Signals part 1.4 – Discrete signals and sampling

Recap - Complex exponential

$$f(t) = Ce^{at}$$

- 'a' is real – exponential

$$f(t) = Ce^{j\omega t}$$

- 'a' is imaginary – phasor on complex plane, sinusoid in the time domain

$$f(t) = C\cos(\omega t) + jC\sin(\omega t)$$

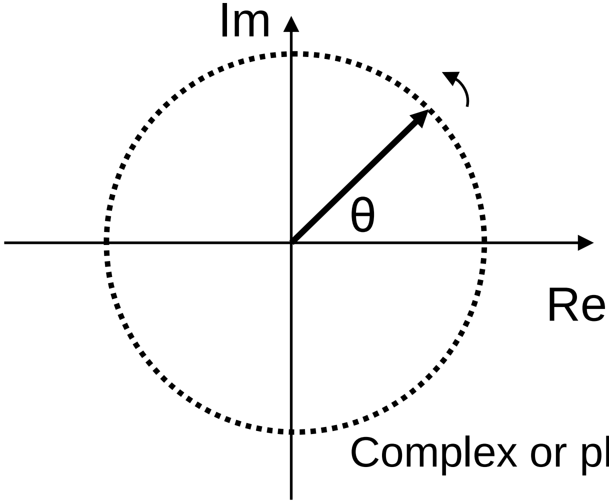
$$f(t) = Ce^{(r+j\omega)t}$$

- 'a' has real and complex parts – phasor decaying or growing exponentially

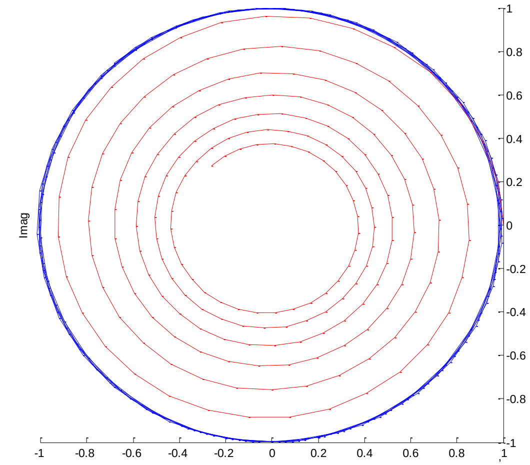
$$f(t) = Ce^{jk\omega t}$$

- Integer 'k' multiplying ω gives harmonic series

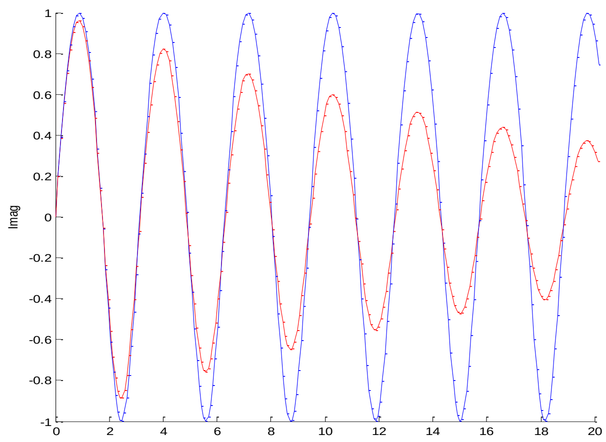
Visualising signals



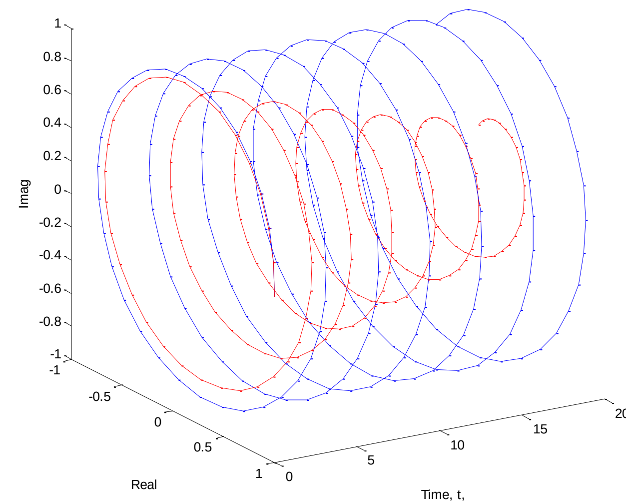
Complex or phase plane



Sinusoid (blue) and decaying sinusoid (red) on phase plane



Sinusoid (blue) and decaying sinusoid (red) in time



Discrete Signals

- Lets first start with some familiar sine waves:

$$f(t) = \sin(\omega t) \qquad\qquad f[n] = \sin(\Omega n)$$

What is this?

- We know that: $x[n] = x(nT)$, *where n is an integer and T is the sampling interval*

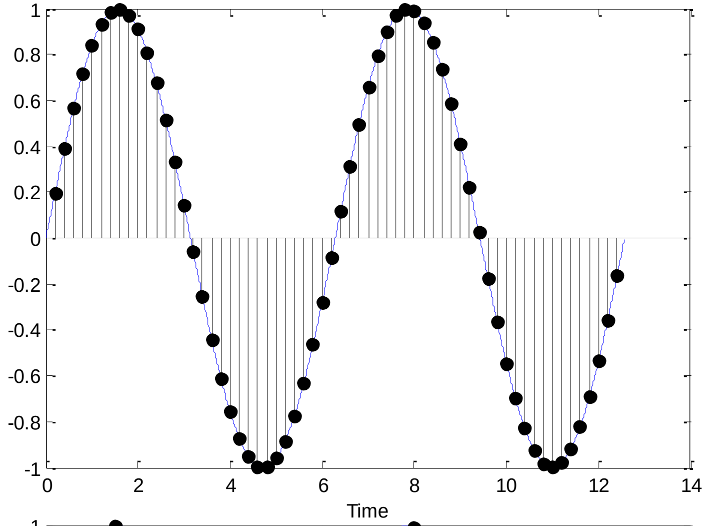
$$\sin(\Omega n) = \sin(\omega n T) \qquad \omega n T = \Omega n \qquad \therefore \omega T = \Omega$$

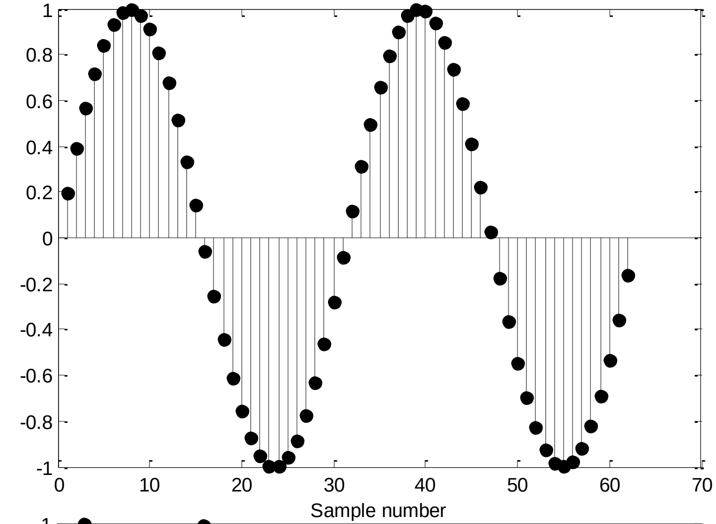
- Hence Ω is 'radians per sample period' – the 'discrete time frequency' or normalised frequency (compare with ω – radians per second)

- So the twist is that 'time' is variable in the discrete world: we can adjust sample period

