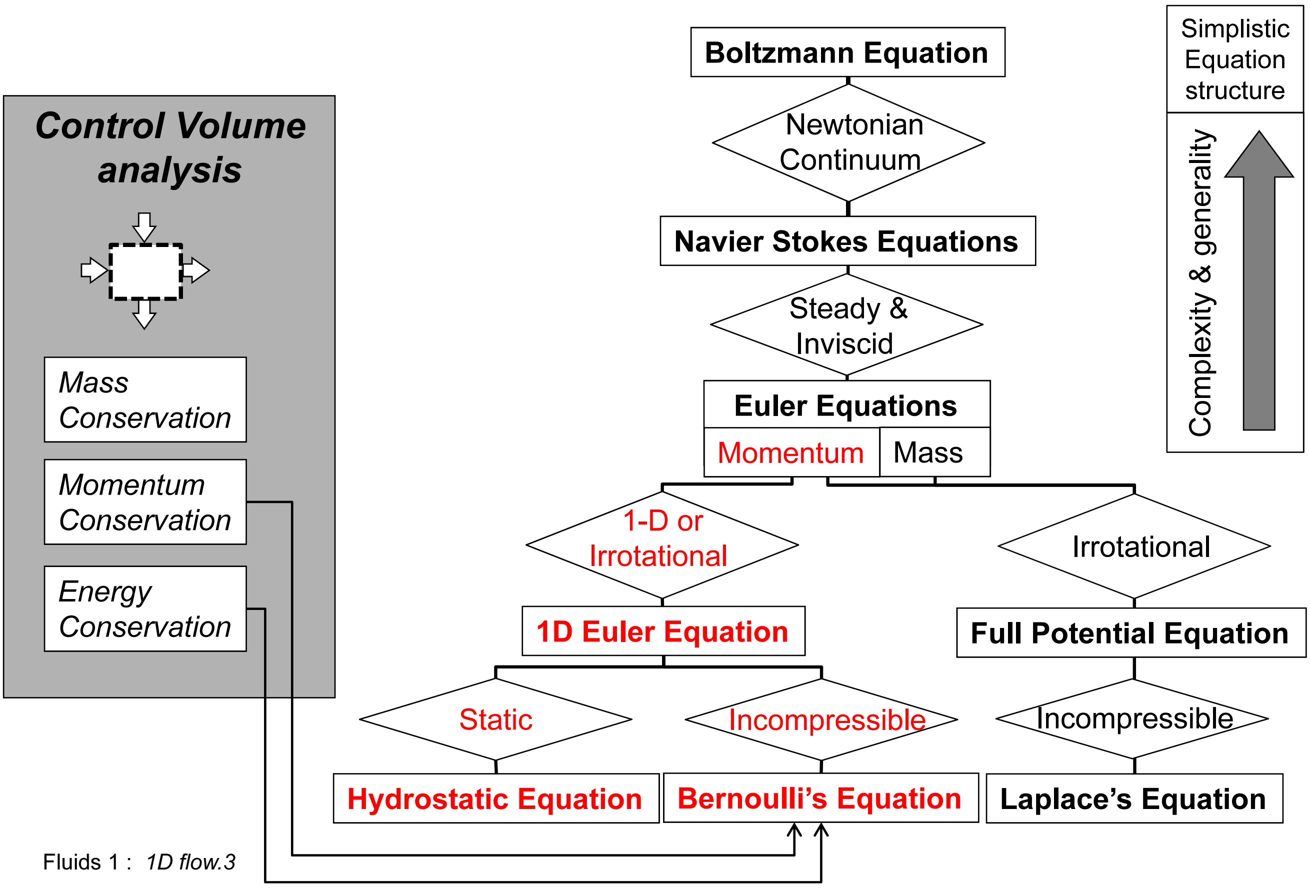
Steady Inviscid Incompressible 1-D Flow

Flow models

Real flows are usually complex and the complete flow equations are generally too complex for analytical solutions. Even the most powerful computers are incapable of computing all the scales involved.

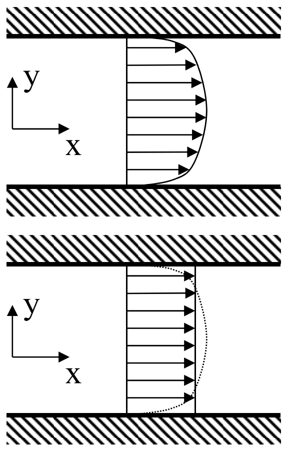
We must make reasonable assumptions to allow analytical solutions

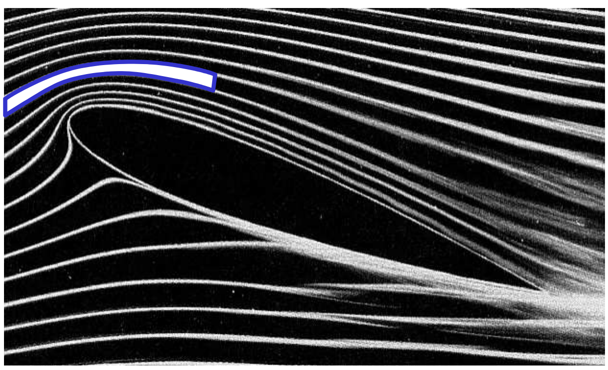
Fluid Equations: Reduction or Construction?

