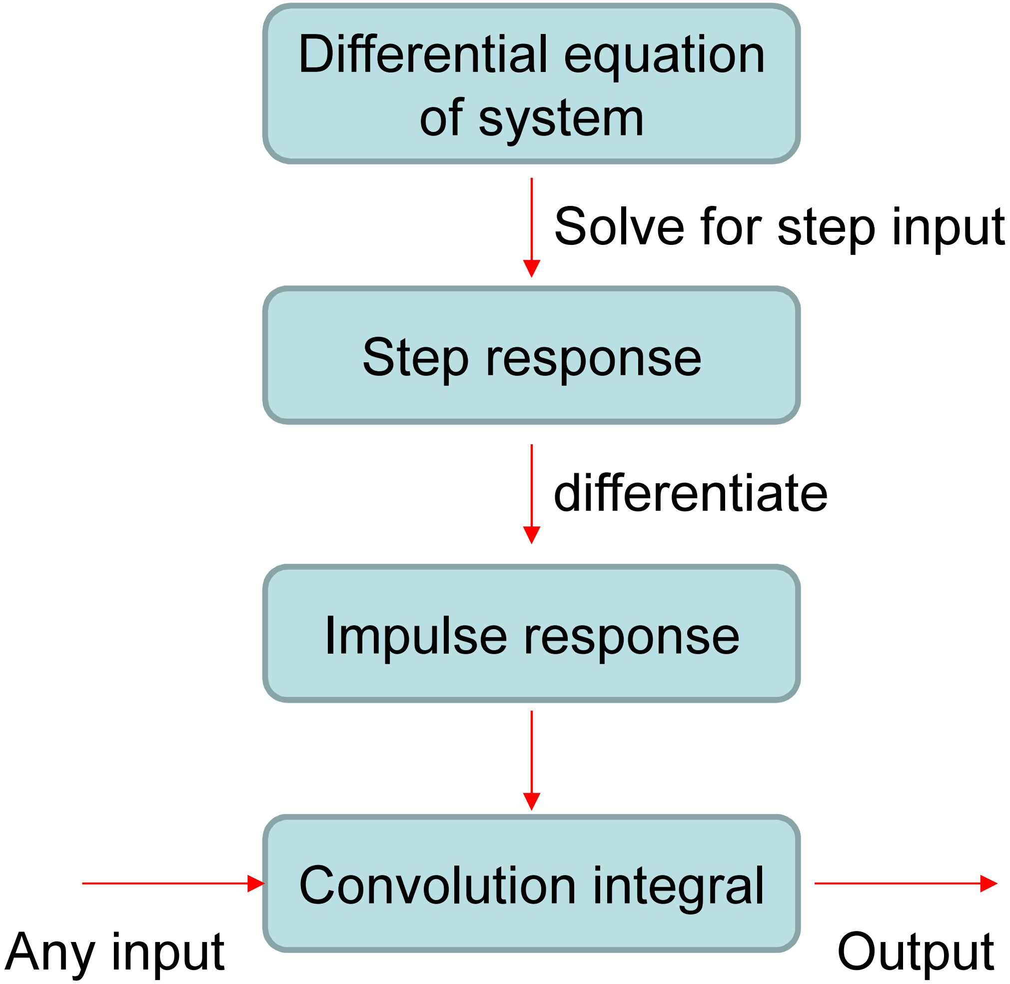
$$(f corr. g)(\tau) = \int f(t) g(t+\tau) dt$$

- Correlation is a function of offset, ‘τ’, but the integral is over time, ‘t’. It is the sum of the product of the functions over time, ‘t’, for various offsets, ‘τ’.
- The integral is solved for one value of offset, ‘τ’,

