Sampling

- Signal Types
- Why sampling
- Sampling Theory
- Nyquist-shannon theorem
- Periodic Sampling
- Aliassing
- Anti-aliassing
- Practical implications

Harry Nyquist and Claude Elwood Shannon

Harry Nyquist
(1889-1976)

Claude Elwood Shannon
(1916 – 2001)

Signal Types

- Analog signals: continuous in time and amplitude
  - Example: voltage, current, temperature etc.
- Digital signals: discrete both in time and amplitude
- Discrete-time signal: discrete in time, continuous in amplitude

Why Sampling?

Analogue signals, in general, are continuous in time.

- For continuous signals to be processed by digital computers, the following two steps are needed:

(1) Convert the analogue signal to a digital signal by sampling.

(2) Process the digital signal.

Nyquist–Shannon sampling theorem

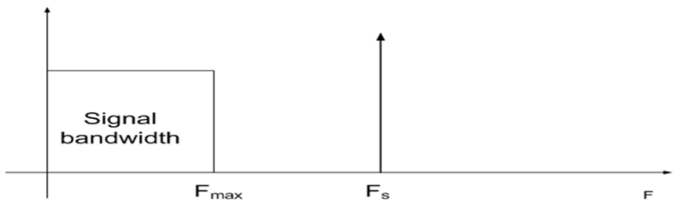
- **Sampling Theorem:**

"When sampling a band-limited signal (e.g., converting from an analog signal to digital), the sampling frequency must be greater than twice the input signal bandwidth *in order to be able to reconstruct the original perfectly from the sampled version"*

i.e., $\mathbf{F_s > 2 \cdot F_{max}}$

Reference:
Claude E. Shannon in 1949 ("*Communication in the presence of noise*")

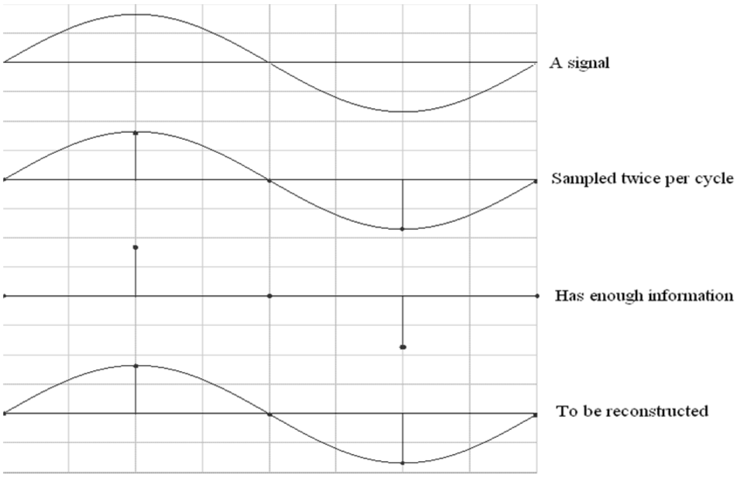
Sampling theorem Nyquist-shannon theorem



- The frequency $\mathbf{2 \cdot F_{max}}$ is called the Nyquist sampling rate.
- And $\mathbf{F_{max}}$, is sometimes called the Nyquist frequency.

- The sampling theorem is considered to have been articulated by Nyquist in 1928 and mathematically proven by Shannon in 1949.
- The term "Nyquist Sampling Theorem" and "Shannon Sampling Theorem" are in fact the same sampling theorem.