Time

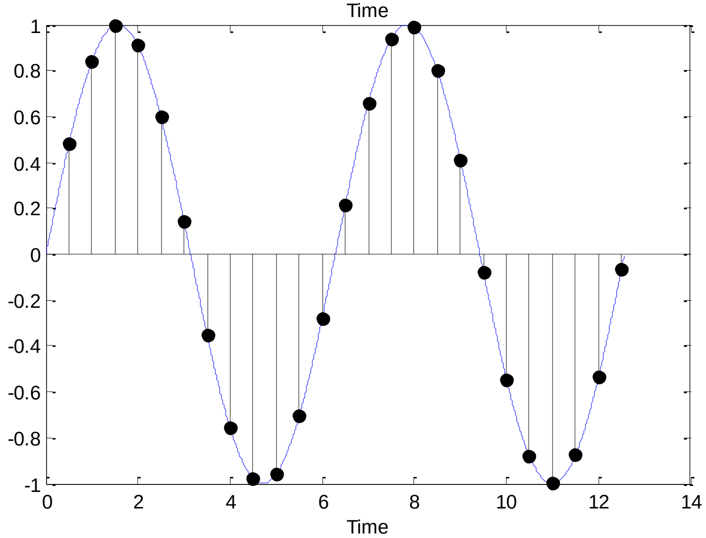
Discrete Time

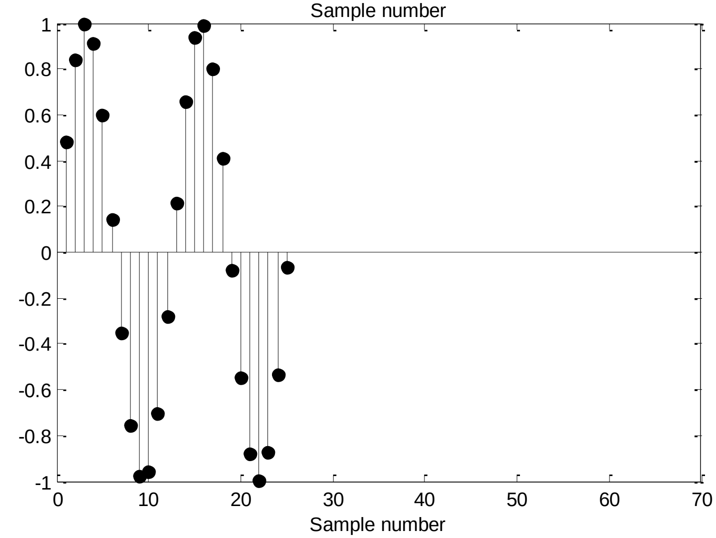




Sin(wt), w=1 sampled at T = 0.2

Ω = 0.2





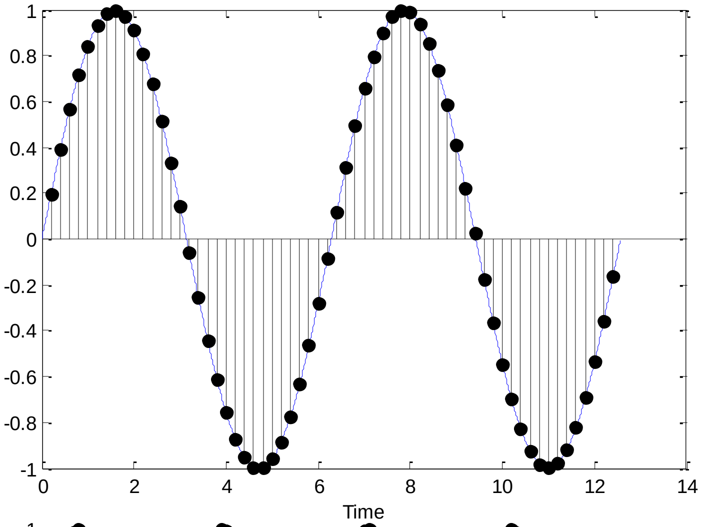
Sin(wt), w=1 sampled at T = 0.5

Ω = 0.5

Increasing the sample interval results in a higher normalised frequency in the discrete time domain

