Vibrations 2, Lecture 16
Forced 2DOF systems with damping

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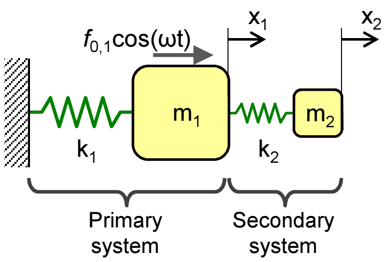
Lecture 15 review

Undamped TVA:



Tuning condition:

$$\omega_0^2 = \frac{k_1}{m_1} = \frac{k_2}{m_2} = \omega_a^2$$

Stiffness and mass ratio and its influence on the natural frequencies:

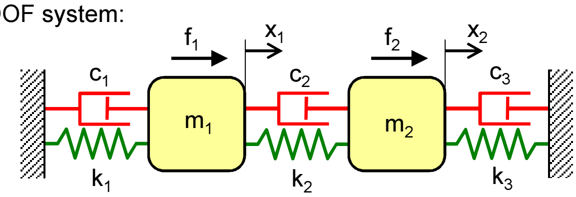
$$\mu = k_2/k_1 = m_2/m_1,\ \ \omega_i = \omega_i(\mu)$$

Lecture 16

- Damped 2DOF system with harmonic forcing
  - Damping matrix
- Complex response and damped TVA
- 2DOF wing example
  - Heaving + Pitching rigid wing with damping
  - Derivation using Newton

General forced damped 2DOF system

Damped 2DOF system:



Equations of dynamic equilibrium:

$$-k_1 x_1 - c_1 \dot{x}_1 - m_1 \ddot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + f_1 = 0$$

$$-k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - m_2 \ddot{x}_2 - k_3 x_2 - c_3 \dot{x}_2 + f_2 = 0$$

EOMs in *matrix form* (note a new term, where **C** is the damping matrix):

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}(t)$$

Steady-state harmonic response

Consider the problem with the steady-state harmonic response and excitation:

$$\mathbf{f}(t) = \begin{bmatrix} f_{0,1} \\ f_{0,2} \end{bmatrix} (\cos(\omega t) + i\,\sin(\omega t)) = \mathbf{f}_0\, e^{i\omega t}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} (\cos(\omega t) + i\,\sin(\omega t)) = \mathbf{x}_0\, e^{i\omega t}$$

$$\mathbf{x}(t) = \mathbf{x}_0\, e^{i\omega t} \Rightarrow \dot{\mathbf{x}}(t) = i\omega\,\mathbf{x}_0\, e^{i\omega t} \Rightarrow \ddot{\mathbf{x}}(t) = -\omega^2\,\mathbf{x}_0\, e^{i\omega t}$$

where $f_{0,i}$ is the *complex* amplitude of the excitation force and $x_{0,i}$ is the *complex* amplitude of the steady-state response. The use of complex numbers enables the capture of all phase relationships caused by damping.

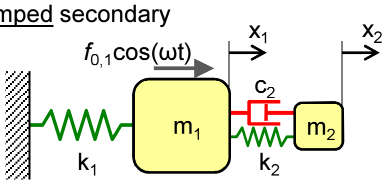
The excitation vector **f**(*t*) and response **x**(*t*) are substituted to EOM to obtain the steady-state response:

$$(-\omega^2\mathbf{M} + i\omega\,\mathbf{C} + \mathbf{K})\,\mathbf{x}_0 = \mathbf{f}_0$$

$$\mathbf{x}_0 = (-\omega^2\mathbf{M} + i\omega\,\mathbf{C} + \mathbf{K})^{-1}\,\mathbf{f}_0$$

Damped TVA

2DOF = primary + damped secondary



EOM in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_{0,1} \\ 0 \end{bmatrix} e^{i\omega t}$$

The steady-state response – complex amplitudes (i.e. the motion magnitude and the phase angle specified relative to the input) – can be found from the following system of two linear *complex* equations:

$$\begin{bmatrix} (k_1 + k_2 - \omega^2 m_1) + i\omega c_2 & -k_2 - i\omega c_2 \\ -k_2 - i\omega c_2 & (k_2 - \omega^2 m_2) + i\omega c_2 \end{bmatrix}\begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = \begin{bmatrix} f_{0,1} \\ 0 \end{bmatrix}$$

Damped TVA

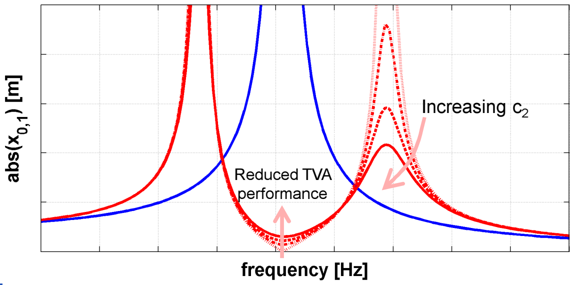
Using the response formula $\mathbf{x}_0=(\ldots)^{-1}\mathbf{f}_0$ and the formula for the 2×2 matrix inverse:

$$\begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = \frac{1}{\det(\ldots)} \begin{bmatrix} (k_2-\omega^2 m_2)+i\omega c_2 & k_2+i\omega c_2 \\ k_2+i\omega c_2 & (k_1+k_2-\omega^2 m_1)+i\omega c_2 \end{bmatrix} \begin{bmatrix} f_{0,1} \\ 0 \end{bmatrix}$$

$$x_{0,1} = \frac{(k_2-\omega^2 m_2)+i\omega c_2}{\det(\ldots)} f_{0,1}$$

An example of response $x_{0,1}$ is illustrated in the following graph:

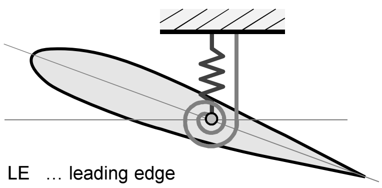
```
» k1=1000;m1=2;m2=0.2*m1;k2=m2*(k1/m1);c2=1.1;
» Nw=301;w=linspace(0,2*pi*10,Nw);x01=zeros(1,Nw);
» B=@(wi,cc)inv([k1+k2-wi^2*m1+i*wi*cc,-k2-i*wi*cc;
-k2-i*wi*cc,k2-wi^2*m2+i*wi*cc]);
» for ii=1:Nw,X=B(w(ii),c2);x01(ii)=X(1);end
» plot(w/2/pi,abs(x01))
```



2DOF wing example

Consider a simplified 2DOF model of a rigid wing suspended on the linear and torsional *springs* and *dampers* (these springs and dampers represent elasticity and dissipative properties of real wings). Derive the matrix EOM for this system.

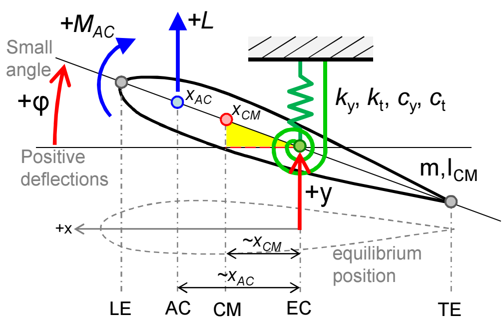
2DOF problem:



LE … leading edge
AC … aerodynamic centre
CM … centre of mass
EC … elastic centre
TE … trailing edge
m … wing mass
I_{CM} … moment of inertia about CM

DOFs: +y, +φ
Small angle φ: cosφ=1, sinφ=φ

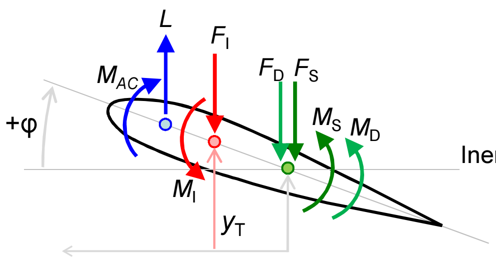
Geometry and applied loads:



2DOF wing example

FBD:

- General motion in 2D
  - vertical CM motion y_T
  - rotational motion φ about CM



$$y_T = y + \varphi x_{CM}$$

$$\ddot{y}_T = \ddot{y} + \ddot{\varphi} x_{CM}$$

Restoring and dissipative loads:

$$F_S = k_y y$$

$$M_S = k_t \varphi$$

$$F_D = c_y \dot{y}$$

$$M_D = c_t \dot{\varphi}$$

Inertial loads:

$$F_I = m\ddot{y}_T = m(\ddot{y} + \ddot{\varphi} x_{CM})$$

$$M_I = \underbrace{I_{CM}\ddot{\varphi}}_{\text{Rotation about CM}} + \underbrace{(m(\ddot{y} + \ddot{\varphi} x_{CM}))x_{CM}}_{\text{Moment about EC due to } F_I}$$

2DOF wing example

Equations of dynamic equilibrium:

$$-F_I - F_S - F_D + L = 0$$
$$-M_I - M_S - M_D + M_{AC} + Lx_{AC} = 0$$

$$m(\ddot{y} + \ddot{\varphi} x_{CM}) + k_y y + c_y \dot{y} = L$$

$$I_{CM}\ddot{\varphi} + (m(\ddot{y} + \ddot{\varphi} x_{CM}))x_{CM} + k_t \varphi + c_t \dot{\varphi} = M_{AC} + Lx_{AC}$$

Matrix form of EOM:

$$\begin{bmatrix} m & mx_{CM} \\ mx_{CM} & I_{CM} + mx_{CM}^2 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\varphi} \end{bmatrix} + \begin{bmatrix} c_y & 0 \\ 0 & c_t \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\varphi} \end{bmatrix} + \begin{bmatrix} k_y & 0 \\ 0 & k_t \end{bmatrix} \begin{bmatrix} y \\ \varphi \end{bmatrix} = \begin{bmatrix} L \\ M_{AC} + Lx_{AC} \end{bmatrix}$$

Notes:

- $I_{EC} = I_{CM} + mx_{CM}^2$, the mass moment of inertia about EC (hinge),
- mx_{CM}, the cross inertia term (inertial coupling, can be negative, zero, positive),
- this model can be further studied using *eigenvalue* and *harmonic* analysis.

Summary

- Damping effects are assembled in the damping matrix **C**
- Damping causes *general* response phase angles and complex numbers are used to capture this information
- It is not possible to tune the primary system with the damped TVA to zero vibration levels at the anti-resonance frequency