AENG21200: Structures & Materials 2

Example 03: Shear Stresses in Bending

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Learning objectives

Shear stress and shear flow

- Stress induced by a shear force applied (*i.e.* a point load)
- Similar to shear stress, but not the same

Shear centre

- Find its position, all shear forces are applied through here to avoid twisting the beam

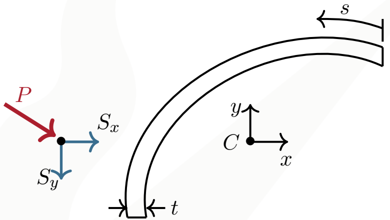
Principal axis method

- Most intuitive method for deformations and stresses, especially when loading axis is unique, i.e. not along a structural axis.

Flanged semi-circular cross-section

- Requires an understanding of the integration to determine moments of area
- Usually seems more difficult than it actually is !!

Shear flow in an arbitrary open-section beam



- Open section beam supports shear loads S_x and S_y.
- There is no twisting or bending of the cross-section.
- Shear loads must both pass through the shear centre.

Equilibrium equation

$$\frac{\partial q_s}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$$

Evaluate each term separately

We want to find the shear flow q_s induced by shear forces S_x and S_y.

$$\frac{\partial q_s}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$$

Let us first evaluate the second term, $\partial \sigma_z / \partial z$.

The shear forces S_x and S_y (generated by the point load P) create bending moments M_y and M_x, respectively.

$$S_x = \frac{\mathrm{d}M_y}{\mathrm{d}z}, \qquad S_y = \frac{\mathrm{d}M_x}{\mathrm{d}z}.$$

Evaluate second term, introduce bending stress equation

We also know that the equation for *general bending stress* is defined as,

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) y$$

rearrange in terms of the bending moments

$$\sigma_z = \left(\frac{I_{yy} y - I_{xy} x}{I_{xx} I_{yy} - I_{xy}^2} \right) M_x + \left(\frac{I_{xx} x - I_{xy} y}{I_{xx} I_{yy} - I_{xy}^2} \right) M_y$$

If we have at least one line of symmetry across our cross-section, then $I_{xy} = 0$ and this equation reduces to

$$\sigma_z = \frac{M_x}{I_{xx}} y + \frac{M_y}{I_{yy}} x$$

Differentiate bending stress equation w.r.t z

We need differentiate the equation for general bending stress with respect to z, i.e. the coordinate along the length of the beam. However, this is a lot easier than exptected, as the only variables that change along the length of the beam are the bending moments M_y and M_x,

$$S_x = \frac{\mathrm{d}M_y}{\mathrm{d}z}, \qquad S_y = \frac{\mathrm{d}M_x}{\mathrm{d}z}.$$

Therefore,

$$\frac{\partial \sigma_z}{\partial z} = \left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) x + \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) y$$

Substituting this back into our equilibrium equation,

$$\frac{\partial q_s}{\partial s} + t \frac{\partial \sigma_z}{\partial z} = 0$$

Substitute back into equilibrium equation

and rearrange for $\partial q/\partial s$,

$$\frac{\partial q_s}{\partial s} + t\left[\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)x + \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)y\right] = 0$$

$$\frac{\partial q_s}{\partial s} = -t\left[\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)x + \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)y\right]$$

$$\frac{\partial q_s}{\partial s} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)tx - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)ty$$

We now need to get rid of the partial derivative $\partial q_s/\partial s$ to recover q_s.

Integrate, and introduce limits

$$\frac{\partial q_s}{\partial s} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) tx - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) ty$$

We integrate the above equation along the arc s between the limits 0 (at the top) and s (at the bottom), because we know at a free edge the shear stress is always zero.

$$q_s = \int_0^s \frac{\partial q}{\partial s}\,\mathrm{d}s$$

Therefore, our **generalised shear flow** equation

$$\int_0^s \frac{\partial q}{\partial s}\mathrm{d}s = q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s tx\,\mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s ty\,\mathrm{d}s$$

If symmetry exists

Most cases will involve at least one line of symmetry, $I_{xy} = 0$, therefore

$$\int_0^s \frac{\partial q}{\partial s} \mathrm{d}s = q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \ \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \ \mathrm{d}s$$

reduces to,

$$q_s = -\frac{S_x}{I_{yy}} \int_0^s tx \ \mathrm{d}s - \frac{S_y}{I_{xx}} \int_0^s ty \ \mathrm{d}s$$

or for a constant thickness,

$$q_s = -\frac{S_x t}{I_{yy}} \int_0^s x \ \mathrm{d}s - \frac{S_y t}{I_{xx}} \int_0^s y \ \mathrm{d}s$$