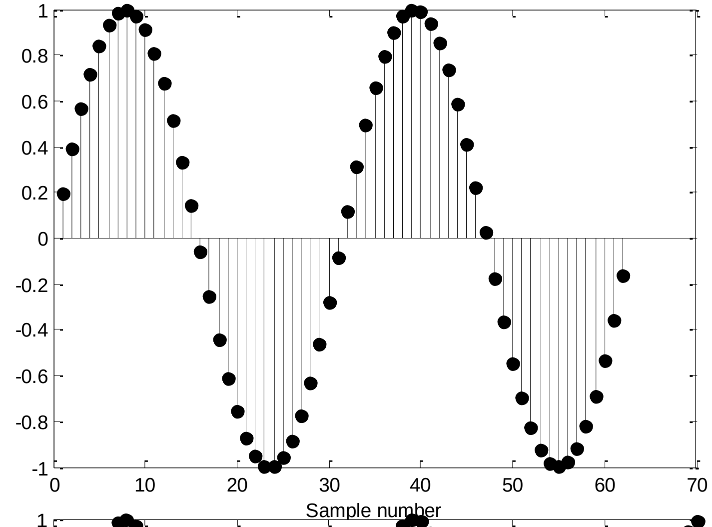
Time

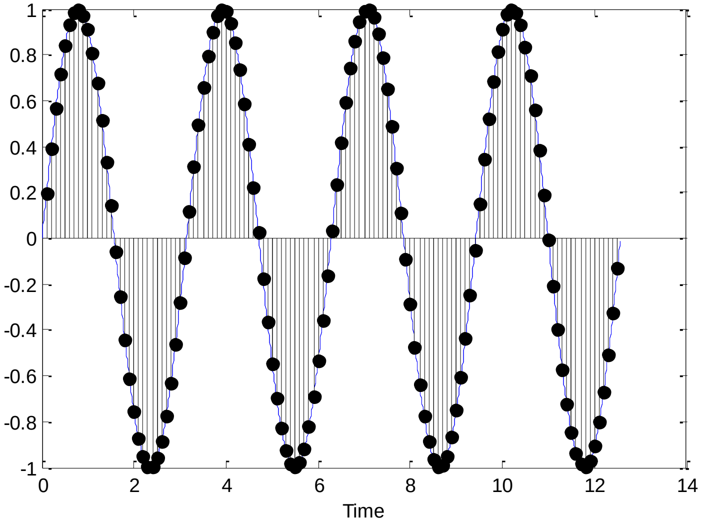
Discrete Time



Sin(wt), w=1 sampled at T = 0.2

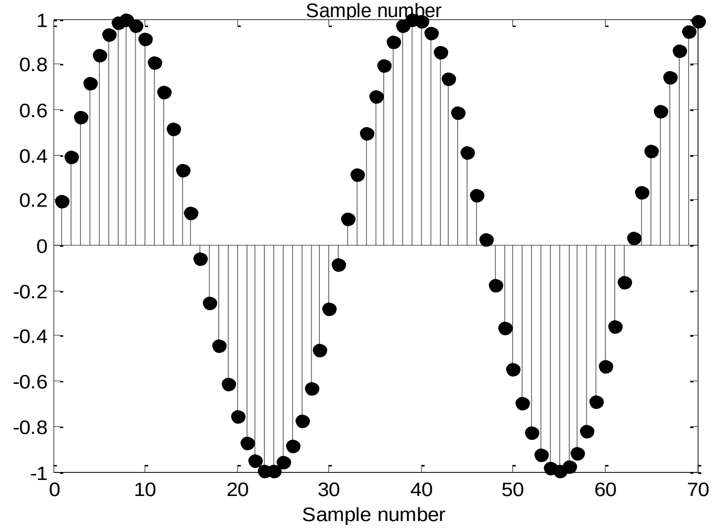
$\Omega = 0.2$





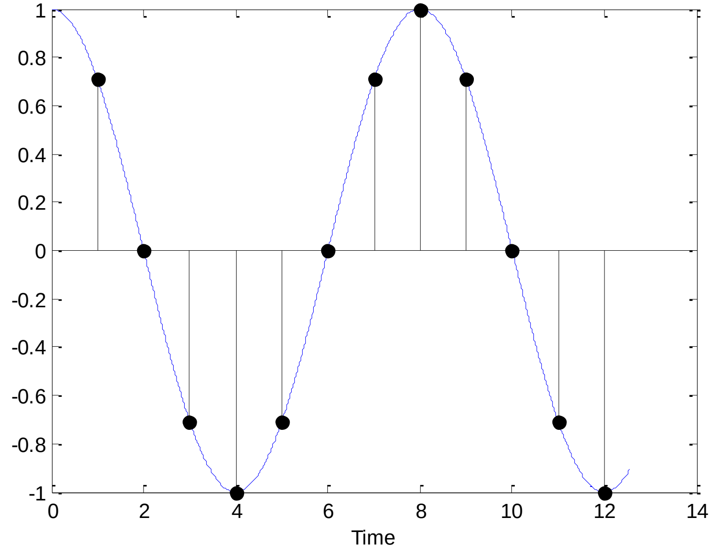
Sin(wt), w=2 sampled at T = 0.1

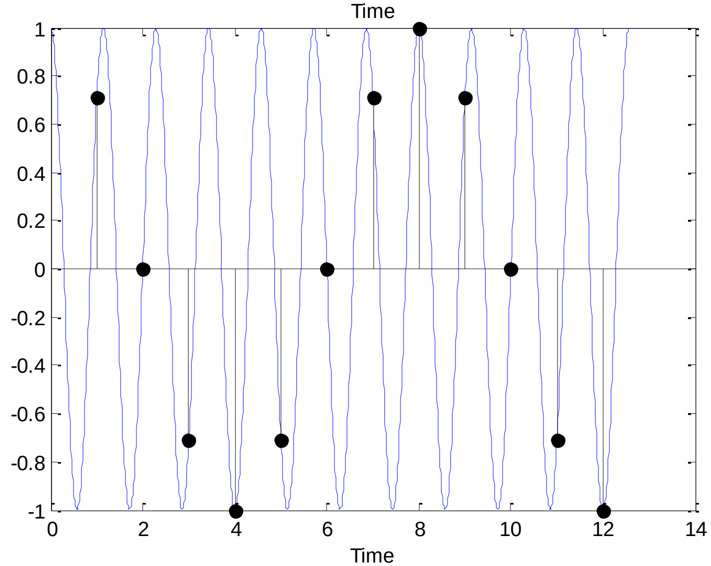
$\Omega = 0.2$



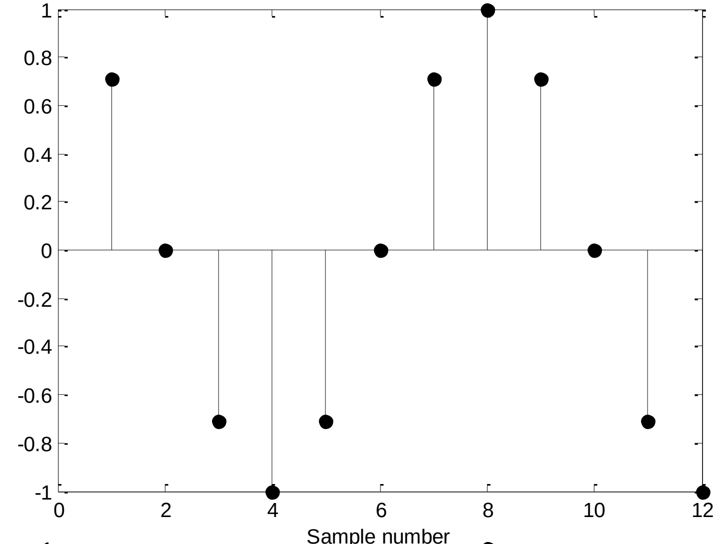
Waveforms with the same normalised frequency can look different in the time domain

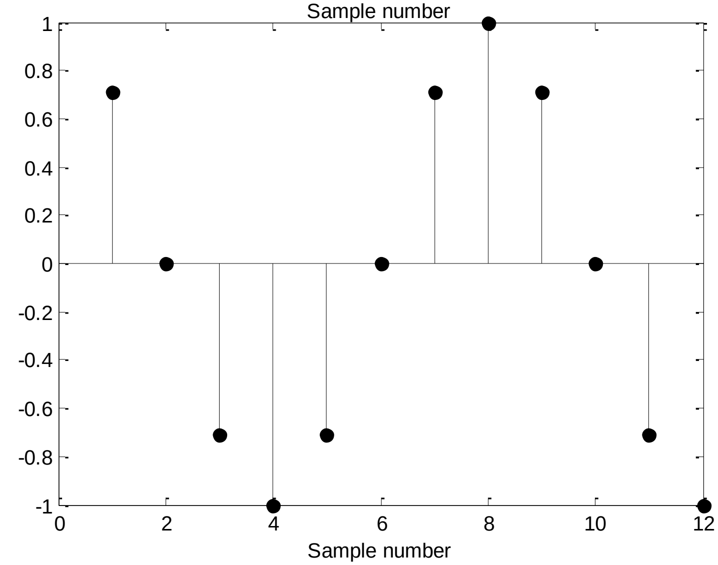
Time





Discrete Time





cos(wt), w=π/4
sampled at T = 1

Ω = π/4

cos(wt), w=7π/4
sampled at T = 1

Ω = 7π/4

π/4 gives the same discrete time sequence as 7π/4

Discrete time complex exponential

$$f[n] = Ce^{\beta n}$$

Where ‘C’ and ‘β’ are complex numbers

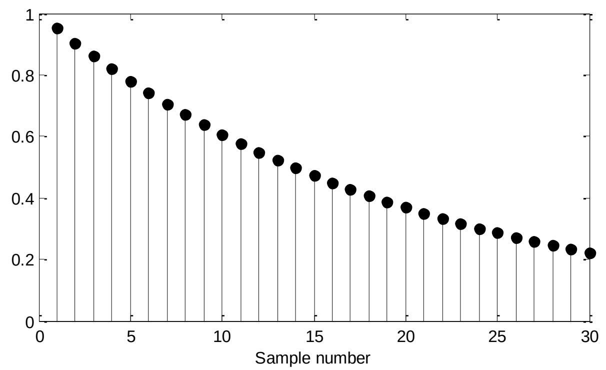
When ‘β’ has an imaginary part the result is a periodic signal;

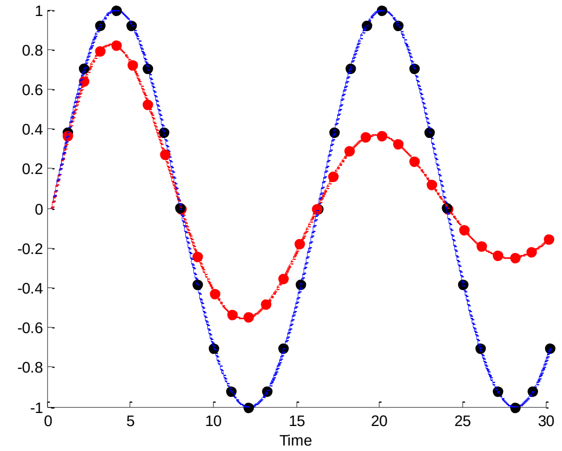
$$f[n] = Ce^{(\varepsilon + j\Omega)n}$$

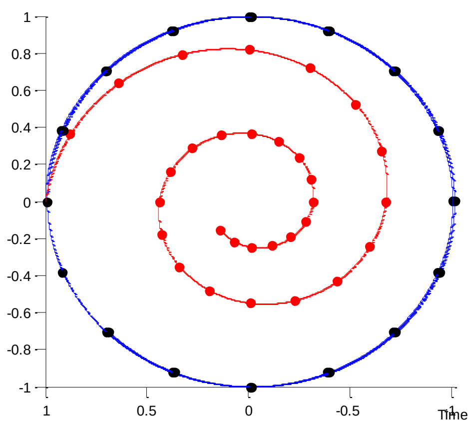
$$\omega T = \Omega$$

- The discrete time complex exponential is analogous to the continuous time version save the substitution of Ω for discrete frequency, (remembering Ω is the radians per sample period).
- Sampled versions of exponential functions, sinusoids, decaying/growing sinusoids, phase shifted sinusoids and harmonic sets can all be reproduced.

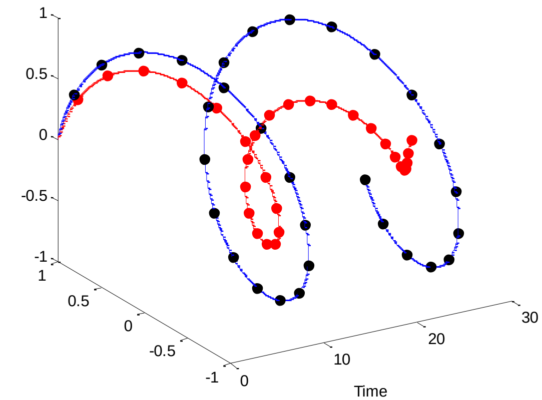
Sampled waveforms



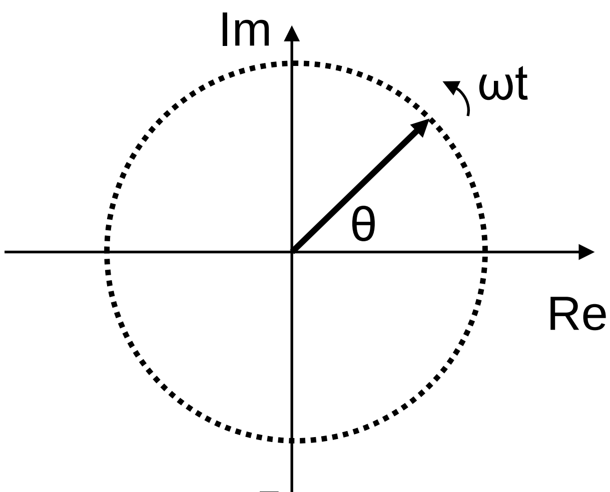


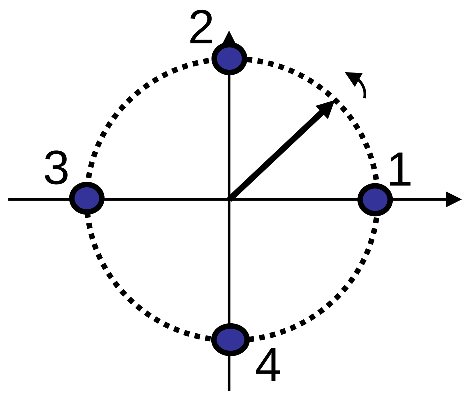


Although these signals show the underlying, continuous waveform, only the individual points exist in the discrete domain