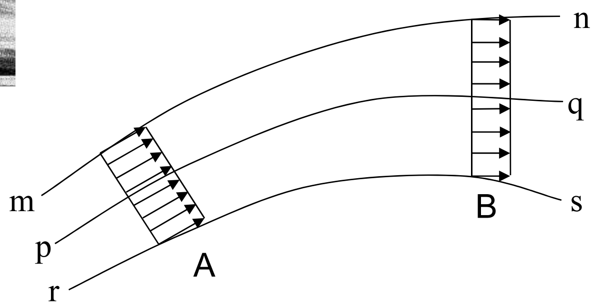


Quasi-one-dimensional Approximation

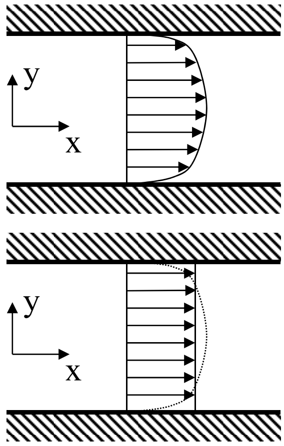
- Consider flow between two parallel flat plates
- Real flow has zero velocity at the wall
- Velocity variation in the cross-stream direction (y)



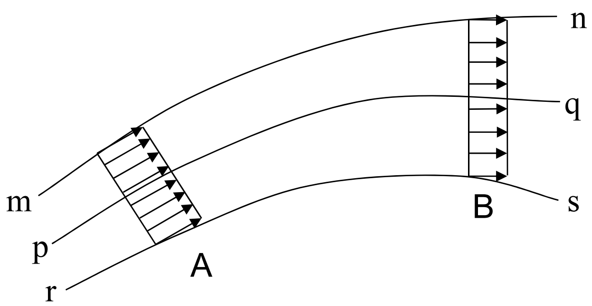




Quasi-one-dimensional Approximation

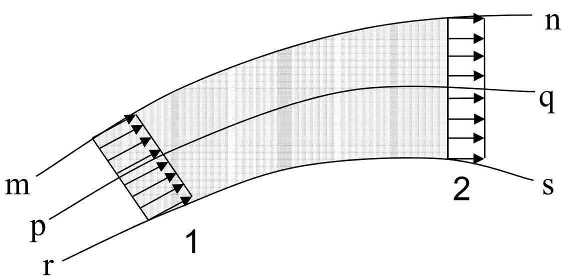


- Consider flow between two parallel flat plates
- Real flow has zero velocity at the wall
- Velocity variation in the cross-stream direction (y)
- 1D-approx takes average velocity across the profile
- General curved stream-tube
- Flow averaged, A and B not necessarily equal
- 1-D approximation good when
  - Area variation is gradual
  - Curvature is small



Conservation of Mass

- Consider 1-D approximation applied to volume defined by "1" and "2"



- Mass flow rate through section 1: $\rho_1 V_1 \; A_1$
- Mass flow rate through section 2: $\rho_2 V_2 \; A_2$
- Principle of mass conservation for quasi 1-D steady flow:

Rate of mass flow in = Rate of mass flow out

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

- For incompressible flow

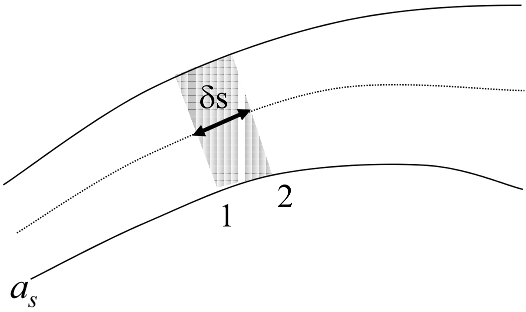
$$A_1 V_1 = A_2 V_2$$

- Valid for viscous as well as inviscid flows

Acceleration of a Fluid Particle

- Velocity in quasi-1D flow is a function of position and time
- As fluid moves from 1 to 2 in time δt, velocity changes because
  - The velocity changes with position
  - Velocity at both 1 and 2 is unsteady
- Total change in velocity for particle

$$\delta V = \frac{\partial V}{\partial s}\delta s + \frac{\partial V}{\partial t}\delta t$$



- As δt tends to 0, stream-wise acceleration a_s

$$\underbrace{a_s = \frac{dV}{dt}}_{Lagrangian} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t} = \underbrace{\frac{\partial V}{\partial s}V + \frac{\partial V}{\partial t}}_{Eulerian}$$

- Lagrangian description follows the particle
- For Steady flow streamline=pathline and $\quad a_s = \frac{\partial V}{\partial s}V$
- For steady flow around a streamline curve (of radius r_s) there is also an acceleration normal to flow a_n

$$a_n = \frac{V^2}{r_s}$$

Assume this is negligible for Bernoulli derivation. Come back to when considering cross stream.

