ODEs - revision

Summary of methods for ODEs

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Definition of ODEs

- An ODE is an equation involving the derivatives of an unknown function/variable:
  - $\frac{\mathrm{d}x}{\mathrm{d}t} = x + t$
  - $x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathrm{e}^x = 0$
  - $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$
- If the equation involves e.g. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ then y is the *dependent* variable and x is the *independent* variable.
- We typically want to solve the ODE to find the dependent variable as a function of the independent variable (e.g. $y(x)$).
- The order of an ODE is the order of the highest-order derivative in the ODE e.g. the above are 1st order, 2nd order and 2nd order respectively.

Classification of ODEs

Example: $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2t\frac{\mathrm{d}x}{\mathrm{d}t} + x + \sin t$

- First look at the derivatives in the ODE to identify the dependent and independent variables (e.g. x and t)
- Also check the order of the derivatives to find the order of the ODE (e.g. 2nd order).
- Try to rearrange into standard form of linear ODE. If you can it's linear, otherwise nonlinear.
- If it is linear we can distinguish between homogeneous or non-homogeneous (this distinction doesn't apply to nonlinear ODEs).

- Standard form for e.g. 2nd order linear ODE is

$$a(t)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b(t)\frac{\mathrm{d}x}{\mathrm{d}t} + c(t)x = f(t)$$

- Given e.g. $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2t\frac{\mathrm{d}x}{\mathrm{d}t} + x + \sin t$ we need to rearrange to get into standard form.
- All terms involving the dependent variable (e.g. x) go on the left and terms not involving the dependent variable go on the right:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}x}{\mathrm{d}t} - x = \sin t$$

- This is in standard form with $a(t) = 1$, $b(t) = -2t$, $c(t) = -1$ and $f(t) = \sin t$ so it's linear.

Homogeneous or non-homogeneous

- If an ODE is linear we can also say whether it is *homogeneous* or *non-homogeneous*.
- put it into standard form for a linear ODE. If the right hand side $f(t) = 0$ (for all t) then it is homogeneous.
- If $f(t) \neq 0$ (for any t) then it is non-homogeneous.
- Example: $\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} = 2t\frac{\mathrm{d} x}{\mathrm{d} t} + x + \sin t$.
  - Rearrange to get
$$\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} - 2t\frac{\mathrm{d} x}{\mathrm{d} t} - x = \sin t$$
  - So $f(t) = \sin t$ which is not zero for all t.
  - This example is non-homogeneous.

Classification examples

$\frac{\mathrm{d} x}{\mathrm{d} t} = \sin t$: 1st order, linear, non-homogeneous.

$\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} = \sin x$: 2nd order, nonlinear.

$\frac{\mathrm{d}^2 y}{\mathrm{d} x^2} = x\frac{\mathrm{d} y}{\mathrm{d} x}$: 2nd order, linear, homogeneous.

$\frac{\mathrm{d}^3 y}{\mathrm{d} t^3} + y\frac{\mathrm{d}^2 y}{\mathrm{d} t^2} = 1$: 3rd order, nonlinear.

$\frac{1}{x}\frac{\mathrm{d} x}{\mathrm{d} t} = 1$: 1st order, linear, homogeneous.

General and particular solutions

- An ODE on its own will not have a unique solution. There will be a family of solutions.
- Example: $\frac{\mathrm{d} x}{\mathrm{d} t} = 2$ has solutions $x = 2t + 1$ or $x = 2t + 2$ etc.
- The *general solution* is a solution with unknown integration constants representing all possible solutions (e.g. $x = 2t + C$).
- To get a unique solution we also need *initial conditions* (e.g. $x(0) = -1$).
- With the initial condition we get the *particular solution* ($x = 2t - 1$).
- The problem where we have both an ODE and initial conditions is known as an *initial value problem* (IVP).
- An IVP has a unique solution if the number of initial conditions matches the order of the ODE (for well-behaved ODEs).

Direct integration

An ODE of order n in the form
$$\frac{\mathrm{d}^n x}{\mathrm{d} t^n} = f(t)$$
can be solved by direct integration. Simply integrate n times.

Example: $\frac{\mathrm{d}^2 x}{\mathrm{d} t^2} = 3$.

Integrate once: $\frac{\mathrm{d} x}{\mathrm{d} t} = 3t + C$.

Integrate again: $x = \frac{3}{2}t^2 + Ct + D$.

Now use initial conditions to find the constants.