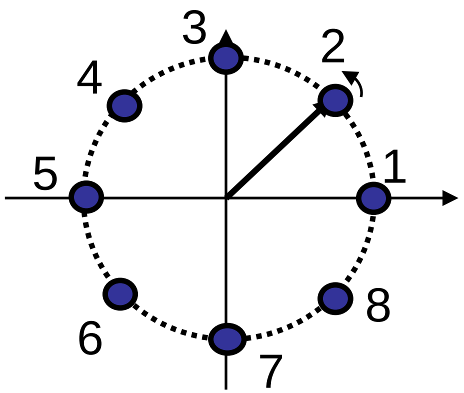


Sampling on the complex plane

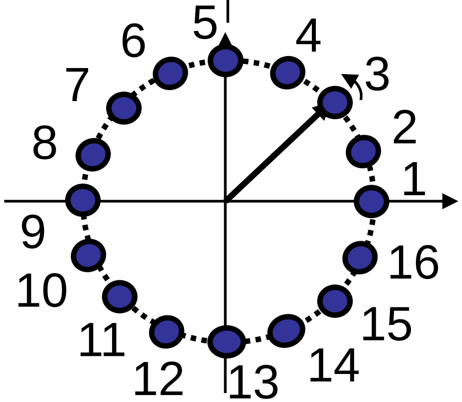




$\Omega = 2\pi/4$



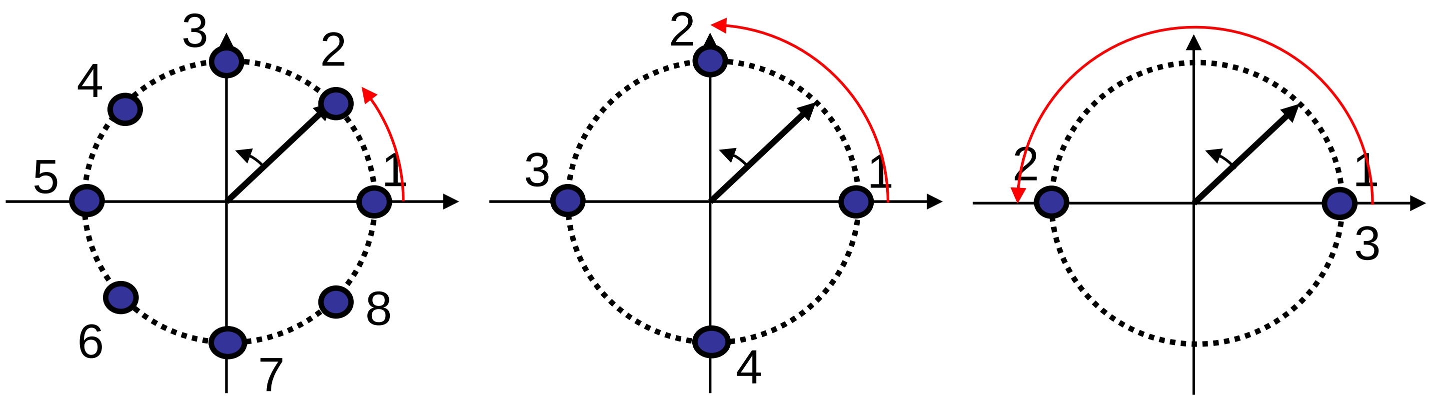
$\Omega = \pi/4$



$\Omega = \pi/8$

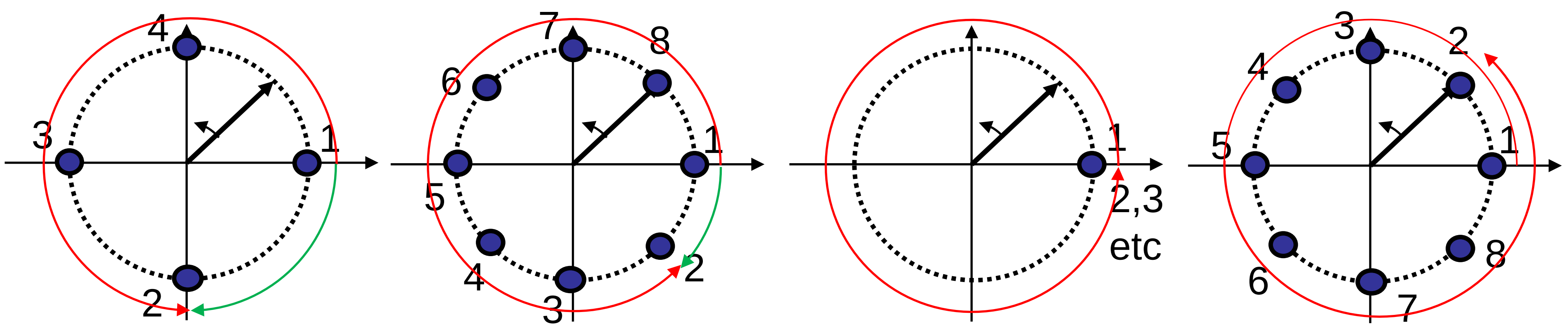
- As the normalised frequency is reduced, the number of times the waveform is sampled per cycle is increased.
- What happens if we increase the normalised frequency?

Sampling on the complex plane



$\Omega = \pi/4$ $\Omega = 2\pi/4$ $\Omega = 4\pi/4$

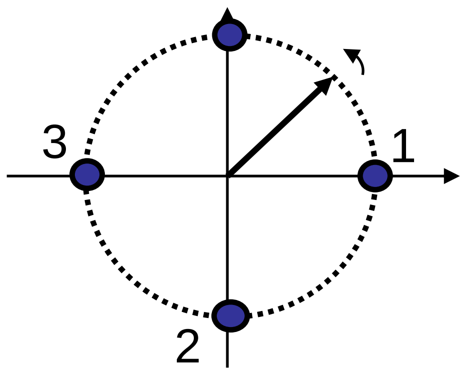
When the normalised frequency is higher than π, the waveform is sampled at the same point on the complex plane as a normalised frequency of Ω-+2π