Conservation of Linear Momentum

- Consider forces in flow direction
- Forces in the flow direction due to pressure on side wall

$(p + k\delta p)\,\delta A \qquad$ where k is a coefficient related dA/ds

- Total force on fluid element in flow direction

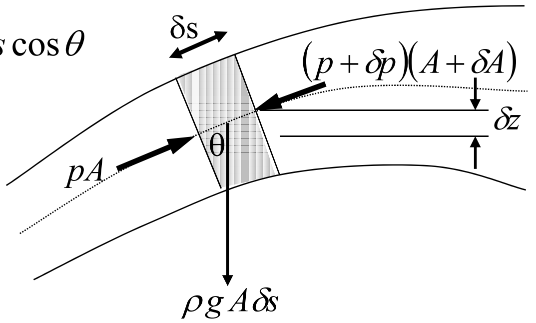
$$pA - (p+\delta p)(A+\delta A) + (p + k\delta p)\,\delta A - \rho g A\delta s\cos\theta$$

- Neglecting products of small terms

$$-A\delta p - \rho g A \delta s \cos\theta$$

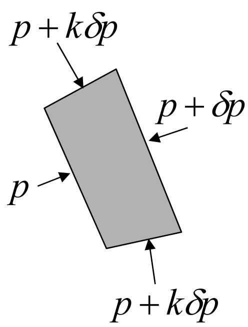
- Force=mass x acceleration Newton's 2[nd] Law

$$-A\delta p - \rho g A\delta s\cos\theta = (\rho A\delta s)\left(V\frac{\partial V}{\partial s}\right)$$



- Substitute $\delta z = \delta s\cos\theta$ and dividing by $(A\delta s)$

$$\frac{dp}{ds} + \rho V\frac{dV}{ds} + \rho g\frac{dz}{ds} = 0$$



- Or in differential form (Euler's equation)

$$dp + \rho V dV + \rho g dz = 0$$

Bernoulli’s equation

- Euler’s equation is valid for steady quasi-1D inviscid flow
- Euler’s equation is valid for compressible and incompressible flow
- Euler’s equation can only be integrated if the variation of density with pressure is known.
- Assuming the density is constant Euler’s equation can be integrated to

$$p + \tfrac{1}{2}\rho V^2 + \rho g z = \text{constant}$$

- Bernoulli’s equation. Valid for steady, inviscid, constant-density flow along a streamline

Conservation of Momentum: Cross Stream

- Consider forces in cross-stream direction due to acceleration normal to streamline with radius of curvature r_s: $a_n = V^2/r_s$ see slide 6 of this handout
- Consider shaded fluid element where r_s and therefore $A_\times$ are constant
- Net force towards centre of curvature due pressure is

$$(p+\delta p)A_\times - pA_\times = \delta p A_\times$$

- Total force on fluid element towards centre of curvature

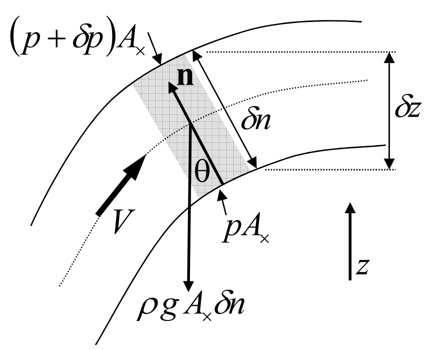
$$\delta p A_\times + \rho g A_\times \delta n \cos\theta = \delta p A_\times + \rho g A_\times \delta z$$

- Force=mass x acceleration Newton’s 2nd Law

$$\delta p A_\times + \rho g A_\times \delta z = (\rho A_\times \delta n) a_n = (\rho A_\times \delta n)\frac{V^2}{r_s}$$

- dividing by $A_\times \delta n$ and taking limit $\delta n \to 0$

$$\frac{\partial p}{\partial n} + \rho g \frac{\partial z}{\partial n} = \rho \frac{V^2}{r_s}$$



- Only assumptions made was steady flow for the equation of acceleration
  Hence Equation valid for: compressible/incompressible & viscous/inviscid steady flows

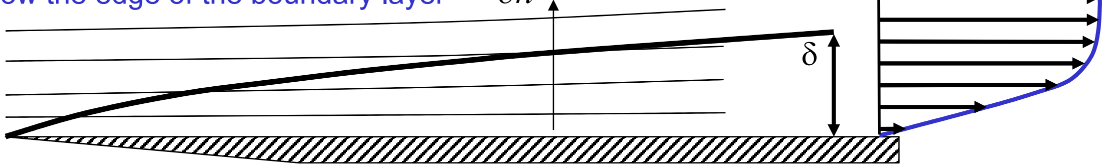
Cross-stream pressure variation- boundary layer

