Handout 1 – Macromechanics of Uni-Directional Lamina

In this handout, we formulate the macromechanical constitutive equations for a unidirectional ply, by modelling it as a *homogeneous* material. First, we recall definitions and properties of stress and strain at a point (see also notes from StM2), before deriving the stress-strain relationship for a specially/generally orthotropic lamina.

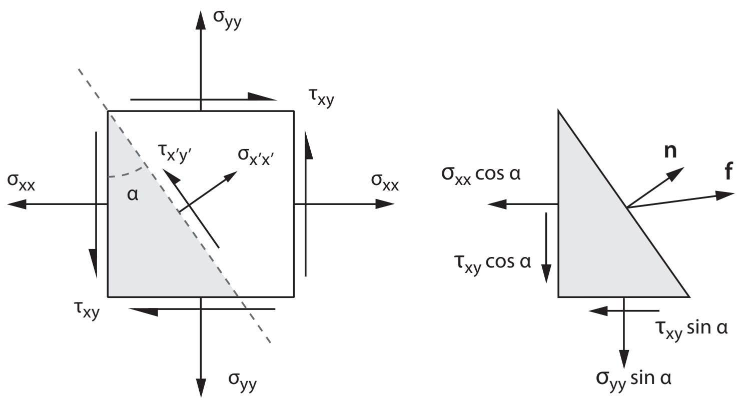
1.1 Stress

Let the stress at a point be defined by the **Cauchy stress tensor**

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

which is symmetric ($\sigma_{ij} = \sigma_{ji}$) due to complementary shear. For *plane stress*, $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$.

Consider the force equilibrium of an infinitesimal element.



Let the cut plane be defined by normal vector $\boldsymbol{n} = [\cos\alpha \;\; \sin\alpha]^T$. From equilibrium, the *traction force* $\boldsymbol{f}$ is:

$$\boldsymbol{f} = \begin{bmatrix} \sigma_{xx}\cos\alpha + \tau_{xy}\sin\alpha \\ \tau_{xy}\cos\alpha + \sigma_{yy}\sin\alpha \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \boldsymbol{\sigma n}$$

To express the stress tensor in a $x'y'$ coordinate system, at a CCW angle θ to the original xy axes, consider the coordinate transformation of vectors $\boldsymbol{f}$ and $\boldsymbol{n}$ given by a rotation matrix:

$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

where $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$, which gives

$$\begin{aligned} \boldsymbol{f} &= \boldsymbol{\sigma n} \\ \boldsymbol{A}^T\boldsymbol{f}' &= \boldsymbol{\sigma A}^T\boldsymbol{n}' \quad \rightarrow \quad \boldsymbol{f}' = \boldsymbol{A\sigma A}^T\boldsymbol{n}' \\ & \qquad\qquad\qquad\qquad\; \boldsymbol{f}' = \boldsymbol{\sigma}'\boldsymbol{n}' \end{aligned}$$

The transformed stress tensor is therefore

$$\boldsymbol{\sigma}' = \boldsymbol{A}\boldsymbol{\sigma}\boldsymbol{A}^T$$

This can also be expressed as a vector transformation:

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{1.1}$$

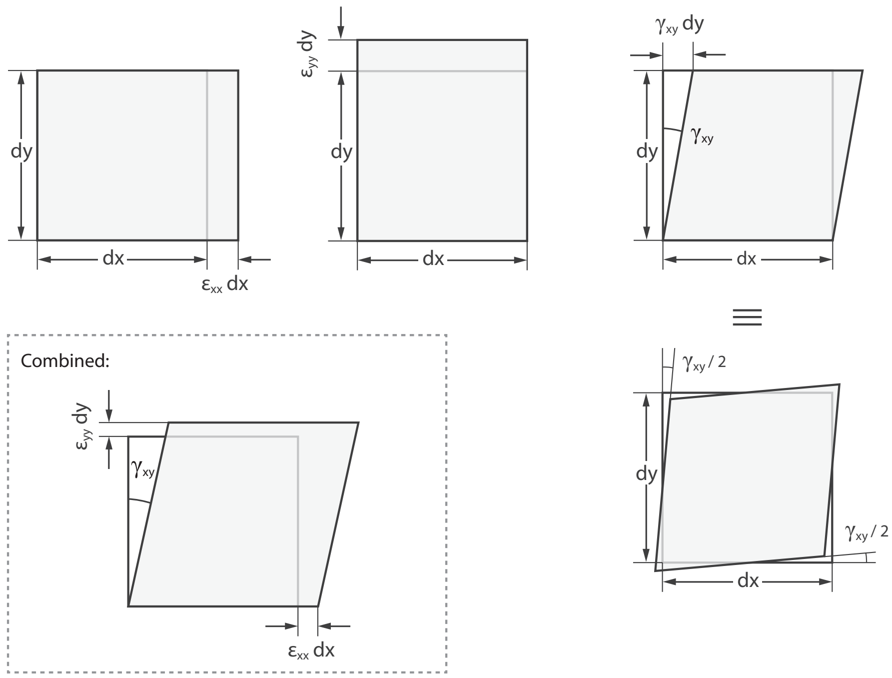
with a transformation matrix $\boldsymbol{T}$. This allows us to convert stress between the structural axes (defined by the applied loads) and the material axes (defined by the fibre orientations).

1.2 Strain

The strain at a point is also expressed as a tensor:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

with the tensorial shear strain $\varepsilon_{xy} = \gamma_{xy}/2$, where γ_{xy} is the engineering shear strain. For *plane strain* $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$; recall that this is not the same as plane stress assumptions, where $\sigma_{zz} = 0$ but $\varepsilon_{zz} \neq 0$.