$\Omega = 6\pi/4$ $\Omega = 7\pi/4$ $\Omega = 8\pi/4$ $\Omega = 9\pi/4$

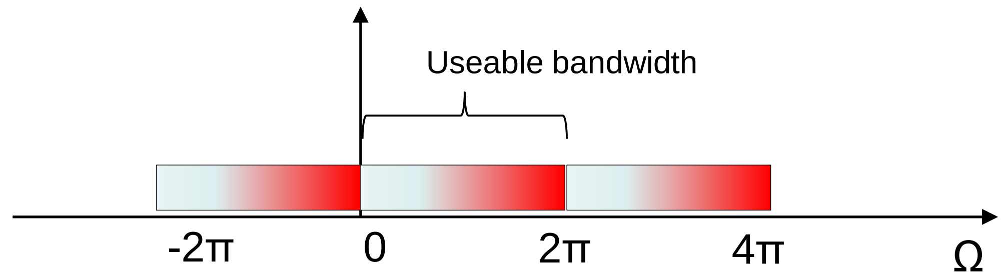
$\Omega = -2\pi/4$ $\Omega = -\pi/4$

Sampling - Aliasing



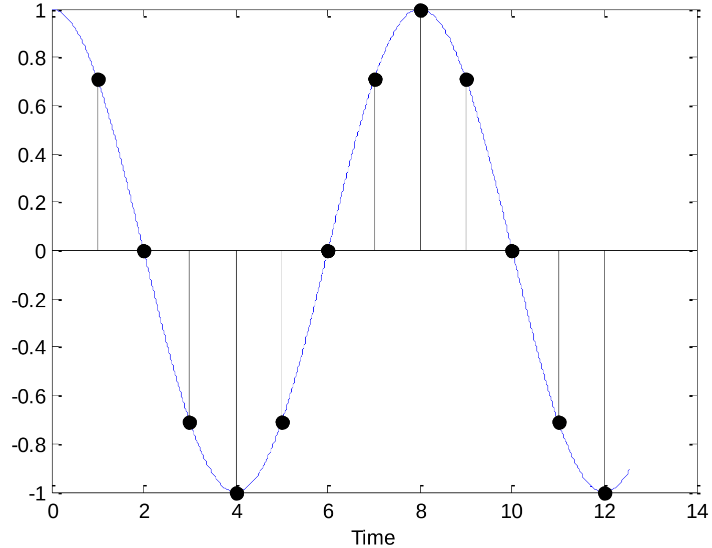
$\Omega = 6\pi/4$

& $\Omega = -2\pi/4$ & $\Omega = 14\pi/4$

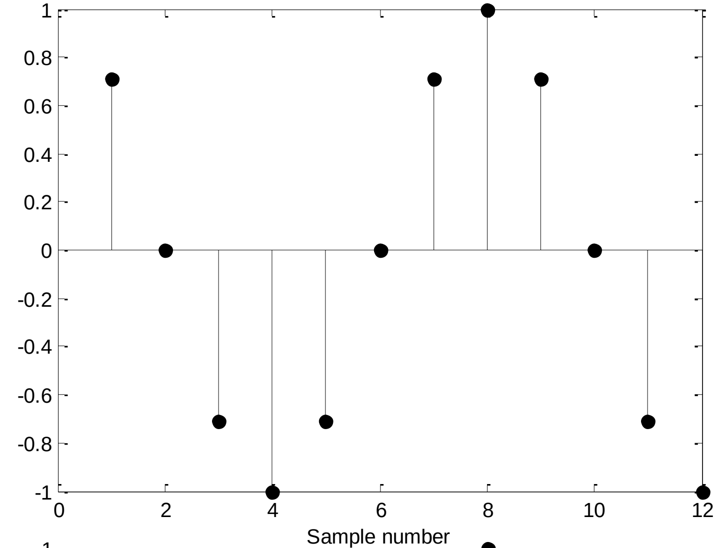


- When a signal is complex valued, as long as it is sampled more than once per cycle, the sample can represent that signal uniquely.
- When a signal is sampled less than once per cycle, the sampled signal appears identical to another in the range $0<\Omega<2\pi$
- When sampling, if signals outside the range $0<\Omega<2\pi$ are present, then they will reflect artefacts back into the range $0<\Omega<2\pi$ causing 'aliasing'
- Remember Ω changes with the frequency of the signal being sampled and the sampling rate

Time

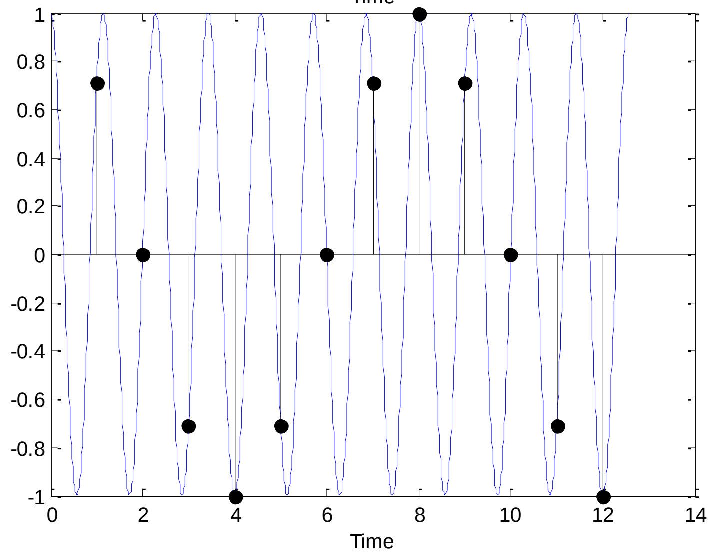


Discrete Time



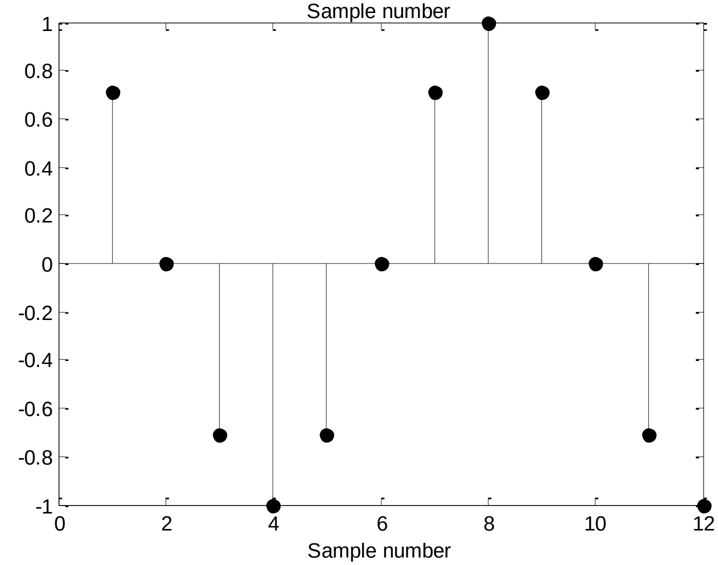
cos(wt), w=π/4
sampled at T = 1

Ω = π/4



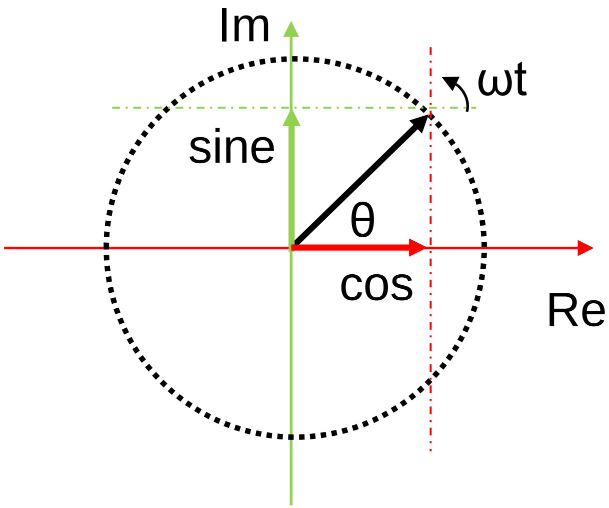
cos(wt), w=7pi/4
sampled at T = 1

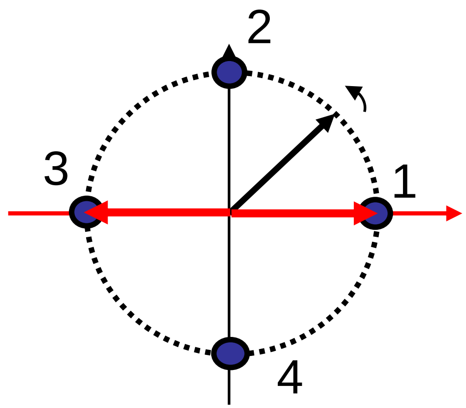
Ω = 7π/4



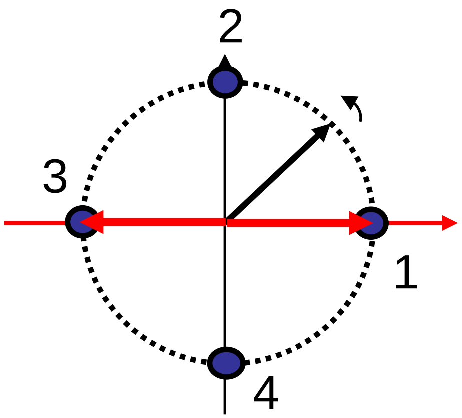
π/4 gives the same discrete time sequence as 7π/4