Shear flow and shear stress

Although similar, there is a distinct difference between *shear flow* and *shear stress*

$$\text{Shear stress} = \tau = -\frac{S_x}{I_{yy}t}\int_0^s x \, \mathrm{d}A$$

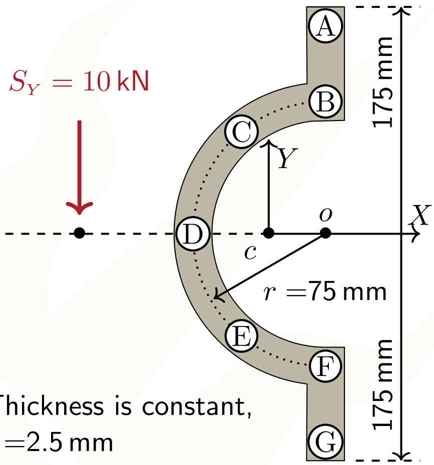
$$\text{Shear flow} = q = -\frac{S_x t}{I_{yy}}\int_0^s x \, \mathrm{d}s$$

however, note the domain at which the integration takes place, for shear stress we integrate over an area, whereas in shear flow we integrate over a length, but if we multiply our length ds by the thickness (another length) we obtain an area which results in,

$$\text{Shear flow} = q = -\frac{S_x}{I_{yy}}\int_0^s x \, \mathrm{d}A$$

now we can see a direct comparison between shear stress and shear flow, thus solidifying our understanding further as $q = \tau t$.

Problem definition,cantilever beam



Thickness is constant,
$t = 2.5$ mm

1) Use "thin wall" assumption to evaluate the shear flow at each of the points A to G

2) Find the position of the shear centre.

Using the *principal axes method*

Equation for shear flow

Shear flow (Learn equation)

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \; \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \; \mathrm{d}s$$

Due to symmetry, $I_{XY} = 0$, this reduces to,

$$q_s = -\frac{S_x}{I_{yy}} \int_0^s tx \; \mathrm{d}s - \frac{S_y}{I_{xx}} \int_0^s ty \; \mathrm{d}s$$

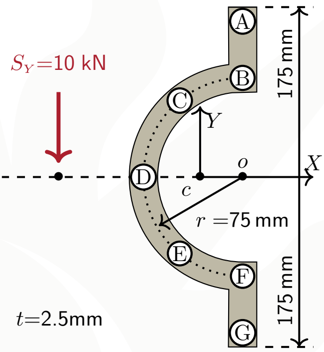
As $S_X = 0$, this reduces further to,

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty \; \mathrm{d}s$$

We therefore need to calculate the second moment of area I_{XX}

Calculate second moment of area

We therefore need to calculate the second moment of area I_{XX}



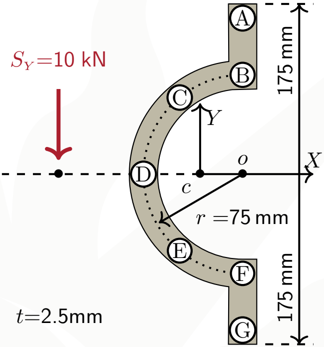
For both flanges,

$$
\begin{aligned}
I_{XX}^{AG-BF} &= \frac{2.5(2\times175)^3}{12} - \frac{2.5(2\times75)}{12} \\
&= \frac{2.5}{12}\left(350^3 - 150^3\right) \\
&= 8.229 \times 10^6 \text{ mm}^4
\end{aligned}
$$

For the semi-circle,

$$
\begin{aligned}
I_{XX}^{BF} &= \frac{tr^3\pi}{2} \\
&= \frac{2.5\times75^3\times\pi}{2} \\
&= 1.657\times10^6 \text{ mm}^4
\end{aligned}
$$

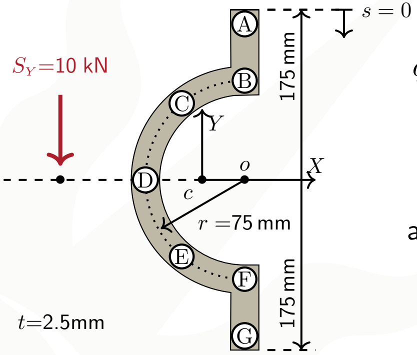
Total second moment of area,



Total second moment of area,

$$I_{XX} = I_{XX}^{AG-BF} + I_{XX}^{BF}$$
$$= (8.229 + 1.657) \times 10^6$$
$$= 9.886 \times 10^6 \text{ mm}^4$$