It can be shown (see StM2) that strain also transforms as a second-rank tensor,

$$\boldsymbol{\varepsilon}' = \boldsymbol{A}\boldsymbol{\varepsilon}\boldsymbol{A}^T$$

or again in vector transformation:

$$\begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{x'y'} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} \tag{1.2}$$

with the identical transformation matrix $\boldsymbol{T}$.

Note that here the *tensorial* shear strain is used, and not the *engineering* shear strain. In order to change coordinates for the engineering strain, we introduce an auxiliary matrix $\boldsymbol{R}$

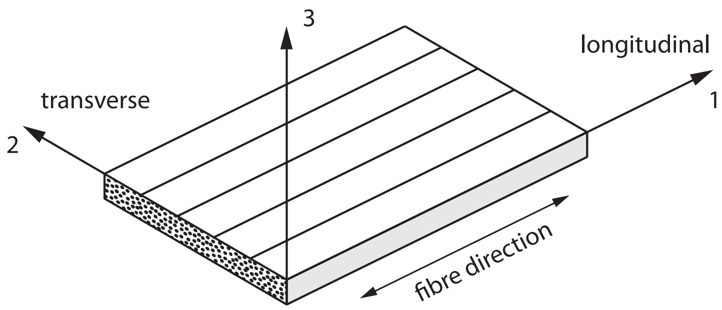
$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{1.3}$$

known as Reuter's matrix. The engineering strain transformation becomes

$$\begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \gamma_{x'y'} \end{bmatrix} = \boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^{-1} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

1.3 Composite Stress-Strain Relationship

In StM2 we derived a generalised Hooke's Law for an isotropic material, where the material properties are the same in all directions. Clearly, a composite is not isotropic, as the mechanical properties are different along the direction of the fibres and perpendicular to the fibres.



sign convention: 123 refers to the natural axes of the material, and xyz refers to structural axes.

A general *anisotropic* linear-elastic material is described by a 6×6 matrix:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$

Note the matrix symmetry $C_{ij} = C_{ji}$ as a result of Maxwell-Betti's reciprocal theorem (see StM2).

Expressed in the material axes, a unidirectional fibre-reinforced composite shows no extension-shear coupling, and is modelled as an *orthotropic* material:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

For a random fibre packing, the material properties transversely to the fibre direction (*i.e.* in the 23-plane) can be assumed identical in all directions. This leads to a *transversely isotropic* material model:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(C_{22} - C_{23}\right)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

where $C_{22} = C_{33}$, $C_{12} = C_{13}$, $C_{55} = C_{66}$, and C_{44} is no longer independent. This leaves five independent material parameters to characterise the composite material.

For *isotropic* materials, with mechanical properties identical in all directions, the material model reduces to:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(C_{11} - C_{12}\right)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(C_{11} - C_{12}\right)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(C_{11} - C_{12}\right)/2 \end{bmatrix}$$

with only two independent material parameters (here, C_{11} and C_{12}). However, composites are *not* isotropic!

1.3.1 Composite Lamina: Plane Stress

In a composite laminate structure, each invidual lamina is assumed to be loaded under *plane stress* ($\sigma_{33} = \tau_{13} = \tau_{23} = 0$). This reduces the **stiffness** matrix to:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

The Q_{ij} components are known as the **reduced stiffnesses**. These are found from C_{ij} by imposing the plane stress condition to obtain an expression for ε_{33}, and simplifying the results to get:

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \qquad i,j = 1,2,6$$

These stiffness components can also be found by inverting the plane stress **compliance** matrix:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix}$$

with

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} \qquad Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}$$

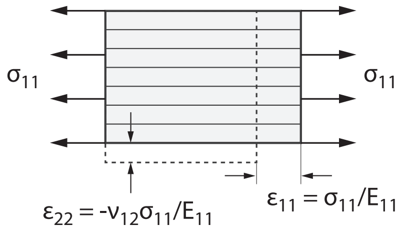
$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2} \qquad Q_{66} = \frac{1}{S_{66}}$$

The compliance matrix can be constructed by considering three load cases:

1. uni-axial stress σ_{11} (with $\sigma_{22} = \tau_{12} = 0$) gives the following strains:

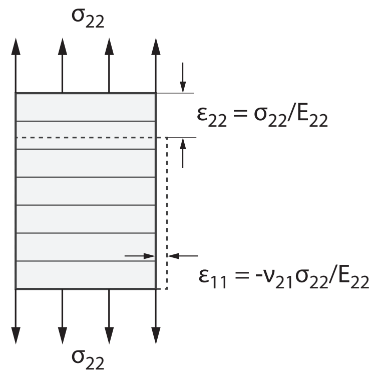
$$\varepsilon_{11} = \frac{\sigma_{11}}{E_{11}} \qquad \varepsilon_{22} = -\nu_{12}\varepsilon_{11} = -\nu_{12}\frac{\sigma_{11}}{E_{11}} \qquad \gamma_{12} = 0$$

Poisson's ratio ν_{ij}: index i is the direction of applied stress, and index j the direction of strain.