Sampling real valued signals

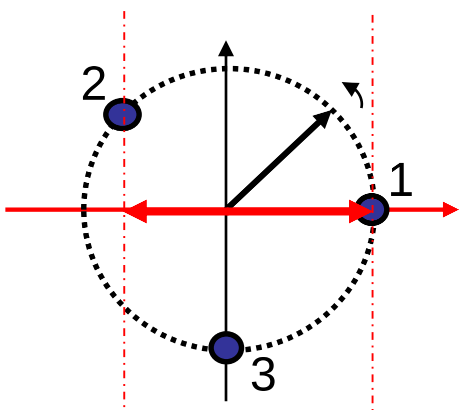




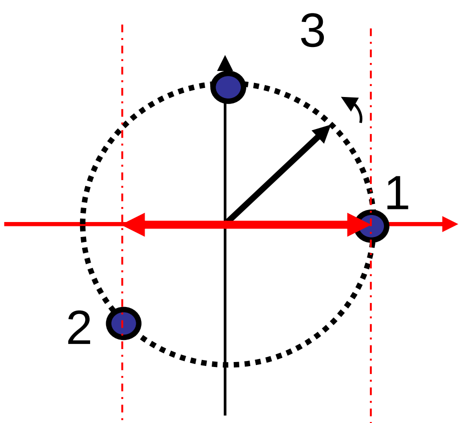
$\Omega = 2\pi/4$



$\Omega = -6\pi/4$ & $10\pi/4$



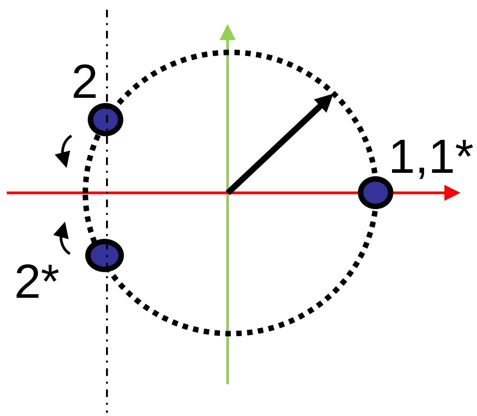
$\Omega = 3\pi/4$



$\Omega = 5\pi/4$

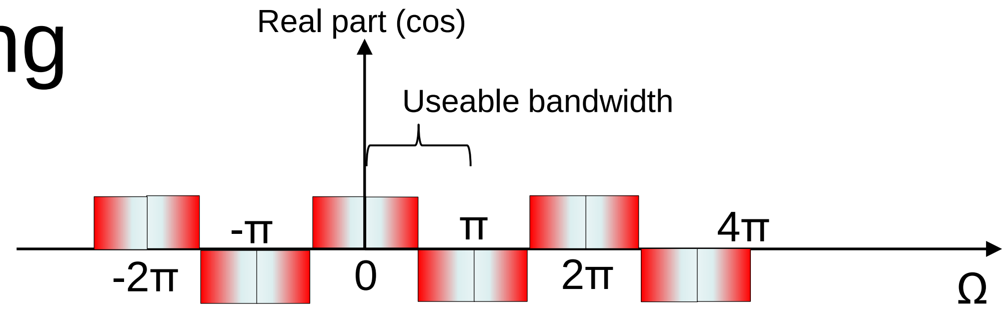
- With real valued signals we still have 2π repetition.
- But we also have another effect due to waveform symmetry

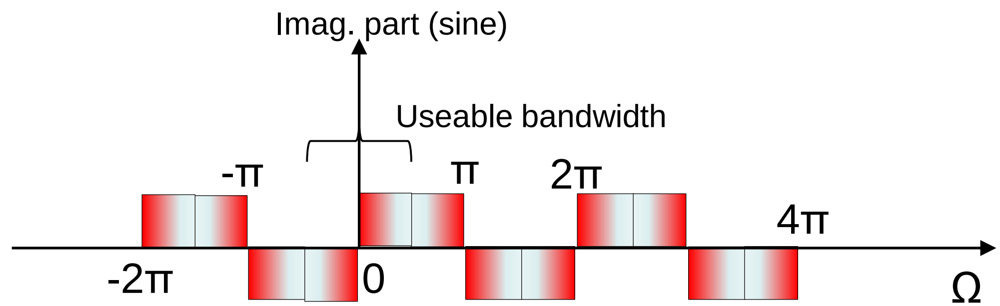
Sampling - Aliasing



Cosine reflects at 0, π

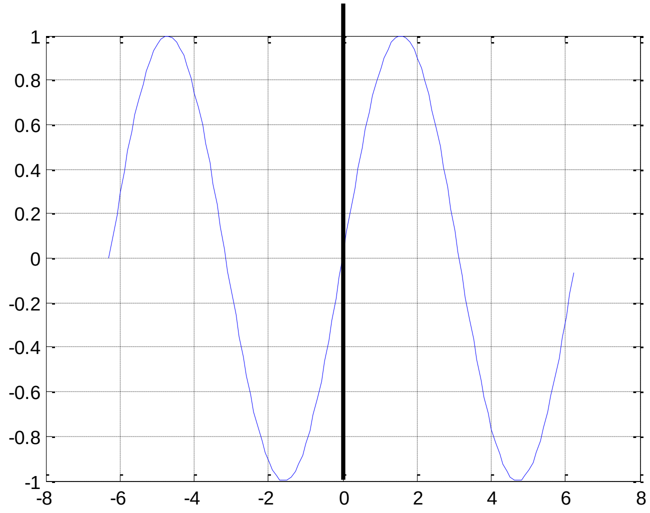
Cosine inverts at π/2, 3π/2

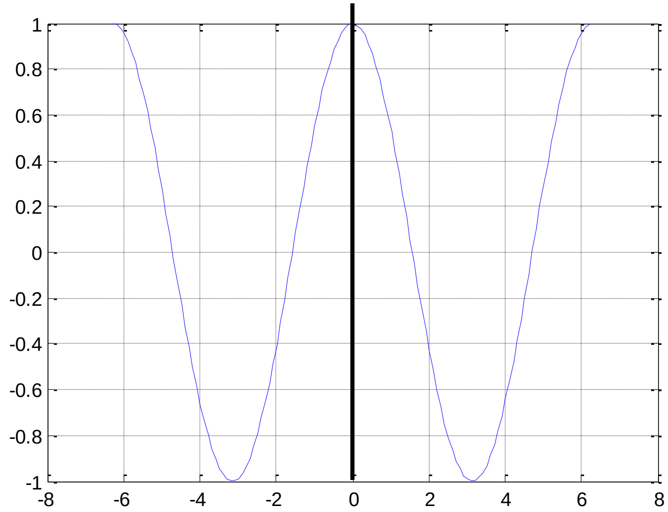




- The case for real valued signals is more complex as the symmetry of cosine and sine causes reflection/inversions at $\Omega = \pi/2$ intervals.
- When a signal has real values only, as long as it is sampled more than twice per cycle, the sample can represent that signal uniquely, however when a signal is sampled less than twice per cycle, the sampled signal appears identical to another in the range $0<\Omega<\pi$
- Signals outside the range $0<\Omega<\pi$ will reflect artefacts back into the range $0<\Omega<\pi$ causing 'aliasing'

Waveform symmetry

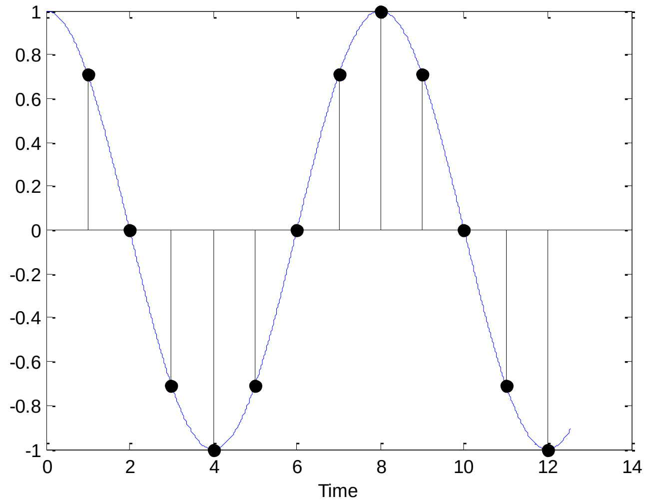




- Sine is an odd function:
  - sin(x) = -sin(-x)
- Sine has quarter wave symmetry
  - It has even symmetry around π/2

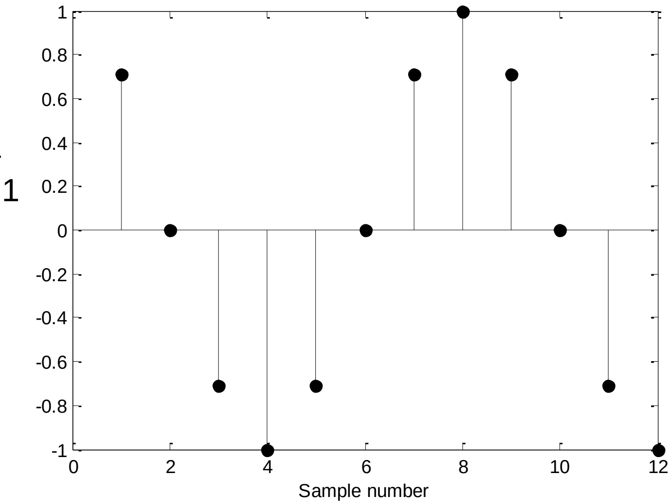
- Cosine is an even function:
  - cos(x) = cos(-x)
- Cosine has quarter wave symmetry
  - It has odd symmetry around π/2