- Consider flows over approximately straight surfaces with negligible hydrostatic pressure variation

$$r_s \rightarrow \infty \qquad \& \qquad \rho g \frac{\partial z}{\partial n} \rightarrow 0 \quad \text{hence} \quad \frac{\partial p}{\partial n} + \underbrace{\rho g \frac{\partial z}{\partial n}}_{\rightarrow 0} = \underbrace{\rho \frac{V^2}{r_s}}_{\rightarrow 0} \qquad \rightarrow \qquad \frac{\partial p}{\partial n} = 0$$

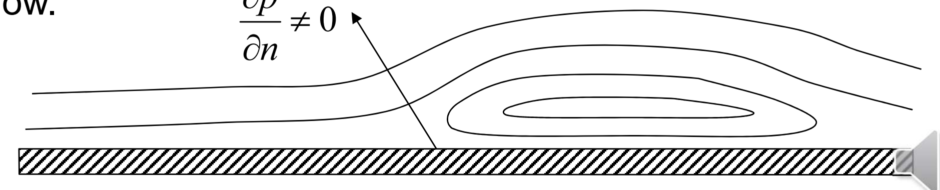
Streamlines thickening slowly as the boundary layer develops (r_s large). Also note the streamlines do not follow the edge of the boundary layer

$$\frac{\partial p}{\partial n} = 0$$



- Not valid for boundary layer separation, streamlines with large curvature
- Pressure gradients normal to the flow.

$$\frac{\partial p}{\partial n} \neq 0$$



Cross-stream pressure variation- parallel flow

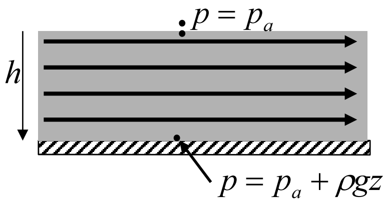
- In a lot of analysis we assume that the stream lines are parallel, hence as before

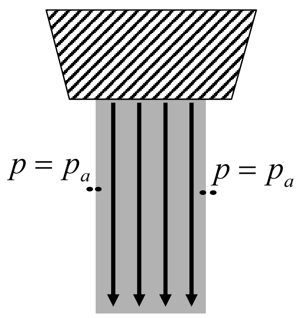
$$r_s \rightarrow \infty \quad \& \quad \rho \frac{V^2}{r_s} \rightarrow 0$$

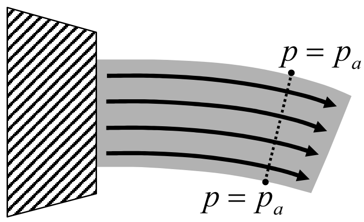
But if we do not neglect the hydrostatic terms

$$\frac{\partial p}{\partial n} + \rho g \frac{\partial z}{\partial n} = \underbrace{\rho \frac{V^2}{r_s}}_{\rightarrow 0} \qquad \rightarrow \qquad \frac{\partial p}{\partial z} = -\rho g$$

- So consider a number of cases shown and remember that the interface of two flows/fluids is at the same pressure



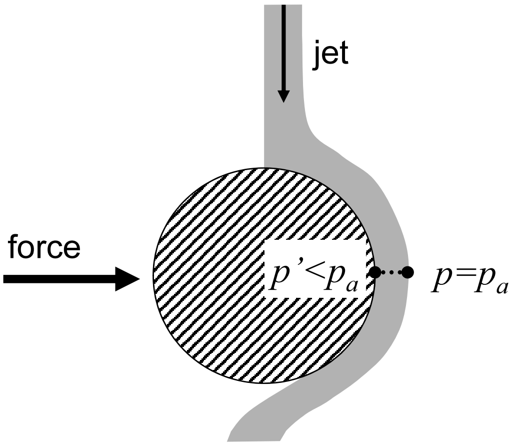




In analysis we assume the radius of curvature is large and the jet width is small: so horizontal jet is straight & parallel -see vena-contracta

Cross-stream pressure variation- Coanda effect

- Previous theory helps to explain why jets of fluid tend to remain attached to convex bodies – the Coanda effect
- Pressure at body surface, p' , lower than atmospheric, p_a, at free surface.

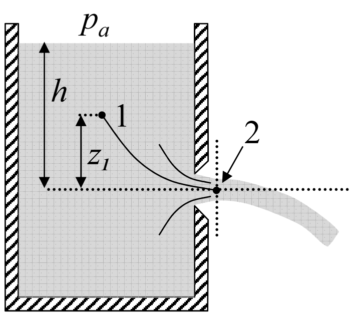


Tangential blowing often used to keep boundary layer attached

- Atmospheric pressure on the unwetted surface but $p'<p_a$ on wetted surface
- Net force on the body from the imbalance in pressure

Application of Bernoulli's Equation: Flow through a sharp-edged orifice

- For a sharp orifice, there is no surface in the jet direction so viscous effects are minimal
- Streamlines are curved as they reach the orifice.
- X-sectional area of jet reduces slightly as it exits:
  Streamline curvature continues through the exit
- X-sectional area reaches minimum, streamlines parallel:
  Condition known as "vena contracta" where
  Pressure in jet uniform & equal to p_a



- Apply Bernoulli's equation between a general point in the container,1, & a point in the centre of the jet at the vena contracta, 2.

