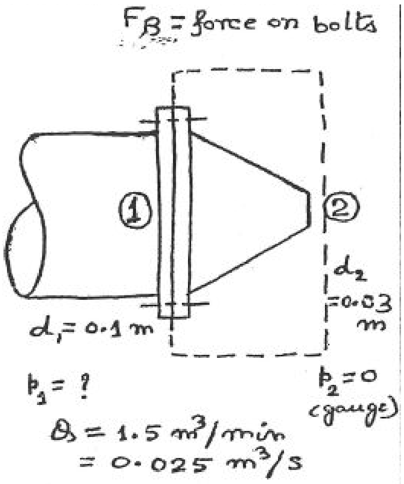
FLUIDS I
Example sheet 4: Control Volume Analysis

SOLUTIONS



• Q1 Steady, incompressible, frictionless.

CONTINUITY: $Q = V_1 A_1 = V_2 A_2 \quad \cdots (1)$

$$V_1 = \frac{Q}{A_1} = \frac{0.025}{\frac{\pi}{4}(0.1)^2}\ m/s = 3.18\ m/s$$

$$V_2 = \frac{Q}{A_2} = \frac{0.025}{\frac{\pi}{4}(.03)^2}\ m/s = 35.37\ m/s$$

BERNOULLI'S EQN.

$$p_1 = p_2 + \frac{1}{2}\rho_0 (V_2^2 - V_1^2)$$
$$= \frac{1}{2} \times 1000 \times [(35.37)^2 - (3.18)^2]\ Pa$$
$$= 6.205 \times 10^5\ Pa\ \text{(gauge)}$$

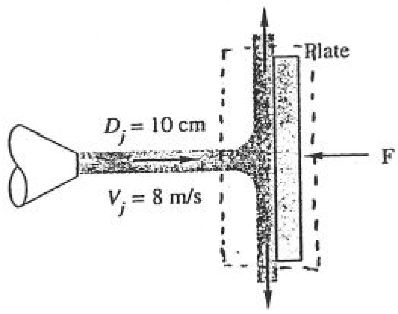
STEADY FLOW MOMENTUM EQN

$$-F_B + p_1 A_1 = \dot{m}(V_2 - V_1)$$

or $F_B = p_1 A_1 - \dot{m}(V_2 - V_1) = 6.205 \times 10^5 \times \frac{\pi}{4}(.1)^2 - 1000 \times 0.025[35.37 - 3.18]\ N$

$$= \underline{4068.6\ N}$$

• Q2



STEADY FLOW MOMENTUM EQUATION

$$F = \rho_0 A_j V_j^2 = 1000 \times \frac{\pi}{4}(0.1)^2 \times (8)^2\ N$$
$$= \underline{502.7\ N}$$

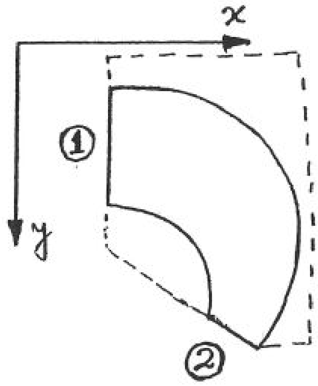
• Q3 Steady, incompressible, frictionless

CONTINUITY : $Q = V_1 A_1 = V_2 A_2$ --- (1)

$V_1 = \frac{Q}{A_1} = \frac{0.23}{\pi/4 (0.3)^2}$ m/s $= 3.254$ m/s

$V_2 = V_1 \cdot (d_1/d_2)^2 = 4V_1 = 13.016$ m/s

$d_1 = 0.3$ m
$d_2 = 0.15$ m
$Q = 0.23$ m³/s
$p_1 = 140$ kPa



BERNOULLI'S EQN between (1) & (2) :

$$\frac{140 \times 10^3}{1000 \times 9.81} + \frac{(3.254)^2}{2 \times 9.81} + 1.4$$

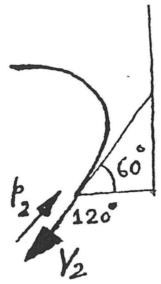
$$= \frac{p_2}{1000 \times 9.81} + \frac{(13.016)^2}{2 \times 9.81}$$

⇒ $p_2 = 74320$ Pa

F_x, F_y → forces on the fluid.

