Vibrations 2, Lecture 7
Forced vibration under harmonic excitation

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Lecture 6

General response of 1 DOF system

$$x = x_H + x_P$$

$$x = X\,e^{-\zeta\omega_0 t}\sin(\omega_D t + \phi) + x_{forced}$$

Response to $F(t)=F_0=\text{const.}$

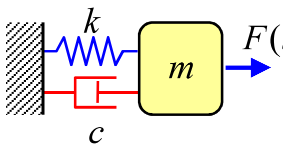
$$x = x_H + x_P$$

$$x = X\,e^{-\zeta\omega_0 t}\sin(\omega_D t + \phi) + F_0/k$$

Lecture 7

- Forced vibration under harmonic excitation
  - Transient response
  - Steady-state response
- Frequency Response Function
  - Magnitude of the FRF
  - Phase angle of the FRF
- Solved example

1 DOF system with harmonic force



$$F(t) = F_0 \sin(\omega t)$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

- find the *particular solution* x_P by "guessing" the trial solution
- the trial solution is $x_P = X_0 \sin(\omega t - \varphi)$, X_0 is the amplitude, φ is the phase angle
- use x_P in the EOM

The total solution is:

$$\boxed{x = x_H + x_P = e^{-\zeta\omega_0 t}(X_1 \sin(\omega_D t) + X_2 \cos(\omega_D t)) + X \sin(\omega t - \varphi)}$$

Response contains the two frequency components, ω_D and ω.

Total response to harmonic excitation

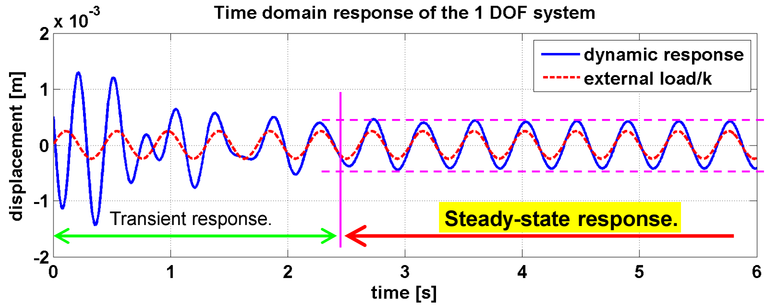
1 DOF damped system with harmonic force:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

$$x(t) = e^{-\zeta\omega_0 t}(X_1 \sin(\omega_D t) + X_2 \cos(\omega_D t)) + X \sin(\omega t - \varphi)$$

Component due to "natural properties" and initial conditions

Component due to forcing function (sin,X,ω,φ)



Steady-state harmonic response

1 DOF damped system with harmonic force:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

$$x(t) \approx x_P(t) = X \sin(\omega t - \varphi), \quad \underline{X, \varphi = ?}$$

Consider only the steady-state solution component:

$$x(t) = X \sin(\omega t - \varphi)$$

$$\dot{x}(t) = \omega X \cos(\omega t - \varphi) = \omega X \sin(\omega t - \varphi + \pi/2)$$

$$\ddot{x}(t) = -\omega^2 X \sin(\omega t - \varphi) = \omega^2 X \sin(\omega t - \varphi + \pi)$$

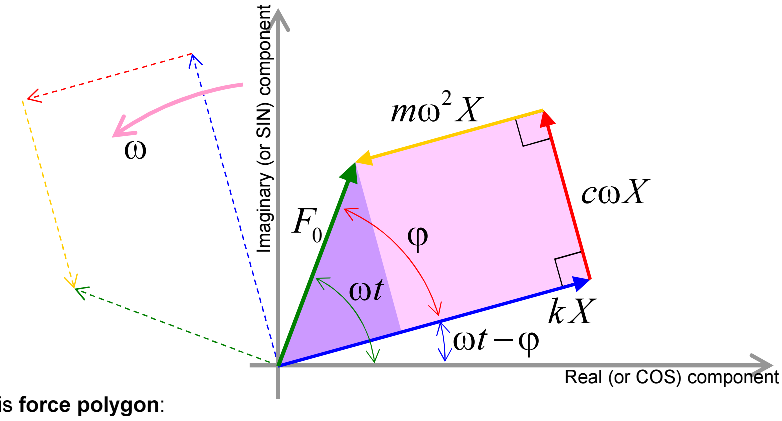
Using EOM we can write the following equation:

$$m\omega^2 X \sin(\omega t - \varphi + \pi) + c\omega X \sin(\omega t - \varphi + \pi/2) + kX \sin(\omega t - \varphi) = F_0 \sin(\omega t)$$

We will look closely at the geometric interpretation of this equation.

Steady-state harmonic response

$$m\omega^2 X\sin(\omega t-\varphi+\pi)+c\,\omega X\sin(\omega t-\varphi+\pi/2)+k\,X\sin(\omega t-\varphi)=F_0\sin(\omega t)$$

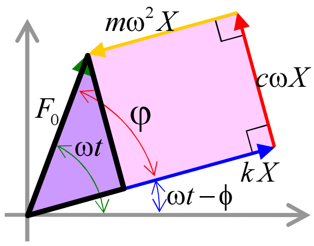


This **force polygon**:

- represents the steady-state dynamic equilibrium under harmonic excitation,
- retains its shape and rotates anti-clockwise with the angular speed ω.