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2 = p_a + \tfrac{1}{2}\rho V_2^2$$

- If container large then change in height of surface is negligible and $V_1 \approx 0$

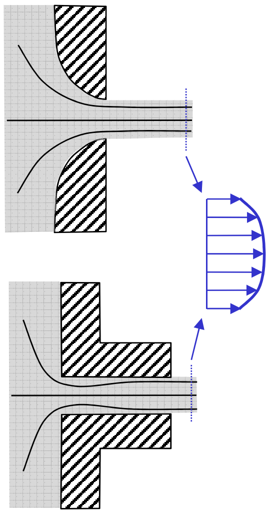
$$p_1 = p_a + \rho g (h - z_1) \qquad p_1 + \rho g z_1 = p_a + \tfrac{1}{2}\rho V_2^2$$

- And so $V_2 = \sqrt{2gh}$

$$p_a + \rho g (h - z_1) + \rho g z_1 = p_a + \tfrac{1}{2}\rho V_2^2$$

$$\rho g h = \tfrac{1}{2}\rho V_2^2$$

Flow through a sharp-edged orifice (2)



- We have only worked out the speed of the jet
- The area of the jet can be found from experiment
- Define volume flow rate Q as

$$Q = V_2 A_2 = C_d \sqrt{2gh} \times A_{\text{orifice}}$$

C_d=discharge coefficient

$$A_2 \approx C_d A_{orifice}$$

- For a sharp edge circular orifice C_d is found from experiment to be between 0.6 and 0.66.
- We can eliminate the contraction of the jet after the orifice by a) a smooth “bell mouth” orifice or b) a short length of tube
- Velocity across the jet then becomes non uniform – viscous losses from boundary layer now present

Bernoulli’s analysis less accurate

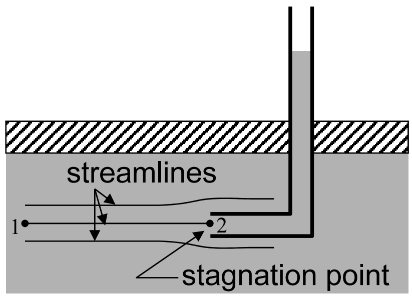
- For a gas where hydrostatic terms are negligible, if we consider a container pressurised to p venting to atmosphere and follow a similar analysis then

$$V_{jet} = \sqrt{\frac{2(p - p_a)}{\rho}}$$

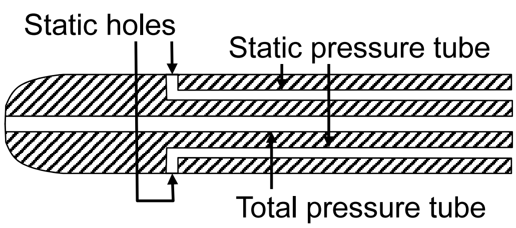
Application of Bernoulli’s Equation: Stagnation pressure and Pitot tube

- Bernoulli’s equation shows that for a steady incompressible flow, brought to rest (frictionless process with no change in hydrostatic pressure)

$$\underbrace{p_1}_{\text{static pressure}} + \underbrace{\tfrac{1}{2}\rho V_1^2}_{\text{dynamic pressure}} = \underbrace{p_2}_{\text{static pressure}} = \underbrace{P_o}_{\text{total pressure}}$$



- Measurement of total (pitot) and static pressure often combined into a Pitot-Static tube
- Around the pitot opening the flow is accelerated and so the static pressure drops. The static pressure holes must be placed far enough away from the pitot opening to allow the static pressure to recover.



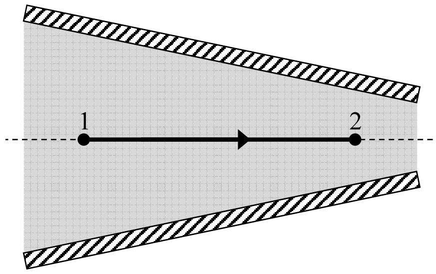
Note that for viscous flows the total pressure falls along a streamline (corresponding to an increase in entropy). However, your calculations will neglect viscous effects as small in this unit.