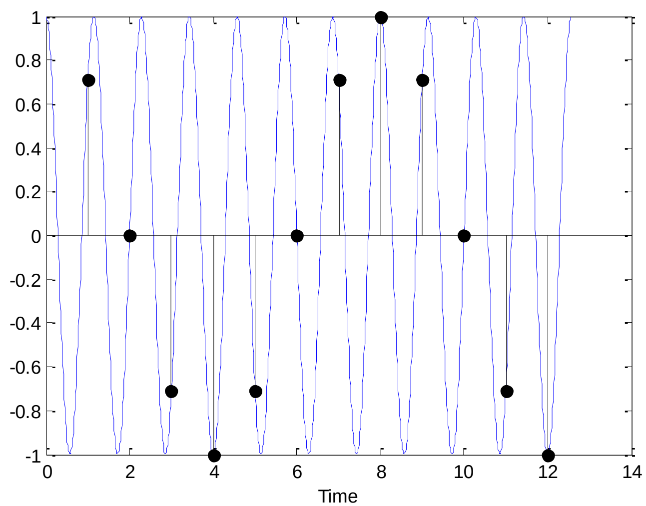
Time	Discrete Time



cos(wt), w=π/4
sampled at T = 1

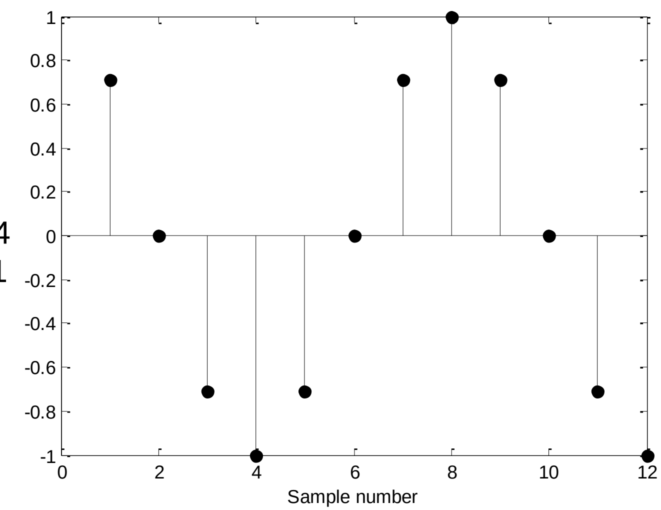
Ω = π/4





cos(wt), w=7pi/4
sampled at T = 1

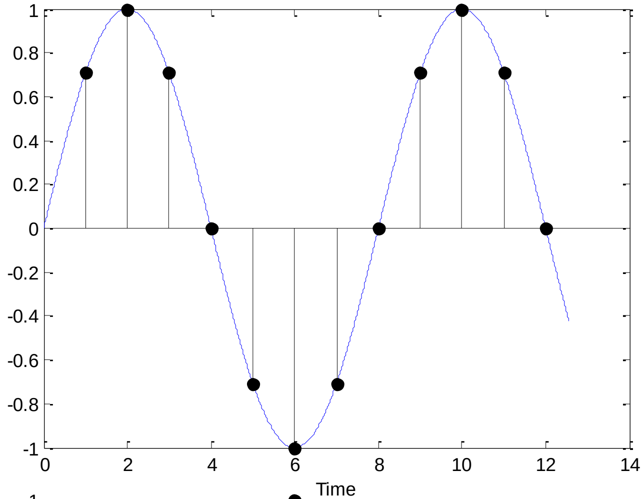
Ω = 7π/4



cos(π/4) is the same as cos(7π/4)