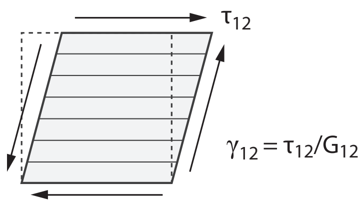
2. uni-axial stress σ_{22} (with $\sigma_{11} = \tau_{12} = 0$) gives the following strains:

$$\varepsilon_{11} = -\nu_{21}\varepsilon_{22} = -\nu_{21}\frac{\sigma_{22}}{E_{22}} \qquad \varepsilon_{22} = \frac{\sigma_{22}}{E_{22}} \qquad \gamma_{12} = 0$$



3. A pure shear τ_{12} (with $\sigma_{11} = \sigma_{22} = 0$) gives:

$$\varepsilon_{11} = 0 \qquad \varepsilon_{22} = 0 \qquad \gamma_{12} = \frac{\tau_{12}}{G_{12}}$$



Each load case provides a column of the compliance matrix $\boldsymbol{S}$. The symmetry, $S_{12} = S_{21}$, subsequently provides a relationship between the two Poisson's ratios:

$$\frac{\nu_{21}}{E_{22}} = \frac{\nu_{12}}{E_{11}} \tag{1.4}$$

This reduces the elastic constants for a specially orthotropic plane stress material to four independent parameters: E_{11}, E_{22}, ν_{12}, and G_{12}. These are either derived from experiments, or micromechanical modelling.

1.3.2 Specially Orthotropic Lamina

In summary, the following equations describe the material behaviour for the laminate material, when the coordinate system is aligned with the natural material axes. The compliance matrix $\boldsymbol{S}$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} \tag{1.5}$$

with

$$\begin{aligned} S_{11} &= \frac{1}{E_{11}} & S_{22} &= \frac{1}{E_{22}} \\ S_{12} &= -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} & S_{66} &= \frac{1}{G_{12}} \end{aligned} \tag{1.6}$$

The stiffness matrix $\boldsymbol{Q}$

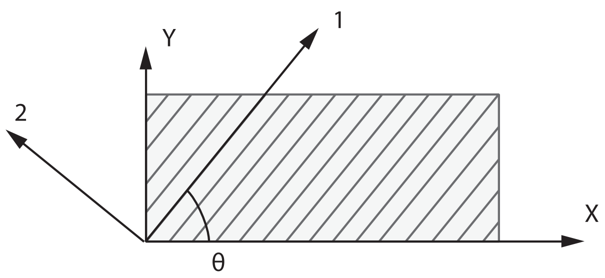
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \tag{1.7}$$

with

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{1-\nu_{12}\nu_{21}} & Q_{22} &= \frac{E_{22}}{1-\nu_{12}\nu_{21}} \\ Q_{12} &= \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} & Q_{66} &= G_{12} \end{aligned} \tag{1.8}$$

1.3.3 Generally Orthotropic Lamina

In general, the natural material axes (123) are not aligned with the structural axes (xyz).



Lamina Stiffness Using the stress and strain transformation equations, we can write:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \boldsymbol{T}^{-1}\boldsymbol{QRTR}^{-1} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{1.9}$$

This gives the required stress-strain equations for a generally inclined orthotropic orthotropic lamina with respect the to the structural axes. This is written in a $\bar{\boldsymbol{Q}}$ matrix:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{1.10}$$

where $\bar{\boldsymbol{Q}} = \boldsymbol{T}^{-1}\boldsymbol{Q}\boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^{-1}$ with components $\bar{Q}_{ij}$

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{12} &= \left(Q_{11}+Q_{22}-4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta+\cos^4\theta\right) \\
\bar{Q}_{66} &= \left(Q_{11}+Q_{22}-2Q_{12}-2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta+\cos^4\theta\right) \\
\bar{Q}_{16} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)\sin\theta\cos^3\theta - \left(Q_{22}-Q_{12}-2Q_{66}\right)\cos\theta\sin^3\theta \\
\bar{Q}_{26} &= \left(Q_{11}-Q_{12}-2Q_{66}\right)\cos\theta\sin^3\theta - \left(Q_{22}-Q_{12}-2Q_{66}\right)\sin\theta\cos^3\theta
\end{aligned} \tag{1.11}$$

The stiffness matrix $\bar{\boldsymbol{Q}}$ is symmetric (as per Maxwell's reciprocal theorem) and is fully populated. The terms $\bar{Q}_{16}$ and $\bar{Q}_{26}$ mean that there is a coupling between the shear strain and normal stresses, and between normal strains and shear stresses (*i.e.* extension-shear coupling).

<u>note</u>: although the $\bar{\boldsymbol{Q}}$ matrix is now fully populated, and thus represents an anisotropic material, it still only contains the four independent mechanical constants of a specially orthotropic material.

Lamina Compliance An identical transformation applies to the compliance matrix:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \tag{1.12}$$

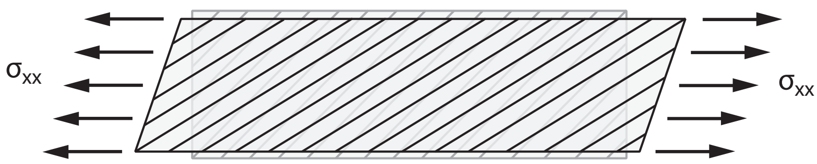
where $\bar{\boldsymbol{S}}$ is given by:

$$\bar{\boldsymbol{S}} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}\boldsymbol{S}\boldsymbol{T} \tag{1.13}$$