STEADY FLOW MOMENTUM EQN. in the x-direction:

$$F_x + p_1 A_1 + p_2 A_2 \cos 60° = \rho_o Q (V_2 \cos 120° - V_1)$$



or $F_x = 1000 \times 0.23 \times [13.016 \cos 120° - 3.254]$

$$- 140 \times 10^3 \times \frac{\pi}{4}(0.3)^2 - 74320 \times \frac{\pi}{4}(0.15)^2 \cos 60° \text{ N}$$

$$= -12798 \text{ N}$$

STEADY FLOW MOMENTUM EQN. in the y-direction

$$F_y - p_2 A_2 \sin 60° + W = \rho_o Q (V_2 \sin 120° - 0)$$

or $F_y = 1000 \times 0.23 \times [13.016 \sin 120°] + 74320 \times \frac{\pi}{4}(0.15)^2 \sin 60°$

$$- 1000 \times 9.81 \times 0.085$$

$$= 2896 \text{ N}$$

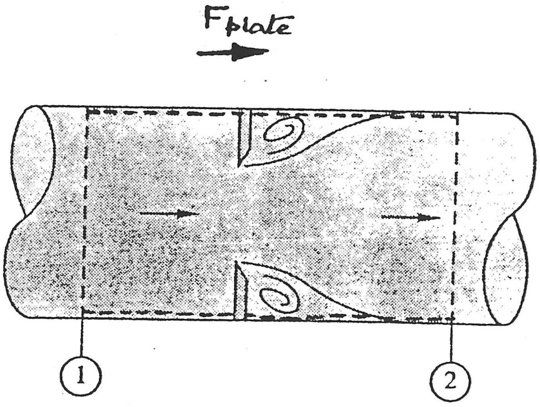
∴ $F = \sqrt{F_x^2 + F_y^2} = \sqrt{(-12798)^2 + (2896)^2}$ N $=$ **13122 N**

$$\tan\theta = \frac{F_y}{F_x} = \frac{2896}{-12798}$$

∴ $\theta = 180° - 12.75°$

Force on bend is equal and opposite to this, i.e.

• Q4



The control volume cuts through the orifice plate.

F_{plate} is the force on the plate.

Force on the fluid is $-F_{plate}$

$A_1 = A_2 = A$

$\therefore$ continuity equation gives $V_1 = V_2$, assuming water is incompressible.

Steady Flow Momentum Equation:

$$-F_{plate} + (p_1 - p_2)A = \dot{m}(V_2 - V_1) = 0$$

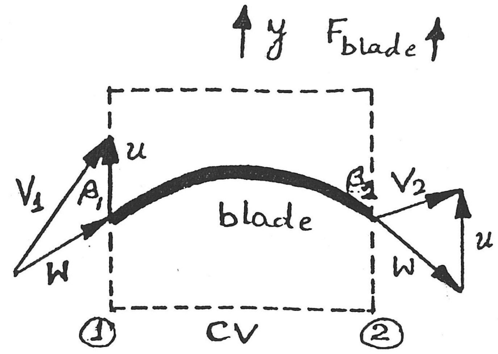
$$\therefore \quad F_{plate} = (p_1 - p_2)A$$

$$= (p_1 - p_2)\frac{\pi}{4}d^2$$

$$= (145 \times 10^3)\ \pi/4\,(0.1)^2 \ \text{N}$$

$$= \underline{\underline{1139\ \text{N}}}$$

- Q5



The CV encloses the blades and moves upward at a speed u so that the flow appears steady in that reference frame.

W is the relative velocity, its magnitude remains unaltered in frictionless flow but the direction changes.

