Autumn Semester 2017

EMAT10100 Engineering Maths I
Lecture 20: Rules for differentiation

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Looking back looking forward

- Last week
  - Abstract definition of continuity and differentiation
  - Higher derivatives and determination of Max, Min & Inflection
  - Tips for graph sketching (more today on rational functions)
  - Introduction to the software Maple
  - Useful as a way of graph plotting and doing maths throughout your Univeristy career & beyond
- This time
  - Rules for differentiation (this lecture) — revision for most
  - Parametric and implicit differentiation
  - Taylor series and L'Hôpital's rule (Lecture 21).

Some rules for differentiation

Suppose $f(x)$, $g(x)$ are smooth functions,

- The product rule

$$\frac{\mathrm{d}}{\mathrm{d}\,x}[f(x)g(x)] = f(x)\frac{\mathrm{d}\,g(x)}{\mathrm{d}\,x} + g(x)\frac{\mathrm{d}\,f(x)}{\mathrm{d}\,x}$$

- The quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \qquad g \neq 0$$

- The chain rule (function of a function)

$$\frac{\mathrm{d}}{\mathrm{d}\,x}f(g(x)) = g'(x)f'(y)\Big|_{y=g(x)}$$

- Example: Note how the quotient rule can be derived from the product rule (by application of the chain rule).

Exercises

- Find the general expression for the derivatives of
  1. $e^{3x^2-2x}\sin(x)$
  2. $[\ln(x)]^2$
  3. $\ln(x^2)$
  4. $\frac{\cos 2x}{x}$
- Note, it's often easier to use product than quotient rule

Rational functions

- A rational function is one that can be expressed as the ratio of two polynomials $f(x) = p(x)/q(x)$
- Defined for all real x, except points x_p where $q(x_p) = 0$
- Such x_p are called singularities or poles and lead to vertical asymptotes in the graph
- The zeros x_0 of the rational function are given by $p(x_0) = 0$, provided also $q(x_0) \neq 0$ (see next lecture)
- Using the quotient rule, the stationary points x^* of the rational function are given by

$$\frac{q(x^*)p'(x^*) - p(x^*)q'(x^*)}{[q(x^*)]^2} = 0, \Rightarrow q(x^*)p'(x^*) - p(x^*)q'(x^*) = 0,$$

provided $q(x^*) \neq 0$

Tips for sketching rational functions

- Piece together the evidence:
  - Find zeros
  - Find stationary points
  - Find out what happens as $x \to \pm\infty$ (horizontal asymptotes)
  - Find singularities $x = x_p$ (vertical asymptotes)
  - Let $x = x_p \pm \Delta x$ to find behaviour near singularities
- Sketch the function

$$f(x) = \frac{x+1}{-x^2 + 5x - 6}.$$

Parametric differentiation

- Find $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ when $y = y(t)$, $x = x(t)$:

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,t}\frac{\mathrm{d}\,t}{\mathrm{d}\,x} = \frac{\frac{\mathrm{d}\,y}{\mathrm{d}\,t}}{\frac{\mathrm{d}\,x}{\mathrm{d}\,t}} = \frac{\dot{y}}{\dot{x}}$$

- Exercise: An expression for a point on a circle of unit radius can be written in terms of a parametric representation

$$x = \cos(t), \quad y = \sin(t), \quad t \in [0, 2\pi]$$

Find an expression for the gradient of the tangent to the circle $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ in terms of the parameter t.

Implicit differentiation

- You are given an expression $F(x, y) = 0$. By considering $y(x)$, find $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ using the chain rule.
- Best illustrated by way of example:
  example find $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ for the unit circle $x^2 + y^2 = 1$

$$F(x,y) = x^2 + y^2 - 1 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\,F(x,y)}{\mathrm{d}\,x} = 0$$
$$\Rightarrow \quad 2x + 2y\frac{\mathrm{d}\,y}{\mathrm{d}\,x} - 0 = 0$$
$$\Rightarrow \quad \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = -\frac{x}{y}$$

- Exercise: Find $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ when $x^3 + 2xy + y^2 + 2 = 0$

Lecture 21: Taylor series

- A way to approximate (very) smooth functions (for which derivatives up to high orders exist and are continuous).
- Suppose everything known about $f(x)$ at $x = x_0$.
- Let $x - x_0 = \Delta x$, then

$$\begin{aligned} f(x_0 + \Delta x) &\approx f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2 + \frac{f'''(x_0)}{6}(\Delta x)^3 + \dots \\ &\approx f(x_0) + \sum_{n=1}^{N} \frac{f^{(n)}(x_0)}{n!}(\Delta x)^n + \dots \end{aligned}$$

where $f^{(n)}$ is the n^{th} derivative and $n! = n \times (n-1) \times \dots \times 3 \times 2 \times 1$.

- This is called the Taylor series approximation of f
- Approximation gets better as we include more terms
- in general only valid for "small" Δx (see Week 22).