with components $\bar{S}_{ij}$

$$\begin{aligned}
\bar{S}_{11} &= S_{11}\cos^4\theta + \left(2S_{12}+S_{66}\right)\sin^2\theta\cos^2\theta + S_{22}\sin^4\theta \\
\bar{S}_{22} &= S_{11}\sin^4\theta + \left(2S_{12}+S_{66}\right)\sin^2\theta\cos^2\theta + S_{22}\cos^4\theta \\
\bar{S}_{12} &= \left(S_{11}+S_{22}-S_{66}\right)\sin^2\theta\cos^2\theta + S_{12}\left(\sin^4\theta+\cos^4\theta\right) \\
\bar{S}_{66} &= 2\left(2S_{11}+2S_{22}-4S_{12}-S_{66}\right)\sin^2\theta\cos^2\theta + S_{66}\left(\sin^4\theta+\cos^4\theta\right) \\
\bar{S}_{16} &= \left(2S_{11}-2S_{12}-S_{66}\right)\sin\theta\cos^3\theta - \left(2S_{22}-2S_{12}-S_{66}\right)\cos\theta\sin^3\theta \\
\bar{S}_{26} &= \left(2S_{11}-2S_{12}-S_{66}\right)\cos\theta\sin^3\theta - \left(2S_{22}-2S_{12}-S_{66}\right)\sin\theta\cos^3\theta
\end{aligned} \tag{1.14}$$

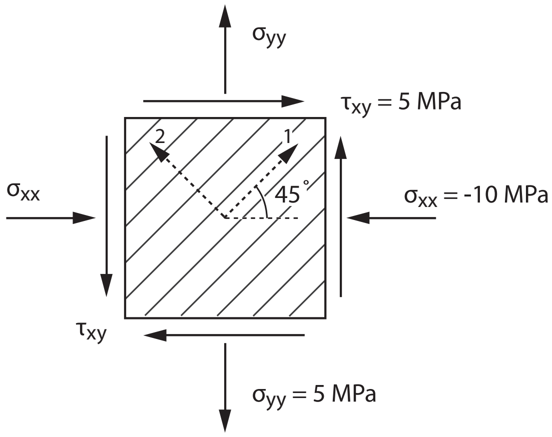
The $\bar{S}_{16}$ and $\bar{S}_{26}$ terms result in extension-shear coupling:



Example 1.1 – Calculate Stress and Strain

A composite lamina with $E_{11} = 180$ GPa, $E_{22} = 10$ GPa, $\nu_{12} = 0.2$, $G_{12} = 5$ GPa, is oriented at $\theta = 45°$ to the xy axes and subject to the following loads:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix} \text{ MPa}$$



Q: what are the resulting strains ε_{xx}, ε_{yy}, γ_{xy} in the structural coordinate system?

A: first, using the transformation matrix $\boldsymbol{T}$ with $\theta = 45°$

$$\boldsymbol{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ -0.5 & 0.5 & 0 \end{bmatrix}$$

the applied stresses are transformed into the material coordinate system:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 2.5 \\ -7.5 \\ 7.5 \end{bmatrix} \text{ MPa}$$

Using the compliance matrix $\boldsymbol{S}$ and Reuter's matrix $\boldsymbol{R}$

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1/E_{11} & -\nu_{21}/E_{22} & 0 \\ -\nu_{12}/E_{11} & 1/E_{22} & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix}$$

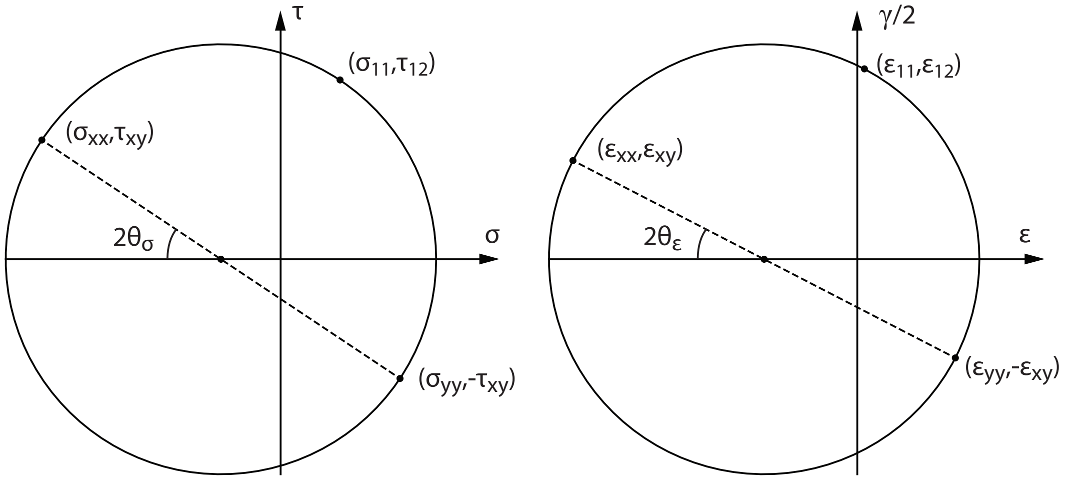
the corresponding strains in the material coordinate system are calculated:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \boldsymbol{S} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 22 \\ -753 \\ 1500 \end{bmatrix} \mu\varepsilon$$

which are then converted back into the structural reference frame:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} -1115 \\ 385 \\ 775 \end{bmatrix} \mu\varepsilon$$

Drawing a Mohr's circle for stress *and* strain is insightful; it illustrates that the principal directions for stress and strain are no longer aligned ($\theta_\sigma \neq \theta_\varepsilon$). This is a result of the $\bar{S}_{16}$ and $\bar{S}_{26}$ coupling terms in the compliance matrix $\bar{\boldsymbol{S}}$ in the structural reference frame.