$$(f conv. g)(t) = \int f(\tau) g(t-\tau) d\tau$$

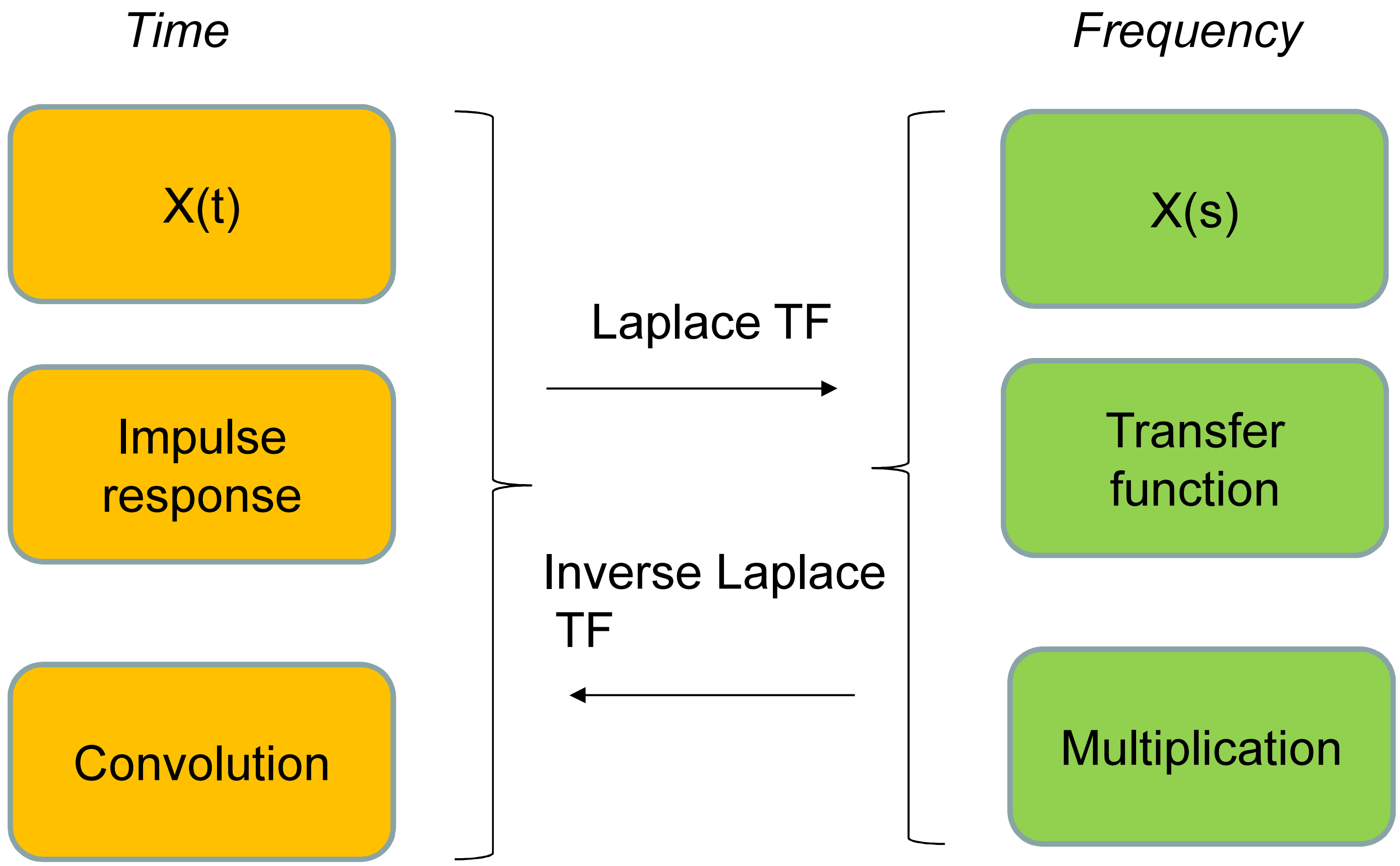
- Convolution is a function of ‘t’, but the integral is over ‘τ’. The way the argument to ‘g’ is expressed ‘reverses’ the response. It is the sum of the product of the functions for over past times ‘τ’, at a particular time, ‘t’.
- The integral is solved for one value of time, ‘t’

Solving in the time domain



- Solving in the time domain involves several steps.
- Typically the differential equation of the system is solved for a step input and this is then differentiated to determine the impulse response.
- Then the impulse response is used in the convolution integral, along with the input signal, to determine the output signal

Solving in the frequency domain



Solving in the frequency domain

- In the frequency domain, convolution becomes multiplication – a much easier task!
- The Transfer Function of a system is easier to derive from the differential equations.
- Not only is it easier to model LTI systems in the frequency domain, it is also a useful tool for solving differential equations.