Separation of variables

A 1st order ODE of the form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = f(x)g(t)$$

can often be solved using separation of variables. Divide by $f(x)$ then "multiply" by $\mathrm{d}t$ and integrate

$$\int \frac{1}{f(x)}\,\mathrm{d}\,x = \int g(t)\,\mathrm{d}\,t$$

then rearrange for x.

Example: $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x^2$.

$$\int \frac{1}{x^2}\,\mathrm{d}\,x = \int \mathrm{d}\,t \implies -\frac{1}{x} = t + C \implies x = -\frac{1}{t+C}$$

Integrating factor

Any 1st order, linear, homogeneous ODE can be written in standard form as

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + p(t)x = q(t)$$

and can be solved using the integrating factor method.

- Put in standard form.
- Multiply through by $I = \exp(\int p(t)\,\mathrm{d}\,t)$.
- Write the LHS as $\frac{\mathrm{d}}{\mathrm{d}\,t}(xI)$.
- Integrate both sides wrt t.
- Rearrange for x.

Integrating factor example

Example: $t^2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = t(1-x)$. First rearrange to standard form

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \frac{x}{t} = \frac{1}{t}$$

We have $p(t) = \dfrac{1}{t}$ so $I = \mathrm{e}^{\int p(t)\,\mathrm{d}\,t} = \mathrm{e}^{\ln t} = t$. Multiply through by I to get

$$t\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x = 1$$

Now rewrite the LHS (reverse product rule), integrate and rearrange:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(xt) = 1 \implies xt = t + C \implies x = 1 + \frac{C}{t}$$

Clever substitution

Sometimes it may not appear possible to apply the previous methods to an ODE but a clever substitution gives a new ODE that can be solved. We have one case in particular.
Any ODE of the form $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = f\left(\frac{x}{t}\right)$ can be made separable by the substitution $y = \frac{x}{t}$.

- $x = yt \qquad \implies \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{\mathrm{d}(yt)}{\mathrm{d}\,t} = t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + y.$
- $t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + y = f(y)$
- $\int \frac{\mathrm{d}\,y}{f(y)-y} = \int \frac{\mathrm{d}\,t}{t}.$

Example: $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{x}{t} + \frac{t}{x} \qquad \implies t\frac{\mathrm{d}\,y}{\mathrm{d}\,t} + y = y + \frac{1}{y} \qquad \implies \int y\,\mathrm{d}\,y = \int \frac{\mathrm{d}\,t}{t}$

So we have $\frac{1}{2}y^2 = \ln t + C \qquad \implies y = \pm\sqrt{2\ln t + C}$

Linear, homogeneous ODEs with constant coefficients

A linear, homogeneous ODE with constant coefficients has the standard form (e.g. for 2nd order):

$$a\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + b\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + cx = 0$$

We can always solve this with the *ansatz* $x = e^{\lambda t}$ giving

$$a\lambda^2\,\mathrm{e}^{\lambda t} + b\lambda\,\mathrm{e}^{\lambda t} + c\,\mathrm{e}^{\lambda t} = 0$$

Since $\mathrm{e}^{\lambda t} \neq 0$ we can divide through to get the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

Use quadratic formula to get λ_1 and λ_2 which gives the general solution

$$x = Ax_1 + Bx_2 = A\,\mathrm{e}^{\lambda_1 t} + B\,\mathrm{e}^{\lambda_2 t}$$

Linear, 2nd order, homogeneous, constant coefficients

A linear, 2nd order, homogeneous ODE with constant coefficients has general solution

$$x = A\,\mathrm{e}^{\lambda_1 t} + B\,\mathrm{e}^{\lambda_2 t} \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have 3 cases:

- $b^2 - 4ac > 0$ real roots, easy case.
- $b^2 - 4ac < 0$ complex roots: $\lambda_{1,2} = \alpha \pm j\beta$ and

$$x = \mathrm{e}^{\alpha t}(A\cos\beta t + B\sin\beta t)$$

- $b^2 - 4ac = 0$ degenerate case (repeated roots). $\lambda = -\frac{b}{2a}$ and

$$x = A\,\mathrm{e}^{\lambda t} + Bt\,\mathrm{e}^{\lambda t}$$

Constant coefficient examples

- $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = -kx \implies x = Ae^{-kt}$
- $\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = k^2 x \implies x = A\,\mathrm{e}^{kt} + B\,\mathrm{e}^{-kt}$
- $\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = -\omega^2 x \implies x = A\sin\omega t + B\cos\omega t$
- $\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t} + 2x \implies x = A\,\mathrm{e}^{-t} + B\,\mathrm{e}^{2t}$
- $\frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} + 2\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x = 0 \implies x = (A + Bt)\,\mathrm{e}^{-t}$
- $\frac{\mathrm{d}^3\,x}{\mathrm{d}\,t^3} + \frac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} - \frac{\mathrm{d}\,x}{\mathrm{d}\,t} - 1 = 0 \implies x = (A + Bt)\,\mathrm{e}^{-t} + C\,\mathrm{e}^{t}$

Always two constants for a 2nd order ODE, three for 3rd etc. You don't need to know how to find roots of a cubic but should be able to follow (not solve) the last example here.

