EMAT10100 Engineering Maths I
Lecture 17: Eigenvalues and Eigenvectors (part 2)

John Hogan & Alan Champneys

Last lecture

- Eigenvalues λ & eigenvectors $\mathbf{v}$ defined by
$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$
- An eigenvector is direction that is held fixed by the matrix transformation
- The corresponding eigenvalue is the stretch factor
- Calculate eigenvalues by setting $\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$
- $\Rightarrow$ characteristic polynomial $P(\lambda) = 0$
- roots are eigenvalues λ
- Calculate eigenvectors by solving $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v} = \mathbf{0}$ for $\mathbf{v}_i$ for the eigenvector $\mathbf{v}_i$ corresponding to eigenvalue λ_i
- should get an under-determined system, so that eigenvector is defined up to a scalar multiple α
- Sometimes choose α to make $\mathbf{v}_i$ a unit vector

The trace of a matrix

- **Definition**: The trace of a matrix is the sum of its diagonal entries:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$
- **Fact**: The trace equals the sum of all eigenvalues:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i, \qquad \text{where} \quad \mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$$
- Exercise 1: Check for the two example matrices: $\begin{pmatrix} 3 & -3 \\ -2 & -2 \end{pmatrix}$ and $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ whose eigenvalues are:
4 & -3 and -1 & -3 respectively

The product of the eigenvalues

- **Fact**: The determinant of a matrix $\mathbf{A} = a_{ij}$ is the product of its eigenvalues. That is:
$$|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i$$
- Exercise 2: Check this for the matrices we have previously used as examples
$$\begin{pmatrix} 3 & -3 \\ -2 & -2 \end{pmatrix}, \qquad \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$
- This make sense because determinant gives the area / volume scaling factor in 2D or 3D, whereas eigenvalues give stretch factors in separate (eigenvector) directions

Other useful properties of eigenvalues

Suppose that a matrix $\mathbf{A} = a_{ij}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

- The eigenvalues of $\mathbf{A^{-1}}$ if it exists are given by

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$$

- The eigenvalues of the transposed matrix $\mathbf{A^T}$ are the same as $\mathbf{A}$.
- If k is a scalar then the eigenvalues of k$\mathbf{A}$ are

$$k\lambda_1, k\lambda_2, \ldots, k\lambda_n$$

- If k is a positive integer then the eigenvalues of $\mathbf{A}^k$ are

$$\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$$

A 3×3 example

- The same principles work for finding eigenvalues and eigenvectors for matrices of any dimension. For an n-dimensional matrix, the characteristic polynomial $P(\lambda)$ is of nth-order and has n independent solutions.
- Exercise 3: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

1. Find its characteristic polynomial $P(\lambda)$
2. Show that the eigenvalues (roots of the polynomial) are $\lambda_{1,2,3}$ = -1, 1 and 2,
3. Find a **unit** eigenvector corresponding to the eigenvalue $\lambda = -1$.

What if $P(\lambda)$ doesn't have real solutions?

- E.g. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Its characteristic $P(\lambda) = \det(\mathbf{A} - \mathbf{I}_2\lambda) \quad = \lambda^2 - 2\lambda + 2,$ which has complex roots $\lambda = 1 \pm j$.

- **Fact**: Complex eigenvalues of a real matrix come in complex conjugate pairs $\mu \pm j\omega$. But what do complex eigenvalues mean?
- Example: Consider first a pure rotation:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

its eigenvalues are $\cos(\theta) \pm j\sin(\theta) = e^{j\theta}$ (can you show this?).

- Hence a pure rotation is given by a unit complex $re^{\pm j\theta}$ with modulus $r = 1$ and argument θ being the rotation angle

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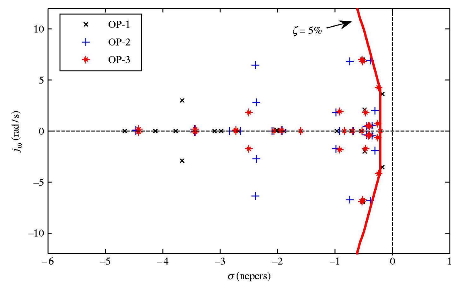
Applications of eignenvalues/vectors:

I. Stablitiy theory

Let $\mathbf{x}(t)$ be a vector of unknowns; consider linear system:

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$$

$\mathbf{x} = \mathbf{0}$ stable if all eigenvalues in left-half of complex plane.