From the velocity triangle at section ① $W^2 = V_1^2 + u^2 - 2V_1 u \cos\beta_1$

" " " " " " ② $W^2 = V_2^2 + u^2 - 2V_2 u \cos\beta_2$.

Eliminating W, and solving for u

$$u = \frac{V_1^2 - V_2^2}{2[V_1\cos\beta_1 - V_2\cos\beta_2]} \quad \text{----} (1)$$

F_{blade} = force on blades in the direction of u

STEADY FLOW MOMENTUM EQUATION

$$-F_{blade} = \dot{m}[W_{2y} - W_{1y}] \quad \text{where } \dot{m} = \tfrac{\pi}{4} D_1^2 \rho_1 V_1$$

$$= \dot{m}[(W_{2y}+u) - (W_{1y}+u)]$$

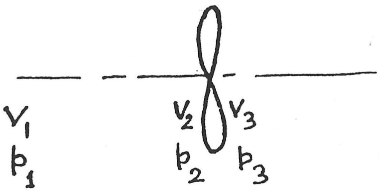
$$= \dot{m}[V_2\cos\beta_2 - V_1\cos\beta_1] \quad \text{---} (2)$$

$$\therefore \quad F_{blade} = \underline{\underline{\dot{m}[V_1\cos\beta_1 - V_2\cos\beta_2]}}$$

$$\text{Power delivered} = F_{blade} \cdot u$$

$$= \underline{\underline{\tfrac{1}{2}\dot{m}(V_1^2 - V_2^2)}} \quad \text{[from equations (1) and (2)]}$$

Q6



See lecture notes for the actuator disc theory.

$V_2 = V_3 \qquad p_1 = p_4$

It was shown:

$$V_2 = \frac{1}{2}(V_4 + V_1) \quad \text{---} \; ①$$

$$\eta = \frac{2V_1}{V_1 + V_4} \quad \text{----} \; ②$$

given

$F = 103 \times 10^3$ N

$\eta = 0.9$

$V_1 = (240 + 48)$ km/h
$= 80$ m/s

(within the range of incompressible flow)

$\rho_o = 1.2$ kg/m^3

From (2)

$$V_4 = \frac{(2-\eta)V_1}{\eta} = \frac{1.1 \times 80}{0.9} \text{ m/s} = 97.778 \text{ m/s}$$

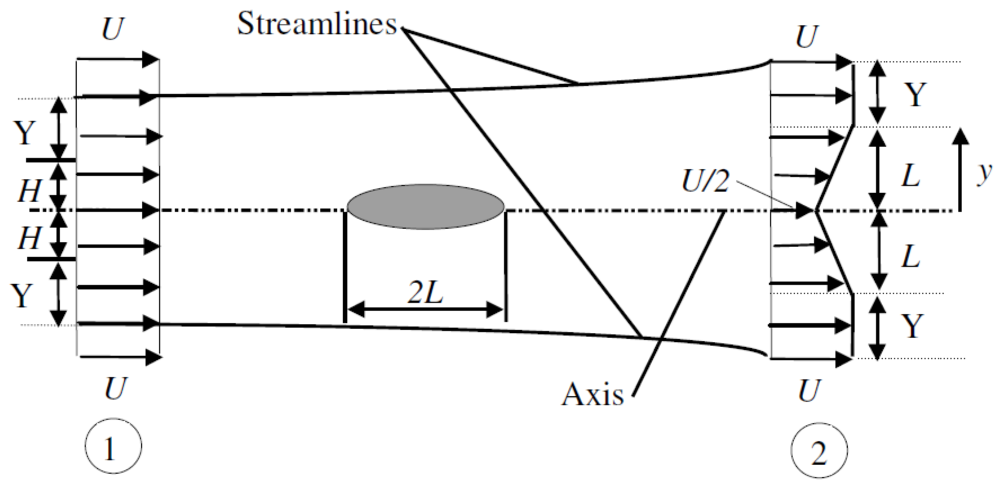
$$F = \rho_o A V_2 (V_4 - V_1)$$
$$= \rho_o A \frac{1}{2}(V_4^2 - V_1^2)$$
$$= \rho_o \pi/4 \, D^2 \frac{1}{2}(V_4^2 - V_1^2)$$