Application of Bernoulli's Equation: Convergent Flow Passage

- Bernoulli's equation and continuity applied to the flow between points 1 and 2

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2 = P_o$$

$$A_1 V_1 = A_2 V_2$$



- Considering flow with negligible hydrostatic component then as $A_1 > A_2$ we have

$$V_1 < V_2 \qquad p_1 > p_2 \qquad p_1 = p_2 + \tfrac{1}{2}\rho V_1^2 \underbrace{\left(\frac{A_1^2}{A_2^2} - 1\right)}_{>0}$$

$$V_1 \tfrac{A_1}{A_2} = V_2$$

- Not intuitive
- Convergent ducts (nozzles) are also useful as any flow non-uniformities tend to be reduced. This is true for large and small nonlinearities, hence convergent sections upstream of test section in wind tunnels

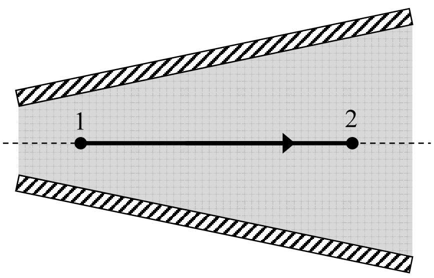
Remember that turbulent flows (small scale non-uniformities) may even become laminar in favourable pressure gradients (flow accelerating)

Application of Bernoulli's Equation: Divergent Flow Passage

- Bernoulli's equation and continuity applied to the flow between points 1 and 2

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2 = P_o$$

$$A_1 V_1 = A_2 V_2$$



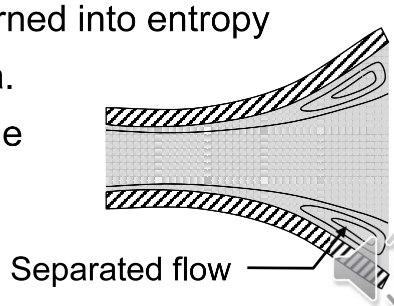
- Considering flow with negligible hydrostatic component again then as $A_1 < A_2$ we have

$$V_1 > V_2 \qquad p_1 < p_2 \qquad p_1 = p_2 + \tfrac{1}{2}\rho V_1^2 \underbrace{\left(\frac{A_1^2}{A_2^2} - 1\right)}_{<0}$$

$$V_1 \tfrac{A_1}{A_2} = V_2$$

- Average velocity defined by continuity so independent of viscous effects but total pressure drops as static pressure losses turned into entropy
- Pressure gradient related to rate of change of area.

If expansion too rapid then flow will separate from the walls leading to large total pressure losses



Application of Bernoulli's Equation: Venturi-meter

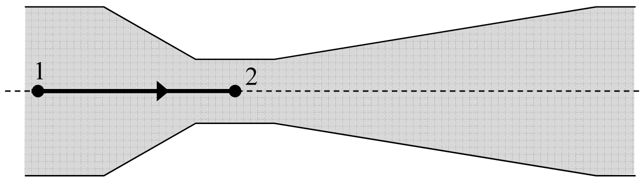
- Can be used to calculate velocity by taking pressure at two points with known cross sectional areas. Long diverging duct (6°) returns the velocity back to original with a small loss in total pressure. Neglected in this situation.
- Bernoulli's equation and continuity applied to the flow between points 1 and 2 (keep heights fixed or ρ small so hydrostatic terms neglected)

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_2^2 = P_o$$

$$A_1 V_1 = A_2 V_2 \qquad V_1 \tfrac{A_1}{A_2} = V_2$$

$$p_1 = p_2 + \tfrac{1}{2}\rho V_1^2 \left(\frac{A_1^2}{A_2^2} - 1 \right)$$

$$V_1 = \sqrt{\frac{2A_2^2 (p_1 - p_2)}{(A_1^2 - A_2^2)\rho}}$$



- We can calculate the difference in static pressure using a manometer

Worked Examples: (1) Vertical Jet of Water

- A vertical jet of water from a 25mm diameter nozzle has a cross section that can be assumed to remain circular. Neglecting any energy loss, what will be the diameter of the jet at a point 4m above the nozzle if the velocity at the nozzle exit is 10m/s?

Sol^n: Flow is steady, incompressible (water) and inviscid (no walls)

- Applying Bernoulli's eqn between 1 & 2 and using $p_1 = p_2 = p_a$

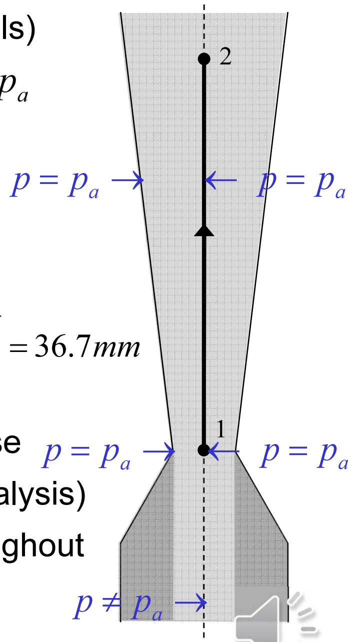
$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2$$

$$\tfrac{1}{2}\rho V_1^2 + \rho g z_1 = \tfrac{1}{2}\rho V_2^2 + \rho g z_2$$

$$V_2 = \sqrt{V_1^2 - 2g(z_2 - z_1)} = \sqrt{10^2 - 2 \times 9.81 \times 4} = 4.64\, m/s$$