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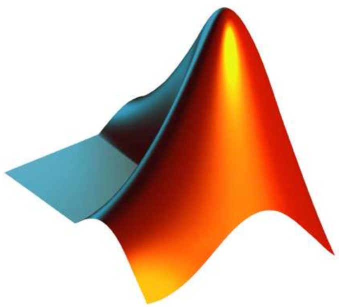
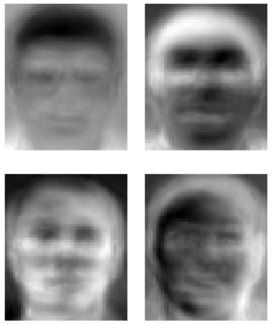
Applications of eignenvalues/vectors:

II. Structural vibration modes are eigenvectors of $M^{-1}K$; where M mass matrix, K stiffness; eigenvalues are frequencies



III. Image processing

Image is (large) vector v of pixels; compare using covariance matrix; decompose using its eigenvectors:



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What about more general complex eigenvalues?

What does a general complex conjugate eigenvalue $\lambda = \mu \pm j\omega$ mean geometrically?

- **Fact** If we express in polar form $\lambda = re^{\pm j\theta}$, then:
  - θ gives the amount of rotation
  - r gives the amount of stretch in the radial direction
- **Fact:** Given complex eigenvalues, the eigenvectors are complex conjugate too:

$$\mathbf{v} = \mathbf{u} \pm j\mathbf{w}$$

- But what does a complex eigenvector this mean ?!!
  - A1. In more than 2D, the two vectors $\mathbf{u}$ and $\mathbf{v}$ define a plane in which the stretch and rotation take place
  - A2. Don't worry about it, we don't really concern ourselves with complex eigenvectors

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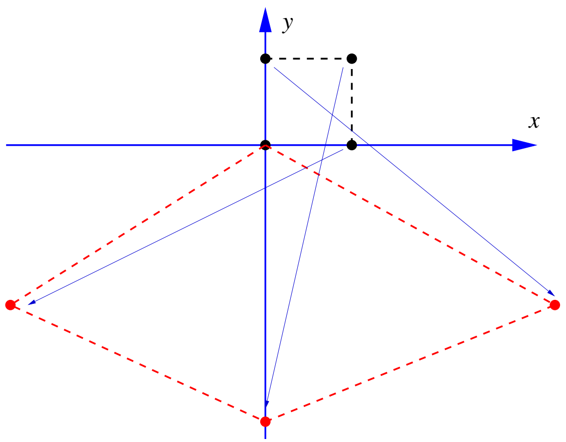
small modification to example from last lecture

- $\mathbf{A} = \begin{pmatrix} -3 & 3 \\ -2 & -2 \end{pmatrix}$
- This is clearly a rotation of some kind!



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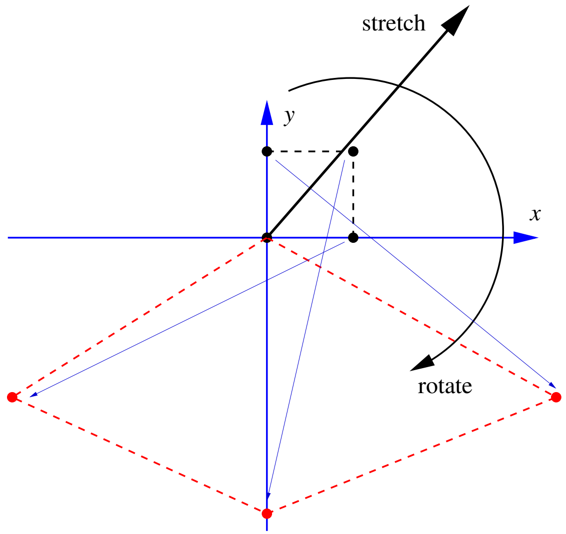
- radial stretch of $2\sqrt{3}$
- rotation of $\theta = 68.1°$ radians.
- Exercise: show this!



Homework

- Read *James* 5.7
- Do exercises 4th edition
  - 5.7.3 Q.96, Q.97 a), b), g), h)
  - 5.7.8 Q.104,105
- Do exercises 5th edition
  - 5.7.3 Q.94, 95 a), b), g), h)
  - 5.7.8 Q.103,104
- Try to get up to date: Don't forget the QMP questions each work
- An Assessed homework will be handed out next Monday (week 7)
  - for completion during the Reading Week (week 8)
  - like the class test, the marks will not count towards the unit mark