$$\therefore \; D = \sqrt{\frac{8F}{\rho_o \pi (V_4^2 - V_1^2)}} = \sqrt{\frac{8 \times 10.3 \times 10^3}{1.2 \, \pi \, [(97.778)^2 - (80)^2]}} \text{ m}$$

$$= \underline{2.63 \text{ m}}$$

$$\text{Power} = \frac{FV_1}{\eta} = \frac{10.3 \times 10^3 \times 80}{0.9} \text{ W} = \underline{915.6 \text{ kW}}$$

Q7



Two-dimensional flow past a thick cylinder of dimension $2L$ in the flow direction

The control volume consists of the cross-sectional planes at stations 1 and 2, and the streamlines. The flow is two dimensional and <u>symmetric</u> about the axis. The wake profile (for positive y in the range $0 \le y \le L$) can be expressed as $V(y) = 0.5\, U\,(1 + y/L)$.

(a) By definition, mass does not cross the streamlines. The fluid is assumed incompressible, i.e. the density ρ remains constant. The mass and momentum flow rate through the cross-sectional area bY are the same at stations 1 and 2, as the velocity at both stations is U. The purpose of adding this arbitrary distance Y in the analysis is that then the streamlines are sufficiently far from the cylinder so that the static pressure can be assumed uniform.

Conservation of mass :

Mass flow rate at station 1 = Mass flow rate at station 2

$$\rho\int_1 V dA = \rho\int_2 V dA$$

or, $2\int_0^H U\,b\,dy = 2\int_0^L \frac{U}{2}\left(1+\frac{y}{L}\right) b\,dy$; or, $2Ub\int_0^H dy = Ub\left[\int_0^L dy + \frac{1}{L}\int_0^L y dy\right]$

or, $2\,[y]_0^H = [y]_0^L + \frac{1}{L}\left[\frac{y^2}{2}\right]_0^L$. Solving, $\underline{\underline{H = 3L/4}}$.

(b) Drag D is the force on the <u>body</u> acting towards right. From Newton's third law of motion, the force on the <u>fluid</u> is $-D$.

Steady Flow Momentum Equation :

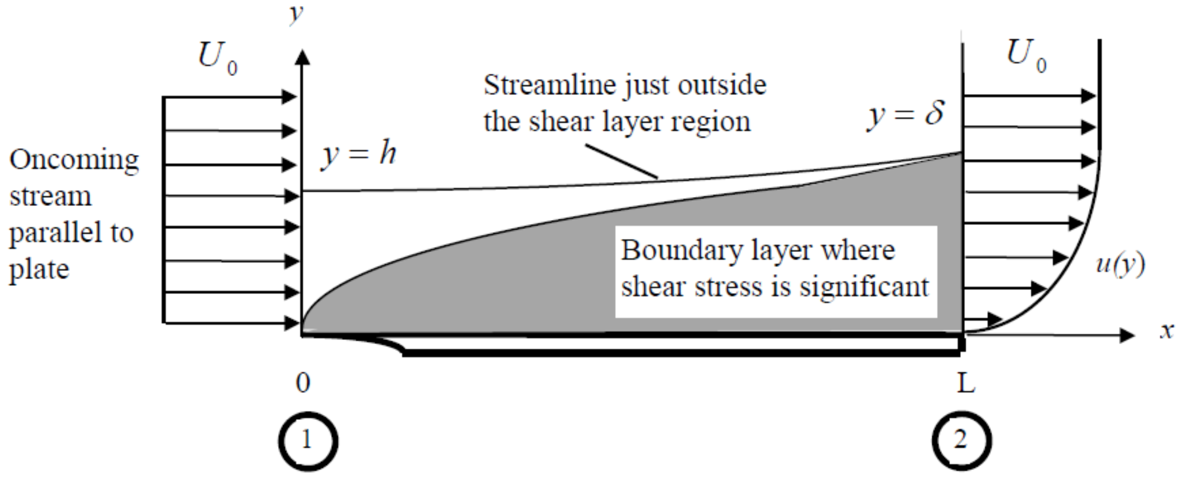
$$-D = \rho\int_2 V^2 dA - \rho\int_1 V^2 dA = 2\rho\int_0^L \left[\frac{U}{2}\left(1+\frac{y}{L}\right)\right]^2 bdy - 2\rho\int_0^H U^2 bdy$$