- Applying continuity to the circular jet

$$A_1 V_1 = A_2 V_2 \quad \rightarrow \quad \tfrac{1}{4}\pi d_1^2 V_1 = \tfrac{1}{4}\pi d_2^2 V_2 \quad \rightarrow \quad d_2 = d_1 \sqrt{V_1 / V_2} = 25\sqrt{10/4.64} = 36.7\, mm$$

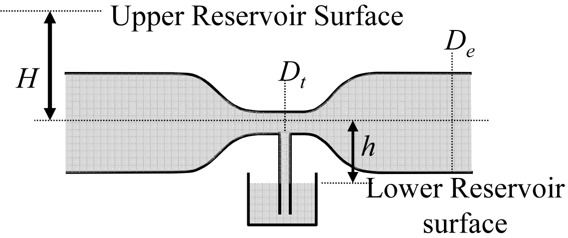


Notes:

- If jet were pointing downward then jet diameter would decrease
- A force is required on the nozzle to hold it in place (see CV analysis)
- Note that the jet adjusts within the nozzle so that $p = p_a$ throughout

Worked Examples: (2) A Suction Device

- A Venturi (convergent-divergent circular duct) produces low pressure at the throat which draws fluid upward from a reservoir. The Venturi centreline is at a height h above this lower reservoir's free surface. The Venturi is supplied by fluid from a second reservoir at a height H above the Venturi centreline and it exhausts to atmosphere. If the throat and exit diameters are D_t & D_e and the flow is assumed inviscid, show that the minimum H that just draws fluid from the lower reservoir into the throat is given by $H_{\min} = \dfrac{h}{\left(D_e^4/D_t^4\right)-1}$



Sol^n: Lower reservoir fluid just in throat when

$p_t = p_a - \rho g h$

- Applying continuity between throat & exit

$V_t/V_e = \left(A_e/A_t\right) = \left(D_e/D_t\right)^2$

- Applying Bernoulli's equation between upper reservoir surface and exit

$p_a + 0 + \rho g H = p_a + \tfrac{1}{2}\rho V_e^2 + 0 \qquad V_e^2 = 2gH$

- Applying Bernoulli's equation between throat and exit

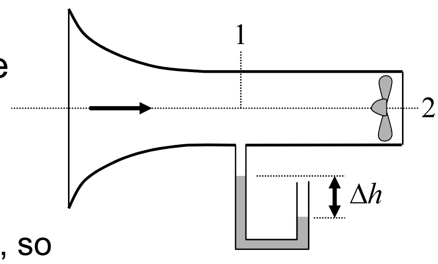
$p_t + \tfrac{1}{2}\rho V_t^2 + 0 = p_a + \tfrac{1}{2}\rho V_e^2 + 0 \quad \rightarrow \quad p_a - p_t = \tfrac{1}{2}\rho\left(V_t^2 - V_e^2\right) = \tfrac{1}{2}\rho V_e^2\left(V_t^2/V_e^2 - 1\right)$

$\rightarrow \quad p_a - p_t = \rho g H\left(\left(D_e/D_t\right)^4 - 1\right) \quad \rightarrow \quad \rho g h = \rho g H\left(\left(D_e/D_t\right)^4 - 1\right)$

$$H_{\min} = \frac{h}{\left(D_e^4/D_t^4\right)-1}$$

Worked Examples: (3) Wind Tunnel

- Calculate the velocity in the working section of a low-speed open circuit wind tunnel, in terms of the air density ρ_a, manometer fluid density ρ_m and manometer height Δh



Sol^n: assume that flow conditions are uniform in the wind tunnel so pressure is constant across width

$p_1 = p_a - \rho_m g \Delta h$

- Note that air density neglected
- Assume the flow is steady incompressible and inviscid, so applying Bernoulli's equation between far upstream & 1

$p_a + 0 + 0 = p_1 + \tfrac{1}{2}\rho_a V_1^2 + 0$

- Combining manometer and Bernoulli's equation $\quad p_a - p_1 = \tfrac{1}{2}\rho_a V_1^2 = \rho_m g \Delta h$

$$V_1 = \sqrt{2\frac{\rho_m}{\rho_a} g \Delta h}$$

Note: The fan sucks air into the working section with no losses.