Sampling theorem Nyquist-shannon theorem (Cnt)



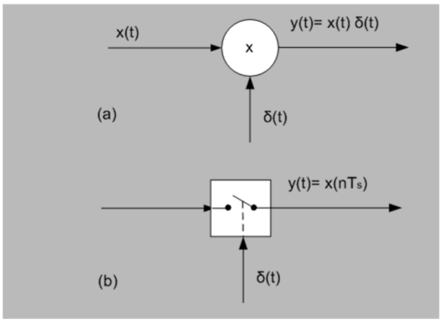
Periodic sampling

- Sampling is a continuous to discrete-time conversion
- Most common sampling is periodic

$$x[n]=x_c(nT) \quad -\infty<n<\infty$$

- T is the sampling period in second.
- F_s = 1/T is the sampling frequency in Hz.
- Sampling frequency in radian-per-second $\Omega_s=2\pi F_s$ rad/sec.
- Use [ ] for discrete-time and ( ) for continuous time signals.

How can we sample?



Principle of sampling, (a) Multiplication and (b) Switching

SAMPLING

- Sampling is treated as a multiplication of the input analog signal with a periodic Delta, Dirac, or impulse function.
- So lets have a look in details at the impulse function.

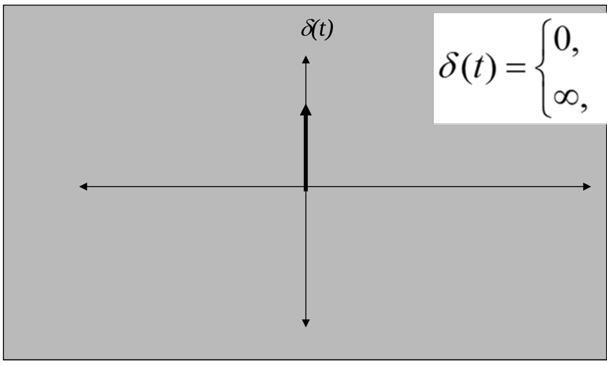
IMPULSE FUNCTION

- The impulse function is represented by δ(t) and is defined for every value of t with the following expression:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

IMPULSE FUNCTION

- The graphical representation of the delta function, δ(t) is given below

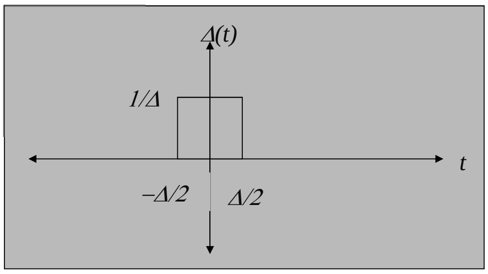


$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

IMPULSE FUNCTION

- To obtain some insight into the δ function, we can consider the function $\Delta(t)$ as shown below.

$$\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & -\dfrac{\Delta}{2} \le t \le \dfrac{\Delta}{2} \\ 0, & otherwise \end{cases}$$



- The function has a height of $1/\Delta$ and a width of Δ, resulting in an area of 1.

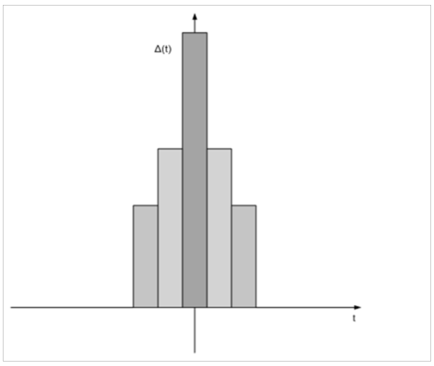
$$\int_{-\infty}^{\infty} \Delta(t)\,dt = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} dt = \frac{1}{\Delta}\left[\frac{\Delta}{2} - \left(-\frac{\Delta}{2}\right)\right] = 1$$

IMPULSE FUNCTION

- The δ function is obtained from $\Delta(t)$ by the limiting process

$$\delta(t) = \lim_{\Delta \to 0} \Delta(t)$$

- The width of function goes to zero as its amplitude goes to infinity.
- The area under the curve to remain constant at unity.