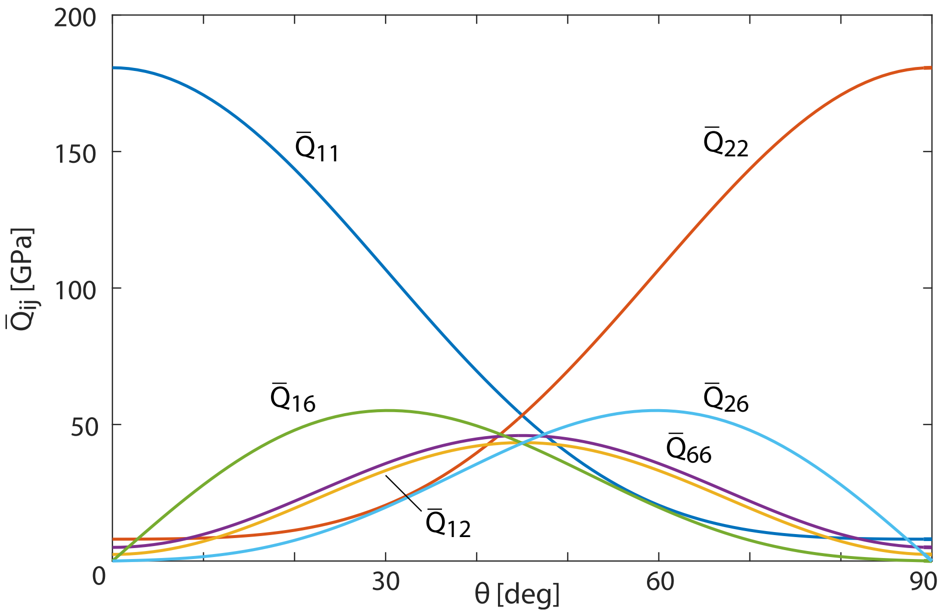


Note: the calculation could have been performed in a single step, using the transformed compliance matrix:

$$\bar{\boldsymbol{S}} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}\boldsymbol{S}\boldsymbol{T}$$

Example 1.2 – Stiffness Transformation

For a carbon/epoxy composite with reduced stiffness $Q_{11} = 180.7$ GPa, $Q_{12} = 2.41$ GPa, $Q_{22} = 8$ GPa and $Q_{66} = 5$ GPa, the transformed reduced stiffnesses are plotted as function of θ.



1.3.4 Engineering Constants

The lamina properties can be more easily interpreted by considering the effective engineering constants in different directions. The compliance in arbitrary axes may be written as:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_{xx}} - \nu_{yx}\frac{\sigma_{yy}}{E_{yy}} + m_{x,xy}\frac{\tau_{xy}}{G_{xy}}$$
$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E_{yy}} - \nu_{xy}\frac{\sigma_{xx}}{E_{xx}} + m_{y,xy}\frac{\tau_{xy}}{G_{xy}}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} + m_{xy,x}\frac{\sigma_{xx}}{E_{xx}} + m_{xy,y}\frac{\sigma_{yy}}{E_{yy}}$$

where $m_{x,xy}$, $m_{y,xy}$, $m_{xy,x}$ and $m_{xy,y}$ are known as the coefficients of mutual influence, and describe the coupling between direct stress and shear strain (and shear stress and direct strain).

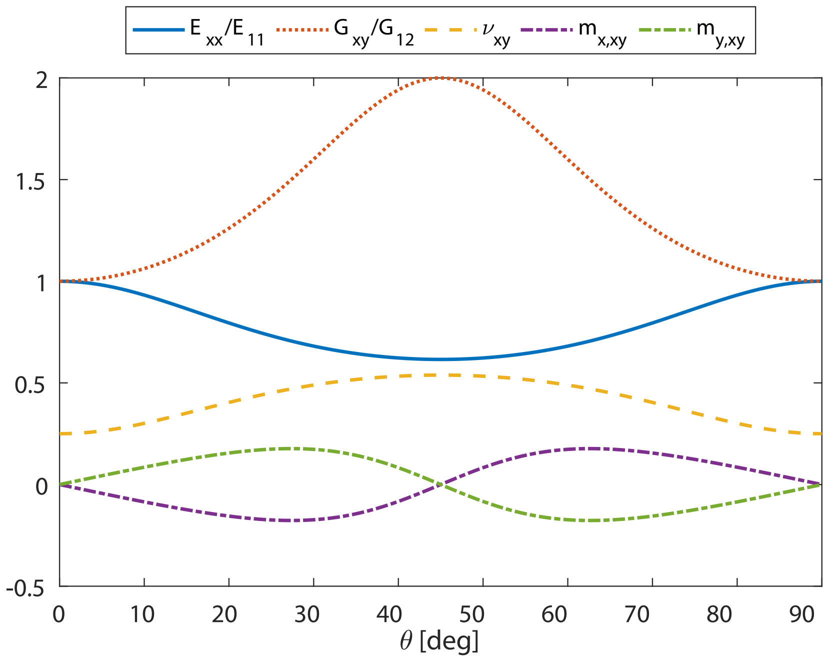
From the compliance matrix $\bar{\boldsymbol{S}}$, the effective engineering properties are found as:

$$\begin{aligned} E_{xx} &= \frac{1}{\bar{S}_{11}} \qquad & E_{yy} &= \frac{1}{\bar{S}_{22}} \\ G_{xy} &= \frac{1}{\bar{S}_{66}} \qquad & \nu_{xy} &= -\frac{\bar{S}_{12}}{\bar{S}_{11}} \end{aligned} \tag{1.15}$$

Note that $G_{xy} > G_{12}$, for all axes other than principal material axes, and will be maximum at $\theta = 45°$ as in that orientation the fibres will be loaded in pure tension/compression.