$$= \frac{\rho U^2 b}{2}\int_0^L \left[1 + 2\frac{y}{L} + \frac{y^2}{L^2}\right]dy - 2\rho U^2 b\int_0^H dy = \frac{\rho U^2 b}{2}\left[y + \frac{y^2}{L} + \frac{y^3}{3L^2}\right]_0^L - 2\rho U^2 b\,[y]_0^H$$

$$= \rho U^2 b\left[\tfrac{7}{6}L - 2H\right] = \rho U^2 b\left[\tfrac{7}{6}L - 2(3L/4)\right] = -\tfrac{1}{3}\rho U^2 bL\,.$$

$\underline{\underline{D = \tfrac{1}{3}\rho U^2 bL}}$ $\qquad$ $\underline{\underline{C_D = \frac{D}{\rho U^2 bL} = \frac{1}{3}}}$

Q8



The control volume consists of the cross-sectional planes at stations 1 and 2, the surface of the flat plate and the streamline. The flow is two dimensional. The wake profile at station 2 is given by $u = U_o\left(2y/\delta - y^2/\delta^2\right)$ *for* $0 \le y \le \delta$. The dimension of the plate in the direction perpendicular to the plane of the figure is b.

(a) By definition, mass does not cross the streamline. The flat plate is impermeable. The fluid is assumed incompressible, i.e. the density ρ remains constant.

Conservation of mass :

Mass flow rate at station 1 = Mass flow rate at station 2

$$\rho\int_1 V dA = \rho\int_2 V dA$$

i.e., $\displaystyle\int_0^h U_0\, b\, dy = \int_0^\delta U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) b\, dy$,

i.e., $\displaystyle U_0 b\int_0^h dy = U_0 b\left[\frac{2}{\delta}\int_0^\delta y dy - \frac{1}{\delta^2}\int_0^\delta y^2 dy\right]$, (constants can be taken outside the integral sign)

i.e., $\displaystyle [y]_0^h = \frac{1}{\delta}\left[y^2\right]_0^\delta - \frac{1}{\delta^2}\left[\frac{y^3}{3}\right]_0^\delta$. Solving, $\underline{\underline{h = 2\delta/3}}$.

(b) Drag D is the force on the body acting towards right. From Newton's third law of motion, the force on the fluid is $-D$. Drag is the only force acting on the control volume in flow direction.

Steady Flow Momentum Equation :

$$-D = \rho\int_2 V^2 dA - \rho\int_1 V^2 dA = \rho\int_0^\delta\left[U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right]^2 bdy - \rho\int_0^h U_0^2 bdy$$

$$= \rho U_0^2 b\int_0^\delta\left[\frac{4y^2}{\delta^2} - \frac{4y^3}{\delta^3} + \frac{y^4}{\delta^4}\right]dy - \rho U_0^2 b\int_0^h dy = \rho U_0^2 b\left[\frac{4y^3}{3\delta^2} - \frac{y^4}{\delta^3} + \frac{y^5}{5\delta^4}\right]_0^\delta - \rho U_0^2 b\,[y]_0^h$$

$$= \rho U_0^2 b\left[\tfrac{8}{15}\delta - h\right] = \rho U_0^2 b\left[\tfrac{8}{15}\delta - \tfrac{2}{3}\delta\right] = -\tfrac{2}{15}\rho U_0^2 b\delta .$$

$\underline{\underline{D = \tfrac{2}{15}\rho U_0^2 b\delta}}$. **Note** You have just learnt von Karman's integral analysis of a boundary layer.