Steady-state harmonic response

Considering the geometry of the force polygon we can write the following relationships for the highlighted triangle (notice the *right-angle triangle*, use the Pythagorean theorem):



$$\tan\varphi=\frac{c\,\omega}{k-m\omega^2}$$

$$F_0^2=(kX-m\omega^2 X)^2+(c\omega X)^2$$

$$F_0=X\sqrt{(k-m\omega^2)^2+(c\omega)^2}$$

Steady-state response – direct solution:

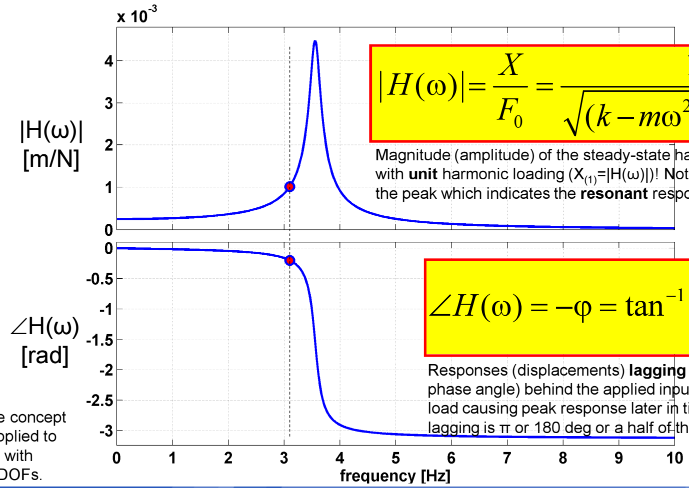
Harmonic excitation $F_0\sin(\omega t)$

$$\boxed{X=\frac{1}{\sqrt{(k-m\,\omega^2)^2+(c\,\omega)^2}}F_0 \qquad \tan\varphi=\frac{c\,\omega}{k-m\omega^2}}$$

Steady-state response $X\sin(\omega t-\varphi)$

Frequency Response Function (FRF)

We characterize the steady-state behavior of systems using the **Frequency Response Function** = output / input. This is a complex function of ω. The ***magnitude*** of FRF is |H(ω)| and the ***phase angle*** of FRF is ∠H(ω).



$$|H(\omega)| = \frac{X}{F_0} = \frac{1}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

Magnitude (amplitude) of the steady-state harmonic response with **unit** harmonic loading ($X_{(1)}=|H(\omega)|$)! Note the presence of the peak which indicates the **resonant** response.

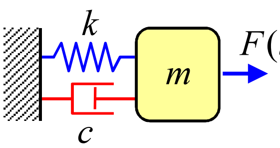
$$\angle H(\omega) = -\varphi = \tan^{-1}\left(\frac{c\,\omega}{k-m\omega^2}\right)$$

Responses (displacements) **lagging** (note the sign of the phase angle) behind the applied inputs (forces), e.g. peak load causing peak response later in time. Maximum lagging is π or 180 deg or a half of the cycle.

The same concept can be applied to problems with **angular** DOFs.

Example: steady state harmonic response

Find the steady-state response $x(t)=X_0\sin(\omega t-\varphi)$ of the system with m=8 kg, c=10 Ns/m, k=4000 N/m and $F(t)=F_0\sin(\omega t)$, where F_0=10 N, f=3.1 Hz.



$$F(t) = F_0\sin(\omega t)$$

$$|H(\omega)| = \left((k-m\omega^2)^2 + (c\omega)^2\right)^{-1/2} = \ldots = 1.016\times10^{-3}\,(kg.s^{-2})^{-1}$$

$$\varphi = \tan^{-1}(c\omega/(k-m\omega^2)) = \ldots = 0.20\,rad$$

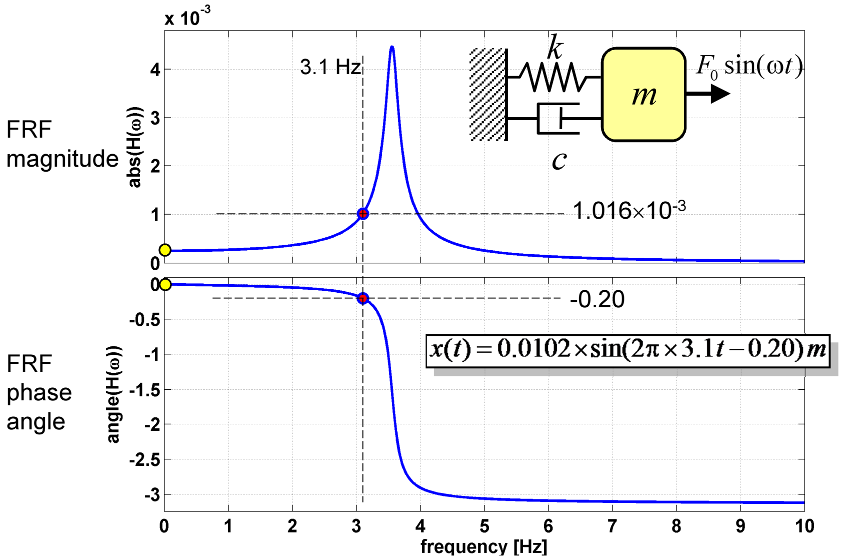
$$X = |H(\omega)|F_0 = 1.016\,(kg.s^{-2})^{-1}\times10\,(kg.m.s^{-2}) = 1.02\,cm$$

$$x(t) = X\sin(\omega t-\varphi) = \boxed{0.0102\times\sin(2\pi\times3.1t-0.20)\,m}$$

$$x_{static} = F_0/k = 2.5\,mm \Rightarrow \boxed{X/x_{static} = 4.1}$$

Ratio of the steady-state and static responses!

Example



Summary

- Steady-state response is determined by the excitation
- FRF characterizes the steady-state harmonic response with harmonic excitation
- FRF can be represented by the FRF magnitude and FRF phase angle