Example 1.3 – Engineering Constants Transformation

Consider a laminate with $E_{11} = E_{22} = 100$ GPa, $\nu_{12} = \nu_{21} = 0.25$, and $G_{12} = 20$ GPa.



Perhaps surprisingly, this does not behave isotropically; can you explain why?

1.4 Lamina Strength Criteria

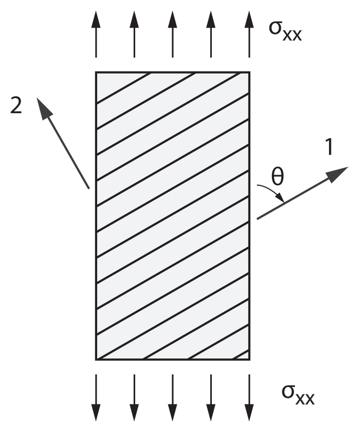
The *stiffness* of a unidirectional lamina is determined by the four elastic constants of a specially orthropic material (E_{11}, E_{22}, G_{12} and ν_{12}). The description of the *strength* of a specially orthotropic lamina requires at least five independent parameters:

$$
\begin{aligned}
X_t &= \text{longitudinal tensile strength} \\
X_c &= \text{longitudinal compressive strength} \\
Y_t &= \text{transverse tensile strength} \\
Y_c &= \text{transverse compressive strength} \\
S &= \text{in-plane intralaminar shear strength}
\end{aligned}
$$

Note that shear strength in the material coordinate system (123) has no associated direction. In a structural coordinate system (xyz), however, the sign of the shear stress matters for strength calculations: depending on the direction of the shear stress, fibres may be loaded in tension or compression, which have different stress allowables.

The formulation of suitable failure criteria for composites is challenging and remains an active area of research. In isotropic materials, the failure criteria rely on calculating the principal stresses (*e.g.* Tresca and Von Mises criteria). For composite materials, however, both the stiffness and strength are highly anisotropic. Thus the composite strength depends on the direction of the loading relative to the material axes. This means that the applied stresses must be converted into the material coordinate system to check against the strength allowables. Moreover, the composite failure modes depend on the direction of loading; for example, X_t is governed by fibre fracture, whereas X_c involves fibre buckling. Formulating a failure criterion for a laminate must attempt to consolidate these different failure mechanisms.

Off-axis Loading An illustrative case is the off-axis loading of a unidirectionally reinforced lamina. The material axes are set at an angle θ to the applied stress σ_{xx}, which generates a biaxial stress state.



The stresses in the material coordinate system follow from the tensor transformation:

$$\sigma_{11} = \sigma_{xx}\cos^2\theta$$

$$\sigma_{22} = \sigma_{xx}\sin^2\theta$$

$$\sigma_{12} = \sigma_{xx}\sin\theta\cos\theta$$

Note that this set-up will not be able to generate combined tension and compression in the material coordinates.

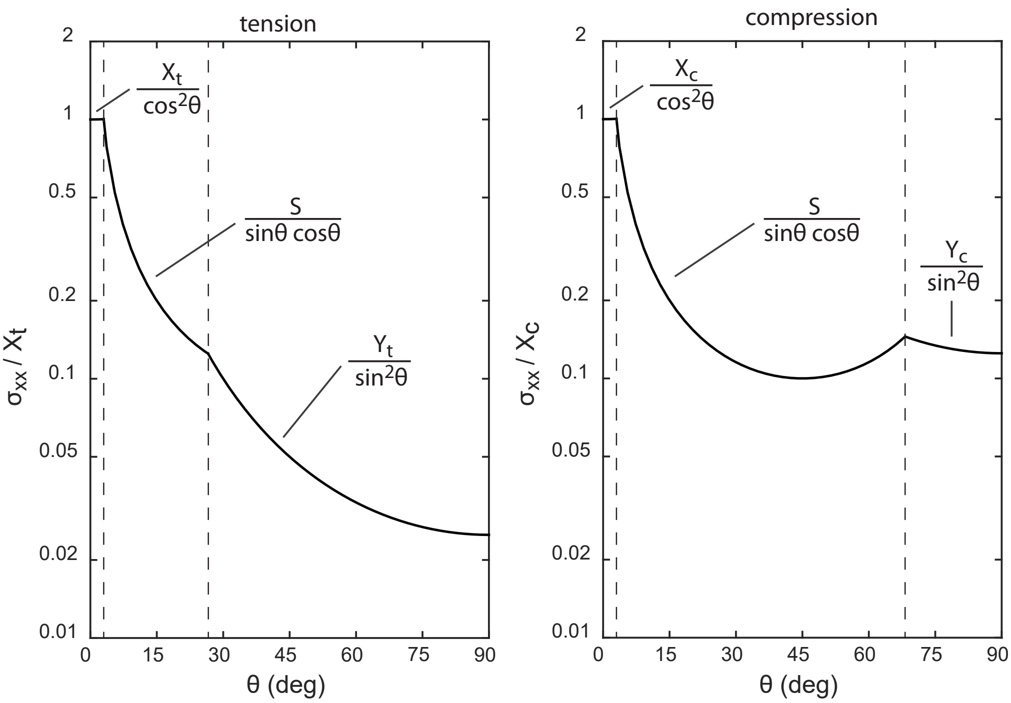
1.4.1 Maximum Stress Criterion

In the maximum stress criterion, failure is assumed to occur if one of the stresses in the natural axes exceeds the corresponding allowable stress:

$$\begin{aligned} -X_c < \sigma_{11} < X_t \\ -Y_c < \sigma_{22} < Y_t \\ |\tau_{12}| < S \end{aligned} \tag{1.16}$$