Higher-order Linear non-homogeneous ODEs

Integrating factor only works for 1st order non-homogeneous ODEs. For higher order we use complementary function and particular integral.
We want to solve e.g.

$$a\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + b\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + cx = f(t) \tag{1}$$

We write our general solution of (1) as $x = x_c + x_p$ where x_c solves the homogeneous equation

$$a\frac{\mathrm{d}^2 x}{\mathrm{d}\,t^2} + b\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + cx = 0$$

and x_p is any particular solution of (1).

Higher order, non-homogeneous example

Example: $\frac{\mathrm{d}^2 x}{\mathrm{d}\, t^2} + 9x = 5\,\mathrm{e}^t$

- $\frac{\mathrm{d}^2 x_c}{\mathrm{d}\, t^2} + 9x_c = 0 \implies x_c = A\sin 3t + B\cos 3t$
- Try $x_p = C\,\mathrm{e}^t$ and find C:

$$\frac{\mathrm{d}^2 x_p}{\mathrm{d}\, t^2} + 9x_p = C\,\mathrm{e}^t + 9C\,\mathrm{e}^t = 10C\,\mathrm{e}^t = 5\,\mathrm{e}^t$$

- Works if $C = \frac{1}{2}$
- $$x = x_c + x_p = A\sin 3t + B\cos 3t + \frac{1}{2}\,\mathrm{e}^t$$

(Note A and B are constants of integration to be determined from initial conditions. C is a parameter whose value is fixed by the ODE for x_p.)

Particular integral

We need to guess the particular integral depending on $f(t)$ so

- $f(t)$ is a polynomial e.g. $f(t) = t^2$
  Try $x_p = Ct^2 + Dt + E$.
- $f(t)$ is trigonometric e.g. $f(t) = \cos 2t$
  Try: $x_p = C\cos 2t + D\sin 2t$.
- $f(t)$ is exponential: x_p should be exponential
- $f(t)$ is a sum of different things: x_p should be a similar sum
- Corner case $f(t)$ has terms in common with x_c: multiply them by t.

It doesn't matter how you find the particular inegral x_p so nothing wrong with guessing.

1st order (state-space) form

Higher order ODEs can be written as systems of 1st-order ODEs e.g.:

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}\, t^2} = -4\sin\theta$$

If we introduce a new variable $\alpha = \frac{\mathrm{d}\,\theta}{\mathrm{d}\, t}$ then we can rewrite this as

$$\frac{\mathrm{d}\,\theta}{\mathrm{d}\, t} = \alpha$$
$$\frac{\mathrm{d}\,\alpha}{\mathrm{d}\, t} = -4\sin\theta$$

All systems of ODEs of *any* order can be rewritten as systems of 1st order ODEs.

State-space for linear, homogeneous ODEs with constant coefficients

Linear, homogeneous ODEs with constant coefficients give a state-space form that can be written as

$$\frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\, t} = \mathbf{M}\,\mathbf{x}$$

with $\mathbf{x}$ a vector of dependent variables and $\mathbf{M}$ a constant matrix.
The general solution (assuming $\mathbf{M}$ is not defective) is given by

$$\mathbf{x} = A_1\mathbf{v}_1\mathrm{e}^{\lambda_1 t} + A_2\mathbf{v}_2\mathrm{e}^{\lambda_2 t} + \ldots$$

where $\mathbf{v}_i$ is an eigenvector of $\mathbf{M}$ with eigenvalue λ_i. The A_i are the constants of integration.

Matrix ODEs example

Example: $\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = x + 2y \quad \frac{\mathrm{d}\,y}{\mathrm{d}\,t} = 2x + y$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Matrix $\mathbf{M}$ has eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ so the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} -1 \\ 1 \end{pmatrix}\mathrm{e}^{-t} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathrm{e}^{3t}$$

So we have

$$x = -A\,\mathrm{e}^{-t} + B\,\mathrm{e}^{3t}, \qquad y = A\,\mathrm{e}^{-t} + B\,\mathrm{e}^{3t}$$