Evaluate the shear flow,



Evaluate each flange individually, consider flange A-B

$$q_s = -\frac{S_y t}{I_{xx}} \int_0^s y \, \mathrm{d}s$$

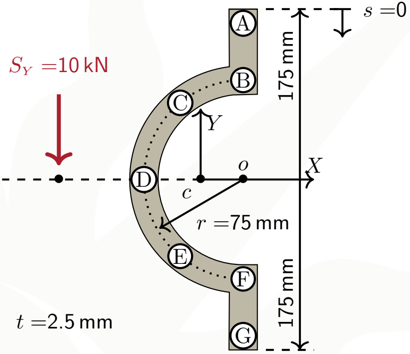
$$= -\frac{S_y t}{I_{xx}} \int_0^s (175 - s) \, \mathrm{d}s$$

at point A, $s = 0$

$$= -\frac{S_y t}{I_{xx}} \left[175s - \frac{s^2}{2} \right]_{s=0}$$

$$q_s^A = 0 \text{ Nmm}^{-1}$$

Evaluate the shear flow,



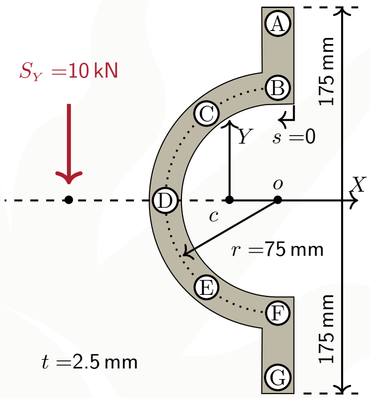
Evaluate each flange individually, consider flange A-B

$$q_s = -\frac{S_y t}{I_{xx}} \int_0^s y \, \mathrm{d}s$$
$$= -\frac{S_y t}{I_{xx}} \int_0^s (175 - s) \, \mathrm{d}s$$

at point B, $s = 100$

$$= -\frac{S_y t}{I_{xx}} \left[175s - \frac{s^2}{2} \right]_0^{100}$$
$$q_s^B = -\frac{S_y t}{I_{xx}} [12,500]$$

Evaluate the shear flow,



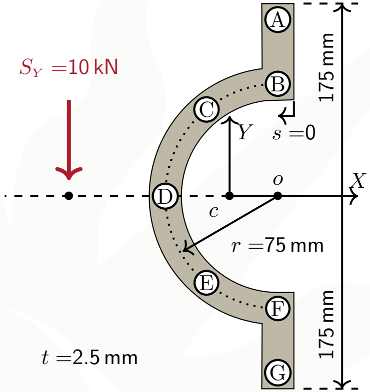
Evaluate each flange individually, consider semi-circular section,

$$q_s = -\frac{S_y t}{I_{xx}} \int_0^s y \; \mathrm{d}s$$
$$= -\frac{S_y t}{I_{xx}} \int_0^s (r \cos\theta) \; \mathrm{d}s$$

we can't integrate this with respect to $\mathrm{d}s$, but $\mathrm{d}s = r \; \mathrm{d}\theta$

$$= -\frac{S_y t}{I_{xx}} \int_0^\phi (r \cos\theta) \; r \; \mathrm{d}\theta$$
$$= -\frac{S_y t}{I_{xx}} \int_0^\phi r^2 \cos\theta \; \mathrm{d}\theta$$

Evaluate the shear flow,



At any point around the arc,

$$q_s^{\text{arc}} = q_s^B - \frac{S_y t}{I_{xx}} \int_0^\phi r^2 \cos\theta \, d\theta$$
$$= -\frac{S_y t}{I_{xx}} \left[12,500 + \int_0^\phi r^2 \cos\theta \, d\theta\right]$$
$$= -\frac{S_y t}{I_{xx}} \left[12,500 + r^2 \sin\phi\right]$$

- C, $\phi = \pi/4$
- D, $\phi = \pi/2$.

The shear flow at $C = E$ and $B = F$ due to symmetry.