Substituting the transformed stresses for the off-axis load case, the maximum uniaxial stress σ_{xx} is found as the smallest value of:

$$\begin{aligned} -\frac{X_c}{\cos^2\theta} < \sigma_{xx} < \frac{X_t}{\cos^2\theta} \\ -\frac{Y_c}{\sin^2\theta} < \sigma_{xx} < \frac{Y_t}{\sin^2\theta} \\ |\sigma_{xx}| < \left|\frac{S}{\sin\theta\cos\theta}\right| \end{aligned}$$



In this and subsequent plots illustrating the failure criteria, representative material properties have been used for a Glass-Epoxy laminate (Jones, 1999).

E_{11}	=	54 GPa	X_t	=	1035 MPa
E_{22}	=	18 GPa	X_c	=	1035 MPa
ν_{12}	=	0.25	Y_t	=	28 MPa
G_{12}	=	9 GPa	Y_c	=	138 MPa
			S	=	55 MPa

1.4.2 Maximum Strain Criterion

Alternatively, the maximum strain criterion compares the strains in the material coordinate system

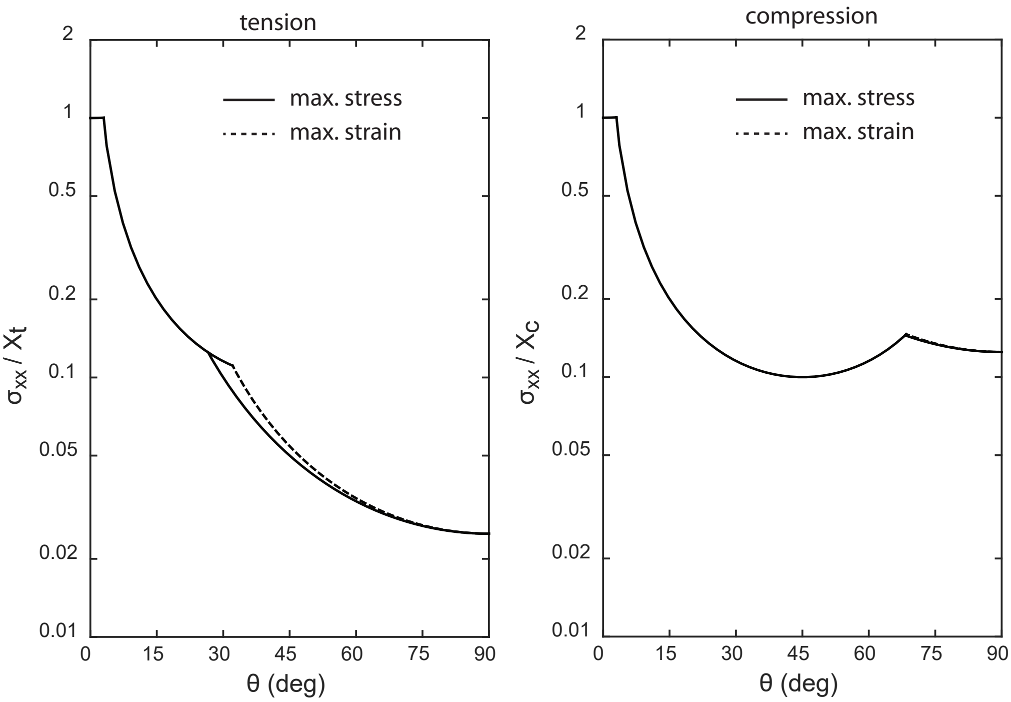
$$\varepsilon_{11} = \frac{1}{E_{11}}\left(\sigma_{11} - \nu_{12}\sigma_{22}\right)$$
$$\varepsilon_{22} = \frac{1}{E_{22}}\left(\sigma_{22} - \nu_{21}\sigma_{11}\right)$$
$$\gamma_{12} = \frac{\tau_{12}}{G_{12}}$$

to the strains at ultimate stresses; the material is assumed linear-elastic up to the point of failure.

Substituting the transformed stresses for the off-axis load case, the maximum uniaxial stress σ_{xx} is found as the smallest value of:

$$-\frac{X_c}{\cos^2\theta - \nu_{12}\sin^2\theta} < \sigma_{xx} < \frac{X_t}{\cos^2\theta - \nu_{12}\sin^2\theta}$$
$$-\frac{Y_c}{\sin^2\theta - \nu_{21}\cos^2\theta} < \sigma_{xx} < \frac{Y_t}{\sin^2\theta - \nu_{21}\cos^2\theta}$$
$$|\sigma_{xx}| < \left|\frac{S}{\sin\theta\cos\theta}\right|$$

This is similar to the maximum stress condition, with the addition of the Poisson's ratio terms.



1.4.3 Tsai-Hill Failure criterion

The previous failure theories neglected any interaction between failure modes, and also showed cusps and discontinuities in the failure curves which are not seen in experimental data. The Tsai-Hill criterion is an extension of the Von Mises criterion to specially orthotropic materials, which captures some of the interaction of the failure modes. Numerous other failure criteria have been proposed; commonly used is the Tsai-Wu criterion, which may provide a better estimate at the expense of requiring more experimental strength variables.