IMPUSE FUNCTION

- As Δ decreases in value, the function becomes narrower and higher. In the limit, as $\Delta \rightarrow 0$, $\Delta t \rightarrow \delta(t)$.
- The integral of $\Delta(t)$ is unity for any value of Δ. It follows that the integral of the δ function is also unity provided that the origin is included in the limits of integration. Otherwise, it is zero.

$$\int_a^b \delta(t)dt = \begin{cases} 1, & a \le 0 \le b \\ 0, & otherwise \end{cases}$$

IMPULSE FUNCTION

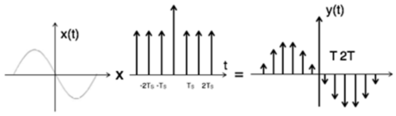
- If we multiply the δ function by a constant, k, and then integrate the product over all time, we get:

$$\int_{-\infty}^{\infty} k\delta(t)dt = k\int_{-\infty}^{\infty} \delta(t)dt = k$$

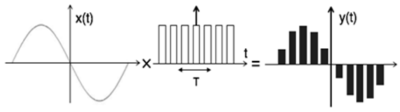
- Multiplication of $\delta(t)$ by a constant is equivalent to choosing $\Delta(t)$ with a height of k/Δ. This results in an area equal to k.

Sampling Functions

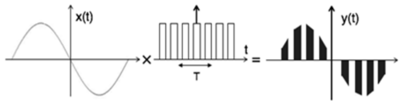
1. ideal impulse



2. flat-topped pulses



3. rectangular pulses



TIME-SHIFTED FUNCTIONS

- If the argument of the δ function is replaced with t-τ, we see that the impulse still occurs when the argument takes on the value of zero (i.e., when t - τ = 0 or t = τ).

$$\delta(t-\tau) = \begin{cases} 0, & t-\tau \neq 0 \\ \infty, & t-\tau = 0 \end{cases}$$

TIME-SIFTING PROPERTY

- If we integrate the product of the δ function and any time function, we can observe the sifting property of δ function

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \int_{-\infty}^{\infty} f(0)\delta(t)dt = f(0)\int_{-\infty}^{\infty} \delta(t)dt$$

$$= f(0).$$

TIME-SIFTING PROPERTY

- So all the values of f(t) are sifted out except the value at the origin, f(0).

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

TIME-SIFTING PROPERTY

- If we use the time-shifted δ function, we obtain:

$$f(\tau) = \int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt$$

SAMPLING

- We can describe our sampling function, s(t), as the sum of all the individual impulse functions:

$$s(t) = \delta(t-\infty) + \ldots + \delta(t-2T_s) + \delta(t-T_s) + \delta(t)$$
$$+\delta(t+T_s) + \delta(t+2T_s) + \ldots + \delta(t+\infty)$$

- Shorter version of the previous expression is:

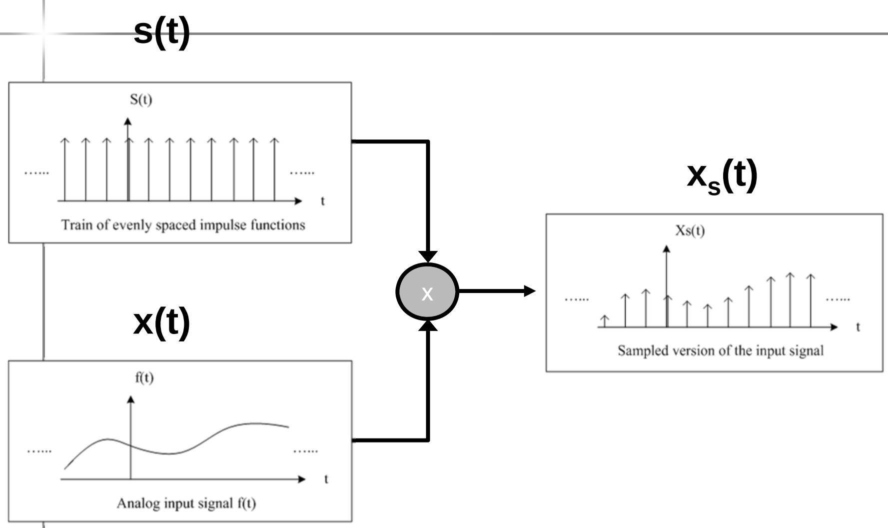
$$s(t) = \sum_{k=\infty}^{k=-\infty} \delta(t-kT_s)$$