Maclaurin series

- Special case of Taylor series with $x_0 = 0$. Then $\Delta x \mapsto x$.

$$f(x) \approx f(0) + \sum_{n=1}^{N} \frac{f^{(n)}(0)}{n!} x^n + \dots$$

- Example Compute the Maclaurin series of $f(x) = \sin(x)$
- We have

$$\begin{aligned} f(x) &= \sin(x) &\Rightarrow\quad f(0) &= 0 \\ f'(x) &= \cos(x) &\Rightarrow\quad f'(0) &= 1 \\ f''(x) &= -\sin(x) &\Rightarrow\quad f''(0) &= 0 \\ f'''(x) &= -\cos(x) &\Rightarrow\quad f'''(0) &= -1 \end{aligned}$$

- Hence $\sin(x) = x - \frac{1}{6}x^3 +$ h.o.t. ("higher-order terms")

Examples of Maclaurin series

- Common examples are given on the formula sheet, e.g.

1. $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + \dots$
2. $\sin(x) \approx x - \frac{x^3}{6} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$
3. $\cos(x) \approx 1 - \frac{x^2}{2} + \dots + (-1)^{n+1} \frac{x^{2n}}{(2n)!} + \dots$
4. $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \dots + \frac{(-1)^{n+1}x^n}{n} + \dots \quad (-1 < x \leq 1)$

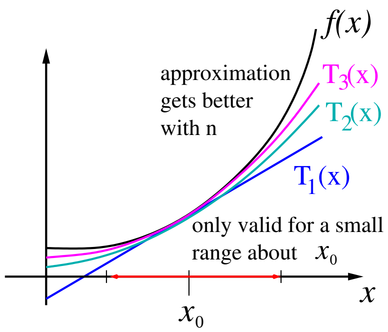
- Exercise: derive 1. and 4. above.

Taylor series — justification

Consider the Maclaurin case for simplicity:

- Let $T_N(x)$ be order-N Maclaurin series approx. to $f(x)$:
  $T_N(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(N)}(0)$
- Hence $T_N(0) = f(0)$,
- and $T_N'(0) = (f'(0) + xf''(0) + \frac{x^2}{2} f'''(0) + \dots)|_{x=0}$,
  hence $T_N'(0) = f'(0)$,
- and $T_N''(0) = (f''(0) + xf'''(0) + \frac{x^2}{2} f'''' + \dots)|_{x=0}$,
  hence $T_N''(0) = f''(0)$,
- etc.
- Hence T_N is the (unique) Nth-order polynomial whose first N derivatives match those of $f(x)$ at $x = 0$

Taylor series — graphical interpretation



Exercise: Compute the first five terms of the Taylor series approximation to $f(x) = \frac{1}{x}$ about the point $x = 1$.

l'Hôpital's rule: An application of Taylor series

- Q. How to evaluate $\lim_{x\to 0} \frac{f(x)}{g(x)}$, if $f(0) = g(0) = 0$?

  e.g. what is $\lim_{x\to 0} \frac{\sin(x)}{x}$?
- A1. Draw a graph of both functions
- A2. Use Taylor (or Maclaurin) series,

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{x - x^3/6 + \dots}{x}$$
$$= \lim_{x\to 0} (1 - x^2/6 + \dots) = 1$$

- A3. Use generalisation of this called l'Hôpital's rule

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{xf'(0) + \dots}{xg'(0) + \dots} = \frac{f'(0)}{g'(0)}$$

l'Hôpital's rule

- Note this is easily generalisable to

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{when} \quad f(a) = g(a) = 0 \quad \text{is} \quad \frac{f'(a)}{g'(a)}$$

- Exercises evaluate the following limits (if they exist!)
  1. $\lim_{x\to 1} \frac{\sin(\pi x)}{2-2x}$
  2. $\lim_{x\to \pi/2} \frac{2x-\pi}{1-\sin(x)}$
  3. $\lim_{x\to \infty} \frac{e^{-x}}{1/x}$
  4. $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$
     (have to differentiate more than once if you get $0/0$)

Homework - for the whole week

- Read the whole of James Sec. 8.3 & Secs. 9.4.1–9.4.3
- Attempt the following exercises (4th edition)
  - Rules for differentiation: Sec. 8.3.8 Qs.25 & 31, Sec. 8.3.11 Q.32, Sec. 8.3.13 Q 35, Sec. 8.3.15 Qs. 44–47, 51
  - Taylor and l'Hôpital: Sec. 9.4.4 Qns. 11–13, 16, Sec. 9.4.4 Qn. 19
- Attempt the following exercises (5th edition)
  - Rules for differentiation: Sec. 8.3.8 Qs.27 & 32, Sec. 8.3.11 Q.34, Sec. 8.3.13 Q 38 Sec. 8.3.15 Qs. 47–49, 56
  - Taylor and l'Hôpital: Sec. 9.4.4 Qns. 11–13, 16, Sec. 9.4.4 Qn. 19
- Remember:
  - Homework: to be handed in in Weds lecture (will accept now also)
  - Marked scripts will be handed back before end of term