The Tsai-Hill criterion is given as:

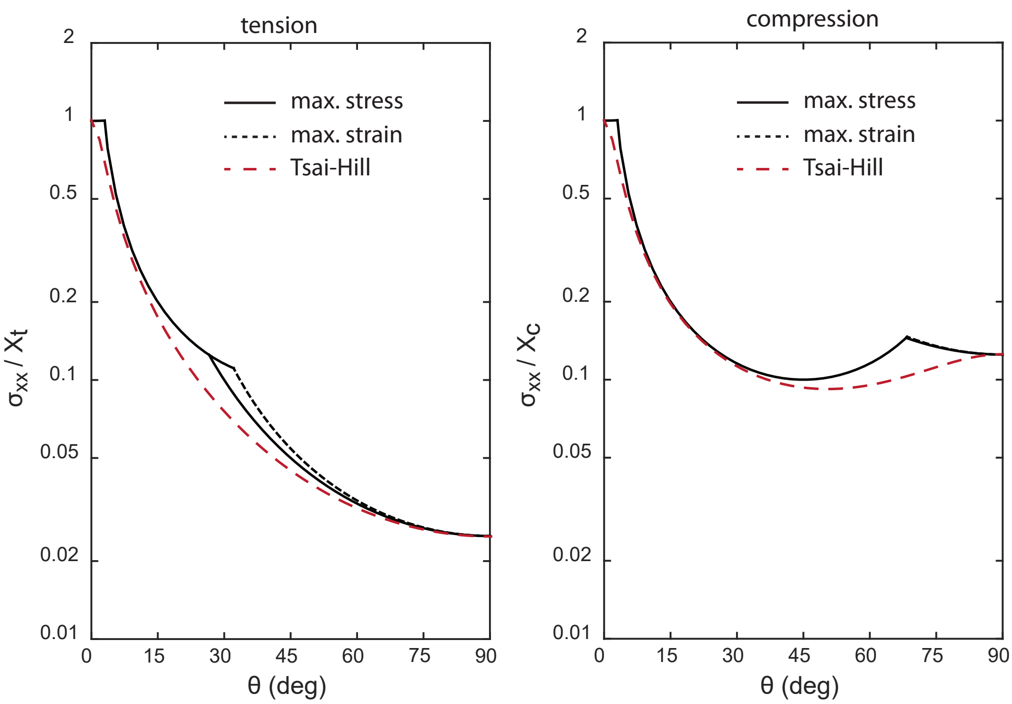
$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 - \frac{\sigma_{11}}{X}\frac{\sigma_{22}}{X} \leq 1 \tag{1.17}$$

where $X = X_t$ or $X = X_c$ and $Y = Y_t$ or Y_c corresponding to sign of σ_{11} and σ_{22}.

Substituting the transformed stresses for the off-axis load case,

$$\frac{\cos^4\theta}{X^2} + \cos^2\theta\sin^2\theta\left[\frac{1}{S^2} - \frac{1}{X^2}\right] + \frac{\sin^4\theta}{Y^2} = \frac{1}{\sigma_{xx}^2}$$

results in a single failure criterion. However, it will be different for each quadrant in the stress plane, due to different tensile/compressive failure strengths.



For the sample Glass-Epoxy laminate the Tsai-Hill criterion provides a better fit to the experimental data, and offers a smoother failure curve by capturing some of the interaction between failure modes. The maximum difference between the failure criteria occurs at the change of failure modes.

1.5 Summary

The structural properties of a single ply are described using a macromechanical model, where the fibre-matrix composite is homogenised and the equivalent elastic moduli are used. In the material axes, the unidirectional laminate corresponds to a specially orthothropic material, but when it is placed at an angle to the stuctural axes, it becomes a generally orthotropic material (with fully populated stiffness/compliance matrices).

The generally orthotropic material model is obtained by transforming the applied stress into the material reference frame, before using the specially orthotropic material model to obtain the strains, which are then converted back into the structural reference frame using the second rank tensor transformation.

Revision Objectives Handout 1:

- apply the second-rank tensor transformation $\boldsymbol{T}$ for plane stress and strain;
- recognise and use the composite subscript conventions $(i, j = 1 \ldots 6)$;
- derive the lamina compliance matrix $\boldsymbol{S}$ in terms of 4 elastic constants (E_{11}, E_{22}, G_{12}, and ν_{12});
- recall relationship between Poisson's ratio and Young's moduli ($\nu_{12}/E_{11} = \nu_{21}/E_{22}$);
- recall the reduced stiffness matrix $\boldsymbol{Q}$ in terms of 4 elastic constants;
- derive the transformed reduced stiffness matrix: $\bar{\boldsymbol{Q}} = \boldsymbol{T}^{-1}\boldsymbol{Q}\boldsymbol{R}\boldsymbol{T}\boldsymbol{R}^{-1}$;
- derive the transformed compliance matrix: $\bar{\boldsymbol{S}} = \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^{-1}\boldsymbol{S}\boldsymbol{T}$;
- calculate in-plane stiffness/compliance in non-fibre directions using $\bar{\boldsymbol{Q}}$ and $\bar{\boldsymbol{S}}$;
- calculate effective engineering constants from ply stiffness matrix;
- recall the five strength parameters for specially orthotropic laminate (X_t, X_c, Y_t, Y_c, S);
- appreciate different failure mechanisms as function of ply angle in angled ply;
- calculate strength of a lamina, using Tsai-Hill or maximum stress failure criteria;