SAMPLING

- If we multiply these by our analog input signal, x(t), we obtain a train of pulses whose amplitudes are equal to the amplitude of x(t) at that moment in time.
- Mathematically, the output sampled waveform, $x_s(t)$, is just the multiplication of s(t) with the input analog signal x(t):

$$x_s(t) = \sum_{k=\infty}^{k=-\infty} x(t)\delta\left(t - kT_s\right)$$

SAMPLING



SAMPLING

- Using the representation of the sampled version of CT signal x(t)

$$x_s(t) = \sum_{k=\infty}^{k=-\infty} x(t)\delta\left(t - kT_s\right)$$

- $x(t)=x(kT_s)$ at the moment $t=kT_s$, therefore:

$$x_s(t) = \sum_{k=\infty}^{k=-\infty} x(kT_s)\delta\left(t - kT_s\right)$$

SAMPLING

- By assuming the kTs =k we can write:

$$x_s(t) = \sum_{k=\infty}^{k=-\infty} x(k)\delta\left(t - k\right)$$

- Actually, the above equation is not correct, t represents time and k is just an integer
- We have already mentioned that the sampled signal $x_s(t)$ exists only as an ordered sequence of numbers – samples
- It is common to use index n instead of time t to indicate this fact.

SAMPLING

- The equation for the sampled signal therefore turns into:

$$x_s(t) = x(n) = \sum_{k=\infty}^{k=-\infty} x(k)\delta\left(n-k\right)$$

No need for index s here, as x(n) already assumes sampled signal

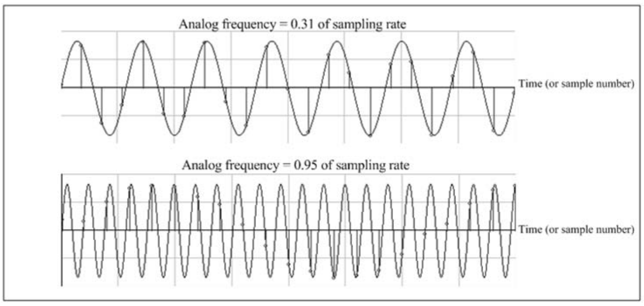
SAMPLING

- If we use index n instead of (continuous) time t in the previous equation, the Impulse function $\delta(t)$ mutates into it's discrete version $\delta(n)$.
- This is usually called unit impulse and is defined as:

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$$

2 Aliasing

- if a signal contains frequencies above the Nyquist frequency.
  - Loss of information
  - Introduces wrong information
  - Loss of phase information (phase shift)



Mathematical Explanation of aliasing

A continuous-time sinusoidal signal can be express as:

$$x(t) = \sin(2\pi ft + \phi)$$

Sampling x(t) (t = nT) will result in discrete-time sequence x[n] as:

$$x[n] = \sin(2\pi fT + \phi)$$

Exploiting the periodicity of the sine wave, we can write:

$$x[n] = \sin(2\pi fTn \pm 2\pi m + \phi) = \sin\left(2\pi T\left(f \pm \frac{m}{Tn}\right)n + \phi\right)$$

Replacing m/n by k will lead to:

$$x[n] = \sin\left[2\pi T\left(f \pm kF_s\right)n + \phi\right]$$

These equation shows that we cannot distinguish between a signal with a frequency f and a signal with a frequency (f +/- kF_s)

Can aliasing be avoided?

- The answer is yes and no!
- Yes, by using a Low-pass analog filter (anti-aliasing filter) before sampling.
- No, in practical situation you can only reduce aliasing because of the characteristic of the anti-aliasing filter (no filter is perfect !).

End of Chapter