Bernoulli's equation can't be applied between 1 & 2 as the fan adds energy

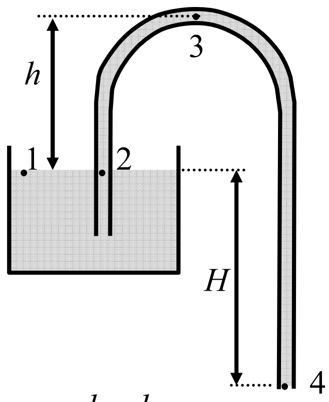
Estimate power output of fan assuming area at 2 the same as at 1

$$\dot{E}_{fan} = \tfrac{1}{2}\dot{m}V_2^2 = \tfrac{1}{2}(\rho_a A_1 V_1)V_2^2$$

Worked Examples: (4) Siphon

- Consider the siphon shown (constant x-section). Assuming the lowest pressure at point 3 is the vapour pressure of the liquid p_v (a lower pressure will cause cavitation and lead to the siphon failing). Show the maximum value of h is

$$h_{\max} = \frac{(p_a - p_v)}{\rho g} - H$$



Sol[n]: Note that we cannot just apply static eqn's between 3 & 4 as the fluid has a velocity.

Applying continuity between 3 & 4

$$A_3 V_3 = A_4 V_4 \qquad \rightarrow \quad V_3 = V_4$$

Applying Bernoulli's equation between 3 &4

$$p_3 + \tfrac{1}{2}\rho V_3^2 + \rho g (H + h) = p_4 + \tfrac{1}{2}\rho V_4^2 + 0$$

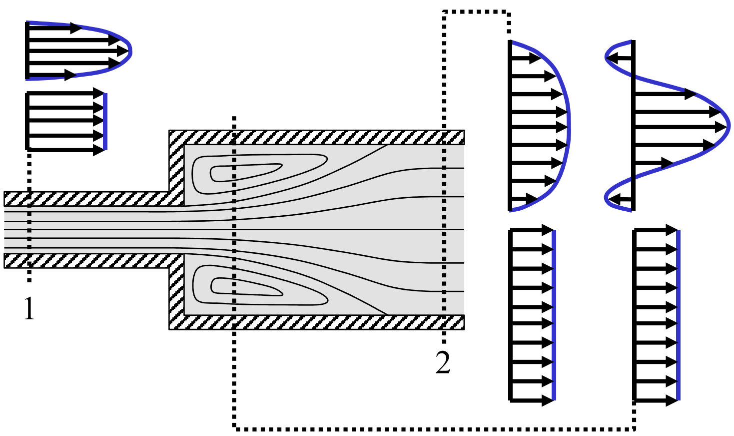
Discharge to atmosphere, $p_4 = p_a$, and we want a solution to $p_3 = p_v\,, h = h_{\max}$

Hence rearranging $h_{\max} = \dfrac{(p_a - p_v)}{\rho g} - H$

Note: Gain more insight by applying Bernoulli between 1 & 2: $p_a = p_2 + \tfrac{1}{2}\rho V_2^2$ so the pressure at 2 is lower than 1 even though they are at the same height. Applying Bernoulli's equation between 1 & 4 gives: $p_a + \rho g H = p_a + \tfrac{1}{2}\rho V_4^2 \quad \rightarrow V_4 = \sqrt{2gH}$ and shows that the discharge is only dependant on the height H and siphon diameter, while h defines the minimum pressure.

When the Flow is quasi-1D

- Consider the flow upstream and downstream of a sudden enlargement (1 & 2). The flow is parallel & steady at 1 & 2 so the static pressure is constant through the depth at each location. We can use Bernoulli's equation in two ways, either using the average velocities ($\overline{V}_1$, $\overline{V}_2$) or using the velocities along a single streamline (V_1 , V_2)



$$p_1 + \tfrac{1}{2}\rho \overline{V}_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho \overline{V}_2^2 + \rho g z_2 + \Delta p_{\text{loss}} \qquad \Delta p_{\text{loss}} \geq 0$$

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2 + \Delta p_{\text{SL loss}} \qquad \Delta p_{\text{SL loss}} \geq 0$$

- Note that the pressure losses are always positive and that the loss associated with average values may be large ($\Delta p_{\text{loss}} \big/ \tfrac{1}{2}\rho \overline{V}_1^2 \geq 0.6$ for a sudden contraction). The losses along streamlines with small cross stream velocity gradients and hence small viscous effects, such as the centreline above, will have negligible total pressure loss
- Consider the Pitot probe measurements inside a boundary layer - the deceleration is inviscid even though the flow is not, so Pitot still measures the correct total pressure.

Learning Outcomes: “What you should have learnt”

- The principle of 1D flow as an approximation to flows in ducts and between streamlines.
- Application of conservation of mass to 1D flow.
- Bernoulli’s Equation: Most importantly you must understand the conditions under which it can be applied. You do NOT need to memorise the derivation of the equation.
- Definition of Total pressure in terms of static dynamic and hydrostatic
- Cross-stream pressure gradient: Again, you do not need to memorise the derivation. You need to understand the main results it leads to: only hydrostatic variation across straight parallel streamlines, pressure decrease on the inside of a curved streamline.
- Understand the Vena Contracta condition.
- Know how a Pitot-Static tube & Venturi can be used to measure flow speed.
- You must be able to apply Bernoulli’s equation and the conservation of mass to a range of engineering situations.

Remember: learn how to apply the Bernoulli & continuity equations, Fluids I is a unit focussed on understanding not memory.