Evaluate the shear flow,

Summary,

$$q_s^A = -\frac{S_y t}{I_{xx}}\left[0\right] \qquad = 0$$

$$q_s^B = -\frac{S_y t}{I_{xx}}\left[12,500\right] \qquad = -31.61$$

$$q_s^C = q_s^B - \frac{S_y t}{I_{xx}}\left[r^2 \sin\left(\pi/4\right)\right] \qquad = -41.67$$

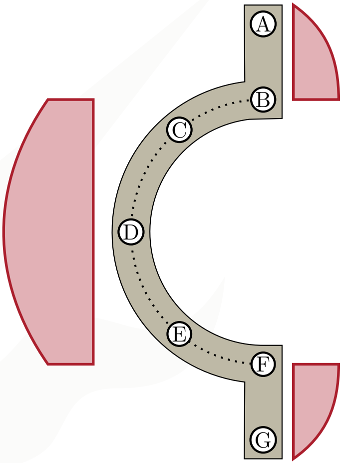
$$q_s^D = q_s^B - \frac{S_y t}{I_{xx}}\left[r^2 \sin\left(\pi/2\right)\right] \qquad = -45.83$$

$$q_s^E = q_s^B - \frac{S_y t}{I_{xx}}\left[r^2 \sin\left(3\pi/4\right)\right] \qquad = -41.67$$

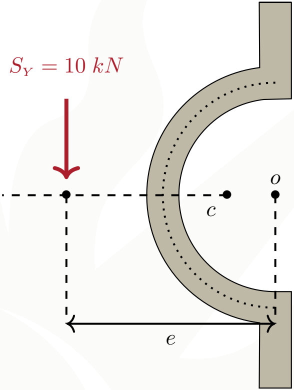
$$q_s^F = q_s^B - \frac{S_y t}{I_{xx}}\left[r^2 \sin\left(\pi\right)\right] \qquad = -31.61$$

$$q_s^G = -\frac{S_y t}{I_{xx}}\left[0\right] \qquad = 0$$

Shear flow distribution,



Find the position of the shear centre



To find the position of the shear centre, e, we need to balance the moments.

$$M = S_y e$$

Rearrange for e,

$$e = \frac{M}{S_y}$$

. . . but we don't know M.
. . . where do we take moments about?

Find the position of the shear centre

Take moments about the origin and not the centroid,

- It is convenient to integrate around the arc from the origin and not the centroid
- It reduces the moment arm of the flanges to zero, thus the moment produced by the two flanges =0.

How do we find this internal moment?

$$\text{Moment} = \text{Force} \times \text{Distance}$$

We have our distance from the origin $\longrightarrow$ radius. So we need the internal force?

$$\text{Force} = \text{Stress} \times \text{Area}$$

We know our area, but we don't know our stress...

Find the position of the shear centre

We can replace stress with shear flow,

$$q = \tau t$$

We can now calculate our moment by integrating around the arc,

$$M_o = \int \mathrm{d}M_o = \int \tau r \,\mathrm{d}A$$

As $\mathrm{d}A = rt\,\mathrm{d}\phi$,

$$\begin{aligned} M_o &= \int_0^\phi \tau r^2 t \,\mathrm{d}\phi && \text{as } q = \tau t, \\ &= \int_0^\phi \frac{q}{t} r^2 t \,\mathrm{d}\phi \\ &= \int_0^\phi q r^2 \,\mathrm{d}\phi \end{aligned}$$

where q is the shear flow for the entire arch.

Find the position of the shear centre

For the shear flow anywhere along the arch,

$$q_s^{\text{arc}} = -\frac{S_y t}{I_{xx}}\left[12,500 + r^2 \sin\phi\right]$$

Substituting back into the equation for moment,

$$\begin{aligned}
M_o &= \int_0^\pi q r^2 \, \mathrm{d}\pi \\
&= -\frac{S_y t}{I_{xx}} \int_0^\pi \left[12,500 + r^2 \sin\phi\right] r^2 \, \mathrm{d}\phi \\
&= -\frac{S_y t r^2}{I_{xx}} \int_0^\pi \left[12,500 + r^2 \sin\phi\right] \, \mathrm{d}\phi \\
&= -\frac{S_y t r^2}{I_{xx}} \left[12,500\pi - r^2 \cos\pi + r^2 \cos 0\right]
\end{aligned}$$

Find the position of the shear centre

The internal moment is therefore,

$$M_o = -\frac{S_y t r^2}{I_{xx}} \left[12,500\pi + 2\,r^2\right]$$

Substituting back into,

$$\begin{aligned} e &= \frac{-\frac{S_y t r^2}{I_{xx}} \left[12,500\pi + 2\,r^2\right]}{S_y} \\ &= -\frac{t r^2}{I_{xx}} \left[12,500\pi + 2\,r^2\right] \\ &= \text{-71.87 mm} \end{aligned}$$