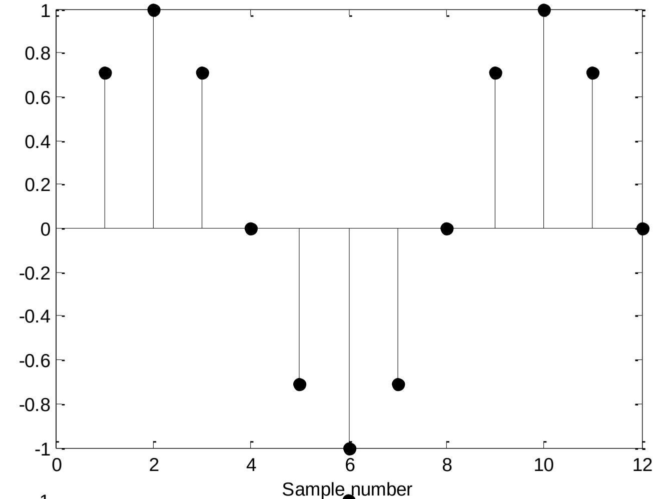
Time

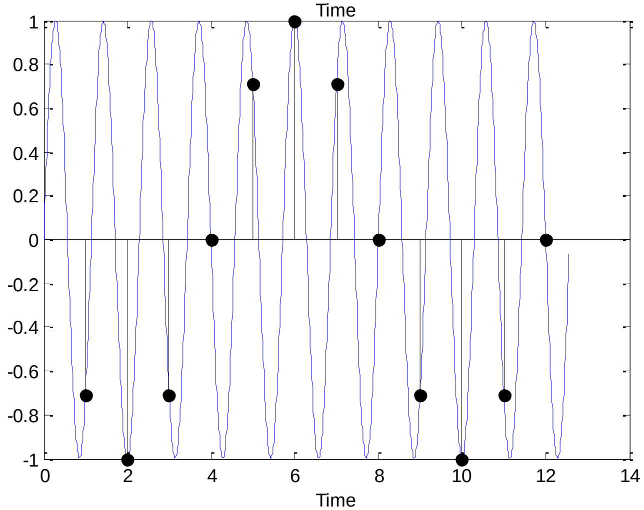
Discrete Time



sin(wt), w=π/4
sampled at T = 1

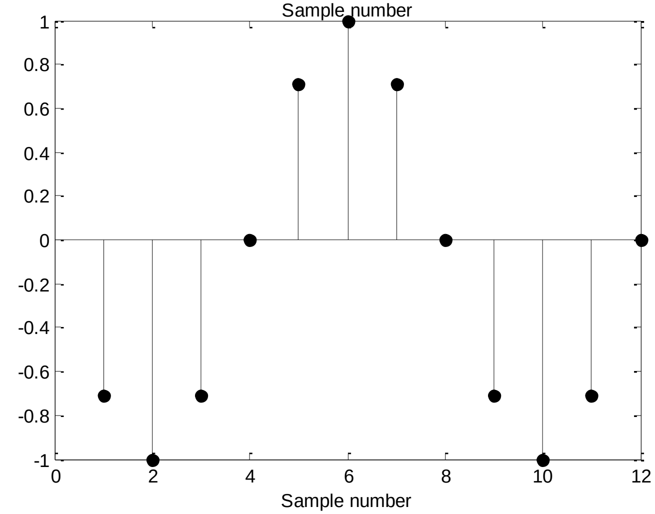
Ω = π/4





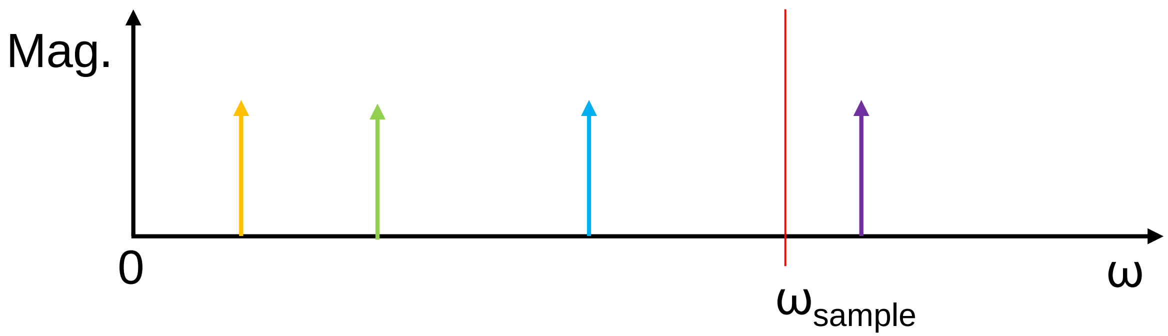
sin(wt), w=7pi/4
sampled at T = 1

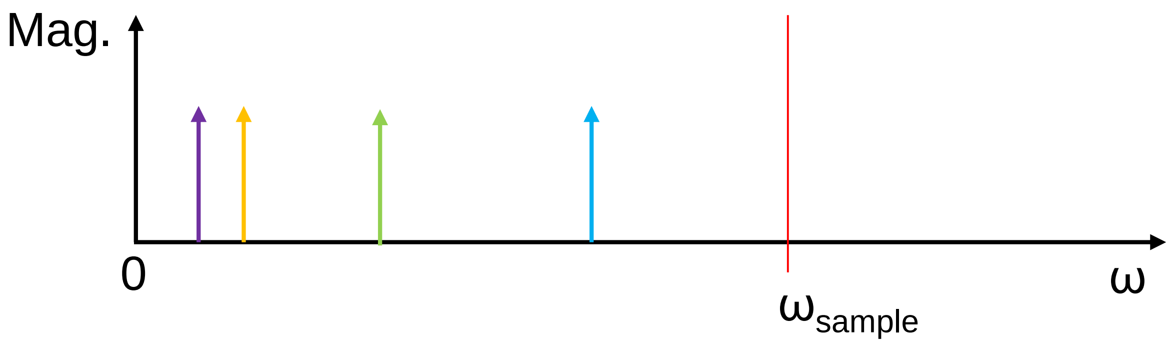
Ω = 7π/4



Sin(π/4) is the same as – sin(7π/4)

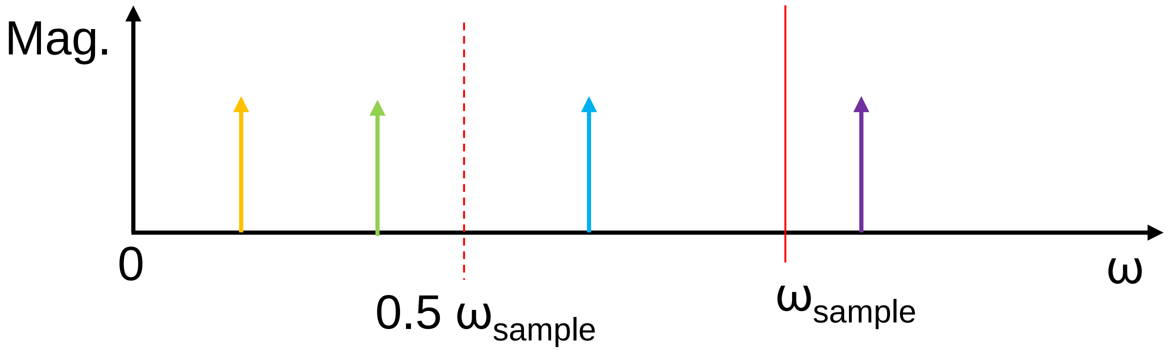
Aliasing illustrated – complex signals

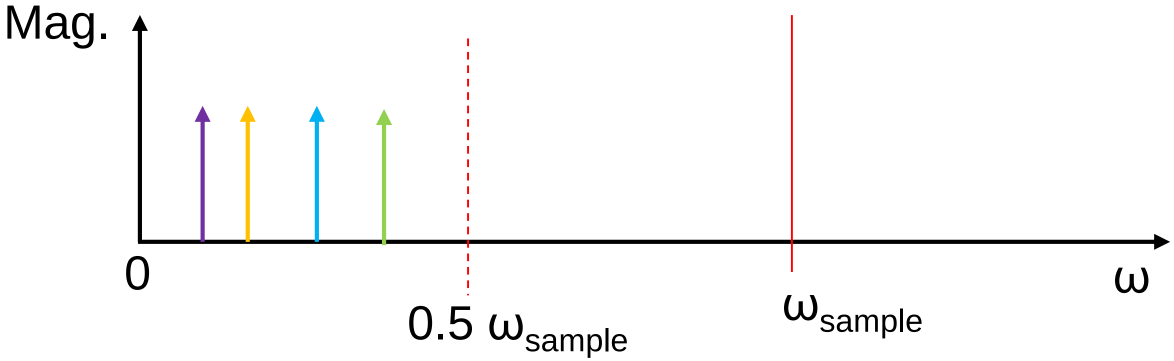




- If a signal has components with frequencies above the sampling frequency, these are reflected back and appears as artefacts at a lower frequency.
- This applies to complex valued signals.
- Here, the upper graph is indistinguishable from the lower once the signal have been sampled.

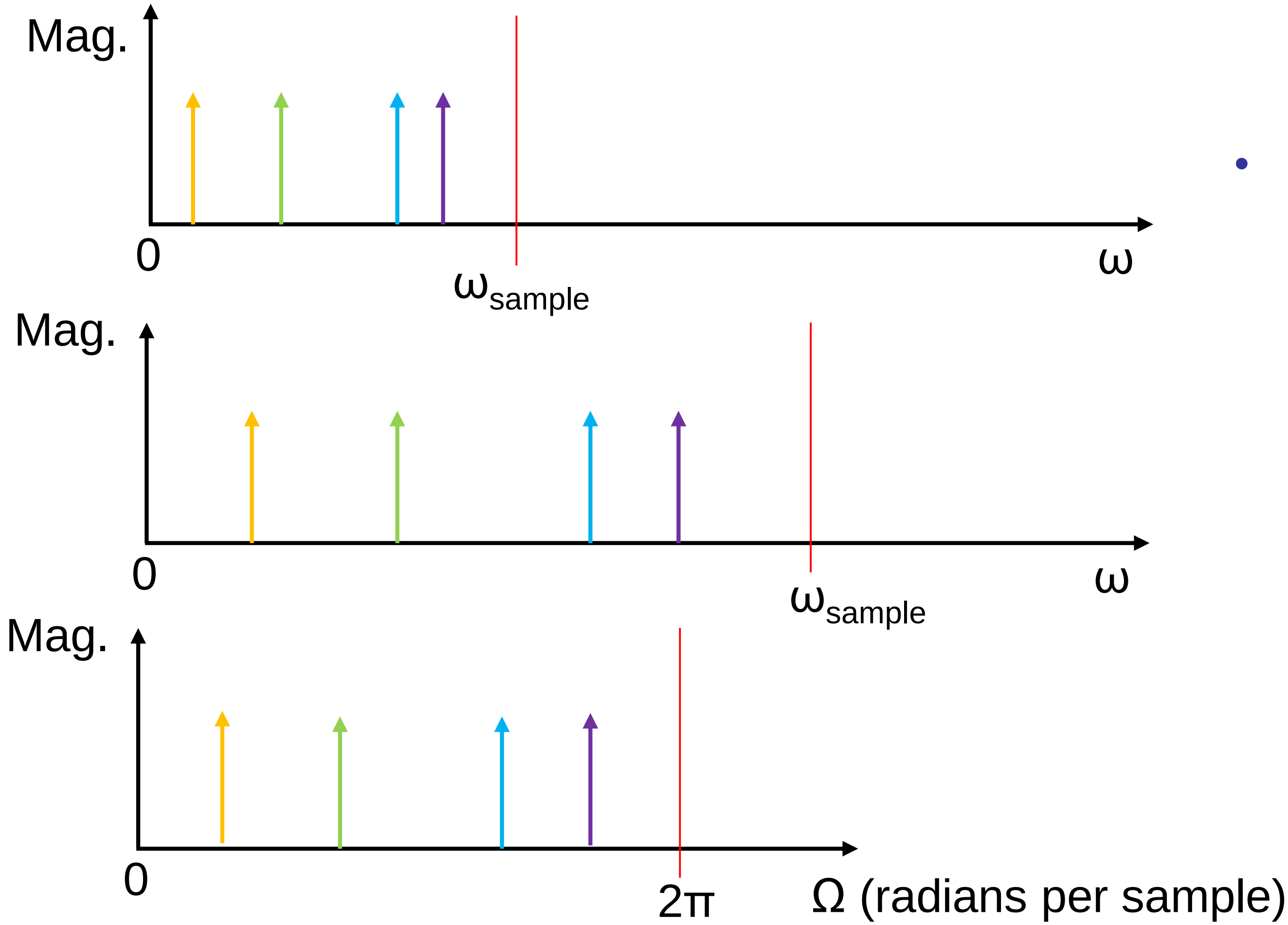
Aliasing illustrated – real-valued signals





- With real valued signals – like most we encounter in real life- it is necessary to sample at twice the maximum frequency
- Again, the upper graph is indistinguishable from the lower once the signal has been sampled.

Aliasing illustrated – normalised frequency



- When we convert to Ω, we normalise frequency to the sample frequency and hence both the upper and middle graphs produce the same graph in the Ω (bottom graph).